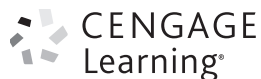


INSTRUCTOR'S SOLUTIONS MANUAL
TO ACCOMPANY

MECHANICS OF MATERIALS

EIGHTH EDITION

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Australia • Brazil • Japan • Korea • Mexico • Singapore • Spain • United Kingdom • United States

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Answers to Problems

CHAPTER 1

- 1.2-1** (a) $A_y = 5 \text{ lb}$, $B_y = -5 \text{ lb}$, $C_x = 50 \text{ lb}$, $C_y = 0$;
(b) $N = 50 \text{ lb}$, $V = 5 \text{ lb}$, $M = 75 \text{ ft-lb}$
- 1.2-2** (a) $M_A = 0$, $C_y = 236 \text{ N}$, $D_y = -75.6 \text{ N}$;
(b) $N = 0$, $V = -70 \text{ N}$, $M = -36.7 \text{ N}\cdot\text{m}$;
(c) $M_A = 0$, $C_y = 236 \text{ N}$, $D_y = -75.6 \text{ N}$;
 $N = 0$, $V = -70 \text{ N}$, $M = -36.7 \text{ N}\cdot\text{m}$
- 1.2-3** (a) $A_x = 12.55 \text{ lb}$, $A_y = -15 \text{ lb}$, $C_y = 104.3 \text{ lb}$,
 $D_x = 11.45 \text{ lb}$, $D_y = -19.83 \text{ lb}$;
(b) Resultant_B = 19.56 lb; (c) $A_x = 42.7 \text{ lb}$,
 $A_y = 37.2 \text{ lb}$, $M_A = 522 \text{ lb}\cdot\text{ft}$, $D_x = -18.67 \text{ lb}$,
 $D_y = 32.3 \text{ lb}$, Resultant_B = 56.6 lb
- 1.2-4** (a) $R_{3x} = 40 \text{ N}$, $R_{3y} = -25 \text{ N}$, $R_{5x} = 20 \text{ N}$;
(b) $F_{11} = 0$, $F_{13} = 28.3 \text{ N}$
- 1.2-5** (a) $A_x = 0$, $A_y = 1.0 \text{ kip}$, $E_y = 5 \text{ kips}$;
(b) $F_{FE} = 1.898 \text{ kips}$
- 1.2-6** (a) $F_x = 0$, $F_y = 12.0 \text{ kN}$, $D_y = 6.0 \text{ kN}$; (b) $F_{FE} = 0$
- 1.2-7** (a) $B_x = -0.8 P$, $B_z = 2.0 P$, $O_z = -1.25 P$;
(b) $F_{AC} = 0.960 P$
- 1.2-8** (a) $A_x = -1.25P$, $B_y = 0$, $B_z = -P$;
(b) $F_{AB} = 1.601P$
- 1.2-9** (a) $A_y = 4.67P$, $A_z = -4.0P$; (b) $F_{AB} = -8.33P$
- 1.2-10** (a) $A_z = 0$, $B_x = -3.75 \text{ kN}$; (b) $F_{AB} = 6.73 \text{ kN}$
- 1.2-11** (a) $T_A = 11,000 \text{ lb}\cdot\text{in.}$;
(b) $T(L_1/2) = -T_A = -11,000 \text{ lb}\cdot\text{in.}$,
 $T(L_1 + L_2/2) = T_2 = 10,000 \text{ lb}\cdot\text{in.}$
- 1.2-12** (a) $T_A = -1225 \text{ N}\cdot\text{m}$; (b) $T(L_1/2) = 62.5 \text{ N}\cdot\text{m}$,
 $T(L_1 + L_2/2) = -T_2 = -1100 \text{ N}\cdot\text{m}$
- 1.2-13** (a) $A_x = -540 \text{ lb}$, $A_y = -55.6 \text{ lb}$,
 $M_A = 4320 \text{ lb}\cdot\text{ft}$, $C_y = 55.6 \text{ lb}$;
(b) $N = 55.6 \text{ lb}$, $V = 506 \text{ lb}$, $M = -2374 \text{ lb}\cdot\text{ft}$
- 1.2-14** (a) $A_x = 280 \text{ N}$, $A_y = 8.89 \text{ N}$, $M_A = -1120 \text{ N}\cdot\text{m}$,
 $D_y = 151.1 \text{ N}$; (b) Resultant_B = 280 N
- 1.2-15** (a) $A_x = 30 \text{ lb}$, $A_y = 140 \text{ lb}$, $C_x = -30 \text{ lb}$,
 $C_y = 60 \text{ lb}$; (b) $N = -23.3 \text{ lb}$, $V = -20 \text{ lb}$,
 $M = 33.3 \text{ lb}\cdot\text{ft}$
- 1.2-16** (a) $A_x = 10.98 \text{ kN}$, $A_y = 29.0 \text{ kN}$,
 $E_x = -8.05 \text{ kN}$, $E_y = -22 \text{ kN}$;
(b) Resultant_C = 23.4 kN
- 1.2-17** (a) $A_y = -1250 \text{ lb}$, $E_x = 0$, $E_y = 1750 \text{ lb}$;
(b) $N = 1750 \text{ lb}$, $V = 500 \text{ lb}$, $M = 575 \text{ lb}\cdot\text{ft}$
- 1.2-18** (a) $A_x = 320 \text{ N}$, $A_y = -240 \text{ N}$, $C_y = 192 \text{ N}$,
 $E_y = -192 \text{ N}$; (b) $N = -312 \text{ N}$, $V = -57.9 \text{ N}$,
 $M = 289 \text{ N}\cdot\text{m}$; (c) Resultant_C = 400 N
- 1.2-19** (a) $A_x = -28.9 \text{ lb}$, $A_y = 50.0 \text{ lb}$,
 $B_x = -65.0 \text{ lb}$; (b) $F_{\text{cable}} = 71.6 \text{ lb}$
- 1.2-20** (a) $A_x = -10 \text{ kN}$, $A_y = -2.17 \text{ kN}$,
 $C_y = 9.83 \text{ kN}$, $E_y = 1.333 \text{ kN}$;
(b) Resultant_D = 12.68 kN
- 1.2-21** (a) $O_x = -48.3 \text{ lb}$, $O_y = 40 \text{ lb}$, $O_z = 12.94 \text{ lb}$,
 $M_{Ox} = 331 \text{ lb}\cdot\text{in.}$, $M_{Oy} = 690 \text{ lb}\cdot\text{in.}$,
 $M_{Oz} = -338 \text{ lb}\cdot\text{in.}$; (b) $N = -40 \text{ lb}$,
 $V = 50 \text{ lb}$, $T = -690 \text{ lb}\cdot\text{in.}$, $M = 473 \text{ lb}\cdot\text{in.}$
- 1.2-22** (a) $A_y = -120 \text{ N}$, $A_z = -60 \text{ N}$,
 $M_{Ax} = -70 \text{ N}\cdot\text{m}$, $M_{Ay} = -142.5 \text{ N}\cdot\text{m}$,
 $M_{Az} = -180 \text{ N}\cdot\text{m}$, $D_x = -60 \text{ N}$,
 $D_y = 120 \text{ N}$, $D_z = 30 \text{ N}$; (b) $N = 120 \text{ N}$,
 $V = 41.3 \text{ N}$, $T = 142.5 \text{ N}\cdot\text{m}$, $M = 180.7 \text{ N}\cdot\text{m}$
- 1.2-23** (a) $A_x = 5.77 \text{ lb}$, $A_y = 47.3 \text{ lb}$,
 $A_z = -2.31 \text{ lb}$, $M_{Az} = 200 \text{ lb}\cdot\text{in.}$;
(b) $T_{DC} = 3.81 \text{ lb}$, $T_{EC} = 6.79 \text{ lb}$
- 1.2-24** $C_x = 120 \text{ N}$, $C_y = -160 \text{ N}$, $C_z = 506 \text{ N}$,
 $D_z = 466 \text{ N}$, $H_y = 320 \text{ N}$, $H_z = 499 \text{ N}$
- 1.2-25** $A_y = 57.2 \text{ lb}$, $B_x = 44.2 \text{ lb}$ (to the left),
 $B_y = 112.4 \text{ lb}$, $C_x = 28.8 \text{ lb}$, $C_y = 5.88 \text{ lb}$
- 1.2-26** (a) $H_B = -104.6 \text{ N}$, $V_B = 516 \text{ N}$, $V_F = 336 \text{ N}$;
(b) $N = -646 \text{ N}$, $V = 176.8 \text{ N}$, $M = 44.9 \text{ kN}\cdot\text{m}$
- 1.3-1** (a) $\sigma_{AB} = 1443 \text{ psi}$; (b) $P_2 = 1487.5 \text{ lbs}$;
(c) $t_{BC} = 0.5 \text{ in.}$
- 1.3-2** (a) $\sigma = 130.2 \text{ MPa}$; (b) $\varepsilon = 4.652 \times 10^{-4}$
- 1.3-3** (a) $R_B = \text{lb}$ (cantilever), 191.3 lb (V-brakes);
 $\sigma_C = 144 \text{ psi}$ (cantilever), 306 psi (V-brakes);
(b) $\sigma_{\text{cable}} = 26,946 \text{ psi}$ (both)
- 1.3-4** (a) $\varepsilon_s = 3.101 \times 10^{-4}$; (b) $\delta = 0.1526 \text{ mm}$;
(c) $P_{\text{max}} = 89.5 \text{ kN}$
- 1.3-5** (a) $\sigma_C = 2.46 \text{ ksi}$; (b) $x_C = 19.56 \text{ in.}$,
 $y_C = 19.56 \text{ in.}$

- 1.3-6** (a) $\sigma_t = 132.7 \text{ MPa}$; (b) $\alpha_{\max} = 34.4^\circ$
- 1.3-7** (a) $\sigma_1 = 34.4 \text{ ksi}$, $\sigma_2 = 30.6 \text{ ksi}$;
(b) $d_{\text{new}} = 3.18 \times 10^{-2} \text{ in.}$;
(c) $\sigma_1 = 19.6 \text{ ksi}$, $\sigma_2 = 18.78 \text{ ksi}$, $\sigma_3 = 22.7 \text{ ksi}$
- 1.3-8** $\sigma_C = 5.21 \text{ MPa}$
- 1.3-9** (a) $T = 184 \text{ lb}$, $\sigma = 10.8 \text{ ksi}$; (b) $\epsilon_{\text{cable}} = 5 \times 10^{-4}$
- 1.3-10** (a) $T = 819 \text{ N}$, $\sigma = 74.5 \text{ MPa}$;
(b) $\epsilon_{\text{cable}} = 4.923 \times 10^{-4}$
- 1.3-11** (a) $T = \begin{pmatrix} 5877 \\ 4679 \\ 7159 \end{pmatrix} \text{ lb}$; (b) $\sigma = \begin{pmatrix} 48,975 \\ 38,992 \\ 59,658 \end{pmatrix} \text{ psi}$;
(c) $T = \begin{pmatrix} 4278 \\ 6461 \\ 3341 \\ 4278 \end{pmatrix} \text{ lb}$, $\sigma = \begin{pmatrix} 35,650 \\ 53,842 \\ 27,842 \\ 35,650 \end{pmatrix} \text{ psi}$
- 1.3-12** (a) $\sigma_x = \gamma\omega^2(L^2 - x^2)/2g$; (b) $\sigma_{\max} = \gamma\omega^2L^2/2g$
- 1.3-13** (a) $T_{AB} = 1620 \text{ lb}$, $T_{BC} = 1536 \text{ lb}$,
 $T_{CD} = 1640 \text{ lb}$; (b) $\sigma_{AB} = 13,501 \text{ psi}$,
 $\sigma_{BC} = 12,799 \text{ psi}$, $\sigma_{CD} = 13,667 \text{ psi}$
- 1.3-14** (a) $T_{AQ} = T_{BQ} = 50.5 \text{ kN}$; (b) $\sigma = 166 \text{ MPa}$
- 1.4-1** (a) $L_{\max} = 11,800 \text{ ft}$; (b) $L_{\max} = 13,500 \text{ ft}$
- 1.4-2** (a) $L_{\max} = 7143 \text{ m}$; (b) $L_{\max} = 8209 \text{ m}$
- 1.4-3** % elongation = 6.5, 24.0, 39.0;
% reduction = 8.1, 37.9, 74.9;
Brittle, ductile, ductile
- 1.4-4** $11.9 \times 10^3 \text{ m}$; $12.7 \times 10^3 \text{ m}$; $6.1 \times 10^3 \text{ m}$;
 $6.5 \times 10^3 \text{ m}$; $23.9 \times 10^3 \text{ m}$
- 1.4-5** $\sigma = 52.3 \text{ ksi}$
- 1.4-6** $\sigma_{pl} \approx 47 \text{ MPa}$, Slope $\approx 2.4 \text{ GPa}$,
 $\sigma_y \approx 53 \text{ MPa}$; Brittle
- 1.4-7** $\sigma_{pl} \approx 65,000 \text{ psi}$, Slope $\approx 30 \times 10^6 \text{ psi}$, $\sigma_y \approx 69,000 \text{ psi}$, $\sigma_U \approx 113,000 \text{ psi}$; Elongation = 6%,
Reduction = 31%
- 1.5-1** 0.13 in. longer
- 1.5-2** 4.0 mm longer
- 1.5-3** (a) $\delta_{\text{pset}} = 1.596 \text{ in.}$; (b) $\sigma_B = 30 \text{ ksi}$
- 1.5-4** (a) $\delta_{\text{pset}} = 4.28 \text{ mm}$; (b) $\sigma_B = 65.6 \text{ MPa}$
- 1.5-5** (b) 0.71 in.; (c) 0.58 in.; (d) 49 ksi
- 1.6-1** $P_{\max} = 157 \text{ k}$
- 1.6-2** $P = 27.4 \text{ kN}$ (tension)
- 1.6-3** $P = -15.708 \text{ kips}$
- 1.6-4** (a) $P = 74.1 \text{ kN}$; (b) $\delta = \epsilon \cdot L = 0.469 \text{ mm}$
shortening; $\frac{A_f - A}{A} = -0.081\%$,
 $\Delta V_l = V_{lf} - \text{Vol}_1 = -207 \text{ mm}^3$;
(c) $d_3 = 65.4 \text{ mm}$
- 1.6-5** $\Delta d = -1.56 \times 10^{-4} \text{ in.}$, $P = 2.154 \text{ kips}$
- 1.6-6** (a) $E = 104 \text{ GPa}$; (b) $\nu = 0.34$
- 1.6-7** (a) $\Delta d_{BC \text{ inner}} = 8 \times 10^{-4} \text{ in.}$;
(b) $\nu_{\text{brass}} = 0.34$; (c) $\Delta t_{AB} = 2.732 \times 10^{-4} \text{ in.}$,
 $\Delta d_{AB \text{ inner}} = 1.366 \times 10^{-4} \text{ in.}$
- 1.6-8** (a) $\Delta L_1 = 12.66 \text{ mm}$; $\Delta L_2 = 5.06 \text{ mm}$;
 $\Delta L_3 = 3.8 \text{ mm}$; (b) $\Delta \text{Vol}_1 = 21,548 \text{ mm}^3$;
 $\Delta \text{Vol}_2 = 21,601 \text{ mm}^3$; $\Delta \text{Vol}_3 = 21,610 \text{ mm}^3$
- 1.7-1** $\sigma_b = 7.04 \text{ ksi}$, $\tau_{\text{ave}} = 10.756 \text{ ksi}$
- 1.7-2** $\sigma_b = 139.86 \text{ MPa}$; $P_{\text{ult}} = 144.45 \text{ kN}$
- 1.7-3** (a) $\tau = 12.732 \text{ ksi}$; (b) $\sigma_{bf} = 20 \text{ ksi}$,
 $\sigma_{bg} = 26.667 \text{ ksi}$
- 1.7-4** (a) $B_x = -252.8 \text{ N}$, $A_x = -B_x$,
 $A_y = 1150.1 \text{ N}$; (b) $A_{\text{resultant}} = 1178 \text{ N}$;
(c) $\tau = 5.86 \text{ MPa}$, $\sigma_{\text{bshoe}} = 7.36 \text{ MPa}$
- 1.7-5** (a) $\tau_{\max} = 2979 \text{ psi}$; (b) $\sigma_{b\max} = 936 \text{ psi}$
- 1.7-6** $T_1 = 13.176 \text{ kN}$, $T_2 = 10.772 \text{ kN}$,
 $\tau_{1\text{ave}} = 25.888 \text{ MPa}$, $\tau_{2\text{ave}} = 21.166 \text{ MPa}$,
 $\sigma_{b1} = 9.15 \text{ MPa}$, $\sigma_{b2} = 7.48 \text{ MPa}$
- 1.7-7** (a) Resultant = 1097 lb; (b) $\sigma_b = 4999 \text{ psi}$;
(c) $\tau_{\text{nut}} = 2793 \text{ psi}$, $\tau_{\text{pl}} = 609 \text{ psi}$
- 1.7-8** $G = 2.5 \text{ MPa}$
- 1.7-9** (a) $\gamma_{\text{aver}} = 0.004$; (b) $V = 89.6 \text{ k}$
- 1.7-10** (a) $\gamma_{\text{aver}} = 0.50$; (b) $\delta = 4.92 \text{ mm}$
- 1.7-11** (a) $\sigma_b = 69.5 \text{ ksi}$, $\sigma_{brg} = 39.1 \text{ ksi}$, $\tau_f = 21 \text{ ksi}$;
(b) $\sigma_b = 60.4 \text{ ksi}$, $\sigma_{brg} = 34 \text{ ksi}$, $\tau_f = 18.3 \text{ ksi}$
- 1.7-12** $\tau_{\text{aver}} = 42.9 \text{ MPa}$
- 1.7-13** (a) $A_x = 0$, $A_y = 170 \text{ lb}$, $M_A = 4585 \text{ in.-lb}$;
(b) $B_x = 253.6 \text{ lb}$, $B_y = 160 \text{ lb}$, $B_{\text{res}} = 299.8 \text{ lb}$,
 $C_x = -B_x$; (c) $\tau_B = 3054 \text{ psi}$, $\tau_C = 1653 \text{ psi}$;
(d) $\sigma_{bB} = 4797 \text{ psi}$, $\sigma_{bC} = 3246 \text{ psi}$
- 1.7-14** For a bicycle with $L/R = 1.8$: (a) $T = 1440 \text{ N}$;
(b) $\tau_{\text{aver}} = 147 \text{ MPa}$
- 1.7-15** (a) $\tau = \frac{P}{2\pi rh}$; (b) $\delta = \frac{P}{2\pi hG} \ln \frac{b}{d}$
- 1.7-16** (a) $\tau_1 = 2.95 \text{ MPa}$, $\tau_4 = 0$;
(b) $\sigma_{b1} = 1.985 \text{ MPa}$, $\sigma_{b4} = 0$;
(c) $\sigma_{b4} = 41 \text{ MPa}$; (d) $\tau = 10.62 \text{ MPa}$;
(e) $\sigma_3 = 75.1 \text{ MPa}$
- 1.7-17** (a) $O_x = 12.68 \text{ lb}$, $O_y = 1.294 \text{ lb}$,
 $O_{\text{res}} = 12.74 \text{ lb}$; (b) $\tau_O = 519 \text{ psi}$,
 $\sigma_{bO} = 816 \text{ psi}$; (c) $\tau = 362 \text{ psi}$
- 1.7-18** (a) $F_x = 153.9 \text{ N}$, $\sigma = 3.06 \text{ MPa}$;
(b) $\tau_{\text{ave}} = 1.96 \text{ MPa}$; (c) $\sigma_b = 1.924 \text{ MPa}$
- 1.7-19** (a) $P = 395 \text{ lb}$; (b) $C_x = 374 \text{ lb}$,
 $C_y = -237 \text{ lb}$, $C_{\text{res}} = 443 \text{ lb}$; (c) $\tau = 18.04 \text{ ksi}$,
 $\sigma_{bC} = 4.72 \text{ ksi}$
- 1.8-1** $P_{\text{allow}} = 3140 \text{ lb}$

- 1.8-2** $T_{\max} = 33.4 \text{ kN}\cdot\text{m}$
1.8-3 $P_{\text{allow}} = 607 \text{ lb}$
1.8-4 (a) $P_{\text{allow}} = 8.74 \text{ kN}$; (b) $P_{\text{allow}} = 8.69 \text{ kN}$;
 (c) $P_{\text{allow}} = 21.2 \text{ kN}$, $P_{\text{allow}} = 8.69 \text{ kN}$
 (shear controls)
1.8-5 $P = 294 \text{ k}$
1.8-6 (a) $F = 1.171 \text{ kN}$; (b) Shear: $F_a = 2.86 \text{ kN}$
1.8-7 $W_{\max} = 5110 \text{ lb}$
1.8-8 (a) $F_A = \sqrt{2}T$, $F_B = 2T$, $F_C = T$;
 (b) Shear at A: $W_{\max} = 66.5 \text{ kN}$
1.8-9 $P_a = 10.21 \text{ kips}$
1.8-10 $C_{\text{ult}} = 5739 \text{ N}$; $P_{\max} = 445 \text{ N}$
1.8-11 $W_{\max} = 0.305 \text{ kips}$
1.8-12 Shear in rivets in *CG* & *CD* controls:
 $P_{\text{allow}} = 45.8 \text{ kN}$
1.8-13 (a) $P_a = \sigma_a (0.587d^2)$; (b) $P_a = 21.6 \text{ kips}$
1.8-14 $P_{\text{allow}} = 96.5 \text{ kN}$
1.8-15 $P_{\max} = 11.98 \text{ psf}$
1.8-16 (a) $P_{\text{allow}} = \sigma_c (\pi d^2/4) \sqrt{1 - (R/L)^2}$;
 (b) $P_{\text{allow}} = 9.77 \text{ kN}$
1.9-1 (a) $d_{\min} = 3.75 \text{ in.}$; (b) $d_{\min} = 4.01 \text{ in.}$
1.9-2 (a) $d_{\min} = 164.6 \text{ mm}$; (b) $d_{\min} = 170.9 \text{ mm}$
1.9-3 (a) $d_{\min} = 0.704 \text{ in.}$; (b) $d_{\min} = 0.711 \text{ in.}$
1.9-4 $d_{\min} = 63.3 \text{ mm}$
1.9-5 $d_{\text{pin}} = 1.029 \text{ in.}$
1.9-6 (b) $A_{\min} = 435 \text{ mm}^2$
1.9-7 $d_{\min} = 0.372 \text{ in.}$
1.9-8 $d_{\min} = 5.96 \text{ mm}$
1.9-9 $n = 11.6$, or 12 bolts
1.9-10 $(d_2)_{\min} = 131 \text{ mm}$
1.9-11 $A_c = 1.189 \text{ in.}^2$
1.9-12 (a) $t_{\min} = 18.8 \text{ mm}$, use $t = 20 \text{ mm}$;
 (b) $D_{\min} = 297 \text{ mm}$
1.9-13 (a) $\sigma_{DF} = 10.38 \text{ ksi} < \sigma_{\text{allow}}$,
 $\sigma_{bF} = 378 \text{ psi} < \sigma_{ba}$; (b) new $\sigma_{BC} = 25 \text{ ksi}$, so
 increase rod *BC* to $\frac{1}{4}$ -in. diameter; required
 diameter of washer $= 1 - \frac{5}{16} \text{ in.} = 1.312 \text{ in.}$
1.9-14 (a) $d_m = 24.7 \text{ mm}$; (b) $P_{\max} = 49.4 \text{ kN}$
1.9-15 $\theta = \arccos 1/\sqrt{3} = 54.7^\circ$

CHAPTER 2

- 2.2-1** (a) $\delta = \frac{6W}{5k}$; (b) $\delta = \frac{4W}{5k}$
2.2-2 (a) $\delta = 12.5 \text{ mm}$; (b) $n = 5.8$

- 2.2-3** (a) $\frac{\delta_a}{\delta_s} = \frac{E_s}{E_a} = \frac{30}{11}$; (b) $\frac{d_a}{d_s} = \sqrt{\frac{E_s}{E_a}} = 1.651$;
 (c) $\frac{L_a}{L_s} = 1.5 \frac{E_a}{E_s} = 0.55$;
 (d) $E_1 = \frac{E_s}{1.7} = 17,647 \text{ ksi}$ (cast iron or copper
 alloy) (see App. I)
2.2-4 $h = 13.4 \text{ mm}$
2.2-5 $h = L - \pi \rho_{\max} d^2/4k$
2.2-6 (a) $x = 102.6 \text{ mm}$; (b) $x = 205 \text{ mm}$;
 (c) $P_{\max} = 12.51 \text{ N}$; (d) $\theta_{\text{init}} = 1.325^\circ$;
 (e) $P = 20.4 \text{ N}$
2.2-7 (a) $\delta_4 = \frac{26P}{3k}$;
 (b) $\delta_4 = \frac{104P}{45k}$, ratio $= \frac{15}{4} = 3.75$
2.2-8 (a) $\delta_B = 1.827 \text{ mm}$; (b) $P_{\max} = 390 \text{ kN}$;
 (c) $\delta_{Bx} = 6.71 \text{ mm}$, $P_{\max} = 106.1 \text{ kN}$
2.2-9 $P_{\max} = 72.3 \text{ lb}$
2.2-10 (a) $x = 134.7 \text{ mm}$; (b) $k_1 = 0.204 \text{ N/mm}$;
 (c) $b = 74.1 \text{ mm}$; (d) $k_3 = 0.638 \text{ N/mm}$
2.2-11 (a) $t_{c,\min} = 0.021 \text{ in.}$; (b) $\delta_r = 0.031 \text{ in.}$;
 (c) $h_{\min} = 0.051 \text{ in.}$
2.2-12 $\delta_A = 0.200 \text{ mm}$, $\delta_D = 0.880 \text{ mm}$
2.2-13 (a) $\delta_D = \frac{P}{16} (28f_2 - 9f_1)$; (b) $\frac{L_1}{L_2} = \frac{27}{16}$
 (c) $\frac{d_1}{d_2} = 1.225$; (d) $x = \frac{365L}{236}$
2.2-14 (a) $\theta = 35.1^\circ$, $\delta = 44.6 \text{ mm}$, $R_A = 25 \text{ N}$,
 $R_C = 25 \text{ N}$; (b) $\theta = 43.3^\circ$, $\delta = 8.19 \text{ mm}$,
 $R_A = 31.5 \text{ N}$, $R_C = 18.5 \text{ N}$,
 $M_A = 1.882 \text{ N}\cdot\text{m}$
2.2-15 (a) $\theta = 35.1^\circ$, $\delta = 1.782 \text{ in.}$, $R_A = 5 \text{ lb}$,
 $R_C = 5 \text{ lb}$; (b) $\theta = 43.3^\circ$, $\delta = 0.327 \text{ in.}$,
 $R_A = 6.3 \text{ lb}$, $R_C = 3.71 \text{ lb}$,
 $M_A = 1.252 \text{ lb}\cdot\text{ft}$
2.3-1 (a) $\delta = 0.0276 \text{ in.}$; (b) $d_B = 1.074 \text{ in.}$
2.3-2 (a) $\delta = 0.675 \text{ mm}$; (b) $P_{\max} = 267 \text{ kN}$
2.3-3 (a) $\delta = 0.01125 \text{ in.}$ (elongation); (b) So new value
 of P_3 is 1690 lb, an increase of 390 lb.
 (c) $A_{AB} = 0.78 \text{ in.}^2$
2.3-4 (a) $\delta = \frac{7PL}{6Ebt}$; (b) $\delta = 0.5 \text{ mm}$; (c) $L_{\text{slot}} = 244 \text{ mm}$

- 2.3-5** (a) $\delta = \frac{7PL}{6Ebt}$; (b) $\delta = 0.021$ in.; (c) $L_{\text{slot}} = 10$ in.
- 2.3-6** (a) $\delta_{AC} = 3.72$ mm; (b) $P_0 = 44.2$ kN
- 2.3-7** (a) $\delta_a = 0.0589$ in.; (b) $\delta_b = 0.0501$ in.;
 (c) $\frac{\delta_c}{\delta_a} = 0.58$, $\frac{\delta_c}{\delta_b} = 0.681$
- 2.3-8** (a) $d_{\text{max}} = 23.9$ mm; (b) $b = 4.16$ mm;
 (c) $x = 183.3$ mm
- 2.3-9** (a) $\delta = \frac{PL}{2EA}$; (b) $\sigma(y) = \frac{P}{A} \left(\frac{y}{L} \right)$;
 (c) $\delta = \frac{PL}{EA} \left(\frac{2}{3} \right)$, $\sigma(y) = \frac{P}{A} \left[\frac{y}{L} \left(2 - \frac{y}{L} \right) \right]$
- 2.3-10** (a) $\delta_{2-4} = 0.024$ mm; (b) $P_{\text{max}} = 8.15$ kN;
 (c) $L_2 = 9.16$ mm
- 2.3-11** (a) $R_1 = -3P/2$; (b) $N_1 = 3P/2$ (tension),
 $N_2 = P/2$ (tension); (c) $x = L/3$;
 (d) $\delta_2 = 2PL/3EA$; (e) $\beta = 1/11$
- 2.3-12** (a) $\delta_c = W(L^2 - h^2)/2EA$;
 (b) $\delta_B = WL/2EA$; (c) $\beta = 3$;
 (d) $\delta = \frac{WL}{2EA} = 359$ mm (in sea water);
 $\delta = \frac{WL}{2EA} = 412$ mm (in air)
- 2.3-13** (b) $\delta = 0.010$ in.
- 2.3-14** $\delta = 2PH/3Eb^2$
- 2.3-15** $\delta = 2WL/\pi d^2 E$
- 2.3-16** (a) $\delta = 2.18$ mm; (b) $\delta = 6.74$ mm
- 2.3-17** (b) $\delta = 11.14$ ft
- 2.4-1** (a) $P = 1330$ lb; (b) $P_{\text{allow}} = 1300$ lb
- 2.4-2** (a) $P = 104$ kN; (b) $P_{\text{max}} = 116$ kN
- 2.4-3** (a) $P_B/P = 3/11$; (b) $\sigma_B/\sigma_A = 1/2$; (c) Ratio = 1
- 2.4-4** (a) If $x \leq L/2$, $R_A = (-3PL)/(2(x + 3L))$,
 $R_B = -P(2x + 3L)/(2(x + 3L))$. If $x \geq L/2$,
 $R_A = (-P(x + L))/(x + 3L)$,
 $R_B = (-2PL)/(x + 3L)$. (b) If $x \leq L/2$,
 $\delta = PL(2x + 3L)/[(x + 3L)E\pi d^2]$. If
 $x \geq L/2$, $\delta = 8PL(x + L)/[3(x + 3L)E\pi d^2]$.
 (c) $x = 3L/10$ or $x = 2L/3$;
 (d) $R_B = \rho g \pi d^2 L/8$, $R_A = 3 \rho g \pi d^2 L/32$
- 2.4-5** (a) 41.7%; (b) $\sigma_M = 32.7$ ksi, $\sigma_O = 51.4$ ksi
- 2.4-6** (a) $\delta = 1.91$ mm; (b) $\delta = 1.36$ mm; (c) $\delta = 2.74$ mm
- 2.4-7** (a) $R_A = 2P/3$, $R_E = -5P/3$;
 (b) $\delta_B = -\frac{LP}{6EA}$, $\delta_C = \frac{LP}{6EA}$, $\delta_D = \frac{5LP}{6EA}$;
 (c) $\delta_{\text{max}} = \frac{5LP}{6EA}$ (to the right), $\delta_A = \delta_E = 0$
 (d) $P_{\text{max}} = 12.37$ kip
- 2.4-8** (a) $R_A = 10.5$ kN (to the left), $R_D = 2.0$ kN (to the right); (b) $F_{BC} = 15.0$ kN (compression)
- 2.4-9** (b) $\sigma_a = 1610$ psi (compression),
 $\sigma_s = 9350$ psi (tension)
- 2.4-10** (a) $P = 13.73$ kN, $R_1 = 9.07$ kN,
 $R_2 = 4.66$ kN, $\sigma_2 = 7$ MPa;
 (b) $\delta_{\text{cap}} = 190.9$ mm, Axial Force Diagram:
 $N(x) = -R_2$ if $x \leq L_2$, $N(x) = R_1$
 if $x > L_2$; Axial Displacement Diagram:
 $\delta(x) = \left[\frac{-R_2}{EA_2} (x) \right]$ if $x \leq L_2$,
 $\delta(x) = \left[\frac{-R_2 L_2}{EA_2} + \frac{R_1}{EA_1} (x - L_2) \right]$ if $x > L_2$;
 (c) $q = 1.552$ kN/m
- 2.4-11** (a) $P_1 = PE_1/(E_1 + E_2)$;
 (b) $e = b(E_2 - E_1)/[2(E_2 + E_1)]$;
 (c) $\sigma_1/\sigma_2 = E_1/E_2$
- 2.4-12** (a) $P_{\text{allow}} = 1504$ N; (b) $P_{\text{allow}} = 820$ N;
 (c) $P_{\text{allow}} = 703$ N
- 2.4-13** $d_2 = 0.338$ in., $L_2 = 48.0$ in.
- 2.4-14** (a) $A_x = -41.2$ kN, $A_y = -71.4$ kN,
 $B_x = -329$ kN, $B_y = 256$ kN;
 (b) $P_{\text{max}} = 233$ kN
- 2.4-15** (a) $\sigma_c = 10,000$ psi, $\sigma_D = 12,500$ psi;
 (b) $\delta_B = 0.0198$ in.
- 2.4-16** $P_{\text{max}} = 1800$ N
- 2.4-17** $\sigma_s = 3.22$ ksi, $\sigma_b = 1.716$ ksi, $\sigma_c = 1.93$ ksi
- 2.5-1** $\sigma = 11,700$ psi
- 2.5-2** $T = 40.3^\circ\text{C}$
- 2.5-3** $\Delta T = 185^\circ\text{F}$
- 2.5-4** (a) $\Delta T = 24^\circ\text{C}$, $\sigma_{\text{rod}} = 57.6$ MPa;
 (b) Clevis: $\sigma_{bc} = 42.4$ MPa, Washer:
 $\sigma_{bw} = 74.1$ MPa; (c) $d_b = 10.68$ mm
- 2.5-5** (a) $\sigma_c = E\alpha(\Delta T_B)/4$;
 (b) $\sigma_c = E\alpha(\Delta T_B)/[4(EA/kL + 1)]$
- 2.5-6** (a) $N = 51.8$ kN, max. $\sigma_c = 26.4$ MPa,
 $\delta_C = -0.314$ mm; (b) $N = 31.2$ kN,
 max. $\sigma_c = 15.91$ MPa, $\delta_C = -0.546$ mm
- 2.5-7** $\delta = 0.123$ in.
- 2.5-8** $\Delta T = 34^\circ\text{C}$
- 2.5-9** $\tau = 15.0$ ksi
- 2.5-10** $P_{\text{allow}} = 39.5$ kN
- 2.5-11** (a) $T_A = 400$ lb, $T_B = 200$ lb;
 (b) $T_A = 454$ lb, $T_B = 92$ lb; (c) $\Delta T = 153^\circ\text{F}$
- 2.5-12** (a) $\sigma = 98$ MPa; (b) $T = 35^\circ\text{C}$
- 2.5-13** (a) $\sigma = -957$ psi; (b) $F_k = 3006$ lbs (C);
 (c) $\sigma = -2560$ psi
- 2.5-14** $s = PL/6EA$

