Dr. Hashem Alkhaldi

Chapter Six

Numerical Differentiation

- Difference Formulas.
- Newton's Divided Difference Differntiation / Gergory /
- Truncation Errors for Differntiation Rules
- Differntiation Formulas Proofs / Taylor series / Lagrange Polynomial

* Richardson Differentiation

Dr. Hashem Alkhaldi

Numerical Differentiation

(First Derivative)

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[2 Points]

Error

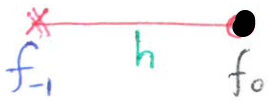
Forward

$$f' = \frac{1}{h}(f_1 - f_0) \longrightarrow$$

$$E = \frac{h}{2} f''$$

Backward

$$f' = \frac{1}{h}(f_0 - f_{-1}) \longrightarrow$$



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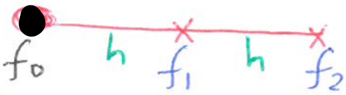
[3 Points]

Forward

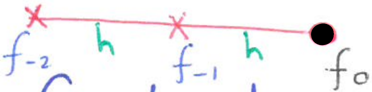
$$f' = \frac{1}{h}\left(-\frac{3}{2} f_0 + 2 f_1 - \frac{1}{2} f_2\right) \longrightarrow$$



$$E = \frac{h^2}{3} f'''$$

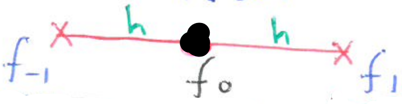
Backward

$$f' = \frac{1}{h}\left(\frac{3}{2} f_0 - 2 f_{-1} + \frac{1}{2} f_{-2}\right) \longrightarrow$$



Central

$$f' = \frac{1}{2h}(f_1 - f_{-1}) \longrightarrow E = \frac{h^2}{6} f'''$$



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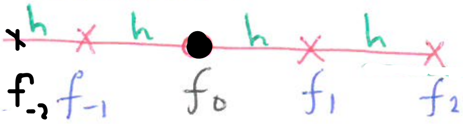
[5 Points]

$$f' = \frac{1}{12h}(f_{-2} - 8 f_{-1} + 8 f_1 - f_2) \longrightarrow E = \frac{h^4}{30} f'''''$$

Central



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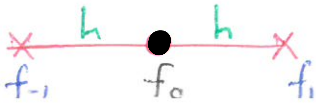
(Second Derivative)

[3 Points]

$$f'' = \frac{1}{h^2}(f_{-1} - 2 f_0 + f_1) \longrightarrow E = \frac{h^2}{12} f''''$$



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[5 Points]   First Derivative

Forward: $$f' = \frac{1}{12h}\left[-25f_0 + 48f_1 - 36f_2 + 16f_3 - 3f_4\right]$$

Backward: $$f' = \frac{1}{12h}\left[25f_0 - 48f_{-1} + 36f_{-2} - 16f_{-3} + 3f_{-4}\right]$$

$E = \frac{h^4}{5} f^{(5)}$

****

Ex: $f(x) = 3\sin x + \frac{1}{x}$

1) Find The $f'(3)$ using $h = 0.15$

2PF ✓ 2PB ✓ 3PF ✓ 3PB ✓ 3PC ✓ $\boxed{\text{5PF}}$ 5PB 5PC

2) Find The $f''(3)$ using $h = 0.15$

3) Find the error in part ① and part ②.

2.4430	2.0652	1.6525	1.2133	0.7566	0.2922	−0.1702	−0.6207	−1.0497
f_{-4}	f_{-3}	f_{-2}	f_{-1}	f_0	f_1	f_2	f_3	f_4
2.40	2.55	2.70	2.85	3	3.15	3.30	3.45	3.60

$\boxed{1}$ 5PF :

$$f' = \frac{1}{12h}\left[-25f_0 + 48f_1 - 36f_2 + 16f_3 - 3f_4\right]$$

$$f' = \frac{1}{12(0.15)}\left[-25(0.7566) + 48(0.2922) - 36(-0.1702) + 16(-0.6207) - 3(-1.0497)\right]$$

$$f' = -3.0801$$

Approximate value

$\boxed{2}$

$$f'' = \frac{1}{h^2}\left(f_{-1} - 2f_0 + f_1\right)$$

$$f'' = \frac{1}{(0.15)^2}\left[1.2133 - 2(0.7566) + 0.2922\right]$$

$$f'' = -0.3422$$

Approximate value

$\boxed{3}$ Calculus :

$$f = 3\sin x + \frac{1}{x}$$

$$f' = 3\cos x - \frac{1}{x^2}$$

$$f'' = -3\sin x + \frac{2}{x^3}$$

f'' و f' عوض في

$x = 3$ عند

$$\Rightarrow \quad f' = 3\cos 3 - \frac{1}{(3)^2}$$

$$\Rightarrow \quad f' = -3.0810$$ Exact value

$$* \quad f'' = -3\sin 3 + \frac{2}{(3)^3}$$

$$\Rightarrow \quad f'' = -0.3492$$ Exact value

$$E_{f'} = \frac{\text{Exact} - \text{Approx}}{\text{Exact}} = \frac{-3.0810 - -3.0801}{-3.0810} = 2.92 \times 10^{-4}$$

$$E_{f''} = \frac{\text{Exact} - \text{Approx}}{\text{Exact}} = \frac{-0.3492 - -0.3422}{-0.3492} = 0.01145$$

given

x	0	0.25	$\boxed{0.5}$	0.75	1
$f(x)$	1	1.5	1.75	1.5	1

Find

① $f'(0.5)$
- Forward , $2P + 3P$
- Back. , $2P + 3P$.
- Centred , $3P + 5P$.

② $f''(0.5)$

③ $f'(0.75)$
- Forward → $2P + 3P$
- Back. → $2P + 3P$
- Centered → $3P + 5P$

****

Sol.

1	1.5	1.75	1.5	1
f_{-2}	f_{-1}	f_0	f_1	f_2
0	0.25	$\boxed{0.5}$	0.75	1

$\boxed{1}$ * $f' = \dfrac{1}{0.25}\left[1.5 - 1.75\right] = -1$ F2P

* $f' = \dfrac{1}{0.25}\left[\dfrac{-3}{2}(1.75) + 2(1.5) - \dfrac{1}{2}(1)\right] = -0.5$

F3P

$$* \quad f' = \frac{1}{0.25}\left[1.75 - 1.5\right] = 1$$ B2P

$$* \quad f' = \frac{1}{0.25}\left[\frac{3}{2}(1.75) + 2(1.5) + \frac{1}{2}(1)\right] = 0.5$$ B3P

$$* \quad f' = \frac{1}{2(0.25)}\left[1.5 - 1.5\right] = 0$$ C3P

$$* \quad f' = \frac{1}{12(0.25)}\left[1 - 8(1.5) + 8(1.5) - 1\right] = 0$$ C5P

$$* \quad f'' = \frac{1}{(0.25)^2}\left[1.5 - 2(1.75) + 1.5\right] = -8$$

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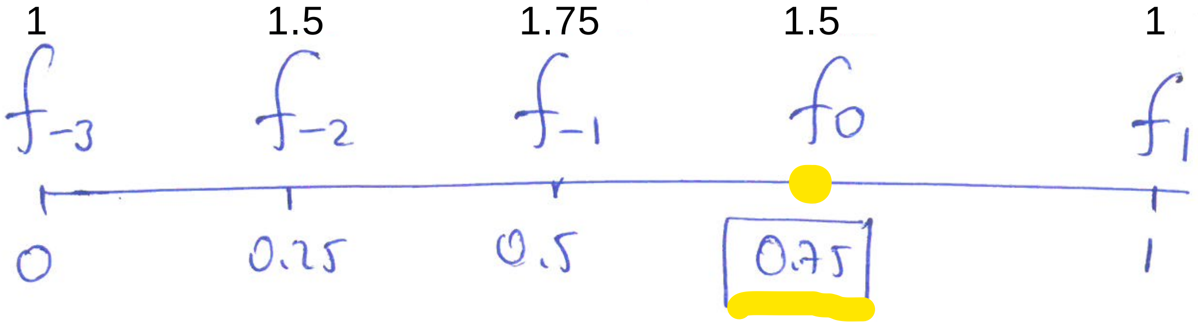


ex) given $f(x) = x^2 e^{2x}$

using $h = 0.02 = \Delta x$

Find Error

Find
1. $f'(2) \rightarrow$ Central 5 points
2. $f'(2) \rightarrow$ Central 3 points
3. $f'(2) \rightarrow$ Forward 3 points
4. $f''(2)$

	f_{-2}	f_{-1}	f_0	f_1	f_2
f	193.6183	205.6537	218.3926	231.8742	246.1397
x	1.96	1.98	$\boxed{2}$	2.02	2.04

Relative Error:

$$E = \left| \frac{\text{exact} - \text{approx}}{\text{exact}} \right|$$

Calculus

$$f'(x) = x^2(2e^{2x}) + (e^{2x})(2x)$$

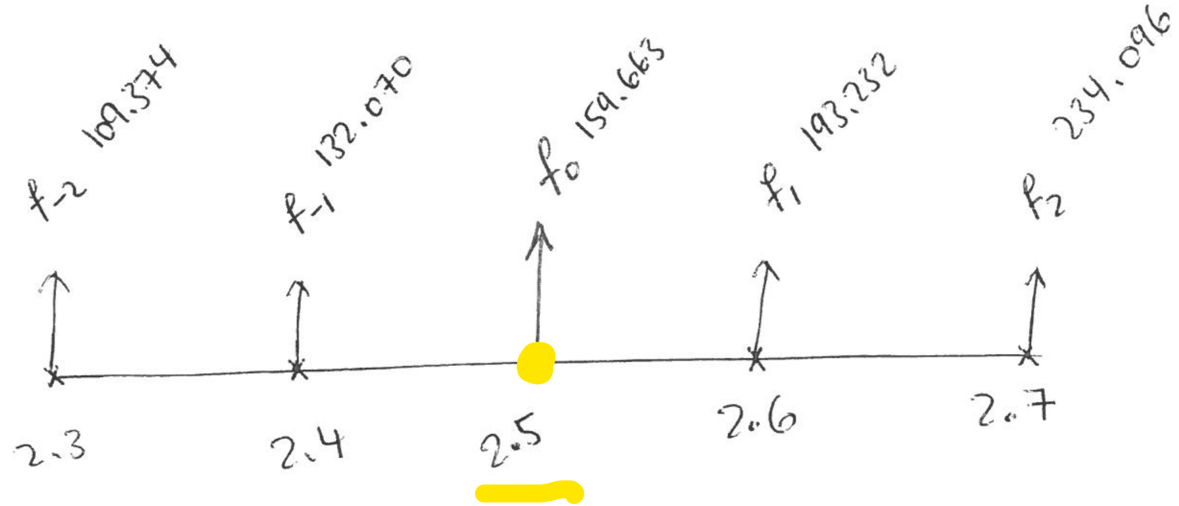
$f'(2) =$ …….. exact value

EX: Find The First and Second derivative For The Function :

$$f(x) = x^2 + 2x + e^{2x}$$

at $x = 2.5$? using $h = 0.10 = \Delta x$

Sol:



(n=1) Forward
$$f' = \frac{1}{0.10}(193.232 - 159.663) =$$

n=1) Backward
$$f' = \frac{1}{0.10}(159.663 - 132.070) =$$

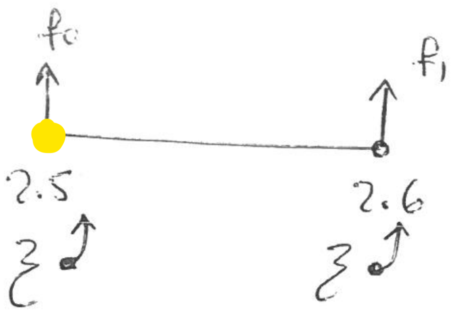
n=2) Centered
$$f' = \frac{1}{2(0.10)}(193.232 - 132.070) =$$

(n=4) Centered
$$f' = \frac{1}{12(0.10)}\left(8(193.232) - 8(132.070) - 234.096 + 109.374\right) =$$

2nd-deriv.
$$f'' = \frac{1}{(0.1)^2}\left(132.070 - 2(159.663) + 193.232\right) =$$

* Find The Error For $(n=1) \Rightarrow$ Forward OR Backward.

- Forward truncation error:



* $E = \frac{h}{2} f''(\xi)$

* $f(x) = x^2 + 2x + e^{2x}$

* $f'(x) = 2x + 2 + 2e^{2x}$ [Calculus]

* $f''(x) = 2 + 4e^{2x}$ [Calculs]

* $E = \frac{0.10}{2} \left[2 + 4e^{2\xi}\right]$

$\Rightarrow E = 0.05 \left[2 + 4e^{2\xi}\right]$

$\xi = 2.5$

$\Rightarrow E = 29.78$

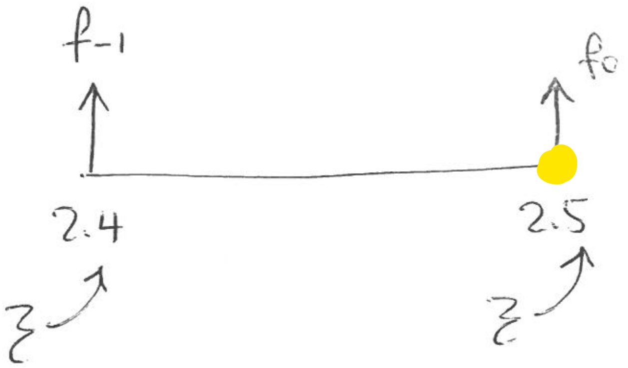
$\xi = 2.6$

$\Rightarrow E = 36.35$

حيث يتم اختيار الخطأ الأكبر كقيمة مطلقة select the largest

$\Rightarrow E = 36.35$ Forward truncation error

- Backward truncation error:



$$E = 0.05\left[2 + 4e^{2\xi}\right]$$

$\xi = 2.4$

$\Rightarrow E = 24.40$

$\xi = 2.5$

$\Rightarrow E = 29.78$

.... select the largest one.

$\Rightarrow E = 29.78$ Backward truncation error

$\Rightarrow$ The Backward is more Accurate Than The Forward For This example at $X = 2.5$.

