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Chapter Four

System of Linear and Nonlinear Equations

Direct Methods

- Cramer's Method
- Gauss Elimination Method without Pivoting
- Gauss Elimination Method with Pivoting
- Gauss-Jordan Method
- Matrix Inverse Method/ Gauss-Jordan Method
- LU Decomposition Method without Pivoting
- LU Decomposition Method with Pivoting
- Positive Definite Matrix
- Cholesky Decomposition Method

Indirect Iterative Methods

- Types of Norms
- Strictly Dominant Matrix
- Jacobi Method
- Gauss-Seidel Method

System of Nonlinear Equations

- Newton's Jacobian Method
- Fixed Point Iterative Method

حل نظام من المعادلات الخطية
Solving system of linear Eqn

Direct Method
الطريقة المباشرة

1. Cramer's Rule.
2. Gauss-Elimination.
3. Gauss- Jordan.
4. LU - Decomposition.
5. Cholesky Method.
6. Inverse Method.

الطريقة غير المباشرة
Indirect Method
(Iterative Method)
الطريقة التكرارية.
(أخطاء)

1. Jacobi Method.
2. Gauss - Seidel Method.

1. Cramer's Rule
2. Gauss- Elimination
3. Inverse Method

1 * Cramer's Rule

Use Cramer to solve :

$$x + 4y - 2z = 3$$
$$x + 3y = 1 - 7z$$
$$2x = 8 - 9y - z$$

س١

$$\Rightarrow \quad \begin{aligned} x + 4y - 2z &= 3 \\ x + 3y + 7z &= 1 \\ 2x + 9y + z &= 8 \end{aligned}$$

$$\begin{bmatrix} 1 & 4 & -2 \\ 1 & 3 & 7 \\ 2 & 9 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix}$$

$$D = \begin{vmatrix} 1 & 4 & -2 \\ 1 & 3 & 7 \\ 2 & 9 & 1 \end{vmatrix} = (1)\begin{vmatrix} 3 & 7 \\ 9 & 1 \end{vmatrix} - 4\begin{vmatrix} 1 & 7 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 2 & 9 \end{vmatrix}$$

$$= (1)(-60) - 4(-13) - 2(3) = -14$$

$$D_1 = \begin{vmatrix} 3 & 4 & -2 \\ 1 & 3 & 7 \\ 8 & 9 & 1 \end{vmatrix} = 70$$

$$D_2 = \begin{vmatrix} 1 & 3 & -2 \\ 1 & 1 & 7 \\ 2 & 8 & 1 \end{vmatrix} = -28$$

$$D_3 = \begin{vmatrix} 1 & 4 & 3 \\ 1 & 3 & 1 \\ 2 & 9 & 8 \end{vmatrix} = 0$$

$$X = \frac{D_1}{D} = \frac{70}{-14} = -5$$

$$y = \frac{D_2}{D} = \frac{-28}{-14} = 2$$

$$z = \frac{D_3}{D} = \frac{0}{-14} = 0$$

$X = -5$

$y = +2$

$z = 0$

ex Using Cramer's

$$2X_2 + 5X_3 = 1$$

$$2X_1 + X_2 + X_3 = 1$$

$$3X_1 + X_2 = 2$$

$$\begin{bmatrix} 0 & 2 & 5 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$D = \begin{vmatrix} 0 & 2 & 5 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{vmatrix} = \qquad X_1 = \frac{D_1}{D}$$

$$D_1 = \begin{vmatrix} 1 & 2 & 5 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = \qquad X_2 = \frac{D_2}{D}$$

$$D_2 = \begin{vmatrix} 0 & 1 & 5 \\ 2 & 1 & 1 \\ 3 & 2 & 0 \end{vmatrix} =$$

$$D_3 = \begin{vmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = \qquad X_3 = \frac{D_3}{D}$$

** Cramer's Rule   نستخدم هذه الطريقة لحل المعادلات الجبرية

Ex solve using Cramer's Rule to solve

$$3x + 7y - 2z = 2$$
$$x - 5y + z = 13$$
$$2x + 3y - 10z = -23$$

ثلاثة معادلات
بثلاثة مجاهيل
(x, y, z)

الحل

$$\begin{bmatrix} 3 & 7 & -2 \\ 1 & -5 & 1 \\ 2 & 3 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ -23 \end{bmatrix}$$

$$D = \begin{vmatrix} 3 & 7 & -2 \\ 1 & -5 & 1 \\ 2 & 3 & -10 \end{vmatrix} = 3[50-3] - 7[-10-2] - 2[3+10]$$
$$= 3(47) - 7(-12) - 2(13) = 199$$

$$D_1 = \begin{vmatrix} 2 & 7 & -2 \\ 13 & -5 & 1 \\ -23 & 3 & -10 \end{vmatrix} = 2[50-3] - 7[-130+23] - 2[39-115]$$
$$= 2[47] - 7[-107] - 2[-76]$$
$$= 995$$

$$D_2 = \begin{vmatrix} 3 & 2 & -2 \\ 1 & 13 & 1 \\ 2 & -23 & -10 \end{vmatrix} = -199$$

$$D_3 = \begin{vmatrix} 3 & 7 & 2 \\ 1 & -5 & 13 \\ 2 & 3 & -23 \end{vmatrix} = 597$$

$$x = \frac{D_1}{D} = \frac{995}{199} = 5$$

$$y = \frac{D_2}{D} = \frac{-199}{199} = -1$$

$$z = \frac{D_3}{D} = \frac{597}{199} = 3$$

ex) Cramer's Rule طريقة كريمر

Using Cramer's Rule To Solve :

$$x_1 - 2x_2 + 3x_3 = 8$$
$$2x_1 + x_2 + x_3 = 1$$
$$-4x_1 - x_2 + 2x_3 = 15$$

الحل

$$D = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ -4 & -1 & 2 \end{bmatrix} \Rightarrow |D| = 25$$

$$D_1 = \begin{bmatrix} 8 & -2 & 3 \\ 1 & 1 & 1 \\ 15 & -1 & 2 \end{bmatrix} \Rightarrow |D_1| = -50$$

$$D_2 = \begin{bmatrix} 1 & 8 & 3 \\ 2 & 1 & 1 \\ -4 & 15 & 2 \end{bmatrix} \Rightarrow |D_2| = 25$$

$$D_3 = \begin{bmatrix} 1 & -2 & 8 \\ 2 & 1 & 1 \\ -4 & -1 & 15 \end{bmatrix} \Rightarrow |D_3| = 100$$

$$x_1 = \frac{D_1}{D} = \frac{-50}{25} = -2$$

$$x_2 = \frac{D_2}{D} = \frac{25}{25} = 1$$

$$x_3 = \frac{D_3}{D} = \frac{100}{25} = 4$$

2 Gauss- Elimination

الهدف هو تصفير الجزء السفلي من المصفوفة

ex use gauss- Elimi. To solve

$$X_1 + 2X_2 - X_3 + 3X_4 = 11$$

$$-2X_1 - X_2 - X_3 + 3X_4 = 8$$

$$4X_1 + 2X_3 + 3X_4 = 13$$

$$-3X_1 + 2X_2 - X_4 = -11$$

الحل

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 11 \\ -2 & -1 & -1 & 3 & 8 \\ 4 & 0 & 2 & 3 & 13 \\ -3 & 2 & 0 & -1 & -11 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 11 \\ 0 & 3 & -3 & 9 & 30 \\ 0 & -8 & 6 & -9 & -31 \\ 0 & 8 & -3 & 8 & 22 \end{array}\right] \quad \begin{array}{l} R_2 - \left(\frac{-2}{1}\right) R_1 \\ R_3 - \left(\frac{4}{1}\right) R_1 \\ R_4 - \left(\frac{-3}{1}\right) R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 11 \\ 0 & 3 & -3 & 9 & 30 \\ 0 & 0 & -2 & 15 & 49 \\ 0 & 0 & 5 & -16 & -58 \end{array}\right] \begin{array}{l} \\ \\ R_3 - \left(\frac{-8}{3}\right) R_2 \\ R_4 - \left(\frac{8}{3}\right) R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 11 \\ 0 & 3 & -3 & 9 & 30 \\ 0 & 0 & -2 & 15 & 49 \\ 0 & 0 & 0 & 21.5 & 64.5 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - \left(\frac{5}{-2}\right) R_3 \end{array}$$

حيث يتم التعويض العكسي منه X_4 ثم X_3 ثم X_2 ثم X_1

$\Rightarrow \quad 21.5 X_4 = 64.5 \Rightarrow X_4 = 3$

$\Rightarrow \quad -2X_3 + 15(3) = 49 \Rightarrow X_3 = -2$

$\Rightarrow \quad 3X_2 - 3(-2) + 9(3) = 30 \Rightarrow X_2 = -1$

$\Rightarrow \quad X_1 + 2(-1) - 1(-2) + 3(3) = 11 \Rightarrow X_1 = 2$

مثال Solve By Gauss Elimination

$$x_1 - x_2 + 2x_3 - x_4 = -8$$

$$2x_1 - 2x_2 + 3x_3 - 3x_4 = -20$$

$$x_1 + x_2 + x_3 = -2$$

$$x_1 - x_2 + 4x_3 + 3x_4 = 4$$

الحل

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 2 & -2 & 3 & -3 & -20 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & -1 & 4 & 3 & 4 \end{array}\right]$$

صفر

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & \boxed{0} & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right] \begin{array}{l} R_2 - \left(\frac{2}{1}\right) R_1 \\ R_3 - \left(\frac{1}{1}\right) R_1 \\ R_4 - \left(\frac{1}{1}\right) R_1 \end{array}$$

Pivoting

بدل الصف الثاني بالصف الثالث

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right] \begin{array}{l} R_3 - \left(\frac{0}{2}\right) R_2 \\ R_4 - \left(\frac{0}{2}\right) R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right] R_4 - \left(\frac{2}{-1}\right) R_3$$

تعويض عكسي

$$\Rightarrow \quad 2X_4 = 4 \Rightarrow X_4 = 2$$

$$\Rightarrow \quad -X_3 - 2 = -4 \Rightarrow X_3 = 2$$

$$\Rightarrow \quad 2X_2 - 2 + 2 = 6 \Rightarrow X_2 = 3$$

$$\Rightarrow \quad X_1 - 3 + 2 - 2 = -8 \Rightarrow X_1 = -7$$

هاشم الخالدي
هندسة ميكانيكية

$$\left[\begin{array}{cccc|c} 2 & 3 & 1 & -1 & 16 \\ 0 & 2 & 1 & -2 & -10 \\ 0 & 0 & 3 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

الصف الأخير كله أصفار

⇒ عدد لا نهائي من الحلول

infinite Number of Solutions

⇒ ∞ – Solutions

$$\left[\begin{array}{cccc|c} -3 & 2 & 1 & 6 & 5 \\ 0 & 4 & -2 & 1 & 3 \\ 0 & 0 & 5 & -2 & -10 \\ 0 & 0 & 0 & 0 & 8 \end{array}\right]$$

الصف الأخير كله أصفار

ما عدا الطرف الأيمن غالباً

(ليس صفراً).

⇒ الحل غير موجود

Solution does Not exist.

