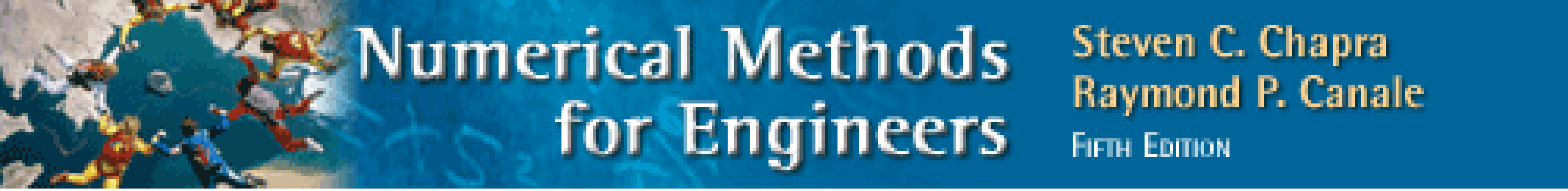


Chapter 5

Roots of Equations Part 2:
Bracketing Methods

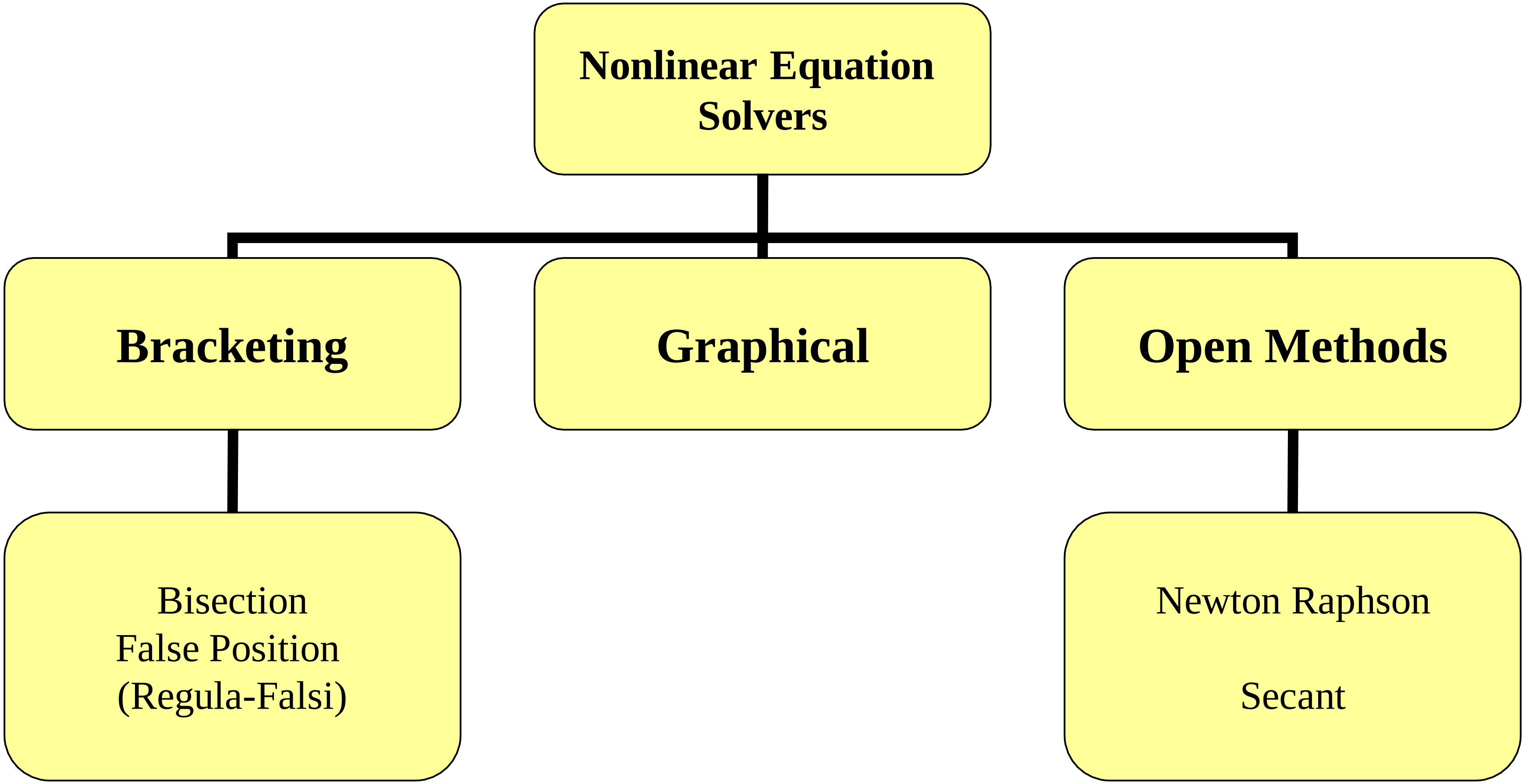
- Why?