- 2.5-15** (a) $P_1 = 231 \text{ k}$, $R_A = -55.2 \text{ k}$, $R_B = 55.2 \text{ k}$;
 (b) $P_2 = 145.1 \text{ k}$, $R_A = -55.2 \text{ k}$, $R_B = 55.2 \text{ k}$;
 (c) For P_1 , $\tau_{\max} = 13.39 \text{ ksi}$, for P_2 ,
 $\tau_{\max} = 19.44 \text{ ksi}$; (d) $\Delta T = 65.8^\circ\text{F}$, $R_A = 0$, $R_B = 0$;
 (e) $R_A = -55.2 \text{ k}$, $R_B = 55.2 \text{ k}$
- 2.5-16** (a) $R_A = [-s + \alpha\Delta T(L_1 + L_2)]/[(L_1/EA_1) + (L_2/EA_2) + (L/k_3)]$, $R_D = -R_A$;
 (b) $\delta_B = \alpha\Delta T(L_1) - R_A(L_1/EA_1)$,
 $\delta_C = \alpha\Delta T(L_1 + L_2) - R_A[(L_1/EA_1) + L_2/EA_2]$
- 2.5-17** $T_B = 660 \text{ lb}$, $T_C = 780 \text{ lb}$
- 2.5-18** $P_{\text{allow}} = 1.8 \text{ MN}$
- 2.5-19** (a) $\sigma_p = -0.196 \text{ ksi}$, $\sigma_r = 3.42 \text{ ksi}$;
 (b) $\sigma_b = 2.74 \text{ ksi}$, $\tau_c = 0.285 \text{ ksi}$
- 2.5-20** $\sigma_p = 25.0 \text{ MPa}$
- 2.5-21** $\sigma_p = 2400 \text{ psi}$
- 2.5-22** (a) $P_B = 25.4 \text{ kN}$, $P_s = -P_B$;
 (b) $S_{\text{reqd}} = 25.7 \text{ mm}$; (c) $\delta_{\text{final}} = 0.35 \text{ mm}$
- 2.5-23** (a) $F_x = -0.174 \text{ k}$; (b) $F_1 = 0.174 \text{ k}$;
 (c) $L_f = 12.01 \text{ in.}$; (d) $\Delta T = 141.9^\circ\text{F}$
- 2.5-24** $\sigma_a = 500 \text{ MPa}$ (tension), $\sigma_c = 10 \text{ MPa}$ (compression)
- 2.5-25** (a) $F_k = 0.174 \text{ k}$; (b) $F_t = -0.174 \text{ k}$;
 (c) $L_f = 11.99 \text{ in.}$; (d) $\Delta T = -141.6^\circ\text{F}$
- 2.6-1** $P_{\max} = 42,600 \text{ lb}$
- 2.6-2** $d_{\min} = 6.81 \text{ mm}$
- 2.6-3** $P_{\max} = 24,000 \text{ lb}$
- 2.6-4** (a) $\Delta T_{\max} = -46^\circ\text{C}$; (b) $\Delta T = +9.93^\circ\text{C}$
- 2.6-5** (a) $\tau_{\max} = 10,800 \text{ psi}$; (b) $\Delta T_{\max} = -49.9^\circ\text{F}$;
 (c) $\Delta T = +75.9^\circ\text{F}$
- 2.6-6** (a) $\sigma_x = 84 \text{ MPa}$; (b) $\tau_{\max} = 42 \text{ MPa}$;
 (c) On rotated x face: $\sigma_{x1} = 42 \text{ MPa}$,
 $\tau_{x1y1} = 42 \text{ MPa}$; On rotated y face:
 $\sigma_{y1} = 42 \text{ MPa}$, (d) On rotated x face:
 $\sigma_{x1} = 71.7 \text{ MPa}$, $\tau_{x1y1} = -29.7 \text{ MPa}$
 On rotated y face: $\sigma_{y1} = 12.3 \text{ MPa}$
- 2.6-7** (a) $\sigma_{\max} = 18,000 \text{ psi}$; (b) $\tau_{\max} = 9000 \text{ psi}$
- 2.6-8** (a) Element A: $\sigma_x = 105 \text{ MPa}$ (compression),
 Element B: $\tau_{\max} = 52.5 \text{ MPa}$; (b) $\theta = 33.1^\circ$
- 2.6-9** (a) $\tau_{\max AC} = \frac{\sigma_{AC}}{2} = 1.859 \text{ ksi}$,
 $\tau_{\max AB} = \frac{\sigma_{AB}}{2} = 7.42 \text{ ksi}$,
 $\tau_{\max DC} = \frac{\sigma_{BC}}{2} = -9.41 \text{ ksi}$;
 (b) $P_{\max} = 36.5 \text{ kip}$
- 2.6-10** (a) (1) $\sigma_x = -945 \text{ kPa}$; (2) $\sigma_\theta = -807 \text{ kPa}$,
 $\tau_\theta = 334 \text{ kPa}$; (3) $\sigma_\theta = -472 \text{ kPa}$,
 $\tau_\theta = 472 \text{ kPa}$, $\sigma_{\max} = -945 \text{ kPa}$,
 $\tau_{\max} = -472 \text{ kPa}$; (b) $\sigma_{\max} = -378 \text{ kPa}$,
 $\tau_{\max} = -189 \text{ kPa}$
- 2.6-11** (a) $\tau_{pq} = 11.54 \text{ psi}$; (b) $\sigma_{pq} = -1700 \text{ psi}$, $\sigma(pq + \pi/2) = -784 \text{ psi}$; (c) $P_{\max} = 14,688 \text{ lb}$
- 2.6-12** (a) $\Delta T_{\max} = 31.3^\circ\text{C}$; (b) $\sigma_{pq} = -21.0 \text{ MPa}$
 (compression), $\tau_{pq} = 30 \text{ MPa}$ (CCW);
 (c) $\beta = 0.62$
- 2.6-13** $N_{AC} = 5.77 \text{ kips}$; $d_{\min} = 1.08 \text{ in.}$
- 2.6-14** (a) $\sigma_\theta = 0.57 \text{ MPa}$, $\tau_\theta = -1.58 \text{ MPa}$;
 (b) $\alpha = 33.3^\circ$; (c) $\alpha = 26.6^\circ$
- 2.6-15** (a) $\theta = 35.26^\circ$, $\tau_\theta = -7070 \text{ psi}$;
 (b) $\sigma_{\max} = 15,000 \text{ psi}$, $\tau_{\max} = 7500 \text{ psi}$
- 2.6-16** $\sigma_{\theta 1} = 54.9 \text{ MPa}$, $\sigma_{\theta 2} = 18.3 \text{ MPa}$,
 $\tau_\theta = -31.7 \text{ MPa}$
- 2.6-17** $\sigma_{\max} = 10,000 \text{ psi}$, $\tau_{\max} = 5000 \text{ psi}$
- 2.6-18** (a) $\theta = 30.96^\circ$; (b) $P_{\max} = 1.53 \text{ kN}$
- 2.6-19** (a) $\tau_\theta = 348 \text{ psi}$, $\theta = 20.1^\circ$;
 (b) $\sigma_{x1} = -950 \text{ psi}$, $\sigma_{y1} = -127.6 \text{ psi}$;
 (c) $k_{\max} = 15,625 \text{ lb/in.}$; (d) $L_{\max} = 1.736 \text{ ft}$;
 (e) $\Delta T_{\max} = 92.8^\circ\text{F}$
- 2.7-1** (a) $U = 23P^2L/12EA$; (b) $U = 125 \text{ in.-lb}$
- 2.7-2** (a) $U = 5P^2L/4\pi Ed^2$; (b) $U = 1.036 \text{ J}$
- 2.7-3** $U = 5040 \text{ in.-lb}$
- 2.7-4** (c) $U = P^2L/2EA + PQL/2EA + Q^2L/4EA$
- 2.7-5** Aluminum: 171 psi , 1740 in.
- 2.7-6** (a) $U = P^2L/EA$; (b) $\delta_B = 2PL/EA$
- 2.7-7** (a) $U_1 = 0.0375 \text{ in.-lb}$; (b) $U_2 = 2.57 \text{ in.-lb}$;
 (c) $U_3 = 2.22 \text{ in.-lb}$
- 2.7-8** (a) $U = 5k\delta^2$; (b) $\delta = W/10k$;
 (c) $F_1 = 3W/10$, $F_2 = 3W/20$, $F_3 = W/10$
- 2.7-9** (a) $U = \frac{P^2L}{2Et(b_2 - b_1)} \ln \frac{b_2}{b_1}$;
 (b) $\delta = \frac{PL}{Et(b_2 - b_1)} \ln \frac{b_2}{b_1}$
- 2.7-10** (a) $P_1 = 270 \text{ kN}$; (b) $\delta = 1.321 \text{ mm}$;
 (c) $U = 243 \text{ J}$
- 2.7-11** (a) $x = 2s$, $P = 2(k_1 + k_2)s$;
 (b) $U_1 = (2k_1 + k_2)s^2$
- 2.7-12** (a) $U = 6.55 \text{ J}$; (b) $\delta_C = 168.8 \text{ mm}$
- 2.8-1** (a) $\delta_{\max} = 0.0361 \text{ in.}$; (b) $\sigma_{\max} = 22,600 \text{ psi}$;
 (c) Impact factor = 113
- 2.8-2** (a) $\delta_{\max} = 6.33 \text{ mm}$; (b) $\sigma_{\max} = 359 \text{ MPa}$;
 (c) Impact factor = 160
- 2.8-3** (a) $\delta_{\max} = 0.0312 \text{ in.}$; (b) $\sigma_{\max} = 26,000 \text{ psi}$;
 (c) Impact factor = 130
- 2.8-4** (a) $\delta_{\max} = 215 \text{ mm}$; (b) Impact factor = 3.9
- 2.8-5** (a) $\delta_{\max} = 9.21 \text{ in.}$; (b) Impact factor = 4.6
- 2.8-6** $v = 13.1 \text{ m/s}$
- 2.8-7** $h_{\max} = 8.55 \text{ in.}$
- 2.8-8** $L_{\min} = 9.25 \text{ m}$

- 2.8-9** $L_{\min} = 500$ in.
2.8-10 $V_{\max} = 5.40$ m/s
2.8-11 $\delta_{\max} = 11.0$ in.
2.8-12 $L = 25.5$ m
2.8-13 (a) Impact factor $= 1 + (1 + 2EA/W)^{1/2}$; (b) 10
2.8-14 $\sigma_{\max} = 33.3$ MPa
2.10-1 (a) $\sigma_{\max} \approx 6.2$ ksi and 6.9 ksi;
(b) $\sigma_{\max} \approx 11.0$ ksi and 9.0 ksi
2.10-2 (a) $\sigma_{\max} \approx 26$ MPa and 29 MPa;
(b) $\sigma_{\max} \approx 25$ MPa and 22 MPa
2.10-3 $P_{\max} = \sigma_b t/3$
2.10-4 $\sigma_{\max} \approx 46$ MPa
2.10-5 $\sigma_{\max} \approx 6100$ psi
2.10-6 (a) No, it makes it weaker: $P_1 = 25.1$ kN,
 $P_2 \approx 14.4$ kN; (b) $d_0 \approx 15.1$ mm
2.10-7 $d_{\max} \approx 0.51$ in.
2.11-2 (a) $\delta_C = 1.67$ mm; (b) $\delta_C = 5.13$ mm;
(c) $\delta_C = 11.88$ mm
2.11-3 (b) $P = 17.7$ k
2.11-4 For $P = 30$ kN: $\delta = 6.2$ mm; for $P = 40$ kN:
 $\delta = 12.0$ mm
2.11-5 For $P = 24$ k: $\delta = 0.18$ in.; for $P = 40$ k:
 $\delta = 0.68$ in.
2.11-6 For $P = 3.2$ kN: $\delta_B = 4.85$ mm; for
 $P = 4.8$ kN: $\delta_B = 17.3$ mm
2.12-1 $P_Y = P_P = 2\sigma_Y A \sin \theta$
2.12-2 $P_P = 201$ kN
2.12-3 (a) $P_P = 5\sigma_Y A$
2.12-4 $P_P = 2\sigma_Y A(1 + \sin \alpha)$
2.12-5 $P_P = 47.9$ k
2.12-6 $P_P = 82.5$ kN
2.12-7 $P_P = 20.4$ k
2.12-8 (a) $P_Y = \sigma_Y A$, $\delta_Y = 3\sigma_Y L/2E$;
(b) $P_P = 4\sigma_Y A/3$, $\delta_P = 3\sigma_Y L/E$
2.12-9 (a) $P_Y = \sigma_Y A$, $\delta_Y = \sigma_Y L/E$;
(b) $P_P = 5\sigma_Y A/4$, $\delta_P = 2\sigma_Y L/E$
2.12-10 (a) $W_Y = 28.8$ kN, $\delta_Y = 125$ mm;
(b) $W_P = 48$ kN, $\delta_P = 225$ mm
2.12-11 (a) $P_Y = 70.1$ k, $\delta_Y = 0.01862$ in.;
(b) $P_P = 104.3$ k, $\delta_P = 0.0286$ in.

CHAPTER 3

- 3.2-1** (a) $d_{\max} = 0.413$ in.; (b) $L_{\min} = 21.8$ in.
3.2-2 (a) $L_{\min} = 162.9$ mm; (b) $d_{\max} = 68.8$ mm
3.2-3 (a) $\gamma_1 = 267 \times 10^{-6}$ rad; (b) $r_{2, \min} = 2.2$ in.
3.2-4 (a) $\gamma_1 = 393 \times 10^{-6}$ rad; (b) $r_{2, \max} = 50.9$ mm
3.2-5 (a) $\gamma_1 = 195 \times 10^{-6}$ rad; (b) $r_{2, \max} = 2.57$ in.

- 3.3-1** (a) $\tau_{\max} = 8344$ psi; (b) $d_{\min} = 0.651$ in.
3.3-2 (a) $\tau_{\max} = 23.8$ MPa; (b) $T_{\max} = 0.402$ N·m;
(c) $\theta = 9.12^\circ/\text{m}$
3.3-3 (a) $\tau_{\max} = 18,300$ psi; (b) $\phi = 3.32^\circ$
3.3-4 (a) $k_T = 2059$ N·m; (b) $\tau_{\max} = 27.9$ MPa,
 $\gamma_{\max} = 997 \times 10^{-6}$ radians;
(c) $\frac{k_{T\text{hollow}}}{k_{T\text{solid}}} = 0.938$, $\frac{\tau_{\max H}}{\tau_{\max S}} = 1.067$;
(d) $d_2 = 32.5$ mm
3.3-5 (a) $L_{\min} = 38.0$ in.; (b) $L_{\min} = 40.9$ in.
3.3-6 $T_{\max} = 6.03$ N·m, $\phi = 2.20^\circ$
3.3-7 (a) $\tau_{\max} = 7965$ psi; $\gamma_{\max} = 0.00255$ radians,
 $G = 3.13 \times 10^6$ psi; (b) $T_{\max} = 5096$ lb·in.
3.3-8 (a) $T_{\max} = 9164$ N·m; (b) $T_{\max} = 7765$ N·m;
3.3-9 $\tau_{\max} = 4840$ psi
3.3-10 (a) $d_{\min} = 63.3$ mm; (b) $d_{\min} = 66$ mm (4.2%
increase in diameter)
3.3-11 (a) $\tau_2 = 5170$ psi; (b) $\tau_1 = 3880$ psi;
(c) $\theta = 0.00898^\circ/\text{in.}$
3.3-12 (a) $\tau_2 = 30.1$ MPa; (b) $\tau_1 = 20.1$ MPa;
(c) $\theta = 0.306^\circ/\text{m}$
3.3-13 (a) $d_{\min} = 2.50$ in.; (b) $k_T = 2941 \frac{\text{in} \cdot \text{kip}}{\text{rad}}$
(c) $d_{\min} = 1.996$ in.
3.3-14 (a) $d_{\min} = 64.4$ mm;
(b) $k_T = 134.9$ kN·m/rad; (c) $d_{\min} = 50$ mm
3.3-15 (a) $T_{1, \max} = 4.60$ in.-k; (b) $T_{1, \max} = 4.31$ in.-k;
(c) Torque: 6.25%, Weight: 25%
3.3-16 (a) $\phi = 5.19^\circ$; (b) $d = 88.4$ mm;
(c) Ratio = 0.524
3.3-17 (a) $r_2 = 1.399$ in. (b) $P_{\max} = 1387$ lb
3.4-1 (a) $\tau_{\max} = \tau_{BC} = 7602$ psi, $\phi_C = 0.16^\circ$;
(b) $d_{BC} = 1.966$ in., $\phi_C = -0.177^\circ$
3.4-2 (a) $\tau_{\text{bar}} = 79.6$ MPa, $\tau_{\text{tube}} = 32.3$ MPa;
(b) $\phi_A = 9.43^\circ$
3.4-3 (a) $\tau_{\max} = \tau_{BC} = 4653$ psi, $\phi_D = 0.978^\circ$;
(b) $d_{AB} = 3.25$ in., $d_{BC} = 2.75$ in., $d_{CD} = 2.16$ in.,
 $\phi_D = 1.303^\circ$
3.4-4 $T_{\text{allow}} = 439$ N·m
3.4-5 $d_1 = 0.818$ in.
3.4-6 (a) $d = 77.5$ mm; (b) $d = 71.5$ mm
3.4-7 (a) $d = 1.78$ in.; (b) $d = 1.83$ in.
3.4-8 (b) $d_B/d_A = 1.45$
3.4-9 Minimum $d_A = 2.52$ in.
3.4-10 Minimum $d_B = 48.6$ mm
3.4-11 (a) $R_1 = -3T/2$; (b) $T_1 = 1.5T$, $T_2 = 0.5T$;
(c) $x = 7L/17$; (d) $\phi_2 = (12/17)(TL/GI_p)$

- 3.4-12** $\phi = 3TL/2\pi Gtd^3_A$
3.4-13 (a) $\phi = 2.79^\circ$; (b) $\phi = 2.21^\circ$
3.4-14 (a) $R_1 = \frac{-T}{2}$; (b) $\varphi_3 = \frac{19}{8} \cdot \frac{TL}{\pi Gtd^3}$
3.4-15 $\phi_D = \left| \frac{4Fd}{\pi G} \right| \frac{L_1}{t_{01}d_{01}^3}$

$$+ \int_0^{L_2} \frac{L_2^A}{(d_{01}L_2 - d_{01}x + d_{03}x)^3 (t_{01}L_2 - t_{01}x + t_{01}x)} dx + \frac{L_3}{t_{03}d_{03}^3},$$
 $\phi_D = 0.142^\circ$
3.4-16 (a) $\tau_{\max} = 16tL/\pi d^3$; (b) $\phi = 16tL^2/\pi Gd^4$
3.4-17 (a) $\tau_{\max} = 8t_AL/\pi d^3$; (b) $\phi = 16t_AL^2/3\pi Gd^4$
3.4-18 (a) $R_A = \frac{T_0}{6}$;
 (b) $T_{AB}(x) = \left(\frac{T_0}{6} - \frac{x^2}{L^2} T_0 \right) 0 \leq x \leq \frac{L}{2}$,
 $T_{BC}(x) = -\left[\left(\frac{x-L}{L} \right)^2 \cdot \frac{T_0}{3} \right]$
 $\frac{L}{2} \leq x \leq L$; (c) $\phi_c = \frac{T_0L}{144GI_p}$;
 (d) $\tau_{\max} = \frac{8}{3\pi} \cdot \frac{T_0}{d_{AB}^3}$
3.4-19 (a) $L_{\max} = 4.42$ m; (b) $\phi = 170^\circ$
3.4-20 (a) $T_{\max} = 875$ N·m; (b) $\tau_{\max} = 25.3$ MPa
3.5-1 (a) $\sigma_{\max} = 6280$ psi; (b) $T = 74,000$ lb-in.
3.5-2 (a) $\varepsilon_{\max} = 320 \times 10^{-6}$; (b) $\sigma_{\max} = 51.2$ MPa;
 (c) $T = 20.0$ kN·m
3.5-3 (a) $d_1 = 2.40$ in.; (b) $\phi = 2.20^\circ$;
 (c) $\gamma_{\max} = 1600 \times 10^{-6}$ rad
3.5-4 $G = 30.0$ GPa
3.5-5 $T = 4200$ lb-in.
3.5-6 (a) $d_{\min} = 37.7$ mm; (b) $T_{\max} = 431$ N·m
3.5-7 (a) $d_1 = 0.6$ in.; (b) $d_{1\max} = 0.661$ in.
3.5-8 (a) $d_2 = 79.3$ mm; (b) $d_2 = 80.5$ mm
3.5-9 (a) $\tau_{\max} = 5090$ psi;
 (b) $\gamma_{\max} = 432 \times 10^{-6}$ rad
3.5-10 (a) $\tau_{\max} = 23.9$ MPa;
 (b) $\gamma_{\max} = 884 \times 10^{-6}$ rad
3.5-11 (a) $T_{1\text{allow}} = 17.84$ k-in., $T_{2\text{allow}} = 13.48$ k-in.;
 (b) $L_{\text{mid}} = 18.54$ in.; (c) $d_{3\text{new}} = 2.58$ in.;
 (d) $T_{\max1} = 17.41$ k-in., $T_{\max2} = 13.15$ k-in.,
 $\varphi_{\max1} = 1.487^\circ$, $\varphi_{\max2} = 1.245^\circ$
3.7-1 (a) $\tau_{\max} = 4950$ psi; (b) $d_{\min} = 3.22$ in.
3.7-2 (a) $\tau_{\max} = 50.0$ MPa; (b) $d_{\min} = 32.3$ mm
3.7-3 (a) $H = 6560$ hp; (b) Shear stress is halved.
3.7-4 (a) $\tau_{\max} = 16.8$ MPa; (b) $P_{\max} = 267$ kW
3.7-5 $d_{\min} = 4.28$ in.
3.7-6 $d_{\min} = 110$ mm
3.7-7 Minimum $d_1 = 1.221d$
3.7-8 $P_{\max} = 91.0$ kW
3.7-9 $d = 2.75$ in.
3.7-10 $d = 53.4$ mm
3.8-1 (a) $\phi_{\max} = 3T_0L/5GI_p$; (b) $\varphi_{\max} = \frac{9LT_0}{25GI_p}$
3.8-2 (a) $x = L/4$; (b) $\phi_{\max} = T_0L/8GI_p$
3.8-3 $\phi_{\max} = 2b\tau_{\text{allow}}/Gd$
3.8-4 $P_{\text{allow}} = 2710$ N
3.8-5 (a) $T_{0,\max} = 3678$ lb-in.;
 (b) $T_{0,\max} = 3898$ lb-in.
3.8-6 (a) $T_{0,\max} = 150$ N·m;
 (b) $T_{0,\max} = 140$ N·m
3.8-7 (a) $a/L = d_A/(d_A + d_B)$;
 (b) $a/L = d_A^4/(d_A^4 + d_B^4)$
3.8-8 (a) $T_A = \frac{Lt_0}{6}$, $T_B = \frac{Lt_0}{3}$,
 (b) $\phi_{\max} = \phi\left(\frac{L}{\sqrt{3}}\right) = -\frac{\sqrt{3}L^2t_0}{27GI_p}$
3.8-9 (a) $x = 30.12$ in.; (b) $\phi_{\max} = -1^\circ$
 (at $x = 30.12$ in.)
3.8-10 (a) $\tau_1 = 32.7$ MPa, $\tau_2 = 49.0$ MPa;
 (b) $\phi = 1.030^\circ$; (c) $k_T = 22.3$ kN·m
3.8-11 (a) $\tau_1 = 1790$ psi, $\tau_2 = 2690$ psi;
 (b) $\phi = 0.354^\circ$; (c) $k_T = 809$ k-in.
3.8-12 (a) $T_{\max} = 1.521$ kN·m; (b) $d_2 = 56.9$ mm
3.8-13 (a) $T_{\max} = 9.13$ k-in.; (b) $d_2 = 2.27$ in.
3.8-14 (a) $T_{1,\text{allow}} = 7.14$ kN·m;
 (b) $T_{2,\text{allow}} = 6.35$ kN·m; (c) $T_{3,\text{allow}} = 7.41$ kN·m;
 (d) $T_{\max} = 6.35$ kN·m;
3.8-15 (a) $T_A = 15,292$ in.-lb, $T_B = 24,708$ in.-lb;
 (b) $T_A = 8734$ in.-lb, $T_B = 31,266$ in.-lb
3.8-16 (a) $R_1 = -0.77T$, $R_2 = -0.23T$;
 (b) $T_{\max} = 2.79$ kN·m; (c) $\phi_{\max} = 7.51^\circ$;
 (d) $T_{\max} = 2.48$ kN·m (shear in flange plate bolts
 controls); (e) $R_2 = \frac{\beta}{f_{T1} + f_{T2}}$,

$$R_1 = -R_2, \text{ with } f_{T1} = \frac{L_1}{G_1I_{p1}}, f_{T2} = \frac{L_2}{G_2I_{p2}};$$
 (f) $\beta_{\max} = 29.1^\circ$
3.9-1 (a) $U = 32.0$ in.-lb; (b) $\phi = 0.775^\circ$
3.9-2 (a) $U = 5.36$ J; (b) $\phi = 1.53^\circ$

- 3.9-3 $U = 22.6 \text{ in.-lb}$
 3.9-4 $U = 1.84 \text{ J}$
 3.9-5 (c) $U_3 = T^2 L / 2GI_P + TtL^2 / 2GI_P + t^2 L^3 / 6GI_P$
 3.9-6 $U = 19T_0^2 L / 32GI_P$
 3.9-7 $\phi = T_0 L_A L_B / [G(L_B I_{PA} + L_A I_{PB})]$
 3.9-8 $U = t_0^2 L^3 / 40GI_P$
 3.9-9 (a) $U = \frac{T^2 L (d_A + d_B)}{\pi G t d_A^2 d_B^2}$;
 (b) $w = \frac{2TL(d_A + d_B)}{\pi G t d_A^2 d_B^2}$
 3.9-10 $U = \frac{\beta^2 G I_{PA} I_{PB}}{2L(I_{PA} + I_{PB})}$
 3.9-11 $\phi = \frac{2n}{15d^2} \sqrt{\frac{2\pi I_m L}{G}}; \tau_{\max} = \frac{n}{15d} \sqrt{\frac{2\pi G J_m}{L}}$
 3.11-1 (a) $\tau_{\text{approx}} = 6310 \text{ psi}$; (b) $\tau_{\text{exact}} = 6830 \text{ psi}$
 3.11-2 $t_{\min} = \pi d / 64$
 3.11-3 (a) $\tau = 1250 \text{ psi}$; (b) $\phi = 0.373^\circ$
 3.11-4 (a) $\tau = 9.17 \text{ MPa}$; (b) $\phi = 0.140^\circ$
 3.11-5 $U_1 / U_2 = 2$
 3.11-6 $\tau = 35.0 \text{ MPa}$, $\phi = 0.570^\circ$
 3.11-7 $\tau = 2390 \text{ psi}$, $\theta = 0.00480^\circ/\text{in.}$
 3.11-8 $\tau = T\sqrt{3}/9b^2 t$, $\theta = 2T/9Gb^3 t$
 3.11-9 (a) $\phi_1 / \phi_2 = 1 + 1/4\beta^2$
 3.11-10 $\tau = 2T(1 + \beta)^2 / tL_m^2 \beta$
 3.11-11 $t_{\min} = 0.140 \text{ in.}$
 3.11-12 (a) $t = 6.66 \text{ mm}$; (b) $t = 7.02 \text{ mm}$
 3.12-1 $T_{\max} \approx 6200 \text{ lb-in.}$
 3.12-2 $R_{\min} \approx 4.0 \text{ mm}$
 3.12-3 For $D_1 = 0.8 \text{ in.}$: $\tau_{\max} \approx 6400 \text{ psi}$
 3.12-4 $D_2 \approx 115 \text{ mm}$; lower limit
 3.12-5 $D_1 \approx 1.31 \text{ in.}$

CHAPTER 4

- 4.3-1 $V = 333 \text{ lb}$, $M = 50,667 \text{ lb-in.}$
 4.3-2 $V = -0.938 \text{ kN}$, $M = 4.12 \text{ kN}\cdot\text{m}$
 4.3-3 $V = 0$, $M = 0$
 4.3-4 $V = 7.0 \text{ kN}$, $M = -9.5 \text{ kN}\cdot\text{m}$
 4.3-5 (a) $V = -190 \text{ lb}$, $M = 16,580 \text{ ft-lb}$;
 (b) $q = 370.4 \text{ lb/ft}$ (upward)
 4.3-6 (a) $V = -1.0 \text{ kN}$, $M = -7 \text{ kN}\cdot\text{m}$;
 (b) $P_2 = 4 \text{ kN}$; (c) $P_1 = -8 \text{ kN}$
 (acts to right)
 4.3-7 $b/L = 1/2$
 4.3-8 $M = 108 \text{ N}\cdot\text{m}$
 4.3-9 $N = P \sin \theta$, $V = P \cos \theta$, $M = Pr \sin \theta$
 4.3-10 $V = -6.04 \text{ kN}$, $M = 15.45 \text{ kN}\cdot\text{m}$
 4.3-11 (a) $P = 1200 \text{ lb}$; (b) $P = 133.3 \text{ lb}$
 4.3-12 $V = -4.17 \text{ kN}$, $M = 75 \text{ kN}\cdot\text{m}$
 4.3-13 (a) $V_B = 6000 \text{ lb}$, $M_B = 9000 \text{ lb-ft}$;
 (b) $V_m = 0$, $M_m = 21,000 \text{ lb-ft}$
 4.3-14 (a) $N = 21.6 \text{ kN}$ (compression), $V = 7.2 \text{ kN}$,
 $M = 50.4 \text{ kN}\cdot\text{m}$; (b) $N = 21.6 \text{ kN}$ (compression),
 $V = -5.4 \text{ kN}$, $M = 0$ (at moment release)
 4.3-15 $V_{\max} = 91wL^2\alpha/30g$, $M_{\max} = 229wL^3\alpha/75g$
 4.5-1 $V_{\max} = P$, $M_{\max} = Pa$
 4.5-2 $V_{\max} = M_0/L$, $M_{\max} = M_0 a/L$
 4.5-3 $V_{\max} = qL/2$, $M_{\max} = -3qL^2/8$
 4.5-4 $V_{\max} = P$, $M_{\max} = PL/4$
 4.5-5 $V_{\max} = -2P/3$, $M_{\max} = PL/9$
 4.5-6 $V_{\max} = 2M_1/L$, $M_{\max} = 7M_1/3$
 4.5-7 (a) $V_{\max} = \frac{P}{2}$ (on AB),
 $M_{\max} = R_C \left(\frac{3L}{4} \right) = \frac{3LP}{8}$ (just right of B);
 (b) $N_{\max} = P$ (tension on AB), $V_{\max} = \frac{P}{5}$,
 $M_{\max} = \frac{-P}{5} \cdot \left(\frac{3L}{4} \right) = -\frac{3LP}{20}$ (just right of B)
 4.5-8 (a) $V_{\max} = P$, $M_{\max} = -Pa$;
 (b) $M = 3Pa$ (CCW); $V_{\max} = 2P$, $M_{\max} = 2Pa$
 4.5-9 $V_{\max} = qL/2$, $M_{\max} = 5qL^2/72$
 4.5-10 (a) $V_{\max} = -q_0 L/2$, $M_{\max} = -q_0 L^2/6$;
 (b) $V_{\max} = -\frac{2Lq_0}{3}$, $M_{\max} = -\frac{4L^2q_0}{15}$ (at B)
 4.5-11 $R_B = 207 \text{ lb}$, $R_A = 73.3 \text{ lb}$,
 $V_{\max} = -207 \text{ lb}$, $M_{\max} = 2933 \text{ lb-in.}$
 4.5-12 $V_{\max} = 1200 \text{ N}$, $M_{\max} = 960 \text{ N}\cdot\text{m}$
 4.5-13 $V_{\max} = 200 \text{ lb}$, $M_{\max} = -1600 \text{ lb-ft}$
 4.5-14 $V_{\max} = 4.5 \text{ kN}$, $M_{\max} = -11.33 \text{ kN}\cdot\text{m}$
 4.5-15 $V_{\max} = -1300 \text{ lb}$, $M_{\max} = -28,800 \text{ lb-in.}$
 4.5-16 $V_{\max} = 15.34 \text{ kN}$, $M_{\max} = 9.80 \text{ kN}\cdot\text{m}$
 4.5-17 The first case has the larger maximum
 moment: $\left(\frac{6}{5} PL \right)$
 4.5-18 The third case has the larger maximum
 moment: $\left(\frac{6}{5} PL \right)$
 4.5-19 $V_{\max} = 900 \text{ lb}$, $M_{\max} = -900 \text{ lb-ft}$
 4.5-20 $V_{\max} = -10.0 \text{ kN}$, $M_{\max} = 16.0 \text{ kN}\cdot\text{m}$
 4.5-21 Two cases have the same maximum moment: (PL) .

4.5-22 $V_{\max} = 33.0 \text{ kN}$, $M_{\max} = -61.2 \text{ kN}\cdot\text{m}$

4.5-23 (a) $V_{\max} = -4P/9$, $M_{\max} = 8PL/3$;

(b) $V_{\max} = -4P/5$, $M_{\max} = -4.8PL$

4.5-24 $M_{Az} = -PL$ (clockwise), $A_x = 0$, $A_y = 0$,

$C_y = \frac{1}{12}P$ (upward), $D_y = \frac{1}{6}P$ (upward),

$V_{\max} = P/12$, $M_{\max} = PL$

4.5-25 (a) $R_A = 1.25 \text{ k}$, $R_B = 13.75 \text{ k}$;

(b) $P = 8 \text{ k}$ (upward)

4.5-26 $V_{\max} = 4.6 \text{ kN}$, $M_{\max} = -6.24 \text{ kN}\cdot\text{m}$

4.5-27 (a) $R_A = 197.1 \text{ lb}$, $R_B = 433 \text{ lb}$;

(b) $a = 4.624 \text{ ft}$; (c) $a = 3.143 \text{ ft}$

4.5-28 $V_{\max} = -2.8 \text{ kN}$, $M_{\max} = 1.450 \text{ kN}\cdot\text{m}$

4.5-29 $a = 0.5858L$, $V_{\max} = 0.2929qL$,

$M_{\max} = 0.02145qL^2$

4.5-30 $V_{\max} = 2.5 \text{ kN}$, $M_{\max} = 5.0 \text{ kN}\cdot\text{m}$

4.5-31 (a) $V_{\max} = -R_B = -\frac{Lq_0}{2}$,

$M_{\max} = -M_A = \frac{L^2q_0}{6}$;

(b) $V_{\max} = -R_B = -\frac{2Lq_0}{3}$,

$M_{\max} = -M_A = \frac{4L^2q_0}{15}$

4.5-32 $M_{\max} = 10 \text{ kN}\cdot\text{m}$

4.5-33 $M_{\max} = M_{\text{pos}} = 897.6 \text{ lb}\cdot\text{ft}$ (at $x = 9.6 \text{ ft}$);

$M_{\text{neg}} = -600 \text{ lb}\cdot\text{ft}$ (at $x = 20 \text{ ft}$)

4.5-34 $V_{\max} = -w_0L/3$, $M_{\max} = -w_0L^2/12$

4.5-35 $M_A = -\frac{w_0}{30}L^2$ (clockwise),

$A_x = -3w_0L/10$ (leftward), $A_y = -3w_0L/20$

(downward), $C_y = w_0L/12$ (upward),

$D_y = w_0L/6$ (upward), $V_{\max} = w_0L/4$,

$M_{\max} = -w_0L^2/24$ at B

4.5-36 (a) $x = 9.6 \text{ m}$, $V_{\max} = 28 \text{ kN}$; (b) $x = 4.0 \text{ m}$,

$M_{\max} = 78.4 \text{ kN}\cdot\text{m}$

4.5-37 (a) $A_x = 50.38 \text{ lb}$ (right), $A_y = 210 \text{ lb}$ (upward),

$B_x = -50.38 \text{ lb}$ (left),

$N_{\max} = -214.8 \text{ lb}$, $V_{\max} = -47.5 \text{ lb}$,

$M_{\max} = 270 \text{ lb}\cdot\text{ft}$; (b) $A_x = 0$,

$A_y = 67.5 \text{ lb}$, $B_x = 0$, $B_y = 142.5 \text{ lb}$,

$N_{\max} = 134.4 \text{ lb}$, $V_{\max} = -47.5 \text{ lb}$,

$M_{\max} = 270 \text{ lb}\cdot\text{ft}$

4.5-38 (a) $A_x = -q_0L/2$ (leftward), $A_y = 17q_0L/18$

(upward), $D_x = -q_0L/2$ (leftward),

$D_y = -4q_0L/9$ (downward), $M_D = 0$,

$N_{\max} = q_0L^2$, $V_{\max} = 17q_0L/18$,

$M_{\max} = q_0L^2$; (b) $B_x = q_0L/2$ (rightward),

$B_y = -q_0L/2 + 5q_0L/3 = 7q_0L/6$ (upward);

$D_x = q_0L/2$ (rightward), $D_y = -5q_0L/3$

(downward), $M_D = 0$, $N_{\max} = 5q_0L/3$

$V_{\max} = 5q_0L/3$, $M_{\max} = q_0L^2$

4.5-39 (a) $M_A = 0$, $R_{Ax} = 0$, $R_{Ay} = q_0L/6$

(upward), $R_{Cy} = q_0L/3$; $N_{\max} = -q_0L/6$,

$V_{\max} = -q_0L/3$, $M_{\max} = 0.06415q_0L^2$;

(b) $M_A = (16/15)q_0L^2$, $R_{Ax} = -4q_0L/3$,

$R_{Ay} = q_0L/6$ (upward), $R_{Cy} = q_0L/3$;

$N_{\max} = -q_0L/6$, $V_{\max} = 4q_0L/3$ (in

column), $V_{\max} = -q_0L/3$ (in beam),

$M_{\max} = -(16/15)q_0L^2$ (in column),

$M_{\max} = 0.06415q_0L^2$ (in beam)

4.5-40 $M_A = 0$, $A_x = 0$, $A_y = -18.41 \text{ kN}$ (downward),

$M_D = 0$, $D_x = -63.0 \text{ kN}$ (leftward), $D_y = 62.1 \text{ kN}$

(upward), $N_{\max} = -62.1 \text{ kN}$, $V_{\max} = 63.0 \text{ kN}$,

$M_{\max} = 756 \text{ kN}\cdot\text{m}$

CHAPTER 5

5.4-1 (a) $\varepsilon_{\max} = 8.67 \times 10^{-4}$; (b) $R_{\min} = 9.35 \text{ in.}$;

(c) $d_{\max} = 0.24 \text{ in.}$

5.4-2 (a) $L_{\min} = 5.24 \text{ m}$; (b) $d_{\max} = 4.38 \text{ mm}$

5.4-3 (a) $\varepsilon_{\max} = 5.98 \times 10^{-3}$; (b) $d_{\max} = 4.85 \text{ in.}$;

(c) $L_{\min} = 51 \text{ ft}$

5.4-4 (a) $\rho = 85 \text{ m}$, $\kappa = 0.0118 \frac{1}{\text{m}}$, $\delta = 23.5 \text{ mm}$;

(b) $h_{\max} = 136 \text{ mm}$; (c) $\delta = 75.3 \text{ mm}$

5.4-5 (a) $\varepsilon = 9.14 \times 10^{-4}$; (b) $t_{\max} = 0.241 \text{ in.}$;

(c) $\delta = 0.744 \text{ in.}$; (d) $L_{\max} = 37.1 \text{ in.}$

5.4-6 (a) $\varepsilon = 4.57 \times 10^{-4}$; (b) $L_{\max} = 2 \text{ m}$

5.5-1 (a) $\sigma_{\max} = 52.4 \text{ ksi}$; (b) 33.3%;

(c) $L_{\text{new}} = 120 \text{ in.}$

5.5-2 (a) $\sigma_{\max} = 250 \text{ MPa}$; (b) -19.98%; (c) +25%

5.5-3 (a) $\sigma_{\max} = 38.2 \text{ ksi}$; (b) +10%; (c) +10%

5.5-4 (a) $\sigma_{\max} = 8.63 \text{ MPa}$; (b) $\sigma_{\max} = 6.49 \text{ MPa}$

5.5-5 $\sigma_{\max} = 21.6 \text{ ksi}$

5.5-6 $\sigma_{\max} = 203 \text{ MPa}$

5.5-7 $\sigma_{\max} = 3420 \text{ psi}$

5.5-8 $\sigma_{\max} = 101 \text{ MPa}$

5.5-9 $\sigma_{\max} = 10.82 \text{ ksi}$

5.5-10 $\sigma_{\max} = 7.0 \text{ MPa}$

5.5-11 (a) $\sigma_{\max} = 432 \text{ psi}$; (b) $s = 0.58579L$,

$\sigma_{\min} = 153.7 \text{ psi}$; (c) $s = 0$ or L , $\sigma_{\max} = 896 \text{ psi}$

5.5-12 $\sigma_{\max} = 2.10 \text{ MPa}$

5.5-13 (a) $\sigma_t = 30.93M/d^3$; (b) $\sigma_t = 360M/(73bh^2)$;