"لا يوجد حل"

$$\left[\begin{array}{cccc|c} -3 & 4 & 1 & 7 & 5 \\ 0 & 8 & 2 & 1 & -3 \\ 0 & 0 & -3 & 4 & 2 \\ 0 & 0 & 0 & 8 & 0 \end{array}\right]$$

"هذه مسألة عادية"

* $8x_4 = 0 \Rightarrow x_4 = 0$

* ثم نصعد تدريجياً

ونحسب كل من

x_3, x_2, x_1

⇒ Unique solution

$$x - y + 2z = 4$$
$$3x - 2y + 9z = 14$$
$$2x - 4y + az = b$$

Find a and b →
- gives unique sol.
- gives ∞ - Sol.
- gives NO Sol.

Gauss - Elimination

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 3 & -2 & 9 & 14 \\ 2 & -4 & a & b \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & a-4 & b-8 \end{array}\right] \begin{array}{l} \\ R_2 - \left(\frac{3}{1}\right) R_1 \\ R_3 - \left(\frac{2}{1}\right) R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & a+2 & b+4 \end{array}\right] \begin{array}{l} \\ \\ R_3 - \left(\frac{-2}{1}\right) R_2 \end{array}$$

1) Unique sol. $\Rightarrow$ $a+2 \neq 0$ $\Rightarrow$ $a \neq -2$
$\Rightarrow$ b = any Number $(-\infty, \infty)$

2) ∞ - Sol. $\Rightarrow$ $a+2=0$ $\Rightarrow$ $a=-2$
$b+4=0$ $\Rightarrow$ $b=-4$

3) No Sol. $\Rightarrow$ $a+2=0 \Rightarrow a=-2$ $\quad b+4 \neq 0 \Rightarrow b \neq -4$

ex)

$$x_1 + 2x_2 = b$$
$$x_1 + a x_2 = 1$$

Find a and b ?

$$\left[\begin{array}{cc|c} 1 & 2 & b \\ 1 & a & 1 \end{array}\right]$$

- * Unique.
- * ∞ - Sol.
- * No Sol.

Find a b For infinite Solutions

$$\begin{bmatrix} 1 & -a^2 \\ 1 & b^2+2b+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -a^2 & 3 \\ 1 & (b+1)^2 \;\; b^2+2b+1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -a^2 & 3 \\ 0 & (b+1)^2+a^2 & 0 \end{array}\right] R_2-\left(\frac{1}{1}\right)R_1$$

$$(b+1)^2+a^2=0$$

$$\therefore \Rightarrow \quad b+1=0 \Rightarrow b=-1$$

$$\Rightarrow \quad a=0$$

$$3x - y = 14$$

$$(a+b)x - 2b\,y = 5a + 2b + 1$$

Gauss- Elimination

$$\left[\begin{array}{cc|c} 3 & -1 & 14 \\ a+b & -2b & 5a+2b+1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -1 & 14 \\ 0 & \frac{1}{3}(a-5b) & \frac{1}{3}(a-8b+3) \end{array}\right] \quad R_2 - \left(\frac{a+b}{3}\right)R_1$$

$\boxed{1}$ Unique sol.

$$\frac{1}{3}(a-5b) \neq 0 \Rightarrow a-5b \neq 0 \Rightarrow a \neq 5b$$

2] ∞ - Sol.

$$a-5b=0 \Rightarrow a-5b=0$$
$$a-8b+3=0 \Rightarrow a-8b=-3$$
$$\Rightarrow a=5, \quad b=1$$

3] No Sol.

$$a-5b=0 \Rightarrow a=5b$$
$$a-8b+3 \neq 0 \Rightarrow 5b-8b+3 \neq 0 \Rightarrow -3b \neq -3$$
$$b \neq 1 \qquad a \neq 5$$

$\boxed{3}$ (Gauss-Jordan)

وهو عبارة عن استكمال طريقة الـ gauss-Elimi:

بحيث نقوم بتصفير الجزء العلوي أيضاً.

مع مراعاة أن جميع عناصر القطر تساوي واحد صحيح

$$\begin{array}{r} \\ \text{بالقسمة على ٤} \\ \text{بالقسمة على -١} \\ \text{بالقسمة على ٤} \end{array}\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -8 \\ 0 & 1 & -0.5 & 0.5 & 3 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 0 & -6 \\ 0 & 1 & -0.5 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]\begin{array}{l} R_1 - \left(\frac{-1}{1}\right)R_4 \\ R_2 - \left(\frac{0.5}{1}\right)R_4 \\ R_3 - \left(\frac{1}{1}\right)R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]\begin{array}{l} R_1 - \left(\frac{2}{1}\right)R_3 \\ R_2 - \left(\frac{-0.5}{1}\right)R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -7 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] R_1 - \left(\frac{-1}{1}\right) R_2$$

$\Leftarrow$ لاحظ أنه تم تصفير الجزء العلوي والجزء السفلي معاً .

$$X_1 = -7$$

$$X_2 = 3$$

$$X_3 = 2$$

$$X_4 = 2$$

4 Matrix Inverse By using (Gauss-Jordan)

وهذه عبارة عن طريقة أخرى (غير طريقة ال Cofactor)
لإيجاد معكوس المصفوفة .

ex Find inverse

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & -1 \\ -2 & -3 & 4 \end{bmatrix} ?$$

الحل

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 4 & -2 & -1 & 0 & 1 & 0 \\ -2 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Pivoting

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 0 & \overset{\text{صفر}}{\boxed{0}} & -7 & -2 & 1 & 0 \\ 0 & -4 & 7 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - \frac{4}{2} R_1 \\ R_3 - \frac{-2}{2} R_1 \end{array}$$

لا بد من التبديل ما بين الصف الثاني والصف الثالث
(Pivoting)

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 0 & -4 & 7 & 1 & 0 & 1 \\ 0 & 0 & -7 & -2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{-1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{-7}{4} & \frac{-1}{4} & 0 & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{2}{7} & \frac{-1}{7} & 0 \end{array}\right]$$

Now $\Rightarrow$ Jordan $\Rightarrow$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{-1}{2} & 0 & \frac{1}{14} & \frac{3}{14} & 0 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{2}{7} & \frac{-1}{7} & 0 \end{array}\right] \begin{array}{l} R_1 - \frac{\frac{3}{2}}{1} R_3 \\ R_2 - \frac{\frac{-7}{4}}{1} R_3 \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{56} & \frac{5}{56} & \frac{-1}{8} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{2}{7} & \frac{-1}{7} & 0 \end{array}\right] \begin{array}{l} R_1 - \frac{\frac{-1}{2}}{1} R_2 \\ \\ \end{array}$$

* the inverse A^{-1} is :-

$$A^{-1} = \begin{bmatrix} \frac{11}{56} & \frac{5}{56} & \frac{-1}{8} \\ \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{2}{7} & \frac{-1}{7} & 0 \end{bmatrix}$$

ويمكن اخراج عامل مشترك

$$A^{-1} = \frac{-1}{56} \begin{bmatrix} -11 & -5 & 7 \\ -14 & 14 & 14 \\ -16 & 8 & 0 \end{bmatrix}$$

* Using inverse Method To solve :-

$$x_1 - 2x_2 + 3x_3 = 8$$

$$2x_1 + x_2 + x_3 = 1$$

$$-4x_1 - x_2 + 2x_3 = 15$$

الحل

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ -4 & -1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 8 \\ 1 \\ 15 \end{bmatrix}$$

Cofactor :- $|A| = 1(3) \overset{\text{غير الإشارة}}{+} 2(8) + 3(2) = 25$

$$A^{-1} = \frac{1}{25}\begin{bmatrix} 3 & \ominus 8 & 2 \\ \ominus -1 & 14 & \ominus -9 \\ -5 & \ominus -5 & 5 \end{bmatrix}^T = \frac{1}{25}\begin{bmatrix} 3 & 1 & -5 \\ -8 & 14 & 5 \\ 2 & 9 & 5 \end{bmatrix}$$

قانون

$$\boxed{X = A^{-1} b}$$

$$X = \frac{1}{25}\begin{bmatrix} 3 & 1 & -5 \\ -8 & 14 & 5 \\ 2 & 9 & 5 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 15 \end{bmatrix}$$

$$X = \frac{1}{25}\begin{bmatrix} -50 \\ 25 \\ 100 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}$$

Use Inverse Method To Solve :

$$2X_1 - 2X_2 + 4X_3 = -10$$

$$X_1 + 3X_2 - X_3 = 8$$

$$-X_1 + 4X_3 = -14$$

$$\underbrace{\begin{bmatrix} 2 & -2 & 4 \\ 1 & 3 & -1 \\ -1 & 0 & 4 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -10 \\ 8 \\ -14 \end{bmatrix}}_{b}$$

$$A \cdot X = b$$

5 (LU-Decomposition) LU-factorization

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}} \cdot \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \underset{b}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}$$

خطوات الحل

الخطوة رقم ①

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 \\ L_{31} & L_{32} & 1 & 0 \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix}} \cdot \underset{U}{\begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & 0 & U_{33} & U_{34} \\ 0 & 0 & 0 & U_{44} \end{bmatrix}}$$

$A = L\,U$

$A\,x = b$
$L\,(U\,x) = b$
$L\,(y) = b$

الخطوة رقم ②

$$L\,y = b$$

حيث يتم ايجاد الـ y عن طريقة التعويض الامامي (من الاعلى الى الاسفل)

الخطوة رقم ③

$$U\,x = y$$

حيث يتم ايجاد الـ x عن طريقة التعويض العكسي من الاسفل الى الاعلى

الخطوة رقم ④

$$|A| = |U|$$

$$|A^{-1}| = \frac{1}{|U|}$$

Use LU Decomposition To Solve :-

ex

هاشم الخالدي
هندسة ميكانيكية

$$2X_1 - 3X_2 + X_3 - 4X_4 = 9$$
$$4X_1 + 2X_2 + 2X_3 + 4X_4 = 22$$
$$-X_1 - X_2 - X_3 + 3X_4 = -12$$
$$5X_1 + 2X_3 - 2X_4 = 28$$

1. Find the lower Matrix L ?
2. Find the upper Matrix U ?
3. Solve these equations (Find X_1, X_2, X_3, X_4) ?
4. Find the determinant $|A|$?
5. Find the determinant of the inverse $|A^{-1}|$?
6. Find the determinant of Transpose $|A^T|$?