- Centered truncation error for (n=4):

$$f = x^2 + 2x + e^{2x}$$

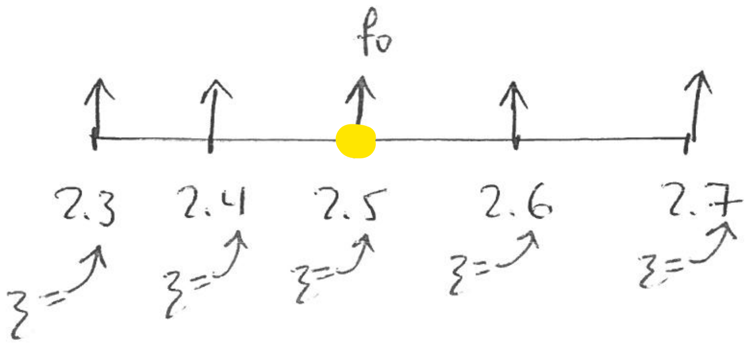
$$f' = 2x + 2 + 2e^{2x}$$

$$f'' = 2 + 4e^{2x}$$

$$f''' = 8e^{2x}$$

$$f'''' = 16e^{2x}$$

$$f^{(5)} = 32e^{2x}$$



$$\Rightarrow E = \frac{h^4}{30} f^{(5)}(\xi)$$

$$E = \frac{(0.10)^4}{30}\left[32\, e^{2\xi}\right]$$

$$\boxed{E = 1.0667 * 10^{-4}\; e^{2\xi}}$$

$\xi = 2.3$ | $\xi = 2.4$ | $\xi = 2.5$ | $\xi = 2.6$ | $\xi = 2.7$

$E =$ | $E =$ | $E =$ | $E =$ | $E =$

.... then select the largest value of E.

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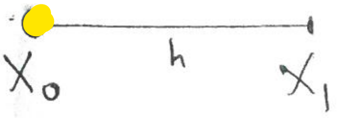
Numerical Differentiation

$h = \Delta x$

الاشتقاق العددي

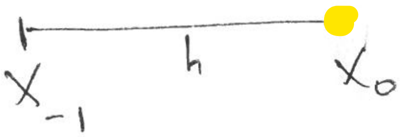
$F \equiv$ Forward
$B \equiv$ Backward
$C \equiv$ Centered
$P \equiv$ points

First Derivative:



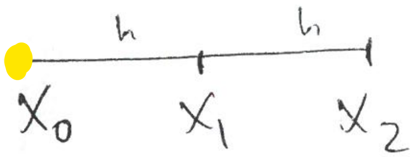
F2P

$$f' = \frac{1}{h}\left(f_1 - f_0\right)$$



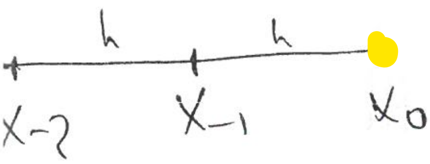
B2P

$$f' = \frac{1}{h}\left(f_0 - f_{-1}\right)$$



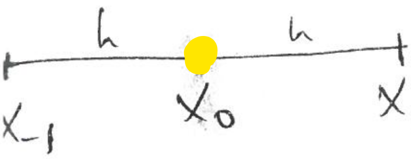
F3P

$$f' = \frac{1}{h}\left(-1.5 f_0 + 2 f_1 - 0.5 f_2\right)$$



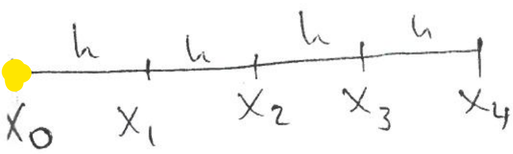
B3P

$$f' = \frac{1}{h}\left(1.5 f_0 - 2 f_{-1} + 0.5 f_{-2}\right)$$



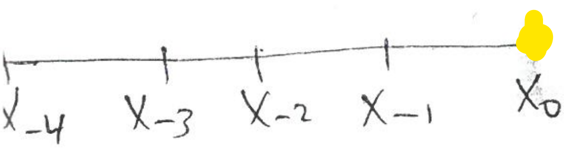
C3P

$$f' = \frac{1}{2h}\left(f_1 - f_{-1}\right)$$



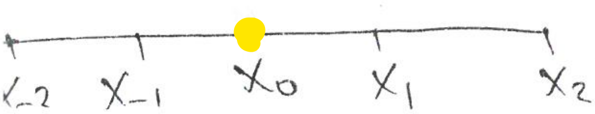
F5P

$$f' = \frac{1}{12h}\left(-25 f_0 + 48 f_1 - 36 f_2 + 16 f_3 - 3 f_4\right)$$



B5P

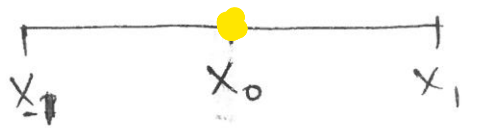
$$f' = \frac{1}{12h}\left(25 f_0 - 48 f_{-1} + 36 f_{-2} - 16 f_{-3} + 3 f_{-4}\right)$$



C5P

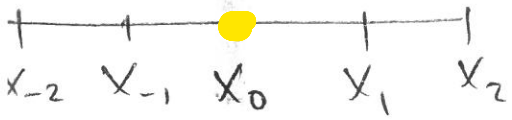
$$f' = \frac{1}{12h}\left(f_{-2} - 8 f_{-1} + 8 f_1 - f_2\right)$$

Second Derivative:



C3P

$$f'' = \frac{1}{h^2}\left(f_{-1} - 2f_0 + f_1\right)$$



C5P

$$f'' = \frac{1}{12h^2}\left(-f_{-2} + 16f_{-1} - 30f_0 + 16f_1 - f_2\right)$$

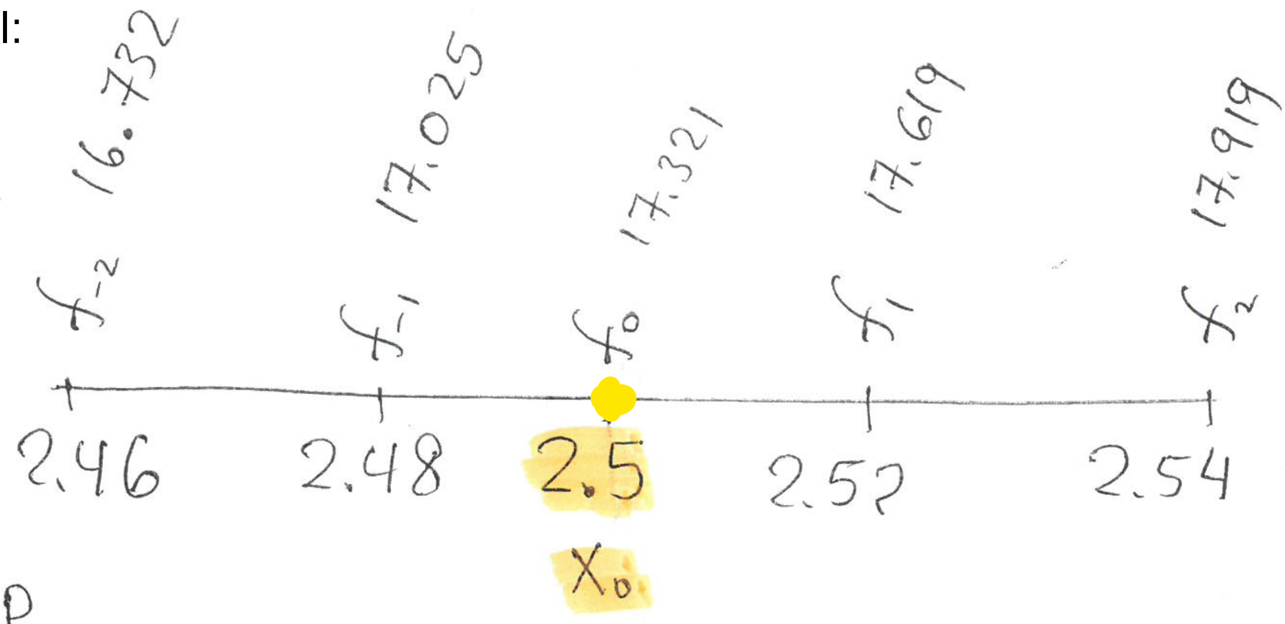
****

<u>example</u> given $f(x) = 3x^2 - \frac{2x}{x+1}$

Find $f'(2.5)$? using $h = 0.02$

Find $f''(2.5)$?

Sol:



C5P

$$f' = \frac{1}{12(0.02)}\left[16.732 - 8(17.025) + 8(17.619) - 17.919\right]$$

$$f' = ----$$

more accurate

F2P $$f' = \frac{1}{0.02}\left[17.619 - 17.321\right]$$

$$f' = ----$$

C3P $$f'' = \frac{1}{(0.02)^2}\left[17.025 - 2(17.321) + 17.619\right]$$

$$f'' = ---$$

EX:

given

x	6.95	7	7.05	7.10
$f(x)$	18.5	18.6	18.72	18.8

Find $f'(7)$

- F2P ✓
- B2P ✓
- F3P ✓
- B3P X
- C3P ✓
- F5P X
- B5P X
- C5P X

Find $f''(7)$

- C3P ✓
- C5P X

$$\boxed{h = 0.05} \Rightarrow \Delta x$$

C3P $\Rightarrow$ $f' = \dfrac{1}{2(0.05)}\left[18.72 - 18.5\right] =$ ✓

C3P $\Rightarrow$ $f'' = \dfrac{1}{(0.05)^2}\left[18.5 - 2(18.6) + 18.72\right] =$ ✓

EX: Find: 1) $f'(3)$ 2) $f''(3)$ to the highest accuracy

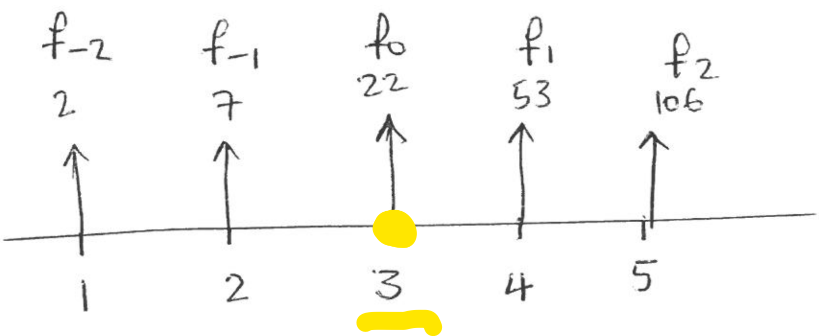
3) $f(3.7)$ & $f'(3.7)$ 4) the best line fit (previous chapter)

x	1	2	3	4	5
f(x)	2	7	22	53	106

Sol:

Five points $(n=4)$

$\boxed{h=1}$



1) $$f'(3) = \frac{1}{12h}\left[8f_1 - 8f_{-1} - f_2 + f_{-2}\right]$$

$$= \frac{1}{12(1)}\left[8(53) - 8(7) - 106 + 2\right] = 22$$

2) $$f''(3) = \frac{1}{h^2}\left[f_{-1} - 2f_0 + f_1\right]$$

$$= \frac{1}{(1)^2}\left[7 - 2(22) + 53\right] = 16$$

3)

x	f	Δ_1	Δ_2	Δ_3	Δ_4
1	$\boxed{2}$ a_0	$\boxed{\cdot}$ a_1	$\boxed{\cdot}$ a_2	$\boxed{\cdot}$ a_3	$\boxed{\cdot}$ a_4
2	7				
3	22				
4	53				
5	106				

Build Newton's table...

EX: For the data points :

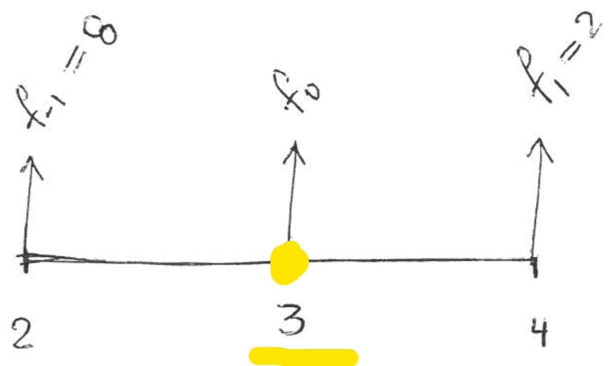
x	0	2	4
$f(x)$	0	8	2

Find For most accurate result :

1) $f'(3)$ 2) $f''(3)$

Sol:

$\boxed{h=1}$



1) Centered (3 points) $n=2$ $\Rightarrow f' = \frac{1}{2h}[f_1 - f_{-1}]$

$\Rightarrow f' = \frac{1}{2(1)}[2-8]$

$\Rightarrow f' = -3$

2) $f'' = \frac{1}{h^2}[f_{-1} - 2f_0 + f_1]$ $\Rightarrow f_0 \equiv$ unknown.

$\downarrow$

Use divided diff.

x	f	Δ_1	Δ_2
0	$\boxed{0}$ a_0		
		$\boxed{4}$ a_1	
2	8		$\boxed{-1.75}$ a_2
		-3	
4	2		

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$f(x) = 0 + 4(x - 0) - 1.75(x - 0)(x - 2)$$

$$f(x) = 4x - 1.75x(x - 2)$$

$$* \quad f(3) = 4(3) - 1.75(3)(3 - 2) = 6.75 \Rightarrow f_0$$

sub. $$f'' = \frac{1}{(1)^2}\left[8 - 2(6.75) + 2\right]$$

$$\Rightarrow f'' = -1.75$$

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Derivation of Difference Formulas (Taylor series)

The Taylor series:

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f''''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots + \boxed{\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi)}$$

$\Delta x = h$

* Forward using n=1 (2points)

$$\Rightarrow f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(\xi)$$

$$h f'(x) = f(x+h) - f(x) - \frac{h^2}{2} f''(\xi)$$

$$\Rightarrow$$

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(\xi)$$

$$\Rightarrow \boxed{f'(x) = \frac{f_1 - f_0}{h}}$$

Forward 2 points (F2P)

Truncation Error E= $\frac{h}{2} f''(\xi)$

****

$\Delta x = -h$

* Backward using n=1 (2points)

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(\xi)$$

$$\Rightarrow h f'(x) = f(x) - f(x-h) + \frac{h^2}{2} f''(\xi)$$

$$\Rightarrow \boxed{f'(x) = \frac{f(x) - f(x-h)}{h}} + \underbrace{\frac{h}{2} f''(\xi)}_{\text{Error}}$$

$$f'(x) = \frac{f_0 - f_{-1}}{h}$$

Backward 2 points (B2P)

****

****

$\Delta x = h$

Now use n=3:

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f''''(\xi) \quad \text{——} (1)$$

$\Delta x = -h$

Use also n=3:

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f''''(\xi) \quad \text{——} (2)$$

(Add Equation 1 and 2 together...)