(c) $\sigma_t = 85.24M/d^3$

- 5.5-14 $\sigma_{\max} = 10.965M/d^3$
- 5.5-15 (a) $\sigma_{\max} = 21.4$ ksi; (b) $L = 20.9$ ft;
(c) $d = 8.56$ ft
- 5.5-16 (a) $\sigma_t = 35.4$ MPa, $\sigma_c = 61$ MPa;
(b) $d_{\max} = \frac{L}{2}$, $\sigma_t = 37.1$ MPa, $\sigma_c = 64.1$ MPa
- 5.5-17 (a) $\sigma_t = 4.34$ ksi, $\sigma_c = 15.96$ ksi;
(b) $P_{\max} = 214$ lb; (c) 4.28 ft
- 5.5-18 (a) $\sigma_c = 1.456$ MPa, $\sigma_t = 1.514$ MPa;
(b) $\sigma_c = 1.666$ MPa (+14%),
 $\sigma_t = 1.381$ MPa (−9%); (c) $\sigma_c = 0.728$ MPa
(−50%), $\sigma_t = 0.757$ MPa (−50%)
- 5.5-19 (a) $\sigma_t = 7810$ psi, $\sigma_c = 13,885$ psi;
(b) $a = 12.73$ ft
- 5.5-20 $\sigma_{\max} = 3pL^2a_0/t$
- 5.5-21 (a) $\sigma_t = 20,360$ psi, $\sigma_c = 13,188$ psi;
(b) $h = 3.20$ in.; (c) $q = 97.2$ lb/ft,
 $P = 675$ lb
- 5.5-22 $\sigma = 25.1$ MPa, 17.8 MPa, -23.5 MPa
- 5.5-23 $d = 3$ ft, $\sigma_{\max} = 171$ psi, $d = 6$ ft,
 $\sigma_{\max} = 830$ psi
- 5.5-24 (a) $c_1 = 91.7$ mm, $c_2 = 108.3$ mm,
 $I_z = 7.969 \times 10^7$ mm⁴; (b) $\sigma_t = 4659$ kPa (top of
beam at C), $\sigma_c = 5506$ kPa (bottom of beam at C)
- 5.5-25 (a) $F_{\text{res}} = 104.8$ lb; (b) $\sigma_{\max} = 36.0$ ksi (compres-
sion at base); (c) $\sigma_{\max} = 32.4$ ksi (tension at base)
- 5.6-1 $d_{\min} = 4.00$ in.
- 5.6-2 (a) $d_{\min} = 12.62$ mm; (b) $P_{\max} = 39.8$ N
- 5.6-3 (a) C 15×33.9 ; (b) S 8×18.4 ; (c) W 8×35
- 5.6-4 (a) W 360×39 ; (b) W 250×89
- 5.6-5 (a) S 10×35 ; (b) $P_{\max} = 3152$ lb
- 5.6-6 (a) $b_{\min} = 161.6$ mm; (b) $b_{\min} = 141.2$ mm,
 $\text{area}_{(b)}/\text{area}_{(a)} = 1.145$
- 5.6-7 (a) 2×12 ; (b) $w_{\max} = 137.3$ lb/ft²
- 5.6-8 (a) $s_{\max} = 429$ mm; (b) $h_{\min} = 214$ mm
- 5.6-9 (a) $q_{0,\text{allow}} = 424$ lb/ft; (b) $q_{0,\text{allow}} = 268$ lb/ft
- 5.6-10 $h_{\min} = 30.6$ mm
- 5.6-11 (a) $S_{\text{reqd}} = 15.37$ in.³; (b) S 8×23
- 5.6-12 (a) $d_{\min} = 37.6$ mm; (b) $d_{\min} = 45.2$ mm,
 $\text{area}_{(b)}/\text{area}_{(a)} = 0.635$
- 5.6-13 (a) 4×12 ; (b) $q_{\max} = 14.2$ lb/ft
- 5.6-14 $b = 152$ mm, $h = 202$ mm
- 5.6-15 $b = 10.25$ in.
- 5.6-16 $t = 13.61$ mm
- 5.6-17 $W_1:W_2:W_3:W_4 = 1:1.260:1.408:0.888$
- 5.6-18 (a) $q_{\max} = 6.61$ kN/m; (b) $q_{\max} = 9.37$ kN/m
- 5.6-19 6.57%
- 5.6-20 (a) $b_{\min} = 11.91$ mm; (b) $b_{\min} = 11.92$ mm
- 5.6-21 (a) $s_{\max} = 49.2$ in.; (b) $d = 12.65$ in.
- 5.6-22 (a) $\beta = 1/9$; (b) 5.35%
- 5.6-23 Increase when $d/h > 0.6861$;
decrease when $d/h < 0.6861$
- 5.7-1 (a) $x = L/4$, $\sigma_{\max} = 4PL/9h_A^3$, $\sigma_{\max}/\sigma_B = 2$;
(b) $x = 0.209L$, $\sigma_{\max} = 0.394PL/h_A^3$,
 $\sigma_{\max}/\sigma_B = 3.54$
- 5.7-2 (a) $x = 4$ m, $\sigma_{\max} = 37.7$ MPa,
 $\sigma_{\max}/\sigma_B = 9/8$; (b) $x = 2$ m,
 $\sigma_{\max} = 25.2$ MPa, $\sigma_{\max}/\sigma_m = 4/3$
- 5.7-3 (a) $x = 8$ in., $\sigma_{\max} = 1250$ psi,
 $\sigma_{\max}/\sigma_B = 1.042$; (b) $x = 4.64$ in.,
 $\sigma_{\max} = 1235$ psi, $\sigma_{\max}/\sigma_m = 1.215$
- 5.7-4 (a) $\sigma_A = 210$ MPa; (b) $\sigma_B = 221$ MPa;
(c) $x = 0.625$ m; (d) $\sigma_{\max} = 231$ MPa;
(e) $\sigma_{\max} = 214$ MPa
- 5.7-5 (a) $1 \leq d_B/d_A \leq 1.5$;
(b) $\sigma_{\max} = \sigma_B = 32PL/\pi d_B^3$
- 5.7-6 $h_x = h_B x/L$
- 5.7-7 $b_y = 2b_B x/L$
- 5.7-8 $h_x = h_B \sqrt{x/L}$
- 5.8-2 (a) $\tau_{\max} = 731$ kPa, $\sigma_{\max} = 4.75$ MPa;
(b) $\tau_{\max} = 1462$ kPa, $\sigma_{\max} = 19.01$ MPa
- 5.8-3 (a) $M_{\max} = 25.4$ k-ft; (b) $M_{\max} = 4.95$ k-ft
- 5.8-4 $\tau_{\max} = 500$ kPa
- 5.8-5 $\tau_{\max} = 2400$ psi
- 5.8-6 (a) $L_0 = h(\sigma_{\text{allow}}/\tau_{\text{allow}})$;
(b) $L_0 = (h/2)(\sigma_{\text{allow}}/\tau_{\text{allow}})$
- 5.8-7 (a) $P_{\max} = 1.914$ kip; (b) $P_{\max} = 2.05$ kip
- 5.8-8 (a) $M_{\max} = 72.2$ N·m; (b) $M_{\max} = 9.01$ N·m
- 5.8-9 (a) 8×12 -in. beam; (b) 8×12 -in. beam
- 5.8-10 (a) $P = 38.0$ kN; (b) $P = 35.6$ kN
- 5.8-11 (a) $w_1 = 121$ lb/ft²; (b) $w_2 = 324$ lb/ft²;
(c) $w_{\text{allow}} = 121$ lb/ft²
- 5.8-12 (a) $b = 89.3$ mm; (b) $b = 87.8$ mm
- 5.9-1 $d_{\min} = 5.70$ in.
- 5.9-2 (a) $W = 28.6$ kN; (b) $W = 38.7$ kN
- 5.9-3 (a) $d = 10.52$ in.; (b) $d = 2.56$ in.
- 5.9-4 (a) $q_{0,\text{max}} = 55.7$ kN/m; (b) $L_{\max} = 2.51$ m
- 5.10-1 (a) $\tau_{\max} = 5795$ psi; (b) $\tau_{\min} = 4555$ psi;
(c) $\tau_{\text{aver}} = 5714$ psi; (d) $V_{\text{web}} = 28.25$ k
- 5.10-2 (a) $\tau_{\max} = 28.43$ MPa; (b) $\tau_{\min} = 21.86$ MPa;
(c) $\tau_{\text{aver}} = 27.41$ MPa; (d) $V_{\text{web}} = 119.7$ kN
- 5.10-3 (a) $\tau_{\max} = 4861$ psi; (b) $\tau_{\min} = 4202$ psi;
(c) $\tau_{\text{aver}} = 4921$ psi; (d) $V_{\text{web}} = 9.432$ k
- 5.10-4 (a) $\tau_{\max} = 32.28$ MPa; (b) $\tau_{\min} = 21.45$ MPa;
(c) $\tau_{\text{aver}} = 29.24$ MPa; (d) $V_{\text{web}} = 196.1$ kN

- 5.10-5** (a) $\tau_{\max} = 2634$ psi; (b) $\tau_{\min} = 1993$ psi;
 (c) $\tau_{\text{aver}} = 2518$ psi; (d) $V_{\text{web}} = 20.19$ k
5.10-6 (a) $\tau_{\max} = 28.40$ MPa; (b) $\tau_{\min} = 19.35$ MPa;
 (c) $\tau_{\text{aver}} = 25.97$ MPa; (d) $V_{\text{web}} = 58.63$ kN
5.10-7 $q_{\max} = 1270$ lb/ft
5.10-8 (a) $q_{\max} = 184.7$ kN/m; (b) $q_{\max} = 247$ kN/m
5.10-9 $S = 8 \times 23$
5.10-10 $V = 273$ kN
5.10-11 $\tau_{\max} = 1.42$ ksi, $\tau_{\min} = 1.03$ ksi
5.10-12 $\tau_{\max} = 19.7$ MPa
5.10-13 $\tau_{\max} = 2221$ psi
5.11-1 $V_{\max} = 676$ lb
5.11-2 $V_{\max} = 1.924$ MN
5.11-3 $F = 1994$ lb/in.
5.11-4 $V_{\max} = 10.7$ kN
5.11-5 (a) $s_{\max} = 5.08$ in.; (b) $s_{\max} = 4.63$ in.
5.11-6 (a) $s_A = 78.3$ mm; (b) $s_B = 97.9$ mm
5.11-7 (a) $s_{\max} = 2.77$ in.; (b) $s_{\max} = 1.85$ in.
5.11-8 $s_{\max} = 92.3$ mm
5.11-9 $V_{\max} = 18.30$ k
5.11-10 $s_{\max} = 236$ mm
5.11-11 (a) Case (1); (b) Case (3); (c) Case (1); (d) Case (3)
5.11-12 $s_{\max} = 180$ mm
5.12-1 $\sigma_t = 14,660$ psi, $\sigma_c = -14,990$ psi
5.12-2 $\sigma_t = 5770$ kPa, $\sigma_c = -6668$ kPa
5.12-3 $t_{\min} = 0.477$ in.
5.12-4 $\sigma_t = -11.83$ MPa, $\sigma_c = -12.33$ MPa,
 $t_{\min} = 12.38$ mm
5.12-5 $\sigma_t = 302$ psi, $\sigma_c = -314$ psi
5.12-6 $T_{\max} = 108.6$ kN
5.12-7 $\alpha = \arctan [(d_2^2 + d_1^2)/(4hd_2)]$
5.12-8 (a) $d_{\min} = 8.46$ cm; (b) $d_{\min} = 8.91$ cm
5.12-9 $H_{\max} = 32.2$ ft
5.12-10 $W = 33.3$ kN
5.12-11 (a) $\sigma_t = 87.6$ psi, $\sigma_c = -99.6$ psi;
 (b) $d_{\max} = 28.9$ in.
5.12-12 (a) $b = \pi - d/6$; (b) $b = \pi - d/3$;
 (c) Rectangular post
5.12-13 (a) $\sigma_t = 1900$ psi, $\sigma_c = -1100$ psi;
 (b) Both stresses increase in magnitude.
5.12-14 (a) $\sigma_t = 8P/b^2$, $\sigma_c = -4P/b^2$;
 (b) $\sigma_t = 9.11P/b^2$, $\sigma_c = 6.36P/b^2$
5.12-15 (a) $\sigma_t = 857$ psi, $\sigma_c = -5711$ psi;
 (b) $y_0 = -4.62$ in.; (c) $\sigma_t = 453$ psi,
 $\sigma_c = -2951$ psi, $y_0 = -6.33$ in.
5.12-16 (a) $\sigma_t = 3.27$ MPa, $\sigma_c = -24.2$ MPa;
 (b) $y_0 = -76.2$ mm; (c) $\sigma_t = 1.587$ MPa,
 $\sigma_c = -20.3$ MPa, $y_0 = -100.8$ mm

- 5.12-17** (a) $\sigma_t = 15.48$ ksi; (b) $\sigma_t = 2.91$ ksi
5.12-18 (a) $y_0 = -21.5$ mm; (b) $P = 67.3$ kN;
 (c) $y_0 = 148.3$ mm, $P = 149.6$ kN
5.13-1 (a) $d = 0.50$ in., $\sigma_{\max} = 15,500$ psi;
 (b) $R = 0.10$ in., $\sigma_{\max} \approx 49,000$ psi
5.13-2 (a) $d = 16$ mm, $\sigma_{\max} = 81$ MPa;
 (b) $R = 4$ mm, $\sigma_{\max} \approx 200$ MPa
5.13-3 $b_{\min} \approx 0.24$ in.
5.13-4 $b_{\min} \approx 0.33$ mm
5.13-5 (a) $R_{\min} \approx 0.45$ in.; (b) $d_{\max} = 4.13$ in.

CHAPTER 6

- 6.2-1** $\sigma_{\text{face}} = \pm 1980$ psi, $\sigma_{\text{core}} = \pm 531$ psi
6.2-2 (a) $M_{\max} = 58.7$ kN·m;
 (b) $M_{\max} = 90.9$ kN·m; (c) $t = 7.08$ mm
6.2-3 (a) $M_{\max} = 172$ k-in.; (b) $M_{\max} = 96$ k-in.
6.2-4 (a) $M_{\text{allow, steel}} = \frac{\pi \sigma_s (E_B d_1^4 - E_s d_1^4 + E_s d_2^4)}{32 E_s d_2}$,
 $M_{\text{allow, brass}} = \frac{\pi \sigma_B (E_B d_1^4 - E_s d_1^4 + E_s d_2^4)}{32 E_s d_1}$;
 (b) $M_{\max, \text{brass}} = 1235$ N·m; (c) $d_1 = 33.3$ mm
6.2-5 (a) $\sigma_w = 666$ psi, $\sigma_s = 13,897$ psi;
 (b) $q_{\max} = 665$ lb/ft; (c) $M_{0, \max} = 486$ lb-ft
6.2-6 (a) $M_{\text{allow}} = 768$ N·m; (b) $\sigma_{sa} = 47.9$ MPa,
 $M_{\max} = 1051$ N·m
6.2-7 (a) $\sigma_{\text{face}} = 3610$ psi, $\sigma_{\text{core}} = 4$ psi;
 (b) $\sigma_{\text{face}} = 3630$ psi, $\sigma_{\text{core}} = 0$
6.2-8 (a) $\sigma_{\text{face}} = 14.1$ MPa, $\sigma_{\text{core}} = 0.214$ MPa;
 (b) $\sigma_{\text{face}} = 14.9$ MPa, $\sigma_{\text{core}} = 0$
6.2-9 $\sigma_a = 4120$ psi, $\sigma_c = 5230$ psi
6.2-10 (a) $\sigma_w = 5.1$ MPa (compression),
 $\sigma_s = 37.6$ MPa (tension); (b) $t_s = 3.09$ mm
6.2-11 (a) $\sigma_{\text{plywood}} = 1131$ psi, $\sigma_{\text{pine}} = 969$ psi;
 (b) $q_{\max} = 95.5$ lb/ft
6.2-12 $Q_{0, \max} = 15.53$ kN/m
6.3-1 (a) $M_{\max} = 442$ k-in.; (b) $M_{\max} = 189$ k-in.
6.3-2 $t_{\min} = 15.0$ mm
6.3-3 (a) $q_{\text{allow}} = 454$ lb/ft; (b) $\sigma_{\text{wood}} = 277$ psi,
 $\sigma_{\text{steel}} = 11,782$ psi
6.3-4 (a) $\sigma_B = 60.3$ MPa, $\sigma_w = 7.09$ MPa;
 (b) $t_B = 25.1$ mm, $M_{\max} = 80$ kN·m
6.3-5 $\sigma_a = 1860$ psi, $\sigma_P = 72$ psi
6.3-6 $\sigma_a = 12.14$ MPa, $\sigma_P = 0.47$ MPa
6.3-7 (a) $q_{\text{allow}} = 264$ lb/ft; (b) $q_{\text{allow}} = 280$ lb/ft
6.3-8 (a) $\sigma_s = 93.5$ MPa; (b) $h_s = 5.08$ mm,
 $h_a = 114.92$ mm
6.3-9 $M_{\max} = 81.1$ k-in.

- 6.3-10** $S_A = 50.6 \text{ mm}^3$; Metal A
6.3-11 $\sigma_s = 13,400 \text{ psi}$ (tension), $\sigma_c = 812 \text{ psi}$ (compression).
6.3-12 (a) $\sigma_c = 8.51 \text{ MPa}$, $\sigma_s = 118.3 \text{ MPa}$;
 (b) $M_{\max} = M_c = 172.9 \text{ kN}\cdot\text{m}$;
 (c) $A_s = 2254 \text{ mm}^2$, $M_{\text{allow}} = 167.8 \text{ kN}\cdot\text{m}$
6.3-13 (a) $\sigma_c = 649 \text{ psi}$, $\sigma_s = 15,246 \text{ psi}$;
 (b) $M_{\text{allow}} = M_s = 207 \text{ kip}\cdot\text{ft}$
6.3-14 (a) $M_{\max} = M_s = 10.59 \text{ kN}\cdot\text{m}$;
 (b) $A_s = 1262 \text{ mm}^2$, $M_{\text{allow}} = 15.79 \text{ kN}\cdot\text{m}$
6.3-15 $M_{\text{allow}} = M_w = 12.58 \text{ kip}\cdot\text{ft}$
6.4-1 $\tan \beta = h/b$, so NA lies along other diagonal
6.4-2 $\beta = 51.8^\circ$, $\sigma_{\max} = 17.5 \text{ MPa}$
6.4-3 $\beta = 42.8^\circ$, $\sigma_{\max} = 1036 \text{ psi}$
6.4-4 $\beta = 78.9^\circ$, $\sigma_A = -\sigma_E = 102 \text{ MPa}$,
 $\sigma_B = -\sigma_D = -48 \text{ MPa}$
6.4-5 $\beta = 72.6^\circ$, $\sigma_A = -\sigma_E = 14,554 \text{ psi}$,
 $\sigma_B = -\sigma_D = -4953 \text{ psi}$
6.4-6 $\beta = -79.3^\circ$, $\sigma_{\max} = 8.87 \text{ MPa}$
6.4-7 $\beta = -78.8^\circ$, $\sigma_{\max} = 1660 \text{ psi}$
6.4-8 $\beta = -81.8^\circ$, $\sigma_{\max} = 69.4 \text{ MPa}$
6.4-9 $\beta = 72.9^\circ$, $\sigma_{\max} = 8600 \text{ psi}$
6.4-10 $\beta = 60.6^\circ$, $\sigma_{\max} = 20.8 \text{ MPa}$
6.4-11 (a) $\sigma_A = 45,420 \sin \alpha + 3629 \cos \alpha$ (psi);
 (b) $\tan \beta = 37.54 \tan \alpha$
6.4-12 $\beta = 79.0^\circ$, $\sigma_{\max} = 16.6 \text{ MPa}$
6.4-13 (a) $\beta = -76.2^\circ$, $\sigma_{\max} = 8469 \text{ psi}$;
 (b) $\beta = -79.4^\circ$, $\sigma_{\max} = 8704 \text{ psi}$
6.5-1 $\beta = 83.1^\circ$, $\sigma_t = 5,060 \text{ psi}$,
 $\sigma_c = -10,420 \text{ psi}$
6.5-2 $\beta = 83.4^\circ$, $\sigma_t = 10.5 \text{ MPa}$,
 $\sigma_c = -23.1 \text{ MPa}$
6.5-3 $\beta = 75.6^\circ$, $\sigma_t = 3,080 \text{ psi}$, $\sigma_c = -3450 \text{ psi}$
6.5-4 $\beta = 75.8^\circ$, $\sigma_t = 31.7 \text{ MPa}$, $\sigma_c = -39.5 \text{ MPa}$
6.5-5 (a) $\beta = -28.7^\circ$, $\sigma_t = 4263 \text{ psi}$,
 $\sigma_c = -4903 \text{ psi}$; (b) $\beta = -38.5^\circ$,
 $\sigma_t = 5756 \text{ psi}$, $\sigma_c = -4868 \text{ psi}$
6.5-6 $\beta = 78.1^\circ$, $\sigma_t = 40.7 \text{ MPa}$,
 $\sigma_c = -40.7 \text{ MPa}$
6.5-7 $\beta = 82.3^\circ$, $\sigma_t = 1397 \text{ psi}$, $\sigma_c = -1157 \text{ psi}$
6.5-8 $\beta = 2.93^\circ$, $\sigma_t = 6.56 \text{ MPa}$, $\sigma_c = -6.54 \text{ MPa}$
6.5-9 For $\theta = 0^\circ$: $\sigma_t = -\sigma_c = 2.546M/r^3$; for
 $\theta = 45^\circ$: $\sigma_t = 4.535M/r^3$, $\sigma_c = -3.955M/r^3$;
 for $\theta = 90^\circ$: $\sigma_t = 3.867M/r^3$,
 $\sigma_c = -5.244M/r^3$;
6.5-10 $\beta = -78.9^\circ$, $\sigma_t = 131.1 \text{ MPa}$,
 $\sigma_c = -148.5 \text{ MPa}$
6.5-11 $\beta = -11.7^\circ$, $\sigma_t = 28.0 \text{ ksi}$, $\sigma_c = -24.2 \text{ ksi}$
6.5-12 $\beta = -56.5^\circ$, $\sigma_t = 31.0 \text{ MPa}$, $\sigma_c = -29.0 \text{ MPa}$

- 6.8-1** (a) $\tau_{\max} = 3584 \text{ psi}$; (b) $\tau_B = 430 \text{ psi}$
6.8-2 (a) $\tau_{\max} = 29.7 \text{ MPa}$; (b) $\tau_B = 4.65 \text{ MPa}$
6.8-3 (a) $\tau_{\max} = 3448 \text{ psi}$; (b) $\tau_{\max} = 3446 \text{ psi}$
6.8-4 (a) $\tau_{\max} = 27.04 \text{ MPa}$; (b) $\tau_{\max} = 27.02 \text{ MPa}$
6.9-1 $e = 1.027 \text{ in.}$
6.9-2 $e = 22.1 \text{ mm}$
6.9-6 (b) $e = \frac{63 \pi r}{24\pi + 38} = 1.745r$
6.9-8 (a) $e = \frac{b}{2} \left(\frac{2h + 3b}{h + 3b} \right)$;
 (b) $e = \frac{b}{2} \left(\frac{43h + 48b}{23h + 48b} \right)$
6.10-1 $f = 2(2b_1 + b_2)/(3b_1 + b_2)$
6.10-2 (a) $f = 16\tau_2(r_2^3 - r_1^3)/3\pi(r_2^4 - r_1^4)$;
 (b) $f = 4/\pi$
6.10-3 $q = 1000 \text{ lb/in.}$
6.10-4 (a) 56.7%; (b) $M = 12.3 \text{ kN}\cdot\text{m}$
6.10-5 $f = 1.12$
6.10-6 $f = 1.15$
6.10-7 $Z = 16.98 \text{ in.}^3$, $f = 1.14$
6.10-8 $Z = 1.209 \times 10^6 \text{ mm}^3$, $f = 1.11$
6.10-9 $M_Y = 525 \text{ k}\cdot\text{ft}$, $M_P = 591 \text{ k}\cdot\text{ft}$, $f = 1.13$
6.10-10 $M_Y = 378 \text{ kN}\cdot\text{m}$, $M_P = 427 \text{ kN}\cdot\text{m}$, $f = 1.13$
6.10-11 $M_Y = 4320 \text{ k}\cdot\text{in.}$, $M_P = 5450 \text{ k}\cdot\text{in.}$, $f = 1.26$
6.10-12 $M_Y = 672 \text{ kN}\cdot\text{m}$, $M_P = 878 \text{ kN}\cdot\text{m}$, $f = 1.31$
6.10-13 $M_Y = 1619 \text{ k}\cdot\text{in.}$, $M_P = 1951 \text{ k}\cdot\text{in.}$, $f = 1.21$
6.10-14 $M_Y = 122 \text{ kN}\cdot\text{m}$, $M_P = 147 \text{ kN}\cdot\text{m}$, $f = 1.20$
6.10-15 (a) $M = 5977 \text{ k}\cdot\text{in.}$; (b) 22.4%
6.10-16 (a) $M = 524 \text{ kN}\cdot\text{m}$; (b) 36%
6.10-17 (a) $M = 2551 \text{ k}\cdot\text{in.}$; (b) 7.7%
6.10-18 $Z = 136 \times 10^3 \text{ mm}^3$, $f = 1.79$
6.10-19 $M_P = 1120 \text{ k}\cdot\text{in.}$
6.10-20 $M_P = 295 \text{ kN}\cdot\text{m}$

CHAPTER 7

- 7.2-1** For $\theta = 55^\circ$: $\sigma_{x1} = 4221 \text{ psi}$, $\sigma_{y1} = 4704 \text{ psi}$,
 $\tau_{x1y1} = -3411 \text{ psi}$
7.2-2 For $\theta = 40^\circ$: $\sigma_{x1} = 117.2 \text{ MPa}$,
 $\sigma_{y1} = 62.8 \text{ MPa}$, $\tau_{x1y1} = -10.43 \text{ MPa}$
7.2-3 For $\theta = 30^\circ$: $\sigma_{x1} = -3041 \text{ psi}$,
 $\sigma_{y1} = -8959 \text{ psi}$, $\tau_{x1y1} = -12,725 \text{ psi}$
7.2-4 For $\theta = 52^\circ$: $\sigma_{x1} = -136.6 \text{ MPa}$,
 $\sigma_{y1} = 16.6 \text{ MPa}$, $\tau_{x1y1} = -84 \text{ MPa}$
7.2-5 For $\theta = 50^\circ$: $\sigma_{x1} = -1243 \text{ psi}$,
 $\sigma_{y1} = -6757 \text{ psi}$, $\tau_{x1y1} = 1240 \text{ psi}$
7.2-6 For $\theta = -40^\circ$: $\sigma_{x1} = -5.5 \text{ MPa}$,
 $\sigma_{y1} = -27 \text{ MPa}$, $\tau_{x1y1} = -28.1 \text{ MPa}$

- 7.2-7** For $\theta = 38^\circ$: $\sigma_{x1} = -13,359$ psi,
 $\sigma_{y1} = -3671$ psi, $\tau_{xlyl} = 4960$ psi
- 7.2-8** For $\theta = -40^\circ$: $\sigma_{x1} = -66.5$ MPa,
 $\sigma_{y1} = -6.52$ MPa, $\tau_{xlyl} = -14.52$ MPa
- 7.2-9** Normal stress on seam, 187 psi tension. Shear stress, 163 psi (clockwise)
- 7.2-10** Normal stress on seam, 1440 kPa tension. Shear stress, 1030 kPa (clockwise)
- 7.2-11** $\sigma_w = -125$ psi, $\tau_w = 375$ psi
- 7.2-12** $\sigma_w = 10.0$ MPa, $\tau_w = -5.0$ MPa
- 7.2-13** $\theta = 51.3^\circ$, $\sigma_{y1} = 500$ psi, $\tau_{xlyl} = -1122$ psi
- 7.2-14** $\theta = 36.6^\circ$, $\sigma_{y1} = -9$ MPa,
 $\tau_{xlyl} = -14.83$ MPa
- 7.2-15** For $\theta = -36^\circ$: $\sigma_{x1} = -12,068$ psi,
 $\sigma_{y1} = -4732$ psi, $\tau_{xlyl} = -4171$ psi
- 7.2-16** For $\theta = -50^\circ$: $\sigma_{x1} = 51.4$ MPa,
 $\sigma_{y1} = -14.4$ MPa, $\tau_{xlyl} = -31.3$ MPa
- 7.2-17** $\sigma_y = 3673$ psi, $\tau_{xy} = 1405$ psi
- 7.2-18** $\sigma_y = -77.7$ MPa, $\tau_{xy} = -27.5$ MPa
- 7.2-19** $\sigma_b = -4700$ psi, $\tau_b = 2655$ psi, $\theta_1 = 48.04^\circ$
- 7.3-1** $\sigma_1 = 5868$ psi, $\sigma_2 = 982$ psi, $\theta_{p1} = 8.94^\circ$
- 7.3-2** $\sigma_1 = 119.2$ MPa, $\sigma_2 = 60.8$ MPa,
 $\theta_{p1} = 29.52^\circ$
- 7.3-3** $\sigma_1 = -6333$ psi, $\sigma_2 = -1167$ psi,
 $\theta_{p1} = -23.68^\circ$
- 7.3-4** $\sigma_1 = 53.6$ MPa, $\theta_{p1} = -14.2^\circ$
- 7.3-5** $\sigma_1 = 5771$ psi, $\sigma_2 = 18,029$ psi,
 $\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = -6129$ psi
 $\theta_{p1} = -14.12^\circ$
- 7.3-6** $\tau_{\max} = 24.2$ MPa, $\sigma_{x1} = -14.25$ MPa,
 $\sigma_{y1} = -14.25$ MPa, $\theta_{s1} = 60.53^\circ$
- 7.3-7** $\tau_{\max} = 6851$ psi, $\theta_{s1} = 61.8^\circ$
- 7.3-8** $\tau_{\max} = 26.7$ MPa, $\theta_{s1} = 19.08^\circ$
- 7.3-9** (a) $\sigma_1 = 180$ psi, $\theta_{p1} = -20.56^\circ$;
 (b) $\tau_{\max} = 730$ psi, $\theta_{s1} = -65.56^\circ$
- 7.3-10** (a) $\sigma_1 = 25$ MPa, $\sigma_2 = -130$ MPa;
 (b) $\tau_{\max} = 77.5$ MPa, $\sigma_{\text{ave}} = -52.5$ MPa
- 7.3-11** (a) $\sigma_1 = 2693$ psi, $\sigma_2 = 732$ psi;
 (b) $\tau_{\max} = 980$ psi, $\sigma_{\text{ave}} = 1713$ psi
- 7.3-12** (a) $\sigma_1 = 2262$ kPa, $\theta_{p1} = -13.70^\circ$;
 (b) $\tau_{\max} = 1000$ kPa, $\theta_{s1} = -58.7^\circ$
- 7.3-13** (a) $\sigma_1 = 14,764$ psi, $\theta_{p1} = 7.90^\circ$;
 (b) $\tau_{\max} = 6979$ psi, $\theta_{s1} = -37.1^\circ$
- 7.3-14** (a) $\sigma_1 = 29.2$ MPa, $\theta_{p1} = -17.98^\circ$;
 (b) $\tau_{\max} = 66.4$ MPa, $\theta_{s1} = -63.0^\circ$
- 7.3-15** (a) $\sigma_1 = -1228$ psi, $\theta_{p1} = 24.7^\circ$;
 (b) $\tau_{\max} = 5922$ psi, $\theta_{s1} = -20.3^\circ$
- 7.3-16** (a) $\sigma_1 = 76.3$ MPa, $\theta_{p1} = 107.5^\circ$;
 (b) $\tau_{\max} = 101.3$ MPa, $\theta_{s1} = -62.5^\circ$
- 7.3-17** 3030 psi $\leq \sigma_y \leq 9470$ psi
- 7.3-18** 18.5 MPa $\leq \sigma_y \leq 85.5$ MPa
- 7.3-19** (a) $\sigma_y = 3961$ psi; (b) $\theta_{p1} = -38.93^\circ$,
 $\sigma_1 = 6375$ psi, $\theta_{p2} = 51.07^\circ$, $\sigma_2 = 2386$ psi
- 7.3-20** (a) $\sigma_y = 23.3$ MPa; (b) $\theta_{p1} = 65.6^\circ$,
 $\sigma_1 = 41$ MPa, $\theta_{p2} = -24.4^\circ$, $\sigma_2 = -62.7$ MPa
- 7.4-1** (a) $\sigma_{x1} = 10,901$ psi, $\sigma_{y1} = 3349$ psi,
 $\tau_{xlyl} = -6042$ psi; (b) $\tau_{\max} = 7125$ psi,
 $\sigma_{\text{ave}} = 7125$ psi
- 7.4-2** (a) $\sigma_{x1} = 40.1$ MPa, $\sigma_{y1} = 16.91$ MPa,
 $\tau_{xlyl} = 26$ MPa; (b) $\tau_{\max} = 28.5$ MPa,
 $\sigma_{\text{ave}} = 28.5$ MPa
- 7.4-3** (a) $\sigma_{x1} = -5400$ psi, $\sigma_{y1} = -1350$ psi,
 $\tau_{xlyl} = 2700$ psi; (b) $\tau_{\max} = -3375$ psi,
 $\sigma_{\text{aver}} = -3375$ psi
- 7.4-4** For $\theta = 25^\circ$: (a) $\sigma_{x1} = -36.0$ MPa,
 $\tau_{xlyl} = 25.7$ MPa; (b) $\tau_{\max} = 33.5$ MPa,
 $\theta_{s1} = 45.0^\circ$
- 7.4-5** For $\theta = 55^\circ$: (a) $\sigma_{x1} = 882$ psi, $\tau_{xlyl} = -3759$ psi,
 $\sigma_{y1} = 3618$ psi, $\sigma_{\text{aver}} = 2250$ psi;
 (b) $\tau_{\max} = 4000$ psi, $\theta_{x1} = -45.0^\circ$
- 7.4-6** For $\theta = 21.80^\circ$: (a) $\sigma_{x1} = -17.1$ MPa,
 $\tau_{xlyl} = 29.7$ MPa; (b) $\tau_{\max} = 43.0$ MPa,
 $\theta_{x1} = 45.0^\circ$
- 7.4-7** For $\theta = 52^\circ$: (a) $\sigma_{x1} = 2620$ psi,
 $\tau_{xlyl} = -653$ psi; (b) $\sigma_1 = 2700$ psi,
 $\theta_{p1} = 45.0^\circ$
- 7.4-8** (a) $\sigma_{x1} = -60.8$ MPa, $\sigma_{y1} = 128.8$ MPa,
 $\tau_{xlyl} = -46.7$ MPa; (b) $\sigma_1 = 139.6$ MPa,
 $\sigma_2 = -71.6$ MPa, $\tau_{\max} = 105.6$ MPa
- 7.4-9** For $\theta = 36.87^\circ$: (a) $\sigma_{x1} = 3600$ psi,
 $\tau_{xlyl} = 1050$ psi; (b) $\sigma_1 = 3750$ psi, $\theta_{p1} = 45.0^\circ$
- 7.4-10** For $\theta = 40^\circ$: $\sigma_{x1} = 27.5$ MPa,
 $\tau_{xlyl} = -5.36$ MPa
- 7.4-11** For $\theta = -51^\circ$: $\sigma_{x1} = 11,982$ psi
 $\tau_{xlyl} = -3569$ psi
- 7.4-12** For $\theta = -33^\circ$: $\sigma_{x1} = 61.7$ MPa,
 $\tau_{xlyl} = -51.7$ MPa, $\sigma_{y1} = -171.3$ MPa
- 7.4-13** For $\theta = 14^\circ$: $\sigma_{x1} = -1509$ psi,
 $\tau_{xlyl} = 527$ psi, $\sigma_{y1} = -891$ psi
- 7.4-14** For $\theta = 35^\circ$: $\sigma_{x1} = 46.4$ MPa,
 $\tau_{xlyl} = -9.81$ MPa
- 7.4-15** For $\theta = 65^\circ$: $\sigma_{x1} = -1846$ psi,
 $\tau_{xlyl} = 3897$ psi
- 7.4-16** (a) $\sigma_1 = 10,865$ kPa, $\theta_{p1} = 115.2^\circ$;
 (b) $\tau_{\max} = 4865$ kPa, $\theta_{s1} = 70.2^\circ$

- 7.4-17 (a) $\sigma_1 = 2565$ psi, $\theta_{p1} = 31.3^\circ$;
(b) $\tau_{\max} = 3265$ psi, $\theta_{s1} = -13.70^\circ$
- 7.4-18 (a) $\sigma_1 = 18.2$ MPa, $\theta_{p1} = 123.3^\circ$;
(b) $\tau_{\max} = 15.4$ MPa, $\theta_{s1} = 78.3^\circ$
- 7.4-19 (a) $\sigma_1 = -6923$ psi, $\theta_{p1} = -32.4^\circ$;
(b) $\tau_{\max} = 7952$ psi, $\theta_{s1} = 102.6^\circ$
- 7.4-20 (a) $\sigma_1 = 40.0$ MPa, $\theta_{p1} = 68.8^\circ$;
(b) $\tau_{\max} = 40.0$ MPa, $\theta_{s1} = 23.8^\circ$
- 7.4-21 (a) $\sigma_1 = 7490$ psi, $\theta_{p1} = 63.2^\circ$;
(b) $\tau_{\max} = 3415$ psi, $\theta_{s1} = -18.20^\circ$
- 7.4-22 (a) $\sigma_1 = 3.43$ MPa, $\theta_{p1} = -19.68^\circ$;
(b) $\tau_{\max} = 15.13$ MPa, $\theta_{s1} = -64.7^\circ$
- 7.4-23 (a) $\sigma_1 = 7525$ psi, $\theta_{p1} = 9.80^\circ$;
(b) $\tau_{\max} = 3875$ psi, $\theta_{s1} = -35.2^\circ$
- 7.5-1 $\sigma_x = 25,385$ psi, $\sigma_y = 19,615$ psi,
 $\Delta t = -2.81 \times 10^{-4}$ in.
- 7.5-2 $\sigma_x = 102.6$ MPa, $\sigma_y = -11.21$ MPa,
 $\Delta t = -1.646 \times 10^{-3}$ mm
- 7.5-3 (a) $\varepsilon_z = -\nu(\varepsilon_x + \varepsilon_y)/(1 - \nu)$;
(b) $e = (1 - 2\nu)(\varepsilon_x + \varepsilon_y)/(1 - \nu)$
- 7.5-4 $\nu = 0.24$, $E = 112.1$ GPa
- 7.5-5 $\nu = 0.3$, $E = 29,560$ ksi
- 7.5-6 (a) $\gamma_{\max} = 5.85 \times 10^{-4}$;
(b) $\Delta t = -1.32 \times 10^{-3}$ mm;
(c) $\Delta V = 387$ mm³
- 7.5-7 (a) $\gamma_{\max} = 1900 \times 10^{-6}$;
(b) $\Delta t = -141 \times 10^{-6}$ in. (decrease);
(c) $\Delta V = 0.0874$ in.³ (increase)
- 7.5-8 (a) $\Delta V_b = -49.2$ mm³, $U_b = 3.52$ J;
(b) $\Delta V_a = -71.5$ mm³, $U_a = 4.82$ J
- 7.5-9 $\Delta V = -0.0377$ in.³, $U = 55.6$ in.-lb
- 7.5-10 (a) $\Delta V = 2766$ mm³, $U = 56$ J;
(b) $t_{\max} = 36.1$ mm; (c) $b_{\min} = 640$ mm
- 7.5-11 (a) $\Delta V = 0.0385$ in.³, $U = 574$ lb-in.;
(b) $t_{\max} = 0.673$ in.; (c) $b_{\min} = 10.26$ in.
- 7.5-12 (a) $\Delta ac = \varepsilon_x d = 0.1296$ mm (increase);
(b) $\Delta bc = \varepsilon_y d = -0.074$ mm (decrease);
(c) $\Delta t = \varepsilon_z t = -2.86 \times 10^{-3}$ mm (decrease); (d) $\Delta V = eV_0 = 430$ mm³;
(e) $U = uV_0 = 71.2$ N·m; (f) $t_{\max} = 22.0$ mm;
(g) $\sigma_{x\max} = 63.9$ MPa
- 7.6-1 (a) $\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 8750$ psi;
(b) $\Delta a = a\varepsilon_x = 7.73 \times 10^{-3}$ in.,
 $\Delta b = \varepsilon_y b = -3.75 \times 10^{-3}$ in.,
 $\Delta c = \varepsilon_z c = -1.3 \times 10^{-3}$ in.;
- (c) $\Delta V = eV_0 = 0.0173$ in.³;
(d) $U = uV_0 = 964$ in.-lb;
(e) $\sigma_{x\max} = 12,824$ psi;
(f) $\sigma_{x\max} = 11,967$ psi
- 7.6-2 (a) $\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 8.5$ MPa;
(b) $\Delta a = a\varepsilon_x = -0.0525$ mm,
 $\Delta b = \varepsilon_y b = -9.67 \times 10^{-3}$ mm,
 $\Delta c = \varepsilon_z c = -9.67 \times 10^{-3}$ mm;
(c) $\Delta V = eV_0 = -2.052 \times 10^3$ mm³;
(d) $U = uV_0 = 56.2$ N·m;
(e) $\sigma_{x\max} = -50$ MPa;
(f) $\sigma_{x\max} = -65.1$ MPa
- 7.6-3 (a) $\sigma_x = -4200$ psi, $\sigma_y = -2100$ psi,
 $\sigma_z = -2100$ psi;
(b) $\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 1050$ psi;
(c) $\Delta V = eV_0 = -0.0192$ in.³;
(d) $U = uV_0 = 35.3$ in.-lb;
(e) $\sigma_{x\max} = -3864$ psi
(f) $\varepsilon_{x\max} = -235 \cdot (10^{-6})$
- 7.6-4 (a) $\sigma_x = -82.6$ MPa, $\sigma_y = -54.7$ MPa,
 $\sigma_z = -54.7$ MPa;
(b) $\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 13.92$ MPa;
(c) $\Delta V = eV_0 = -846$ mm³;
(d) $U = uV_0 = 29.9$ N·m;
(e) $\sigma_{x\max} = -73$ MPa
(f) $\varepsilon_{x\max} = -741 \cdot (10^{-6})$
- 7.6-5 (a) $K_{A1} = 1 \times 10^7$ psi; (b) $E = 6139$ ksi,
 $\nu = 0.35$
- 7.6-6 (a) $K = 4.95$ GPa; (b) $E = 1.297$ GPa, $\nu = 0.40$
- 7.6-7 (a) $p = \nu F/[A(1 - \nu)]$;
(b) $\delta = FL(1 + \nu)(1 - 2\nu)/EA(1 - \nu)$
- 7.6-8 (a) $p = \nu p_0$; (b) $e = -p_0(1 + \nu)(1 - 2\nu)/E$;
(c) $u = p_0^2(1 - \nu^2)/2E$
- 7.6-9 (a) $\Delta d = 1.472 \times 10^{-3}$ in., $\Delta V = 0.187$ in.³,
 $U = 332$ in.-lb; (b) $h = 5282$ ft
- 7.6-10 (a) $p = 700$ MPa; (b) $K = 175$ GPa;
(c) $U = 2470$ J
- 7.6-11 $\varepsilon_0 = 276 \times 10^{-6}$, $e = 828 \times 10^{-6}$,
 $u = 4.97$ psi
- 7.7-1 (a) $\Delta d = 1.878 \times 10^{-3}$ in.;
(b) $\Delta \phi = -\alpha = 1.425 \times 10^{-4}$ (decrease, radians);
(c) $\Delta \psi = -\alpha = 1.425 \times 10^{-4}$ (increase, radians)