الحل

$$\begin{bmatrix} 2 & -3 & 1 & -4 \\ 4 & 2 & 2 & 4 \\ -1 & -1 & -1 & 3 \\ 5 & 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ -12 \\ 28 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 2 & -3 & 1 & -4 \\ 4 & 2 & 2 & 4 \\ -1 & -1 & -1 & 3 \\ 5 & 0 & 2 & -2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \overset{2}{\cancel{L_{21}}} & 1 & 0 & 0 \\ \overset{-0.5}{\cancel{L_{31}}} & \overset{-0.3125}{\cancel{L_{32}}} & 1 & 0 \\ \overset{2.5}{\cancel{L_{41}}} & \underset{0.9375}{\cancel{L_{42}}} & \overset{1}{\cancel{L_{43}}} & 1 \end{bmatrix}} \cdot \underset{U}{\begin{bmatrix} \overset{2}{\cancel{U_{11}}} & \overset{-3}{\cancel{U_{12}}} & \overset{1}{\cancel{U_{13}}} & \overset{-4}{\cancel{U_{14}}} \\ 0 & \overset{8}{\cancel{U_{22}}} & \overset{0}{\cancel{U_{23}}} & \overset{12}{\cancel{U_{24}}} \\ 0 & 0 & \overset{-0.5}{\cancel{U_{33}}} & \overset{4.75}{\cancel{U_{34}}} \\ 0 & 0 & 0 & \overset{-8}{\cancel{U_{44}}} \end{bmatrix}}$$

* $U_{11} = 2$
* $U_{12} = -3$
* $U_{13} = 1$
* $U_{14} = -4$

* $2L_{21} = 4 \Rightarrow L_{21} = 2$ ✓
* $-6 + U_{22} = 2 \Rightarrow U_{22} = 8$
* $2 + U_{23} = 2 \Rightarrow U_{23} = 0$
* $-8 + U_{24} = 4 \Rightarrow U_{24} = 12$

* $2L_{31} = -1 \Rightarrow L_{31} = -0.5$ ✓
* $1.5 + 8L_{32} = -1 \Rightarrow L_{32} = -0.3125$
* $-0.5 + U_{33} = -1 \Rightarrow U_{33} = -0.5$
* $2 - 3.75 + U_{34} = 3 \Rightarrow U_{34} = 4.75$

* $2L_{41} = 5 \Rightarrow L_{41} = 2.5$ ✓
* $-7.5 + 8L_{42} = 0 \Rightarrow L_{42} = 0.9375$
* $2.5 + 0 - 0.5L_{43} = 2 \Rightarrow L_{43} = 1$
* $-10 + 11.25 + 4.75 + U_{44} = -2 \Rightarrow U_{44} = -8$

① $$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -0.5 & -0.3125 & 1 & 0 \\ 2.5 & 0.9375 & 1 & 1 \end{bmatrix}$$

② $$U = \begin{bmatrix} 2 & -3 & 1 & -4 \\ 0 & 8 & 0 & 12 \\ 0 & 0 & -0.5 & 4.75 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

* $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -0.5 & -0.3125 & 1 & 0 \\ 2.5 & 0.9375 & 1 & 1 \end{bmatrix} \begin{bmatrix} \cancel{y_1}\,9 \\ \cancel{y_2}\,4 \\ \cancel{y_3}\,-6.214 \\ \cancel{y_4}\,7.964 \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ -12 \\ 28 \end{bmatrix}$$

عوض من الاعلى الى الاسفل

* $y_1 = 9$
* $18 + y_2 = 22 \Rightarrow y_2 = 4$
* $-4.5 - 1.286 + y_3 = -12 \Rightarrow y_3 = -6.214$
* $22.5 + 3.75 - 6.214 + y_4 = 28 \Rightarrow y_4 = 7.964$

* $Ux = y$

③

$$\begin{bmatrix} 2 & -3 & 1 & -4 \\ 0 & 8 & 0 & 12 \\ 0 & 0 & -0.5 & 4.75 \\ 0 & 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} \overset{4}{\cancel{x_1}} \\ \cancel{x_2}^{2} \\ \cancel{x_3}^{3} \\ \underset{-1}{\cancel{x_4}} \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -6.214 \\ 7.964 \end{bmatrix}$$

عوّض من الاسفل الى الاعلى

* $-8x_4 = 7.964 \Rightarrow x_4 = -1$

* $-0.5x_3 - 4.75 = -6.214 \Rightarrow X_3 = 3$

* $8X_2 - 12 = 4 \Rightarrow X_2 = 2$

* $2x_1 - 6 + 3 + 4 = 9 \Rightarrow x_1 = 4$

④

* $|A| = |U| = 2(8)(-0.5)(-8) = 64$

⑤

* $|A^{-1}| = \dfrac{1}{|U|} = \dfrac{1}{64}$

⑥

* $|A^T| = |A| = |U| = 64$

* (LU Decomposition)

ex) Solve :

$$4X_1 + X_2 + X_3 = 4$$
$$2X_1 + X_2 - X_3 = -2$$
$$6X_1 - X_2 - 2X_3 = 4$$

الحل

* ابدأ من السؤال وكأنه مسألة Gauss Elimination

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & \cancel{4} \\ 2 & 1 & -1 & \cancel{-2} \\ 6 & -1 & -2 & \cancel{4} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & \\ 0 & 0.5 & -1.5 & \\ 0 & -2.5 & -3.5 & \end{array}\right] \begin{array}{l} R_2 - \left(\frac{2}{4}\right) R_1 \\ R_3 - \left(\frac{6}{4}\right) R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & \\ 0 & 0.5 & -1.5 & \\ 0 & 0 & -11 & \end{array}\right] R_3 - \left(\frac{-2.5}{0.5}\right) R_2$$

الخطوة الأولى

$$A = LU$$

$$U = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 0.5 & -1.5 \\ 0 & 0 & -11 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{4} & 1 & 0 \\ \frac{6}{4} & \frac{-2.5}{0.5} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -5 & 1 \end{bmatrix}$$

الخطوة الثانية

$$Ly = b$$

تعويض أمامي

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 1.5 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix} \Rightarrow \begin{array}{l} y_1 = 4 \\ y_2 = -4 \\ y_3 = -22 \end{array}$$

الخطوة الثالثة

$$Ux = y$$

تعويض عكسي

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & 0.5 & -1.5 \\ 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -22 \end{bmatrix} \Rightarrow \begin{array}{l} X_3 = 2 \\ X_2 = -2 \\ X_1 = 1 \end{array}$$

كاشف

Find $|A| = |U| = 4(0.5)(-11) = -22$

ex) LU-decomposition (with pivoting) "Pivoting"

سؤال سابق

$$x_1 - x_2 + 2x_3 - x_4 = -8$$
$$2x_1 - 2x_2 + 3x_3 - 3x_4 = -20$$
$$x_1 + x_2 + x_3 = -2$$
$$x_1 - x_2 + 4x_3 + 3x_4 = 4$$

هاشم الخالدي
هندسة ميكانيكية

الحل

يتم الحل كـ Gauss elimination

كما هو في السابق حيث نحصل على

$$U = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

pivoting حيث تم تبديل الصف الثاني مع الثالث

حيث تكون الـ L هي عبارة عن المعاملات !!

$$L^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{bmatrix}$$

نعيد عليه التبديل ما بين العنصر الثاني والثالث إلى أصلها

$$Ly = b$$

تبديل الصفوف في ٢ ، ٣

$$L \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -8 \\ -20 \\ -2 \\ 4 \end{bmatrix}$$

* $y_1 = -8$
* $-16 + y_3 = -20 \Rightarrow y_3 = -4$
* $-8 + y_2 = -2 \Rightarrow y_2 = 6$
* $-8 + 8 + y_4 = 4 \Rightarrow y_4 = 4$

$$Ux = y$$

$$\begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \\ -4 \\ 4 \end{bmatrix}$$

* $2x_4 = 4 \Rightarrow x_4 = 2$
* $-x_3 - 2 = -4 \Rightarrow x_3 = 2$
* $2x_2 - 2 + 2 = 6$

$x_2 = 3$

* $x_1 - 3 + 4 - 2 = -8$

$x_1 = -7$

6 Cholesky Decomposition

وهي عبارة عن حالة خاصة من طريقة LU

** حيث يتم استخدامها اذا تحقق شرطان :

⟸ الشرط الاول : Symmetric Matrix

⟸ الشرط الثاني : positive definite.

خطوات الحل

1. $L = U^T$, $U = U$ $\Rightarrow$ $A = \overset{L}{U^T}\,\overset{U}{U}$

2. $U^T y = b$

3. $U x = y$

4. $|A| = |U|^2$

ex) use Cholesky To Solve :-

$$4X_1 - 2X_2 - X_3 = 4$$
$$-2X_1 + 6X_2 = -6$$
$$-X_1 + 3X_3 = 0$$

الحل

$$A = \begin{bmatrix} 4 & -2 & -1 \\ -2 & 6 & 0 \\ -1 & 0 & 3 \end{bmatrix} \Rightarrow \boxed{A = A^T}$$

symmetric matrix

$$* \quad |4| = 4 > 0$$

$$* \quad \begin{vmatrix} 4 & -2 \\ -2 & 6 \end{vmatrix} = 24 - 4 = 20 > 0$$

$$* \quad \begin{vmatrix} 4 & -2 & -1 \\ -2 & 6 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 4(18) + 2(-6) - 1(6) = 72 - 12 - 6 = 54 > 0$$

Positive Definite

$\Rightarrow$ O.K. $\Rightarrow$ use Cholesky.

$$A = U^T \cdot U$$

$$\begin{bmatrix} 4 & -2 & -1 \\ -2 & 6 & 0 \\ -1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \overset{2}{u_{11}} & 0 & 0 \\ \overset{-1}{u_{12}} & \overset{\sqrt{5}}{u_{22}} & 0 \\ \underset{-0.5}{u_{13}} & \underset{\frac{-1}{2\sqrt{5}}}{u_{23}} & \underset{\sqrt{2.7}}{u_{33}} \end{bmatrix} \cdot \begin{bmatrix} \overset{2}{u_{11}} & \overset{-1}{u_{12}} & \overset{-0.5}{u_{13}} \\ 0 & \overset{\sqrt{5}}{u_{22}} & \overset{\frac{-1}{2\sqrt{5}}}{u_{23}} \\ 0 & 0 & \underset{\sqrt{2.7}}{u_{33}} \end{bmatrix}$$

حيث تتم عملية الضرب :-

* $u_{11}^2 = 4 \Rightarrow u_{11} = 2$

* $2u_{12} = -2 \Rightarrow u_{12} = -1$

* $2u_{13} = -1 \Rightarrow u_{13} = -0.5$

* $1 + u_{22}^2 = 6 \Rightarrow u_{22} = \sqrt{5} = 2.236$

* $0.5 + \sqrt{5}\, u_{23} = 0 \Rightarrow u_{23} = \frac{-0.5}{\sqrt{5}} = \frac{-1}{2\sqrt{5}}$

* $+0.25 + 0.05 + u_{33}^2 = 3 \Rightarrow u_{33} = \sqrt{2.7}$

السطر الاول بالاعمده الثلاثه
السطر الثاني بآخر عامودين
والسطر الثالث بالعامود الثالث

* $U^T y = b \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ -1 & \sqrt{5} & 0 \\ -0.5 & \frac{-1}{2\sqrt{5}} & \sqrt{2.7} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 0 \end{bmatrix}$

$\Rightarrow y_1 = 2 \qquad y_2 = \frac{-4}{\sqrt{5}} \qquad y_3 = 0.36515$

* $UX = y \quad \begin{bmatrix} 2 & -1 & -0.5 \\ 0 & \sqrt{5} & -1/2\sqrt{5} \\ 0 & & \sqrt{2.7} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4/\sqrt{5} \\ 0.36515 \end{bmatrix}$

$\Rightarrow X_1 = 0.6666 \qquad X_2 = -0.7777 \qquad X_3 = 0.2222$

Cholesky Factorization

** هذه الطريقه شبيهه تماماً لطريقه ال LU

مع الفرق الوحيد انها تستخدم لكل المصفوفات

المتماثله (Symmetric Matrix)

$A = A^T$

** ويجب ايضاً ان تكون المصفوفه A عباره عن Positive definite

Step 1

$$A = \underbrace{U^T}_{L}\,\underbrace{U}_{U}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$A = A^T$

حيث يتم ايجاد هذه العناصر

و العناصر داخل المصفوفه

عن طريق ضرب المصفوفات مباشرةً "

Step 2 $U^T y = b$

Step 3 $UX = y$

use Cholesky factorization to solve :-

$$4x_1 - 2x_2 - x_3 = 4$$

$$-2x_1 + 6x_2 = -6$$

$$-x_1 + 3x_3 = 0$$

الحل

Symmetrical Matrix.