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \frac{h^4}{12} f''''(\xi)$$

$$\Rightarrow \quad h^2 f''(x) = f(x+h) + f(x-h) - 2f(x) - \frac{h^4}{12} f''''(\xi)$$

$$\Rightarrow \quad f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{h^2}{12} f''''(\xi)$$

Centered 3 points (C3P) 2nd derivative

$$\Rightarrow \quad \boxed{f''(x) = \frac{f_1 - 2f_0 + f_{-1}}{h^2}} - \frac{h^2}{12} f''''(\xi)$$

****

(Subtract Equation 1 and 2 ...)

$$f(x+h) - f(x-h) = 2h f'(x) + \frac{h^3}{3} f'''(\xi)$$

$$\Rightarrow \quad \boxed{f'(x) = \frac{f(x+h) - f(x-h)}{2h}} - \frac{h^2}{6} f'''(\xi)$$

Centered 3 points (C3P) 1st derivative

** Dr. Hashem Alkhaldi

$\Delta x = h$

Now use n=5: $\Rightarrow$ $$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f''''(x) + \frac{h^5}{120} f^{(5)}(x) + \frac{h^6}{720} f^{(6)}(\xi) \quad \text{—— (1)}$$

$\Delta x = -h$

Use also n=5: $\Rightarrow$ $$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f''''(x) - \frac{h^5}{120} f^{(5)}(x) + \frac{h^6}{720} f^{(6)}(\xi) \quad \text{—— (2)}$$

$\Delta x = 2h$

Use n=5: $\Rightarrow$ $$f(x+2h) = f(x) + 2h f'(x) + 2h^2 f''(x) + \frac{4h^3}{3} f'''(x) + \frac{2h^4}{3} f''''(x) + \frac{4h^5}{15} f^{(5)}(x) + \frac{4h^6}{45} f^{(6)}(\xi) \quad \text{—— (3)}$$

$\Delta x = -2h$

again use n=5: $\Rightarrow$ $$f(x-2h) = f(x) - 2h f'(x) + 2h^2 f''(x) - \frac{4h^3}{3} f'''(x) + \frac{2h^4}{3} f''''(x) - \frac{4h^5}{15} f^{(5)}(x) + \frac{4h^6}{45} f^{(6)}(\xi) \quad \text{—— (4)}$$

(Subtract Equation 1 and 2 ...)

$$\Rightarrow f(x+h) - f(x-h) = 2h f'(x) + \frac{h^3}{3} f'''(x) + \frac{h^5}{60} f^{(5)}(\xi) \quad \text{—— (5)}$$

(Subtract Equation 3 and 4 ...)

$$\Rightarrow f(x+2h) - f(x-2h) = 4h f'(x) + \frac{8h^3}{3} f'''(x) + \frac{8h^5}{15} f^{(5)}(\xi) \quad \text{—— (6)}$$

To get rid of the 2nd derivative, multiply Eq(5) by 8 and then subtract it from Eq(6):

$$\Rightarrow 8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h) = 12h\,f'(x) - \frac{6h^5}{15} f^{(5)}(\varepsilon)$$

$$\Rightarrow f'(x) = \frac{8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h)}{12h} - \frac{\frac{6h^5}{15} f^{(5)}(\varepsilon)}{12h}$$

$$\Rightarrow f'(x) = \frac{8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h)}{12h} - \frac{h^4}{30} f^{(5)}(\varepsilon)$$

$$\Rightarrow \boxed{f'(x) = \frac{8f_1 - 8f_{-1} - f_2 + f_{-2}}{12h}} - \frac{h^4}{30} f^{(5)}(\varepsilon)$$

Centered
5 points
(C5P)
1st derivative

Exercise:

- Finding $f''(x) \Rightarrow$
  (1) + (2)
  (3) + (4)
  ? ___ × (5) + (6)

- Finding $f'''(x) \Rightarrow$
  (1) − (2)
  (3) − (4)
  2×(5) − (6)

- Finding $f''''(x)$
  $\Downarrow$
  (1) + (2)
  (3) + (4)
  ? × (5) + (6)

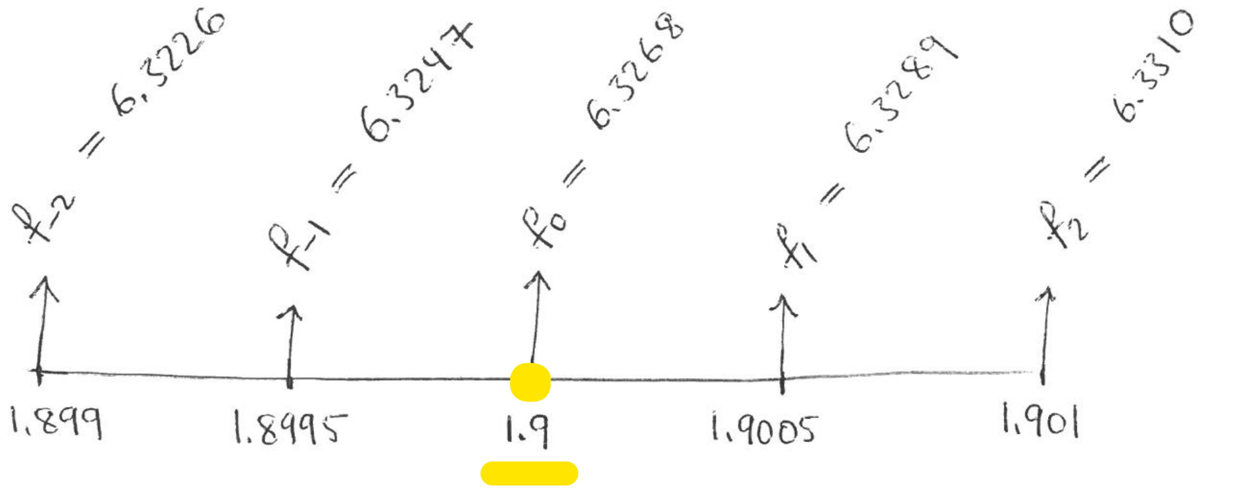
EX: Let $f(x) = e^{x}\sin x$, using $h = 0.0005$

1) Find $f'(1.9)$ using Central diff. Formula 3 points

2) Find $f'(1.9)$ " " " " 5 points

3) Find $f''(1.9)$

Sol:



* Be careful... change your calculator to RADIANS.

1) (3 points) $\Rightarrow$

$$f' = \frac{1}{2h}\left[f_1 - f_{-1}\right]$$

$$= \frac{1}{2(0.0005)}\left[6.3289 - 6.3247\right]$$

$$= 4.16538189$$

2) (5 points) $\Rightarrow$

$$f' = \frac{1}{12h}\left[8f_1 - 8f_{-1} - f_2 + f_{-2}\right]$$

$$= \frac{1}{12(0.0005)}\left[8(6.3289) - 8(6.3247) - 6.3310 + 6.3226\right]$$

$$= 4.165382595$$

3) $f'' = \frac{1}{h^2}\left[f_{-1} - 2f_0 + f_1\right]$

$$= \frac{1}{(0.0005)^2}\left[6.3247 - 2(6.3268) + 6.3289\right]$$

$$= \text{-----} \checkmark$$

4) For check [error using Calculus] :

$$* \quad f'(x) = e^x \cos x + e^x \sin x = e^x\left[\sin x + \cos x\right]$$

$$f''(x) = e^x\left[\cos x - \sin x\right] + e^x\left[\sin x + \cos x\right]$$

$$* \quad f''(x) = 2e^x \cos x$$

for x=1.9 $\Rightarrow$ $f'(1.9) = e^{1.9}\left[\sin 1.9 + \cos 1.9\right] = 4.16538258$ Exact.

and $f''(1.9) = 2e^{1.9}\cos 1.9 = \text{---}$

Relative Error For (5points) $= \left|\frac{\text{Exact} - \text{Approx.}}{\text{Exact}}\right| * 100\%$

$$= \frac{4.16538258 - 4.165382595}{4.16538258} * 100\%$$

$$= \text{----}\ \%$$

$\boxed{2}$ Derivative using Divided Difference Tables

(used for Non-Equal spaced data points)

$$f'(x) = \Delta_1 + \Delta_2 \left[(x-x_1) + (x-x_2) \right]$$

$$+ \Delta_3 \left[(x-x_1)(x-x_2) + (x-x_1)(x-x_3) + (x-x_2)(x-x_3) \right]$$

$$+ \Delta_4 \left[(x-x_1)(x-x_2)(x-x_3) + (x-x_1)(x-x_2)(x-x_4) + (x-x_1)(x-x_3)(x-x_4) + (x-x_2)(x-x_3)(x-x_4) \right]$$

EX:

1) Find $f(2)$ using Divided Diff. Tables
2) Find $f'(2)$ using Divided Diff Tables

⇒ For The Function

$$f(x) = x e^{x}$$

using The data points

$x = 0$, $x = 0.5$, $x = 1.5$

$x = 2.5$, $x = 3$

Sol:

X	f	Δ_1	Δ_2	Δ_3	Δ_4
	a_0				
0	$\boxed{0}$				
		a_1 $\boxed{1.64}$ Δ_1			
			a_2 $\boxed{2.84}$ Δ_2		
0.5	0.82				
				a_3 $\boxed{2.43}$ Δ_3	
		5.9^{Δ_1}			a_4 $\boxed{1.18}$ Δ_4
			8.92^{Δ_2}		
1.5	6.72				
		23.74^{Δ_1}		5.98^{Δ_3}	
2.5	30.46		23.87^{Δ_2}		
		59.54^{Δ_1}			
3	60.23				

$$
\begin{aligned}
f(x) = \; & a_0 + a_1(x-x_0) \\
& + a_2(x-x_0)(x-x_1) \\
& + a_3(x-x_0)(x-x_1)(x-x_2) \\
& + a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3)
\end{aligned}
$$

1)

$$
\begin{aligned}
f(2) = \; & 0 + 1.64(2) + 2.84(2)(2-0.5) \\
& + 2.43(2)(2-0.5)(2-1.5) \\
& + 1.18(2)(2-0.5)(2-1.5)(2-2.5) = \text{-----}
\end{aligned}
$$

Around $(x = x_0 = 0)$

2)
$$f'(x_2) = 1.64 + 2.84\left[(2-0.5)+(2-1.5)\right]$$
$$+ 2.43\left[(2-0.5)(2-1.5) + (2-0.5)(2-2.5) + (2-1.5)(2-2.5)\right]$$
$$+ 1.18\left[(2-0.5)(2-1.5)(2-2.5) + (2-0.5)(2-1.5)(2-3)\right.$$
$$\left. + (2-0.5)(2-2.5)(2-3) + (2-1.5)(2-2.5)(2-3)\right] = ---$$

"more Accurate"

* Around $x = x_1 = 0.5$

$$f'(2) = 5.9 + 8.92\left[(2-0.5)+(2-1.5)\right]$$
$$+ 5.98\left[(2-0.5)(2-1.5) + (2-0.5)(2-2.5) + (2-1.5)(2-2.5)\right]$$
$$= 22.245$$

* Around $x = x_2 = 1.5$

$$f'(2) = 23.74 + 23.87\left[(2-0.5)+(2-1.5)\right] = ----$$

* Around $x = x_3 = 2.5$

$$f'(2) = 59.54$$

3 Derivative using Ordinary Diff. Tables (Gergory) used For equally spaced data points.

$$f'(x) = \frac{1}{h}\left[\Delta_1 - \frac{1}{2}\Delta_2 + \frac{1}{3}\Delta_3 - \frac{1}{4}\Delta_4 + \frac{1}{5}\Delta_5 - \frac{1}{6}\Delta_6 + \cdots\right]$$

* حيث تم إيجاد المشتقة الأولى عند نقاط الـ X المحددة المعطاة في السؤال

EX: Let $f(X) = xe^x$, $[X = 0, 0.5, 1, 1.5, 2, 2.5]$ using Gergory To find

① $f'(0)$ ② $f'(0.5)$ ③ $f'(1)$

④ $f'(1.5)$ ⑤ $f'(2)$ ⑥ $f'(1.75)$?

Sol:

X	f	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
0	0	0.82	1.08	1.02	0.94	0.66
0.5	0.82	1.90	2.10	1.96	1.60	
1	2.72	4	4.06	3.56		
1.5	6.72	8.06	7.62			
2	14.78	15.68				
2.5	30.46					

$\boxed{h = 0.5}$ "Constant"

① $f'(0) = \frac{1}{0.5}\left[0.82 - \frac{1}{2}(1.08) + \frac{1}{3}(1.02) - \frac{1}{4}(0.94) + \frac{1}{5}(0.66)\right]$

$= \text{-----}$ ✓

② $f'(0.5) = \frac{1}{0.5}\left[1.90 - \frac{1}{2}(2.10) + \frac{1}{3}(1.96) - \frac{1}{4}(1.60)\right]$

$= 2.2066$ ✓

③ $f'(1) = \frac{1}{0.5}\left[4 - \frac{1}{2}(4.06) + \frac{1}{3}(3.56)\right]$

$= 6.30$

④ $f'(1.5) = \frac{1}{0.5}\left[8.06 - \frac{1}{2}(7.62)\right]$

$= \text{-----}$ ✓

⑤ $f'(2) = \frac{1}{0.5}\left[15.68\right]$

$= \text{-----}$ ✓

⑥ $f'(1.75) \Rightarrow$ point $X = 1.75$ ~~is~~ not the Table

$\Rightarrow$ use Divided Diff. Table

$\Rightarrow$ Build New Table and find $f'(1.75)$!!

Richardson Differentiation

$$A_n(h) = \frac{2^P A_{n-1}\left(\frac{h}{2}\right) - A_{n-1}(h)}{2^P - 1}$$ (Richardson formula)

$P=1 \Rightarrow$ $$A_2(h) = 2A_1\left(\frac{h}{2}\right) - A_1(h)$$

$P=2 \Rightarrow$ $$A_3 = \frac{4}{3}A_2\left(\frac{h}{2}\right) - \frac{1}{3}A_2(h)$$

$P=3 \Rightarrow$ $$A_4 = \frac{8}{7}A_3\left(\frac{h}{2}\right) - \frac{1}{7}A_3(h)$$

$\boxed{\text{example}}$ Using Richadson Diff.