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

- But

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \quad \Rightarrow x = ?$$

$$\sin x + x = 0 \quad \Rightarrow x = ?$$



All Iterative

Bracketing Methods

(Or, two point methods for finding roots)

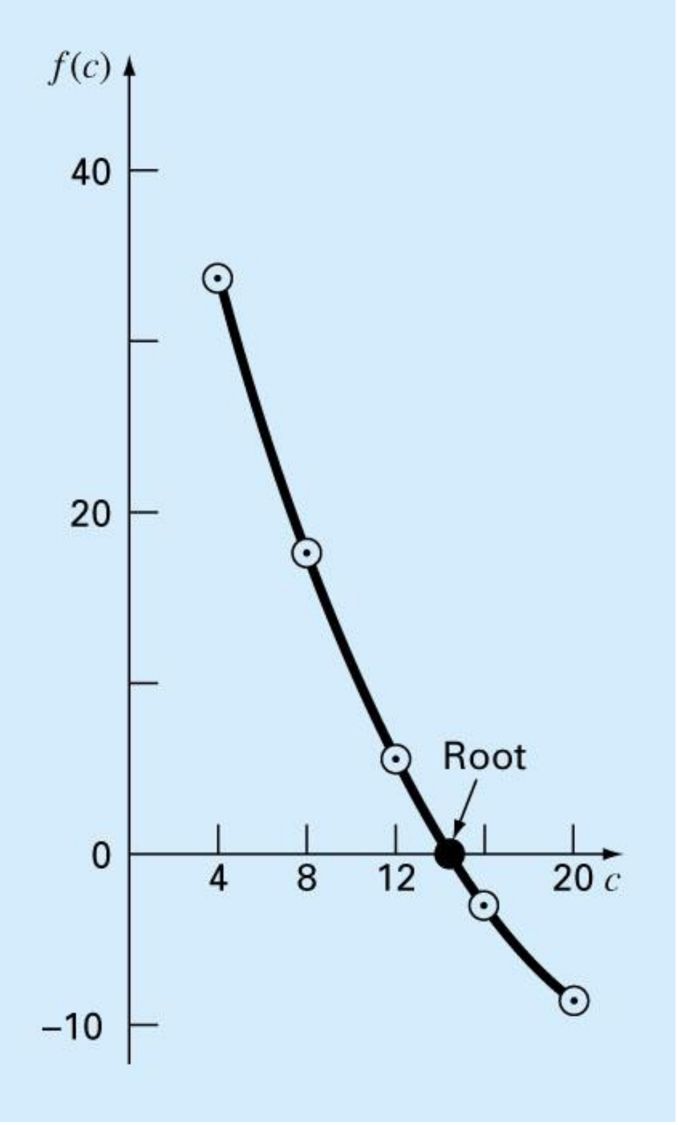
- Two initial guesses for the root are required. These guesses must "bracket" or be on either side of the root.

