

Chapter Seven

Numerical Integration

- Trapezoidal Method/ Basic and Composite Rules
- Simpson's $1/3$ Method/ Basic and Composite Rules
- Simpson's $3/8$ Method/ Basic and Composite Rules
- Simpson's $2/45$ (Boole's) Method/ Basic and Composite Rules
- Truncation Errors for Integration Rules
- Integration Formulas Proofs
- Romberg Integration Method
- Gaussian Quadrature Integration Method
- Double Integration Method/ Simpson's $1/3$
- Triple Integration Method/ Simpson's $1/3$

Numerical Integration

التكامل العددي

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-1- Trapezoidal Rule (2 points)

✓ Basic القاعدة الأساسية
✓ Composite المركبة

-2- Simpson's $\frac{1}{3}$ Rule (3 points)

✓ Basic
✓ Composite

-3- Simpson's $\frac{3}{8}$ Rule (4 points)

✓ Basic
Composite

-4- Simpson's Rule "Booles" (5 points)

✓ Basic
Composite

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-5- Romberg integration

-6- Gaussian Quadrature

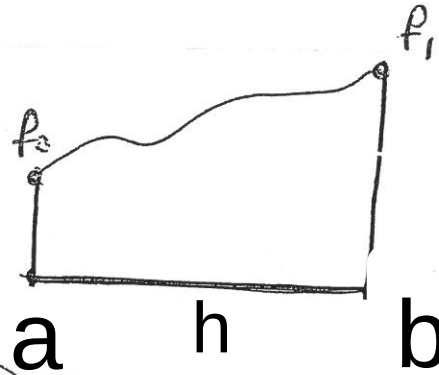
-7- Double Integration (Simpson's $\frac{1}{3}$)

Trapezoidal Rule

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* Basic rule: ($n=1$)

$$h = b - a$$



$$\int_a^b f(x) dx = \frac{h}{2} [f_0 + f_1]$$

$$E = \frac{h^3}{12} f''(\xi)$$

$$a \leq \xi \leq b$$

* Composite rule: ($n > 1$)

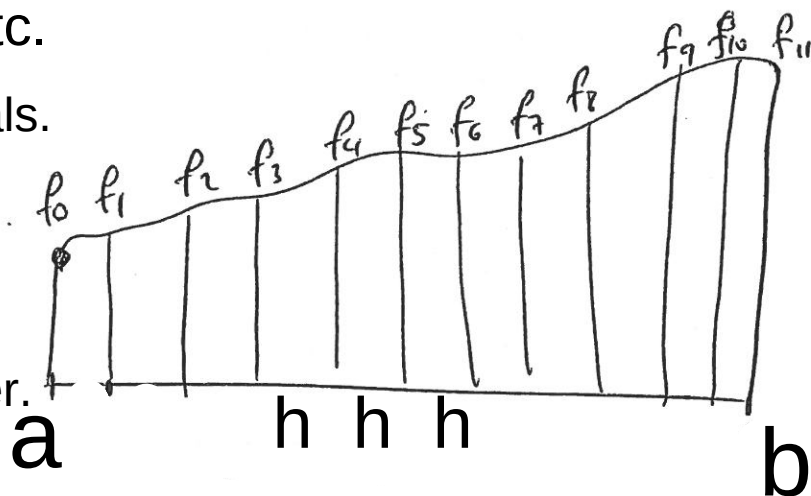
$n = 2, 3, 4, 5, 6, \dots$ etc.

n = number of intervals.

h = interval length.

$$h = \frac{b-a}{n}$$

(n) is even or odd number.



$$\int_a^b f(x) dx = \frac{h}{2} [f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 + 2f_5 + 2f_6 + 2f_7 + 2f_8 + 2f_9 + 2f_{10} + f_{11}]$$

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$$E = \frac{h^2(b-a)}{12} f''(\xi)$$

EX: Find the integral $\int_0^2 \sqrt{4+x^3} dx$

using trapezoidal Rule for $n=2$

for two intervals

for two strips

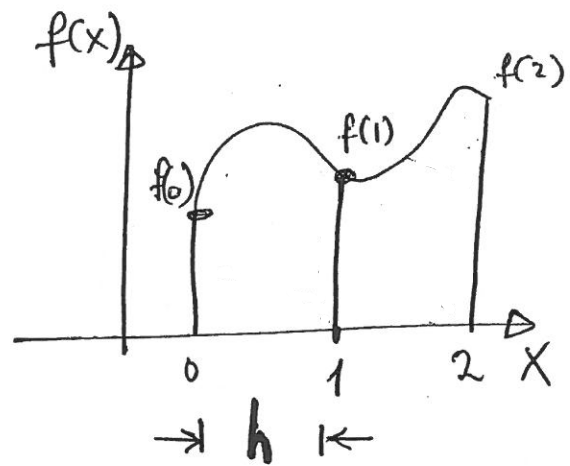
for two panels

Sol

$$h = \frac{2-0}{2} = 1$$

$$A = \frac{h}{2} [f_0 + 2f_1 + f_2]$$

$$= \frac{1}{2} [f(0) + 2f(1) + f(2)]$$



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$$* f(0) = \sqrt{4+(0)^3} = ?$$

$$* f(1) = \sqrt{4+(1)^3} = ??$$

$$* f(2) = \sqrt{4+(2)^3} = ???$$

$$\text{sub: } A = \frac{1}{2} [? + ?? + ???]$$

$$A = 4.968$$

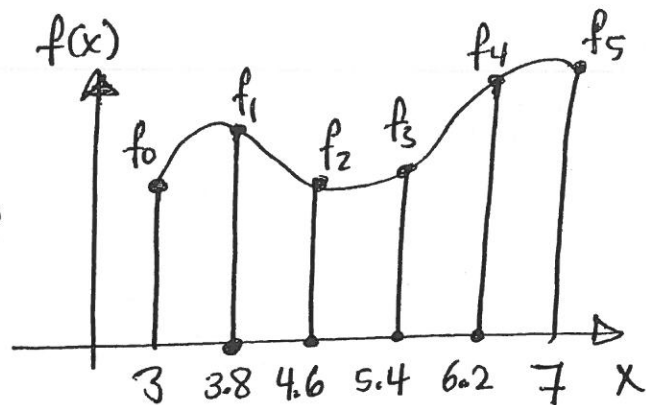
ex) estimate $\int_3^7 \sqrt{4+x^3} dx$

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use 5 strips using Trapezoidal Rule

Sol.

$$h = \frac{7-3}{5} = \frac{4}{5} = 0.8$$



$$A = \frac{0.8}{2} [f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 + f_5]$$

$$= 0.4 [f(3) + 2f(3.8) + 2f(4.6) + 2f(5.4) + 2f(6.2) + f(7)]$$

$$* f(3) = \sqrt{4+(3)^3} = *$$

$$* f(3.8) = \sqrt{4+(3.8)^3} = **$$

$$* f(4.6) \quad \text{---} \quad = ***$$

$$* f(5.4) \quad \text{---} \quad = \vdots$$

$$* f(6.2) \quad \text{---} \quad = \vdots$$

$$* f(7) \quad \text{---} \quad = \vdots$$

$$\Rightarrow A = 0.4 [* + 2(**) + 2(***) + \dots]$$

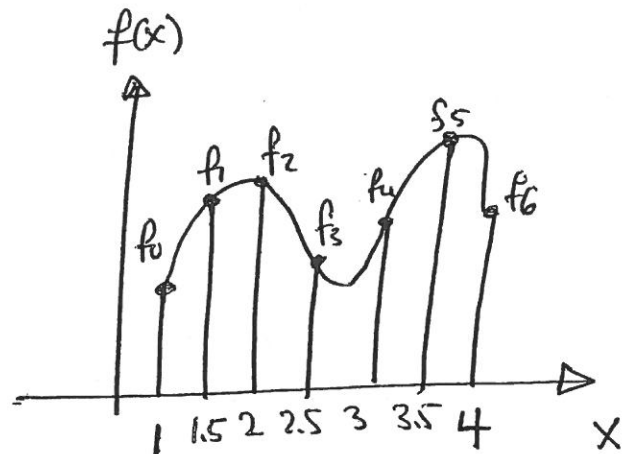
ex) use Trapezoidal for n=6

To find $\int_1^4 x^{1/3} dx$

6 intervals.

Sol)

$$* h = \frac{4-1}{6} = \frac{3}{6} = 0.5$$



$$A = \frac{0.5}{2} [f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 + 2f_5 + f_6]$$

$$* A = \frac{0.5}{2} [f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)]$$

$$\Rightarrow f(1) = (1)^{1/3} = 1$$

$$f(1.5) = (1.5)^{1/3} = 1.144$$

$$f(2) = (2)^{1/3} = 1.259$$

$$f(2.5) = (2.5)^{1/3} = 1.357$$

$$f(3) = (3)^{1/3} = 1.442$$

$$f(3.5) = (3.5)^{1/3} = 1.518$$

$$f(4) = 1.587$$

$$A = \frac{0.5}{2} [\dots] = \underline{\underline{4.006}}$$

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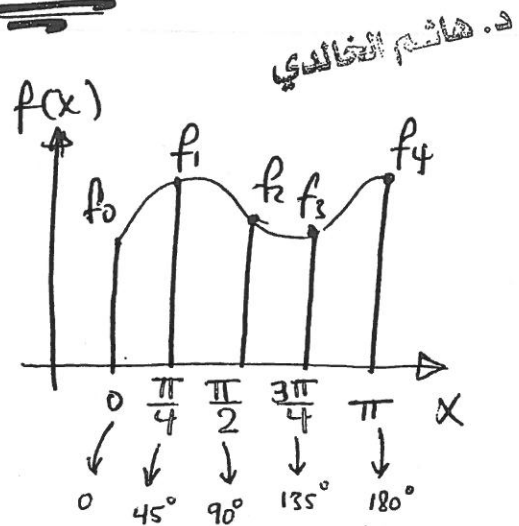
ex) use Trapezoidal Rule to find

$$\int_0^{\pi} \sin^2 x \, dx$$

use $n=4$

Sol)

$$h = \frac{\pi - 0}{4} = \frac{\pi}{4}$$



$$A = \frac{\pi}{8} \left[f(0) + 2f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 2f\left(\frac{3\pi}{4}\right) + f(\pi) \right]$$

$$A = \frac{\pi}{4} \left[f(0) + 2f(45^\circ) + 2f(90^\circ) + 2f(135^\circ) + f(180^\circ) \right]$$

in radians

$$* f(0) = \sin^2 0 = 0$$

$$* f\left(\frac{\pi}{4}\right) = \sin^2 \frac{\pi}{2} = 1$$

$$* f\left(\frac{\pi}{2}\right) = \sin^2 \pi = 0$$

*

⋮

$$A = \checkmark$$

in degree

$$f(0) = \sin^2 0 = 0$$

$$f(45^\circ) = \sin^2 90^\circ = 1$$

$$f(90^\circ) = \sin^2 180^\circ = 0$$

Numerical integration

modified trapezoidal rule

$$I \approx (b-a) \left(\frac{f(x_0) + 2 \sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n} \right)$$

$$E = \frac{(b-a)^2}{12n^2} f''$$

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ex. $f(x) = 2x^4 + x^3 - 5x^2 + 2$

integrate between 0, 5 $\therefore n=5$

① $f(x) = 2x^4 + x^3 - 5x^2 + 2$

② determine $a=0$ $b=5$ $h = \frac{b-a}{n} = \frac{5-0}{5} = 1$

x_i	0	1	2	3	4	5
$f(x)$	2	0	22	146	498	1252

③
$$I \approx (b-a) \left(\frac{f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b)}{2n} \right)$$

$$(5-0) \left[\frac{2 + 2\{0 + 22 + 146 + 498\} + 1252}{(2)(5)} \right] = 1293$$

④ $E = -\frac{(b-a)^2}{12h^2} f''$ $f''(x) = 24x^2 + 6x - 10$ $\bar{f}'' = \int_a^b \frac{f''(x) dx}{b-a}$