- 7.7-2 (a) $\Delta d = \varepsilon_{x1} L_d = 0.062$ mm;
 (b) $\Delta \phi = -\alpha = 1.89 \times 10^{-4}$ (decrease, radians);
 (c) $\Delta \psi = -\alpha = 1.89 \times 10^{-4}$ (increase, radians)

- 7.7-3 (a) $\Delta d = 0.00458$ in. (increase);
 (b) $\Delta \phi = 157 \times 10^{-6}$ rad (decrease);
 (c) $\gamma = -314 \times 10^{-6}$ rad (angle *ced* increases)

- 7.7-4 (a) $\Delta d = 0.168$ mm (increase);
 (b) $\Delta \phi = 317 \times 10^{-6}$ rad (decrease);
 (c) $\gamma = -634 \times 10^{-6}$ rad (angle *ced* increases)

- 7.7-5 $\varepsilon_{x1} = 3.97 \times 10^{-4}$, $\varepsilon_{y1} = 3.03 \times 10^{-4}$,
 $\gamma_{x1y1} = 1.829 \times 10^{-4}$

- 7.7-6 $\varepsilon_{x1} = 9.53 \times 10^{-5}$, $\varepsilon_{y1} = -1.353 \times 10^{-4}$,
 $\gamma_{x1y1} = -3.86 \times 10^{-4}$

- 7.7-7 $\varepsilon_1 = 554 \times 10^{-6}$, $\theta_{p1} = -22.9^\circ$,
 $\gamma_{\max} = 488 \times 10^{-6}$

- 7.7-8 $\varepsilon_1 = 172 \times 10^{-6}$, $\theta_{p1} = 163.9^\circ$,
 $\gamma_{\max} = 674 \times 10^{-6}$

- 7.7-9 For $\theta = 75^\circ$: (a) $\varepsilon_{x1} = 202 \times 10^{-6}$,
 $\gamma_{x1y1} = -569 \times 10^{-6}$; (b) $\varepsilon_1 = 568 \times 10^{-6}$,
 $\theta_{p1} = 22.8^\circ$; (c) $\gamma_{\max} = 587 \times 10^{-6}$

- 7.7-10 For $\theta = 45^\circ$: (a) $\varepsilon_{x1} = -385 \times 10^{-6}$,
 $\gamma_{x1y1} = 690 \times 10^{-6}$;
 (b) $\varepsilon_1 = -254 \times 10^{-6}$, $\theta_{p1} = 65.7^\circ$;
 (c) $\gamma_{\max} = 1041 \times 10^{-6}$

$$7.7-11 \quad \tau_{\max xy} = \frac{\sigma_x - \sigma_y}{2} = 4076 \text{ psi},$$

$$\gamma_{xy\max} = 2\sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 6.83 \times 10^{-4},$$

$$\gamma_{xz\max} = 2\sqrt{\left(\frac{\varepsilon_x - \varepsilon_z}{2}\right)^2 + \gamma_{xz}^2} = 8.96 \times 10^{-4},$$

$$\gamma_{yz\max} = 2\sqrt{\left(\frac{\varepsilon_y - \varepsilon_z}{2}\right)^2 + \gamma_{yz}^2} = 2.13 \times 10^{-4}$$

$$7.7-12 \quad \tau_{\max xy} = \frac{\sigma_x - \sigma_y}{2} = 33.7 \text{ MPa},$$

$$\gamma_{xy\max} = 2\sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 1.244 \times 10^{-3},$$

$$\gamma_{xz\max} = 2\sqrt{\left(\frac{\varepsilon_x - \varepsilon_z}{2}\right)^2 + \gamma_{xz}^2}$$

$$= 1.459 \times 10^{-3},$$

$$\gamma_{yz\max} = 2\sqrt{\left(\frac{\varepsilon_y - \varepsilon_z}{2}\right)^2 + \gamma_{yz}^2}$$

$$= 2.15 \times 10^{-4}$$

- 7.7-13 For $\theta = 30^\circ$: (a) $\varepsilon_{x1} = -756 \times 10^{-6}$,
 $\gamma_{x1y1} = 868 \times 10^{-6}$; (b) $\varepsilon_1 = 426 \times 10^{-6}$,
 $\theta_{p1} = 99.8^\circ$; (c) $\gamma_{\max} = 1342 \times 10^{-6}$

- 7.7-14 For $\theta = 50^\circ$: (a) $\varepsilon_{x1} = -1469 \times 10^{-6}$,
 $\gamma_{x1y1} = -717 \times 10^{-6}$; (b) $\varepsilon_1 = -732 \times 10^{-6}$,
 $\theta_{p1} = 166.0^\circ$; (c) $\gamma_{\max} = 911 \times 10^{-6}$

- 7.7-15 $\varepsilon_1 = 551 \times 10^{-6}$, $\theta_{p1} = 12.5^\circ$,
 $\gamma_{\max} = 662 \times 10^{-6}$

- 7.7-16 $\varepsilon_1 = 332 \times 10^{-6}$, $\theta_{p1} = 12.0^\circ$,
 $\gamma_{\max} = 515 \times 10^{-6}$

- 7.7-17 (a) $P = 5154$ lb, $T = -978$ in.-lb;
 (b) $\gamma_{\max} = 2.84 \times 10^{-4}$, $\tau_{\max} = 3304$ psi

- 7.7-18 $P = 121.4$ kN, $\alpha = 56.7^\circ$

- 7.7-19 $P = 9726$ lb, $\alpha = 75.2^\circ$

- 7.7-20 $\varepsilon_x = \varepsilon_a$, $\varepsilon_y = (2\varepsilon_b + 2\varepsilon_c - \varepsilon_a)/3$,
 $\gamma_{xy} = 2(\varepsilon_b - \varepsilon_c)/\sqrt{3}$

- 7.7-21 For $\theta_{p1} = 30^\circ$: $\varepsilon_1 = 1550 \times 10^{-6}$,
 $\varepsilon_2 = -250 \times 10^{-6}$, $\sigma_1 = 10,000$ psi,
 $\sigma_2 = 2000$ psi

- 7.7-22 $\sigma_x = 91.6$ MPa

- 7.7-23 $\varepsilon_{x1} = 3.97 \times 10^{-4}$, $\varepsilon_{y1} = 3.03 \times 10^{-4}$,
 $\gamma_{x1y1} = 1.829 \times 10^{-4}$

- 7.7-24 $\varepsilon_{x1} = 9.53 \times 10^{-5}$, $\varepsilon_{y1} = -1.353 \times 10^{-4}$,
 $\gamma_{x1y1} = -3.86 \times 10^{-4}$

- 7.7-25 $\varepsilon_1 = 554 \times 10^{-6}$, $\theta_{p1} = 157.1^\circ$,
 $\gamma_{\max} = 488 \times 10^{-6}$

- 7.7-26 $\varepsilon_1 = 172 \times 10^{-6}$, $\theta_{p1} = 163.9^\circ$,
 $\gamma_{\max} = 674 \times 10^{-6}$

- 7.7-27 For $\theta = 75^\circ$: (a) $\varepsilon_{x1} = 202 \times 10^{-6}$,
 $\gamma_{x1y1} = -569 \times 10^{-6}$; (b) $\varepsilon_1 = 568 \times 10^{-6}$,
 $\theta_{p1} = 22.8^\circ$; (c) $\gamma_{\max} = 587 \times 10^{-6}$

- 7.7-28 For $\theta = 45^\circ$: (a) $\varepsilon_{x1} = -385 \times 10^{-6}$,
 $\gamma_{x1y1} = 690 \times 10^{-6}$; (b) $\varepsilon_1 = -254 \times 10^{-6}$,
 $\theta_{p1} = 65.7^\circ$; (c) $\gamma_{\max} = 1041 \times 10^{-6}$

CHAPTER 8

- 8.2-1 (a) Use $t = 2.5$ in. (b) $p_{\max} = 381$ psi

- 8.2-2 (a) Use $t = 98$ mm. (b) $p_{\max} = 3.34$ MPa

- 8.2-3 (a) $F = 1073$ lb, $\sigma = 255$ psi;
 (b) $d_b = 0.286$ in.; (c) $r = 7.35$ in.

- 8.2-4 (a) $\sigma_{\max} = 3.12$ MPa, $\varepsilon_{\max} = 0.438$;
 (b) $t_{\text{reqd}} = 1.29$ mm

- 8.2-5 (a) $\sigma_{\max} = 425$ psi, $\varepsilon_{\max} = 1.105$;
 (b) $p_{\max} = 7.77$ psi

- 8.2-6 (a) $p_{\max} = 3.51$ MPa; (b) $p_{\max} = 2.93$ MPa

- 8.2-7 (a) $f = 26.4$ k/in.; (b) $\tau_{\max} = 7543$ psi;
 (c) $\varepsilon_{\max} = 3.57 \times 10^{-4}$

- 8.2-8 (a) $f = 5.5$ MN/m; (b) $\tau_{\max} = 57.3$ MPa;
 (c) $\varepsilon_{\max} = 3.87 \times 10^{-4}$

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- 8.2-9 (a) $t_{\min} = 0.291$ in.; (b) $p = 1904$ psi
 8.2-10 (a) $t_{\min} = 7.17$ mm; (b) $p = 19.25$ MPa
 8.2-11 $D_0 = 90$ ft
 8.3-1 (a) $t_{\min} = 0.289$ in.; (b) $p_{\max} = 2286$ psi
 8.3-2 (a) $h = 22.2$ m; (b) zero
 8.3-3 $n = 2.25$
 8.3-4 (a) $F = 3\pi pr^2$; (b) $t_{\text{reqd}} = 10.91$ mm
 8.3-5 (a) $p = 55$ psi; (b) $\varepsilon_r = 9.18 \times 10^{-4}$
 8.3-6 (a) $\varepsilon_{\max} = 6.67 \times 10^{-5}$; (b) $\varepsilon_r = 2.83 \times 10^{-4}$
 8.3-7 $t_{\min} = 0.113$ in.
 8.3-8 $t_{\min} = 3.71$ mm
 8.3-9 (a) $h = 25$ ft; (b) $\sigma_1 \approx 125$ psi
 8.3-10 (a) $\sigma_h = 24.9$ MPa; (b) $\sigma_c = 49.7$ MPa;
 (c) $\sigma_w = 24.9$ MPa; (d) $\tau_h = 12.43$ MPa;
 (e) $\tau_c = 24.9$ MPa
 8.3-11 (a) $t_{\min} = 0.675$ in.; (b) $t_{\min} = 0.338$ in.
 8.3-12 (a) $\sigma_1 = 93.3$ MPa, $\sigma_2 = 46.7$ MPa;
 (b) $\tau_1 = 23.2$ MPa, $\tau_2 = 46.7$ MPa;
 (c) $\varepsilon_1 = 3.97 \times 10^{-4}$, $\varepsilon_2 = 9.33 \times 10^{-5}$;
 (d) $\theta = 35^\circ$, $\sigma_{x_1} = 62.0$ MPa,
 $\sigma_{y_1} = 78.0$ MPa, $\tau_{x_1y_1} = 21.9$ MPa
 8.3-13 (a) $\sigma_1 = 7015$ psi, $\sigma_2 = 3508$ psi;
 (b) $\tau_1 = 1754$ psi, $\tau_2 = 3508$ psi;
 (c) $\varepsilon_1 = 1.988 \times 10^{-4}$, $\varepsilon_2 = 4.68 \times 10^{-5}$;
 (d) $\theta = 28^\circ$, $\sigma_{x_1} = 4281$ psi, $\sigma_{y_1} = 6242$ psi,
 $\tau_{x_1y_1} = 1454$ psi
 8.4-1 (a) $\sigma_y = 0$, $\sigma_x = \frac{-M\left(d - \frac{h}{2}\right)}{I} = -5$ ksi,
 $\tau_{xy} = \frac{VQ}{Ib} = 1.111$ ksi, $\sigma_1 = -5.24$ ksi,
 $\theta_{p1} = -11.98^\circ$, $\sigma_2 = 0.236$ ksi, $\theta_{p2} = 78.02^\circ$;
 $\tau_{\max} = 2.74$ ksi
 (b) $\sigma_y = 0$,
 $\sigma_x = \frac{-N}{A} - \frac{M\left(d - \frac{h}{2}\right)}{I}$
 $= -5.95$ ksi, $\tau_{xy} = \frac{VQ}{Ib} = 1.11$ ksi,
 $\sigma_1 = -6.15$ ksi, $\theta_{p1} = -10.24^\circ$,
 $\sigma_2 = 0.201$ ksi, $\theta_{p2} = 79.76^\circ$; $\tau_{\max} = 3.18$ ksi
 8.4-2 (a) $\sigma_y = 0$,
 $\sigma_x = \frac{-M\left(d - \frac{h}{2}\right)}{I} = -37.4$ MPa,

- $\tau_{xy} = \frac{VQ}{Ib} = 7.49$ MPa,
 $\sigma_1 = -38.9$ MPa, $\theta_{p1} = -10.9^\circ$,
 $\sigma_2 = 1.442$ MPa, $\theta_{p2} = 79.1^\circ$,
 $\tau_{\max} = 20.2$ MPa;
 (b) $\sigma_y = 0$,
 $\sigma_x = \frac{-N}{A} - \frac{M\left(d - \frac{h}{2}\right)}{I} = -44.4$ MPa,
 $\tau_{xy} = \frac{VQ}{Ib} = 7.49$ MPa,
 $\sigma_1 = -45.7$ MPa, $\theta_{p1} = -9.3^\circ$,
 $\sigma_2 = 1.227$ MPa, $\theta_{p2} = 80.7^\circ$,
 $\tau_{\max} = 23.4$ MPa
 8.4-3 (a) $\sigma_1 = 219$ psi, $\sigma_2 = -219$ psi,
 $\tau_{\max} = 219$ psi; (b) $\sigma_1 = 49.6$ psi,
 $\sigma_2 = -762$ psi, $\tau_{\max} = 406$ psi;
 (c) $\sigma_1 = 0$ psi, $\sigma_2 = -2139$ psi,
 $\tau_{\max} = 1069$ psi
 8.4-4 $P = 20$ kN
 8.4-5 $P = 2.91$ k
 8.4-6 (b) $\sigma_1 = 4.5$ MPa, $\sigma_2 = -76.1$ MPa,
 $\tau_{\max} = 40.3$ MPa
 8.4-7 (b) $\sigma_1 = 14,100$ psi, $\sigma_2 = -220$ psi,
 $\tau_{\max} = 7160$ psi
 8.4-8 (b) $\sigma_1 = 8.27$ MPa, $\sigma_2 = -64.3$ MPa,
 $\tau_{\max} = 36.3$ MPa
 8.4-9 (b) $\sigma_1 = 159.8$ psi, $\sigma_2 = -3393$ psi,
 $\tau_{\max} = 1777$ psi
 8.4-10 $\sigma_1 = 17.86$ MPa, $\sigma_2 = -0.145$ MPa,
 $\tau_{\max} = 9.00$ MPa
 8.4-11 $\frac{\sigma_1}{\sigma_2} = -184$
 8.4-12 $\frac{\sigma_1}{\sigma_2} = -663$
 8.5-1 $t_{\min} = 0.125$ in.
 8.5-2 $p_{\max} = 9.60$ MPa
 8.5-3 (a) $\sigma_{\max} = \sigma_1 = 11.09$ ksi, $\tau_{\max} = 3.21$ ksi;
 (b) $T_{\max} = 178$ k-ft; (c) $t_{\min} = 0.519$ in.
 8.5-4 (a) $P_{\max} = 52.7$ kN; (b) $p_{\max} = 6$ MPa
 8.5-5 $\sigma_t = 10,680$ psi: No compressive stresses.
 $\tau_{\max} = 5340$ psi
 8.5-6 $\phi_{\max} = 0.552$ rad = 31.6°
 8.5-7 $\sigma_t = 3963$ psi, $\sigma_c = -8791$ psi, $\tau_{\max} = 6377$ psi
 8.5-8 $\sigma_t = 16.93$ MPa, $\sigma_c = -41.4$ MPa,
 $\tau_{\max} = 28.9$ MPa
 8.5-9 $P = 194.2$ k

- 8.5-10 (a) $\sigma_{\max} = \sigma_1 = 35.8 \text{ MPa}$, $\tau_{\max} = 18.05 \text{ MPa}$;
 (b) $P_{\max} = 6.73 \text{ kN}$
- 8.5-11 (a) $\sigma_x = 0 \text{ psi}$, $\sigma_y = 6145 \text{ psi}$, $\tau_{xy} = 345 \text{ psi}$;
 (b) $\sigma_1 = 6164 \text{ psi}$, $\sigma_2 = -19.30 \text{ psi}$, $\tau_{\max} = 3092 \text{ psi}$
- 8.5-12 $\tau_A = 76.0 \text{ MPa}$, $\tau_B = 19.94 \text{ MPa}$, $\tau_C = 23.7 \text{ MPa}$
- 8.5-13 $\sigma_1 = 1094 \text{ psi}$ (max. tensile stress at base of pole),
 $\sigma_2 = -7184 \text{ psi}$ (max. compressive stress at base of pole), $\tau_{\max} = 3731 \text{ psi}$ (max. shear stress at base of pole)
- 8.5-14 $d_{\min} = 48.4 \text{ mm}$
- 8.5-15 $\sigma_t = 39,950 \text{ psi}$, $\sigma_c = -2226 \text{ psi}$, $\tau_{\max} = 21,090 \text{ psi}$
- 8.5-16 (a) $\sigma_t = 29.15 \frac{qR^2}{d^3}$, $\sigma_c = -8.78 \frac{qR^2}{d^3}$,
 $\tau_{\max} = 18.97 \frac{qR^2}{d^3}$; (b) $\sigma_t = 14.04 \frac{qR^2}{d^3}$,
 $\sigma_c = -2.41 \frac{qR^2}{d^3}$, $\tau_{\max} = 8.22 \frac{qR^2}{d^3}$
- 8.5-17 $\sigma_t = 4320 \text{ psi}$, $\sigma_c = -1870 \text{ psi}$, $\tau_{\max} = 3100 \text{ psi}$
- 8.5-18 Pure shear: $\tau_{\max} = 0.804 \text{ MPa}$
- 8.5-19 (a) $d_{\min} = 1.65 \text{ in.}$; (b) $P_{\max} = 19.25 \text{ lb}$
- 8.5-20 (a) $\sigma_1 = 29.3 \text{ MPa}$, $\sigma_2 = -175.9 \text{ MPa}$,
 $\tau_{\max} = 102.6 \text{ MPa}$; (b) $\sigma_1 = 156.1 \text{ MPa}$,
 $\sigma_2 = -33 \text{ MPa}$, $\tau_{\max} = 94.5 \text{ MPa}$
- 8.5-21 (a) $\sigma_1 = 0 \text{ psi}$, $\sigma_2 = -20,730 \text{ psi}$,
 $\tau_{\max} = 10,365 \text{ psi}$; (b) $\sigma_1 = 988 \text{ psi}$,
 $\sigma_2 = -21,719 \text{ psi}$, $\tau_{\max} = 11,354 \text{ psi}$
- 8.5-22 Maximum: $\sigma_t = 18.35 \text{ MPa}$,
 $\sigma_c = -18.35 \text{ MPa}$, $\tau_{\max} = 9.42 \text{ MPa}$
- 8.5-23 Top of beam: $\sigma_1 = 8591 \text{ psi}$, $\sigma_2 = 0 \text{ psi}$,
 $\tau_{\max} = 4295 \text{ psi}$
- 8.5-24 (a) $d_{AI} = 26.3 \text{ mm}$; (b) $d_{II} = 21.4 \text{ mm}$
- 8.5-25 $\sigma_y = 0$, $\sigma_x = \frac{(FL) \frac{d_2}{2}}{I_p} = 1943 \text{ psi}$,
 $\tau_{xy} = \frac{-T \frac{d_2}{2}}{I_p} = -547 \text{ psi}$, $\sigma_1 = 2087 \text{ psi}$,
 $\sigma_2 = -143.2 \text{ psi}$, $\tau_{\max} = 1115 \text{ psi}$
- 8.5-26 (a) $\sigma_1 = 0$, $\sigma_2 = \sigma_x = -108.4 \text{ MPa}$,
 $\tau_{\max} = \frac{\sigma_x}{2} = -54.2 \text{ MPa}$;
 (b) $\sigma_1 = 0.703 \text{ MPa}$, $\sigma_2 = -1.153 \text{ MPa}$,
 $\tau_{\max} = 0.928 \text{ MPa}$; (c) $P_{\max} = 348 \text{ N}$
- 8.5-27 $\sigma_x = -18.6 \text{ ksi}$, $\sigma_y = 0$, $\tau_{xy} = 4.45 \text{ ksi}$,
 $\sigma_1 = 1.012 \text{ ksi}$, $\sigma_2 = -19.62 \text{ ksi}$, $\tau_{\max} = 10.31 \text{ ksi}$

CHAPTER 9

- 9.2-1 $q = q_0 x/L$; Triangular load, acting downward
- 9.2-2 (a) $q = q_0 \sin \pi x/L$, Sinusoidal load;
 (b) $R_A = R_B = q_0 L/\pi$; (c) $M_{\max} = q_0 L^2/\pi^2$
- 9.2-3 $q = q_0(1 - x/L)$; Triangular load, acting downward
- 9.2-4 (a) $q = q_0(L^2 - x^2)/L^2$; Parabolic load, acting downward; (b) $R_A = 2q_0 L/3$,
 $M_A = -q_0 L^2/A$
- 9.3-1 $\delta_{\max} = 0.182 \text{ in.}$, $\theta = 0.199^\circ$
- 9.3-2 $h = 96 \text{ mm}$
- 9.3-3 $L = 120 \text{ in.} = 10 \text{ ft}$
- 9.3-4 $\delta_{\max} = 15.4 \text{ mm}$
- 9.3-5 $\delta/L = 1/400$
- 9.3-6 $E_g = 80.0 \text{ GPa}$
- 9.3-7 Let $\beta = a/L$: $\frac{\delta_C}{\delta_{\max}} = 3\sqrt{3(-1 + 8\beta - 4\beta^2)}$
 The deflection at the midpoint is close to the maximum deflection. The maximum difference is only 2.6%.
- 9.3-11 $v = -mx^2(3L - x)/6EI$, $\delta_B = mL^2/3EI$,
 $\theta_B = mL^2/2EI$
- 9.3-12 $v(x) = -\frac{q}{48EI}(2x^4 - 12x^2L^2 + 11L^4)$,
 $\delta_B = \frac{qL^4}{48EI}$
- 9.3-13 See Table H-2, Case 9.
- 9.3-14 See Table H-1, Case 2.
- 9.3-15 $v(x) = \frac{q_0 L}{24 EI}(x^3 - 2Lx^2)$ for $0 \leq x \leq \frac{L}{2}$,
 $v(x) = \frac{-q_0}{960LEI}(-160L^2x^3 + 160L^3x^2 + 80Lx^4 - 16x^5 - 25L^4x + 3L^5)$ for
 $\frac{L}{2} \leq x \leq L$, $\delta_B = \frac{7}{160} \frac{q_0 L^4}{EI}$, $\delta_C = \frac{1}{64} \frac{q_0 L^4}{EI}$
- 9.3-16 $v(x) = \frac{q_0 x}{5760LEI}(200x^2L^2 - 240x^3L + 96x^4 - 53L^4)$ for $0 \leq x \leq \frac{L}{2}$,
 $v(x) = \frac{-q_0 L}{5760EI}(40x^3 - 120Lx^2 + 83L^2x - 3L^3)$
 for $\frac{L}{2} \leq x \leq L$, $\delta_C = \frac{3q_0 L^4}{1280EI}$

$$\begin{aligned}
 9.3-17 \quad v(x) &= -\frac{PL}{10,368EI}(-4104x^2 + 3565L^2) \\
 \text{for } 0 \leq x \leq \frac{L}{3}, v(x) &= -\frac{P}{1152EI} \\
 &(-648Lx^2 + 192x^3 + 64L^2x + 389L^3) \\
 \text{for } \frac{L}{3} \leq x \leq \frac{L}{2}, v(x) &= -\frac{P}{144EI} \\
 &(-72L^2x^2 + 12Lx^3 + 6x^4 + 5L^3x + 49L^4) \\
 \text{for } \frac{L}{2} \leq x \leq L, \delta_A &= \frac{3565PL^3}{10,368EI}, \delta_C = \frac{3109PL^3}{10,368EI}
 \end{aligned}$$

$$\begin{aligned}
 9.4-3 \quad v &= -M_0x(L-x)^2/2LEI, \\
 \delta_{\max} &= 2M_0L^2/27EI \text{ (downward)}
 \end{aligned}$$

$$\begin{aligned}
 9.4-4 \quad v(x) &= -\frac{q}{48EI}(2x^4 - 12x^2L^2 + 11L^4), \\
 \theta_B &= -\frac{qL^3}{3EI}
 \end{aligned}$$

9.4-5 See Table H-1, Case 10.

$$\begin{aligned}
 9.4-6 \quad v &= -q_0x^2(45L^4 - 40L^3x + 15L^2x^2 - x^4)/ \\
 &360L^2EI, \delta_B = 19q_0L^4/360EI, \theta_B = q_0L^3/15EI
 \end{aligned}$$

$$\begin{aligned}
 9.4-7 \quad v &= -q_0x(3L^5 - 5L^3x^2 + 3Lx^4 - x^5)/90L^2EI, \\
 \delta_{\max} &= 61q_0L^4/5760EI
 \end{aligned}$$

$$\begin{aligned}
 9.4-8 \quad v(x) &= \frac{q_0}{120EIL}(x^5 - 5Lx^4 + 20L^3x^2 - 16L^5), \\
 \delta_{\max} &= \frac{2q_0L^4}{15EI}
 \end{aligned}$$

$$\begin{aligned}
 9.4-9 \quad v(x) &= -\frac{qL^2}{16EI}(x^2 - L^2) \text{ for } 0 \leq x \leq L, \\
 v(x) &= -\frac{q}{48EI}(-20L^3x + 27L^2x^2 - \\
 &12Lx^3 + 2x^4 + 3L^4) \text{ for } L \leq x \leq \frac{3L}{2}, \\
 \delta_C &= \frac{9qL^4}{128EI}, \theta_C = \frac{7qL^3}{48EI}
 \end{aligned}$$

$$\begin{aligned}
 9.4-10 \quad v(x) &= -\frac{q_0L^2}{480EI}(-20x^2 + 19L^2) \text{ for } 0 \leq x \leq \frac{L}{2}, \\
 v(x) &= -\frac{q_0}{960EIL}(80Lx^4 - 16x^5 - \\
 &120L^2x^3 + 40L^3x^2 - 25L^4x + 41L^5) \\
 \text{for } \frac{L}{2} \leq x \leq L, \\
 \delta_A &= \frac{19q_0L^4}{480EI}, \theta_B = -\frac{13q_0L^3}{192EI}, \delta_C = \frac{7q_0L^4}{240EI}
 \end{aligned}$$

$$9.5-1 \quad \theta_B = 7PL^2/9EI, \delta_B = 5PL^3/9EI$$

$$\begin{aligned}
 9.5-2 \quad (a) \delta_1 &= 11PL^3/144EI; (b) \delta_2 = 25PL^3/384EI; \\
 (c) \delta_1/\delta_2 &= 88/75 = 1.173
 \end{aligned}$$

$$9.5-3 \quad (a) a/L = 2/3; (b) a/L = 1/2$$

$$\begin{aligned}
 9.5-4 \quad (a) \delta_C &= 6.25 \text{ mm (upward)} \\
 (b) \delta_C &= 18.36 \text{ mm (downward)}
 \end{aligned}$$

$$9.5-5 \quad y = Px^2(L-x)^2/3LEI$$

$$9.5-6 \quad \theta_B = 7qL^3/162EI, \delta_B = 23qL^4/648EI$$

$$9.5-7 \quad \delta_C = 0.0905 \text{ in.}, \delta_B = 0.293 \text{ in.}$$

$$\begin{aligned}
 9.5-8 \quad (a) M &= PL/2; (b) M = 5PL/24, \\
 \theta_B &= PL^2/12EI; (c) M = PL/8, \\
 \delta_B &= -PL^3/24EI
 \end{aligned}$$

$$9.5-9 \quad M = (19/180)q_0L^2$$

$$\begin{aligned}
 9.5-10 \quad (a) \delta_A &= PL^2(10L - 9a)/324EI \text{ (positive upward);} \\
 (b) \text{ Upward when } a/L &< 10/9, \text{ downward when } a/L > 10/9
 \end{aligned}$$

$$9.5-11 \quad (a) \delta_C = PH^2(L+H)/3EI; (b) \delta_{\max} = PHL^2/9\sqrt{3}EI$$

$$9.5-12 \quad \delta_C = 3.5 \text{ mm}$$

$$9.5-13 \quad \theta_B = q_0L^3/10EI, \delta_B = 13q_0L^4/180EI$$

$$\begin{aligned}
 9.5-14 \quad \theta_A &= q(L^3 - 6La^2 + 4a^3)/24EI, \\
 \delta_{\max} &= q(5L^4 - 24L^2a^2 + 16a^4)/384EI
 \end{aligned}$$

$$\begin{aligned}
 9.5-15 \quad (a) P/Q &= 9a/4L; (b) P/Q = 8a(3L+a)/9L^2; \\
 (c) P/qa &= 9a/8L \text{ for } \delta_B = 0, \\
 P/qa &= a(4L+a)/3L^2 \text{ for } \delta_D = 0
 \end{aligned}$$

$$9.5-16 \quad \delta = 19WL^3/31,104EI$$

$$9.5-17 \quad k = 3.33 \text{ lb/in.}$$

$$9.5-18 \quad M_1 = 7800 \text{ N}\cdot\text{m}, M_2 = 4200 \text{ N}\cdot\text{m}$$

$$9.5-19 \quad \delta = \frac{6Pb^3}{EI}$$

$$9.5-20 \quad \delta_E = \frac{47Pb^3}{12EI}$$

$$9.5-21 \quad \delta_C = 0.120 \text{ in.}$$

$$9.5-22 \quad q = 16cEI/7L^4$$

$$9.5-23 \quad \delta_h = Pcb^2/2EI, \delta_v = Pc^2(c+3b)/3EI$$

$$9.5-24 \quad \delta = PL^2(2L+3a)/3EI$$

$$9.5-25 \quad (a) H_B = 0, V_B = \frac{M}{L}, V_C = -V_B;$$

$$\begin{aligned}
 (b) \theta_A &= \frac{5ML}{6EI}, \theta_B = \frac{ML}{3EI}, \theta_C = \frac{-ML}{6EI}, \\
 \theta_D &= \theta_C;
 \end{aligned}$$

$$\begin{aligned}
 (c) \delta_A &= (7/24)ML^2/EI \text{ (to the left),} \\
 \delta_D &= (1/12)ML^2/EI \text{ (to the right);}
 \end{aligned}$$

$$(d) L_{CD} = \frac{\sqrt{14}}{2}L = 1.871L$$

$$9.5-26 \quad (a) H_B = 0, V_B = \frac{P}{3}, V_C = \frac{2P}{3};$$

$$(b) \theta_A = \left(\frac{-4}{81}\right)\frac{ML}{EI}, \theta_B = \theta_A,$$

$$\theta_C = \left(\frac{5}{81}\right)\frac{ML}{EI}, \theta_D = \theta_C;$$

- (c) $\delta_A = -\theta_B \left(\frac{L}{2} \right) = \frac{2L^2 M}{81EI}$ (to the right),
 $\delta_D = \theta_C \left(\frac{L}{2} \right) = \frac{5L^2 M}{162EI}$ (to the left);
 (d) $L_{CD} = \frac{2\sqrt{5}L}{5} = 0.894L$
- 9.5-27** (a) $b/L = 0.403$; (b) $\delta_C = 0.00287qL^4/EI$
- 9.5-28** $\alpha = 22.5^\circ, 112.5^\circ, -67.5^\circ$, or -157.5°
- 9.6-4** $\theta_B = 7qL^3/162EI$, $\delta_B = 23qL^4/648EI$
- 9.6-5** $\delta_B = 0.443$ in., $\delta_C = 0.137$ in.
- 9.6-6** $\delta_B = 11.8$ mm, $\delta_C = 4.10$ mm
- 9.6-8** $P = 64$ kN
- 9.6-9** $\theta_A = M_0L/6EI$, $\theta_B = M_0L/3EI$,
 $\delta = M_0L^2/16EI$
- 9.6-10** $\theta_A = Pa(L - a)(L - 2a)/6LEI$,
 $\delta_1 = Pa^2(L - 2a)^2/6LEI$, $\delta_2 = 0$
- 9.6-11** $\theta_A = M_0L/6EI$, $\theta_B = 0$, $\delta = M_0L^2/27EI$
 (downward)
- 9.7-1** (a) $\delta_B = PL^3(1 + 7I_1/I_2)/24EI_1$;
 (b) $r = (1 + 7I_1/I_2)/8$
- 9.7-2** (a) $\delta_B = qL^4(1 + 15I_1/I_2)/128EI_1$;
 (b) $r = (1 + 15I_1/I_2)/16$
- 9.7-3** (a) $\delta_C = 0.31$ in. (upward);
 (b) $\delta_C = 0.75$ in. (downward)
- 9.7-4** $v = -qx(21L^3 - 64Lx^2 + 32x^3)/768EI$ for
 $0 \leq x \leq L/4$,
 $v = -q(13L^4 + 256L^3x - 512Lx^3 + 256x^4)/12,288EI$ for $L/4 \leq x \leq L/2$,
 $\theta_A = 7qL^3/256EI$, $\delta_{\max} = 31qL^4/4096EI$
- 9.7-5** $\theta_A = 8PL^2/243EI$, $\delta_B = 8PL^3/729EI$,
 $\delta_{\max} = 0.01363PL^3/EI$
- 9.7-6** $v = -2Px(19L^2 - 27x^2)/729EI$ for
 $0 \leq x \leq L/3$,
 $v = P(13L^3 - 175L^2x + 243Lx^2 - 81x^3)/1458EI$ for $L/3 \leq x \leq L$,
 $\theta_A = 38PL^2/729EI$, $\theta_C = 34PL^2/729EI$,
 $\delta_B = 32PL^3/2187EI$
- 9.7-7** $v = \frac{PL^3}{EI_A} \left[\frac{L}{2(L+x)} - \frac{3x}{8L} + \frac{1}{8} + \ln \left(\frac{L+x}{2L} \right) \right]$,
 $\delta_A = \frac{PL^3}{8EI_A} (8 \ln 2 - 5)$
- 9.7-8** $v = \frac{PL^3}{24EI_A} \left[7 - \frac{4L(2L+3x)}{(L+x)^2} - \frac{2x}{L} \right]$,
 $\delta_A = \frac{PL^3}{24EI_A}$
- 9.7-9** $v = \frac{8PL^3}{EI_A} \left[\frac{L}{2L+x} - \frac{2x}{9L} - \frac{1}{9} + \ln \left(\frac{2L+x}{3L} \right) \right]$,
 $\delta_A = \frac{8PL^3}{EI_A} \left(\ln \frac{3}{2} - \frac{7}{18} \right)$
- 9.7-10** $v(x) = \frac{19,683PL^3}{2000EI_A} \left(\frac{81L}{81L+40x} + 2 \ln \left(\frac{81}{121} + \frac{40x}{121L} \right) - \left(\frac{6440x}{14,641L} - \frac{3361}{14,641} \right) \right)$,
 $\delta_A = \frac{19,683PL^3}{7,320,500EI_A} \left(-2820 + 14,641 \ln \left(\frac{11}{9} \right) \right)$
- 9.7-11** $v(x) = -\frac{19,683PL^3}{2000EI_A} \left(\frac{81L}{81L+40x} + 2 \ln \left(1 + \frac{40x}{81L} \right) - \frac{6440x}{14,641L} - 1 \right)$,
 $\delta_B = \frac{19,683PL^3}{7,320,500EI_A} \left(-2820 + 14,641 \ln \left(\frac{11}{9} \right) \right)$
- 9.7-12** (a) $v' = -\frac{qL^3}{16EI_A} \left[1 - \frac{8Lx^2}{(L+x)^3} \right]$
 for $0 \leq x \leq L$,
 $v = -\frac{qL^4}{2EI_A} \left[\frac{(9L^2 + 14Lx + x^2)x}{8L(L+x)^2} - \ln \left(1 + \frac{x}{L} \right) \right]$ for $0 \leq x \leq L$;
 (b) $\theta_A = \frac{qL^3}{16EI_A}$, $\delta_C = \frac{qL^4(3 - 4 \ln 2)}{8EI_A}$
- 9.8-1** $U = 4bhL\sigma_{\max}^2/45E$
- 9.8-2** (a) and (b) $U = P^2L^3/96EI$;
 (c) $\delta = PL^3/48EI$
- 9.8-3** (a) and (b) $U = \frac{q^2L^3}{15EI}$
- 9.8-4** (a) $U = 32EI\delta^2/L^3$; (b) $U = \pi^4EI\delta^2/4L^3$
- 9.8-5** (a) $U = P^2a^2(L+a)/6EI$;
 (b) $\delta_C = Pa^2(L+a)/3EI$;
 (c) $U = 241$ in.-lb, $\delta_C = 0.133$ in.