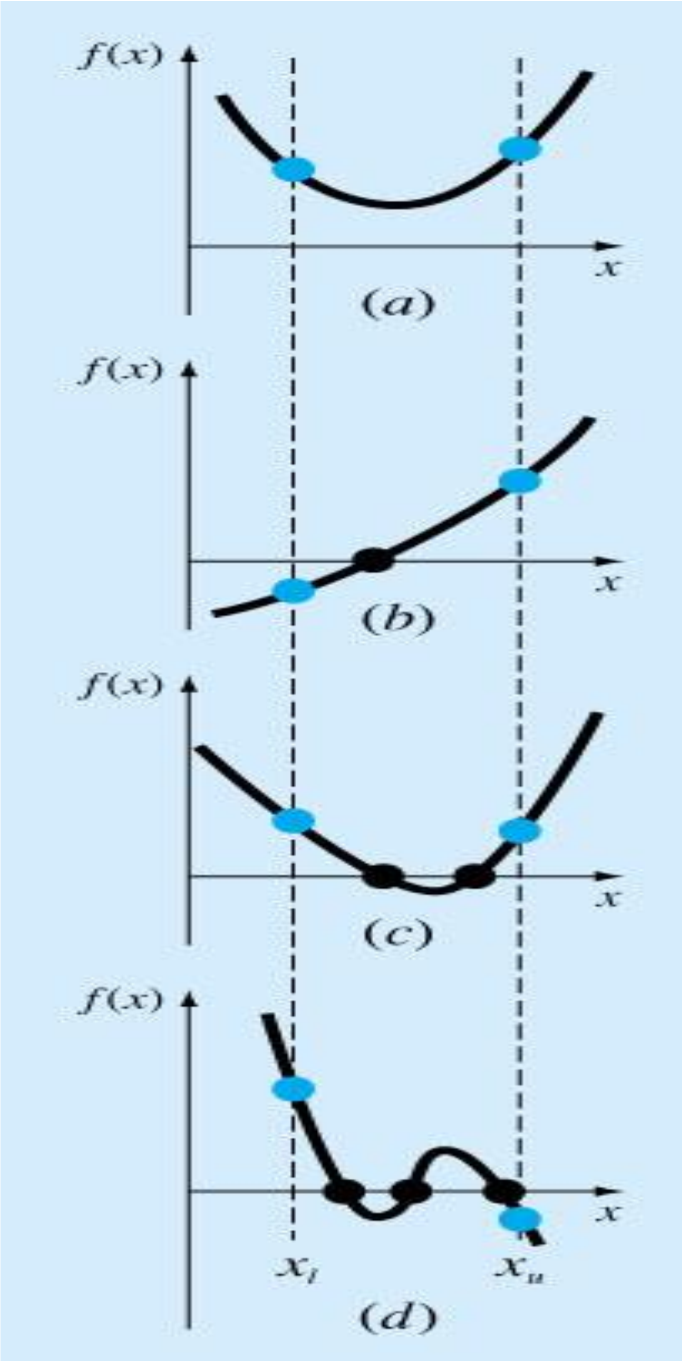
== > From The Figure

- If one root of a real and continuous function, $f(x)=0$, is bounded by values $x = x_l$, $x = x_u$ then

  $f(x_l) \cdot f(x_u) < 0$.

(The function changes sign on opposite sides of the root)