Error :

$$E = \frac{-(b-a)^3}{12n^2} \bar{f}'' \quad f(x) = 2x^4 + x^3 - 5x^2 + 2$$

$$f'(x) = 8x^3 + 3x^2 - 10x$$

$$f''(x) = 24x^2 + 6x - 10$$

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$$\bar{f}'' = \int_a^b \frac{f''(x)}{b-a} dx = \int_0^5 (24x^2 + 6x - 10) dx$$

$$= \frac{8x^3 + 3x^2 - 10x}{5} = \frac{8(5)^3 + 3(5)^2 - 10(5) - (0+0-0)}{5}$$

$$= \frac{1000 + 75 - 50 - (0)}{5} = \frac{1025}{5} = 205$$

$$E = \frac{-(b-a)^3}{n} \bar{f}'' = \frac{-(5-0)^3}{12(5)^2} \bar{f}''$$

$$= \frac{-(125)(205)}{300} = -85.42$$

Ex.Numerically integrate $f(x)$ from 2 to 4① state the function $f(x) = x^3 + 2x + 5$ ② state interval $a=2$ $b=4$ ③ determine $f(a)$ $f(b)$

$$f(2) = 2^3 + (2)(2) + 5 = 17$$

$$f(4) = 4^3 + (2)(4) + 5 = 77$$

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④ state Appx. Rule $I \approx \text{base} \times \text{Avg. height}$

$$= (b-a) \left(\frac{f(b) + f(a)}{2} \right)$$

$$= (4-2) \left(\frac{17+77}{2} \right)$$

$$E_t = \left| \frac{\text{true value} - \text{Apprx. value}}{\text{true value}} \right| \times 100 = \left| \frac{82 - 94}{82} \right| \times 100 = 14.6\%$$

⑤ determine error (when true value is not available)

$$E_a = -\frac{1}{12} \bar{f}''(b-a)^3$$

$$f'(x) = 3x^2 + 2 \quad f''(x) = 6x$$

$$\bar{f}'' = \frac{\int_2^4 6x \, dx}{4-2} = \frac{3x^2 \Big|_2^4}{2} = \frac{3[4^2 - 2^2]}{2} = 18$$

$$E_a = -\frac{1}{12} (18)(4-2)^3 = -\frac{18(8)}{12} = -12$$

ExNumerically integrate $f(x)$ from 0 to $\pi/4$

①

State function $f(x) = x^2 + 2\cos x$

②

State intervals $a=0$ $b=\pi/4$

③

determine $f(a)$, $f(b)$

$$f(a) = 2 \quad f(b) = (\pi/4)^2 + 2\cos \pi/4 = 2.03$$

④

State Appx. rule

$$I \approx \text{base} \times \text{Avg height}$$

$$= (b-a) \left(\frac{f(b) + f(a)}{2} \right) = \left(\frac{\pi}{4} - 0 \right) \left(\frac{2.03 + 2}{2} \right)$$

$$= 1.5757$$

⑤

determine error

$$E_a = -\frac{1}{12} \bar{f}''(b-a)^3$$

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$$f(x) = x^2 + 2\cos x$$

$$f'(x) = 2x - 2\sin x$$

$$f''(x) = 2 - 2\cos x$$

$$\bar{f}'' = \frac{\int_0^{\pi/4} (2 - 2\cos x) dx}{\pi/4 - 0} = \frac{2x - 2\sin x \Big|_0^{\pi/4}}{\pi/4} = \frac{\pi/2 - (2)(0.707)}{\pi/4}$$

$$= 0.1996$$

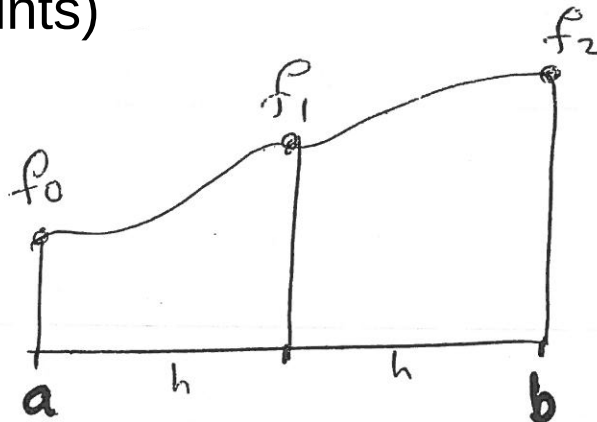
$$E_a = -\frac{1}{12} (0.1996) \left(\frac{\pi}{4} \right)^3 = -0.008058$$

Simpson's Rule 1/3

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* Basic rule: (n=2) (3 points)

$$h = \frac{b-a}{2}$$



$$\int_a^b f(x) dx = \frac{h}{3} [f_0 + 4f_1 + f_2]$$

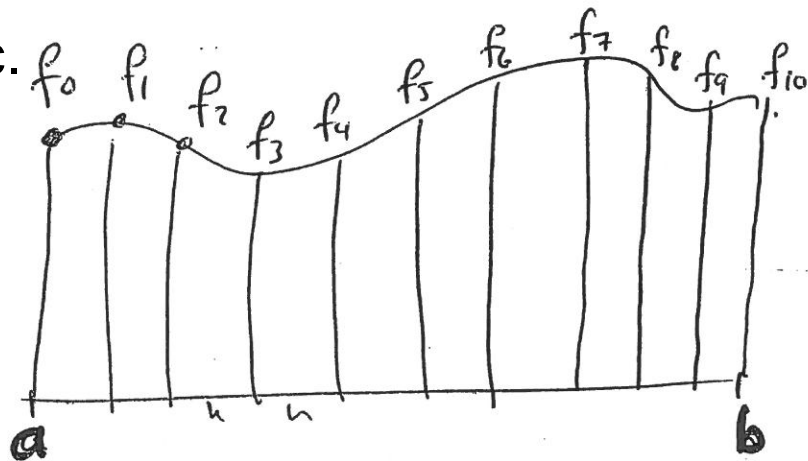
$$E = \frac{h^5}{90} f'''(\xi)$$

$$a \leq \xi \leq b$$

* Composite rule: (even n>2)

n = 4, 6, 8, 10, 12 etc.

$$h = \frac{b-a}{n}$$



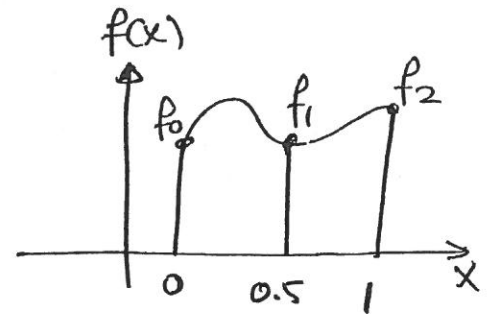
$$\int_a^b f(x) dx = \frac{h}{3} [f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + 2f_6 + 4f_7 + 2f_8 + 4f_9 + f_{10}]$$

$$E = \frac{h^4(b-a)}{180} f'''(\xi)$$

ex) use Simpson's Rule to solve.

$$\int_0^1 (1 + e^{-x} \sin 4x) dx \quad \begin{matrix} a, b \\ [0, 1] \end{matrix}$$

→ Soll * $h = \frac{1-0}{2} = 0.5$



$$\begin{aligned} A &= \frac{0.5}{3} [f_0 + 4f_1 + f_2] \\ &= \frac{0.5}{3} [f(0) + 4f(0.5) + f(1)] \end{aligned}$$

$$f(0) = 1 + e^0 \sin 0 = 1$$

$$f(0.5) = 1 + e^{-0.5} \sin(2) = ? \quad 1.552$$

$$f(1) = 1 + e^{-1} \sin(4) = ?? \quad 0.721$$

$$A = \frac{0.5}{3} [1 + 4(?) + ??]$$

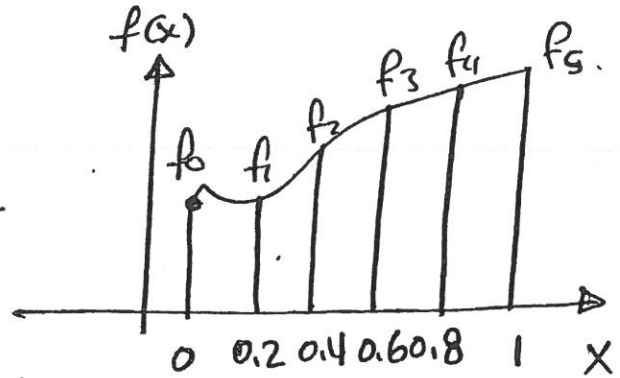
$$= 1.32128$$

ex1

same... the previous problem

use 5 strips (n=5)Sol.

$$h = \frac{1-0}{5} = \frac{1}{5} = 0.2$$



$$A = \frac{0.2}{3} [f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + f_5]$$

$$\rightarrow A = 1.2556$$

$$f(0) = 1$$

$$f(0.2) = 1.587$$

$$f(0.4) = 1.670$$

$$f(0.6) = 1.370$$

$$f(0.8) = 0.973$$

$$f(1) = 0.721$$

Ex:

Given :-

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x	0	3	6
f	?	?	8.25

$$* \int_0^6 f(x) dx = 13.2 \rightarrow \text{using Trapezoidal Rule}$$

$$* \int_0^6 f(x) dx = 11.4 \rightarrow \text{using Simpson's Rule (3 points)}$$

→ Find 1) $f(0)$?
2) $f(3)$?