$$\begin{bmatrix} 4 & -2 & -1 \\ -2 & 6 & 0 \\ -1 & 0 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} \overset{2}{u_{11}} & 0 & 0 \\ \overset{-1}{u_{12}} & \overset{2.24}{u_{22}} & 0 \\ \overset{-0.5}{u_{13}} & \overset{-0.223}{u_{23}} & \overset{1.64}{u_{33}} \end{bmatrix}}_{U^T} \cdot \underbrace{\begin{bmatrix} \overset{2}{u_{11}} & \overset{-1}{u_{12}} & \overset{0.5}{u_{13}} \\ 0 & \overset{2.24}{u_{22}} & \overset{-0.223}{u_{23}} \\ 0 & 0 & \overset{1.64}{u_{33}} \end{bmatrix}}_{U}$$

$* \quad u_{11}^2 = 4 \Rightarrow u_{11} = 2$

$\leftarrow \quad 2u_{12} = -2 \Rightarrow u_{12} = -1$

$\leftarrow \quad 2u_{13} = -1 \Rightarrow u_{13} = -0.5$

$\leftarrow \quad 1 + u_{22}^2 = 6 \Rightarrow u_{22} = 2.24$

$\leftarrow \quad 0.5 + 2.24u_{23} = 0 \Rightarrow u_{23} = -0.223$

$0.25 + (0.223)^2 + u_{33}^2 = 3 \Rightarrow u_{33} = 1.64$

$U^T y = b$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 2.24 & 0 \\ -0.5 & -0.223 & 1.64 \end{bmatrix} \begin{array}{c} 2 \\ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \begin{array}{l} \\ -1.79 \\ \\ \end{array} \\ -0.365 \end{array} = \begin{bmatrix} 4 \\ -6 \\ 0 \end{bmatrix}$$

* $2y_1 = 4 \Rightarrow y_1 = 2$

* $-2 + 2.24 y_2 = -6 \Rightarrow y_2 = -1.79$

* $-0.5(2) + (-0.223)(-1.79) + 1.64 y_3 = 0 \Rightarrow y_3 = -0.365$

$Ux = y$

$$\begin{bmatrix} 2 & -1 & -0.5 \\ 0 & 2.24 & -0.223 \\ 0 & 0 & 1.64 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 0.533 \\ x_1 \\ -0.822 \\ x_2 \\ x_3 \end{bmatrix} \\ -0.222 \end{array} = \begin{bmatrix} 2 \\ -1.79 \\ -0.365 \end{bmatrix}$$

* $1.64\, x_3 = -0.365 \Rightarrow x_3 = -0.222$

* $2.24 x_2 + (-0.223)(-0.222) = -1.79 \Rightarrow x_2 = -0.822$

$2x_1 + 0.822 + 0.111 = 2 \Rightarrow x_1 = 0.533$

Solve using Cholesky Method :-

$$6x + 15y + 55z = 76$$

$$15x + 55y + 225z = 295$$

$$55x + 225y + 979z = 1259$$

Matrix A is symmetrical because $a_{ij} = a_{ji}$. Its leading principal submatrices are

$$A_1 = [2] \qquad A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\det(A_1) = 2 \geq 0, \qquad \det(A_2) = 3 \geq 0, \qquad \det(A_3) = 4 \geq 0$$

Therefore matrix A positive definite.

Cholesky Decomposition for Symmetrical Matrices

The cholesky decomposition is used primarily for symmetrical positive definite matrices.

Let S denote a symmetrical matrix and U denote upper triangular matrix according to Cholesky S can be decomposed as follows:

$$S = U^T \times U$$

And hence can be written as follows

$$\begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1j} \\ s_{21} & s_{22} & \cdots & s_{2j} \\ \vdots & \vdots & & \vdots \\ s_{j1} & s_{j2} & \cdots & s_{jj} \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & \cdots & 0 \\ u_{12} & u_{22} & \cdots & 0 \\ \vdots & \vdots & & 0 \\ u_{1j} & u_{2j} & \cdots & u_{jj} \end{bmatrix} \times \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1j} \\ 0 & u_{22} & \cdots & u_{2j} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{jj} \end{bmatrix}$$

It can easily be shown by using matrix multiplication that

$s_{11} = u_{11} \times u_{11}$, and therefore, $u_{11} = \sqrt{s_{11}}$,

Also

$s_{1j} = u_{11} \times u_{1j}$, hence $u_{1j} = \dfrac{s_{1j}}{u_{11}}$

Next lest solve for the second row (note that we already know $u_{21} = 0$)

$$s_{22} = u_{12} \times u_{12} + u_{22} \times u_{22},$$
$$s_{23} = u_{12} \times u_{13} + u_{22} \times u_{23}$$
$$s_{2j} = u_{12} \times u_{1j} + u_{22} \times u_{2j}$$

If we solve for the third row we obtain:

$$s_{33} = u_{13} \times u_{13} + u_{23} \times u_{23} + u_{33} \times u_{33}$$
$$s_{3j} = u_{13} \times u_{1j} + u_{23} \times u_{2j} + u_{33} \times u_{3j}$$

Now it is evident that we can write the general format for the matrix S as

$$s_{ii} = \sum_{k=1} (u_{ki})^2 \text{ , while } s_{ij} = \sum_{k=1} (u_{ki})(u_{kj}) \qquad j = i+1, i+2, \cdots, n$$

The elements in row i of U are therefore

$$u_{ii} = \left[s_{ii} - \sum_{k=1}^{i-1} (u_{ki})^2 \right]^{\frac{1}{2}} \qquad i = 2,3,\cdots,n$$

$$u_{ij} = \left[\frac{1}{u_{ii}} \right] \left[s_{ij} - \sum_{k=1}^{i-1} (u_{ki})(u_{kj}) \right] \qquad j = i+1, i+2, \cdots, n$$

Example:

$$A = \begin{bmatrix} 4 & -2 & -1 \\ -2 & 6 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_{31} \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 0 \end{bmatrix}$$

$$A = U^T U$$

$$\begin{bmatrix} 4 & -2 & -1 \\ -2 & 6 & 0 \\ -1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$4 = (u_{11})^2 \Rightarrow u_{11} = \sqrt{4} = 2$$

$$-2 = u_{12} \times u_{11} \quad \Rightarrow u_{12} = \frac{-2}{u_{11}} = \frac{-2}{2} = -1$$

$$-1 = u_{13} \times u_{11} \quad \Rightarrow u_{13} = \frac{-1}{u_{11}} = \frac{-1}{2}$$

$$6 = u_{12} \times u_{12} + u_{22} \times u_{22} = u_{12}^2 + u_{22}^2$$

$$u_{22}^2 = 6 - (-1)^2 = 5 \Rightarrow u_{22} = \sqrt{5}$$

$$0 = u_{12} \times u_{13} + u_{22} \times u_{23} + 0 = (-1)\left(-\frac{1}{2}\right) + \left(\sqrt{5}\right)u_{23} \Rightarrow u_{23} = \frac{-1}{2\sqrt{5}}$$

$$3 = u_{13}^2 + u_{23}^2 + u_{33}^2 = \left(-\frac{1}{2}\right)^2 + \left(\frac{-1}{2\sqrt{5}}\right)^2 + u_{33}^2$$

$$u_{33}^2 = 3 - \frac{1}{4} - \frac{1}{20} = \frac{27}{10} \Rightarrow u_{33} = \sqrt{\frac{27}{10}}$$

Therefore we get

$$U = \begin{bmatrix} 2 & -1 & -\frac{1}{2} \\ 0 & \sqrt{5} & \frac{-1}{2\sqrt{5}} \\ 0 & 0 & \sqrt{\frac{27}{10}} \end{bmatrix}$$

Since

$Ax = U^T U x = b$, let $Ux = y$, then $U^T y = b$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & \sqrt{5} & 0 \\ \frac{-1}{2} & \frac{-1}{2\sqrt{5}} & \sqrt{\frac{27}{10}} \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 0 \end{bmatrix}$$

$$4 = 2y_1 \qquad \Rightarrow y_1 = 2$$

$$-y_1 + \sqrt{5}y_2 = -6 \qquad \Rightarrow y_2 = \frac{-4}{\sqrt{5}}$$

$$\frac{-1}{2}y_1 - \frac{1}{2\sqrt{5}}y_2 + \sqrt{\frac{27}{10}}y_3 = 0 \qquad \Rightarrow y_3 = 0.36515$$

$$y = U \times x$$

$$\begin{bmatrix} 2 \\ \dfrac{-4}{\sqrt{5}} \\ 0.36515 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -\dfrac{1}{2} \\ 0 & \sqrt{5} & \dfrac{-1}{2\sqrt{5}} \\ 0 & 0 & \sqrt{\dfrac{27}{10}} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\sqrt{\frac{27}{10}}x_3 = 0{,}36515 \qquad \Rightarrow x_3 = 0.2222$$

$$\sqrt{5}x_2 - \frac{1}{2\sqrt{5}}x_3 = -\frac{4}{\sqrt{5}} \qquad \Rightarrow x_2 = -0.7777$$

$$2x_1 - x_2 - \frac{1}{2}x_3 = 2 \qquad \Rightarrow x_1 = 0.6666$$

Norms of Vectors and Matrices

We need to measure the distance between vectors to determine whether a sequence of vectors converges to a solution of the system.

Let R^n denote the set of all n-dimensional column vectors with real-number coefficients. To define a distance in R^n we use the notion of a norm.

A vector norm on R^n is a function, $\| \ \|$ from R^n into R with the following properties

1. $\|x\| \geq 0$ for all x.
2. $\|x\| = 0$ if and only if **x=0.**
3. $\|\alpha x\| = |\alpha| \times \|x\|$ for all α=constant and all x.
4. $\|x + y\| \leq \|x\| + \|y\|$ for all x and y.

The l_1, l_2, l_∞ norms for the vector $x = (x_1, x_2, \cdots, x_n)'$ are defined by

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \qquad \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

l_2 norm is called the Euclidean norm of the vector x

Example: $x = (-1, 1, -2)'$ $\qquad \|x\|_1 = |-1| + |1| + |-2| = 4$

$$\|x\|_2 = \sqrt{(-1)^2 + (1)^2 + (-2)^2} = \sqrt{6}$$

$$\|x\|_\infty = \max(|-1|, |1|, |-2|) = 2$$

Note that for each x $\|x\|_\infty \le \|x\|_2$

If $x = (x_1, x_2, \cdots, x_n)'$ and $y = (y_1, y_2, \cdots, y_n)'$ the l_1, l_2, l_∞ distances between x and y are defined by:

$$\|x - y\|_1 = \sum_{i=1}^{n} |x_i - y_i| \qquad \|x - y\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \qquad \|x - y\|_\infty = \max_{1 \le i \le n}(|x_i - y_i|)$$

Example: $x = (1, 1, 1)'$ and $y = (1.2001, 0.99991, 0.92538)'$

$$\|x - y\|_1 = \sum_{i=1}^{n} |x_i - y_i| = |1 - 1.2001| + |1 - 0.99991| + |1 - 0.92538| = 0.27281$$

$$\|x - y\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \sqrt{(1 - 1.2001)^2 + (1 - 0.9991)^2 + (1 - 0.92538)} = 0.21356$$

$$\|x - y\|_\infty = \max_{1 \le i \le n}(|x_i - y_i|) = \max(|1 - 1.2001|, |1 - 0.99991|, |1 - 0.92538|) = 0.2001$$

The $n \times n$ matrix A is said to be strictly diagonally dominant when

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|$$

Example

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix} \qquad \begin{array}{l} |7| > |2| + |0| \ \ yes \\ |5| > |3| + |-1| \ \ yes \\ |-6| > |0| + |5| \ \ yes \end{array}$$

$$B = \begin{bmatrix} 6 & 4 & 3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix} \qquad \begin{array}{ll} |6| > |4| + |3| & no \\ |-2| > |4| + |0| & no \\ |1| > |-3| + |0| & no \end{array}$$

Cholesky Decomposition For Symmetrical Matrices

* وهذه الطريقة تستخدم لحل النظام من المعادلات الخطية

بشرط أن تكون المصفوفة Positive definite

$$Ax = b$$

$$A = LU$$

$$\Rightarrow A = U^T U$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$ given معطاه

الطريقة العامة

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$ المطلوب حساب

بالترتيب

1. $u_{11} = \sqrt{a_{11}}$
2. $u_{12} = \dfrac{a_{12}}{u_{11}}$
3. $u_{13} = \dfrac{a_{13}}{u_{11}}$
4. $u_{22} = \sqrt{a_{22} - u_{12}^2}$
5. $u_{23} = \dfrac{a_{23} - u_{12}u_{13}}{u_{22}}$
6. $u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2}$

given $\Rightarrow$ System of equations ?