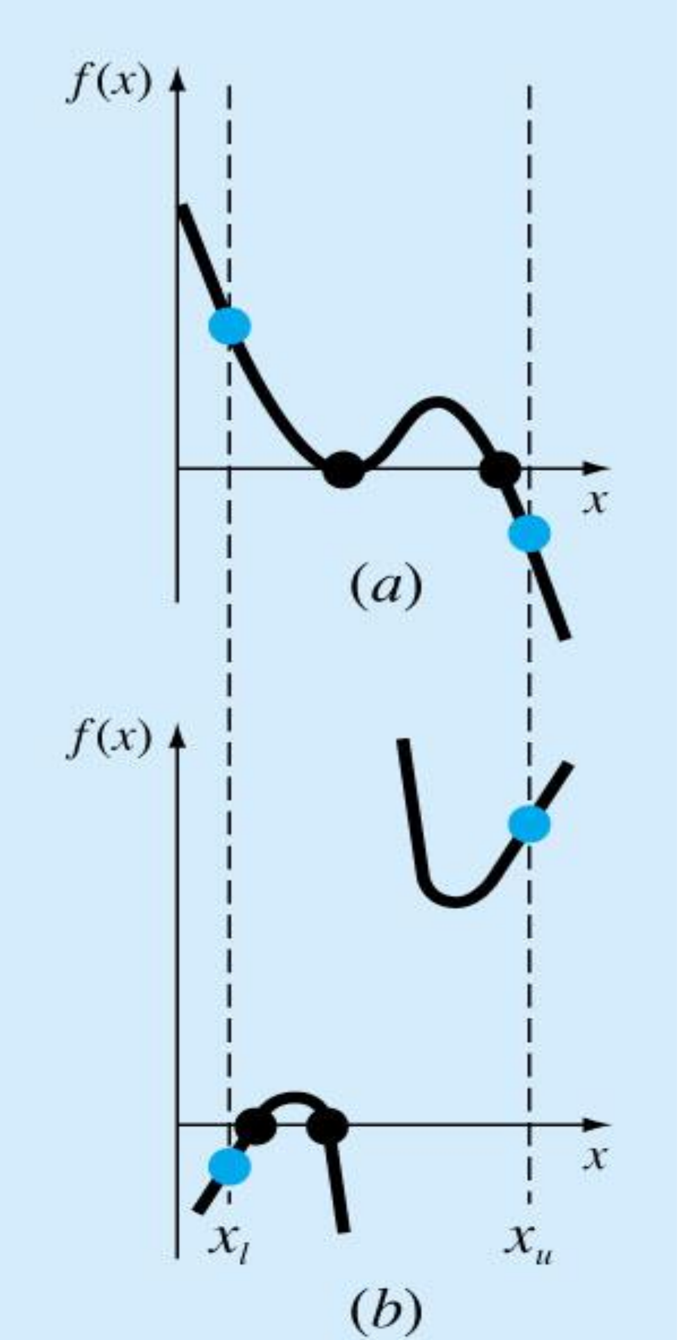


No answer (No root)

Nice case (one root)

Oops!! (two roots!!)

Three roots(Might work for a while!!)



Two roots(Might work for a while!!)

Discontinuous function. Need special method

MANY-MANY roots. What do we do?