To find $f'(1)$ using $h = 0.5$

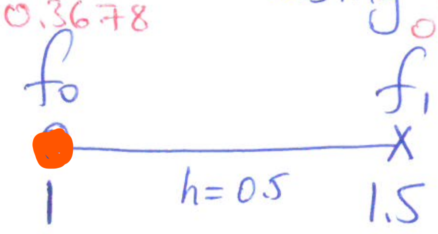
for $f(x) = e^{-x^2}$

Table 2x2

$A_1(h)$	
$A_1\left(\frac{h}{2}\right)$	$A_2(h)$

Sol:

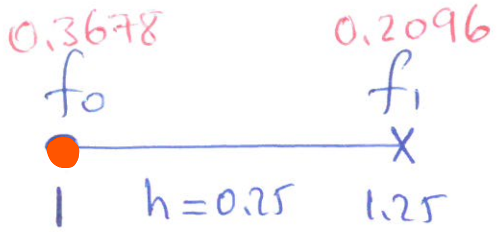
Using Forward (2points)



$$A_1(h) = \frac{1}{h}\left[f_1 - f_0\right]$$

$$= \frac{1}{0.5}\left[0.1053 - 0.3678\right]$$

$$= -0.525$$



re-calculate at h/2 = 0.25

$$A_1\left(\frac{h}{2}\right) = \frac{1}{h}\left[f_1 - f_0\right]$$

$$= \frac{1}{0.25}\left[0.2096 - 0.3678\right]$$

$$= -0.6328$$

Apply Richardson:

$$A_2 = 2\,A_1\left(\frac{h}{2}\right) - A_1(h)$$

$$A = 2(-0.6328) - (-0.525)$$

$$A = -0.7406$$

Error:

Calculus : $f' = -2x\,e^{-x^2}$

$$\Rightarrow f' = -2(1)\,e^{-1} = -0.7357 \quad \text{exact}$$

$$E_h = \frac{-0.7357 - -0.525}{-0.7357} = 0.286 = 28.6\%$$

$$E_{\frac{h}{2}} = \frac{-0.7357 - -0.6328}{-0.7357} = 0.139 = 13.9\%$$

$$E_{Rich.} = \frac{-0.7357 - -0.7406}{-0.7357} = 7.2 \times 10^{-3} = 0.72\%$$

Richardson Tables

use Difference formulas	use Richardson formula (P=1)	use Richardson formula (P=2)	use Richardson formula (P=3)
$A_1(h)$	$A_2(h)$	$A_3(h)$	$A_4(h)$
$A_1\left(\frac{h}{2}\right)$	$A_2\left(\frac{h}{2}\right)$	$A_3\left(\frac{h}{2}\right)$	
$A_1\left(\frac{h}{4}\right)$	$A_2\left(\frac{h}{4}\right)$		
$A_1\left(\frac{h}{8}\right)$			

$$A_2(h) = 2\,A_1\left(\frac{h}{2}\right) - A_1(h)$$

$$A_2\left(\frac{h}{2}\right) = 2\,A_1\left(\frac{h}{4}\right) - A_1\left(\frac{h}{2}\right)$$

$$A_2\left(\frac{h}{4}\right) = 2\,A_1\left(\frac{h}{8}\right) - A_1\left(\frac{h}{4}\right)$$

$$A_3(h) = \frac{4}{3}A_2\left(\frac{h}{2}\right) - \frac{1}{3}A_2(h)$$

$$A_3\left(\frac{h}{2}\right) = \frac{4}{3}A_2\left(\frac{h}{4}\right) - \frac{1}{3}A_2\left(\frac{h}{2}\right)$$

$$A_4(h) = \frac{8}{7}A_3\left(\frac{h}{2}\right) - \frac{1}{7}A_3(h)$$

ex) given: $f(x) = x^2 - 3x + e^{2x}$

Find the first derivative $f'(1.4)$?

[using two points]

→ So that the error will not exceed

$E = 0.08$

SOL.

$$f(x) = x^2 - 3x + e^{2x}$$
$$f'(x) = 2x - 3 + 2e^{2x}$$
$$f''(x) = 2 + 4e^{2x}$$

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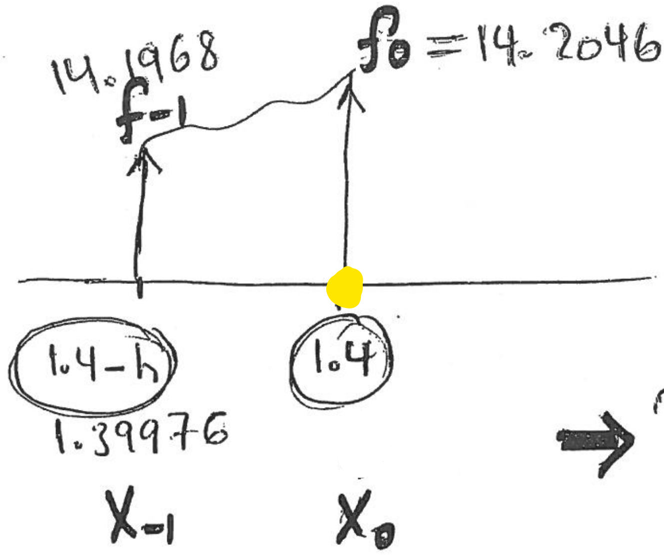
هندسة ميكانيكية

Two points:

$$E = \frac{h}{2} f''(\xi)$$
$$E = \frac{h}{2} [2 + 4e^{2x}]$$
$$0.08 = \frac{h}{2} [2 + 4e^{2(1.4)}]$$
$$0.16 = h(67.7789)$$
$$\Rightarrow h = 2.361 * 10^{-3} = 0.002361$$



$\rightarrow$ Backward

$$f' = \frac{f_0 - f_{-1}}{h}$$

$$f' = \frac{14.2046 - 14.1968}{0.002361}$$

$$f' = 3.3067$$

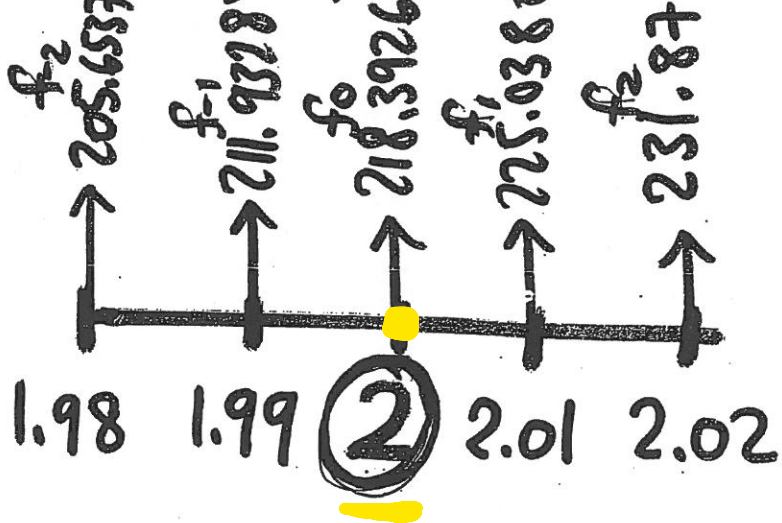
Ex Find the value of derivative

$$f(x) = x^2 e^{2x}$$

using $h = 0.01 \equiv \Delta x$

→ Find $f'(2)$, $f''(2)$

Sol.



* Forward (2 points) $f'(2) = \dfrac{225.0380 - 218.3926}{0.01} = 664.54$

* Backward (2 points) $f'(2) = \dfrac{218.3926 - 211.9328}{0.01} = 645.98$

* Centered (3 points) $f'(2) = \dfrac{225.0380 - 211.9328}{2(0.01)} = 655.26$

* Centered (5 points)

$$f'(2) = \frac{8(225.0380) + 8(211.9328) - 231.8742 + 205.6537}{12(0.01)}$$

more accurate

$$= 655.176$$

$$* \; f''(2) = \frac{211.9328 - 2(218.3926) + 225.0380}{(0.01)^2}$$

$$= 1856$$

$$f(x) = x^2 e^{2x}$$

→ Calculus :

$$f'(x) = [x^2][2e^{2x}] + [e^{2x}][2x]$$

المشتقة الأولى

$$f'(x) = 2x^2 e^{2x} + 2x e^{2x}$$

$$\boxed{f'(2) = 655.177}$$ Exact

المشتقة الثانية

$$f''(x) = 2x^2[2e^{2x}] + [e^{2x}][4x] + [2x][2e^{2x}] + [e^{2x}][2]$$

$$\rightarrow \; f''(x) = 4x^2 e^{2x} + 8x e^{2x} + 2e^{2x}$$

$$\boxed{f''(2) = 1856.337}$$ Exact

Finite Difference Formula Using

Lagrangian Polynomials

* using 3 points:

x	x_0	x_1	x_2
$f(x)$	f_0	f_1	f_2

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

$$R = \frac{f'''(\varepsilon)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

Lagrangian Error

$$f'(x) = \frac{f_0}{(x_0-x_1)(x_0-x_2)}\left[(x-x_1)(1) + (x-x_2)(1)\right]$$

$$+ \frac{f_1}{(x_1-x_0)(x_1-x_2)}\left[(x-x_0)(1) + (x-x_2)(1)\right]$$

$$+ \frac{f_2}{(x_2-x_0)(x_2-x_1)}\left[(x-x_0)(1) + (x-x_1)(1)\right]$$

$$\Rightarrow f'(x) = \frac{(2x-x_1-x_2)f_0}{(x_0-x_1)(x_0-x_2)} + \frac{(2x-x_0-x_2)f_1}{(x_1-x_0)(x_1-x_2)} + \frac{(2x-x_0-x_1)f_2}{(x_2-x_0)(x_2-x_1)}$$

**

* Forward $\xrightarrow{\text{sub.}}$ $x = x_0$

* Centered $\xrightarrow{\text{sub.}}$ $x = x_1$

* Backward $\xrightarrow{\text{sub.}}$ $x = x_2$

$\underline{X = X_0}$ (Forward)

$X_1 - X_0 = h$
$X_2 - X_1 = h$

$X_2 - X_0 = 2h$
$X_0 - X_2 = -2h$

$X_0 - X_1 = -h$
$X_2 - X_1 = +h$

$X_0 \qquad X_1 \qquad X_2$

$$\rightarrow f'(X_0) = \frac{(2X_0 - X_1 - X_2) f_0}{(X_0 - X_1)(X_0 - X_2)} + \frac{(\cancel{2X_0}^{\,X_0} - \cancel{X_0} - X_2) f_1}{(X_1 - X_0)(X_1 - X_2)}$$

$$+ \frac{(\cancel{2X_0}^{\,X_0} - \cancel{X_0} - X_1) f_2}{(X_2 - X_0)(X_2 - X_1)}$$

$$f' = \frac{(-3h) f_0}{(-h)(-2h)} + \frac{(-2h) f_1}{(h)(-h)}$$

$$+ \frac{(-h) f_2}{(2h)(h)}$$

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$$f' = \frac{(3) f_0}{2h} + \frac{2}{h} f_1 - \frac{1}{2h} f_2$$

$$f' = \frac{1}{h} \left[\frac{3}{2} f_0 + 2 f_1 - \frac{1}{2} f_2 \right]$$

وهو المطلوب

Error:

$$R' = \frac{f'''(\xi)}{6}\left[(x-x_0)(x-x_1)(1) + (x-x_2)\left[(x-x_0)(1) + (x-x_1)(1)\right]\right]$$

$x = x_0$

Forward

$$= \frac{f'''(\xi)}{6}\left[\cancel{(x_0-x_0)}(x_0-x_1) + (x_0-x_2)\left[\cancel{(x_0-x_0)} + (x_0-x_1)\right]\right]$$

$$= \frac{f'''(\xi)}{6}\left[(x_0-x_2)(x_0-x_1)\right]$$

$$= \frac{f'''(\xi)}{6}\left[(-2h)(-h)\right]$$

$$= \frac{f'''(\xi)}{\cancel{6}\,3}\left(\cancel{2}^{\,1}h^2\right)$$

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$$\boxed{E = \frac{h^2}{3} f'''(\xi)}$$

وهو المطلوب

Lagrange...

* using 2 points (forward):

x	x_0	x_1
$f(x)$	f_0	f_1

$$\Rightarrow f(x) = \frac{(x - x_1)}{x_0 - x_1} f_0 + \frac{(x - x_0)}{x_1 - x_0} f_1$$

$$f'(x) = \frac{f_0}{x_0 - x_1} + \frac{f_1}{x_1 - x_0}$$

$$f' = \frac{1}{h}[f_1 - f_0]$$

ودي المطلوب

(Backword)
Two points:

x	x_{-1}	x_0
$f(x)$	f_{-1}	f_0

من (**) السابقة
اشتقة المعادلة
صفحة 270 s

* for the second derivative:

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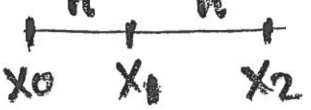
$$\Rightarrow f''(x) = \frac{2f_0}{(x_0-x_1)(x_0-x_2)} + \frac{2f_1}{(x_1-x_0)(x_1-x_2)} + \frac{2f_2}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{2f_0}{(-h)(-2h)} + \frac{2f_1}{(h)(-h)} + \frac{2f_2}{(2h)(h)}$$

$$= \frac{\cancel{2}f_0}{\cancel{2}h^2} + \frac{2f_1}{-h^2} + \frac{\cancel{2}f_2}{\cancel{2}h^2}$$

$$f''(x_1) = \frac{1}{h^2}\left[f_0 - 2f_1 + f_2\right]$$ وهو المطلوب



م. قاسم الخالدي
هندسة ميكانيكية

Proove :- //

(1) Use the Taylor series $(n=4)$ $\rightarrow$ (5 points)
to derive the following
Central difference formula :-
For the second derivative $\Rightarrow$

$$\boxed{f''(x) = \frac{-f(x+2h)+16f(x+h)-30f(x)+16f(x-h)-f(x-2h)}{12h^2}}$$

(2) Given $f(x) = x^2 \tan x$

Find $f''(1)$ by using above
Central difference formula :-
with $h = 0.01$?