$$9.8-6 \quad U = \frac{L}{15,360EI} (17L^4q^2 + 280qL^2M_0 + 2560M_0^2)$$

$$9.8-7 \quad \delta_B = 2PL^3/3EI + 8\sqrt{2}PL/EA$$

$$9.9-2 \quad \delta_D = Pa^2b^2/3LEI$$

$$9.9-3 \quad \delta_C = Pa^2(L+a)/3EI$$

$$9.9-6 \quad \delta_C = L^3(2P_1 + 5P_2)/48EI,$$

$$\delta_B = L^3(5P_1 + 16P_2)/48EI$$

$$9.9-7 \quad \theta_A = 7qL^3/48EI$$

$$9.9-8 \quad \delta_C = Pb^2(b+3h)/3EI,$$

$$\theta_C = Pb(b+2h)/2EI$$

$$9.9-9 \quad \delta_C = 31qL^4/4096EI$$

$$9.9-10 \quad \theta_A = M_A(L+3a)/3EI,$$

$$\delta_A = M_Aa(2L+3a)/6EI$$

$$9.9-11 \quad \delta_C = Pa^2(L+a)/3EI + P(L+a)^2/kL^2$$

$$9.9-12 \quad \delta_D = 37qL^4/6144EI \text{ (upward)}$$

$$9.10-1 \quad \sigma_{\max} = \sigma_{st}[1 + (1 + 2h/\delta_{st})^{1/2}]$$

$$9.10-2 \quad \sigma_{\max} = \sqrt{18WEh/AL}$$

$$9.10-3 \quad \delta_{\max} = 0.302 \text{ in.}, \sigma_{\max} = 21,700 \text{ psi}$$

$$9.10-4 \quad d = 281 \text{ mm}$$

$$9.10-5 \quad W 14 \times 53$$

$$9.10-6 \quad h = 360 \text{ mm}$$

$$9.10-7 \quad R = \sqrt{3EI\omega^2/L^3}$$

$$9.11-1 \quad v = -\alpha(T_2 - T_1)(x)(L-x)/2h$$

(pos. upward),

$$\theta_A = \alpha L(T_2 - T_1)/2h \text{ (clockwise),}$$

$$\delta_{\max} = \alpha L^2(T_2 - T_1)/8h \text{ (downward)}$$

$$9.11-2 \quad v = -\alpha(T_2 - T_1)(x^2)/2h \text{ (upward),}$$

$$\theta_B = \alpha L(T_2 - T_1)/h \text{ (counterclockwise),}$$

$$\delta_B = \alpha L^2(T_2 - T_1)/2h \text{ (upward)}$$

$$9.11-3 \quad v(x) = \frac{\alpha(T_2 - T_1)(x^2 - L^2)}{2h},$$

$$\theta_C = \frac{\alpha(T_2 - T_1)(L+a)}{h}$$

(counterclockwise),

$$\delta_C = \frac{\alpha(T_2 - T_1)(2La + a^2)}{2h} \text{ (upward)}$$

$$9.11-4 \quad (a) \delta_{\max} = \frac{\alpha T_0 L^3}{9\sqrt{3}h} \text{ (downward);}$$

$$(b) \delta_{\max} = \frac{\alpha T_0 L^4(2\sqrt{2} - 1)}{48h} \text{ (downward)}$$

$$9.11-5 \quad (a) \delta_{\max} = \frac{\alpha T_0 L^3}{6h} \text{ (downward);}$$

$$(b) \delta_{\max} = \frac{\alpha T_0 L^4}{12h} \text{ (downward);}$$

$$(c) \delta_{\max} = \frac{\alpha T_0 L^3}{6h} \text{ (downward),}$$

$$\delta_{\max} = \frac{\alpha T_0 L^4}{12h} \text{ (downward)}$$

CHAPTER 10

$$10.3-1 \quad R_A = -R_B = 3M_0/2L, M_A = M_0/2,$$

$$v = -M_0x^2(L-x)/4LEI$$

$$10.3-2 \quad R_A = R_B = qL/2, M_A = M_B = qL^2/12,$$

$$v = -qx^2(L-x)^2/24EI$$

$$10.3-3 \quad R_A = R_B = 3EI\delta_B/L^3, M_A = 3EI\delta_B/L^2,$$

$$v = -\delta_Bx^2(3L-x)/2L^3$$

$$10.3-4 \quad \theta_B = \frac{qL^3}{6(k_RL - EI)},$$

$$\delta_B = -\frac{1}{8}qL^4 + \frac{k_RqL^5}{12(k_RL - EI)}$$

$$10.3-5 \quad R_A = V(0) = \frac{9}{40}q_0L,$$

$$R_B = -V(L) = \frac{11}{40}q_0L,$$

$$M_A = \frac{7}{120}q_0L^2$$

$$10.3-6 \quad (a) R_A = V(0) = \frac{7}{60}q_0L,$$

$$R_B = -V(L) = \frac{13}{60}q_0L,$$

$$M_A = \frac{1}{30}q_0L^2,$$

$$v = \frac{q_0}{360L^2EI}(-x^6 + 7L^3x^3 - 6q_0L^4x^2);$$

$$(b) R_A = V(0) = 0.31q_0L$$

$$= \left(\frac{2}{\pi} - 6 \frac{\pi^2 - 4\pi + 8}{\pi^4} \right) q_0L,$$

$$R_B = -V(L) = 0.327q_0L$$

$$= \left(6 \frac{\pi^2 - 4\pi + 8}{\pi^4} \right) q_0L,$$

$$M_A = -2q_0L^2 \frac{\pi^2 - 12\pi + 24}{\pi^4},$$

$$v = \frac{1}{EI}$$

$$\left[-q_0 \left(\frac{2L}{\pi} \right)^4 \sin \left(\frac{\pi x}{2L} \right) - 6q_0L \frac{\pi^2 - 4\pi + 8}{\pi^4} \frac{x^3}{6} \right. \\ \left. + 2q_0L^2 \frac{\pi^2 - 12\pi + 24}{\pi^4} \frac{x^2}{2} + q_0 \left(\frac{2L}{\pi} \right)^3 x \right]$$

$$10.3-7 \quad (a) \quad R_A = \frac{61Lq_0}{120}, \quad M_A = \frac{11L^2q_0}{120},$$

$$R_B = \frac{19Lq_0}{120},$$

$$v(x) = -\frac{q_0x^2(33L^4 - 61L^3x + 30L^2x^2 - 2x^4)}{720EI L^2};$$

$$(b) \quad R_A = \frac{48Lq_0}{\pi^4},$$

$$R_B = \int_0^L q(x) dx - R_A = \frac{2Lq_0}{\pi} - \frac{48Lq_0}{\pi^4},$$

$$M_A = \int_0^L q(x)x dx - R_B L = \frac{2L^2q_0(\pi - 2)}{\pi^2}$$

$$- L\left(\frac{2Lq_0}{\pi} - \frac{48Lq_0}{\pi^4}\right),$$

$$v(x) = \frac{16L^4q_0 - 24L^2q_0x^2 + 8Lq_0x^3 - 16L^4q_0 \cos\left(\frac{\pi x}{2L}\right)}{\pi^4 EI}$$

$$10.3-8 \quad (a) \quad R_A = V(0) = \frac{24}{\pi^4} q_0L,$$

$$R_B = -V(L) = -\frac{24}{\pi^4} q_0L,$$

$$M_A = \left(\frac{12}{\pi^4} - \frac{1}{\pi^2}\right) q_0L^2 \quad (\text{counterclockwise}),$$

$$M_B = \left(\frac{12}{\pi^4} - \frac{1}{\pi^2}\right) q_0L^2 \quad (\text{counterclockwise}),$$

$$v = \frac{1}{\pi^4 EI} [-q_0L^4 \cos\left(\frac{\pi x}{L}\right)$$

$$+ 4q_0Lx^3 - 6q_0L^2x^2 + q_0L^4];$$

$$(b) \quad R_A = R_B = q_0L/\pi, \quad M_A = M_B = 2q_0L^2/\pi^3,$$

$$v = -q_0L^2(L^2 \sin \pi x/L + \pi x^2 - \pi Lx)/\pi^4 EI$$

$$10.3-9 \quad (a) \quad R_A = V(0) = \frac{48(4 - \pi)}{\pi^4} q_0L,$$

$$R_B = -V(L) = \left(\frac{2}{\pi} - \frac{48(4 - \pi)}{\pi^4}\right) q_0L,$$

$$M_A = -q_0\left(\frac{2L}{\pi}\right)^2 + \frac{16(6 - \pi)}{\pi^4} q_0L^2,$$

$$M_B = -\frac{32(\pi - 3)}{\pi^4} q_0L^2,$$

$$v = \frac{1}{\pi^4 EI} [-16q_0L^4 \cos\left(\frac{\pi x}{2L}\right)$$

$$+ 8(4 - \pi)q_0Lx^3$$

$$- 8(6 - \pi)q_0L^2x^2 + 16q_0L^4];$$

$$(b) \quad R_A = V(0) = \frac{13}{30} q_0L,$$

$$R_B = -V(L) = \frac{7}{30} q_0L,$$

$$M_A = \frac{1}{15} q_0L^2 \quad (\text{counterclockwise}),$$

$$M_B = -\frac{1}{20} q_0L^2 \quad (\text{counterclockwise}),$$

$$v = \frac{q_0}{360L^2 EI} [x^6 - 15L^2x^4 + 26L^3x^3$$

$$- 12L^4x^2]$$

$$10.3-10 \quad R_A = V(0) = \frac{3}{20} q_0L,$$

$$R_B = -V(L) = \frac{7}{20} q_0L,$$

$$M_A = \frac{1}{30} q_0L^2,$$

$$v = \frac{1}{120LEI} (-q_0x^5 + 3q_0Lx^3 - 2q_0L^2x^2)$$

$$10.3-11 \quad R_A = -R_B = 3M_0/2L, \quad M_A = -M_B = M_0/4,$$

$$v = -M_0x^2(L - 2x)/8LEI \text{ for } 0 \leq x \leq L/2$$

$$10.3-12 \quad R_B = -\frac{9}{8} \frac{M_0}{L},$$

$$R_A = \frac{9}{8} \frac{M_0}{L},$$

$$M_A = \frac{1}{8} \frac{M_0}{L},$$

$$v = \frac{1}{EI} \left(\frac{9M_0}{48L} x^3 - \frac{M_0}{16} x^2 \right) \left(0 \leq x \leq \frac{L}{2} \right),$$

$$v = \frac{1}{EI}$$

$$\left(\frac{9M_0}{48L} x^3 - \frac{9M_0}{16} x^2 + \frac{M_0L}{2} x - \frac{M_0L^2}{8} \right)$$

$$\left(\frac{L}{2} \leq x \leq L \right)$$

$$10.4-1 \quad R_A = Pb(3L^2 - b^2)/2L^3,$$

$$R_B = Pa^2(3L - a)/2L^3, \quad M_A = Pab(L + b)/2L^2$$

$$10.4-2 \quad R_A = qL, \quad M_A = \frac{qL^2}{3}, \quad M_B = \frac{qL^2}{6}$$

$$10.4-3 \quad R_A = -\frac{1}{8} qL, \quad R_B = \frac{17}{8} qL, \quad M_A = -\frac{1}{8} qL^2$$

$$10.4-4 \quad (a) \quad R_A = M_0/3L, \quad H_A = -4M_0/3L,$$

$$R_B = -R_A, \quad R_C = -H_A;$$

$$(b) \quad \theta_A = -M_0L/18EI, \quad \theta_B = M_0L/9EI,$$

$$\theta_C = \theta_A; \quad (c) \quad L_{BC} = 2L$$

$$10.4-5 \quad (a) R_A = \frac{4M_0}{3L}, H_A = \frac{2M_0}{3L}, R_B = -\frac{4M_0}{3L},$$

$$R_C = -\frac{2M_0}{3L};$$

$$(b) \theta_A = \frac{5}{18} \frac{M_0 L}{EI}, \theta_B = \frac{-M_0 L}{18EI},$$

$$\theta_C = \frac{M_0 L}{36EI};$$

$$(c) L_{AB} = 2.088L$$

$$10.4-6 \quad (a) R_A = \frac{M_0}{L} + \frac{M_0 k_R}{2(3EI + Lk_R)},$$

$$R_B = -R_A,$$

$$M_B = \frac{LM_0 k_R}{6EI + 2Lk_R} \text{ (CCW);}$$

$$(b) \theta_A = \frac{LM_0}{4EI} + \frac{LM_0}{4(3EI + Lk_R)}$$

For k_R goes to zero:

$$\theta_A = \frac{LM_0}{4EI} + \frac{LM_0}{4(3EI)} = \frac{LM_0}{3EI}$$

For k_R goes to infinity: $\theta_A = \frac{M_0 L}{4EI}$

For k_R goes to $6EI/L$:

$$\theta_A = \frac{LM_0}{4EI} + \frac{LM_0}{4\left[3EI + L\left(\frac{6EI}{L}\right)\right]} = \frac{5LM_0}{18EI}$$

$$10.4-7 \quad (a) H_A = \frac{3}{2} \frac{M_0}{L},$$

$$H_B = 0, V_B = 0, V_C = 0, H_D = -H_A;$$

$$(b) \theta_A = \frac{-M_0 L}{16EI}, \theta_D = -\theta_A,$$

$$\theta_B = \frac{M_0 L}{8EI}, \theta_C = -\theta_B;$$

$$(c) H_A = \frac{M_0}{L}, H_B = -2 \frac{M_0}{L},$$

$$V_B = \frac{M_0}{L}, V_C = \frac{-M_0}{L}, H_D = H_A,$$

$$\theta_A = \frac{-M_0 L}{24EI}, \theta_D = \theta_A, \theta_B = \frac{M_0 L}{12EI},$$

$$\theta_C = \theta_B$$

$$10.4-8 \quad t_{AB}/t_{CD} = L_{AB}/L_{CD}$$

$$10.4-9 \quad R_A = \frac{7}{12} qL, R_B = \frac{17}{12} qL, M_A = \frac{7}{12} qL^2$$

$$10.4-10 \quad R_A = 2qL, M_B = \frac{7}{12} qL^2$$

$$10.4-11 \quad R_A = R_B = q_0 L/4, M_A = M_B = 5q_0 L^2/96$$

$$10.4-12 \quad R_A = qL/8, R_B = 33qL/16,$$

$$R_C = 13qL/16$$

$$10.4-13 \quad R_A = 1100 \text{ lb (downward)}, R_B = 2800 \text{ lb (upward)},$$

$$M_A = 30,000 \text{ lb-in. (clockwise)}$$

$$10.4-14 \quad R_B = 6.44 \text{ kN}$$

$$10.4-15 \quad (a) \text{ The tension force in the tie rod =}$$

$$R_D = 604 \text{ lb; (b) } R_A = 796 \text{ lb,}$$

$$M_A = 1308 \text{ lb-ft} = 1.567 \times 10^4 \text{ lb-in.}$$

$$10.4-16 \quad R_A = 31qL/48, R_B = 17qL/48,$$

$$M_A = 7qL^2/48$$

$$10.4-17 \quad (a) R_A = -23P/17, R_D = R_E = 20P/17,$$

$$M_A = 3PL/17; (b) M_{\max} = PL/2$$

$$10.4-18 \quad R_A = R_D = 2qL/5, R_B = R_C = 11qL/10$$

$$10.4-19 \quad M_B(q) = (-800 \cdot q) \text{ lb-in. for } q < 250 \text{ lb/in., } M_B(q)$$

$$= (-200 \cdot q - 150,000) \text{ lb-in. for}$$

$$q \geq 250 \text{ lb/in.}$$

$$10.4-20 \quad R_A = -R_B = 6M_0 ab/L^3,$$

$$M_A = M_0 b(3a - L)/L^2,$$

$$M_B = -M_0 a(3b - L)/L^2$$

$$10.4-21 \quad \sigma = 509 \text{ psi}$$

$$10.4-22 \quad (M_{AB})_{\max} = 121qL^2/2048 = 6.05 \text{ kN}\cdot\text{m};$$

$$(M_{CD})_{\max} = 5qL^2/64 = 8.0 \text{ kN}\cdot\text{m}$$

$$10.4-23 \quad F = 3160 \text{ lb, } M_{AB} = 18,960 \text{ lb-ft,}$$

$$M_{DE} = 7320 \text{ lb-ft}$$

$$10.4-24 \quad k = 48EI(6 + 5\sqrt{2})/7L^3 = 89.63EI/L^3$$

$$10.4-25 \quad (a) V_A = V_C = 3P/32, H_A = P,$$

$$M_A = 13PL/32; (b) M_{\max} = 13PL/32$$

$$10.4-26 \quad H_A = -\frac{35}{64} P, H_C = -\frac{29}{64} P,$$

$$M_{\max} = \frac{35}{128} PL$$

$$10.4-27 \quad R_A = R_B = 3000 \text{ lb, } R_C = 0$$

$$10.4-28 \quad (a) M_A = M_B = qb(3L^2 - b^2)/24L;$$

$$(b) b/L = 1.0, M_A = qL^2/12;$$

$$(c) \text{ For } a = b = L/3, (M_{\max})_{\text{pos}} = 19qL^2/648$$

$$10.4-29 \quad (a) d_2/d_1 = \sqrt[4]{48} = 1.682;$$

$$(b) M_{\max} = qL^2(3 - 2\sqrt{2})/2 = 0.08579qL^2;$$

$$(c) \text{ Point C is below points A and B by the amount } 0.01307qL^4/EI$$

$$10.4-30 \quad M_{\max} = 19q_0 L^2/256, \sigma_{\max} = 13.4 \text{ MPa,}$$

$$\sigma_{\max} = 19q_0 L^4/7680EI = 0.00891 \text{ mm}$$

$$10.5-1 \quad S = \frac{243E_S E_W I A H \alpha (\Delta T)}{4AL^3 E_S + 243I H E_W}$$

10.5-2 (a) $R_A = -\frac{\alpha(T_2 - T_1)L^2}{2h} \cdot \left(\frac{3EI \cdot k}{3EI + L^3 \cdot k} \right)$,
 $R_B = \frac{\alpha(T_2 - T_1)L^2}{2h} \cdot \left(\frac{3EI \cdot k}{3EI + L^3 \cdot k} \right)$,
 $M_A = R_B L$
 $= \frac{\alpha(T_2 - T_1)L^3}{2h} \cdot \left(\frac{3EI \cdot k}{3EI + L^3 \cdot k} \right)$;
 (b) $R_A = -R_B = -\frac{3EI\alpha(T_2 - T_1)}{2hL}$ (upward),
 $R_B = \frac{3EI\alpha(T_2 - T_1)}{2hL}$ (downward),
 $M_A = R_B L = \frac{3EI\alpha(T_2 - T_1)}{2h}$ (counterclockwise)

10.5-3 $R_A = -R_B = -\frac{\alpha(T_2 - T_1)L^2}{2h}$
 $\cdot \left(\frac{3EI \cdot k}{3EI + L^3 \cdot k} \right)$ (upward),
 $R_B = \frac{\alpha(T_2 - T_1)L^2}{2h} \cdot \left(\frac{3EI \cdot k}{3EI + L^3 \cdot k} \right)$ (downward),
 $M_A = R_B L = \frac{\alpha(T_2 - T_1)L^3}{2h}$
 $\cdot \left(\frac{3EI \cdot k}{3EI + L^3 \cdot k} \right)$ (counterclockwise)

10.5-4 (a) $R_B = -\frac{\alpha(T_1 - T_2)L^2}{h}$
 $\left(\frac{6EI \cdot k}{36EI + L^3 \cdot k} \right)$ (downward),
 $R_A = -\frac{1}{4}R_B = \frac{\alpha(T_1 - T_2)L^3}{2h}$
 $\left(\frac{3EI \cdot k}{36EI + L^3 \cdot k} \right)$ (upward),
 $R_C = -\frac{3}{4}R_B = \frac{\alpha(T_1 - T_2)L^2}{2h}$
 $\left(\frac{9EI \cdot k}{36EI + L^3 \cdot k} \right)$ (upward);
 (b) $R_B = -\frac{6El\alpha(T_1 - T_2)}{Lh}$ (downward),
 $R_A = \frac{3El\alpha(T_1 - T_2)}{2Lh}$ (upward),
 $R_C = \frac{9El\alpha(T_1 - T_2)}{2Lh}$ (upward)

10.5-5 $R_B = -\frac{\alpha(T_1 - T_2)L^2}{h} \left(\frac{6EI \cdot k}{36EI + L^3 \cdot k} \right)$
 (downward),
 $R_A = -\frac{1}{4}R_B = \frac{\alpha(T_1 - T_2)L^2}{2h}$
 $\left(\frac{3EI \cdot k}{36EI + L^3 \cdot k} \right)$ (upward),
 $R_C = -\frac{3}{4}R_B = \frac{\alpha(T_1 - T_2)L^2}{2h}$
 $\left(\frac{9EI \cdot k}{36EI + L^3 \cdot k} \right)$ (upward)

10.6-1 (a) $H = \pi^2 EA \delta^2 / 4L^2$, $\sigma_t = \pi^2 E \delta^2 / 4L^2$;
 (b) $\sigma_t = 617, 154$, and 69 psi

10.6-2 (a) $\lambda = 17q^2 L^7 / 40,320E^2 I^2$, $\sigma_b = qhL^2 / 16I$;
 (b) $\sigma_t = 17q^2 L^6 / 40,320EI^2$;
 (c) $\lambda = 0.01112$ mm, $\sigma_b = 117.2$ MPa,
 $\sigma_t = 0.741$ MPa

CHAPTER 11

11.2-1 $P_{cr} = \beta_R / L$

11.2-2 (a) $P_{cr} = \frac{\beta a^2 + \beta_R}{L}$; (b) $P_{cr} = \frac{\beta a^2 + 2\beta_R}{L}$

11.2-3 $P_{cr} = 6\beta_R / L$

11.2-4 (a) $P_{cr} = \frac{(L - a)(\beta a^2 + \beta_R)}{aL}$;
 (b) $P_{cr} = \frac{\beta L^2 + 20\beta_R}{4L}$

11.2-5 $P_{cr} = \frac{3\beta_R}{L}$

11.2-6 $P_{cr} = \frac{3}{5}\beta L$

11.2-7 $P_{cr} = \frac{7}{4}\beta L$

11.3-1 (a) $P_{cr} = 453$ k; (b) $P_{cr} = 152$ k

11.3-2 (a) $P_{cr} = 2803$ kN; (b) $P_{cr} = 953$ kN

11.3-3 (a) $P_{cr} = 650$ k; (b) $P_{cr} = 140$ k

11.3-4 $M_{allow} = 1143$ kN·m

11.3-5 $Q_{allow} = 23.8$ k

11.3-6 (a) $Q_{cr} = \frac{\pi^2 EI}{L^2}$; (b) $Q_{cr} = \frac{2\pi^2 EI}{9L^2}$

11.3-7 (a) $Q_{cr} = \frac{2\pi^2 EI}{L^2}$; (b) $M_{cr} = \frac{3d\pi^2 EI}{L^2}$

- 11.3-8** $\Delta T = \pi^2 I / \alpha A L^2$
11.3-9 $h/b = 2$
11.3-10 (a) $P_{cr} = 3\pi^3 E r^4 / 4L^2$; (b) $P_{cr} = 11\pi^3 E r^4 / 4L^2$
11.3-11 $P_1 : P_2 : P_3 = 1.000 : 1.047 : 1.209$
11.3-12 $P_{allow} = 604 \text{ kN}$
11.3-13 $F_{allow} = 54.4 \text{ k}$
11.3-14 $W_{max} = 124 \text{ kN}$
11.3-15 $t_{min} = 0.165 \text{ in.}$
11.3-16 $P_{cr} = 497 \text{ kN}$
11.3-17 $W_{cr} = 51.9 \text{ k}$
11.3-18 $\theta = \arctan 0.5 = 26.57^\circ$
11.3-19 (a) $q_{max} = 142.4 \text{ lb/ft}$; (b) $I_{b,min} = 38.5 \text{ in.}^4$;
(c) $s = 0.264 \text{ ft}$, 2.42 ft
11.3-20 $P_{cr} = 3.56 \text{ kN}$
11.3-21 $P_{cr} = 16.28 \text{ k}$
11.4-1 $P_{cr} = 235 \text{ k}$, 58.7 k , 480 k , 939 k
11.4-2 $P_{cr} = 62.2 \text{ kN}$, 15.6 kN , 127 kN , 249 kN
11.4-3 $P_{allow} = 253 \text{ k}$, 63.2 k , 517 k , 1011 k
11.4-4 $P_{allow} = 678 \text{ kN}$, 169.5 kN , 1387 kN , 2712 kN
11.4-5 $P_{cr} = 229 \text{ k}$
11.4-6 $T_{allow} = 18.1 \text{ kN}$
11.4-7 (a) $Q_{cr} = 4575 \text{ lb}$; (b) $Q_{cr} = 10,065 \text{ lb}$,
 $a = 0 \text{ in.}$
11.4-8 $P_{cr} = 447 \text{ kN}$, 875 kN , 54.7 kN , 219 kN
11.4-9 $P_{cr} = 4\pi^2 EI / L^2$, $v = \delta(1 - \cos 2\pi x / L) / 2$
11.4-10 $t_{min} = 10.0 \text{ mm}$
11.4-11 (b) $P_{cr} = 413.89 EI / L^2$
11.5-1 $\delta = 0.112 \text{ in.}$, $M_{max} = 1710 \text{ lb-in.}$
11.5-2 $\delta = 8.87 \text{ mm}$, $M_{max} = 2.03 \text{ kN}\cdot\text{m}$
11.5-3 For $P = 0.3P_{cr}$:
 $M/Pe = 1.162 (\sin 1.721 x/L) + \cos 1.721 x/L$
11.5-4 $P = 583.33 \{ \arccos [5/(5 + \delta)] \}^2$, in which
 $P = \text{kN}$ and $\delta = \text{mm}$; $P = 884 \text{ kN}$ when
 $\delta = 10 \text{ mm}$
11.5-5 $P = 125.58 \{ \arccos [0.2/(0.2 + \delta)] \}^2$, in which
 $P = \text{k}$ and $\delta = \text{in.}$; $P = 190 \text{ k}$ when
 $\delta = 0.4 \text{ in.}$
11.5-6 $P_{allow} = 49.9 \text{ kN}$
11.5-7 $L_{max} = 150.5 \text{ in.} = 12.5 \text{ ft}$
11.5-8 $L_{max} = 3.14 \text{ m}$
11.5-9 $\delta = e(\sec kL - 1)$, $M_{max} = Pe \sec kL$
11.5-10 $L_{max} = 2.21 \text{ m}$
11.5-11 $L_{max} = 130.3 \text{ in.} = 10.9 \text{ ft}$
11.5-12 $T_{max} = 8.29 \text{ kN}$
11.5-13 (a) $q_0 = 2230 \text{ lb/ft} = 186 \text{ lb/in.}$;
(b) $M_{max} = 37.7 \text{ k}\cdot\text{in.}$, ratio = 0.47
11.6-1 (a) $\sigma_{max} = 17.3 \text{ ksi}$; (b) $L_{max} = 46.2 \text{ in.}$
11.6-2 $P_{allow} = 37.2 \text{ kN}$

- 11.6-3** $b_{min} = 4.10 \text{ in.}$
11.6-4 (a) $\sigma_{max} = 38.8 \text{ MPa}$; (b) $L_{max} = 5.03 \text{ m}$
11.6-5 (a) $\sigma_{max} = 9.65 \text{ ksi}$; (b) $P_{allow} = 3.59 \text{ k}$
11.6-6 $d_2 = 131 \text{ mm}$
11.6-7 (a) $\sigma_{max} = 10.9 \text{ ksi}$; (b) $P_{allow} = 160 \text{ k}$
11.6-8 (a) $\sigma_{max} = 104.5 \text{ MPa}$; (b) $L_{max} = 3.66 \text{ m}$
11.6-9 (a) $\sigma_{max} = 9.60 \text{ ksi}$; (b) $P_{allow} = 53.6 \text{ k}$
11.6-10 (a) $\sigma_{max} = 47.6 \text{ MPa}$; (b) $n = 2.30$
11.6-11 (a) $\sigma_{max} = 13.4 \text{ ksi}$; (b) $n = 2.61$
11.6-12 (a) $\sigma_{max} = 120.4 \text{ MPa}$; (b) $P_2 = 387 \text{ kN}$
11.6-13 (a) $\sigma_{max} = 17.6 \text{ ksi}$; (b) $n = 1.89$
11.6-14 (a) $\sigma_{max} = 115.2 \text{ MPa}$; (b) $P_2 = 193 \text{ kN}$

$$11.9-1 \quad P_{allow} = \begin{pmatrix} 254 \\ 177 \\ 97 \\ 55 \end{pmatrix} \text{ k for } L = \begin{pmatrix} 8 \\ 16 \\ 24 \\ 32 \end{pmatrix} \text{ ft}$$

$$11.9-2 \quad P_{allow} = \begin{pmatrix} 3019 \\ 2193 \\ 1285 \\ 723 \end{pmatrix} \text{ kN for } \begin{pmatrix} 3 \text{ m} \\ 6 \text{ m} \\ 9 \text{ m} \\ 12 \text{ m} \end{pmatrix}$$

$$11.9-3 \quad P_{allow} = \begin{pmatrix} 338 \\ 240 \\ 135 \\ 76 \end{pmatrix} \text{ k for } L = \begin{pmatrix} 10 \\ 20 \\ 30 \\ 40 \end{pmatrix} \text{ ft}$$

$$11.9-4 \quad W 250 \times 67$$

$$11.9-5 \quad W 12 \times 87$$

$$11.9-6 \quad W 360 \times 122$$

$$11.9-7 \quad P_{allow} = \begin{pmatrix} 60.7 \\ 42.4 \\ 23.3 \\ 13.1 \end{pmatrix} \text{ k for } L = \begin{pmatrix} 6 \\ 12 \\ 18 \\ 24 \end{pmatrix} \text{ ft}$$

$$11.9-8 \quad P_{allow} = \begin{pmatrix} 1104 \\ 919 \\ 678 \\ 441 \end{pmatrix} \text{ kN for } L = \begin{pmatrix} 2.5 \\ 5.0 \\ 7.5 \\ 10.0 \end{pmatrix} \text{ m}$$

$$11.9-9 \quad P_{allow} = \begin{pmatrix} 96.9 \\ 73.9 \\ 50.6 \\ 32.6 \end{pmatrix} \text{ k for } L = \begin{pmatrix} 6.0 \\ 9.0 \\ 12.0 \\ 15.0 \end{pmatrix} \text{ ft}$$

$$11.9-10 \quad P_{allow} = \begin{pmatrix} 229 \\ 207 \\ 185 \\ 164 \end{pmatrix} \text{ kN for } L = \begin{pmatrix} 2.6 \\ 2.8 \\ 3.0 \\ 3.2 \end{pmatrix} \text{ m}$$

$$11.9-11 \quad L_{\max} = 5.13 \text{ ft}$$

$$11.9-12 \quad L_{\max} = 3.52 \text{ m}$$

$$11.9-13 \quad L_{\max} = 13.9 \text{ ft}$$

$$11.9-14 \quad P_{\text{allow}} = 5520 \text{ kN}$$

$$11.9-15 \quad (a) L_{\max} = 21.25 \text{ ft; (b) } L_{\max} = 14.10 \text{ ft}$$

$$11.9-16 \quad (a) L_{\max} = 6.44 \text{ m; (b) } L_{\max} = 4.68 \text{ m}$$

$$11.9-17 \quad d = 4.88 \text{ in.}$$

$$11.9-18 \quad d = 99.8 \text{ mm}$$

$$11.9-19 \quad d = 5.25 \text{ in.}$$

$$11.9-20 \quad d = 190 \text{ mm}$$

$$11.9-21 \quad P_{\text{allow}} = \begin{pmatrix} 143.5 \\ 124.0 \\ 85.5 \\ 59.4 \end{pmatrix} \text{ k for } L = \begin{pmatrix} 6.0 \\ 8.0 \\ 10.0 \\ 12.0 \end{pmatrix} \text{ ft}$$

$$11.9-22 \quad P_{\text{allow}} = \begin{pmatrix} 315 \\ 245 \\ 127 \\ 71 \end{pmatrix} \text{ kN for } L = \begin{pmatrix} 1.0 \\ 2.0 \\ 3.0 \\ 4.0 \end{pmatrix} \text{ m}$$

$$11.9-23 \quad P_{\text{allow}} = \begin{pmatrix} 18.1 \\ 14.8 \\ 8.4 \\ 5.4 \end{pmatrix} \text{ k for } L = \begin{pmatrix} 2.0 \\ 3.0 \\ 4.0 \\ 5.0 \end{pmatrix} \text{ ft}$$

$$11.9-24 \quad P_{\text{allow}} = \begin{pmatrix} 96.5 \\ 84.1 \\ 61.0 \\ 42.4 \end{pmatrix} \text{ kN for } L = \begin{pmatrix} 0.6 \\ 0.8 \\ 1.0 \\ 1.2 \end{pmatrix} \text{ m}$$

$$11.9-25 \quad (a) L_{\max} = 25.2 \text{ in.; (b) } d_{\min} = 2.11 \text{ in.}$$

$$11.9-26 \quad (a) L_{\max} = 468 \text{ mm; (b) } d_{\min} = 42.8 \text{ mm}$$

$$11.9-27 \quad (a) L_{\max} = 14.9 \text{ in.; (b) } d_{\min} = 1.12 \text{ in.}$$

$$11.9-28 \quad (a) L_{\max} = 473 \text{ mm; (b) } d_{\min} = 33.4 \text{ mm}$$

$$11.9-29 \quad P_{\text{allow}} = \begin{pmatrix} 34.6 \\ 28.0 \\ 19.9 \end{pmatrix} \text{ k for } L = \begin{pmatrix} 5.0 \\ 7.5 \\ 10.0 \end{pmatrix} \text{ ft}$$

$$11.9-30 \quad P_{\text{allow}} = \begin{pmatrix} 193.8 \\ 177.3 \\ 153.5 \end{pmatrix} \text{ kN for } L = \begin{pmatrix} 1.5 \\ 2.0 \\ 2.5 \end{pmatrix} \text{ m}$$

$$11.9-31 \quad P_{\text{allow}} = \begin{pmatrix} 22.8 \\ 20.2 \\ 16.7 \end{pmatrix} \text{ k for } L = \begin{pmatrix} 6.0 \\ 8.0 \\ 10.0 \end{pmatrix} \text{ ft}$$

$$11.9-32 \quad P_{\text{allow}} = \begin{pmatrix} 310 \\ 255 \\ 190 \end{pmatrix} \text{ kN for } L = \begin{pmatrix} 2.5 \\ 3.5 \\ 4.5 \end{pmatrix} \text{ m}$$

$$11.9-33 \quad (a) L_{\max} = 10.37 \text{ ft; (b) } b_{\min} = 5.59 \text{ in.}$$

$$11.9-34 \quad (a) L_{\max} = 3.45 \text{ m; (b) } b_{\min} = 154.9 \text{ mm}$$

$$11.9-35 \quad (a) L_{\max} = 10.25 \text{ ft; (b) } b_{\min} = 3.47 \text{ in.}$$

$$11.9-36 \quad (a) L_{\max} = 2.50 \text{ m; (b) } b_{\min} = 134.8 \text{ mm}$$