Figure 5.4a

$$f(x)=\sin 10x+\cos 3x$$

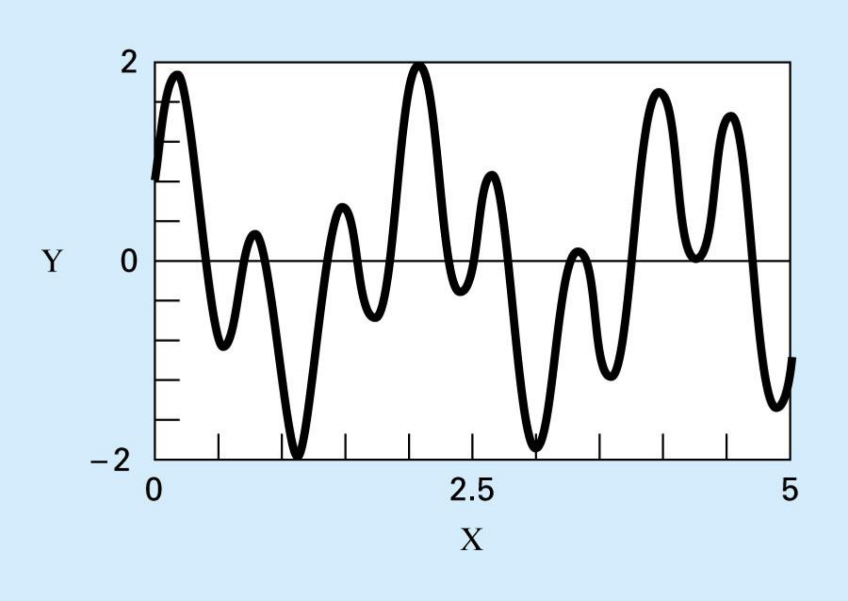


Figure 5.4b

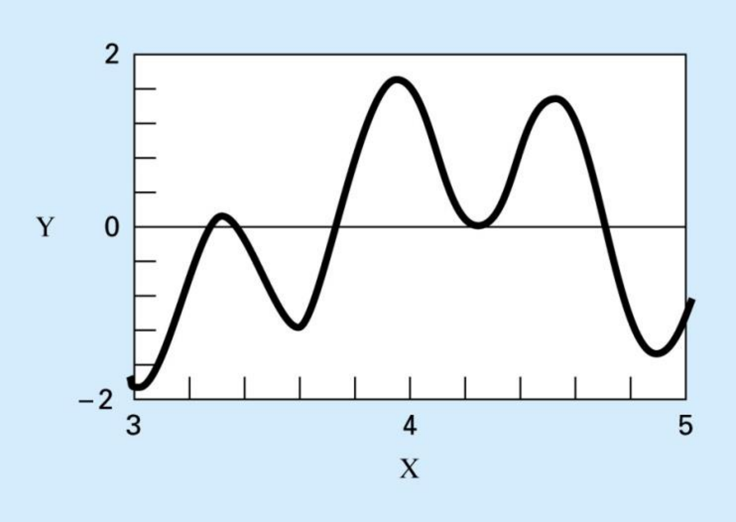
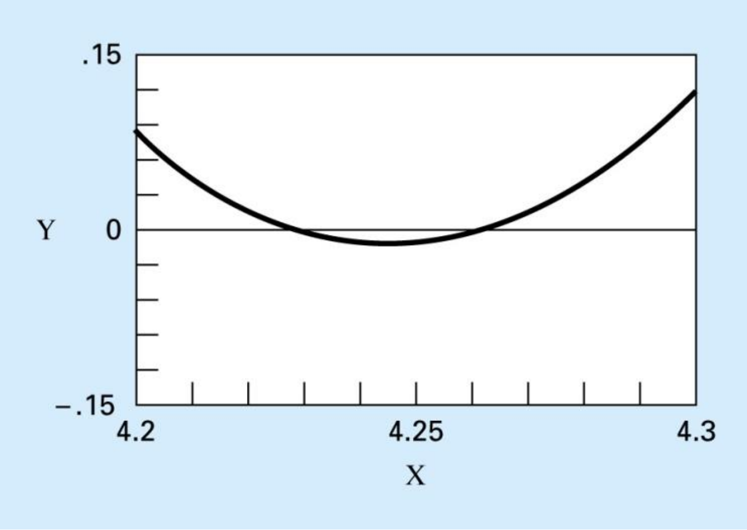


Figure 5.4c



The Bisection Method

For the arbitrary equation of one variable, $f(x)=0$

1. Pick x_l and x_u such that they bound the root of interest, check if $f(x_l).f(x_u) <0$.

2. Estimate the root by evaluating $f[(x_l+x_u)/2]$.

3. Find the pair
   - If $f(x_l). f[(x_l+x_u)/2]<0$, root lies in the lower interval, then $x_u=(x_l+x_u)/2$ and go to step 2.

- If $f(x_l) . f[(x_l+x_u)/2]>0$, root lies in the upper interval, then $x_l = [(x_l+x_u)/2$, go to step 2.

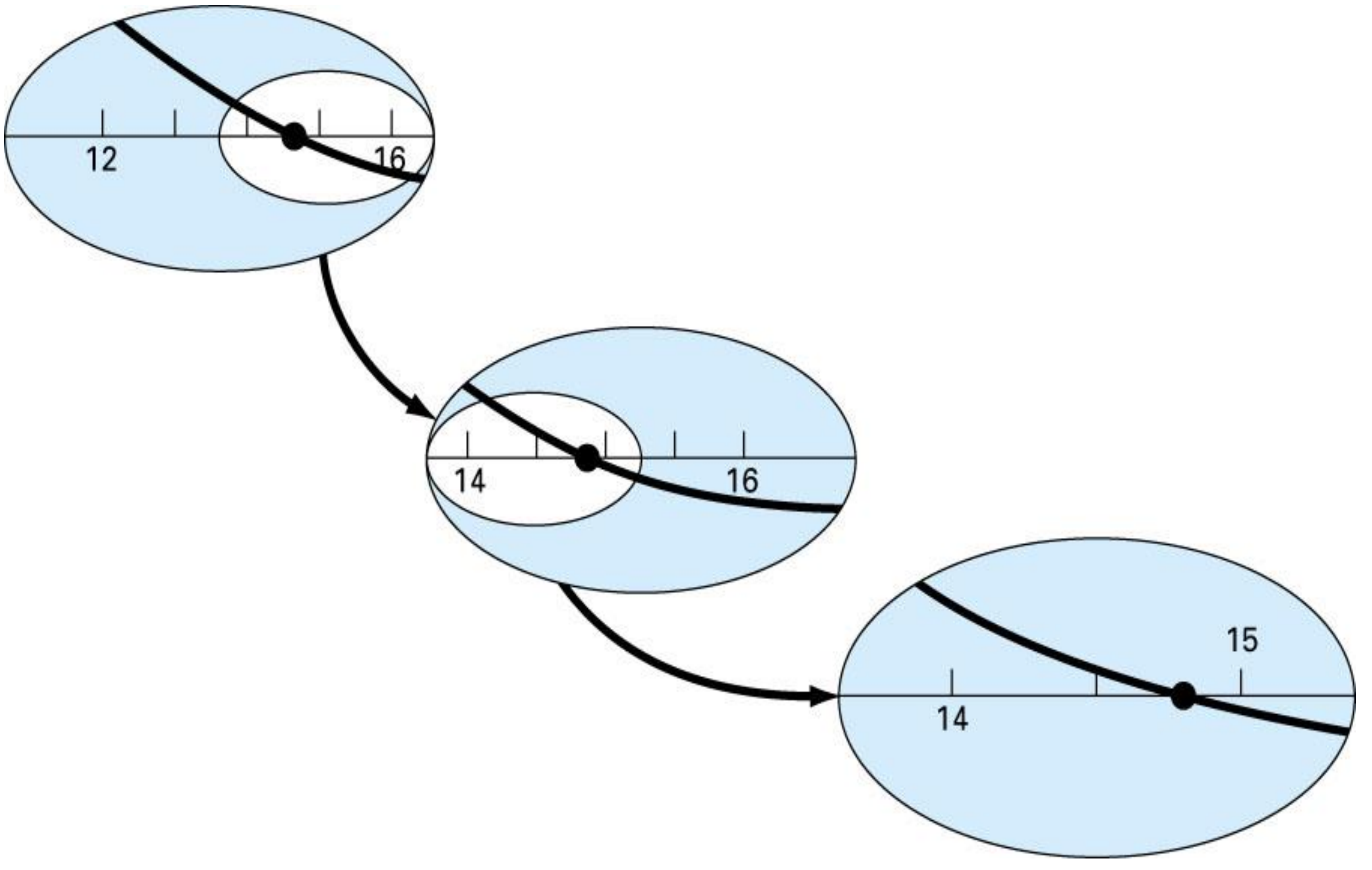
- If $f(x_l) . f[(x_l+x_u)/2]=0$, then root is $(x_l+x_u)/2$ and terminate.