المعادلة الاولى ________ $= z_1$

المعادلة الثانية ________ $= z_2$

الثالثة ________ $= z_3$

الرابعة ________ $= z_4$

$\Downarrow$

خطر

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 5 & 90 \\ 0 & \boxed{0} & -2 & 3 & 80 \\ 0 & 2 & 1 & -2 & -40 \\ 0 & 6 & -1 & 1 & 20 \end{array}\right] \begin{array}{l} \\ R_2-(a)R_1 \\ R_3-(b)R_1 \\ R_4-(c)R_1 \end{array}$$

Pivoting الصف الثاني مع الصف الرابع

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 5 & 90 \\ 0 & 6 & -1 & 1 & 20 \\ 0 & 2 & 1 & -2 & -40 \\ 0 & 0 & -2 & 3 & 80 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 5 & 90 \\ 0 & 6 & -1 & 1 & 20 \\ 0 & 0 & 4/3 & -7/3 & -140/3 \\ 0 & 0 & -2 & 3 & 80 \end{array}\right] \begin{array}{l} \\ \\ R_3-\left(\frac{2}{6}\right)R_2 \\ R_4-\left(\frac{0}{6}\right)R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 5 & 90 \\ 0 & 6 & -1 & 1 & 20 \\ 0 & 0 & \frac{4}{3} & \frac{-7}{3} & \frac{-140}{3} \\ 0 & 0 & 0 & -\frac{1}{2} & 10 \end{array}\right] R_4 - \left(\frac{-2}{4/3}\right) R_3$$

$$U = \begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 6 & -1 & 1 \\ 0 & 0 & \frac{4}{3} & -\frac{7}{3} \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$L^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ c & 1 & 0 & 0 \\ b & 2/6 & 1 & 0 \\ a & 0/6 & {}^{-2}/_{4/3} & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 0/6 & {}^{-2}/_{4/3} & 1 \\ b & 2/6 & 1 & 0 \\ c & 1 & 0 & 0 \end{bmatrix}$$

$$Ly = b$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 0/6 & -2/4/3 & 1 \\ b & 2/6 & 1 & 0 \\ c & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix}$$

Z هون في السؤال الأصلي بنفس الترتيب

$$Ux = y$$

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 6 & -1 & 1 \\ 0 & 0 & 4/3 & -7/3 \\ 0 & 0 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}$$

هاشم الخالدي
هندسة ميكانيكية

Tridiagonal Matrix

Tridiagonal Matrix
(Thomas Algorithm)

Finding L & U decomposed matrices.

المصفوفة ثلاثية الأقطار

(Thomas Algorithm)

كيف نجد ال L وال U لهذه المصفوفة كالآتي

$$A = \begin{bmatrix} f_1 & g_1 & 0 & 0 \\ e_2 & f_2 & g_2 & 0 \\ 0 & e_3 & f_3 & g_3 \\ 0 & 0 & e_4 & f_4 \end{bmatrix}$$

* لا يوجد تغيير على f_1 ولا على g_1 g_2 g_3

* $e_2 = e_2 / f_1 \Rightarrow f_2 = f_2 - e_2 \cdot g_1$
* $e_3 = e_3 / f_2 \Rightarrow f_3 = f_3 - e_3 \, g_2$
* $e_4 = e_4 / f_3 \Rightarrow f_4 = f_4 - e_4 \, g_3$
* $e_5 = e_5 / f_4 \Rightarrow f_5 = f_5 - e_5 \, g_4$

⋮

وهكذا

* $e_9 = e_9 / f_8 \Rightarrow f_9 = f_9 - e_9 \, g_8$

Find: L U For the Tridiagonal Matrix A :

$$A = \begin{bmatrix} 2^{\text{ f1}} & -1^{\text{ g1}} & 0 & 0 \\ -1^{\text{ e2}} & 2^{\text{ f2}} & -1^{\text{ g2}} & 0 \\ 0 & -1^{\text{ e3}} & 2^{\text{ f3}} & -1^{\text{ g3}} \\ 0 & 0 & -1^{\text{ e4}} & 2^{\text{ f4}} \end{bmatrix}$$

$$e_2 = \frac{-1}{2} \Rightarrow f_2 = 2 - \left(\frac{-1}{2}\right)(-1) = \frac{3}{2}$$

$$e_3 = {}^{-1}\!\big/\tfrac{3}{2} = \frac{-2}{3} \Rightarrow f_3 = 2 - \left(\frac{-2}{3}\right)(-1) = \frac{4}{3}$$

$$e_4 = {}^{-1}\!\big/\tfrac{4}{3} = \frac{-3}{4} \Rightarrow f_4 = 2 - \left(\frac{-3}{4}\right)(-1) = \frac{5}{4}$$

$$A_{\text{new}} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ \frac{-1}{2} & \frac{3}{2} & -1 & 0 \\ 0 & \frac{-2}{3} & \frac{4}{3} & -1 \\ 0 & 0 & \frac{-3}{4} & \frac{5}{4} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}$$

Solve (LU Decomposition)

$$x_1 + x_2 + x_3 = 3$$

$$x_1 + 2x_2 + 2x_3 = 5$$

$$x_1 + 2x_2 + 3x_3 = 5$$

1. Write system $AX = B$?

2. Find L and U ?

3. Solve the solution x ?

4. Determinant $|A|$?

Solve :   (LU Decomposition)

$$X_1 + 2X_2 + 3X_3 = 35$$

$$2X_1 + 2X_2 - X_3 = 0$$

$$4X_1 + 6X_2 + 5X_3 = 70$$

1. $AX = B$

2. L & U

3. Solution X

4. $|A|$

Matrix Inverse : (A^{-1})
Using $\underline{\underline{L}}$ & $\underline{\underline{U}}$

⇒ Starting By (L) & (U) matricies

Step (1)

$[L]\{d\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$ ⇒ Solve for $\{d\}$

$[U]\{x\} = \{d\}$ ⇒ Solve for $\{x\}$

⇒ $\{X\}$ is the first Column of $A^{-1} = \begin{bmatrix} x_1 & - & - \\ x_2 & - & - \\ x_3 & - & - \end{bmatrix}$

Step (2)

$[L]\{d\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$ ⇒ Solve for $\{d\}$

$[U]\{x\} = \{d\}$ ⇒ Solve for $\{x\}$

⇒ $\{x\}$ is the Second Column of $A^{-1} = \begin{bmatrix} - & x_1 & - \\ - & x_2 & - \\ - & x_3 & - \end{bmatrix}$

Step (3)

$[L]\{d\} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ⇒ Solve for $\{d\}$

$[U]\{x\} = \{d\}$ ⇒ Solve for $\{x\}$

⇒ $\{x\}$ is the third Column of $A^{-1} = \begin{bmatrix} - & - & x_1 \\ - & - & x_2 \\ - & - & x_3 \end{bmatrix}$

Ex Given: $L = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 1 & 0 \\ 0.3 & -0.5 & 1 \end{bmatrix}$ & $U = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0.5 & 0.7 \\ 0 & 0 & -0.8 \end{bmatrix}$

Find the inverse of the system matrix (A^{-1})

where $A = LU$

Solution:

Step (1) $\begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 1 & 0 \\ 0.3 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ Solve for $\{d\}$ $\Rightarrow$ $d_1 = 1$, $d_2 = -0.1$, $d_3 = -0.35$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 0.5 & 0.7 \\ 0 & 0 & -0.8 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.1 \\ -0.35 \end{bmatrix}$$

$\Rightarrow$ Solve for $\{x\}$ $\Rightarrow$ $X_3 = 0.43$, $X_2 = -0.8$, $X_1 = 0.72$

$$\text{So, } A^{-1} = \begin{bmatrix} 0.72 & - & - \\ -0.8 & - & - \\ 0.43 & - & - \end{bmatrix}$$

Step (2) $\Rightarrow$ Repeat step (1) to find the second Column of A^{-1}.

Step (3) $\Rightarrow$ Repeat step (1) to find the third Column of A^{-1}.

Iterative method

① Jaccobi Method

② Gauss-Seidel Method

Norm $\|x\|$

$\|x\|_1 =$ مجموع القيم المطلقة للمتجه

$\|x\|_2 = \sqrt{\text{مجموع مربعات القيم للمتجه}}$

$\|x\|_\infty =$ اكبر قيمة مطلقة في المتجه

الاكثر استخداما لحسابات الخطأ error

ex

$x = \{2, -3, 0, 5, -2, -6\}$ Vetor

Find : $\|x\|_1 = 2+3+0+5+2+6 = 18$

Find : $\|x\|_2 = \sqrt{4+9+0+25+4+36} = \sqrt{78}$

Find : $\|x\|_\infty = 6$

ex

$x_1 = \{2.56, -3.63, 4.95\}$

$x_2 = \{2.55, -3.61, 4.92\}$

الخطأ

$\text{Error} = \|x_2 - x_1\|_\infty = \{-0.01, 0.02, -0.03\} = 0.03$

الكبيرة

* Strictlly ~~diagullauly~~ Dominant Matrix

$$A = \begin{bmatrix} 8 & -2 & 1 \\ -4 & 7 & 2 \\ -5 & -3 & -10 \end{bmatrix} \begin{array}{l} \rightarrow 8 \geqslant 2+1 \checkmark \\ \rightarrow 7 \geqslant 4+2 \checkmark \\ \rightarrow 10 \geqslant 5+3 \checkmark \end{array}$$

Matrix A is diagonally Dominant

المصفوفه A قطرياً مسيطره

$$B = \begin{bmatrix} 4 & 2 & 0 \\ -3 & 9 & 6 \\ -8 & 4 & 8 \end{bmatrix} \begin{array}{l} \rightarrow 4 \geqslant 2+0 \checkmark \\ \rightarrow 9 \geqslant 3+6 \checkmark \\ \rightarrow 8 \geqslant 8+4 \ X \end{array}$$

Matrix B is **Not** diag. Dominant

$$C = \begin{bmatrix} 4 & -2 & 9 \\ 6 & 5 & -1 \\ -2 & 6 & 3 \end{bmatrix} \begin{array}{l} \rightarrow 4 \geqslant 2+9 \ X \end{array}$$

Not Dominant

هل يمكن تحويل C الى Dominant ← نعم عن طريق تبديل الصفوف

$$\Rightarrow \begin{bmatrix} 6 & 5 & -1 \\ -2 & 6 & 3 \\ 4 & -2 & 9 \end{bmatrix} \begin{array}{l} \rightarrow 6 \geqslant 6 \\ \rightarrow 6 \geqslant 5 \\ \rightarrow 9 \geqslant 6 \end{array}$$

تحولت الى Dominant

* Jacobi Method
* and Siedel Method

الشرط

* حتى نستخدم هاتان الطريقتان لابد أن تكون

المصفوفة Diagonally Dominant

* اذا لم تكن المصفوفة Dominant فلابد

من تحويلها الى Dominant اذا كان بالامكان

عن طريق تبديل الاسطر والصفوف .