CHAPTER 12

$$12.3-2 \quad \bar{x} = \bar{y} = 5a/12$$

$$12.3-3 \quad \bar{y} = 1.10 \text{ in.}$$

$$12.3-4 \quad 2c^2 = ab$$

$$12.3-5 \quad \bar{y} = 13.94 \text{ in.}$$

$$12.3-6 \quad \bar{y} = 52.5 \text{ mm}$$

$$12.3-7 \quad \bar{x} = 0.99 \text{ in., } \bar{y} = 1.99 \text{ in.}$$

$$12.3-8 \quad \bar{x} = 137 \text{ mm., } \bar{y} = 132 \text{ mm}$$

$$12.4-6 \quad I_x = 518 \times 10^3 \text{ mm}^4$$

$$12.4-7 \quad I_x = 36.1 \text{ in.}^4, I_y = 10.9 \text{ in.}^4$$

$$12.4-8 \quad I_x = I_y = 194.6 \times 10^6 \text{ mm}^4,$$

$$r_x = r_y = 80.1 \text{ mm}$$

$$12.4-9 \quad I_1 = 1480 \text{ in.}^4, I_2 = 186 \text{ in.}^4, r_1 = 7.10 \text{ in.,}$$

$$r_2 = 2.52 \text{ in.}$$

$$12.5-1 \quad I_b = 940 \text{ in.}^4$$

$$12.5-2 \quad I_c = 11a^4/192$$

$$12.5-3 \quad I_{x_c} = 7.23 \text{ in.}^4$$

$$12.5-4 \quad I_2 = 405 \times 10^3 \text{ mm}^4$$

$$12.5-5 \quad I_{x_c} = 6050 \text{ in.}^4, I_{y_c} = 475 \text{ in.}^4$$

$$12.5-6 \quad I_{x_c} = 106 \times 10^6 \text{ mm}^4$$

$$12.5-7 \quad I_{x_c} = 17.40 \text{ in.}^4, I_{y_c} = 6.27 \text{ in.}^4$$

$$12.5-8 \quad b = 250 \text{ mm}$$

$$12.6-1 \quad I_P = bh(b^2 + 12h^2)/48$$

$$12.6-2 \quad (I_P)_C = r^4(9\alpha^2 - 8\sin^2\alpha)/18\alpha$$

$$12.6-3 \quad I_P = 233 \text{ in.}^4$$

$$12.6-4 \quad I_P = bh(b^2 + h^2)/24$$

$$12.6-5 \quad (I_P)_C = r^4(176 - 84\pi + 9\pi^2)/[72(4 - \pi)]$$

$$12.7-2 \quad I_{xy} = r^4/24$$

$$12.7-3 \quad b = 2r$$

$$12.7-4 \quad I_{xy} = t^2(2b^2 - t^2)/4$$

$$12.7-5 \quad I_{12} = -20.5 \text{ in.}^4$$

$$12.7-6 \quad I_{xy} = 24.3 \times 10^6 \text{ mm}^4$$

$$12.7-7 \quad I_{x_c y_c} = -6.079 \text{ in.}^4$$

$$12.8-1 \quad I_{x_1} = I_{y_1} = b^4/12, I_{x_1 y_1} = 0$$

$$12.8-2 \quad I_{x_1} = \frac{b^3 h^3}{6(b^2 + h^2)}, I_{y_1} = \frac{bh(b^4 + h^4)}{12(b^2 + h^2)},$$

$$I_{x_1 y_1} = \frac{b^2 h^2 (h^2 - b^2)}{12(b^2 + h^2)}$$

1082 Answers to Problems

- 12.8-3** $I_d = 159 \text{ in.}^4$
- 12.8-4** $I_{x_1} = 12.44 \times 10^6 \text{ mm}^4$,
 $I_{y_1} = 9.68 \times 10^6 \text{ mm}^4$, $I_{x_1 y_1} = 6.03 \times 10^6 \text{ mm}^4$
- 12.8-5** $I_{x_1} = 13.50 \text{ in.}^4$, $I_{y_1} = 3.84 \text{ in.}^4$,
 $I_{x_1 y_1} = 4.76 \text{ in.}^4$
- 12.8-6** $I_{x_1} = 8.75 \times 10^6 \text{ mm}^4$, $I_{y_1} = 1.02 \times 10^6 \text{ mm}^4$,
 $I_{x_1 y_1} = -0.356 \times 10^6 \text{ mm}^4$
- 12.9-1** (a) $c = \sqrt{a^2 - b^2}/2$; (b) $a/b = \sqrt{5}$;
(c) $1 \leq a/b < \sqrt{5}$
- 12.9-2** Shows that two different sets of principal axes exist at each point.
- 12.9-3** $\theta_{p1} = -29.87^\circ$, $\theta_{p2} = 60.13^\circ$, $I_1 = 311.1 \text{ in.}^4$,
 $I_2 = 88.9 \text{ in.}^4$
- 12.9-4** $\theta_{p1} = -8.54^\circ$, $\theta_{p2} = 81.46^\circ$,
 $I_1 = 17.24 \times 10^6 \text{ mm}^4$, $I_2 = 4.88 \times 10^6 \text{ mm}^4$
- 12.9-5** $\theta_{p1} = 37.73^\circ$, $\theta_{p2} = 127.73^\circ$, $I_1 = 15.45 \text{ in.}^4$,
 $I_2 = 1.89 \text{ in.}^4$
- 12.9-6** $\theta_{p1} = 32.63^\circ$, $\theta_{p2} = 122.63^\circ$,
 $I_1 = 8.76 \times 10^6 \text{ mm}^4$, $I_2 = 1.00 \times 10^6 \text{ mm}^4$
- 12.9-7** $\theta_{p1} = 16.85^\circ$, $\theta_{p2} = 106.85^\circ$, $I_1 = 0.2390b^4$,
 $I_2 = 0.0387b^4$
- 12.9-8** $\theta_{p1} = 74.08^\circ$, $\theta_{p2} = -15.92^\circ$,
 $I_1 = 8.29 \times 10^6 \text{ mm}^4$, $I_2 = 1.00 \times 10^6 \text{ mm}^4$
- 12.9-9** $\theta_{p1} = 75.73^\circ$, $\theta_{p2} = -14.27^\circ$, $I_1 = 20.07 \text{ in.}^4$,
 $I_2 = 2.12 \text{ in.}^4$

APPENDIX A

A1.1	C	A2.15	D	A5.7	B	A8.12	A
A1.2	C	A2.16	C	A5.8	C	A8.13	C
A1.3	D	A3.1	D	A5.9	C	A8.14	B
A1.4	A	A3.2	A	A5.10	A	A8.15	C
A1.5	B	A3.3	C	A5.11	B	A9.1	C
A1.6	A	A3.4	A	A5.12	B	A9.2	C
A1.7	A	A3.5	D	A6.1	B	A9.3	B
A1.8	D	A3.6	D	A6.2	C	A9.4	C
A1.9	A	A3.7	B	A6.3	B	A9.5	B
A1.10	C	A3.8	B	A6.4	B	A9.6	C
A1.11	D	A3.9	C	A6.5	D	A9.7	D
A1.12	D	A3.10	B	A7.1	C	A10.1	B
A1.13	D	A3.11	D	A7.2	C	A10.2	B
A1.14	A	A3.12	B	A7.3	D	A10.3	B
A1.15	B	A3.13	B	A7.4	A	A10.4	D
A1.16	C	A3.14	D	A7.5	B	A10.5	B
A2.1	D	A3.15	B	A7.6	A	A10.6	D
A2.2	B	A4.1	D	A7.7	C	A10.7	A
A2.3	A	A4.2	C	A7.8	D	A11.1	D
A2.4	A	A4.3	D	A8.1	A	A11.2	B
A2.5	D	A4.4	A	A8.2	C	A11.3	D
A2.6	A	A4.5	A	A8.3	D	A11.4	A
A2.7	B	A4.6	C	A8.4	B	A11.5	D
A2.8	C	A4.7	B	A8.5	C	A11.6	A
A2.9	A	A5.1	A	A8.6	A	A11.7	B
A2.10	D	A5.2	C	A8.7	D	A11.8	B
A2.11	D	A5.3	D	A8.8	D	A11.9	B
A2.12	C	A5.4	D	A8.9	A	A11.10	C
A2.13	A	A5.5	A	A8.10	D	A11.11	D
A2.14	C	A5.6	B	A8.11	D		

Appendix A

FE Exam Review Problems

A-1.1: A plane truss has downward applied load P at joint 2 and another load P applied leftward at joint 5. The force in member 3–5 is:

- (A) 0
- (B) $-P/2$
- (C) $-P$
- (D) $+1.5 P$

Solution

$$\Sigma M_1 = 0$$

$$V_6(3L) + PL - PL = 0$$

$$\text{so } V_6 = 0$$

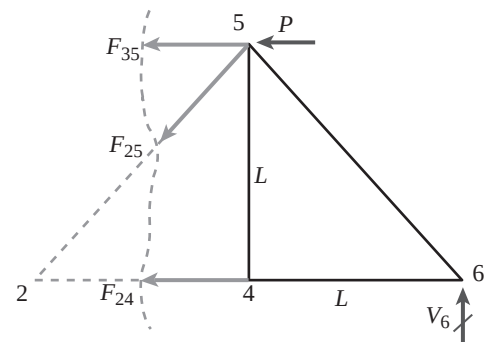
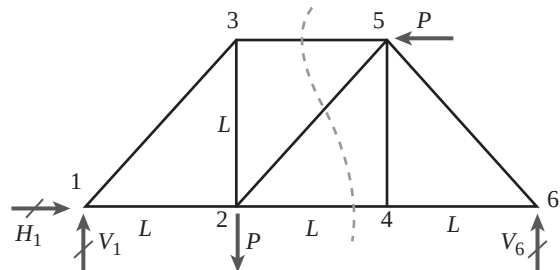
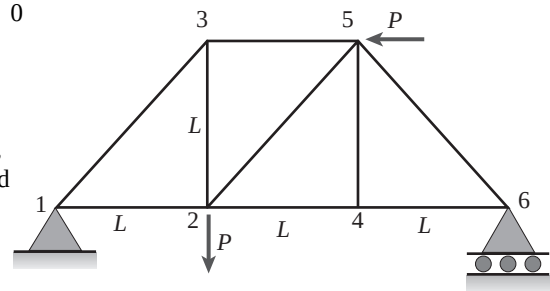
Method of sections

Cut through members 3-5, 2-5 and 2-4; use right hand FBD

$$\Sigma M_2 = 0$$

$$F_{35}L + PL = 0$$

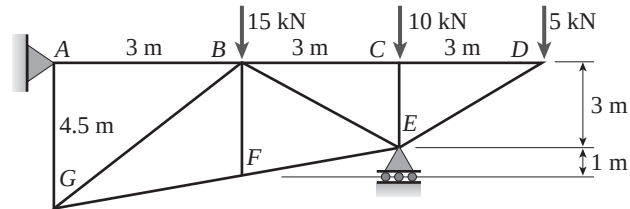
$$F_{35} = -P$$



A-1.2: The force in member FE of the plane truss below is approximately:

- (A) -1.5 kN
- (B) -2.2 kN
- (C) 3.9 kN
- (D) 4.7 kN

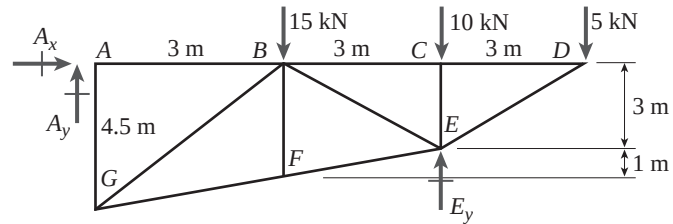
Solution



Statics

$$\sum M_A = 0 \quad E_y(6 \text{ m}) - 15 \text{ kN}(3 \text{ m}) - 10 \text{ kN}(6 \text{ m}) - 5 \text{ kN}(9 \text{ m}) = 0$$

$$E_y = 25 \text{ kN}$$



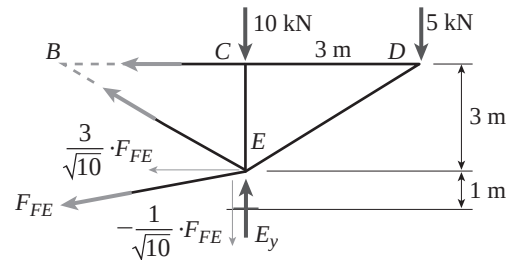
Method of sections: cut through BC, BE and FE; use right-hand FBD; sum moments about B

$$\frac{-3}{\sqrt{10}} F_{FE}(3 \text{ m}) - \frac{1}{\sqrt{10}} F_{FE}(3 \text{ m}) - 10 \text{ kN}(3 \text{ m}) - 5 \text{ kN}(6 \text{ m}) + E_y(3 \text{ m}) = 0$$

Solving

$$F_{FE} = \frac{5}{4} \sqrt{10} \text{ kN}$$

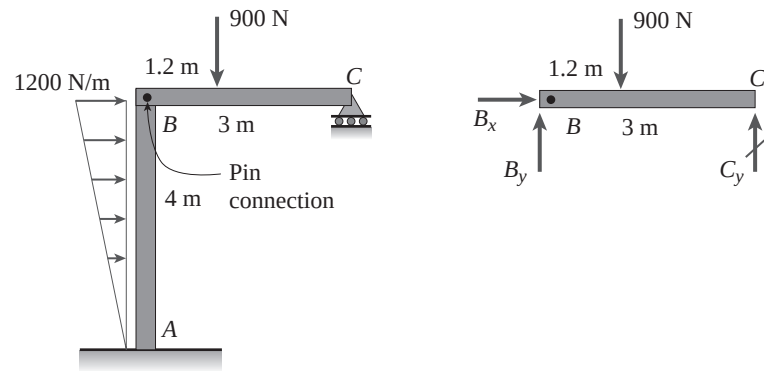
$$F_{FE} = 3.95 \text{ kN}$$



A-1.3: The moment reaction at A in the plane frame below is approximately:

- (A) +1400 N·m
- (B) -2280 N·m
- (C) -3600 N·m
- (D) +6400 N·m

Solution



Statics: use FBD of member BC to find reaction C_y

$$\Sigma M_B = 0 \quad C_y(3 \text{ m}) - 900 \text{ N}(1.2 \text{ m}) = 0$$

$$C_y = \frac{900 \text{ N}(1.2 \text{ m})}{3 \text{ m}} = 360 \text{ N}$$

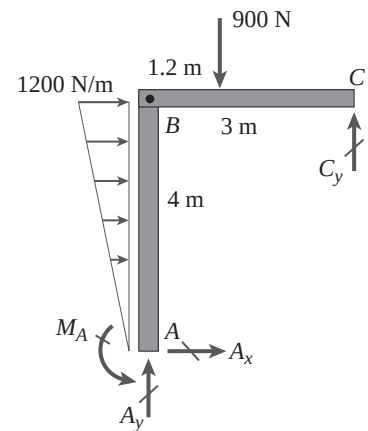
Sum moments about A for entire structure

$$\Sigma M_A = 0$$

$$M_A + C_y(3 \text{ m}) - 900 \text{ N}(1.2 \text{ m}) - \frac{1}{2} \left(1200 \frac{\text{N}}{\text{m}} \right) 4 \text{ m} \left(\frac{2}{3} 4 \text{ m} \right) = 0$$

Solving for M_A

$$M_A = 6400 \text{ N} \cdot \text{m} \quad \leftarrow$$



A-1.4: A hollow circular post ABC (see figure) supports a load $P_1 = 16$ kN acting at the top. A second load P_2 is uniformly distributed around the cap plate at B . The diameters and thicknesses of the upper and lower parts of the post are $d_{AB} = 30$ mm, $t_{AB} = 12$ mm, $d_{BC} = 60$ mm, and $t_{BC} = 9$ mm, respectively. The lower part of the post must have the same compressive stress as the upper part. The required magnitude of the load P_2 is approximately:

- (A) 18 kN
- (B) 22 kN
- (C) 28 kN
- (D) 46 kN

Solution

$$P_1 = 16 \text{ kN} \quad d_{AB} = 30 \text{ mm} \quad t_{AB} = 12 \text{ mm}$$

$$d_{BC} = 60 \text{ mm} \quad t_{BC} = 9 \text{ mm}$$

$$A_{AB} = \frac{\pi}{4} [d_{AB}^2 - (d_{AB} - 2t_{AB})^2] = 679 \text{ mm}^2$$

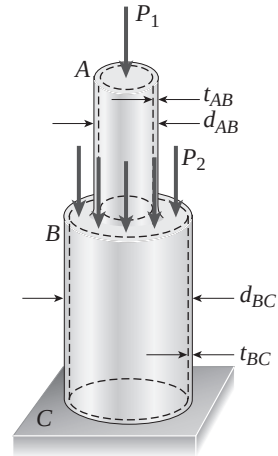
$$A_{BC} = \frac{\pi}{4} [d_{BC}^2 - (d_{BC} - 2t_{BC})^2] = 1442 \text{ mm}^2$$

Stress in AB : $\sigma_{AB} = \frac{P_1}{A_{AB}} = 23.6 \text{ MPa}$

Stress in BC : $\sigma_{BC} = \frac{P_1 + P_2}{A_{BC}} < \text{must equal } \sigma_{AB}$

Solve for P_2 : $P_2 = \sigma_{AB} A_{BC} - P_1 = 18.00 \text{ kN}$ ←

Check: $\sigma_{BC} = \frac{P_1 + P_2}{A_{BC}} = 23.6 \text{ MPa} < \text{same as in } AB$



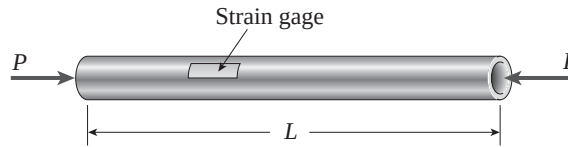
A-1.5: A circular aluminum tube of length $L = 650$ mm is loaded in compression by forces P . The outside and inside diameters are 80 mm and 68 mm, respectively. A strain gage on the outside of the bar records a normal strain in the longitudinal direction of 400×10^{-6} . The shortening of the bar is approximately:

- (A) 0.12 mm
- (B) 0.26 mm
- (C) 0.36 mm
- (D) 0.52 mm

Solution

$$\epsilon = 400 (10^{-6}) \quad L = 650 \text{ mm}$$

$$\delta = \epsilon L = 0.260 \text{ mm} \quad \leftarrow$$



A-1.6: A steel plate weighing 27 kN is hoisted by a cable sling that has a clevis at each end. The pins through the clevises are 22 mm in diameter. Each half of the cable is at an angle of 35° to the vertical. The average shear stress in each pin is approximately:

- (A) 22 MPa
- (B) 28 MPa
- (C) 40 MPa
- (D) 48 MPa

Solution

$$W = 27 \text{ kN} \quad d_p = 22 \text{ mm} \quad \theta = 35^\circ$$

Cross sectional area of each pin:

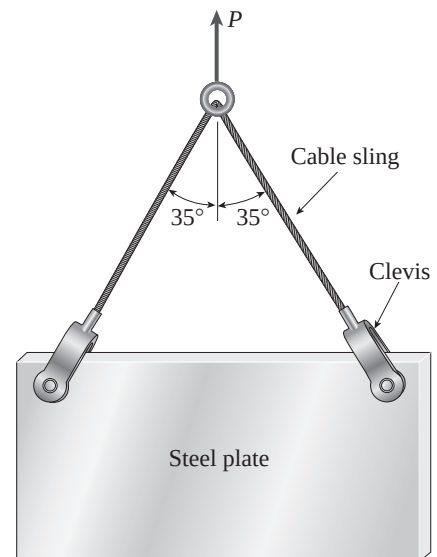
$$A_p = \frac{\pi}{4} d_p^2 = 380 \text{ mm}^2$$

Tensile force in cable:

$$T = \frac{\left(\frac{W}{2}\right)}{\cos(\theta)} = 16.48 \text{ kN}$$

Shear stress in each clevis pin (double shear):

$$\tau = \frac{T}{2A_p} = 21.7 \text{ MPa} \quad \leftarrow$$



A-1.7: A steel wire hangs from a high-altitude balloon. The steel has unit weight 77 kN/m^3 and yield stress of 280 MPa. The required factor of safety against yield is 2.0. The maximum permissible length of the wire is approximately:

- (A) 1800 m
- (B) 2200 m
- (C) 2600 m
- (D) 3000 m

Solution

$$\gamma = 77 \frac{\text{kN}}{\text{m}^3} \quad \sigma_Y = 280 \text{ MPa} \quad FS_Y = 2$$

Allowable stress: $\sigma_{\text{allow}} = \frac{\sigma_Y}{FS_Y} = 140.0 \text{ MPa}$

Weight of wire of length L : $W = \gamma AL$

Max. axial stress in wire of length L : $\sigma_{\text{max}} = \frac{W}{A} \quad \sigma_{\text{max}} = \gamma L$

Max. length of wire: $L_{\text{max}} = \frac{\sigma_{\text{allow}}}{\gamma} = 1818 \text{ m}$ ←

A-1.8: An aluminum bar ($E = 72 \text{ GPa}$, $\nu = 0.33$) of diameter 50 mm cannot exceed a diameter of 50.1 mm when compressed by axial force P . The maximum acceptable compressive load P is approximately:

- (A) 190 kN
- (B) 200 kN
- (C) 470 kN
- (D) 860 kN

Solution

$$E = 72 \text{ GPa} \quad d_{\text{init}} = 50 \text{ mm} \quad d_{\text{final}} = 50.1 \text{ mm} \quad \nu = 0.33$$

Lateral strain: $\epsilon_L = \frac{d_{\text{final}} - d_{\text{init}}}{d_{\text{init}}} \quad \epsilon_L = 0.002$

Axial strain: $\epsilon_a = \frac{-\epsilon_L}{\nu} = -0.006$

Axial stress: $\sigma = E \epsilon_a = -436.4 \text{ MPa} \quad < \text{below yield stress of } 480 \text{ MPa}$
so Hooke's Law applies

Max. acceptable compressive load:

$$P_{\text{max}} = \sigma \left(\frac{\pi}{4} d_{\text{init}}^2 \right) = 857 \text{ kN} \quad \leftarrow$$

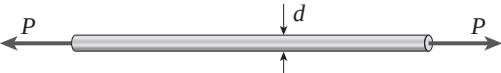
A-1.9: An aluminum bar ($E = 70 \text{ GPa}$, $\nu = 0.33$) of diameter 20 mm is stretched by axial forces P , causing its diameter to decrease by 0.022 mm. The load P is approximately:

- (A) 73 kN
- (B) 100 kN
- (C) 140 kN
- (D) 339 kN

Solution

$$E = 70 \text{ GPa} \quad d_{\text{init}} = 20 \text{ mm} \quad \Delta d = -0.022 \text{ mm} \quad \nu = 0.33$$

Lateral strain: $\varepsilon_L = \frac{\Delta d}{d_{\text{init}}}$



$$\varepsilon_L = -0.001$$

Axial strain: $\varepsilon_a = \frac{-\varepsilon_L}{\nu} = 3.333 \times 10^{-3}$

Axial stress: $\sigma = E \varepsilon_a = 233.3 \text{ MPa}$ < below yield stress of 270 MPa
so Hooke's Law applies

Max. acceptable load:

$$P_{\text{max}} = \sigma \left(\frac{\pi}{4} d_{\text{init}}^2 \right) = 73.3 \text{ kN} \quad \leftarrow$$

A-1.10: A polyethylene bar ($E = 1.4 \text{ GPa}$, $\nu = 0.4$) of diameter 80 mm is inserted in a steel tube of inside diameter 80.2 mm and then compressed by axial force P . The gap between steel tube and polyethylene bar will close when compressive load P is approximately:

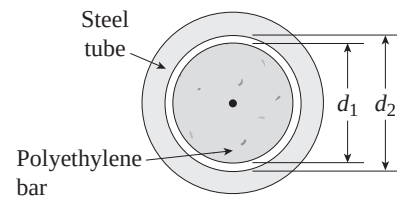
- (A) 18 kN
- (B) 25 kN
- (C) 44 kN
- (D) 60 kN

Solution

$$E = 1.4 \text{ GPa} \quad d_1 = 80 \text{ mm} \quad \Delta d_1 = 0.2 \text{ mm} \quad \nu = 0.4$$

Lateral strain: $\varepsilon_L = \frac{\Delta d_1}{d_1}$

$$\varepsilon_L = 0.003$$



Axial strain: $\varepsilon_a = \frac{-\varepsilon_L}{\nu} = -6.250 \times 10^{-3}$

Axial stress: $\sigma = E \varepsilon_a = -8.8 \text{ MPa}$ < well below ultimate stress of 28 MPa so Hooke's Law applies

Max. acceptable compressive load:

$$P_{\text{max}} = \sigma \left(\frac{\pi}{4} d_1^2 \right) = 44.0 \text{ kN} \quad \leftarrow$$

A-1.11: A pipe ($E = 110$ GPa) carries a load $P_1 = 120$ kN at A and a uniformly distributed load $P_2 = 100$ kN on the cap plate at B. Initial pipe diameters and thicknesses are: $d_{AB} = 38$ mm, $t_{AB} = 12$ mm, $d_{BC} = 70$ mm, $t_{BC} = 10$ mm. Under loads P_1 and P_2 , wall thickness t_{BC} increases by 0.0036 mm. Poisson's ratio ν for the pipe material is approximately:

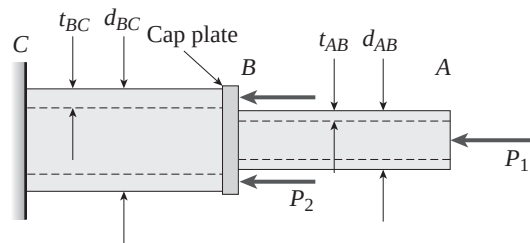
- (A) 0.27
(B) 0.30
(C) 0.31
(D) 0.34

Solution

$$E = 110 \text{ GPa} \quad d_{AB} = 38 \text{ mm} \quad t_{AB} = 12 \text{ mm} \quad d_{BC} = 70 \text{ mm}$$

$$t_{BC} = 10 \text{ mm} \quad P_1 = 120 \text{ kN} \quad P_2 = 100 \text{ kN}$$

$$A_{BC} = \frac{\pi}{4} [d_{BC}^2 - (d_{BC} - 2t_{BC})^2] = 1885 \text{ mm}^2$$



$$\text{Axial strain of BC: } \epsilon_{BC} = \frac{-(P_1 + P_2)}{EA_{BC}} = -1.061 \times 10^{-3}$$

$$\text{Axial stress in BC: } \sigma_{BC} = E\epsilon_{BC} = -116.7 \text{ MPa}$$

(well below yield stress of 550 MPa so Hooke's Law applies)

$$\text{Lateral strain of BC: } \Delta t_{BC} = 0.0036 \text{ mm}$$

$$\epsilon_L = \frac{\Delta t_{BC}}{t_{BC}} = 3.600 \times 10^{-4}$$

$$\text{Poisson's ratio: } \nu = \frac{-\epsilon_L}{\epsilon_{BC}} = 0.34$$

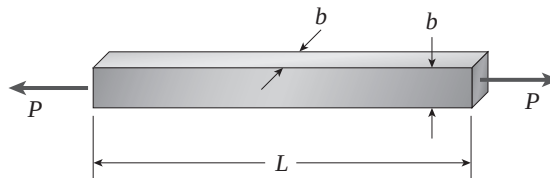
← < confirms value for **brass** given in properties table

A-1.12: A titanium bar ($E = 100$ GPa, $\nu = 0.33$) with square cross section ($b = 75$ mm) and length $L = 3.0$ m is subjected to tensile load $P = 900$ kN. The increase in volume of the bar is approximately:

- (A) 1400 mm³
- (B) 3500 mm³
- (C) 4800 mm³
- (D) 9200 mm³

Solution

$$E = 100 \text{ GPa} \quad b = 75 \text{ mm} \quad L = 3.0 \text{ m} \quad P = 900 \text{ kN} \quad \nu = 0.33$$



$$\text{Initial volume of bar:} \quad V_{\text{init}} = b^2 L = 1.6875000 \times 10^7 \text{ mm}^3$$

$$\text{Normal strain in bar:} \quad \varepsilon = \frac{P}{Eb^2} = 1.60000 \times 10^{-3}$$

$$\text{Lateral strain in bar:} \quad \varepsilon_L = -\nu\varepsilon = -5.28000 \times 10^{-4}$$

$$\text{Final length of bar:} \quad L_f = L + \varepsilon L = 3004.800 \text{ mm}$$

$$\text{Final lateral dimension of bar:} \quad b_f = b + \varepsilon_L b = 74.96040 \text{ mm}$$

$$\text{Final volume of bar:} \quad V_{\text{final}} = b_f^2 L_f = 1.68841562 \times 10^7 \text{ mm}^3$$

$$\text{Increase in volume of bar:} \quad \Delta V = V_{\text{final}} - V_{\text{init}} = 9156 \text{ mm}^3 \quad \leftarrow$$

$$\frac{\Delta V}{V_{\text{init}}} = 0.000543$$

A-1.13: An elastomeric bearing pad is subjected to a shear force V during a static loading test. The pad has dimensions $a = 150$ mm and $b = 225$ mm, and thickness $t = 55$ mm. The lateral displacement of the top plate with respect to the bottom plate is 14 mm under a load $V = 16$ kN. The shear modulus of elasticity G of the elastomer is approximately:

- (A) 1.0 MPa
- (B) 1.5 MPa
- (C) 1.7 MPa
- (D) 1.9 MPa

Solution

$$V = 16 \text{ kN} \quad a = 150 \text{ mm} \quad b = 225 \text{ mm} \quad d = 14 \text{ mm} \quad t = 55 \text{ mm}$$

Ave. shear stress:

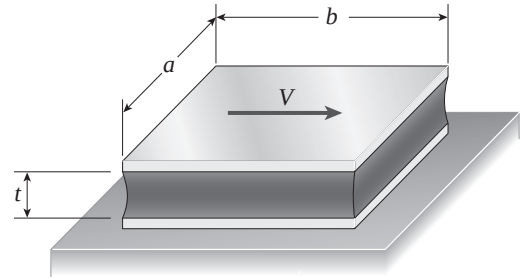
$$\tau = \frac{V}{ab} = 0.474 \text{ MPa}$$

Ave. shear strain:

$$\gamma = \arctan\left(\frac{d}{t}\right) = 0.249$$

Shear modulus of elastomer:

$$G = \frac{\tau}{\gamma} = 1.902 \text{ MPa}$$

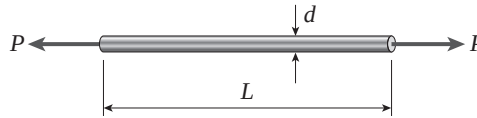


A-1.14: A bar of diameter $d = 18 \text{ mm}$ and length $L = 0.75 \text{ m}$ is loaded in tension by forces P . The bar has modulus $E = 45 \text{ GPa}$ and allowable normal stress of 180 MPa . The elongation of the bar must not exceed 2.7 mm . The allowable value of forces P is approximately:

- (A) 41 kN
- (B) 46 kN
- (C) 56 kN
- (D) 63 kN

Solution

$$d = 18 \text{ mm} \quad L = 0.75 \text{ m} \quad E = 45 \text{ GPa} \quad \sigma_a = 180 \text{ MPa} \\ \delta_a = 2.7 \text{ mm}$$



(1) allowable value of P based on elongation

$$\epsilon_a = \frac{\delta_a}{L} = 3.600 \times 10^{-3} \quad \sigma_{\max} = E \epsilon_a = 162.0 \text{ MPa}$$

$$P_{a1} = \sigma_{\max} \left(\frac{\pi}{4} d^2 \right) = 41.2 \text{ kN} \quad \leftarrow < \text{elongation governs}$$

(2) allowable load P based on tensile stress

$$P_{a2} = \sigma_a \left(\frac{\pi}{4} d^2 \right) = 45.8 \text{ kN}$$

A-1.15: Two flanged shafts are connected by eight 18 mm bolts. The diameter of the bolt circle is 240 mm . The allowable shear stress in the bolts is 90 MPa . Ignore friction between the flange plates. The maximum value of torque T_0 is approximately:

- (A) $19 \text{ kN}\cdot\text{m}$
- (B) $22 \text{ kN}\cdot\text{m}$
- (C) $29 \text{ kN}\cdot\text{m}$
- (D) $37 \text{ kN}\cdot\text{m}$

Solution

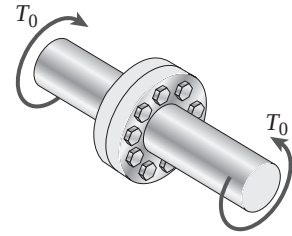
$$d_b = 18 \text{ mm} \quad d = 240 \text{ mm} \quad \tau_a = 90 \text{ MPa} \quad n = 8$$

Bolt shear area:

$$A_s = \frac{\pi d_b^2}{4} = 254.5 \text{ mm}^2$$

Max. torque:

$$T_{\max} = n(\tau_a A_s) \frac{d}{2} = 22.0 \text{ kN} \cdot \text{m} \quad \leftarrow$$



A-1.16: A copper tube with wall thickness of 8 mm must carry an axial tensile force of 175 kN. The allowable tensile stress is 90 MPa. The minimum required outer diameter is approximately:

- (A) 60 mm
- (B) 72 mm
- (C) 85 mm
- (D) 93 mm

Solution

$$t = 8 \text{ mm} \quad P = 175 \text{ kN} \quad \sigma_a = 90 \text{ MPa}$$



Required area based on allowable stress:

$$A_{\text{reqd}} = \frac{P}{\sigma_a} = 1944 \text{ mm}^2$$

Area of tube of thickness t but unknown outer diameter d :

$$A = \frac{\pi}{4}[d^2 - (d - 2t)^2] \quad A = \pi t(d - t)$$

Solving for $d_{\min}$:

$$d_{\min} = \frac{P}{\sigma_a} + t = 85.4 \text{ mm} \quad \leftarrow \quad \text{so} \quad d_{\text{inner}} = d_{\min} - 2t = 69.4 \text{ mm}$$

A-2.1: Two wires, one copper and the other steel, of equal length stretch the same amount under an applied load P . The moduli of elasticity for each is: $E_s = 210$ GPa, $E_c = 120$ GPa. The ratio of the diameter of the copper wire to that of the steel wire is approximately:

- (A) 1.00
- (B) 1.08
- (C) 1.19
- (D) 1.32

Solution

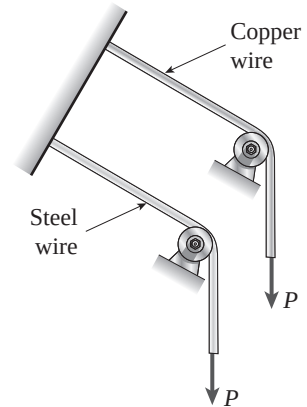
$$E_s = 210 \text{ GPa} \quad E_c = 120 \text{ GPa}$$

Displacements are equal: $\delta_s = \delta_c$

$$\text{or} \quad \frac{PL}{E_s A_s} = \frac{PL}{E_c A_c}$$

$$\text{so} \quad E_s A_s = E_c A_c$$

$$\text{and} \quad \frac{A_c}{A_s} = \frac{E_s}{E_c}$$



Express areas in terms of wire diameters then find ratio:

$$\frac{\frac{\pi d_c^2}{4}}{\left(\frac{\pi d_s^2}{4}\right)} = \frac{E_s}{E_c} \quad \text{so} \quad \frac{d_c}{d_s} = \sqrt{\frac{E_s}{E_c}} = 1.323 \quad \leftarrow$$

A-2.2: A plane truss with span length $L = 4.5$ m is constructed using cast iron pipes ($E = 170$ GPa) with cross sectional area of 4500 mm^2 . The displacement of joint B cannot exceed 2.7 mm. The maximum value of loads P is approximately:

- (A) 340 kN
- (B) 460 kN
- (C) 510 kN
- (D) 600 kN

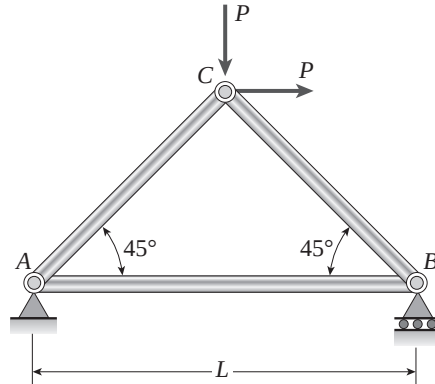
Solution

$$L = 4.5 \text{ m} \quad E = 170 \text{ GPa}$$

$$A = 4500 \text{ mm}^2 \quad \delta_{\max} = 2.7 \text{ mm}$$

Statics: sum moments about A to find reaction at B

$$R_B = \frac{P \frac{L}{2} + P \frac{L}{2}}{L} \quad R_B = P$$



Method of Joints at B:

$$F_{AB} = P \quad (\text{tension})$$

Force-displ. relation:

$$P_{\max} = \frac{EA}{L} \delta_{\max} = 459 \text{ kN} \quad \leftarrow$$

Check normal stress in bar AB: $\sigma = \frac{P_{\max}}{A} = 102.0 \text{ MPa}$

< well below yield stress of
290 MPa in tension

A-2.3: A brass rod ($E = 110 \text{ GPa}$) with cross sectional area of 250 mm^2 is loaded by forces $P_1 = 15 \text{ kN}$, $P_2 = 10 \text{ kN}$, and $P_3 = 8 \text{ kN}$. Segment lengths of the bar are $a = 2.0 \text{ m}$, $b = 0.75 \text{ m}$, and $c = 1.2 \text{ m}$. The change in length of the bar is approximately:

- (A) 0.9 mm
- (B) 1.6 mm
- (C) 2.1 mm
- (D) 3.4 mm

Solution

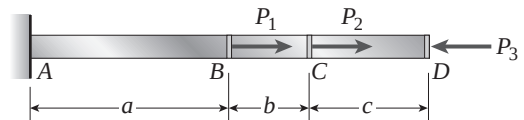
$$E = 110 \text{ GPa} \quad A = 250 \text{ mm}^2$$

$$a = 2 \text{ m} \quad b = 0.75 \text{ m}$$

$$c = 1.2 \text{ m}$$

$$P_1 = 15 \text{ kN} \quad P_2 = 10 \text{ kN}$$

$$P_3 = 8 \text{ kN}$$



Segment forces (tension is positive): $N_{AB} = P_1 + P_2 - P_3 = 17.00 \text{ kN}$

$$N_{BC} = P_2 - P_3 = 2.00 \text{ kN}$$

$$N_{CD} = -P_3 = -8.00 \text{ kN}$$

Change in length:

$$\delta_D = \frac{1}{EA} (N_{AB}a + N_{BC}b + N_{CD}c) = 0.942 \text{ mm} \quad \leftarrow$$

$$\frac{\delta_D}{a + b + c} = 2.384 \times 10^{-4}$$

$\wedge$ positive so elongation

Check max. stress:

$$\frac{N_{AB}}{A} = 68.0 \text{ MPa} < \text{well below yield stress for brass so OK}$$

A-2.4: A brass bar ($E = 110 \text{ GPa}$) of length $L = 2.5 \text{ m}$ has diameter $d_1 = 18 \text{ mm}$ over one-half of its length and diameter $d_2 = 12 \text{ mm}$ over the other half. Compare this nonprismatic bar to a prismatic bar of the same volume of material with constant diameter d and length L . The elongation of the prismatic bar under the same load $P = 25 \text{ kN}$ is approximately:

- (A) 3 mm
- (B) 4 mm
- (C) 5 mm
- (D) 6 mm

Solution

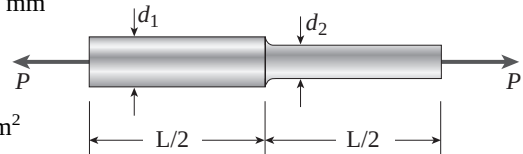
$$L = 2.5 \text{ m} \quad P = 25 \text{ kN}$$

$$d_1 = 18 \text{ mm} \quad d_2 = 12 \text{ mm}$$

$$E = 110 \text{ GPa}$$

$$A_1 = \frac{\pi}{4} d_1^2 = 254.469 \text{ mm}^2$$

$$A_2 = \frac{\pi}{4} d_2^2 = 113.097 \text{ mm}^2$$



Volume of nonprismatic bar:

$$Vol_{\text{nonprismatic}} = (A_1 + A_2) \frac{L}{2} = 459458 \text{ mm}^3$$

$$\text{Diameter of prismatic bar of same volume: } d = \sqrt{\frac{Vol_{\text{nonprismatic}}}{\frac{\pi}{4} L}} = 15.30 \text{ mm}$$

$$A_{\text{prismatic}} = \frac{\pi}{4} d^2 = 184 \text{ mm}^2$$

$$V_{\text{prismatic}} = A_{\text{prismatic}} L = 459458 \text{ mm}^3$$

Elongation of prismatic bar:

$$\delta = \frac{PL}{EA_{\text{prismatic}}} = 3.09 \text{ mm} \quad \leftarrow < \text{less than } \delta \text{ for nonprismatic bar}$$