Ex:

x	2.1	2.4	2.7	3	3.3	3.6
f(x)	2.745	0.241	0.914	1.546	2.12	1.843

Find 1) $\int_{2.1}^{3.6} f(x) dx \rightarrow \text{using Simpson's Rule .}$

2) $\int_{2.1}^3 f(x) dx \rightarrow \text{using Trapezoidal .}$

3) $f'(2.7) \rightarrow \text{Taylor Divided diff .}$

4) $f''(2.7)$

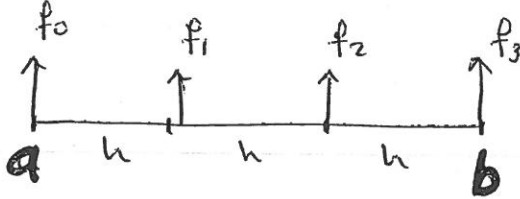
Simpson's Rule 3/8

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* Basic rule: (n=3) (4 points)

$$\int_a^b f(x) dx = \frac{3h}{8} [f_0 + 3f_1 + 3f_2 + f_3]$$

$n=3 \rightarrow h = \frac{b-a}{3}$

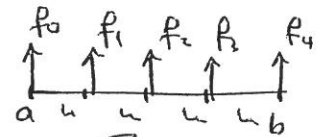


* Composite rule: (n>3)

n = 6, 9, 12, 15, 18, etc.

$$\int_a^b f(x) dx = \frac{3h}{8} [f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + 2f_6 + 3f_7 + 3f_8 + f_9]$$

Boole's Rule: (Simpson's 2/45)



* Basic: $\int_a^b f(x) dx = \frac{2h}{45} [7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4]$ (n=4) (5 points)

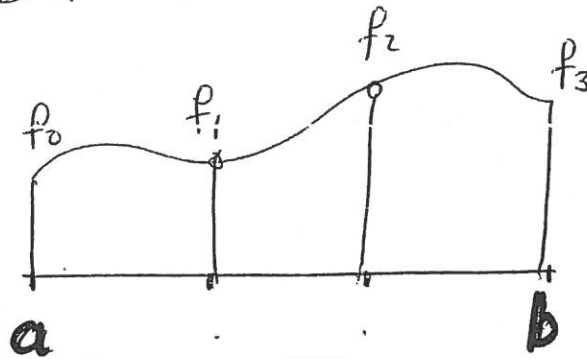
* Composite $\int_a^b f(x) dx = \frac{2h}{45} [7f_0 + 32f_1 + 12f_2 + 32f_3 + 14f_4 + 32f_5 + 12f_6 + 32f_7 + 7f_8]$

* Composite rule: (n>4)

n = 8, 12, 16, 20, etc.

Simpson's Rule $\frac{3}{8}$ [4 points]

$$h = \frac{b-a}{3}$$



$$\int_a^b f(x) dx \approx \frac{3h}{8} [f_0 + 3f_1 + 3f_2 + f_3]$$

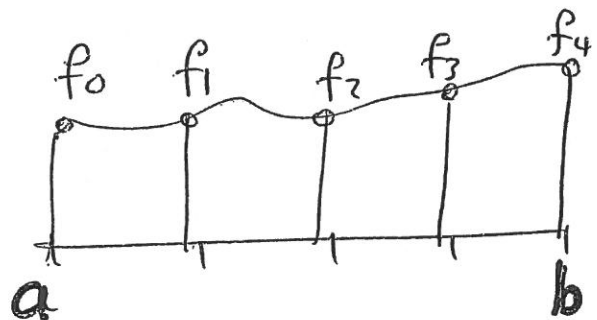
$$E = \frac{3h^5}{80} f'''(\xi)$$

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$$a \leq \xi \leq b$$

Simpson's Rule [5 points]

$$h = \frac{b-a}{4}$$



$$\int_a^b f(x) dx \approx \frac{2h}{45} [7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4]$$

$$E = \frac{8h^7}{945} f^{(7)}(\xi)$$

$$a \leq \xi \leq b$$

* Simpson's $\frac{3}{8}$ (4 points, $n=3$ strips) Composite Rule

$$A = \frac{3h}{8} \left[f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + 2f_6 + 3f_7 + 3f_8 + 2f_9 + 3f_{10} + 3f_{11} + 2f_{12} + \dots + f_n \right]$$

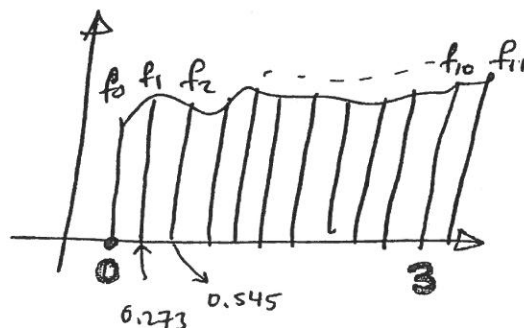
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ex) Use Simpson's $\frac{3}{8}$ to solve $\rightarrow \int_0^3 (x^2 + e^x) dx$
use $n=11$

Sol.)

$$h = \frac{3-0}{11} = 0.273$$

$$A = \frac{3(0.273)}{8} \left[f_0 + 3f_1 + 3f_2 + 2f_3 + 3f_4 + 3f_5 + 2f_6 + 3f_7 + 3f_8 + 2f_9 + 3f_{10} + f_{11} \right]$$



* **Boole's Rule** (Simpson's Rule $n=4$) (5 points $n=4$ Strips) **Composite Rule**

نقطة 5، 4 شريط، 4 نقطة

$$A = \frac{2h}{45} \left[\begin{aligned} &7f_0 + 32f_1 + 12f_2 + 32f_3 + 14f_4 \\ &+ 32f_5 + 12f_6 + 32f_7 + 14f_8 \\ &+ 32f_9 + 12f_{10} + 32f_{11} + 14f_{12} \\ &+ \dots \\ &+ \dots \dots \dots + 7f_n \end{aligned} \right]$$

$$h = \frac{b-a}{n}$$

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Ex)

Solve

$$\int_2^8 \sqrt{x^2+3} \, dx$$

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① Trap. ($n=1$)

② Simpson's (3 points) ($n=2$)

③ Simpson's (5 points) ($n=4$)

④ Trap. ($n=4$)

⑤ Simpson's (3 points) ($n=10$)

⑥ Simpson's (3 points) ($n=4$)

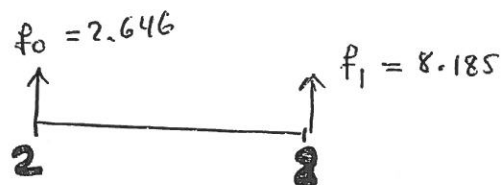
⑦ Simpson's (4 points) ($n=3$)

$$f(x) = \sqrt{x^2+3}$$

SOL)

①

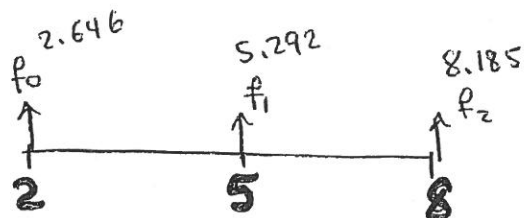
$$h = 8 - 2 = 6$$



$$A = \frac{6}{2} [2.646 + 8.185] = 32.494$$

②

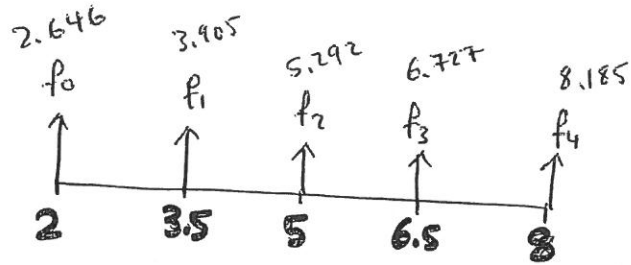
$$h = \frac{8-2}{2} = \frac{6}{2} = 3$$



$$A = \frac{3}{3} [2.646 + 4(5.292) + 8.185] = 31.999$$

③

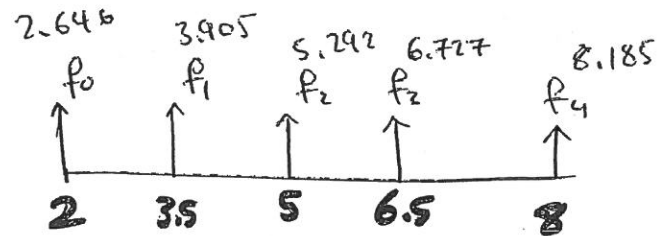
$$h = \frac{8-2}{4} = \frac{6}{4} = 1.5$$



$$A = \frac{2(1.5)}{45} \left[7(2.646) + 32(3.905) + 12(5.292) + 32(6.727) + 7(8.185) \right] = \dots$$

④

$$h = \frac{8-2}{4} = \frac{6}{4} = 1.5$$

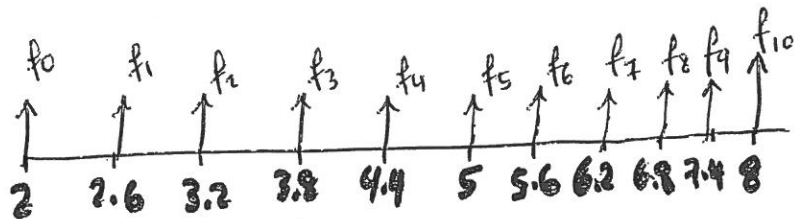


$$A = \frac{1.5}{2} \left[2.646 + 2(3.905) + 2(5.292) + 2(6.727) + 8.185 \right]$$

$$= 32.0096$$

⑤

$$h = \frac{8-2}{10} = \frac{6}{10} = 0.6$$



$$A = \frac{0.6}{3} \left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + 4f_5 + 2f_6 + 4f_7 + 2f_8 + 4f_9 + f_{10} \right]$$

$$\Rightarrow A = \dots$$

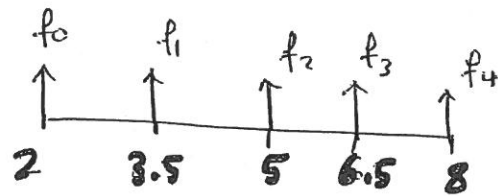
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⑥