* لحساب الخطأ بين الاشواط لا بد من

استخدام الـ Norm

$$\| \underbrace{X^{(n+1)}}_{\text{new}} - \underbrace{X^{(n)}}_{\text{old}} \|_\infty \leq \varepsilon$$

Stop.

ex

* Initial guess: $\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array}$ $[0, 0, 0, 0]$

$\varepsilon = 1\times10^{-5}$

$$8X_1 - 2X_2 + 4X_3 - X_4 = 8$$

$$-2X_1 + 9X_2 - 3X_3 = -20$$

$$4X_1 - X_2 - 10X_3 + X_4 = 0$$

$$-2X_1 + 3X_3 + 6X_4 = -7$$

الحل

$$\begin{bmatrix} 8 & -2 & 4 & -1 \\ -2 & 9 & -3 & 0 \\ 4 & -1 & -10 & 1 \\ -2 & 0 & 3 & 6 \end{bmatrix}$$

Dominant

O.K

$$X_1 = \frac{1}{8}\left[8 + 2X_2 - 4X_3 + X_4\right]$$

$$X_2 = \frac{1}{9}\left[-20 + 2X_1 + 3X_3\right]$$

$$X_3 = \frac{-1}{10}\left[0 - 4X_1 + X_2 - X_4\right]$$

$$X_4 = \frac{1}{6}\left[-7 + 2X_1 - 3X_3\right]$$

هنضفي المعادلات كتيرة في
حل أي له وتنفيذ الأسواء

Jacobi : $[0, 0, 0, 0]$

الشوط الاول
$(n=1)$

$$X_1 = 1 = \frac{1}{8}\left[8 + 2(0) - 4(0) + 0\right]$$
$$X_2 = -2.222 = \frac{1}{9}\left[-20 + 2(0) + 3(0)\right]$$
$$X_3 = 0 = \frac{-1}{10}\left[0 - 4(0) + 0 - 0\right]$$
$$X_4 = -1.167 = \frac{1}{6}\left[-7 + 2(0) - 3(0)\right]$$

Error : $E = \| 1, -2.222, 0, -1.167 \|_\infty = 2.222 > \varepsilon$

Cont.

$[1, -2.222, 0, -1.167]$

الشوط الثاني
$(n=2)$

$$X_1 = \frac{1}{8}\left[8 + 2(-2.222) - 4(0) - 1.167\right] = 0.299$$
$$X_2 = \frac{1}{9}\left[-20 + 2(1) + 3(0)\right] = -2$$
$$X_3 = \frac{-1}{10}\left[0 - 4(1) + \overset{-2.222}{\cancel{-0.222}} - -1.167\right] = 0.306$$
$$X_4 = \frac{1}{6}\left[-7 + 2(1) - 3(0)\right] = -0.8333$$

$$E = \| -0.701, 0.222, 0.306, 0.333 \|_\infty$$
$$= 0.701 > \varepsilon \quad \text{Cont.}$$

$[0.299, -2, 0.306, -0.833]$

الشوط الثالث
$(n=3)$

$X_1 = --$ $\quad X_2 = --$ $\quad X_3 = --$ $\quad X_4 = --$

Gauss-Siedel $[0, 0, 0, 0]$

الدور الاول
$(n=1)$

$$X_1 = \frac{1}{8}[8 + 2(0) - 4(0) + 0] = 1$$

$$X_2 = \frac{1}{9}[-20 + 2(1) + 3(0)] = -2$$

$$X_3 = \frac{-1}{10}[0 - 4(1) + (-2) - 0] = 0.6$$

$$X_4 = \frac{1}{6}[-7 + 2(1) - 3(0.6)] = -1.133$$

Error: E = ---

الدور الثاني
$(n=2)$

$[1, -2, 0.6, -1.133]$

$$X_1 = \frac{1}{8}[8 + 2(-2) - 4(0.6) - 1.133] = 0.05833$$

$$X_2 = \frac{1}{9}[-20 + 2(0.05833) + 3(0.6)] = -2.009$$

$$X_3 = \frac{-1}{10}[0 - 4(0.05833) + -2.009 - -1.133] = 0.119$$

$$X_4 = \frac{1}{6}[-7 + 2(0.05833) - 3(0.119)] = -1.207$$

Error E = ---

ex) Using <u>Jacobi</u> and <u>Siedel</u> :-

$$5X_1 - 2X_2 + 8X_3 = 10$$
$$-4X_1 - X_2 + 3X_3 = -6$$
$$2X_1 + 7X_2 - 2X_3 = -1$$

$[0, 0, 0]$

$\varepsilon = 1 \times 10^{-2}$

$$\begin{bmatrix} 5 & -2 & 8 \\ -4 & -1 & 3 \\ 2 & 7 & -2 \end{bmatrix}$$ **Not Dominant**

حول الى Dominat بتبديل الأسطر

$$\begin{bmatrix} -4 & -1 & 3 \\ 2 & 7 & -2 \\ 5 & -2 & 8 \end{bmatrix} \Rightarrow \begin{aligned} -4X_1 - X_2 + 3X_3 &= -6 \\ 2X_1 + 7X_2 - 2X_3 &= -1 \\ 5X_1 - 2X_2 + 8X_3 &= 10 \end{aligned}$$

$$X_1 = \frac{-1}{4}\left[-6 + X_2 - 3X_3\right]$$
$$X_2 = \frac{1}{7}\left[-1 - 2X_1 + 2X_3\right]$$
$$X_3 = \frac{1}{8}\left[10 - 5X_1 + 2X_2\right]$$

"Reference"
هذا هو المرجع في التعويض

1 Jaccobi Method $X = [0, 0, 0]$

Solve
$$2X_1 - 4X_2 + X_3 = -14$$
$$X_1 + X_2 + 4X_3 = -15$$
$$-3X_1 - X_2 + 2X_3 = -7$$

Check:

الحل

Not Diag. Dominant $\varepsilon = 0.05$

نعيد ترتيب الصفوف

لتكون Dominant $\Rightarrow$

$$-3X_1 - X_2 + 2X_3 = -7 \longrightarrow X_1 = \frac{-1}{3}[-7 + X_2 - 2X_3]$$
$$2X_1 - 4X_2 + X_3 = -14 \longrightarrow X_2 = \frac{-1}{4}[-14 - 2X_1 - X_3]$$
$$X_1 + X_2 + 4X_3 = -15 \longrightarrow X_3 = \frac{1}{4}[-15 - X_1 - X_2]$$

$$\begin{array}{ccc} X_1 & X_2 & X_3 \\ \end{array}$$
$$[0, 0, 0]$$

أول الحلول

$$\Rightarrow X_1 = \frac{-1}{3}[-7 + 0 - 0] = 2.333$$
$$\Rightarrow X_2 = \frac{-1}{4}[-14 - 0 - 0] = 3.5$$
$$X_3 = \frac{1}{4}[-15 - 0 - 0] = -3.75$$

error $= 3.75 \geq \varepsilon$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$
$$\left[\begin{array}{ccc} 2.333, & 3.5, & -3.75 \end{array}\right]$$

الخطوة الثانية

$$x_1 = \frac{-1}{3}\left[-7 + 3.5 - 2(-3.75)\right] = -1.333$$

$$x_2 = \frac{-1}{4}\left[-14 - 2(2.333) - (-3.75)\right] = 3.729$$

$$x_3 = \frac{1}{4}\left[-15 - 2.333 - 3.5\right] = -5.208$$

$$\underline{\underline{\text{error}}} = \left\{ \overset{\text{الأكبر}}{-3.666},\ 0.229,\ -1.458 \right\} = 3.666 > \varepsilon$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc} -1.333, & 3.729, & -5.208 \end{array}\right] \qquad \text{Cont.}$$

الخطوة الثالثة

$$x_1 = \frac{-1}{3}\left[-7 + 3.729 - 2(-5.208)\right] = -2.382$$

$$x_2 = \frac{-1}{4}\left[-14 - 2(-1.333) - (-5.208)\right] = --$$

$$x_3 = \frac{1}{4}\left[-15 - (-1.333) - (3.729)\right] = --$$

2 Gauss_Seidel

نفس السؤال السابق

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$
$$\begin{bmatrix} 0, & 0, & 0 \end{bmatrix}$$