4. Compare ε_s with ε_a

5. If $\varepsilon_a < \varepsilon_s$, stop. Otherwise repeat the process.

$$\left\{ \begin{array}{l} \dfrac{\left| x_l - \dfrac{x_l + x_u}{2} \right|}{\left| \dfrac{x_l + x_u}{2} \right|} \prec 100\% \\ or \\ \dfrac{\left| x_u - \dfrac{x_l + x_u}{2} \right|}{\left| \dfrac{x_l + x_u}{2} \right|} \prec 100\% \end{array} \right\}$$

Figure 5.6



Evaluation of the Bisection Method

Properties

- Easy
- Always find root
- Number of iterations required to attain an absolute error can be computed a priori.

Constrains

- Slow
- Know a and b that bound root
- Multiple roots
- No account is taken of $f(x_l)$ and $f(x_u)$, if $f(x_l)$ is closer to zero, it is likely that root is closer to x_l .

How Many Iterations will It Take?

- Length of the first Interval $L_o = b - a$
- After 1 iteration $L_1 = L_o/2$
- After 2 iterations $L_2 = L_o/4$

- After k iterations $L_k = L_o/2^k$

$$\varepsilon_a \leq \frac{L_k}{x} \times 100\% \qquad \varepsilon_a \leq \varepsilon_s$$