• Proof for $f''(x)$

$$f_1 = f_0 + hf' + \frac{h^2}{2} f'' \quad \text{— (A)}$$

$$f_{-1} = f_0 - hf' + \frac{h^2}{2} f'' \quad \text{— (B)}$$

$$f_2 = f_0 + 2hf' + 2h^2 f'' \quad \text{— (C)}$$

$$f_{-2} = f_0 - 2hf' + 2h^2 f'' \quad \text{— (D)}$$

مسألة ميكانيكية
بالقسم العالي (ع)

$$(A) + (B) \Rightarrow f_1 + f_{-1} = 2f_0 + h^2 f'' \quad \text{— (1)}$$

$$(C) + (D) \Rightarrow f_2 + f_{-2} = 2f_0 + 4h^2 f'' \quad \text{— (2)}$$

$$16 * (1) \rightarrow 16f_1 + 16f_{-1} = 32f_0 + 16h^2 f''$$

$$(2) \rightarrow f_2 + f_{-2} = 2f_0 + 4h^2 f''$$

$$16f_1 + 16f_{-1} - f_2 - f_{-2} = 30f_0 + 12h^2 f''$$

وهو المطلوب

$$\Rightarrow f'' = \frac{16f_1 + 16f_{-1} - f_2 - f_{-2} - 30f_0}{12h^2}$$

Forward Finite-Difference

First Derivative	Error
$f'(x_i) = \dfrac{f(x_{i+1}) - f(x_i)}{h}$	$O(h)$
$f'(x_i) = \dfrac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h}$	$O(h^2)$
Second Derivative	
$f''(x_i) = \dfrac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$	$O(h)$
$f''(x_i) = \dfrac{-f(x_{i+3}) + 4f(x_{i+2}) - 5f(x_{i+1}) + 2f(x_i)}{h^2}$	$O(h^2)$
Third Derivative	
$f'''(x_i) = \dfrac{f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)}{h^3}$	$O(h)$
$f'''(x_i) = \dfrac{-3f(x_{i+4}) + 14f(x_{i+3}) - 24f(x_{i+2}) + 18f(x_{i+1}) - 5f(x_i)}{2h^3}$	$O(h^2)$
Fourth Derivative	
$f''''(x_i) = \dfrac{f(x_{i+4}) - 4f(x_{i+3}) + 6f(x_{i+2}) - 4f(x_{i+1}) + f(x_i)}{h^4}$	$O(h)$
$f''''(x_i) = \dfrac{-2f(x_{i+5}) + 11f(x_{i+4}) - 24f(x_{i+3}) + 26f(x_{i+2}) - 14f(x_{i+1}) + 3f(x_i)}{h^4}$	$O(h^2)$

Backward Finite-Difference

	Error
First Derivative	
$f'(x_i) = \dfrac{f(x_i) - f(x_{i-1})}{h}$	$O(h)$
$f'(x_i) = \dfrac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h}$	$O(h^2)$
Second Derivative	
$f''(x_i) = \dfrac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2}$	$O(h)$
$f''(x_i) = \dfrac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3})}{h^2}$	$O(h^2)$
Third Derivative	
$f'''(x_i) = \dfrac{f(x_i) - 3f(x_{i-1}) + 3f(x_{i-2}) - f(x_{i-3})}{h^3}$	$O(h)$
$f'''(x_i) = \dfrac{5f(x_i) - 18f(x_{i-1}) + 24f(x_{i-2}) - 14f(x_{i-3}) + 3f(x_{i-4})}{2h^3}$	$O(h^2)$
Fourth Derivative	
$f''''(x_i) = \dfrac{f(x_i) - 4f(x_{i-1}) + 6f(x_{i-2}) - 4f(x_{i-3}) + f(x_{i-4})}{h^4}$	$O(h)$
$f''''(x_i) = \dfrac{3f(x_i) - 14f(x_{i-1}) + 26f(x_{i-2}) - 24f(x_{i-3}) + 11f(x_{i-4}) - 2f(x_{i-5})}{h^4}$	$O(h^2)$

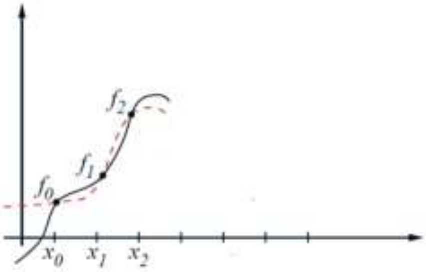
Centered Finite-Difference

First Derivative	Error
$f'(x_i) = \dfrac{f(x_{i+1}) - f(x_{i-1})}{2h}$	$O(h^2)$
$f'(x_i) = \dfrac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h}$	$O(h^4)$
Second Derivative	
$f''(x_i) = \dfrac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$	$O(h^2)$
$f''(x_i) = \dfrac{-f(x_{i+2}) + 16f(x_{i+1}) - 30f(x_i) + 16f(x_{i-1}) - f(x_{i-2})}{12h^2}$	$O(h^4)$
Third Derivative	
$f'''(x_i) = \dfrac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2h^3}$	$O(h^2)$
$f'''(x_i) = \dfrac{-f(x_{i+3}) + 8f(x_{i+2}) - 13f(x_{i+1}) + 13f(x_{i-1}) - 8f(x_{i-2}) + f(x_{i-3})}{8h^3}$	$O(h^4)$
Fourth Derivative	
$f''''(x_i) = \dfrac{f(x_{i+2}) - 4f(x_{i+1}) + 6f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{h^4}$	$O(h^2)$
$f''''(x_i) = \dfrac{-f(x_{i+3}) + 12f(x_{i+2}) + 39f(x_{i+1}) + 56f(x_i) - 39f(x_{i-1}) + 12f(x_{i-2}) + f(x_{i-3})}{6h^4}$	$O(h^4)$

Mostly used three-point formula (see Figure)

Let x_0, x_1, and x_2 be equally spaced and the grid spacing be h.

Thus $x_1 = x_0 + h$; and $x_2 = x_0 + 2h$.



1. $f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_1) - f(x_2)] + \frac{h^2}{3} f^{(3)}(\xi(x_0))$

(three-point endpoint formula with error)

2. $f'(x_1) = \frac{1}{2h}[-f(x_0) + f(x_2)] + \frac{h^2}{6} f^{(3)}(\xi(x_1))$

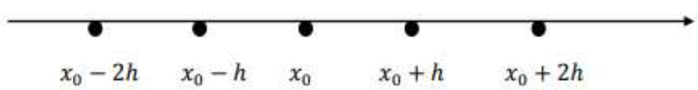
(three-point midpoint formula with error)

3. $f'(x_2) = \frac{1}{2h}[f(x_0) - 4f(x_1) + 3f(x_2)] + \frac{h^2}{3} f^{(3)}(\xi(x_2))$

(three-point endpoint formula with error)

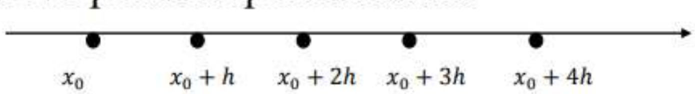
Mostly used five-point formula

1.Five-point midpoint formula



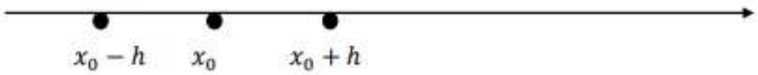
$$f'(x_0) = \frac{1}{12h}[f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)] + \frac{h^4}{30} f^{(5)}(\xi)$$

2.Five-point endpoint formula



$$f'(x_0) = \frac{1}{12h}[-25f(x_0) + 48f(x_0 + h) - 36f(x_0 + 2h) + 16f(x_0 + 3h) - 3f(x_0 + 4h)] + \frac{h^4}{5} f^{(5)}(\xi)$$

2^{nd} derivative approximation (obtained by Taylor polynomial)



Approximate $f(x_0 + h)$ by expansion about x_0:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \frac{1}{6}f'''(x_0)h^3 + \frac{1}{24}f^{(4)}(\xi_1)h^4 \qquad (3)$$

Approximate $f(x_0 - h)$ by expansion about x_0:

$$f(x_0 - h) = f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 - \frac{1}{6}f'''(x_0)h^3 + \frac{1}{24}f^{(4)}(\xi_2)h^4 \qquad (4)$$

Add Eqns. (3) and (4): $f(x_0 - h) + f(x_0 + h) = 2f(x_0) + f''(x_0)h^2 + [\frac{1}{24}f^{(4)}(\xi_1)h^4 + \frac{1}{24}f^{(4)}(\xi_2)h^4]$

Thus

Second derivative midpoint formula

$$\boxed{f''(x_0) = \frac{1}{h^2}[f(x_0 - h) - 2f(x_0) + f(x_0 + h)] - \frac{h^2}{12}f^{(4)}(\xi)}$$

Example Values for $f(x) = xe^x$ are given in the following table. Use all applicable 3-point and 5-point formulas to approximate $f'(2.0)$.

x	1.8	1.9	2.0	2.1	2.2
$f(x)$	10.889365	12.703199	14.778112	17.148957	19.855030

BASIC NUMERICAL DIFFERENTIATION

FORWARD FINITE DIFFERENCE

Let $f : [a,b] \to \mathbb{R}$ be differentiable and let $a \leq x_i < x_{i+1} \leq b$,then, using Taylor theorem:

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \mathcal{O}h^2$$

where $h = x_{i+1} - x_i$. In that case, the forward finite-difference can be used to approximate $f'(x_i)$ as follows:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} + \mathcal{O}(h) \tag{1}$$

where $\mathcal{O}h$ indicates that the error term is directly proportional to the chosen step size h.

BACKWARD FINITE DIFFERENCE

Let $f : [a,b] \to \mathbb{R}$ be differentiable and let $a \leq x_{i-1} < x_i \leq b$, then, using Taylor theorem:

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \mathcal{O}(h^2)$$

where $h = x_i - x_{i-1}$. In this case, the backward finite-difference can be used to approximate $f'(x_i)$ as follows:

$$f'(x_i) = \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i} + \mathcal{O}(h) \tag{2}$$

where $\mathcal{O}(h)$ indicates that the error term is directly proportional to the chosen step size h.

CENTRED FINITE DIFFERENCE

The centred finite difference can provide a better estimate for the derivative of a function at a particular point. If the values of a function f are known at the points $x_{i-1} < x_i < x_{i+1}$ and $x_{i+1} - x_i = x_i - x_{i-1} = h$, then, we can use the Taylor series to find a good approximation for the derivative as follows:

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \mathcal{O}(h^3)$$
$$f(x_{i-1}) = f(x_i) + f'(x_i)(-h) + \frac{f''(x_i)}{2!}h^2 + \mathcal{O}(h^3)$$

subtracting the above two equations and dividing by h gives the following:

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + \mathcal{O}(h^2)$$

where $\mathcal{O}(h^2)$ indicates that the error term is directly proportional to the square of the chosen step size h. I.e., the centred finite difference provides a better estimate for the derivative when the step size h is reduced compared to the forward and backward finite differences. Notice that when $x_{i+1} - x_i = x_i - x_{i-1}$, the centred finite difference is the average of the forward and backward finite difference!

BASIC NUMERICAL DIFFERENTIATION FORMULAS FOR HIGHER DERIVATIVES

The formulas presented in the previous section can be extended naturally to higher-order derivatives as follows.

FORWARD FINITE DIFFERENCE

Let $f:[a,b]\to\mathbb{R}$ be differentiable and let $a\leq x_i<x_{i+1}<x_{i+2}\leq b$, with $h=x_{i+1}-x_i=x_i-x_{i-1}$, then, using the basic forward finite difference formula for the second derivative, we have:

$$\begin{aligned} f''(x_i) &= \frac{f'(x_{i+1})-f'(x_i)}{h}+\mathcal{O}(h)\\ &= \frac{\frac{f(x_{i+2})-f(x_{i+1})}{h}-\frac{f(x_{i+1})-f(x_i)}{h}}{h}+\mathcal{O}(h)\\ &= \frac{f(x_{i+2})-2f(x_{i+1})+f(x_i)}{h^2}+\mathcal{O}(h)\end{aligned}\tag{3}$$

Notice that in order to calculate the second derivative at a point x_i using forward finite difference, the values of the function at two additional points x_{i+2} and x_{i+1} are needed.
Similarly, for the third derivative, the value of the function at another point x_{i+3} with $x_{i+2}<x_{i+3}\leq b$ is required (with the same spacing h). Then, the third derivative can be calculated as follows:

$$\begin{aligned} f'''(x_i) &= \frac{f''(x_{i+1})-f''(x_i)}{h}+\mathcal{O}(h)\\ &= \frac{\frac{f(x_{i+3})-2f(x_{i+2})+f(x_{i+1})}{h^2}-\frac{f(x_{i+2})-2f(x_{i+1})+f(x_i)}{h^2}}{h}+\mathcal{O}(h)\\ &= \frac{f(x_{i+3})-3f(x_{i+2})+3f(x_{i+1})-f(x_i)}{h^3}+\mathcal{O}(h)\end{aligned}$$

Similarly, for the fourth derivative, the value of the function at another point x_{i+4} with $x_{i+3}<x_{i+4}\leq b$ is required (with the same spacing h). Then, the fourth derivative can be calculated as follows:

$$\begin{aligned} f''''(x_i) &= \frac{f'''(x_{i+1})-f'''(x_i)}{h}+\mathcal{O}(h)\\ &= \frac{\frac{f(x_{i+4})-3f(x_{i+3})+3f(x_{i+2})-f(x_{i+1})}{h^3}-\frac{f(x_{i+3})-3f(x_{i+2})+3f(x_{i+1})-f(x_i)}{h^3}}{h}+\mathcal{O}(h)\\ &= \frac{f(x_{i+4})-4f(x_{i+3})+6f(x_{i+2})-4f(x_{i+1})+f(x_i)}{h^4}+\mathcal{O}(h)\end{aligned}$$

BACKWARD FINITE DIFFERENCE

Let $f:[a,b]\to\mathbb{R}$ be differentiable and let $a\le x_{i-2}<x_{i-1}<x_i\le b$, with $h=x_i-x_{i-1}=x_{i-1}-x_{i-2}$, then, using the basic backward finite difference formula for the second derivative, we have:

$$\begin{aligned} f''(x_i) &= \frac{f'(x_i)-f'(x_{i-1})}{h}+\mathcal{O}(h) \\ &= \frac{\frac{f(x_i)-f(x_{i-1})}{h}-\frac{f(x_{i-1})-f(x_{i-2})}{h}}{h}+\mathcal{O}(h) \\ &= \frac{f(x_i)-2f(x_{i-1})+f(x_{i-2})}{h^2}+\mathcal{O}(h) \end{aligned} \tag{4}$$