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$$h = \frac{8-2}{4} = \frac{6}{4} = 1.5$$

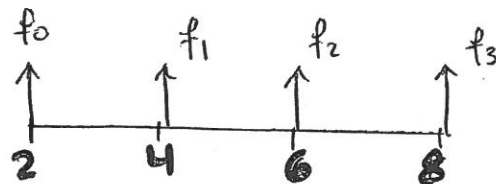
"Table"



$$A = \frac{1.5}{3} [f_0 + 4f_1 + 2f_2 + 4f_3 + f_4] = \text{-----} \checkmark$$

⑦

$$* h = \frac{8-2}{3} = \frac{6}{3} = 2$$



$$A = \frac{3(2)}{8} [f_0 + 3f_1 + 3f_2 + f_3] = \text{-----}$$

$$* \text{ Rel. Error} = \frac{\text{Exact} - \overset{A}{\text{Approx.}}}{\text{Exact}} * 100\%$$

$$* \text{ Exact} \begin{cases} \text{From Calculus} \\ \text{Calculator} \rightarrow \int_2^8 \sqrt{x^2+3} dx = \underline{\underline{31.9678}} \end{cases}$$

ex] Using Trapezoidal Rule $\rightarrow$

Find the approximate value for the integral

$$\int_0^1 x e^x dx$$

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مستشار أكاديمي

$\Rightarrow$ So that the error will not exceed 0.01.

$\Leftarrow$ أريد لتقريب القيمة التقريبية خطأ لا يتجاوز

خطأ 1% عند 0.01

الـ Sol.

$$f(x) = x e^x$$

$$f'(x) = x e^x + e^x (1)$$

$$\rightarrow f'(x) = x e^x + e^x$$

$$\rightarrow f'' = x e^x + e^x + e^x = x e^x + 2 e^x$$

* Error: $E = \frac{h^2(b-a)^3}{12} f'''(\xi)$

$$E = \frac{(b-a)h^2}{12} [xe^x + 2e^x]$$

$$0 < x \leq 1$$

$0 + 2(1) = 2$

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(max) الوارد يفتي، (نقطة) (الآلة)

$$\Rightarrow (1)e' + 2e' = 3e'$$

* $0.01 = \frac{h^2}{12} [(1)e^{(1)} + 2e^{(1)}]$

$$\Rightarrow 0.01 = \frac{h^2}{12} (3e) \Rightarrow h^2 = \frac{(12)(0.01)}{3e}$$

$$\Rightarrow h = 0.1213$$

* $h = \frac{b-a}{n} \Rightarrow 0.1213 = \frac{1-0}{n}$

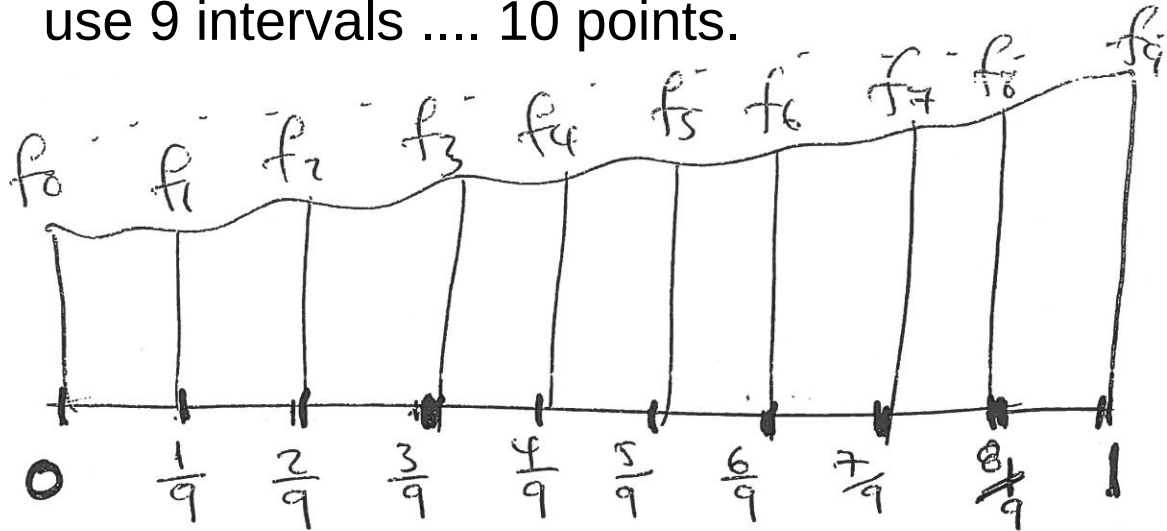
$$\Rightarrow n = 8.2438$$

حيث يتم التقريب دائماً للأعلى $\rightarrow$ عدد فرداً فردياً أو زوجياً

(Trapezoidal) $\Rightarrow$ use $n = 9$

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use 9 intervals 10 points.



$$h = \frac{b-a}{n} \rightarrow h = \frac{1-0}{9} \rightarrow \underline{\underline{h = \frac{1}{9}}}$$

$$\int_0^1 f(x) dx = \frac{h}{2} \left[f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 + 2f_5 + 2f_6 + 2f_7 + 2f_8 + f_9 \right]$$

$$= \frac{1/9}{2} \left[+ \dots \dots \dots \right]$$

$$= 1.0046$$



$$\int_0^1 x e^x dx$$

Using
Simpson's Rule $\frac{1}{3}$

So that

$$\text{Error } E \leq 0.001$$

SOL

$$f = x e^x$$

$$f' = x e^x + e^x$$

$$f'' = x e^x + 2e^x$$

$$f''' = x e^x + e^x + 2e^x = x e^x + 3e^x$$

$$f^{(4)} = x e^x + e^x + 3e^x = x e^x + 4e^x$$

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$$\Rightarrow E = \frac{(b-a)h^4}{180} f^{(4)}(\xi)$$

$$E = \frac{(1-0)h^4}{180} (x e^x + 4e^x)$$

$$0 \leq x \leq 1$$

gives max error

$$0.001 = (1-0) \frac{h^4}{180} (1e^{(1)} + 4e^{(0)})$$

$$0.001 = \frac{h^4}{180} (5e)$$

$$h^4 = \frac{180(0.001)}{5e}$$

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$$h = \sqrt[4]{\frac{180(0.001)}{5e}}$$

$$h = 0.3392$$

$$h = \frac{b-a}{n} \Rightarrow 0.3392 = \frac{1-0}{n}$$

$$\Rightarrow n = 2.9477$$

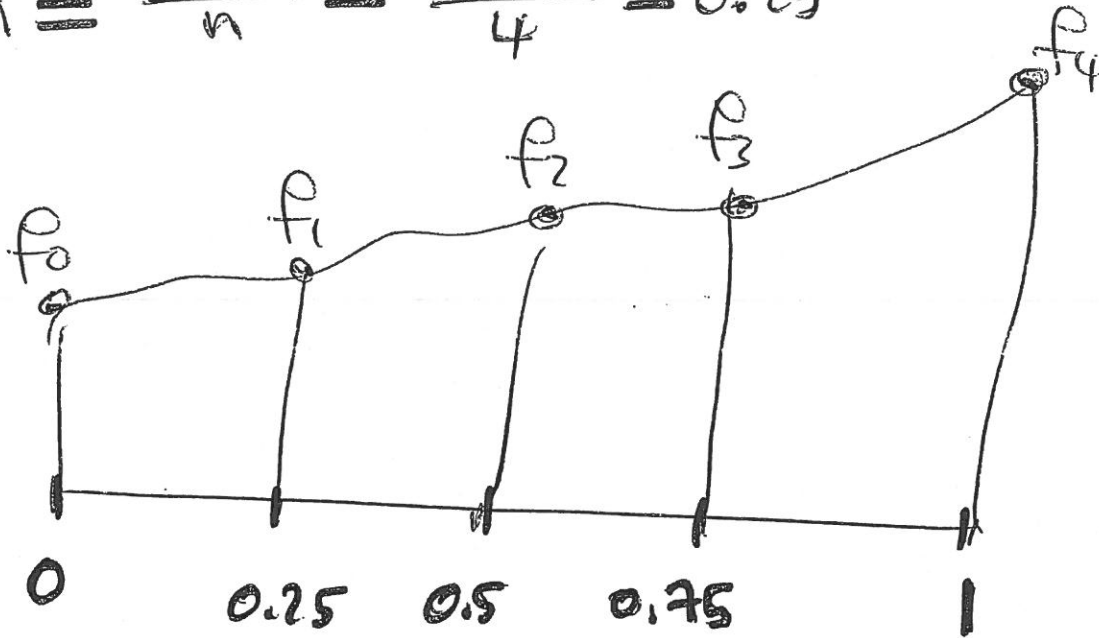
← يجب أن يتم التقريب للأعلى واختيار العدد الزوجي

* Since (Simpson's $\frac{1}{3}$) $\Rightarrow$

$$\Rightarrow \text{use } n=4$$

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$$* h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$



$$\int_0^1 f(x) dx = \frac{h}{3} \left[f_0 + 4f_1 + 2f_2 + 4f_3 + f_4 \right]$$

$$= 1.0002$$

ex

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x	1	2	3	4	5
f	2	7	22	53	106

→ use "Simpson's Rule" to highest accuracy!!

$$\Rightarrow \int_1^5 f(x) dx \quad ?$$

Sol.

$$h=1$$

f_0	f_1	f_2	f_3	f_4
2	7	22	53	106
↑	↑	↑	↑	↑
1	2	3	4	5

$$\int_1^5 = \frac{1}{3} [2 + 4(7) + 2(22) + 4(53) + 106]$$

$$= 130.6667$$

ex)

Find:

$$\int_{\frac{\pi}{2}}^{2\pi} \frac{\sin 3x}{x^2} dx$$

معرفة
لحساب
القيمة
الراديان

Using ① Trapezoidal Rule

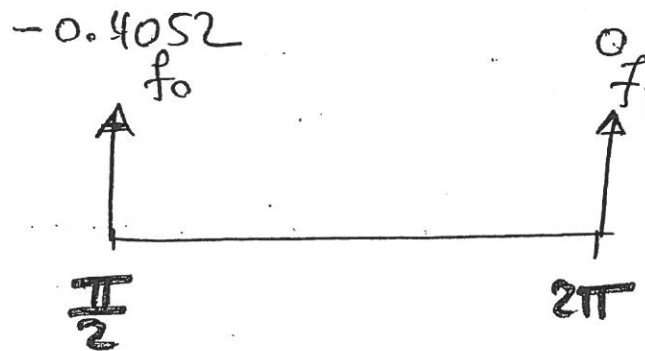
② Simpson's Rule.