الشوط الأول

$$X_1 = \frac{-1}{3}\left[-7+0-0\right] = 2.333$$

$$X_2 = \frac{-1}{4}\left[-14-2(2.333)-0\right] = 4.667$$

$$X_3 = \frac{1}{4}\left[-15-2.333-4.667\right] = -5.5$$

$$\left[2.333 \;,\; 4.667 \;,\; -5.5\right]$$

الشوط الثاني

$$X_1 = \frac{-1}{3}\left[-7+4.667-2(-5.5)\right] = -2.889$$

$$X_2 = \frac{-1}{4}\left[-14-2(-2.889)-(-5.5)\right] = 0.681$$

$$X_3 = \frac{1}{4}\left[-15-(-2.889)-0.681\right] = \cdots$$

* Electrical Applications:-

Given

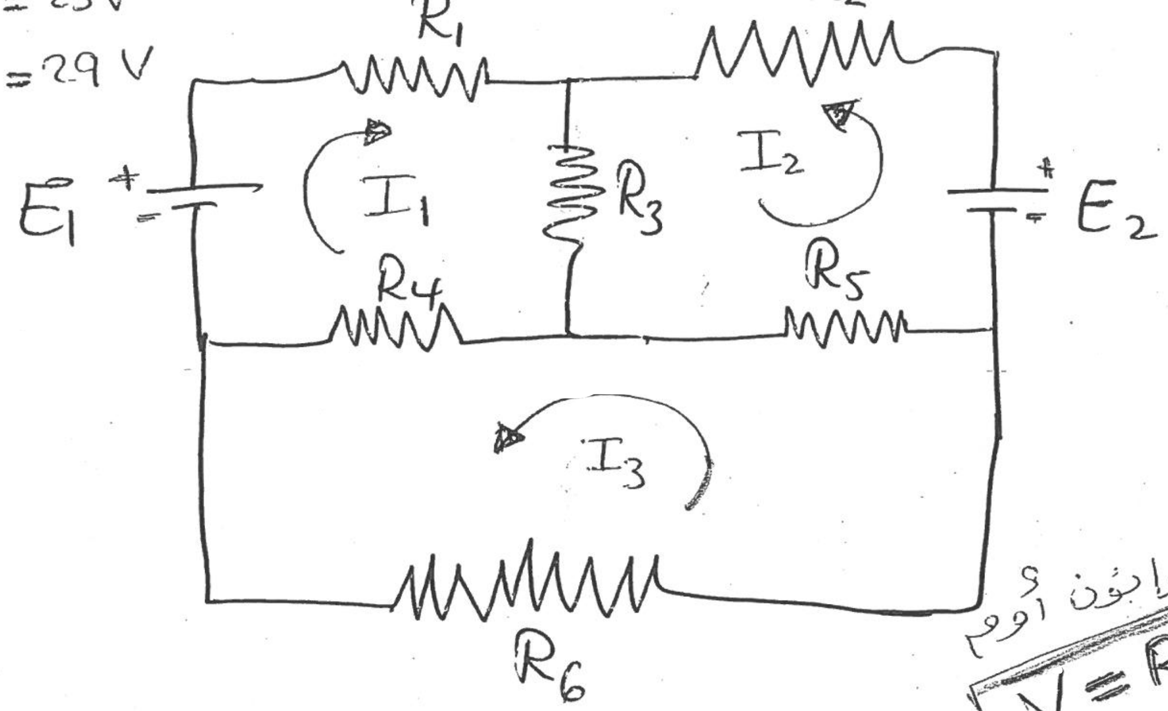
$R_1 = R_2 = R_4 = 1\,\Omega$

$R_3 = R_5 = 2\,\Omega$

$R_6 = 4\,\Omega$

$E_1 = 23\,V$

$E_2 = 29\,V$



قانون أوم

$$\boxed{V = RI}$$

المعادلة الأولى

loop ①

$$(R_1 + R_3 + R_4)\,I_1 + R_3 I_2 + R_4 I_3 = E_1$$

المعادلة الثانية

Loop ②

$$(R_2 + R_3 + R_5)\,I_2 + R_3 I_1 - R_5 I_3 = E_2$$

المعادلة الثالثة

loop ③

$$(R_4 + R_5 + R_6)\,I_3 - R_5 I_2 + R_4 I_1 = 0$$

$$4I_1 + 2I_2 + I_3 = 23 \quad \text{---} (1)$$

$$5I_2 + 2I_1 - 2I_3 = 29 \quad \text{---} (2)$$

$$7I_3 - 2I_2 + I_1 = 0 \quad \text{---} (3)$$

يجب ترتيب المعادلات

$$\boxed{\begin{aligned} 4I_1 + 2I_2 + I_3 &= 23 \\ 2I_1 + 5I_2 - 2I_3 &= 29 \\ I_1 - 2I_2 + 7I_3 &= 0 \end{aligned}}$$

يمكن حل هذه المعادلات الثلاثة بالطرق التالية

Direct Method ⇓

1. $\boxed{\text{Gauss - Elimination}}$
2. Cramer's Rule
3. Inverse Method.
4. LU decomposition

Iterative Method ⇓

1. Gauss - Jacobi
2. Gauss Siedel

* Using Gauss- Elimination :

$$\left[\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ 2 & 5 & -2 & 29 \\ 4 & 2 & 1 & 23 \end{array}\right] \quad -2R_1+R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ 0 & 9 & -16 & 29 \\ 4 & 2 & 1 & 23 \end{array}\right] \quad R_2/9$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ 0 & 1 & -1.7777 & 3.222 \\ 4 & 2 & 1 & 23 \end{array}\right] \quad -4R_1+R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ 0 & 1 & -1.777 & 3.222 \\ 0 & 10 & -27 & 23 \end{array}\right] \quad -10R_2+R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ 0 & 1 & -1.7777 & 3.222 \end{array}\right. \quad R_3/-9.23$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ 0 & 1 & -1.7777 & 3.222 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Using Back substitution.

$$\boxed{I_3 = 1}$$

$$I_2 - 1.7777 = 3.2222$$

$$\boxed{I_2 = 5}$$

$$I_1 - 2(5) + 7(1) = 0$$

$$I_1 = 10 - 7$$

$$\boxed{I_1 = 3}$$

Mechanical Applications

→ 4 springs in series being depress with a force of

القوة $\boxed{F = 2000\ Ib}$ and Expressed
باوند

in the following equations قانون هوك

$\boxed{F = KX}$ Hooke's law

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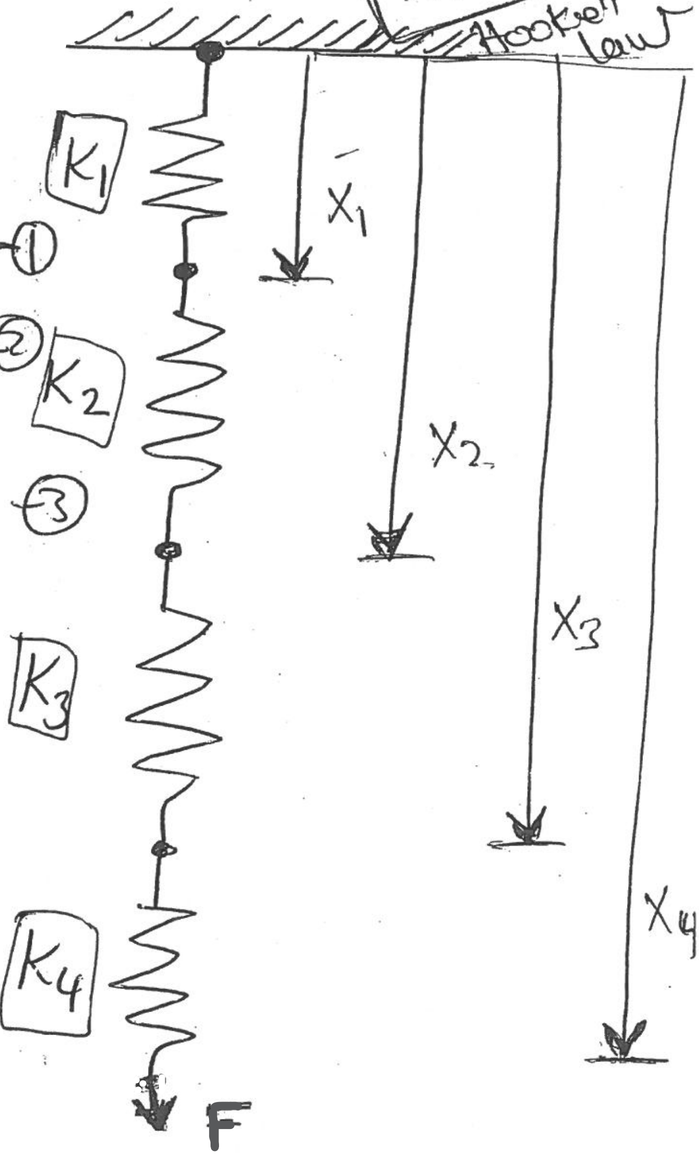
$$K_1 X_1 = K_2(X_2 - X_1) \quad (1)$$

$$K_2(X_2 - X_1) = K_3(X_3 - X_2) \quad (2)$$

$$K_3(X_3 - X_2) = K_4(X_4 - X_3) \quad (3)$$

$$K_4(X_4 - X_3) = F \quad (4)$$



معطى في السؤال

$K_1 = 100$ Ib/in

$K_2 = 50$ Ib/in

$K_3 = 75$ Ib/in

$K_4 = 200$ Ib/in

$$100 X_1 = 50(X_2 - X_1)$$

$$100 X_1 = 50 X_2 - 50 X_1$$

$$150 X_1 - 50 X_2 = 0$$

$$3 X_1 - X_2 = 0 \quad \text{——} \quad (1)$$

$$50(X_2 - X_1) = 75(X_3 - X_2)$$

$$X_2 - X_1 = 1.5 X_3 - 1.5 X_2$$

$$X_1 - X_2 + 1.5 X_3 - 1.5 X_2 = 0$$

$$X_1 - 2.5 X_2 + 1.5 X_3 = 0$$

$$\Rightarrow 2X_1 - 5X_2 + 3X_3 = 0 \quad \text{——} \quad (2)$$

$$75(X_3 - X_2) = 200(X_4 - X_3)$$

$$X_3 - X_2 = 2.666(X_4 - X_3)$$

$$X_2 - X_3 + 2.667 X_4 - 2.667 X_3 = 0$$

$$\Rightarrow X_2 - 3.667 X_3 + 2.667 X_4 = 0$$

$$\Rightarrow 3X_2 - 11 X_3 + 8 X_4 = 0 \quad \text{——} \quad (3)$$

$$200(X_4 - X_3) = 2000$$

$$X_4 - X_3 = 10$$

$$\Rightarrow \quad X_3 - X_4 = -10 \quad \text{———} \quad (4)$$

$$\Rightarrow \begin{bmatrix} 3 & -1 & 0 & 0 \\ 2 & -5 & 3 & 0 \\ 0 & 3 & -11 & 8 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -10 \end{bmatrix}$$

أربع معادلات بـ ٤ مجاهيل

$$[A][X] = [b]$$

* Using inverse method $[X] = [A]^{-1}[b]$

*** Using Matlab :

```
A = [3 -1 0 0; 2 -5 3 0;
     0 3 -11 8; 0 0 1 -1];
C = inv(A);
b = [0; 0; 0; -10];
X = [C*b]
```

Answers:-

$$X = \begin{bmatrix} 20 \\ 60 \\ 86.6667 \\ 96.6667 \end{bmatrix}$$

* Solving Two Nonlinear Equations

حل معادلتين غير خطيتين بطريقة نيوتن

(Newton's Methods)

لديك معادلتان غير خطيتان هما :

$$f(x,y) = 0$$

$$g(x,y) = 0$$

خطوات الحل

① Find Jacobian Matrix $J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$

②
$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -f \\ -g \end{bmatrix}$$

⟸ حيث يتم حل هاتان المعادلتان إما بالآلة الحاسبة مباشرة أو باستخدام طريقة Cramer أو إستخدام الطريقة المعكوس inverse

③ $\Delta x = x_{new} - x_{old} \Rightarrow \boxed{x_{new} = x_{old} + \Delta x}$

④ $\Delta y = y_{new} - y_{old} \Rightarrow \boxed{y_{new} = y_{old} + \Delta y}$

⑤ Repeat until reaches Tolerance $|\Delta x| \leq \varepsilon$ & $|\Delta y| \leq \varepsilon$

حيث نستمر في الاشواط حتى نحقق قيمة الخطأ المطلوب

Example use Newton's Method to solve :

$$x^2 - 2x - y = -0.5$$

$$x^2 + 4y^2 = 4$$

using the initial guess $x_0 = 2$, $y_0 = 0.25$

and do only Two iterations .