Notice that in order to calculate the second derivative at a point x_i using backward finite difference, the values of the function at two additional points x_{i-2} and x_{i-1} are needed.
Similarly, for the third derivative the value of the function at another point x_{i-3} with $a\le x_{i-3}<x_{i-2}$ is required (with the same spacing h). Then, the third derivative can be calculated as follows:

$$\begin{aligned} f'''(x_i) &= \frac{f''(x_i)-f''(x_{i-1})}{h}+\mathcal{O}(h) \\ &= \frac{\frac{f(x_i)-2f(x_{i-1})+f(x_{i-2})}{h^2}-\frac{f(x_{i-1})-2f(x_{i-2})+f(x_{i-3})}{h^2}}{h}+\mathcal{O}(h) \\ &= \frac{f(x_i)-3f(x_{i-1})+3f(x_{i-2})-f(x_{i-3})}{h^3}+\mathcal{O}(h) \end{aligned}$$

Similarly, for the fourth derivative, the value of the function at another point x_{i-4} with $a\le x_{i-4}<x_{i-3}$ is required (with the same spacing h). Then, the fourth derivative can be calculated as follows:

$$\begin{aligned} f''''(x_i) &= \frac{f'''(x_i)-f'''(x_{i-1})}{h}+\mathcal{O}(h) \\ &= \frac{\frac{f(x_i)-3f(x_{i-1})+3f(x_{i-2})-f(x_{i-3})}{h^3}-\frac{f(x_{i-1})-3f(x_{i-2})+3f(x_{i-3})-f(x_{i-4})}{h^3}}{h}+\mathcal{O}(h) \\ &= \frac{f(x_i)-4f(x_{i-1})+6f(x_{i-2})-4f(x_{i-3})+f(x_{i-4})}{h^4}+\mathcal{O}(h) \end{aligned}$$

CENTRED FINITE DIFFERENCE

Let $f:[a,b]\to\mathbb{R}$ be differentiable and let $a\leq x_{i-1}<x_i<x_{i+1}\leq b$, with a constant spacing h, then, we can use the Taylor theorem for $f(x_{i+1})$ and $f(x_{i-1})$ as follows:

$$
\begin{aligned}
f(x_{i+1}) &= f(x_i)+f'(x_i)h+\frac{f''(x_i)}{2!}h^2+\frac{f'''(x_i)}{3!}h^3+\mathcal{O}(h^4)\\
f(x_{i-1}) &= f(x_i)+f'(x_i)(-h)+\frac{f''(x_i)}{2!}h^2+\frac{f'''(x_i)}{3!}h^3+\mathcal{O}(h^3)
\end{aligned}
$$

Adding the above two equations and dividing by h^2 gives the following:
\being{equation}\label{eq:eq1}
f''(x_i)=\frac{f(x_{i+1})-2f(x_i)+f(x_{i-1})}{h^2}+\mathcal O (h^2)
\end{equation}
which provides a better approximation for the second derivative than that provided by the forward or backward finite difference as the error is directly proportional to the square of the step size.
For the third derivative, the value of the function is required at the points x_{i-2} and x_{i+2}. Assuming all the points to be equidistant with a spacing h, then, the third derivative can be calculated using Equation ?? as follows:

$$
f'''(x_i)=\frac{f'(x_{i+1})-2f'(x_i)+f'(x_{i-1})}{h^2}+\mathcal{O}(h^2)
$$

Replacing the first derivatives with the centred finite difference value for those:

$$
\begin{aligned}
f'''(x_i) &= \frac{\frac{f(x_{i+2})-f(x_i)}{2h}-2\frac{f(x_{i+1})-f(x_{i-1})}{2h}+\frac{f(x_i)-f(x_{i-2})}{2h}}{h^2}+\mathcal{O}(h^2)\\
&= \frac{f(x_{i+2})-2f(x_{i+1})+2f(x_{i-1})-f(x_{i-2})}{2h^3}+\mathcal{O}(h^2)
\end{aligned} \tag{5}
$$

For the fourth derivative, the value of the function at the points x_{i+2} and x_{i-2} is required. Assuming all the points to be equidistant with a spacing h, then, the fourth derivative can be calculated using Equation ?? as follows:

$$
f''''(x_i)=\frac{f''(x_{i+1})-2f''(x_i)+f''(x_{i-1})}{h^2}+\mathcal{O}(h^2)
$$

Using the centred finite difference for the second derivatives (Equation ??) yields:

$$
\begin{aligned}
f''''(x_i) &= \frac{\frac{f(x_{i+2})-2f(x_{i+1})+f(x_i)}{h^2}-2\frac{f(x_{i+1})-2f(x_i)+f(x_{i-1})}{h^2}+\frac{f(x_i)-2f(x_{i-1})+f(x_{i-2})}{h^2}}{h^2}+\mathcal{O}(h^2)\\
&= \frac{f(x_{i+2})-4f(x_{i+1})+6f(x_i)-4f(x_{i-1})+f(x_{i-2})}{h^4}+\mathcal{O}(h^2)
\end{aligned}
$$

HIGH-ACCURACY NUMERICAL DIFFERENTIATION FORMULAS

The formulas presented in the previous sections for the forward and backward finite difference have an error term of $\mathcal{O}(h)$ while those for the centred finite difference scheme have an error term of $\mathcal{O}(h^2)$. It is possible to provide formulas with less error by utilizing more terms in the Taylor approximation. In essence, by increasing the number of terms in the Taylor series approximation, we assume a higher-order polynomial for the approximation which increases the accuracy of the derivatives. In the following presentation, the function f is assumed to be smooth and the points are equidistant with a step size of h.

FORWARD FINITE DIFFERENCE

According to Taylor theorem, we have:

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \mathcal{O}(h^3)$$

Using the forward finite difference equation (Equation 3) for $f''(x_i)$ yields:

$$f'(x_i)h = f(x_{i+1}) - f(x_i) - \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{2} + \mathcal{O}(h^3)$$

Therefore:

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + \mathcal{O}(h^2)$$

In comparison with Equation 1, this equation provides an error term that is directly proportional to the square of the step size indicating higher accuracy.

Using the same procedure, the following equations can be obtained for the second, third, and fourth derivatives:

$$\begin{aligned}
f''(x_i) &= \frac{-f(x_{i+3}) + 4f(x_{i+2}) - 5f(x_{i+1}) + 2f(x_i)}{h^2} + \mathcal{O}(h^2) \\
f'''(x_i) &= \frac{-3f(x_{i+4}) + 14f(x_{i+3}) - 24f(x_{i+2}) + 18f(x_{i+1}) - 5f(x_i)}{2h^3} + \mathcal{O}(h^2) \\
f''''(x_i) &= \frac{-2f(x_{i+5}) + 11f(x_{i+4}) - 24f(x_{i+3}) + 26f(x_{i+2}) - 14f(x_{i+1}) + 3f(x_i)}{h^4} + \mathcal{O}(h^2)
\end{aligned}$$

BACKWARD FINITE DIFFERENCE

According to Taylor theorem, we have:

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \mathcal{O}(h^3)$$

Using the backward finite difference equation (Equation 4) for $f''(x_i)$ yields:

$$f'(x_i)h = -f(x_{i-1}) + f(x_i) + \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{2} + \mathcal{O}(h^3)$$

Therefore:

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h} + \mathcal{O}(h^2)$$

In comparison with Equation 2, this equation provides an error term that is directly proportional to the square of the step size indicating higher accuracy.
Using the same procedure, the following equations can be obtained for the second, third, and fourth derivatives:

$$\begin{aligned}
f''(x_i) &= \frac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3})}{h^2} + \mathcal{O}(h^2) \\
f'''(x_i) &= \frac{5f(x_i) - 18f(x_{i-1}) + 24f(x_{i-2}) - 14f(x_{i-3}) + 3f(x_{i-4})}{2h^3} + \mathcal{O}(h^2) \\
f''''(x_i) &= \frac{3f(x_i) - 14f(x_{i-1}) + 26f(x_{i-2}) - 24f(x_{i-3}) + 11f(x_{i-4}) - 2f(x_{i-5})}{h^4} + \mathcal{O}(h^2)
\end{aligned}$$

CENTRED FINITE DIFFERENCE

According to Taylor theorem, we have:

$$\begin{aligned}
f(x_{i+1}) &= f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!}h^3 + \frac{f''''(x_i)}{4!}h^4 + \mathcal{O}(h^5) \\
f(x_{i-1}) &= f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \frac{f'''(x_i)}{3!}h^3 + \frac{f''''(x_i)}{4!}h^4 + \mathcal{O}(h^5)
\end{aligned}$$

Subtracting the above two equations and using the centred finite difference equation (Equation 5) for $f'''(x_i)$ yields:

$$f'(x_i)2h = f(x_{i+1}) - f(x_{i-1}) - \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{6} + \mathcal{O}(h^5)$$

Therefore:

$$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h} + \mathcal{O}(h^4)$$

Using the same procedure, the following equations can be obtained for the second, third, and fourth derivatives:

$$\begin{aligned}
f''(x_i) &= \frac{-f(x_{i+2}) + 16f(x_{i+1}) - 30f(x_i) + 16f(x_{i-1}) - f(x_{i-2})}{12h^2} + \mathcal{O}(h^4) \\
f'''(x_i) &= \frac{-f(x_{i+3}) + 8f(x_{i+2}) - 13f(x_{i+1}) + 13f(x_{i-1}) - 8f(x_{i-2}) + f(x_{i-3})}{8h^3} + \mathcal{O}(h^4) \\
f''''(x_i) &= \frac{-f(x_{i+3}) + 12f(x_{i+2}) - 39f(x_{i+1}) + 56f(x_i) - 39f(x_{i-1}) + 12f(x_{i-2}) - f(x_{i-3})}{6h^4} + \mathcal{O}(h
\end{aligned}$$

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NUMERICAL ANALYSIS

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CONTENTS

Differentiation

1. Symbolic Method

- Since our problem is Central difference then :-

** $$\boxed{\delta f_i = f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}$$

** $$\delta^2 f_i = \delta f_{i+\frac{1}{2}} - \delta f_{i-\frac{1}{2}}$$
$$= \left(f_{i+\frac{1}{2}+\frac{1}{2}} - f_{i+\frac{1}{2}-\frac{1}{2}}\right) - \left(f_{i-\frac{1}{2}+\frac{1}{2}} - f_{i-\frac{1}{2}-\frac{1}{2}}\right)$$
$$= f_{i+1} - f_i - f_i + f_{i-1}$$

$$\therefore \boxed{\delta^2 f_i = f_{i+1} - 2f_i + f_{i-1}}$$

** $$\delta^3 f_i = \delta\left(f_{i+1} - 2f_i + f_{i-1}\right)$$
$$= \left(f_{i+\frac{3}{2}} - f_{i+\frac{1}{2}}\right) - 2\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) + \left(f_{i-\frac{1}{2}} - f_{i-\frac{3}{2}}\right)$$

$$\therefore \boxed{\delta^3 f_i = f_{i+\frac{3}{2}} - 3f_{i+\frac{1}{2}} + 3f_{i-\frac{1}{2}} - f_{i-\frac{3}{2}}}$$

** $$\delta^4 f_i = \delta\left(f_{i+\frac{3}{2}} - 3f_{i+\frac{1}{2}} + 3f_{i-\frac{1}{2}} - f_{i-\frac{3}{2}}\right)$$
$$= \left(f_{i+2} - f_{i+1}\right) - 3\left(f_{i+1} - f_i\right) + 3\left(f_i - f_{i-1}\right) - \left(f_{i-1} - f_{i-2}\right)$$

$$\therefore \boxed{\delta^4 f_i = f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2}}$$

** $$\delta^5 f_i = \delta\left(f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2}\right)$$
$$= \left(f_{i+\frac{5}{2}} - f_{i+\frac{3}{2}}\right) - 4\left(f_{i+\frac{3}{2}} - f_{i+\frac{1}{2}}\right) + 6\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right)$$
$$- 4\left(f_{i-\frac{1}{2}} - f_{i-\frac{3}{2}}\right) + \left(f_{i-\frac{3}{2}} - f_{i-\frac{5}{2}}\right)$$

$$\therefore \boxed{\delta^5 f_i = f_{i+\frac{5}{2}} - 5f_{i+\frac{3}{2}} + 10f_{i+\frac{1}{2}} - 10f_{i-\frac{1}{2}} + 5f_{i-\frac{3}{2}} - f_{i-\frac{5}{2}}}$$

** $\delta^6 f_i = \delta\left(f_{i+\frac{5}{2}} - 5 f_{i+\frac{3}{2}} + 10 f_{i+\frac{1}{2}} - 10 f_{i-\frac{1}{2}} + 5 f_{i-\frac{3}{2}} - f_{i-\frac{5}{2}} \right)$

$$= \left(f_{i+3} - f_{i+2}\right) - 5\left(f_{i+2} - f_{i+1}\right) + 10\left(f_{i+1} - f_i\right) - 10\left(f_i - f_{i-1}\right) + 5\left(f_{i-1} - f_{i-2}\right) - \left(f_{i-2} - f_{i-3}\right)$$

$$\therefore \boxed{\delta^6 f_i = f_{i+3} - 6 f_{i+2} + 15 f_{i+1} - 20 f_i + 15 f_{i-1} - 6 f_{i-2} + f_{i-3}}$$

** First Derivative

$$hD = \mu\left(\delta - \frac{\delta^3}{6} + \frac{\delta^5}{30} + \mathcal{O}(h^6)\right)$$

$$f_i' = Df_i = \frac{\mu}{h}\Big(f_{i+\frac{1}{2}} + \frac{3}{6} f_{i+\frac{1}{2}} + \frac{10}{30} f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} - \frac{3}{6} f_{i-\frac{1}{2}} - \frac{10}{30} f_{i-\frac{1}{2}} - \frac{1}{6} f_{i+\frac{3}{2}} - \frac{5}{30} f_{i+\frac{3}{2}} + \frac{1}{6} f_{i-\frac{3}{2}} + \frac{5}{30} f_{i-\frac{3}{2}} + \frac{1}{30} f_{i+\frac{5}{2}} - \frac{1}{30} f_{i-\frac{5}{2}} \Big)$$

$$= \frac{\mu}{30h}\left[55\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) - 10\left(f_{i+\frac{3}{2}} - f_{i-\frac{3}{2}}\right) + \left(f_{i+\frac{5}{2}} - f_{i-\frac{5}{2}}\right)\right]$$