sol)

①

$$h = 2\pi - \frac{\pi}{2}$$

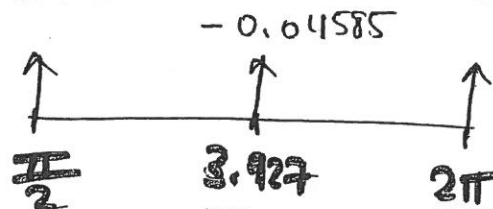
$$h = 4.7124$$



$$\int = \frac{4.7124}{2} \left[\underset{\substack{f_0 \\ -0.4052}}{-0.4052} + \underset{\substack{f_1 \\ 0}}{0} \right] = \boxed{0.9547}$$

②

$$h = 2.3562$$



$$\int = \frac{2.3562}{3} \left[-0.4052 + 4(-0.04585) + 0 \right] = \boxed{-0.4623}$$

ex

X	0	2	4
f(x)	0	8	2

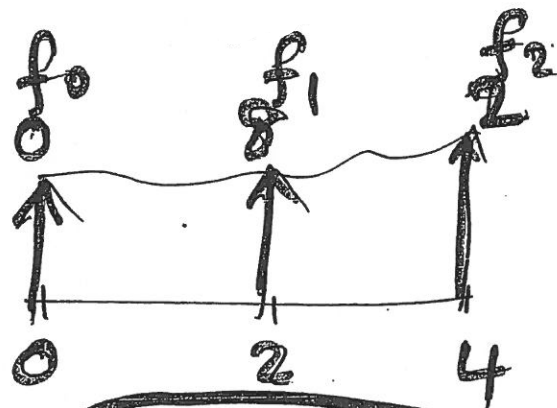
Find

$$\int_0^4 5 f(x)$$

?

لَا نَدْرُكُ!ال

$$h=2$$



$$\int_0^4 5 f(x) = 5 \left[\frac{2}{3} (0 + 4(8) + 2) \right]$$

$$= 5 \left[\frac{2(34)}{3} \right]$$

$$= 113.3333$$

Derivation of Integration Rule: using Lagrange

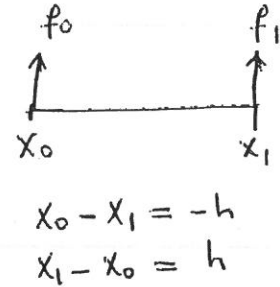
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** Derivation of Trapezoidal. (2 points)

* Proofs

* Use Lagrange:

x	x_0	x_1
$f(x)$	f_0	f_1



$$* f(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1$$

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$$* \int_{x_0}^{x_1} f(x) dx = \frac{f_0}{x_0 - x_1} \int_{x_0}^{x_1} (x - x_1) dx + \frac{f_1}{x_1 - x_0} \int_{x_0}^{x_1} (x - x_0) dx$$

$$= \frac{f_0}{x_0 - x_1} \left[\frac{(x - x_1)^2}{2} \right]_{x_0}^{x_1} + \frac{f_1}{x_1 - x_0} \left[\frac{(x - x_0)^2}{2} \right]_{x_0}^{x_1}$$

$$= \frac{f_0}{2(x_0 - x_1)} [0 - (x_0 - x_1)^2] + \frac{f_1}{2(x_1 - x_0)} [(x_1 - x_0)^2 - 0]$$

$$= \frac{f_0}{2h} [-h^2] + \frac{f_1}{2h} [h^2]$$

$$= -\frac{f_0 h}{2} + \frac{f_1 h}{2}$$

$$\boxed{\int_{x_0}^{x_1} f(x) dx = \frac{h}{2} [f_0 + f_1]}$$

Trapezoidal

Rule. (Basic)

Trapezoidal / Composite

$$\int_{x_0}^{x_1} f(x) dx = \frac{h}{2} \left[(f_0 + f_1) + (f_1 + f_2) + (f_2 + f_3) + (f_3 + f_4) \right. \\ \left. + (f_4 + f_5) + (f_5 + f_6) + (f_6 + f_7) \right. \\ \left. + (f_7 + f_8) + (f_8 + f_9) \right]$$

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$$= \frac{h}{2} \left[f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 \right. \\ \left. + 2f_5 + 2f_6 + 2f_7 + 2f_8 + f_9 \right]$$

⇒ Proof :-

* Derivation Simpson's Rule $\frac{1}{3}$ (3 points)

$$x_0 - x_2 = -2h \\ x_2 - x_0 = 2h$$

Lagrange :-

x	x ₀	x ₁	x ₂
f(x)	f ₀	f ₁	f ₂

$$\begin{array}{c} x_0 \quad x_1 \quad x_2 \\ | \quad | \quad | \end{array}$$

$$x_0 - x_1 = -h \\ x_1 - x_2 = -h \\ x_2 - x_1 = h \\ x_1 - x_0 = h$$

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

$$\int_{x_0}^{x_2} f(x) dx = \frac{f_0}{(-h)(-2h)} \int_{x_0}^{x_2} (x-x_1)(x-x_2) dx + \frac{f_1}{(h)(-h)} \int_{x_0}^{x_2} (x-x_0)(x-x_2) dx + \frac{f_2}{(2h)(h)} \int_{x_0}^{x_2} (x-x_0)(x-x_1) dx$$

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$$A = \frac{f_0}{2h^2} \int_{x_0}^{x_2} (x-x_1)(x-x_2) - \frac{f_1}{h^2} \int_{x_0}^{x_2} (x-x_0)(x-x_2) + \frac{f_2}{2h^2} \int_{x_0}^{x_2} (x-x_0)(x-x_1)$$

$$\begin{aligned} * \int_{x_0}^{x_2} (x-x_1)(x-x_2) dx &= \int_{x_0}^{x_2} (x^2 - x_2x - x_1x + x_1x_2) dx \\ &= \left(\frac{x^3}{3} - \frac{x_2x^2}{2} - \frac{x_1x^2}{2} + x_1x_2x \right)_{x_0}^{x_2} \end{aligned}$$

$$\begin{aligned} &= \frac{x_2^3}{3} - \frac{x_0^3}{3} - \frac{x_2^3}{2} + \frac{x_2x_0^2}{2} \\ &\quad - \frac{x_1x_2^2}{2} + \frac{x_1x_0^2}{2} + x_1x_2^2 - x_1x_2x_0 \end{aligned}$$

.....

$$\Rightarrow A = \frac{h}{3} [f_0 + 4f_1 + f_2]$$

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** Proofs

Trapezoidal Rule [2 points]

$$\overline{x_0 \quad x_1}$$

$$x_1 - x_0 = h$$

$$x_0 - x_1 = -h$$

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نقطة مكانية

x	x_0	x_1
$f(x)$	f_0	f_1

* Lagrange function:-

$$f(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1$$

$$R = \frac{f''(\xi)}{2} (x - x_0)(x - x_1)$$

$$\begin{aligned}
 * \int_{x_0}^{x_1} f(x) dx &= \frac{f_0}{x_0 - x_1} \int_{x_0}^{x_1} (x - x_1) dx + \frac{f_1}{x_1 - x_0} \int_{x_0}^{x_1} (x - x_0) dx \\
 &= \left(\frac{f_0}{x_0 - x_1} \right) \left(\frac{(x - x_1)^2}{2} \right) \Big|_{x_0}^{x_1} + \left(\frac{f_1}{x_1 - x_0} \right) \left(\frac{(x - x_0)^2}{2} \right) \Big|_{x_0}^{x_1} \\
 &= \left(\frac{f_0}{x_0 - x_1} \right) \left(0 - \frac{(x_0 - x_1)^2}{2} \right) + \left(\frac{f_1}{x_1 - x_0} \right) \left[\frac{(x_1 - x_0)^2}{2} - 0 \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{f_0}{-h} \left(-\frac{(-h)^2}{2} \right) + \frac{f_1}{h} \left[\frac{h^2}{2} \right] \\
 &= \frac{f_0 h}{2} + \frac{f_1 h}{2} \\
 &= \left(\frac{h}{2} [f_0 + f_1] \right)
 \end{aligned}$$

* Error Proof : $E = \int_{x_0}^{x_1} R \, dx$

$$E = \int_{x_0}^{x_1} \frac{f''(\xi)}{2} (x-x_0)(x-x_1) \, dx$$

$$E = \frac{f''(\xi)}{2} \int_{x_0}^{x_1} (x^2 - x_0 x - x_1 x + x_0 x_1) \, dx$$

$$E = \frac{f''(\xi)}{2} \left[\frac{x^3}{3} + x_0 \frac{x^2}{2} - x_1 \frac{x^2}{2} + x_0 x_1 x \right]_{x_0}^{x_1}$$

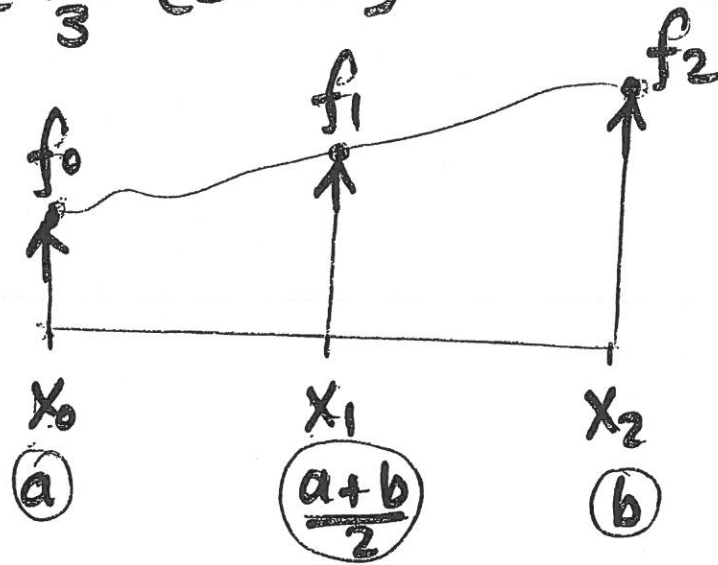
$$E = \frac{f''(\xi)}{2} \left[\left(\frac{X_1^3}{3} + \frac{X_0 X_1^2}{2} - \frac{X_1^3}{2} + X_0 X_1^2 \right) - \left(\frac{X_0^3}{3} + \frac{X_0^3}{2} - \frac{X_1 X_0^2}{2} + X_0^2 X_1 \right) \right]$$

$$= \frac{f''(\xi)}{2} \left[\frac{X_1^3 - X_0^3}{3} + \frac{X_0 X_1^2 - X_1 X_0^2}{2} - \frac{X_1^3 - X_0^3}{2} + X_0 X_1^2 + X_1 X_0^2 \right]$$

$$E = \frac{h^3}{12} f''(\xi)$$

Simpson's Rule $\frac{1}{3}$ (Basic)