الحل

$$f = x^2 - 2x - y + 0.5$$

$$g = x^2 + 4y^2 - 4$$

$$\left.\begin{aligned} * \ f_x &= 2x - 2 \\ * \ f_y &= -1 \\ * \ g_x &= 2x \\ * \ g_y &= 8y \end{aligned}\right\} \Rightarrow J = \begin{bmatrix} 2x-2 & -1 \\ 2x & 8y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x-2 & -1 \\ 2x & 8y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -x^2 + 2x + y - 0.5 \\ -x^2 - 4y^2 + 4 \end{bmatrix}$$

الشوط الأول

عوض بالالة الحاسبة عن

$x = 2$
$y = 0.25$

Mode 5 1

$$\begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -0.25 \\ -0.25 \end{bmatrix}$$

حل هاتان المعادلتان بالآلة الحاسبة [illegible] (Mode 5/1)

$$\Delta x = -0.09375$$

$$\Delta y = 0.0625$$

$$\Rightarrow \quad x_1 = x_0 + \Delta x \quad و \quad y_1 = y_0 + \Delta y$$

$$x_1 = 2 + -0.09375 \quad و \quad y_1 = 0.25 + 0.0625$$

$$x_1 = 1.90625 \qquad y_1 = 0.3125$$

الخط الثاني

عوض

$$\begin{bmatrix} 1.8125 & -1 \\ 3.8125 & 2.5 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -8.79 * 10^{-3} \\ -0.02441 \end{bmatrix}$$

إستخدم الآلة الحاسبة (Mode 5/1)

$$\Delta x = -5.559 * 10^{-3}$$

$$\Delta y = -1.286 * 10^{-3}$$

$$\Rightarrow \quad x_2 = x_1 + \Delta x \quad و \quad y_2 = y_1 + \Delta y$$

$$\Rightarrow \quad x_2 = 1.90625 - 5.559 * 10^{-3} \quad و \quad y_2 = 0.3125 - 1.286 * 10^{-3}$$

$$x_2 = 1.9007 \qquad y_2 = 0.3112$$

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هندسة ميكانيكية

ex

سؤال امتحان

Use Newtons Method :

$$16x^2 + 5\sqrt{y} = 4$$

$$2\sqrt{xy} - \sqrt{y} = 0$$

for point Q $(x = y = 0.25)$

(a) Find Jacobian Matrix at point Q ? → J ?

(b) Find the inverse of the Jacobian Matrix? → J^{-1} ?

(c) do Two iterations to solve the problem ? → $x = ?$ $y = ?$

الحل

$$f = 16x^2 + 5\sqrt{y} - 4$$

$$g = 2\sqrt{xy} - \sqrt{y}$$

$$f_x = 32x$$

$$f_y = \frac{5}{2\sqrt{y}}$$

$$g_x = \frac{\cancel{2}y}{\cancel{2}\sqrt{xy}} = \frac{y}{\sqrt{xy}}$$

$$g_y = \frac{\cancel{2}x}{\cancel{2}\sqrt{xy}} - \frac{1}{2\sqrt{y}} = \frac{x}{\sqrt{xy}} - \frac{1}{2\sqrt{y}}$$

هاشم الخالدي
هندسة ميكانيكية

(a)

$$J = \begin{bmatrix} 32x & \frac{5}{2\sqrt{y}} \\ \frac{y}{\sqrt{xy}} & \frac{x}{\sqrt{xy}} - \frac{1}{2\sqrt{y}} \end{bmatrix} \xrightarrow[\substack{x=0.25 \\ y=0.25}]{\text{عوض}} J = \begin{bmatrix} 8 & 5 \\ 1 & 0 \end{bmatrix}$$

(b) $|J| = (8)(0) - (5)(1) = 0 - 5 = -5$

$$\Rightarrow J^{-1} = \frac{-1}{5}\begin{bmatrix} 0 & -5 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{5} & -\frac{8}{5} \end{bmatrix}$$

$$\Rightarrow J^{-1} = \begin{bmatrix} 0 & 1 \\ 0.2 & -1.6 \end{bmatrix}$$

(c) الشوط الأول

$$\begin{bmatrix} 8 & 5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \Delta X \\ \Delta y \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

بالآلة الحاسبة $\Leftarrow$

$\Delta X = 0 \Rightarrow X_1 = X_0 + \Delta X = 0.25$

$\Delta = 0.1 \Rightarrow y_1 = y_0 + \Delta y = 0.35$

الشوط الثاني

$$\begin{bmatrix} 8 & 4.2257 \\ 1.1832 & 0 \end{bmatrix}\begin{bmatrix} \Delta X \\ \Delta y \end{bmatrix} = \begin{bmatrix} 0.04196 \\ 0 \end{bmatrix}$$

$\Delta X = 0 \Rightarrow X_2 = X_1 + \Delta X \Rightarrow X_1 = 0.25$

$\Delta y = 9.929 * 10^{-3} \Rightarrow y_2 = y_1 + \Delta y \Rightarrow y_2 = 0.35993$

ex

سؤال امتحان

Use Newton's to solve :-

$$\sin(x_1) - \cos(x_2) = 0$$
$$\cos(x_1) + \sin(x_2) = 2$$

* شروط بداية

$x_1 = \pi/2$
$x_2 = \pi/2$

الحل

حوّل شكل المسألة الى x, y
بدل من x_1، x_2

$$\Rightarrow \quad \sin x - \cos y = 0 \quad , \quad x_0 = \pi/2$$
$$\cos x + \sin y = 2 \quad , \quad y_0 = \pi/2$$

$$f = \sin x - \cos y$$
$$g = \cos x + \sin y - 2$$

$$\left.\begin{aligned} f_x &= \cos x \\ f_y &= \sin y \\ g_x &= -\sin x \\ g_y &= \cos y \end{aligned}\right\} \Rightarrow J = \begin{bmatrix} \cos x & \sin y \\ -\sin x & \cos y \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \cos x & \sin y \\ -\sin x & \cos y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -\sin x + \cos y \\ -\cos x - \sin y + 2 \end{bmatrix}}$$

$$\begin{bmatrix} \cos x & \sin y \\ -\sin x & \cos y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -\sin x + \cos y \\ -\cos x - \sin y + 2 \end{bmatrix}$$

الخطوة الاول

$$x_0 = \pi/2$$
$$y_0 = \pi/2$$

الالة بالراديان

مهم

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Delta x = -1$$
$$\Delta y = -1$$

$$x_1 = x_0 + \Delta x = \frac{\pi}{2} - 1 = 0.57079$$
$$y_1 = y_0 + \Delta x = \frac{\pi}{2} - 1 = 0.57079$$

الشوابني

$$x_1 = 0.57079$$
$$y_1 = 0.57079$$

$$\begin{bmatrix} 0.84147 & 0.54029 \\ -0.54029 & 0.84147 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 0.30117 \\ 1.00046 \end{bmatrix}$$

$$\Delta x = -0.2868$$
$$\Delta y = 1.0042$$

$$x_2 = x_1 + \Delta x = \boxed{0.28399}$$

$$y_2 = y_1 + \Delta y = \boxed{1.57499}$$

2) Fixed point iteration method for system of nonlinear equations

F_i is non linear for some $1 \le i \le n$ to solve System I

$$x_1 = g_1(x_1, x_2, x_3, \ldots, x_n)$$
$$x_2 = g_2(x_1, x_2, x_3, \ldots, x_n)$$
$$x_n = g_n(x_1, x_2, x_3, \ldots, x_n)$$

let $x_{10}, x_{20}, \ldots, x_{n0}$ be given

$$x_{1\,i+1} = g_1(x_{1i}, x_{2i}, \ldots, x_{ni})$$
$$x_{2\,i+1} = g_2(x_{1i}, x_{2i}, \ldots, x_{ni})$$
$$\vdots$$
$$x_{n\,i+1} = g_n(x_{1i}, x_{2i}, \ldots, x_{ni})$$

العملية أدق في استخدام القيمة السابقة المستخرجة للاستخراج القيم اللاحقة
يحق بتقدم X_0 مرة واحدة وبعدها بتقدم اي بعدها

$$x_{1\,i+1} = g_1(x_{1i}, x_{2i}, \ldots, x_{ni})$$
$$x_{2\,i+1} = g_2(x_{1\,i+1}, x_{2i}, \ldots, x_{ni})$$
$$x_{3\,i+1} = g_3(x_{1\,i+1}, x_{2\,i+1}, \ldots, x_{ni})$$
$$x_{n\,i+1} = g_n(x_{1\,i+1}, x_{2\,i+1}, x_{3\,i+1}, \ldots, x_{n-1\,i+1}, x_{ni})$$

أدق

هنا استخدمنا القيم السابقة في استخراج القيم اللاحقة

Ex8 locate the solution of the system

$$3x - \cos(yz) - \frac{1}{2} = 0$$

$$x^2 - 81(y+0.1)^2 + \sin z + 1.06 = 0$$

$$e^{-xy} + 20z + \frac{10\pi - 3}{3} = 0$$

with $x_0 = 0.1$, $y_0 = 0.1$, $z_0 = 0.1$

Sol8

$$x = \frac{\cos(yz) + \frac{1}{2}}{3}$$

$$y = \frac{1}{9}\sqrt{x^2 + \sin z + 1.06} - 0.1$$

$$z = \frac{3 - 10\pi}{60} - \frac{e^{-xy}}{20}$$

$$x_{i+1} = \frac{\cos(0.1 * -0.1) + \frac{1}{2}}{3} = 0.49998333$$

$$y_{i+1} = \frac{1}{9}\sqrt{(0.1)^2 + \sin(-0.1) + 1.06} - 0.1 = 9.4411496 * 10^{-3}$$

$$z_{i+1} = \frac{30 - 10\pi}{60} - \frac{e^{-0.1 * 0.1}}{20} = -0.073101267$$

i	x_i	y_i	z_i
0	x_0	y_0	z_0
1	x_1	y_1	z_1
2	x_2	y_2	z_2

$E_a = |\text{Max}_{i2} - \text{Max}_{i1}|$

بحث في الجدول عن أكبر متغير
في كل عمود وبطرحهم
لأعرف مقدار الخطأ

Fixed-Point Iteration

The fixed-point-iteration approach (Sec. 6.1) can be modified to solve two simultaneous, nonlinear equations. This approach will be illustrated in the following example.

Fixed-Point Iteration for a Nonlinear System

Problem Statement. Use fixed-point iteration to determine the roots of Eq. (6.19). Note that a correct pair of roots is $x = 2$ and $y = 3$. Initiate the computation with guesses of $x = 1.5$ and $y = 3.5$.

$x^2 + xy - 10 = 0$

$y + 3xy^2 - 57 = 0$

Solution. Equation (6.19a) can be solved for

$$\boxed{x_{i+1} = \frac{10 - x_i^2}{y_i}} \tag{E6.11.1}$$

diverges

and Eq. (6.19b) can be solved for

$$\boxed{y_{i+1} = 57 - 3x_i y_i^2} \tag{E6.11.2}$$

Note that we will drop the subscripts for the remainder of the example.

On the basis of the initial guesses, Eq. (E6.11.1) can be used to determine a new value of x:

$$x = \frac{10 - (1.5)^2}{3.5} = 2.21429$$

This result and the initial value of $y = 3.5$ can be substituted into Eq. (E6.11.2) to determine a new value of y:

$$y = 57 - 3(2.21429)(3.5)^2 = -24.37516$$

Thus, the approach seems to be diverging. This behavior is even more pronounced on the second iteration:

$$x = \frac{10 - (2.21429)^2}{-24.37516} = -0.20910$$

$$y = 57 - 3(-0.20910)(-24.37516)^2 = 429.709$$

Obviously, the approach is deteriorating.

Now we will repeat the computation but with the original equations set up in a different format. For example, an alternative formulation of Eq. (6.19a) is

$$\boxed{x = \sqrt{10 - xy}}$$

and of Eq. (6.19b) is

Convergas

$$\boxed{y = \sqrt{\frac{57 - y}{3x}}}$$

Now the results are more satisfactory:

$$x = \sqrt{10 - 1.5(3.5)} = 2.17945$$

$$y = \sqrt{\frac{57 - 3.5}{3(2.17945)}} = 2.86051$$

$$x = \sqrt{10 - 2.17945(2.86051)} = 1.94053$$

$$y = \sqrt{\frac{57 - 2.86051}{3(1.94053)}} = 3.04955$$

Thus, the approach is converging on the true values of $x = 2$ and $y = 3$.