But $f_{i+h} = E^h f_i$

Also, the average $\mu = \frac{FWD + BWD}{2} = \frac{\Delta + \nabla}{2}$

$$\mu f_i = \frac{E^{1/2} + E^{-1/2}}{2}$$

subsitute :

$$f_i' = \frac{1}{60h}\left[55\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) + 10\left(E^{1/2} + E^{-1/2}\right)\left(E^{3/2} - E^{-3/2}\right) + \left(E^{1/2} + E^{-1/2}\right)\left(E^{-5/2} - E^{-5/2}\right)\right] f_i$$

$$= \frac{1}{60h}\left(55E - 10E - 55E^{-1} + 10E^{1} - 10E^{2} + E^{2} + 10E^{-2} - E^{-2} + E^{3} - E^{-3}\right) f_i$$

$$\therefore \quad f_i' = \frac{1}{60h}\left(-E^{-3} + 9E^{-2} - 45E^{-1} + 45E - 9E^{2} + E^{3}\right) f_i$$

$$= \frac{1}{60h}\left(-f_{i-3} + 9f_{i-2} - 45f_{i-1} + 45f_{i+1} - 9f_{i+2} + f_{i+3}\right)$$

or

$$\boxed{f'(x) = \frac{1}{60h}\Big(-f(x-3h) + 9f(x-2h) - 45f(x-h) + 45f(x+h) - 9f(x+2h) + f(x+3h)\Big)}$$

** Second Derivative

$$h^2D^2 = \delta^2 - \frac{\delta^4}{12} + \frac{\delta^6}{90} + O(h^6)$$

$$f_i'' = D^2 f_i = \frac{1}{h^2}\left(\delta^2 - \frac{\delta^4}{12} + \frac{\delta^6}{90}\right)$$

$$= \frac{1}{h^2}\Big[f_{i+1} + \frac{4}{12}f_{i+1} + \frac{15}{90}f_{i+1} - 2f_i - \frac{6}{12}f_i - \frac{20}{90}f_i + f_{i-1} - \frac{4}{12}f_{i-1} + \frac{15}{90}f_{i-1} - \frac{1}{12}f_{i+2} - \frac{6}{90}f_{i+2} - \frac{1}{12}f_{i-2} - \frac{6}{90}f_{i-2} + \frac{1}{90}f_{i+3} + \frac{1}{90}f_{i-3}\Big]$$

$$= \frac{1}{180h^2}\Big[2f_{i-3} - 27f_{i-2} + 270f_{i-1} - 490f_i + 270f_{i+1} - 27f_{i+2} + 2f_{i+3}\Big]$$

or

$$\boxed{f''(x) = \frac{1}{180h^2}\Big[2f(x-3h) - 27f(x-2h) + 270f(x-h) - 490f(x) + 270f(x+h) - 27f(x+2h) + 2f(x+3h)\Big]}$$

2. Lozenge Diagram

* $\Delta f_i = f_{i+1} - f_i$
* $\Delta^2 f_i = f_{i+2} - 2 f_{i+1} + f_i$
* $\Delta^3 f_i = f_{i+3} - 3 f_{i+2} + 3 f_{i+1} - f_i$
* $\Delta^4 f_i = f_{i+4} - 4 f_{i+3} + 6 f_{i+2} - 4 f_{i+1} + f_i$
* $\Delta^5 f_i = f_{i+5} - 5 f_{i+4} + 10 f_{i+3} - 10 f_{i+2} + 5 f_{i+1} - f_i$
* $\Delta^6 f_i = f_{i+6} - 6 f_{i+5} + 15 f_{i+4} - 20 f_{i+3} + 15 f_{i+2} - 6 f_{i+1} + f_i$

** First Derivative

In Lozenge diagram for derivative , if we extend the horizontal path until we reach the column of sixth differences [Central difference] , then :-

$$f_i' = \frac{1}{h}\left[\frac{\Delta f_{i-1} + \Delta f_i}{2} - \left(\frac{1}{6}\right)\frac{\Delta^3 f_{i-2} + \Delta^3 f_{i-1}}{2} + \left(\frac{1}{30}\right)\frac{\Delta^5 f_{i-3} + \Delta^5 f_{i-2}}{2}\right]$$

$$= \frac{1}{h}\Big[\frac{1}{2}\left(f_i - f_{i-1} + f_{i+1} - f_i\right) - \frac{1}{12}\big(f_{i+1} - 3f_i + 3f_{i-1} - f_{i-2} + f_{i+2} - 3f_{i+1} + 3f_i - f_{i-1}\big) + \frac{1}{60}\big(f_{i+2} - 5f_{i+1} + 10f_i - 10f_{i-1} + 5f_{i-2} - f_{i-3} + f_{i+3} - 5f_{i+2} + 10f_{i+1} - 10f_i + 5f_{i-1} - f_{i-2}\big)\Big]$$

$$= \frac{1}{60h}\Big[- f_{i-3} + (5+5-1) f_{i-2} + (-30-15+5-10+5) f_{i-1} + (30-30+15-15+10-10) f_i + (30-5+15-5+10) f_{i+1} + (-5+1-5) f_{i+2} + f_{i+3}\Big]$$

$$\therefore \quad f_i' = \frac{1}{60h}\left[-f_{i-3} + 9 f_{i-2} - 45 f_{i-1} + 45 f_{i+1} - 9 f_{i+2} + f_{i+3}\right]$$

OR

$$\boxed{f'(x) = \frac{1}{60h}\left[-f(x-3h) + 9 f(x-2h) - 45 f(x-h) + 45 f(x+h) - 9 f(x+2h) + f(x+3h)\right]}$$

** second Derivative

Similarily, from Lozenge diagram :-

$$f_i'' = \frac{1}{h^2}\left[\Delta^2 f_{i-1} - \frac{1}{12}\Delta^4 f_{i-2} + \frac{1}{90}\Delta^6 f_{i-3}\right]$$

$$= \frac{1}{h^2}\Big[(f_{i+1} - 2f_i + f_{i-1}) - \left(\tfrac{1}{12}\right)(f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2})$$
$$+ \frac{1}{90}(f_{i+3} - 6f_{i+2} + 15 f_{i+1} - 20 f_i + 15 f_{i-1} - 6 f_{i-2} + f_{i-3})\Big]$$

$$= \frac{1}{180h^2}\Big[2 f_{i-3} + (-15-12) f_{i-2} + (180+60+30) f_{i-1}$$
$$+ (-360 - 90 - 40) f_i + (180 + 60 + 30) f_{i+1}$$
$$+ (-15-12) f_{i+2} + 2 f_{i+3}\Big]$$

$$f_i'' = \frac{1}{180h^2}\Big[2 f_{i-3} - 27 f_{i-2} + 270 f_{i-1} - 490 f_i + 270 f_{i+1} - 27 f_{i+2} + 2 f_{i+3}\Big]$$

OR

$$\boxed{f''(x) = \frac{1}{180h^2}\left[2 f(x-3h) - 27 f(x-2h) + 270 f(x-h) - 490 f(x) + 270 f(x+h) - 27 f(x+2h) + 2 f(x+3h)\right]}$$

3. Undetermined Coefficients

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5$$

$$f''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + 30 a_6 x^4$$

$$f'''(x) = 6a_3 + 24 a_4 x + 60 a_5 x^2 + 120 a_6 x^3$$

$$f^{IV}(x) = 24 a_4 + 120 a_5 x + 360 a_6 x^2$$

$$f^{V}(x) = 120 a_5 + 720 a_6 x$$

$$f^{VI}(x) = 720 a_6$$

** At $x_i = 0$, The values of derivatives become :-

$f = a_0$	$f^{IV} = 24 a_4$
$f' = a_1$	$f^{V} = 120 a_5$
$f'' = 2a_2$	$f^{VI} = 720 a_6$
$f''' = 6a_3$	

** At $x_{i+1} = x_i + h = h$, then For Central difference

- at $x_i = 0 \Rightarrow f_i = a_0$
- at $x_{i+1} = h \Rightarrow f_{i+1} = a_0 + h a_1 + h^2 a_2 + h^3 a_3 + h^4 a_4 + h^5 a_5 + h^6 a_6$
- at $x_{i+2} = 2h \Rightarrow f_{i+2} = a_0 + 2h a_1 + 4h^2 a_2 + 8h^3 a_3 + 16 h^4 a_4 + 32 h^5 a_5 + 64 h^6 a_6$
- at $x_{i+3} = 3h \Rightarrow f_{i+3} = a_0 + 3h a_1 + 9h^2 a_2 + 27 h^3 a_3 + 81 h^4 a_4$
  $+ 243 h^5 a_5 + 729 h^6 a_6$
- at $x_{i-1} = -h \Rightarrow f_{i-1} = a_0 - h a_1 + h^2 a_2 - h^3 a_3 + h^4 a_4 - h^5 a_5 + h^6 a_6$
- at $x_{i-2} = -2h \Rightarrow f_{i-2} = a_0 - 2h a_1 + 4h^2 a_2 - 8h^3 a_3 + 16 h^4 a_4 - 32 h^5 a_5 + 64 h^6 a_6$
- at $x_{i-3} = -3h \Rightarrow f_{i-3} = a_0 - 3h a_1 + 9h^2 a_2 - 27 h^3 a_3 + 81 h^4 a_4$
  $- 243 h^5 a_5 + 729 h^6 a_6$

** put the previous eqn's in matrix form then :-

$$
\begin{bmatrix}
1 & -3 & (3)^2 & -(3)^3 & (3)^4 & -(3)^5 & (3)^6 \\
1 & -2 & (2)^2 & -(2)^3 & (2)^4 & -(2)^5 & (2)^6 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & (2)^2 & (2)^3 & (2)^4 & (2)^5 & (2)^6 \\
1 & 3 & (3)^2 & (3)^3 & (3)^4 & (3)^5 & (3)^6
\end{bmatrix}
\begin{bmatrix}
a_0 \\ a_1 h \\ a_2 h^2 \\ a_3 h^3 \\ a_4 h^4 \\ a_5 h^5 \\ a_6 h^6
\end{bmatrix}
=
\begin{bmatrix}
f_{i-3} \\ f_{i-2} \\ f_{i-1} \\ f_i \\ f_{i+1} \\ f_{i+2} \\ f_{i+3}
\end{bmatrix}
$$

** The solution the matrix equations gives :-

$$
\begin{bmatrix}
a_0 \\ a_1 h \\ a_2 h^2 \\ a_3 h^3 \\ a_4 h^4 \\ a_5 h^5 \\ a_6 h^6
\end{bmatrix}
= \frac{1}{720}
\begin{bmatrix}
0 & 0 & 0 & 720 & 0 & 0 & 0 \\
-12 & 108 & -540 & 0 & 540 & -108 & 12 \\
4 & -54 & 540 & -980 & 540 & -54 & 4 \\
15 & -120 & 195 & 0 & -195 & 120 & -15 \\
-5 & 60 & -195 & 280 & -195 & 60 & -5 \\
-3 & 12 & -15 & 0 & 15 & -12 & 3 \\
1 & -6 & 15 & -20 & 15 & -6 & 1
\end{bmatrix}
\begin{bmatrix}
f_{i-3} \\ f_{i-2} \\ f_{i-1} \\ f_i \\ f_{i+1} \\ f_{i+2} \\ f_{i+3}
\end{bmatrix}
$$

*** After simplification , the solution becomes :-

$$\boxed{f_i' = a_1 = \frac{1}{60h}\left[-f_{i-3} + 9f_{i-2} - 45f_{i-1} + 45f_{i+1} - 9f_{i+2} + f_{i+3}\right]}$$

$$\boxed{f_i'' = 2a_2 = \frac{1}{180h^2}\left[2f_{i-3} - 27f_{i-2} + 270f_{i-1} - 490f_i + 270f_{i+1} - 27f_{i+2} + 2f_{i+3}\right]}$$

$$\boxed{f_i''' = 6a_3 = \frac{1}{8h^3}\left[f_{i-3} - 8f_{i-2} + 13f_{i-1} - 13f_{i+1} + 8f_{i+2} - f_{i+3}\right]}$$

$$\boxed{f_i^{IV} = 24a_4 = \frac{1}{6h^4}\left[-f_{i-3} + 12f_{i-2} - 39f_{i-3} + 50f_i - 39f_{i+1} + 12f_{i+2} - f_{i+3}\right]}$$

$$\boxed{f_i^{V} = 120a_5 = \frac{1}{2h^5}\left[-f_{i-3} + 4f_{i-2} - 5f_{i-1} + 5f_{i+1} - 4f_{i+2} + f_{i+3}\right]}$$

$$\boxed{f_i^{VI} = 720a_6 = \frac{1}{h^6}\left[f_{i-3} - 6f_{i-2} + 15f_{i-1} - 20f_i + 15f_{i+1} - 6f_{i+2} + f_{i+3}\right]}$$

4. Taylor Series

4.A Central Difference

* $f_{i-3} = f_i - 3hf_i' + \frac{9}{2}h^2 f_i'' - \frac{9}{2}h^3 f_i''' + \frac{27}{8}h^4 f_i^{IV} - \frac{81}{40}h^5 f_i^{V} + \frac{81}{80}h^6 f_i^{VI}$
* $f_{i-2} = f_i - 2hf_i' + 2h^2 f_i'' - \frac{4}{3}h^3 f_i''' + \frac{2}{3}h^4 f_i^{IV} - \frac{4}{15}h^5 f_i^{V} + \frac{4}{45}h^6 f_i^{VI}$
* $f_{i-1} = f_i - hf_i' + \frac{h^2}{2} f_i'' - \frac{h^3}{6} f_i''' + \frac{h^4}{24} f_i^{IV} + \frac{h^5}{120} f_i^{V} + \frac{h^6}{720} f_i^{VI}$
* $f_{i+1} = f_i + hf_i' + \frac{h^2}{2} f_i'' + \frac{h^3}{6} f_i''' + \frac{h^4}{24} f_i^{IV} + \frac{h^5}{120} f_i^{V} + \frac{h^6}{720} f_i^{VI}$
* $f_{i+2} = f_i + 2hf_i' + 2h^2 f_i'' + \frac{4}{3}h^3 f_i''' + \frac{2}{3}h^4 f_i^{IV} + \frac{4}{15}h^5 f_i^{V} + \frac{4}{45}h^6 f_i^{VI}$
* $f_{i+3} = f_i + 3hf_i' + \frac{9}{2}h^2 f_i'' + \frac{9}{2}h^3 f_i''' + \frac{27}{8}h^4 f_i^{IV} + \frac{81}{40}h^5 f_i^{V} + \frac{81}{80}h^6 f_i^{VI}$