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Using $\Rightarrow$

Taylor Series about x_1 ?

$$\int_{x_0}^{x_2} f(x) dx = \int_{x_0}^{x_2} f(x_1) + \int_{x_0}^{x_2} (x-x_1) f'(x_1) + \int_{x_0}^{x_2} \frac{1}{2} (x-x_1)^2 f''(x_1) + \int_{x_0}^{x_2} \frac{1}{6} f'''(x_1) (x-x_1)^3$$

$$* \int_{x_0}^{x_2} f(x_1) dx = f(x_1) (x_2 - x_0) = f_1 (2h) = 2h f_1$$

$$* \int_{x_0}^{x_2} f'(x_1) (x-x_1) dx = f'(x_1) \frac{(x-x_1)^2}{2} \Big|_{x_0}^{x_2} = \frac{f'(x_1)}{2} [(x_2-x_1)^2 - (x_0-x_1)^2] = 0$$

$$\begin{aligned}
 * \int_{x_0}^{x_2} \frac{1}{2} f''(x_1) (x-x_1)^2 dx \\
 &= \frac{1}{2} f''(x_1) \left[\frac{(x-x_1)^3}{3} \right]_{x_0}^{x_2} \\
 &= \frac{1}{6} f''(x_1) [h^3 - (-h)^3] \\
 &= \frac{1}{6} f''(x_1) [h^3 - (-h^3)] \\
 &= \frac{1}{6} f''(x_1) (2h^3) \\
 &= \frac{h^3}{3} f''(x_1)
 \end{aligned}$$

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Also

$$* \int_{x_0}^{x_2} \frac{1}{6} f'''(x_1) (x-x_1)^3 dx = 0$$

sum all terms together:

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$$\int f(x) dx = 2hf_1 + 0 + \frac{h^3}{3} f''(x_1) + 0$$

$$= 2hf_1 + \frac{h^3}{3} f''(x_1)$$



from before....

$$\Rightarrow f'' = \frac{1}{h^2} [f_0 - 2f_1 + f_2]$$

$$\Rightarrow \int f(x) dx = 2hf_1 + \frac{h^3}{3} \frac{1}{h^2} [f_0 - 2f_1 + f_2]$$

$$= 2hf_1 + \frac{h}{3} f_0 - \frac{2h}{3} f_1 + \frac{h}{3} f_2$$

$$= h \left[\frac{1}{3} f_0 + \frac{4}{3} f_1 + \frac{1}{3} f_2 \right]$$

$$= \frac{h}{3} [f_0 + 4f_1 + f_2]$$

Romberg Integration

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→ Two steps :

① Finding R_1 using "Composite Trapezoidal" ($h = \frac{b-a}{n}$)

$$\rightarrow R_1 = \int_a^b f(x) dx = \frac{h}{2} [f_0 + 2f_1 + 2f_2 + 2f_3 + \dots + f_n]$$

② Extrapolation Rule

$$\rightarrow R_n(h) = \frac{1}{\frac{2^{(n-1)}}{2} - 1} \left[\frac{2^{(n-1)}}{2} R_{n-1}\left(\frac{h}{2}\right) - R_{n-1}(h) \right]$$

*** Construct the "Romberg Table" ($n \times n$)

($n=2$)

$R_1(h)$	
$R_1(\frac{h}{2})$	$R_2(h)$

($n=3$)

$R_1(h)$		
$R_1(\frac{h}{2})$	$R_2(h)$	
$R_1(\frac{h}{4})$	$R_2(\frac{h}{2})$	$R_3(h)$

($n=4$)

$R_1(h)$			
$R_1(\frac{h}{2})$	$R_2(h)$		
$R_1(\frac{h}{4})$	$R_2(\frac{h}{2})$	$R_3(h)$	
$R_1(\frac{h}{8})$	$R_2(\frac{h}{4})$	$R_3(\frac{h}{2})$	$R_4(h)$

⇒ Explanation of The "Extrapolation rule"

$$\underline{n=2} : R_2(h) = \frac{1}{3} \left[4 R_1\left(\frac{h}{2}\right) - R_1(h) \right]$$

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$$\underline{n=2} : R_2\left(\frac{h}{2}\right) = \frac{1}{3} \left[4 R_1\left(\frac{h}{4}\right) - R_1\left(\frac{h}{2}\right) \right]$$

$$\underline{n=2} : R_2\left(\frac{h}{100}\right) = \frac{1}{3} \left[4 R_1\left(\frac{h}{200}\right) - R_1\left(\frac{h}{100}\right) \right]$$

$$\underline{n=3} : R_3(h) = \frac{1}{15} \left[16 R_2\left(\frac{h}{2}\right) - R_2(h) \right]$$

$$\underline{n=3} : R_3\left(\frac{h}{2}\right) = \frac{1}{15} \left[16 R_2\left(\frac{h}{4}\right) - R_2\left(\frac{h}{2}\right) \right]$$

$$\underline{n=4} : R_4(h) = \frac{1}{63} \left[64 R_3\left(\frac{h}{2}\right) - R_3(h) \right]$$

$$\underline{n=4} : R_4\left(\frac{h}{4}\right) = \frac{1}{63} \left[64 R_3\left(\frac{h}{8}\right) - R_3\left(\frac{h}{4}\right) \right]$$

ex) use Romberg Integ. To find $\int_0^1 \frac{1}{1+x} dx$

$$\int_0^1 \frac{1}{1+x} dx, \text{ use } h=0.3$$

Sol

$$h = \frac{b-a}{n} = \frac{1-0}{3} = 0.3$$

→ Romberg Table (3x3)

$$R_1(h)$$

$$R_1(\frac{1}{3})$$

$$R_1(\frac{1}{6})$$

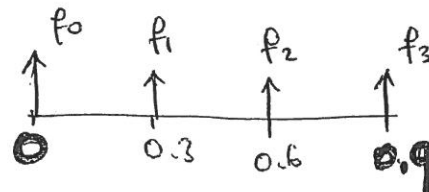
$$R_2(h)$$

$$R_2(\frac{1}{6})$$

$$R_3(h)$$

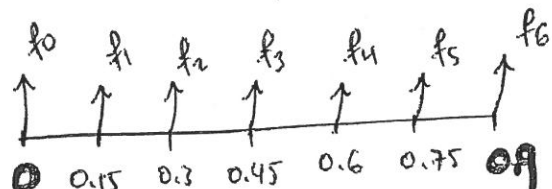
$$\frac{1}{1+x}$$

* $h=0.3 \rightarrow$



$$R_1(h) = \frac{0.3}{2} [f_0 + 2f_1 + 2f_2 + f_3] = \boxed{0.647217}$$

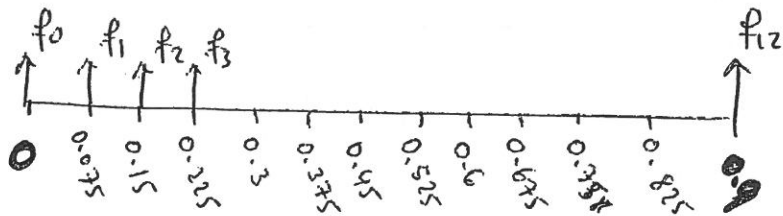
* $h=0.15$



$$R_1(\frac{h}{2}) = \frac{0.15}{2} [f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 + 2f_5 + f_6]$$

$$= \boxed{0.643206}$$

$$* h = \frac{0.3}{4} = 0.075$$



$$R_1\left(\frac{h}{4}\right) = \frac{0.075}{2} [f_0 + 2f_1 + 2f_2 + 2f_3 + \dots + f_{12}]$$

$$= \boxed{0.6428925}$$

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$$* R_2(h) = \frac{1}{3} [4R_1\left(\frac{h}{2}\right) - R_1(h)]$$

$$= \frac{1}{3} [4(0.643206) - 0.647217] = \boxed{0.641869}$$

$$* R_2\left(\frac{h}{2}\right) = \frac{1}{3} [4R_1\left(\frac{h}{4}\right) - R_1\left(\frac{h}{2}\right)]$$

$$= \frac{1}{3} [4(0.621925) - 0.643206] = \boxed{0.641855}$$

$$* R_3(h) = \frac{1}{15} [16R_2\left(\frac{h}{2}\right) - R_2(h)]$$

$$= \frac{1}{15} [16(0.641855) - 0.641869] = \boxed{0.64184066}$$

* الجواب النهائي

$$\int_0^{0.9} \frac{1}{1+x} dx = 0.64184066$$

$$\text{Error} = \frac{\text{Exact} - \text{Approx.}}{\text{Exact}}$$

$$\text{Exact} = 0.6418538862$$

Gaussian Quadrature

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Find $\int_a^b f(x) dx$ $[a, b]$

المعادلة

Let $x = \left(\frac{b+a}{2}\right) + \left(\frac{b-a}{2}\right)t$

التقريبية

$$\int_a^b f(x) dx = \frac{b-a}{2} \left[w_1 f(t_1) + w_2 f(t_2) + w_3 f(t_3) + w_4 f(t_4) + \dots \right]$$

w = weight factor معامل الوزن

حيث t قبل على x , جدول التالي :-

Number of terms

Value of t

w

2 terms

$$\begin{aligned} -\frac{\sqrt{3}}{3} &= t_1 = -0.57735 \\ \frac{\sqrt{3}}{3} &= t_2 = +0.57735 \end{aligned}$$

$$w_1 = 1$$

$$w_2 = 1$$

3 terms

$$\begin{aligned} -\sqrt{0.6} &= t_1 = -0.7746 \\ 0 &= t_2 = 0 \\ \sqrt{0.6} &= t_3 = +0.7746 \end{aligned}$$

$$w_1 = 0.55556 = \frac{5}{9}$$

$$w_2 = 0.88889 = \frac{8}{9}$$

$$w_3 = 0.55556 = \frac{5}{9}$$

4 terms

$$\begin{aligned} t_1 &= \\ t_2 &= \\ t_3 &= \\ t_4 &= \end{aligned}$$

$$w_1 =$$

$$w_2 =$$

$$w_3 =$$

$$w_4 =$$

Gauss-Quadrature :-

$$\int_a^b f(x) dx \quad ??$$

الافتراض
Let

$$x = \frac{b+a}{2} + \left(\frac{b-a}{2}\right)t$$

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 f\left[\frac{b+a}{2} + \left(\frac{b-a}{2}\right)t\right] dt$$

$$\int_a^b f(x) dx = \frac{b-a}{2} \left[w_1 f(t_1) + w_2 f(t_2) + w_3 f(t_3) + \dots \right]$$