- If the absolute magnitude of the error is

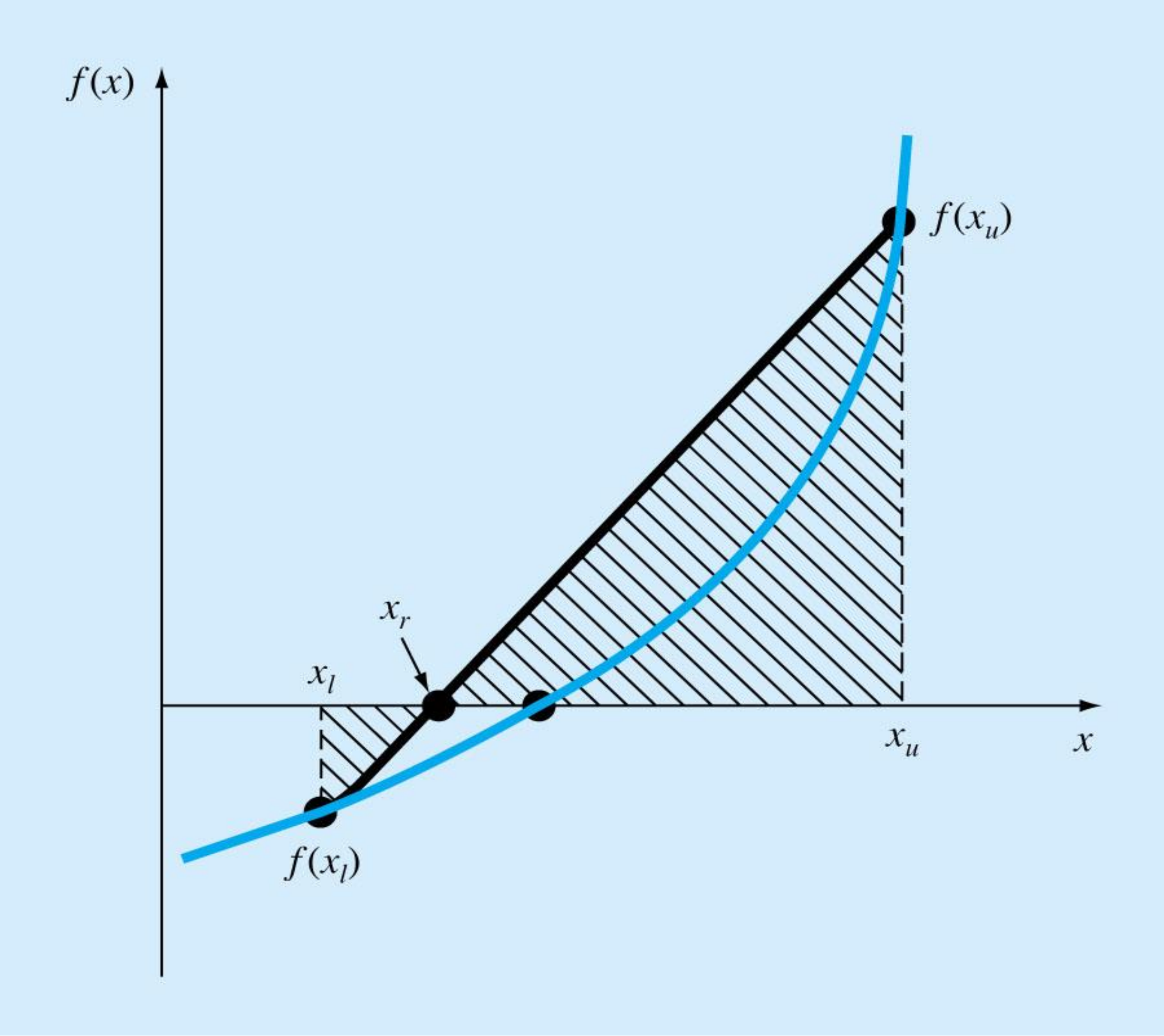
$$\frac{\varepsilon_s \cdot x}{100\%} = 10^{-4}$$

and $L_0=2$, how many iterations will you have to do to get the required accuracy in the solution?

$$10^{-4} = \frac{2}{2^k} \quad \Rightarrow 2^k = 2 \times 10^4 \quad \Rightarrow k \cong 14.3 = 15$$

The False-Position Method (Regula-Falsi)

- If a real root is bounded by x_l and x_u of $f(x)=0$, then we can approximate the solution by doing a linear interpolation between the points $[x_l, f(x_l)]$ and $[x_u, f(x_u)]$ to find the x_r value such that $l(x_r)=0$, $l(x)$ is the linear approximation of $f(x)$.



Procedure

1. Find a pair of values of x, x_l and x_u such that $f_l=f(x_l) <0$ and $f_u=f(x_u) >0$.

2. Estimate the value of the root from the following formula (Refer to Box 5.1)

$$x_r = \frac{x_l f_u - x_u f_l}{f_u - f_l}$$

and evaluate $f(x_r)$.

3. Use the new point to replace one of the original points, keeping the two points on opposite sides of the x axis.

If $f(x_r)<0$ then $x_l = x_r$ == > $f_l = f(x_r)$

If $f(x_r)>0$ then $x_u = x_r$ == > $f_u = f(x_r)$

If $f(x_r) = 0$ then you have found the root and need go no further!

4. See if the new x_l and x_u are close enough for convergence to be declared. If they are not go back to step 2.

- Why this method?
  - Faster
  - Always converges for a single root.

➔ See Sec.5.3.1, Pitfalls of the False-Position Method

Note: Always check by substituting estimated root in the original equation to determine whether $f(x_r) \approx 0$.