* Put the previous equations in matrix form, then :-

$$\begin{bmatrix} -3 & 9/2 & -9/2 & 27/8 & -81/40 & 81/80 \\ -2 & 2 & -4/3 & 2/3 & -4/15 & 4/45 \\ -1 & 1/2 & -1/6 & 1/24 & -1/120 & 1/720 \\ 1 & 1/2 & 1/6 & 1/24 & 1/120 & 1/720 \\ 2 & 2 & 4/3 & 2/3 & 4/15 & 4/45 \\ 3 & 9/2 & 9/2 & 27/8 & 81/40 & 81/80 \end{bmatrix} \cdot \begin{bmatrix} h f_i' \\ h^2 f_i'' \\ h^3 f_i''' \\ h^4 f_i^{IV} \\ h^5 f_i^{V} \\ h^6 f_i^{VI} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{i-3} \\ f_{i-2} \\ f_{i-1} \\ f_i \\ f_{i+1} \\ f_{i+2} \\ f_{i+3} \end{bmatrix}$$

** The solution of the matrix equations is :

$$
\begin{bmatrix} h f_i' \\ h^2 f_i'' \\ h^3 f_i''' \\ h^4 f_i^{IV} \\ h^5 f_i^{V} \\ h^6 f_i^{VI} \end{bmatrix}
=
\begin{bmatrix}
-1/60 & 9/60 & -45/60 & 0 & 45/60 & -9/60 & 1/60 \\
2/180 & -27/180 & 270/180 & -490/180 & 270/180 & -27/180 & 2/180 \\
1/8 & -8/8 & 13/8 & 0 & -13/8 & 8/8 & -1/8 \\
-1/6 & 12/6 & -39/6 & 56/6 & -39/6 & 12/6 & -1/6 \\
-1/2 & 4/2 & -5/2 & 0 & 5/2 & -4/2 & 1/2 \\
1 & -6 & 15 & -20 & 15 & -6 & 1
\end{bmatrix}
\cdot
\begin{bmatrix} f_{i-3} \\ f_{i-2} \\ f_{i-1} \\ f_i \\ f_{i+1} \\ f_{i+2} \\ f_{i+3} \end{bmatrix}
$$

$$\boxed{f_i' = \frac{1}{60h}\left[-f_{i-3} + 9 f_{i-2} - 45 f_{i-1} + 45 f_{i+1} - 9 f_{i+2} + f_{i+3}\right]}$$

$$\boxed{f_i'' = \frac{1}{180h^2}\left[2 f_{i-3} - 27 f_{i-2} + 270 f_{i-1} - 490 f_i + 270 f_{i+1} - 27 f_{i+2} + 2 f_{i+3}\right]}$$

$$\boxed{f_i''' = \frac{1}{8h^3}\left[f_{i-3} - 8 f_{i-2} + 13 f_{i-1} - 13 f_{i+1} + 8 f_{i+2} - f_{i+3}\right]}$$

$$\boxed{f_i^{IV} = \frac{1}{6h^4}\left[-f_{i-3} + 12 f_{i-2} - 39 f_{i-1} + 56 f_i - 39 f_{i+1} + 12 f_{i+2} - f_{i+3}\right]}$$

$$\boxed{f_i^{V} = \frac{1}{2h^5}\left[-f_{i-3} + 4 f_{i-2} - 5 f_{i-1} + 5 f_{i+1} - 4 f_{i+2} + f_{i+3}\right]}$$

$$\boxed{f_i^{VI} = \frac{1}{h^6}\left[f_{i-3} - 6 f_{i-2} + 15 f_{i-1} - 20 f_i + 15 f_{i+1} - 6 f_{i+2} + f_{i+3}\right]}$$

4.B Forward Difference

* $f_{i+1} = f_i + h f_i' + \frac{h^2}{2} f_i'' + \frac{h^3}{3} f_i''' + \frac{h^4}{24} f_i^{IV} + \frac{h^5}{120} f_i^{V} + \frac{h^6}{720} f_i^{VI}$
* $f_{i+2} = f_i + 2h f_i' + 2h^2 f_i'' + \frac{4}{3} h^3 f_i''' + \frac{2}{3} h^4 f_i^{IV} + \frac{4}{15} h^5 f_i^{V} + \frac{4}{45} h^6 f_i^{VI}$
* $f_{i+3} = f_i + 3h f_i' + \frac{9}{2} h^2 f_i'' + \frac{9}{2} h^3 f_i''' + \frac{27}{8} h^4 f_i^{IV} + \frac{81}{40} h^5 f_i^{V} + \frac{81}{80} h^6 f_i^{VI}$
* $f_{i+4} = f_i + 4h f_i' + 8 h^2 f_i'' + \frac{32}{3} h^3 f_i''' + \frac{32}{3} h^4 f_i^{IV} + \frac{128}{15} h^5 f_i^{V} + \frac{256}{45} h^6 f_i^{VI}$
* $f_{i+5} = f_i + 5h f_i' + \frac{25}{2} h^2 f_i'' + \frac{125}{6} h^3 f_i''' + \frac{625}{24} h^4 f_i^{IV} + \frac{625}{24} h^5 f_i^{V} + \frac{3125}{144} h^6 f_i^{VI}$
* $f_{i+6} = f_i + 6h f_i' + 18 h^2 f_i'' + 36 h^3 f_i''' + 54 h^4 f_i^{IV} + \frac{324}{5} h^5 f_i^{V} + \frac{324}{5} h^6 f_i^{VI}$

* put the previous equations in matrix form , then :-

$$\begin{bmatrix} 1 & 1/2 & 1/6 & 1/24 & 1/120 & 1/720 \\ 2 & 2 & 4/3 & 2/3 & 4/15 & 4/45 \\ 3 & 9/2 & 9/2 & 27/8 & 81/40 & 81/80 \\ 4 & 8 & 32/3 & 32/3 & 128/15 & 256/45 \\ 5 & 25/2 & 125/6 & 625/24 & 625/24 & 3125/144 \\ 6 & 18 & 36 & 54 & 324/5 & 324/5 \end{bmatrix} \cdot \begin{bmatrix} h f_i' \\ h^2 f_i'' \\ h^3 f_i''' \\ h^4 f_i^{IV} \\ h^5 f_i^{V} \\ h^6 f_i^{VI} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} f_i \\ f_{i+1} \\ f_{i+2} \\ f_{i+3} \\ f_{i+4} \\ f_{i+5} \\ f_{i+6} \end{bmatrix}$$

** The Solution of the matrix equations is :

$$\begin{bmatrix} h f_i' \\ h^2 f_i'' \\ h^3 f_i''' \\ h^4 f_i^{IV} \\ h^5 f_i^{V} \\ h^6 f_i^{VI} \end{bmatrix} = \begin{bmatrix} -147/60 & 360/60 & -450/60 & 400/60 & -225/60 & 72/60 & -10/60 \\ 812/180 & -3132/180 & 5265/180 & -5080/180 & 2970/180 & -972/180 & 137/180 \\ -49/8 & 232/8 & -461/8 & 496/8 & -307/8 & 104/8 & -15/8 \\ 35/6 & -186/6 & 411/6 & -484/6 & 321/6 & -114/6 & 17/6 \\ -7/2 & 40/2 & -95/2 & 120/2 & -85/2 & 32/2 & -5/2 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \end{bmatrix} \cdot \begin{bmatrix} f_i \\ f_{i+1} \\ f_{i+2} \\ f_{i+3} \\ f_{i+4} \\ f_{i+5} \\ f_{i+6} \end{bmatrix}$$

$$\boxed{f_i' = \frac{1}{60h}\left[-147 f_i + 360 f_{i+1} - 450 f_{i+2} + 400 f_{i+3} - 255 f_{i+4} + 72 f_{i+5} - 10 f_{i+6}\right]}$$

$$\boxed{f_i'' = \frac{1}{180 h^2}\left[812 f_i - 3132 f_{i+1} + 5265 f_{i+2} - 5080 f_{i+3} + 2970 f_{i+4} - 972 f_{i+5} + 137 f_{i+6}\right]}$$

$$\boxed{f_i''' = \frac{1}{8 h^3}\left[-49 f_i + 232 f_{i+1} - 461 f_{i+2} + 496 f_{i+3} - 307 f_{i+4} + 104 f_{i+5} - 15 f_{i+6}\right]}$$

$$\boxed{f_i^{IV} = \frac{1}{6 h^4}\left[35 f_i - 186 f_{i+1} + 411 f_{i+2} - 484 f_{i+3} + 321 f_{i+4} - 114 f_{i+5} + 17 f_{i+6}\right]}$$

$$\boxed{f_i^{V} = \frac{1}{2 h^5}\left[-7 f_i + 40 f_{i+1} - 95 f_{i+2} + 120 f_{i+3} - 85 f_{i+4} + 32 f_{i+5} - 5 f_{i+6}\right]}$$

$$\boxed{f_i^{VI} = \frac{1}{h^6}\left[f_i - 6 f_{i+1} + 15 f_{i+2} - 20 f_{i+3} + 15 f_{i+4} - 6 f_{i+5} + f_{i+6}\right]}$$

4.C Backward Difference

* $f_{i-1} = f_i - h f_i' + \frac{h^2}{2} f_i'' - \frac{h^3}{6} f_i''' + \frac{h^4}{24} f_i^{IV} - \frac{h^5}{120} f_i^{V} + \frac{h^6}{720} f_i^{VI}$
* $f_{i-2} = f_i - 2h f_i' + 2h^2 f_i'' - \frac{4}{3} h^3 f_i''' + \frac{2}{3} h^4 f_i^{IV} - \frac{4}{15} h^5 f_i^{V} + \frac{4}{45} h^6 f_i^{VI}$
* $f_{i-3} = f_i - 3h f_i' + \frac{9}{2} h^2 f_i'' - \frac{9}{2} h^3 f_i''' + \frac{27}{8} h^4 f_i^{IV} - \frac{81}{40} h^5 f_i^{V} + \frac{81}{80} h^6 f_i^{VI}$
* $f_{i-4} = f_i - 4h f_i' + 8h^2 f_i'' - \frac{32}{3} h^3 f_i''' + \frac{32}{3} h^4 f_i^{IV} - \frac{128}{15} h^5 f_i^{V} + \frac{256}{45} h^6 f_i^{VI}$
* $f_{i-5} = f_i - 5h f_i' + \frac{25}{2} h^2 f_i'' - \frac{125}{6} h^3 f_i''' + \frac{625}{24} h^4 f_i^{IV} - \frac{625}{24} h^5 f_i^{V} + \frac{3125}{144} h^6 f_i^{VI}$
* $f_{i-6} = f_i - 6h f_i' + 18h^2 f_i'' - 36h^3 f_i''' + 54h^4 f_i^{IV} - \frac{324}{5} h^5 f_i^{V} + \frac{324}{5} h^6 f_i^{VI}$

* Put the previous equations in matrix form, then :-

$$\begin{bmatrix} -1 & 1/2 & -1/6 & 1/24 & -1/120 & 1/720 \\ -2 & 2 & -4/3 & 2/3 & -4/15 & 4/45 \\ -3 & 9/2 & -9/2 & 27/8 & -81/40 & 81/80 \\ -4 & 8 & -32/3 & 32/3 & -128/15 & 256/45 \\ -5 & 25/2 & -125/6 & 625/24 & -625/24 & 3125/144 \\ -6 & 18 & -36 & 54 & -324/5 & 324/5 \end{bmatrix} \cdot \begin{bmatrix} h f_i' \\ h^2 f_i'' \\ h^3 f_i''' \\ h^4 f_i^{IV} \\ h^5 f_i^{V} \\ h^6 f_i^{VI} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} f_i \\ f_{i-1} \\ f_{i-2} \\ f_{i-3} \\ f_{i-4} \\ f_{i-5} \\ f_{i-6} \end{bmatrix}$$

** The solution of the matrix equations is :-

$$
\begin{bmatrix} h f_i' \\ h^2 f_i'' \\ h^3 f_i''' \\ h^4 f_i^{IV} \\ h^5 f_i^{V} \\ h^6 f_i^{VI} \end{bmatrix}
=
\begin{bmatrix}
147/60 & -360/60 & 450/60 & -400/60 & 225/60 & -72/60 & 10/60 \\
812/180 & -3132/180 & 5265/180 & -5080/180 & 2970/180 & -972/180 & 137/180 \\
49/8 & -232/8 & 461/8 & -496/8 & 307/8 & -104/8 & 15/8 \\
35/6 & -186/6 & 411/6 & -484/6 & 321/6 & -114/6 & 17/6 \\
7/2 & -40/2 & 95/2 & -120/2 & 85/2 & -32/2 & 5/2 \\
1 & -6 & 15 & -20 & 15 & -6 & 1
\end{bmatrix}
\cdot
\begin{bmatrix} f_i \\ f_{i-1} \\ f_{i-2} \\ f_{i-3} \\ f_{i-4} \\ f_{i-5} \\ f_{i-6} \end{bmatrix}
$$

$$f_i' = \frac{1}{60h}\left[147 f_i - 360 f_{i-1} + 450 f_{i-2} - 400 f_{i-3} + 225 f_{i-4} - 72 f_{i-5} + 10 f_{i-6}\right]$$

$$f_i'' = \frac{1}{180h^2}\left[812 f_i - 3132 f_{i-1} + 5265 f_{i-2} - 5080 f_{i-3} + 2970 f_{i-4} - 972 f_{i-5} + 137 f_{i-6}\right]$$

$$f_i''' = \frac{1}{8h^3}\left[49 f_i - 232 f_{i-1} + 461 f_{i-2} - 496 f_{i-3} + 307 f_{i-4} - 104 f_{i-5} + 15 f_{i-6}\right]$$

$$f_i^{IV} = \frac{1}{6h^4}\left[35 f_i - 186 f_{i-1} + 411 f_{i-2} - 484 f_{i-3} + 321 f_{i-4} - 114 f_{i-5} + 17 f_{i-6}\right]$$

$$f_i^{V} = \frac{1}{2h^5}\left[7 f_i - 40 f_{i-1} + 95 f_{i-2} - 120 f_{i-3} + 85 f_{i-4} - 32 f_{i-5} + 5 f_{i-6}\right]$$

$$f_i^{VI} = \frac{1}{h^6}\left[f_i - 6 f_{i-1} + 15 f_{i-2} - 20 f_{i-3} + 15 f_{i-4} - 6 f_{i-5} + f_{i-6}\right]$$