From table....get values of (t) and (w)

2 terms Gauss-Quadrature $w_1 = 1$, $t_1 = -0.577350269$
 $w_2 = 1$, $t_2 = 0.577350269$

3 terms $f_1 \rightarrow \begin{matrix} w_1 \\ w_2 \\ w_3 \end{matrix} \quad \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix}$

4 terms $f_1 \rightarrow \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix} \quad \begin{matrix} t_1 \\ t_2 \\ t_3 \\ t_{..} \end{matrix}$

It turns out that the t_i 's for a given n are the roots of the n th-degree Legendre polynomial. The Legendre polynomials are defined by recursion:

$$(n+1)L_{n+1}(x) - (2n+1)xL_n(x) + nL_{n-1}(x) = 0,$$

with $L_0(x) = 1$, $L_1(x) = x$.

Then $L_2(x)$ is

$$L_2(x) = \frac{3xL_1(x) - (1)L_0(x)}{2} = \frac{3}{2}x^2 - \frac{1}{2},$$

whose zeros are $\pm\sqrt{\frac{1}{3}} = \pm 0.5773$, precisely the t -values for the two-term formula.

By using the recursion relation, we find

$$L_3(x) = \frac{5x^3 - 3x}{2},$$

$$L_4(x) = \frac{35x^4 - 30x^2 + 3}{8}, \quad \text{and so on.}$$

The methods of Chapter 1 allow us to find the roots of these polynomials. After they have been determined, the set of equations analogous to Eqs. (5.41) can easily be solved for the weighting factors because the equations are linear with respect to these unknowns.

Table 5.14 lists the zeros of Legendre polynomials up to degree 5, giving values that we need for Gaussian quadrature where the equivalent polynomial is up to degree 9. For example, $L_3(x)$ has zeros at $x = 0$, $+0.77459667$, and -0.77459667 .

Table 5.14 Values for Gaussian quadrature

Number of terms	Values of t	Weighting factor	Valid up to degree
2	$t_1 = -0.57735027$ $t_2 = 0.57735027$	$w_1 = 1.0$ $w_2 = 1.0$	3
3	$t_1 = -0.77459667$ $t_2 = 0.0$ $t_3 = 0.77459667$	$w_1 = 0.55555555$ $w_2 = 0.88888889$ $w_3 = 0.55555555$	5
4	$t_1 = -0.86113631$ $t_2 = -0.33998104$ $t_3 = 0.33998104$ $t_4 = 0.86113631$	$w_1 = 0.34785485$ $w_2 = 0.65214515$ $w_3 = 0.65214515$ $w_4 = 0.34785485$	7
5	$t_1 = -0.90617975$ $t_2 = -0.53846931$ $t_3 = 0.0$ $t_4 = 0.53846931$ $t_5 = 0.90617975$	$w_1 = 0.23692689$ $w_2 = 0.47862867$ $w_3 = 0.56888889$ $w_4 = 0.47862867$ $w_5 = 0.23692689$	9

ex) Use **Two** terms Gauss Quadrature to estimate the integral

$$\int_{-1}^1 \frac{dx}{x+2}, \quad f(x) = \frac{1}{x+2}$$

Sol.) $x = \left(\frac{1+(-1)}{2}\right) + \left(\frac{1-(-1)}{2}\right)t$

$$x = 0 + \frac{2}{2}t$$

$$\boxed{x = t}$$

$$\int_{-1}^1 \frac{dx}{x+2} = \frac{1-(-1)}{2} [w_1 f(t_1) + w_2 f(t_2)]$$

$$= \frac{2}{2} [1 f(-0.57735027) + 1 f(0.57735027)]$$

$$= \frac{1}{-0.57735027 + 2} + \frac{1}{0.57735027 + 2}$$

$$= 1.091489172$$

* To find the error :

$$\begin{aligned}
 \int_{-1}^1 \frac{dx}{x+2} &= \ln(x+2) \Big|_{-1}^1 \\
 &= \ln 3 - \ln 1 \\
 &= \ln \frac{3}{1} = \ln 3 \\
 &= 1.098612289
 \end{aligned}$$

exact
value.

$$\begin{aligned}
 \text{Error} &= |1.098612289 - 1.091489172| \\
 &= 0.007123117
 \end{aligned}$$

* * * * *



ex 11

use "Three terms" Gauss Quadrature
to estimate the integral

$$\int_{0.2}^{1.5} e^{-x^2} dx$$

Sol. 1 $X = \left(\frac{1.5+0.2}{2} \right) + \left(\frac{1.5-0.2}{2} \right) t$

$$X = 0.85 + 0.65t$$

$$\int_{0.2}^{1.5} e^{-x^2} dx = \frac{1.5-0.2}{2} \int_{-1}^1 e^{-(0.85+0.65t)^2} dt$$

$$= 0.65 \left[0.555555 e^{-[0.85+0.65(0.77459667)]^2} + 0.888889 e^{-[0.85+0]^2} + 0.555555 e^{-[0.85+0.65(-0.77459667)]^2} \right]$$

$$= 0.65860$$

$$\text{exact value} = \int_{0.2}^{1.5} e^{-x^2} dx = 0.6588233$$

$$\text{Error} = |0.65860 - 0.6588233|$$

$$= 0.000223362$$

فاطم الخالدي
مهندسة ميكانيكية

Example Use "two Terms" Quadrature Integration to estimate the

$$\int_0^{\pi/2} \sin x \, dx \quad ?$$

Sol.

$$x = \frac{\frac{\pi}{2} + 0}{2} + \left(\frac{\frac{\pi}{2} - 0}{2} \right) t$$

$$x = \frac{\pi}{4} + \frac{\pi}{4} t$$

$$\int_0^{\pi/2} \sin x \, dx = \frac{\pi}{4} \int_{-1}^1 f\left(\frac{\pi}{4} + \frac{\pi}{4} t\right) dt$$

$$= \frac{\pi}{4} [w_1 f(t_1) + w_2 f(t_2)]$$

$$= \frac{\pi}{4} [1 f(0.57735027) + 1 f(0.57735027)]$$

$$x = \frac{\pi}{4} + \frac{\pi}{4}t = \frac{\pi}{4}(1+t)$$

$$f(x) = f\left(\frac{\pi}{4}(1+t)\right) = \sin\left[\frac{\pi}{4}(1+t)\right]$$

$$\begin{aligned} \Rightarrow \int_0^{\pi/2} \sin x \, dx &= \frac{\pi}{4} \left[\sin \frac{\pi}{4}(1+0.57735027) \right. \\ &\quad \left. + \sin \frac{\pi}{4}(1+0.57735027) \right] \\ &= 0.998472 \end{aligned}$$

Error

$$\begin{aligned} \text{Exact value} &= \int_0^{\pi/2} \sin x \, dx \\ &= -\cos x \Big|_0^{\pi/2} \\ &= -\left[\cos \frac{\pi}{2} - \cos 0\right] \\ &= -[0 - 1] = 1 \end{aligned}$$

$$\text{Error} = |1 - 0.998472| = \underline{\underline{0.001528}}$$

Example Use Gauss Quadrature to find the integral:-

$$\int_1^4 \sqrt{x^2+3} dx \quad \text{using } \underline{\text{two terms!}}$$

Sol.

$$X = \left(\frac{4+1}{2}\right) + \left(\frac{4-1}{2}\right)t$$

$$X = 2.5 + 1.5t$$

$$f(t) = \sqrt{(2.5+1.5t)^2+3}$$

$$f(x) = f(2.5+1.5t) = \sqrt{(2.5+1.5t)^2+3}$$

$$\int_1^4 \sqrt{x^2+3} dx = 1.5 [w_1 f(t_1) + w_2 f(t_2)]$$

$$= 1.5 \left[1 \sqrt{(2.5+1.5(0.57735027))^2+3} \right.$$

$$\left. + 1 \sqrt{(2.5+1.5(0.57735027))^2+3} \right]$$

...



[illegible]

Dr. Hashem Alkhaldi

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* مثال : أوجد القيمة التقريبية للتكامل $\iint_R (x^2 \cdot y) dA$ ، حيث R هو الحقل المحدد بـ :
 * use 4 intervals

$$1 \leq x \leq 2$$

$$0 \leq y \leq 1$$

الحل: نقسم الفترة في المحور الأفقي الى عدد من الفترات الجزئية $2n = 4$ طول منها:

$$h = \frac{2-1}{4} = \frac{1}{4}$$

نقسم الفترة في المحور العمودي الى عدد من الفترات الجزئية $2m = 4$ طول منها:

$$k = \frac{1-0}{4} = \frac{1}{4}$$

يا سيدي راجع

ونحصل بذلك على الشبكة المبينة في الشكل التالي:

		1	4	2	4	1
$y_4=1$		4	16	8	16	4
0.75		2	8	4	8	2
0.5		4	16	8	16	4
0.25		1	4	2	4	1
$y_0=0$						
	$x_0=1$	1.25	1.5	1.75	$x_4=2$	

$$x_i = 1 + ih = 1 + 0.25i$$

$$y_i = 0.25j$$

$$\iint_{\mathcal{R}} f(x, y) dA = \frac{hk}{9} \left\{ \begin{aligned} &f(x_0, y_0) + 4f(x_1, y_0) + 2f(x_2, y_0) + 4f(x_3, y_0) + f(x_4, y_0) \\ &+ 4f(x_0, y_1) + 16f(x_1, y_1) + 8f(x_2, y_1) + 16f(x_3, y_1) + 4f(x_4, y_1) \\ &+ 2f(x_0, y_2) + 8f(x_1, y_2) + 4f(x_2, y_2) + 8f(x_3, y_2) + 2f(x_4, y_2) \\ &+ 4f(x_0, y_3) + 16f(x_1, y_3) + 8f(x_2, y_3) + 16f(x_3, y_3) + 4f(x_4, y_3) \\ &+ f(x_0, y_4) + 4f(x_1, y_4) + 2f(x_2, y_4) + 4f(x_3, y_4) + f(x_4, y_4) \end{aligned} \right\}$$

$$\begin{aligned} \iint_{\mathcal{R}} f(x, y) dA &= \frac{1}{144} \left\{ \begin{aligned} &f(1,0) + 4f(1.25,0) + 2f(1.5,0) + 4f(1.75,0) + f(2,0) \\ &+ 4f(1,0.25) + 16f(1.25,0.25) + 8f(1.5,0.25) + 16f(1.75,0.25) + 4f(2,0.25) \\ &+ 2f(1,0.5) + 8f(1.25,0.5) + 4f(1.5,0.5) + 8f(1.75,0.5) + 2f(2,0.5) \\ &+ 4f(1,0.75) + 16f(1.25,0.75) + 8f(1.5,0.75) + 16f(1.75,0.75) + 4f(2,0.75) \\ &+ f(1,1) + 4f(1.25,1) + 2f(1.5,1) + 4f(1.75,1) + f(2,1) \end{aligned} \right\} \\ &= 1.1667 \end{aligned}$$

بما أننا نريد

Triples Integrals :-

(The Tripple integrals) التكاملات الثلاثية

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سنبحث الآن في كيفية إيجاد قيمة التكامل الثلاثي $\iiint_V f(x, y, z) dV$ ،

حيث V هو حيزاً مكعباً، فيه:

$$a \leq x \leq b$$

$$c \leq y \leq d$$

$$e \leq z \leq g$$

فباستخدام قاعدة سمبسون، نفرض أن الفترة في المحور الأول (محور المتغير x) قد تم تقسيمها الى عدد من الفترات الجزئية مقداره $2n$ ، أي أن طول الفترة الجزئية في هذا المحور هو:

$$h = \frac{b-a}{2n}$$

والفترة في المحور الثاني (محور المتغير y) قد تم تقسيمها الى عدد من الفترات الجزئية مقداره $2m$ ، أي أن طول الفترة الجزئية في هذا المحور هو:

$$k = \frac{d-c}{2m}$$

وإن الفترة في المحور الثالث (محور المتغير z) قد تم تقسيمها الى عدد من الفترات الجزئية مقداره $2p$ ، أي أن طول الفترة الجزئية في هذا المحور هو:

$$l = \frac{g-e}{2p}$$

وكما قمنا ببناء شبكة من نقاط التقاطع التي تنتج من تجزئة الفترتين $[a, b]$ و $[c, d]$ لحساب قيمة النحاصر الثاني فإننا، ولحساب قيمة التكامل

سأستعمل الجدول

الثلاثي، نتخيل شبكة ثلاثية الأبعاد ناتجة من نقاط تقاطع الفترات $[a, b]$ ، $[c, d]$ و $[e, g]$ ومنها نحدد المعاملات $\lambda_{i,j,q}$.

ولإيجاد قيم المعاملات $\lambda_{i,j,q}$ ، فإننا نتخيل الشبكة ثلاثية الأبعاد كعدد من الشبكات ثنائية الأبعاد مرتبة فوق بعضها (عددتها $2p$) وبالمحور الثالث، وتكون قيمة التكامل $\iiint_V f(x, y, z) dV$ هي:

$$\begin{aligned} \iiint_V f(x, y, z) dV = \frac{hkl}{27} & \left[\begin{aligned} & f(x_0, y_0, z_0) + 4f(x_1, y_0, z_0) + 2f(x_2, y_0, z_0) + \dots + f(x_{2n}, y_0, z_0) \\ & + 4f(x_0, y_1, z_0) + 16f(x_1, y_1, z_0) + 8f(x_2, y_1, z_0) + \dots + 4f(x_{2n}, y_1, z_0) \\ & + 2f(x_0, y_2, z_0) + 8f(x_1, y_2, z_0) + 4f(x_2, y_2, z_0) + \dots + 2f(x_{2n}, y_2, z_0) + \\ & \dots \dots \dots \\ & + f(x_0, y_{2m}, z_0) + 4f(x_1, y_{2m}, z_0) + 2f(x_2, y_{2m}, z_0) + \dots + f(x_{2n}, y_{2m}, z_0) \\ & + 4f(x_0, y_0, z_1) + 16f(x_1, y_0, z_1) + 8f(x_2, y_0, z_1) + \dots + 4f(x_{2n}, y_0, z_1) \\ & + 16f(x_0, y_1, z_1) + 64f(x_1, y_1, z_1) + 32f(x_2, y_1, z_1) + \dots + 16f(x_{2n}, y_1, z_1) \\ & + 8f(x_0, y_2, z_1) + 32f(x_1, y_2, z_1) + 16f(x_2, y_2, z_1) + \dots + 8f(x_{2n}, y_2, z_1) + \\ & \dots \dots \dots \\ & + 4f(x_0, y_{2m}, z_1) + 16f(x_1, y_{2m}, z_1) + 8f(x_2, y_{2m}, z_1) + \dots + 4f(x_{2n}, y_{2m}, z_1) \\ & + 2f(x_0, y_0, z_2) + 8f(x_1, y_0, z_2) + 4f(x_2, y_0, z_2) + \dots + 2f(x_{2n}, y_0, z_2) \\ & + 8f(x_0, y_1, z_2) + 32f(x_1, y_1, z_2) + 16f(x_2, y_1, z_2) + \dots + 8f(x_{2n}, y_1, z_2) \\ & + 4f(x_0, y_2, z_2) + 16f(x_1, y_2, z_2) + 8f(x_2, y_2, z_2) + \dots + 4f(x_{2n}, y_2, z_2) + \\ & \dots \dots \dots \\ & + 2f(x_0, y_{2m}, z_2) + 8f(x_1, y_{2m}, z_2) + 4f(x_2, y_{2m}, z_2) + \dots + 2f(x_{2n}, y_{2m}, z_2) + \\ & \dots \dots \dots \\ & + f(x_0, y_0, z_{2p}) + 4f(x_1, y_0, z_{2p}) + 2f(x_2, y_0, z_{2p}) + \dots + f(x_{2n}, y_0, z_{2p}) \\ & + 4f(x_0, y_1, z_{2p}) + 16f(x_1, y_1, z_{2p}) + 8f(x_2, y_1, z_{2p}) + \dots + 4f(x_{2n}, y_1, z_{2p}) \\ & + 2f(x_0, y_2, z_{2p}) + 8f(x_1, y_2, z_{2p}) + 4f(x_2, y_2, z_{2p}) + \dots + 2f(x_{2n}, y_2, z_{2p}) + \\ & \dots \dots \dots \\ & + f(x_0, y_{2m}, z_{2p}) + 4f(x_1, y_{2m}, z_{2p}) + 2f(x_2, y_{2m}, z_{2p}) + \dots + f(x_{2n}, y_{2m}, z_{2p}) \end{aligned} \right] \end{aligned}$$

طريقة التوليد

Double integrals $\iint$

$$\int_a^b \int_c^d f(x,y) dy dx$$

تقسيم المجال
مستطوي

$$a \leq x \leq b \rightarrow h = \frac{b-a}{n}$$

$$c \leq y \leq d \rightarrow k = \frac{d-c}{m}$$

ex)

Find : $\int_{5.2}^{5.6} \int_0^6 (x^2y + 2x^2 + 5xy) dy dx$

$$\boxed{n=6, m=4}$$

No. of intervals in x and y directions

Sol

5.6	1	4	2	4	2	4	1
5.5	4	16	8	16	8	16	4
5.4	2	8	4	8	4	8	2
5.3	4	16	8	16	8	16	4
5.2	1	4	2	4	2	4	1
	0	1	2	3	4	5	6

$h=1$
 $k=0.1$

* السطر الأول (1, 4, 2, 4, 2, 4, 1) وهو نفس السطر

* السطر الثاني (4, 16, 8, 16, 8, 16, 4) وهو نفس السطر الثاني

* السطر الثالث (2, 8, 4, 8, 4, 8, 2) وهو نفس السطر الثالث

$$f(x,y) = x^2y + 3x^3 + \sin y$$

$$\iint = \frac{kh}{9} \left[\begin{aligned} &[1] f(0, 5.6) \\ &+ [4] f(1, 5.6) \\ &+ [2] f(2, 5.6) \\ &+ [4] f(3, 5.6) \\ &+ [2] f(4, 5.6) \\ &+ \vdots \end{aligned} \right]$$

Using all cells in the Table
As the Coefficients ...

** Find The error ?

كلمة الخالد
د. هاشم الخالدي

ex)
Find $\int_2^3 \int_1^2 (y^x + x^y - x^x) dx dy$

using $h=k=0.5$

sol.)

	3	1	4	1
y	2.5	4	16	4
	2	1	4	1
		1	1.5	2
			x	

$$f(x,y) = y^x + x^y - x^x$$

$\Rightarrow f(1,2) = 2 + 1 - 1 = 2$	$\Rightarrow f(1,3) =$
$\Rightarrow f(1.5,2) = 2^{1.5} + 1.5^2 - 1.5^{1.5} =$	$\Rightarrow f(1.5,3) =$
$\Rightarrow f(2,2) = 2^2 + 2^2 - 2^2 = 4$	$\Rightarrow f(2,3) =$
$\Rightarrow f(1,2.5) =$	
$\Rightarrow f(1.5,2.5) =$	
$\Rightarrow f(2,2.5) =$	

هاشم الخالدي
مكتبة ميكانيكية

$$\int_2^3 \int_1^2 (y^x + x^y - x^x) dx dy$$

$$= \frac{(0.5)(0.5)}{9} \left[1 f(1,2) + 4 f(1.5,2) + 1 f(2,2) \right. \\ \left. + 4 f(1,2.5) + 16 f(1.5,2.5) + 4 f(2,2.5) \right. \\ \left. + 1 f(1,3) + 4 f(1.5,3) + 1 f(2,3) \right] \\ = \dots \checkmark$$

ex|| Using Simpson's Rule $\frac{1}{3}$ To
estimate :- $\int_{-0.5}^{0.4} \int_1^2 (x+3y) e^x dy dx$

For: $h = 0.3 \equiv \Delta x$

$K = 0.5 \equiv \Delta y$

⇒ Find The Absolute error ?