Elongation of nonprismatic bar shown in fig. above:

$$\Delta = \frac{PL}{2E} \left(\frac{1}{A_1} + \frac{1}{A_2} \right) = 3.63 \text{ mm}$$

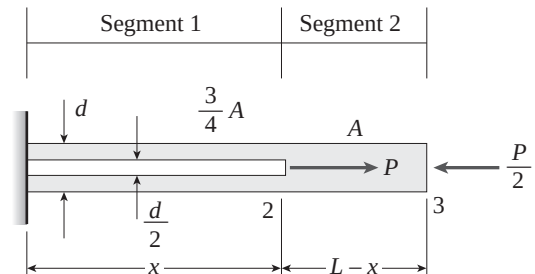
A-2.5: A nonprismatic cantilever bar has an internal cylindrical hole of diameter $d/2$ from 0 to x , so the net area of the cross section for Segment 1 is $(3/4)A$. Load P is applied at x , and load $-P/2$ is applied at $x = L$. Assume that E is constant. The length of the hollow segment, x , required to obtain axial displacement $\delta = PL/EA$ at the free end is:

- (A) $x = L/5$
- (B) $x = L/4$
- (C) $x = L/3$
- (D) $x = 3L/5$

Solution

Forces in Segments 1 & 2:

$$N_1 = \frac{3P}{2} \quad N_2 = \frac{-P}{2}$$



Displacement at free end:

$$\delta_3 = \frac{N_1 x}{E \left(\frac{3}{4} A \right)} + \frac{N_2 (L - x)}{EA}$$

$$\delta_3 = \frac{\frac{3P}{2} x}{E \left(\frac{3}{4} A \right)} + \frac{\frac{-P}{2} (L - x)}{EA} = -\frac{P(L - 5x)}{2AE}$$

Set δ_3 equal to PL/EA and solve for x

$$-\frac{P(L - 5x)}{2AE} = \frac{PL}{EA} \quad \text{or}$$

$$-\frac{P(L - 5x)}{2AE} - \frac{PL}{EA} = 0 \quad \text{simplify} \rightarrow -\frac{P(3L - 5x)}{2AE} = 0$$

$$\text{So } x = 3L/5 \quad \leftarrow$$

A-2.6: A nylon bar ($E = 2.1 \text{ GPa}$) with diameter 12 mm, length 4.5 m, and weight 5.6 N hangs vertically under its own weight. The elongation of the bar at its free end is approximately:

- (A) 0.05 mm
- (B) 0.07 mm
- (C) 0.11 mm
- (D) 0.17 mm

Solution

$$E = 2.1 \text{ GPa} \quad L = 4.5 \text{ m} \quad d = 12 \text{ mm}$$

$$A = \frac{\pi d^2}{4} = 113.097 \text{ mm}^2$$

$$\gamma = 11 \frac{\text{kN}}{\text{m}^3}$$

$$W = \gamma LA = 5.598 \text{ N}$$

$$\delta_B = \frac{WL}{2EA} \quad \text{or} \quad \delta_B = \frac{(\gamma LA)L}{2EA}$$

$$\text{so} \quad \delta_B = \frac{\gamma L^2}{2E} = 0.053 \text{ mm} \quad \leftarrow$$



$$\text{Check max. normal stress at top of bar} \quad \sigma_{\max} = \frac{W}{A} = 0.050 \text{ MPa}$$

< ok - well below ult.
stress for nylon

A-2.7: A monel shell ($E_m = 170 \text{ GPa}$, $d_3 = 12 \text{ mm}$, $d_2 = 8 \text{ mm}$) encloses a brass core ($E_b = 96 \text{ GPa}$, $d_1 = 6 \text{ mm}$). Initially, both shell and core are of length 100 mm. A load P is applied to both shell and core through a cap plate. The load P required to compress both shell and core by 0.10 mm is approximately:

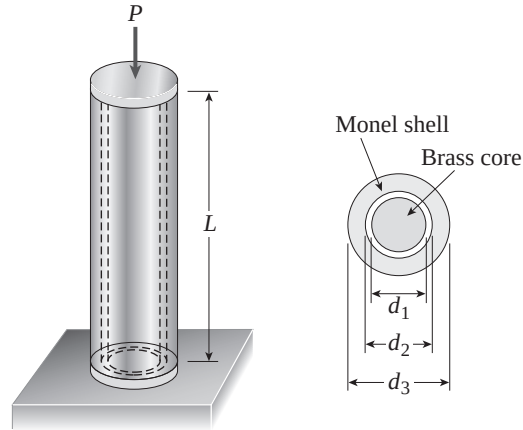
- (A) 10.2 kN
- (B) 13.4 kN
- (C) 18.5 kN
- (D) 21.0 kN

Solution

$$E_m = 170 \text{ GPa} \quad E_b = 96 \text{ GPa}$$

$$d_1 = 6 \text{ mm} \quad d_2 = 8 \text{ mm}$$

$$d_3 = 12 \text{ mm} \quad L = 100 \text{ mm}$$



$$A_m = \frac{\pi}{4}(d_3^2 - d_2^2) = 62.832 \text{ mm}^2$$

$$A_b = \frac{\pi}{4}d_1^2 = 28.274 \text{ mm}^2$$

Compatibility: $\delta_m = \delta_b$

$$\frac{P_m L}{E_m A_m} = \frac{P_b L}{E_b A_b} \quad P_m = \frac{E_m A_m}{E_b A_b} P_b$$

Statics: $P_m + P_b = P$ so $P_b = \frac{P}{\left(1 + \frac{E_m A_m}{E_b A_b}\right)}$

Set δ_b equal to 0.10 mm and solve for load P :

$$\delta_b = \frac{P_b L}{E_b A_b} \quad \text{so} \quad P_b = \frac{E_b A_b}{L} \delta_b \quad \text{with} \quad \delta_b = 0.10 \text{ mm}$$

$$\text{and then} \quad P = \frac{E_b A_b}{L} \delta_b \left(1 + \frac{E_m A_m}{E_b A_b}\right) = 13.40 \text{ kN} \quad \leftarrow$$

A-2.8: A steel rod ($E_s = 210 \text{ GPa}$, $d_r = 12 \text{ mm}$, $\alpha_s = 12 \times 10^{-6}/^\circ\text{C}$) is held stress free between rigid walls by a clevis and pin ($d_p = 15 \text{ mm}$) assembly at each end. If the allowable shear stress in the pin is 45 MPa and the allowable normal stress in the rod is 70 MPa, the maximum permissible temperature drop ΔT is approximately:

- (A) 14 °C
- (B) 20 °C
- (C) 28 °C
- (D) 40 °C

Solution

$$E_s = 210 \text{ GPa}$$

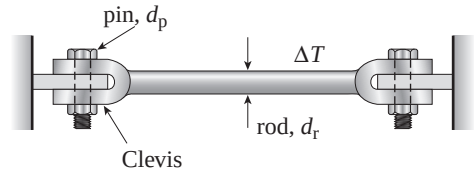
$$d_r = 12 \text{ mm} \quad d_p = 15 \text{ mm}$$

$$A_r = \frac{\pi}{4} d_r^2 = 113.097 \text{ mm}^2$$

$$A_p = \frac{\pi}{4} d_p^2 = 176.715 \text{ mm}^2$$

$$\alpha_s = 12(10^{-6})/^{\circ}\text{C}$$

$$\tau_a = 45 \text{ MPa} \quad \sigma_a = 70 \text{ MPa}$$



Force in rod due to temperature drop ΔT : and normal stress in rod:

$$F_r = E_s A_r (\alpha_s) \Delta T \quad \sigma_r = \frac{F_r}{A_r}$$

So $\Delta T_{\max}$ associated with normal stress in **rod**

$$\Delta T_{\max \text{ rod}} = \frac{\sigma_a}{E_s \alpha_s} = 27.8 \quad \leftarrow \text{degrees Celsius (decrease) } < \text{Controls}$$

Now check ΔT based on shear stress in **pin** (in double shear): $\tau_{\text{pin}} = \frac{F_r}{2A_p}$

$$\Delta T_{\max \text{ pin}} = \frac{\tau_a (2A_p)}{E_s A_r \alpha_s} = 55.8$$

A-2.9: A threaded steel rod ($E_s = 210 \text{ GPa}$, $d_r = 15 \text{ mm}$, $\alpha_s = 12 \times 10^{-6}/^{\circ}\text{C}$) is held stress free between rigid walls by a nut and washer ($d_w = 22 \text{ mm}$) assembly at each end. If the allowable bearing stress between the washer and wall is 55 MPa and the allowable normal stress in the rod is 90 MPa, the maximum permissible temperature drop ΔT is approximately:

- (A) 25 $^{\circ}\text{C}$
- (B) 30 $^{\circ}\text{C}$
- (C) 38 $^{\circ}\text{C}$
- (D) 46 $^{\circ}\text{C}$

Solution

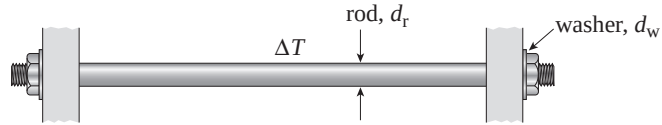
$$E_s = 210 \text{ GPa} \quad d_r = 15 \text{ mm} \quad d_w = 22 \text{ mm}$$

$$A_r = \frac{\pi}{4} d_r^2 = 176.7 \text{ mm}^2$$

$$A_w = \frac{\pi}{4} (d_w^2 - d_r^2) = 203.4 \text{ mm}^2$$

$$\alpha_s = 12(10^{-6})/^{\circ}\text{C}$$

$$\sigma_{\text{ba}} = 55 \text{ MPa} \quad \sigma_a = 90 \text{ MPa}$$



Force in rod due to temperature drop ΔT : and normal stress in rod:

$$F_r = E_s A_r (\alpha_s) \Delta T \quad \sigma_r = \frac{F_r}{A_r}$$

So $\Delta T_{\max}$ associated with normal stress in **rod**

$$\Delta T_{\max \text{ rod}} = \frac{\sigma_a}{E_s \alpha_s} = 35.7 \quad \text{degrees Celsius (decrease)}$$

Now check ΔT based on bearing stress beneath **washer**: $\sigma_b = \frac{F_r}{A_w}$

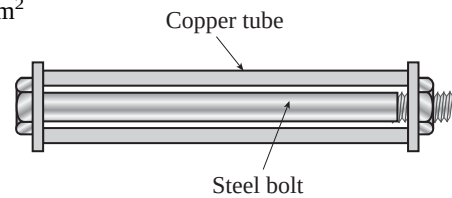
$$\Delta T_{\max \text{ washer}} = \frac{\sigma_{ba}(A_w)}{E_s A_r \alpha_s} = 25.1 \quad \leftarrow \text{degrees Celsius (decrease)} < \text{Controls}$$

A-2.10: A steel bolt (area = 130 mm^2 , $E_s = 210 \text{ GPa}$) is enclosed by a copper tube (length = 0.5 m , area = 400 mm^2 , $E_c = 110 \text{ GPa}$) and the end nut is turned until it is just snug. The pitch of the bolt threads is 1.25 mm . The bolt is now tightened by a quarter turn of the nut. The resulting stress in the bolt is approximately:

- (A) 56 MPa
- (B) 62 MPa
- (C) 74 MPa
- (D) 81 MPa

Solution

$$\begin{aligned} E_s &= 210 \text{ GPa} & E_c &= 110 \text{ GPa} & L &= 0.5 \text{ m} \\ A_c &= 400 \text{ mm}^2 & A_s &= 130 \text{ mm}^2 \\ n &= 0.25 & p &= 1.25 \text{ mm} \end{aligned}$$



Compatibility: shortening of tube and elongation of bolt = applied displacement of $n \times p$

$$\frac{P_c L}{E_c A_c} + \frac{P_s L}{E_s A_s} = np$$

Statics: $P_c = P_s$

Solve for P_s

$$\frac{P_s L}{E_c A_c} + \frac{P_s L}{E_s A_s} = np \quad \text{or} \quad P_s = \frac{np}{L \left(\frac{1}{E_c A_c} + \frac{1}{E_s A_s} \right)} = 10.529 \text{ kN}$$

Stress in steel bolt:

$$\sigma_s = \frac{P_s}{A_s} = 81.0 \text{ MPa} \quad \leftarrow < \text{tension}$$

Stress in copper tube:

$$\sigma_c = \frac{P_s}{A_c} = 26.3 \text{ MPa} \quad < \text{compression}$$

A-2.11: A steel bar of rectangular cross section ($a = 38 \text{ mm}$, $b = 50 \text{ mm}$) carries a tensile load P . The allowable stresses in tension and shear are 100 MPa and 48 MPa respectively. The maximum permissible load $P_{\max}$ is approximately:

- (A) 56 kN
- (B) 62 kN
- (C) 74 kN
- (D) 91 kN

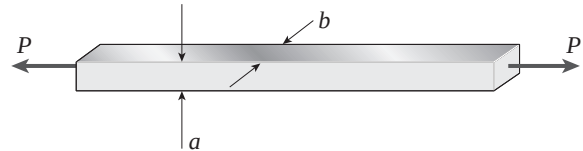
Solution

$$a = 38 \text{ mm} \quad b = 50 \text{ mm}$$

$$A = ab = 1900 \text{ mm}^2$$

$$\sigma_a = 100 \text{ MPa}$$

$$\tau_a = 48 \text{ MPa}$$



Bar is in uniaxial tension so $T_{\max} = \sigma_{\max}/2$; since $2\tau_a < \sigma_a$, shear stress governs

$$P_{\max} = \tau_a A = 91.2 \text{ kN} \quad \leftarrow$$

A-2.12: A brass wire ($d = 2.0 \text{ mm}$, $E = 110 \text{ GPa}$) is pretensioned to $T = 85 \text{ N}$. The coefficient of thermal expansion for the wire is $19.5 \times 10^{-6}/^\circ\text{C}$. The temperature change at which the wire goes slack is approximately:

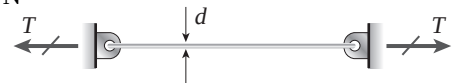
- (A) $+5.7^\circ\text{C}$
- (B) -12.6°C
- (C) $+12.6^\circ\text{C}$
- (D) -18.2°C

Solution

$$E = 110 \text{ GPa} \quad d = 2.0 \text{ mm}$$

$$\alpha_b = 19.5 (10^{-6})/^\circ\text{C} \quad T = 85 \text{ N}$$

$$A = \frac{\pi}{4} d^2 = 3.14 \text{ mm}^2$$



Normal tensile stress in wire due to pretension T and temperature increase ΔT :

$$\sigma = \frac{T}{A} - E\alpha_b \Delta T$$

Wire goes slack when normal stress goes to zero; solve for ΔT

$$\Delta T = \frac{\frac{T}{A}}{E\alpha_b} = +12.61 \quad \leftarrow \text{degrees Celsius (increase in temperature)}$$

A-2.13: A copper bar ($d = 10$ mm, $E = 110$ GPa) is loaded by tensile load $P = 11.5$ kN. The maximum shear stress in the bar is approximately:

- (A) 73 MPa
- (B) 87 MPa
- (C) 145 MPa
- (D) 150 MPa

Solution

$$E = 110 \text{ GPa} \quad d = 10 \text{ mm}$$

$$A = \frac{\pi}{4}d^2 = 78.54 \text{ mm}^2$$

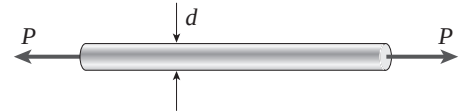
$$P = 11.5 \text{ kN}$$

Normal stress in bar:

$$\sigma = \frac{P}{A} = 146.4 \text{ MPa}$$

For bar in uniaxial stress, max. shear stress is on a plane at 45 deg. to axis of bar and equals 1/2 of normal stress:

$$\tau_{\max} = \frac{\sigma}{2} = 73.2 \text{ MPa} \quad \leftarrow$$



A-2.14: A steel plane truss is loaded at B and C by forces $P = 200$ kN. The cross sectional area of each member is $A = 3970 \text{ mm}^2$. Truss dimensions are $H = 3$ m and $L = 4$ m. The maximum shear stress in bar AB is approximately:

- (A) 27 MPa
- (B) 33 MPa
- (C) 50 MPa
- (D) 69 MPa

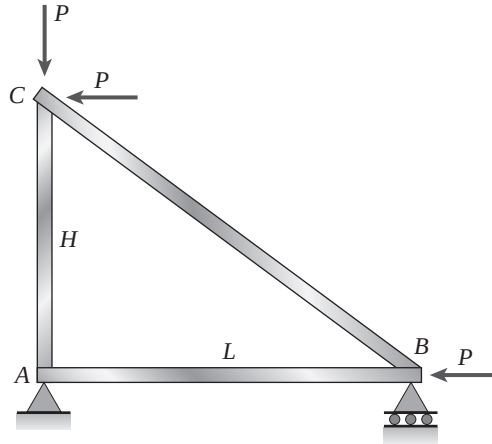
Solution

$$P = 200 \text{ kN} \quad A = 3970 \text{ mm}^2 \quad H = 3 \text{ m} \quad L = 4 \text{ m}$$

Statics: sum moments about A to find vertical reaction at B

$$B_{\text{vert}} = \frac{-PH}{L} = -150.000 \text{ kN}$$

(downward)



Method of Joints at B:

$$CB_{\text{vert}} = -B_{\text{vert}} \quad CB_{\text{horiz}} = \frac{L}{H} CB_{\text{vert}} = 200.0 \text{ kN}$$

So bar force in AB is: $AB = P + CB_{\text{horiz}} = 400.0 \text{ kN}$ (compression)

$$\text{Max. normal stress in AB: } \sigma_{AB} = \frac{AB}{A} = 100.8 \text{ MPa}$$

Max. shear stress is 1/2 of max. normal stress for bar in uniaxial stress and is on plane at 45 deg. to axis of bar:

$$\tau_{\text{max}} = \frac{\sigma_{AB}}{2} = 50.4 \text{ MPa} \quad \leftarrow$$

A-2.15: A plane stress element on a bar in uniaxial stress has tensile stress of $\sigma_\theta = 78 \text{ MPa}$ (see fig.). The maximum shear stress in the bar is approximately:

- (A) 29 MPa
- (B) 37 MPa
- (C) 50 MPa
- (D) 59 MPa

Solution

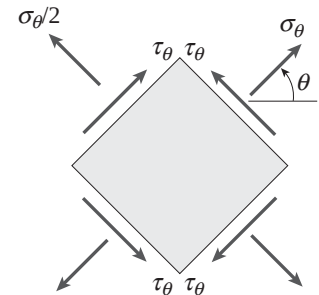
$$\sigma_\theta = 78 \text{ MPa}$$

Plane stress transformation formulas for uniaxial stress:

$$\sigma_x = \frac{\sigma_\theta}{\cos(\theta)^2} \quad \text{and} \quad \sigma_x = \frac{\sigma_\theta}{\sin(\theta)^2}$$

$\wedge$ on element face at angle θ

$\wedge$ on element face at angle $\theta + 90$



Equate above formulas and solve for σ_x

$$\tan(\theta)^2 = \frac{1}{2}$$

$$\text{so } \theta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 35.264^\circ$$

$$\sigma_x = \frac{\sigma_\theta}{\cos(\theta)^2} = 117.0 \text{ MPa} \quad \text{also } \tau_\theta = -\sigma_x \sin(\theta) \cos(\theta) = -55.154 \text{ MPa}$$

Max. shear stress is 1/2 of max. normal stress for bar in uniaxial stress and is on plane at 45 deg. to axis of bar:

$$\tau_{\max} = \frac{\sigma_x}{2} = 58.5 \text{ MPa} \quad \leftarrow$$

A-2.16: A prismatic bar (diameter $d_0 = 18 \text{ mm}$) is loaded by force P_1 . A stepped bar (diameters $d_1 = 20 \text{ mm}$, $d_2 = 25 \text{ mm}$, with radius R of fillets $= 2 \text{ mm}$) is loaded by force P_2 . The allowable axial stress in the material is 75 MPa. The ratio P_1/P_2 of the maximum permissible loads that can be applied to the bars, considering stress concentration effects in the stepped bar, is:

- (A) 0.9
- (B) 1.2
- (C) 1.4
- (D) 2.1

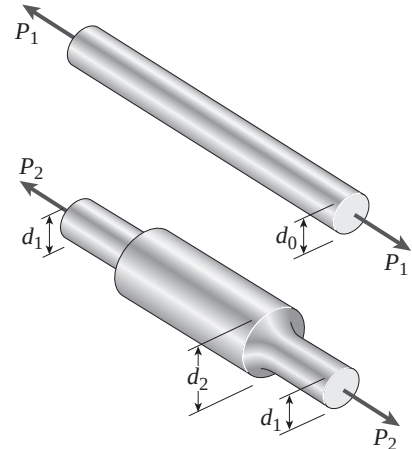
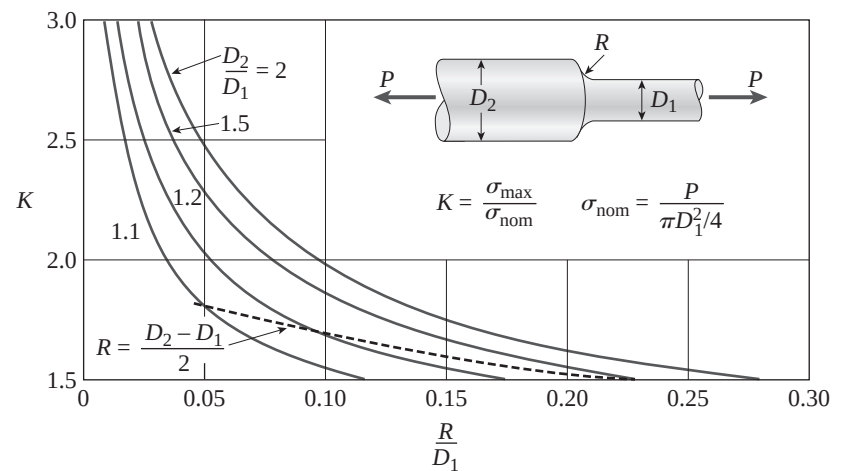


FIG. 2-66 Stress-concentration factor K for round bars with shoulder fillets. The dashed line is for a full quarter-circular fillet.

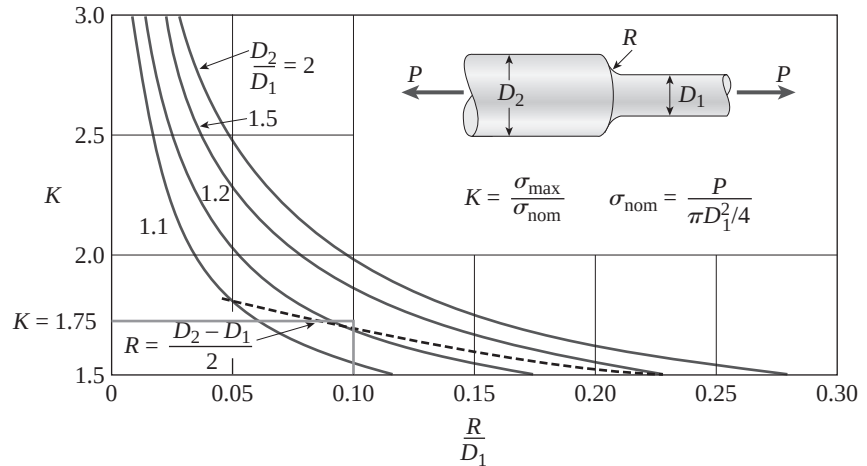


Solution

Prismatic bar $P_{1 \max} = \sigma_{\text{allow}} \left(\frac{\pi d_0^2}{4} \right) = (75 \text{ MPa}) \left[\frac{\pi (18 \text{ mm})^2}{4} \right] = 19.1 \text{ kN}$

Stepped bar $\frac{R}{d_1} = \frac{2 \text{ mm}}{20 \text{ mm}} = 0.100$ $\frac{d_2}{d_1} = \frac{25 \text{ mm}}{20 \text{ mm}} = 1.250$ so $K = 1.75$

from stress conc. Fig. 2-66



$$P_{2 \max} = \frac{\sigma_{\text{allow}}}{K} \left(\frac{\pi d_1^2}{4} \right) = \left(\frac{75 \text{ MPa}}{K} \right) \left[\frac{\pi (20 \text{ mm})^2}{4} \right] = 13.5 \text{ kN}$$

$$\frac{P_{1 \max}}{P_{2 \max}} = \frac{19.1 \text{ kN}}{13.5 \text{ kN}} = 1.41 \quad \leftarrow$$

A-3.1: A brass rod of length $L = 0.75 \text{ m}$ is twisted by torques T until the angle of rotation between the ends of the rod is 3.5° . The allowable shear strain in the copper is 0.0005 rad . The maximum permissible diameter of the rod is approximately:

- (A) 6.5 mm
- (B) 8.6 mm
- (C) 9.7 mm
- (D) 12.3 mm

Solution

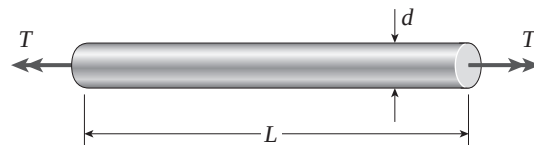
$$L = 0.75 \text{ m}$$

$$\phi = 3.5^\circ$$

$$\gamma_a = 0.0005$$

Max. shear strain:

$$\gamma_{\max} = \left(\frac{d}{2} \phi \right) / L \quad \text{so} \quad d_{\max} = \frac{2 \gamma_a L}{\phi} = 12.28 \text{ mm} \quad \leftarrow$$



A-3.2: The angle of rotation between the ends of a nylon bar is 3.5° . The bar diameter is 70 mm and the allowable shear strain is 0.014 rad. The minimum permissible length of the bar is approximately:

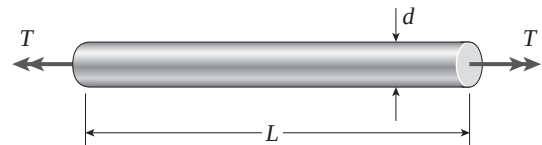
- (A) 0.15 m
- (B) 0.27 m
- (C) 0.40 m
- (D) 0.55 m

Solution

$$d = 70 \text{ mm}$$

$$\phi = 3.5^\circ$$

$$\gamma_a = 0.014$$



Max. shear strain:

$$\gamma = \frac{r\phi}{L} \quad \text{so} \quad L_{\min} = \frac{d\phi}{2\gamma_a} = 0.15 \text{ m} \quad \leftarrow$$

A-3.3: A brass bar twisted by torques T acting at the ends has the following properties: $L = 2.1 \text{ m}$, $d = 38 \text{ mm}$, and $G = 41 \text{ GPa}$. The torsional stiffness of the bar is approximately:

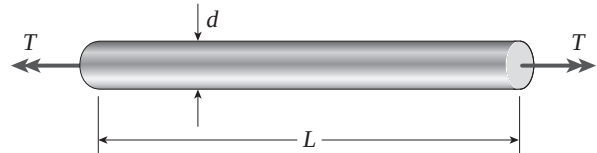
- (A) 1200 N·m
- (B) 2600 N·m
- (C) 4000 N·m
- (D) 4800 N·m

Solution

$$G = 41 \text{ GPa}$$

$$L = 2.1 \text{ m}$$

$$d = 38 \text{ mm}$$



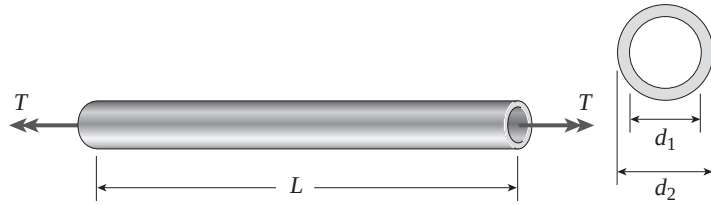
$$\text{Polar moment of inertia, } I_p: \quad I_p = \frac{\pi}{32} d^4 = 2.047 \times 10^5 \text{ mm}^4$$

Torsional stiffness, k_T :

$$k_T = \frac{GI_p}{L} = 3997 \text{ N·m} \quad \leftarrow$$

A-3.4: A brass pipe is twisted by torques $T = 800 \text{ N·m}$ acting at the ends causing an angle of twist of 3.5 degrees. The pipe has the following properties: $L = 2.1 \text{ m}$, $d_1 = 38 \text{ mm}$, and $d_2 = 56 \text{ mm}$. The shear modulus of elasticity G of the pipe is approximately:

- (A) 36.1 GPa
- (B) 37.3 GPa
- (C) 38.7 GPa
- (D) 40.6 GPa

Solution

$$L = 2.1 \text{ m} \quad d_1 = 38 \text{ mm} \quad d_2 = 56 \text{ mm} \quad \phi = 3.5^\circ \quad T = 800 \text{ N}\cdot\text{m}$$

$$\text{Polar moment of inertia: } I_p = \frac{\pi}{32}(d_2^4 - d_1^4) = 7.608 \times 10^5 \text{ mm}^4$$

Solving torque-displacement relation for shear modulus G :

$$G = \frac{TL}{\phi I_p} = 36.1 \text{ GPa} \quad \leftarrow$$

A-3.5: An aluminum bar of diameter $d = 52 \text{ mm}$ is twisted by torques T_1 at the ends. The allowable shear stress is 65 MPa . The maximum permissible torque T_1 is approximately:

- (A) $1450 \text{ N}\cdot\text{m}$
- (B) $1675 \text{ N}\cdot\text{m}$
- (C) $1710 \text{ N}\cdot\text{m}$
- (D) $1800 \text{ N}\cdot\text{m}$

Solution

$$d = 52 \text{ mm}$$

$$\tau_a = 65 \text{ MPa}$$



$$I_p = \frac{\pi}{32}d^4 = 7.178 \times 10^5 \text{ mm}^4$$

From shear formula:

$$T_{1\max} \frac{\tau_a I_p}{\left(\frac{d}{2}\right)} = 1795 \text{ N}\cdot\text{m} \quad \leftarrow$$

A-3.6: A steel tube with diameters $d_2 = 86 \text{ mm}$ and $d_1 = 52 \text{ mm}$ is twisted by torques at the ends. The diameter of a solid steel shaft that resists the same torque at the same maximum shear stress is approximately:

- (A) 56 mm
- (B) 62 mm
- (C) 75 mm
- (D) 82 mm

Solution

$$d_2 = 86 \text{ mm} \quad d_1 = 52 \text{ mm}$$

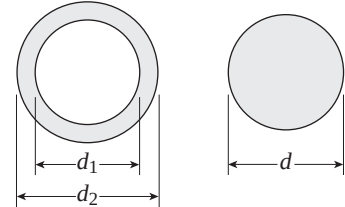
$$I_{p_{\text{pipe}}} = \frac{\pi}{32}(d_2^4 - d_1^4) = 4.652 \times 10^6 \text{ mm}^4$$

Shear formula for hollow pipe:

$$\tau_{\max} = \frac{T \left(\frac{d_2}{2} \right)}{I_{p_{\text{pipe}}}}$$

Shear formula for solid shaft:

$$\tau_{\max} = \frac{T \left(\frac{d}{2} \right)}{\frac{\pi}{32} d^4} = \frac{16 T}{\pi d^3}$$



Equate and solve for d of solid shaft:
$$d = \left[\frac{\frac{16 T}{\pi}}{\frac{T \left(\frac{d_2}{2} \right)}{I_{p_{\text{pipe}}}}} \right]^{\frac{1}{3}}$$

$$d = \left(\frac{32 I_{p_{\text{pipe}}}}{\pi d_2} \right)^{\frac{1}{3}} = 82.0 \text{ mm} \quad \leftarrow$$

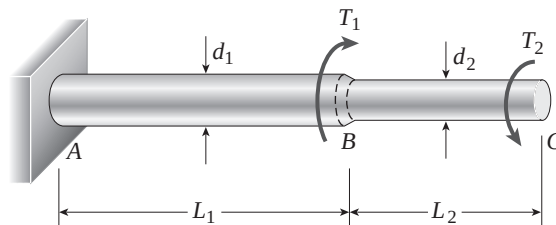
A-3.7: A stepped steel shaft with diameters $d_1 = 56 \text{ mm}$ and $d_2 = 52 \text{ mm}$ is twisted by torques $T_1 = 3.5 \text{ kN}\cdot\text{m}$ and $T_2 = 1.5 \text{ kN}\cdot\text{m}$ acting in opposite directions. The maximum shear stress is approximately:

- (A) 54 MPa
- (B) 58 MPa
- (C) 62 MPa
- (D) 79 MPa

Solution

$$d_1 = 56 \text{ mm} \quad d_2 = 52 \text{ mm}$$

$$T_1 = 3.5 \text{ kN}\cdot\text{m} \quad T_2 = 1.5 \text{ kN}\cdot\text{m}$$



Polar moments of inertia:

$$I_{p1} = \frac{\pi}{32} d_1^4 = 9.655 \times 10^5 \text{ mm}^4$$

$$I_{p2} = \frac{\pi}{32} d_2^4 = 7.178 \times 10^5 \text{ mm}^4$$

Shear formula - max. shear stresses in segments 1 & 2:

$$\tau_{\max 1} = \frac{(T_1 - T_2) \frac{d_1}{2}}{I_{p1}} = 58.0 \text{ MPa} \quad \leftarrow \quad \tau_{\max 2} = \frac{T_2 \left(\frac{d_2}{2} \right)}{I_{p2}} = 54.3 \text{ MPa}$$

A-3.8: A stepped steel shaft ($G = 75 \text{ GPa}$) with diameters $d_1 = 36 \text{ mm}$ and $d_2 = 32 \text{ mm}$ is twisted by torques T at each end. Segment lengths are $L_1 = 0.9 \text{ m}$ and $L_2 = 0.75 \text{ m}$. If the allowable shear stress is 28 MPa and maximum allowable twist is 1.8 degrees , the maximum permissible torque is approximately:

- (A) $142 \text{ N}\cdot\text{m}$
- (B) $180 \text{ N}\cdot\text{m}$
- (C) $185 \text{ N}\cdot\text{m}$
- (D) $257 \text{ N}\cdot\text{m}$

Solution

$$d_1 = 36 \text{ mm}$$

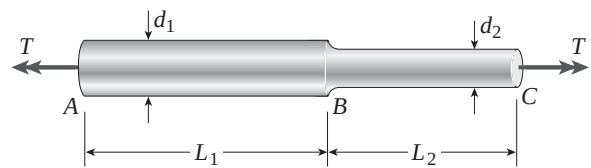
$$d_2 = 32 \text{ mm}$$

$$G = 75 \text{ GPa}$$

$$\tau_a = 28 \text{ MPa}$$

$$L_1 = 0.9 \text{ m} \quad L_2 = 0.75 \text{ m}$$

$$\phi_a = 1.8^\circ$$



Polar moments of inertia:

$$I_{p1} = \frac{\pi}{32} d_1^4 = 1.649 \times 10^5 \text{ mm}^4$$

$$I_{p2} = \frac{\pi}{32} d_2^4 = 1.029 \times 10^5 \text{ mm}^4$$

Max torque based on allowable shear stress - use shear formula:

$$\tau_{\max 1} = \frac{T \frac{d_1}{2}}{I_{p1}} \quad \tau_{\max 2} = \frac{T \left(\frac{d_2}{2} \right)}{I_{p2}}$$

$$T_{\max 1} = \tau_a \left(\frac{2I_{p1}}{d_1} \right) = 257 \text{ N}\cdot\text{m} \quad T_{\max 2} = \tau_a \left(\frac{2I_{p2}}{d_2} \right) = 180 \text{ N}\cdot\text{m} < \text{controls}$$

Max. torque based on max. rotation & torque-displacement relation:

$$\phi = \frac{T}{G} \left(\frac{L_1}{I_{p1}} + \frac{L_2}{I_{p2}} \right)$$

$$T_{\max} = \frac{G \phi_a}{\left(\frac{L_1}{I_{p1}} + \frac{L_2}{I_{p2}} \right)} = 185 \text{ N}\cdot\text{m}$$

A-3.9: A gear shaft transmits torques $T_A = 975 \text{ N}\cdot\text{m}$, $T_B = 1500 \text{ N}\cdot\text{m}$, $T_C = 650 \text{ N}\cdot\text{m}$ and $T_D = 825 \text{ N}\cdot\text{m}$. If the allowable shear stress is 50 MPa, the required shaft diameter is approximately:

- (A) 38 mm
- (B) 44 mm
- (C) 46 mm
- (D) 48 mm

Solution

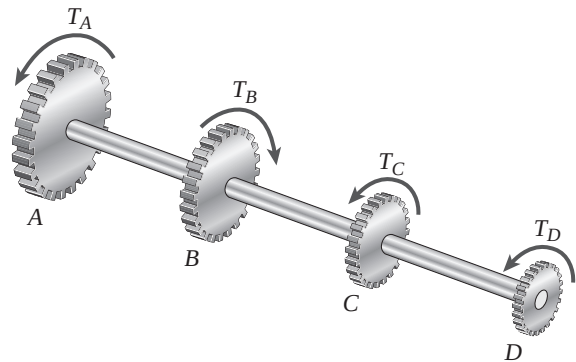
$$\tau_a = 50 \text{ MPa}$$

$$T_A = 975 \text{ N}\cdot\text{m}$$

$$T_B = 1500 \text{ N}\cdot\text{m}$$

$$T_C = 650 \text{ N}\cdot\text{m}$$

$$T_D = 825 \text{ N}\cdot\text{m}$$



Find torque in each segment of shaft:

$$T_{AB} = T_A = 975.0 \text{ N}\cdot\text{m} \quad T_{BC} = T_A - T_B = -525.0 \text{ N}\cdot\text{m}$$

$$T_{CD} = T_D = 825.0 \text{ N}\cdot\text{m}$$

Shear formula:
$$\tau = \frac{T \left(\frac{d}{2} \right)}{\frac{\pi}{32} d^4} = \frac{16T}{\pi d^3}$$

Set τ to $\tau_{\text{allowable}}$ and T to torque in each segment; solve for required diameter d (largest controls)

Segment AB:
$$d = \left(\frac{16|T_{AB}|}{\pi \tau_a} \right)^{\frac{1}{3}} = 46.3 \text{ mm} \quad \leftarrow$$

Segment BC:
$$d = \left(\frac{16|T_{BC}|}{\pi \tau_a} \right)^{\frac{1}{3}} = 37.7 \text{ mm}$$

Segment CD:
$$d = \left(\frac{16|T_{CD}|}{\pi \tau_a} \right)^{\frac{1}{3}} = 43.8 \text{ mm}$$

A-3.10: A hollow aluminum shaft ($G = 27 \text{ GPa}$, $d_2 = 96 \text{ mm}$, $d_1 = 52 \text{ mm}$) has an angle of twist per unit length of $1.8^\circ/\text{m}$ due to torques T . The resulting maximum tensile stress in the shaft is approximately:

- (A) 38 MPa
- (B) 41 MPa
- (C) 49 MPa
- (D) 58 MPa

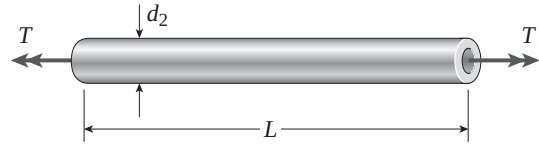
Solution

$$G = 27 \text{ GPa}$$

$$d_2 = 96 \text{ mm}$$

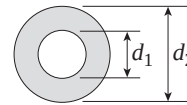
$$d_1 = 52 \text{ mm}$$

$$\theta = 1.8^\circ/\text{m}$$



Max. shear strain due to twist per unit length:

$$\gamma_{\max} = \left(\frac{d_2}{2}\right)\theta = 1.508 \times 10^{-3} \text{ radians}$$



Max. shear stress: $\tau_{\max} = G\gamma_{\max} = 40.7 \text{ MPa}$

Max. tensile stress on plane at 45 degrees & equal to max. shear stress:

$$\sigma_{\max} = \tau_{\max} = 40.7 \text{ MPa}$$

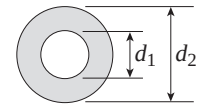
A-3.11: Torques $T = 5.7 \text{ kN}\cdot\text{m}$ are applied to a hollow aluminum shaft ($G = 27 \text{ GPa}$, $d_1 = 52 \text{ mm}$). The allowable shear stress is 45 MPa and the allowable normal strain is 8.0×10^{-4} . The required outside diameter d_2 of the shaft is approximately:

- (A) 38 mm
- (B) 56 mm
- (C) 87 mm
- (D) 91 mm

Solution

$$T = 5.7 \text{ kN}\cdot\text{m} \quad G = 27 \text{ GPa} \quad d_1 = 52 \text{ mm}$$

$$\tau_{a1} = 45 \text{ MPa} \quad \epsilon_a = 8.0(10^{-4})$$



Allowable shear strain based on allowable normal strain for pure shear

$$\gamma_a = 2\epsilon_a = 1.600 \times 10^{-3} \quad \text{so resulting allow. shear stress is:}$$

$$\tau_{a2} = G\gamma_a = 43.2 \text{ MPa}$$

So allowable shear stress based on normal strain governs $\tau_a = \tau_{a2}$

Use torsion formula to relate required d_2 to allowable shear stress:

$$\tau_{\max} = \frac{T \left(\frac{d_2}{2} \right)}{\frac{\pi}{32} (d_2^4 - d_1^4)} \quad \text{and rearrange equation to get } d_2^4 - d_1^4 = \frac{16}{\pi} \frac{T}{\tau_a} d_2$$

Solve resulting 4th order equation numerically, or use a calculator and trial & error

$$T = 5700000 \text{ N}\cdot\text{mm} \quad d_1 = 52 \text{ mm} \quad \tau_a = 43.2 \text{ MPa}$$

$$f(d_2) = d_2^4 - \left(\frac{16}{\pi} \frac{T}{\tau_a} \right) d_2 - d_1^4 \quad \text{gives } d_2 = 91 \text{ mm} \quad \leftarrow$$

A-3.12: A motor drives a shaft with diameter $d = 46 \text{ mm}$ at $f = 5.25 \text{ Hz}$ and delivers $P = 25 \text{ kW}$ of power. The maximum shear stress in the shaft is approximately:

- (A) 32 MPa
- (B) 40 MPa
- (C) 83 MPa
- (D) 91 MPa

Solution

$$f = 5.25 \text{ Hz} \quad d = 46 \text{ mm}$$

$$P = 25 \text{ kW}$$

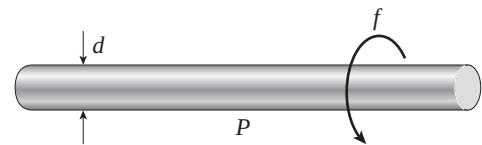
$$I_p = \frac{\pi}{32} d^4 = 4.396 \times 10^5 \text{ mm}^4$$

Power in terms of torque T :

$$P = 2\pi f T$$

Solve for torque T :

$$T = \frac{P}{2\pi f} = 757.9 \text{ N}\cdot\text{m}$$



Max. shear stress using torsion formula:

$$\tau_{\max} = \frac{T \left(\frac{d}{2} \right)}{I_p} = 39.7 \text{ MPa} \quad \leftarrow$$

A-3.13: A motor drives a shaft at $f = 10 \text{ Hz}$ and delivers $P = 35 \text{ kW}$ of power. The allowable shear stress in the shaft is 45 MPa. The minimum diameter of the shaft is approximately:

- (A) 35 mm
- (B) 40 mm
- (C) 47 mm
- (D) 61 mm

Solution

$$f = 10 \text{ Hz} \quad P = 35 \text{ kW}$$

$$\tau_a = 45 \text{ MPa}$$

Power in terms of torque T :

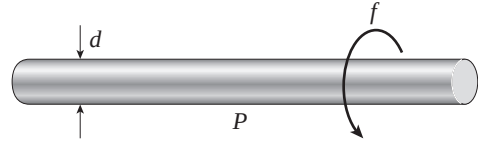
$$P = 2\pi f T$$

Solve for torque T :

$$T = \frac{P}{2\pi f} = 557.0 \text{ N}\cdot\text{m}$$

Shear formula: $\tau = \frac{T\left(\frac{d}{2}\right)}{\frac{\pi}{32}d^4}$ or $\tau = \frac{16T}{\pi d^3}$

Solve for diameter d : $d = \left(\frac{16T}{\pi\tau_a}\right)^{\frac{1}{3}} = 39.8 \text{ mm}$ ←



A-3.14: A drive shaft running at 2500 rpm has outer diameter 60 mm and inner diameter 40 mm. The allowable shear stress in the shaft is 35 MPa. The maximum power that can be transmitted is approximately:

- (A) 220 kW
- (B) 240 kW
- (C) 288 kW
- (D) 312 kW

Solution

$$n = 2500 \text{ rpm}$$

$$\tau_a = 35 (10^6) \frac{\text{N}}{\text{m}^2}$$

$$d_2 = 0.060 \text{ m}$$

$$d_1 = 0.040 \text{ m}$$

$$I_p = \frac{\pi}{32}(d_2^4 - d_1^4) = 1.021 \times 10^{-6} \text{ m}^4$$

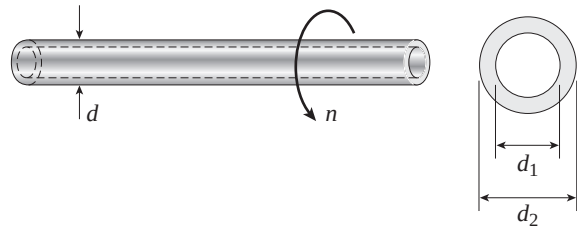
Shear formula:

$$\tau = \frac{T\left(\frac{d_2}{2}\right)}{I_p} \quad \text{or} \quad T_{\max} = \frac{2\tau_a I_p}{d_2} = 1191.2 \text{ N}\cdot\text{m}$$

Power in terms of torque T :

$$P = 2\pi f T = 2\pi(n/60) T$$

$$P_{\max} = \frac{2\pi n}{60} T_{\max} = 3.119 \times 10^5 \text{ W} \quad P_{\max} = 312 \text{ kW} \quad \leftarrow$$



A-3.15: A prismatic shaft (diameter $d_0 = 19$ mm) is loaded by torque T_1 . A stepped shaft (diameters $d_1 = 20$ mm, $d_2 = 25$ mm, radius R of fillets = 2 mm) is loaded by torque T_2 . The allowable shear stress in the material is 42 MPa. The ratio T_1/T_2 of the maximum permissible torques that can be applied to the shafts, considering stress concentration effects in the stepped shaft is:

- (A) 0.9
- (B) 1.2
- (C) 1.4
- (D) 2.1

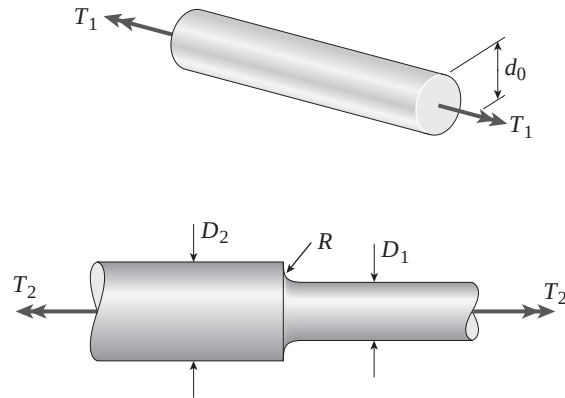
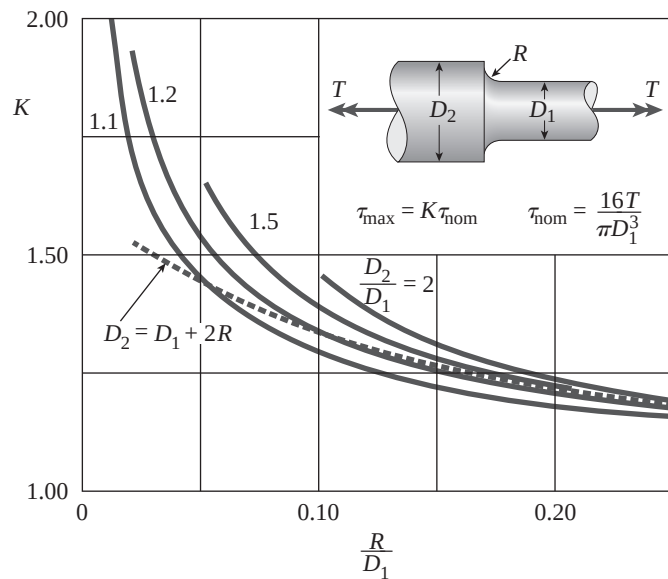


FIG. 3-59 Stress-concentration factor K for a stepped shaft in torsion. (The dashed line is for a full quarter-circular fillet.)



SolutionPrismatic shaft

$$T_{1\max} = \frac{\tau_{\text{allow}} I_P}{\frac{d_0}{2}} = \tau_{\text{allow}} \left(\frac{\pi d_0^3}{16} \right) = 42 \text{ MPa} \left[\frac{\pi}{16} (19 \text{ mm})^3 \right] = 56.6 \text{ N}\cdot\text{m}$$

Stepped shaft

$$\frac{d_2}{d_1} = \frac{25 \text{ mm}}{20 \text{ mm}} = 1.250 \quad \frac{R}{d_1} = \frac{2 \text{ mm}}{20 \text{ mm}} = 0.100 \quad \text{so from graph (see Fig. 3-59)}$$

$$K = 1.35$$

$$T_{2\max} = \frac{\tau_{\text{allow}}}{K} \left(\frac{\pi d_1^3}{16} \right) = \frac{42 \text{ MPa}}{1.35} \left[\frac{\pi}{16} (20 \text{ mm})^3 \right] = 48.9 \text{ N}\cdot\text{m}$$

$$\frac{T_{1\max}}{T_{2\max}} = \frac{56.6}{48.9} = 1.16 \quad \leftarrow$$

A-4.1: A simply supported beam with proportional loading ($P = 4.1 \text{ kN}$) has span length $L = 5 \text{ m}$. Load P is 1.2 m from support A and load $2P$ is 1.5 m from support B . The bending moment just left of load $2P$ is approximately:

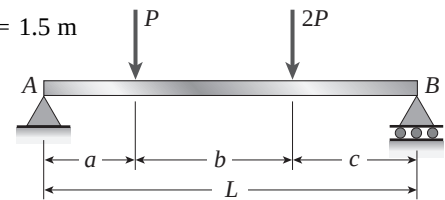
- (A) 5.7 kN·m
- (B) 6.2 kN·m
- (C) 9.1 kN·m
- (D) 10.1 kN·m

Solution

$$a = 1.2 \text{ m} \quad b = 2.3 \text{ m} \quad c = 1.5 \text{ m}$$

$$L = a + b + c = 5.00 \text{ m}$$

$$P = 4.1 \text{ kN}$$



Statics to find reaction force at B :

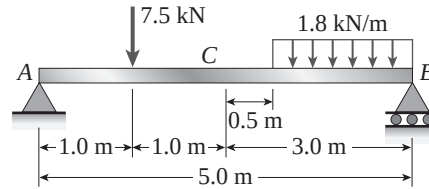
$$R_B = \frac{1}{L} [Pa + 2P(a + b)] = 6.724 \text{ kN}$$

Moment just left of load $2P$:

$$M = R_B c = 10.1 \text{ kN}\cdot\text{m} \quad \leftarrow < \text{compression on top of beam}$$

A-4.2: A simply-supported beam is loaded as shown in the figure. The bending moment at point C is approximately:

- (A) 5.7 kN·m
- (B) 6.1 kN·m
- (C) 6.8 kN·m
- (D) 9.7 kN·m

Solution

Statics to find reaction force at A:

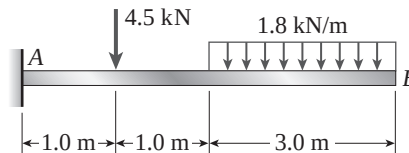
$$R_A = \frac{1}{5 \text{ m}} \left[\left[1.8 \frac{\text{kN}}{\text{m}} \frac{(3 \text{ m} - 0.5 \text{ m})^2}{2} \right] + 7.5 \text{ kN}(3 \text{ m} + 1 \text{ m}) \right] = 7.125 \text{ kN}$$

Moment at point C, 2 m from A:

$$M = R_A(2 \text{ m}) - 7.5 \text{ kN}(1.0 \cdot \text{m}) = 6.75 \text{ kN} \cdot \text{m} \quad \leftarrow < \text{compression on top of beam}$$

A-4.3: A cantilever beam is loaded as shown in the figure. The bending moment at 0.5 m from the support is approximately:

- (A) 12.7 kN·m
- (B) 14.2 kN·m
- (C) 16.1 kN·m
- (D) 18.5 kN·m

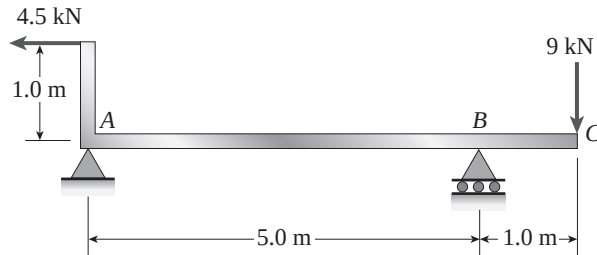
Solution

Cut beam at 0.5 m from support; use statics and right-hand FBD to find internal moment at that point

$$\begin{aligned} M &= 0.5 \text{ m}(4.5 \text{ kN}) + \left(0.5 \text{ m} + 1.0 \text{ m} + \frac{3.0 \text{ m}}{2} \right) 1.8 \frac{\text{kN}}{\text{m}}(3.0 \text{ m}) \\ &= 18.5 \text{ kN} \cdot \text{m} \quad \leftarrow (\text{tension on top of beam}) \end{aligned}$$

A-4.4: An L-shaped beam is loaded as shown in the figure. The bending moment at the midpoint of span AB is approximately:

- (A) 6.8 kN·m
- (B) 10.1 kN·m
- (C) 12.3 kN·m
- (D) 15.5 kN·m

Solution

Use statics to find reaction at B; sum moments about A

$$R_B = \frac{1}{5 \text{ m}} [9 \text{ kN}(6 \text{ m}) - 4.5 \text{ kN}(1. \text{ m})] = 9.90 \text{ kN}$$

Cut beam at midpoint of AB; use right hand FBD, sum moments

$$M = R_B \left(\frac{5 \text{ m}}{2} \right) - 9 \text{ kN} \left(\frac{5 \text{ m}}{2} + 1 \text{ m} \right) = 6.75 \text{ kN} \cdot \text{m} \quad \leftarrow \text{tension on top of beam}$$

A-4.5: A T-shaped simple beam has a cable with force P anchored at B and passing over a pulley at E as shown in the figure. The bending moment just left of C is $1.25 \text{ kN} \cdot \text{m}$. The cable force P is approximately:

- (A) 2.7 kN
- (B) 3.9 kN
- (C) 4.5 kN
- (D) 6.2 kN

Solution

$$M_C = 1.25 \text{ kN} \cdot \text{m}$$

Sum moments about D to find vertical reaction at A:

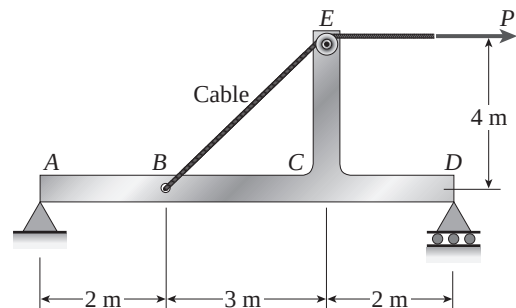
$$V_A = \frac{-1}{7 \text{ m}} [P(4 \text{ m})]$$

$$V_A = \frac{-4}{7} P \quad (\text{downward})$$

Now cut beam & cable just left of CE & use left FBD; show V_A downward & show vertical cable force component of $(4/5)P$ upward at B; sum moments at C to get M_C and equate to given numerical value of M_C to find P :

$$M_C = \frac{4}{5}P(3) + V_A(2 + 3)$$

$$M_C = \frac{4}{5}P(3) + \left(\frac{-4}{7}P \right)(2 + 3) = -\frac{16P}{35}$$



Solve for P :

$$P = \frac{35}{16} (1.25) = 2.73 \text{ kN} \quad \leftarrow$$

A-4.6: A simple beam ($L = 9 \text{ m}$) with attached bracket BDE has force $P = 5 \text{ kN}$ applied downward at E . The bending moment just right of B is approximately:

- (A) $6 \text{ kN}\cdot\text{m}$
- (B) $10 \text{ kN}\cdot\text{m}$
- (C) $19 \text{ kN}\cdot\text{m}$
- (D) $22 \text{ kN}\cdot\text{m}$

Solution

Sum moments about A to find reaction at C :

$$R_C = \frac{1}{L} \left[P \left(\frac{L}{6} + \frac{L}{3} \right) \right] = \frac{P}{2}$$

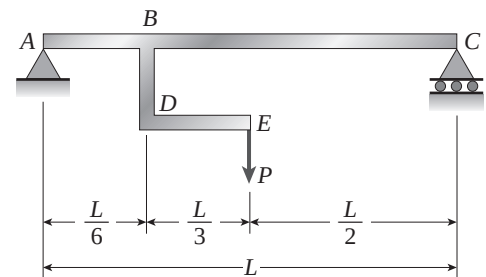
Cut through beam just right of B , then use FBD of BC to find moment at B :

$$M_B = R_C \left(\frac{L}{2} + \frac{L}{3} \right) = \frac{5LP}{12}$$

Substitute numbers for L and P :

$$L = 9 \text{ m} \quad P = 5 \text{ kN}$$

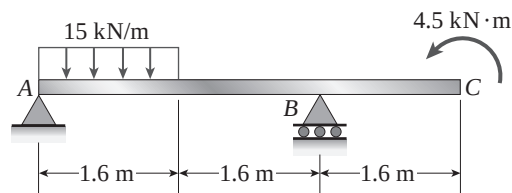
$$M_B = \frac{5LP}{12} = 18.8 \text{ kN}\cdot\text{m} \quad \leftarrow$$



A-4.7: A simple beam AB with an overhang BC is loaded as shown in the figure. The bending moment at the midspan of AB is approximately:

- (A) $8 \text{ kN}\cdot\text{m}$
- (B) $12 \text{ kN}\cdot\text{m}$
- (C) $17 \text{ kN}\cdot\text{m}$
- (D) $21 \text{ kN}\cdot\text{m}$

Solution



Sum moments about B to
get reaction at A :

$$R_A = \frac{1}{3.2} \left[15(1.6) \left(1.6 + \frac{1.6}{2} \right) + 4.5 \right] = 19.40625 \text{ kN}$$

Cut beam at midspan, use left FBD & sum moments to find moment at midspan:

$$M_{\text{mspan}} = R_A(1.6) - 15(1.6) \left(\frac{1.6}{2} \right) = 11.85 \text{ kN}\cdot\text{m} \quad \leftarrow$$

A-5.1: A copper wire ($d = 1.5 \text{ mm}$) is bent around a tube of radius $R = 0.6 \text{ m}$. The maximum normal strain in the wire is approximately:

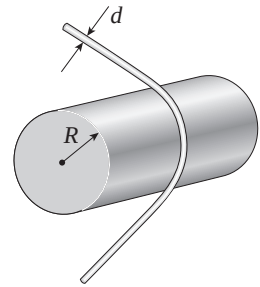
- (A) 1.25×10^{-3}
- (B) 1.55×10^{-3}
- (C) 1.76×10^{-3}
- (D) 1.92×10^{-3}

Solution

$$\epsilon_{\text{max}} = \frac{\frac{d}{2}}{R + \frac{d}{2}} = \frac{d}{2 \left(R + \frac{d}{2} \right)}$$

$$d = 1.5 \text{ mm} \quad R = 0.6 \text{ m}$$

$$\epsilon_{\text{max}} = \frac{d}{2 \left(R + \frac{d}{2} \right)} = 1.248 \times 10^{-3} \quad \leftarrow$$



A-5.2: A simply supported wood beam ($L = 5 \text{ m}$) with rectangular cross section ($b = 200 \text{ mm}$, $h = 280 \text{ mm}$) carries uniform load $q = 6.5 \text{ kN/m}$ which includes the weight of the beam. The maximum flexural stress is approximately:

- (A) 8.7 MPa
- (B) 10.1 MPa
- (C) 11.4 MPa
- (D) 14.3 MPa

Solution

$$L = 5 \text{ m} \quad b = 200 \text{ mm} \quad h = 280 \text{ mm}$$

$$q = 9.5 \frac{\text{kN}}{\text{m}}$$

Section modulus:

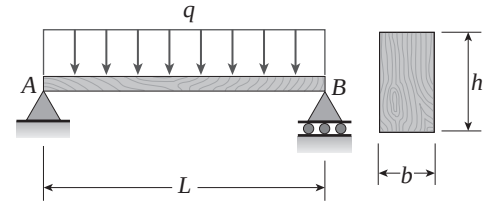
$$S = \frac{bh^2}{6} = 2.613 \times 10^6 \text{ m}^3$$

Max. moment at midspan:

$$M_{\max} = \frac{qL^2}{8} = 29.7 \text{ kN}\cdot\text{m}$$

Max. flexural stress at midspan:

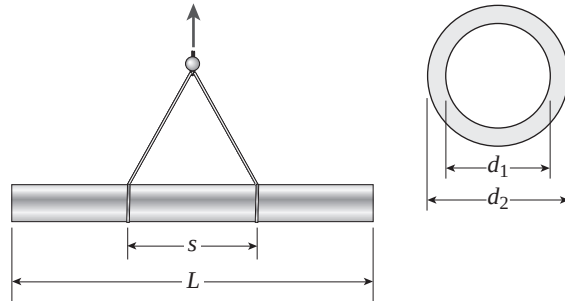
$$\sigma_{\max} = \frac{M_{\max}}{S} = 11.4 \text{ MPa} \quad \leftarrow$$



A-5.3: A cast iron pipe ($L = 12 \text{ m}$, weight density $= 72 \text{ kN/m}^3$, $d_2 = 100 \text{ mm}$, $d_1 = 75 \text{ mm}$) is lifted by a hoist. The lift points are 6 m apart. The maximum bending stress in the pipe is approximately:

- (A) 28 MPa
- (B) 33 MPa
- (C) 47 MPa
- (D) 59 MPa

Solution



$$L = 12 \text{ m} \quad s = 4 \text{ m} \quad d_2 = 100 \text{ mm} \quad d_1 = 75 \text{ mm} \quad \gamma_{CI} = 72 \frac{\text{kN}}{\text{m}^3}$$

Pipe cross sectional properties:

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 3436 \text{ mm}^2 \quad I = \frac{\pi}{64}(d_2^4 - d_1^4) = 3.356 \times 10^6 \text{ mm}^4$$

$$\text{Uniformly distributed weight of pipe, } q: \quad q = \gamma_{CI} A = 0.247 \frac{\text{kN}}{\text{m}}$$

$$\text{Vertical force at each lift point:} \quad F = \frac{qL}{2} = 1.484 \text{ kN}$$

Max. moment is either at lift points (M_1) or at midspan (M_2):

$$M_1 = -q \left(\frac{L-s}{2} \right) \left(\frac{L-s}{2} \right) = -3.958 \text{ kN}\cdot\text{m} < \text{controls, tension on top}$$

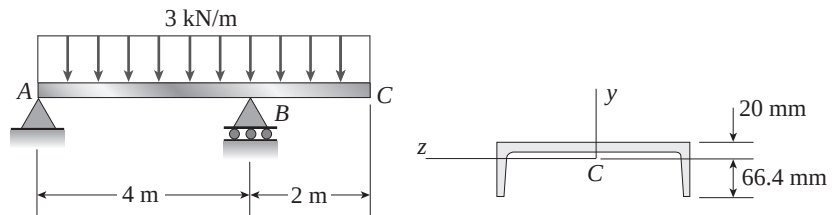
$$M_2 = F \frac{s}{2} - q \frac{L}{2} \left(\frac{L}{4} \right) = -1.484 \text{ kN}\cdot\text{m} < \text{tension on top}$$

Max. bending stress at lift point: $\sigma_{\max} = \frac{|M_1| \left(\frac{d_2}{2} \right)}{I} = 59.0 \text{ MPa}$ ←

A-5.4: A beam with an overhang is loaded by a uniform load of 3 kN/m over its entire length. Moment of inertia $I_z = 3.36 \times 10^6 \text{ mm}^4$ and distances to top and bottom of the beam cross section are 20 mm and 66.4 mm, respectively. It is known that reactions at A and B are 4.5 kN and 13.5 kN, respectively. The maximum bending stress in the beam is approximately:

- (A) 36 MPa
- (B) 67 MPa
- (C) 102 MPa
- (D) 119 MPa

Solution



$$R_A = 4.5 \text{ kN} \quad I_z = 3.36 (10^6) \text{ mm}^4 \quad q = 3 \frac{\text{kN}}{\text{m}}$$

Location of max. positive moment in AB (cut beam at location of zero shear & use left FBD):

$$x_{\max} \frac{R_A}{q} = 1.5 \text{ m} \quad M_{\text{pos}} = R_A x_{\max} - 3 \frac{\text{kN}}{\text{m}} \frac{x_{\max}^2}{2} = 3.375 \text{ kN}\cdot\text{m}$$

< compression on top of beam

Compressive stress on top of beam at $x_{\max}$:

$$\sigma_{c1} = \frac{M_{\text{pos}} (20 \text{ mm})}{I_z} = 20.1 \text{ MPa}$$

Tensile stress at bottom of beam at $x_{\max}$:

$$\sigma_{t1} = \frac{M_{\text{pos}} (66.4 \text{ mm})}{I_z} = 66.696 \text{ MPa}$$

Max. negative moment at B (use FBD of BC to find moment; compression on bottom of beam):

$$M_{\text{neg}} = \left(3 \frac{\text{kN}}{\text{m}} \right) \frac{(2 \text{ m})^2}{2} = 6.000 \text{ kN} \cdot \text{m}$$

$$\sigma_{c2} = \frac{M_{\text{neg}} (66.4 \text{ mm})}{I_z} = 118.6 \text{ MPa} \quad \leftarrow$$

$$\sigma_{t2} = \frac{M_{\text{neg}} (20 \text{ mm})}{I_z} = 35.7 \text{ MPa}$$

A-5.5: A steel hanger with solid cross section has horizontal force $P = 5.5 \text{ kN}$ applied at free end D . Dimension variable $b = 175 \text{ mm}$ and allowable normal stress is 150 MPa . Neglect self weight of the hanger. The required diameter of the hanger is approximately:

- (A) 5 cm
- (B) 7 cm
- (C) 10 cm
- (D) 13 cm

Solution

$$P = 5.5 \text{ kN} \quad b = 175 \text{ mm}$$

$$\sigma_a = 150 \text{ MPa}$$

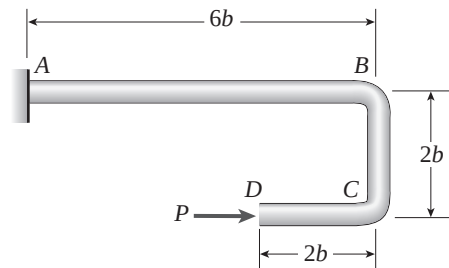
Reactions at support:

$$N_A = P$$

(leftward)

$$M_A = P(2b) = 1.9 \text{ kN} \cdot \text{m}$$

(tension on bottom)



Max. normal stress at bottom of cross section at A:

$$\sigma_{\text{max}} = \frac{P}{\left(\frac{\pi d^2}{4} \right)} + \frac{(2 Pb) \left(\frac{d}{2} \right)}{\left(\frac{\pi d^4}{64} \right)} \quad \sigma_{\text{max}} = \frac{4P(16b + d)}{\pi d^3}$$

Set $\sigma_{\text{max}} = \sigma_a$ and solve for required diameter d :

$$(\pi \sigma_a) d^3 - (4P)d - 64Pb = 0 \quad < \text{solve numerically or by trial \& error to find}$$

$$d_{\text{reqd}} = 5.11 \text{ cm} \quad \leftarrow$$

A-5.6: A cantilever wood pole carries force $P = 300$ N applied at its free end, as well as its own weight (weight density $= 6$ kN/m³). The length of the pole is $L = 0.75$ m and the allowable bending stress is 14 MPa. The required diameter of the pole is approximately:

- (A) 4.2 cm
- (B) 5.5 cm
- (C) 6.1 cm
- (D) 8.5 cm

Solution

$$P = 300 \text{ N} \quad L = 0.75 \text{ m}$$

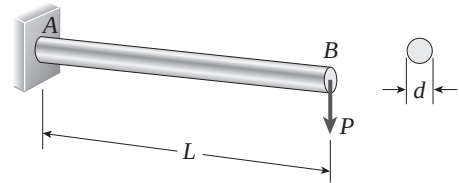
$$\sigma_a = 14 \text{ MPa} \quad \gamma_w = 6 \frac{\text{kN}}{\text{m}^3}$$

Uniformly distributed weight of pole:

$$w = \gamma_w \left(\frac{\pi d^2}{4} \right)$$

Max. moment at support:

$$M_{\max} = PL + wL \frac{L}{2}$$



Section modulus of pole cross section: $S = \frac{I}{\left(\frac{d}{2}\right)} \quad S = \frac{\frac{\pi d^4}{64}}{\left(\frac{d}{2}\right)} = \frac{\pi d^3}{32}$

Set $M_{\max}$ equal to $\sigma_a \times S$ and solve for required min. diameter d :

$$PL + \left[\gamma_w \left(\frac{\pi d^2}{4} \right) \right] L \frac{L}{2} - \sigma_a \left(\frac{\pi d^3}{32} \right) = 0$$

Or

$$\left(\frac{\pi \sigma_a}{32} \right) d^3 - \left(\frac{\pi \gamma_w L^2}{8} \right) d^2 - PL = 0 \quad < \text{solve numerically or by trial}$$

& error to find

$$d_{\text{reqd}} = 5.50 \text{ cm}$$

Since wood pole is light, try simpler solution which ignores self weight:

$$PL = \sigma_a S \quad \text{Or} \quad \left(\frac{\pi \sigma_a}{32} \right) d^3 = PL$$

$$d_{\text{reqd}} = \left[PL \left(\frac{32}{\pi \sigma_a} \right) \right]^{\frac{1}{3}} = 5.47 \text{ cm} \quad \leftarrow$$

A-5.7: A simply supported steel beam of length $L = 1.5$ m and rectangular cross section ($h = 75$ mm, $b = 20$ mm) carries a uniform load of $q = 48$ kN/m, which includes its own weight. The maximum transverse shear stress on the cross section at 0.25 m from the left support is approximately:

- (A) 20 MPa
- (B) 24 MPa
- (C) 30 MPa
- (D) 36 MPa

Solution

$$L = 1.5 \text{ m} \quad q = 48 \frac{\text{kN}}{\text{m}}$$

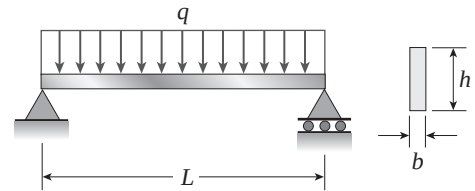
$$h = 75 \text{ mm} \quad b = 20 \text{ mm}$$

Cross section properties:

$$A = bh = 1500 \text{ mm}^2$$

$$Q = \left(b \frac{h}{2}\right) \frac{h}{4} = 14062 \text{ mm}^3$$

$$I = \frac{bh^3}{12} = 7.031 \times 10^5 \text{ mm}^4$$



Support reactions:

$$R = \frac{qL}{2} = 36.0 \text{ kN}$$

Transverse shear force at 0.25 m from left support:

$$V_{0.25} = R - q(0.25 \text{ m}) = 24.0 \text{ kN}$$

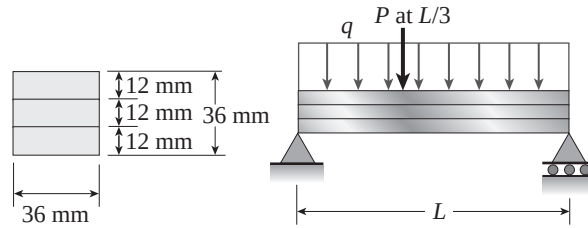
Max. shear stress at NA at 0.25 m from left support:

$$\tau_{\max} = \frac{V_{0.25}Q}{Ib} = 24.0 \text{ MPa} \quad \text{Or more simply . . .}$$

$$\tau_{\max} = \frac{3V_{0.25}}{2A} = 24.0 \text{ MPa} \quad \leftarrow$$

A-5.8: A simply supported laminated beam of length $L = 0.5$ m and square cross section weighs 4.8 N. Three strips are glued together to form the beam, with the allowable shear stress in the glued joint equal to 0.3 MPa. Considering also the weight of the beam, the maximum load P that can be applied at $L/3$ from the left support is approximately:

- (A) 240 N
- (B) 360 N
- (C) 434 N
- (D) 510 N

Solution

$$L = 0.5 \text{ m} \quad W = 4.8 \text{ N} \quad q = \frac{W}{L} = 9.60 \frac{\text{N}}{\text{m}}$$

$$h = 36 \text{ mm} \quad b = 36 \text{ mm} \quad \tau_a = 0.3 \text{ MPa}$$

Cross section properties:

$$A = bh = 1296 \text{ mm}^2 \quad Q_{\text{joint}} = \left(b \frac{h}{3}\right) \left(\frac{h}{2} - \frac{h}{6}\right) = 5184 \text{ mm}^3$$

$$I = \frac{bh^3}{12} = 1.400 \times 10^5 \text{ mm}^4$$

Max. shear force at left support: $V_{\text{max}} = \frac{qL}{2} + P\left(\frac{2}{3}\right)$

Shear stress on glued joint at left support; set $\tau = \tau_a$ then solve for P_{max} :

$$\tau = \frac{V_{\text{max}} Q_{\text{joint}}}{Ib} \quad \text{Or} \quad \tau = \frac{V_{\text{max}} \left(\frac{bh^2}{9}\right)}{\left(\frac{bh^3}{12}\right)b} \quad \text{Or} \quad \tau_a = \frac{4 V_{\text{max}}}{3bh}$$

$$\tau_a = \frac{4}{3bh} \left[\frac{qL}{2} + P\left(\frac{2}{3}\right) \right] \quad \text{so for } \tau_a = 0.3 \text{ MPa}$$

$$P_{\text{max}} = \frac{3}{2} \left(\frac{3bh\tau_a}{4} - \frac{qL}{2} \right) = 434 \text{ N} \quad \leftarrow$$

A-5.9: An aluminum cantilever beam of length $L = 0.65 \text{ m}$ carries a distributed load, which includes its own weight, of intensity $q/2$ at A and q at B. The beam cross section has width 50 mm and height 170 mm. Allowable bending stress is 95 MPa and allowable shear stress is 12 MPa. The permissible value of load intensity q is approximately:

- (A) 110 kN/m
- (B) 122 kN/m
- (C) 130 kN/m
- (D) 139 kN/m

Solution

$$L = 0.65 \text{ m} \quad b = 50 \text{ mm} \quad h = 170 \text{ mm}$$

$$\sigma_a = 95 \text{ MPa} \quad \tau_a = 12 \text{ MPa}$$

Cross section properties:

$$A = bh = 8500 \text{ mm}^2$$

$$I = \frac{bh^3}{12} = 2.047 \times 10^7 \text{ mm}^4 \quad S = \frac{bh^2}{6} = 2.408 \times 10^5 \text{ mm}^3$$

Reaction force and moment at A:

$$R_A = \frac{1}{2} \left(\frac{q}{2} + q \right) L \quad R_A = \frac{3}{4} qL \quad M_A = \frac{qL}{2} \frac{L}{2} + \frac{1}{2} \frac{qL}{2} \frac{2L}{3}$$

$$M_A = \frac{5}{12} qL^2$$

Compare max. permissible values of q based on shear and moment allowable stresses; smaller value controls

$$\tau_{\max} = \frac{3}{2} \frac{R_A}{A} \quad \tau_a = \frac{3}{2} \left(\frac{\frac{3}{4} qL}{A} \right) \quad \text{So, since } \tau_a = 12 \text{ MPa}$$

$$q_{\max 1} = \frac{8}{9} \frac{\tau_a A}{L} = 139 \frac{\text{kN}}{\text{m}}$$

$$\sigma_{\max} = \frac{M_A}{S} \quad \sigma_a = \frac{\frac{5}{12} qL^2}{S} \quad \text{So, since } \sigma_a = 95 \text{ MPa}$$

$$q_{\max 2} = \frac{12}{5} \frac{\sigma_a S}{L^2} = 130.0 \frac{\text{kN}}{\text{m}} \quad \leftarrow$$

A-5.10: An aluminum light pole weighs 4300 N and supports an arm of weight 700 N, with arm center of gravity at 1.2 m left of the centroidal axis of the pole. A wind force of 1500 N acts to the right at 7.5 m above the base. The pole cross section at the base has outside diameter 235 mm and thickness 20 mm. The maximum compressive stress at the base is approximately:

- (A) 16 MPa
- (B) 18 MPa
- (C) 21 MPa
- (D) 24 MPa

Solution

$$H = 7.5 \text{ m} \quad B = 1.2 \text{ m}$$

$$W_1 = 4300 \text{ N} \quad W_2 = 700 \text{ N}$$

$$P_1 = 1500 \text{ N}$$

$$d_2 = 235 \text{ mm} \quad t = 20 \text{ mm}$$

$$d_1 = d_2 - 2t = 195 \text{ mm}$$

Pole cross sectional properties at base:

$$A = \frac{\pi}{4} (d_2^2 - d_1^2) = 13509 \text{ mm}^2$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) = 7.873 \times 10^7 \text{ mm}^4$$

Compressive (downward) force at base of pole:

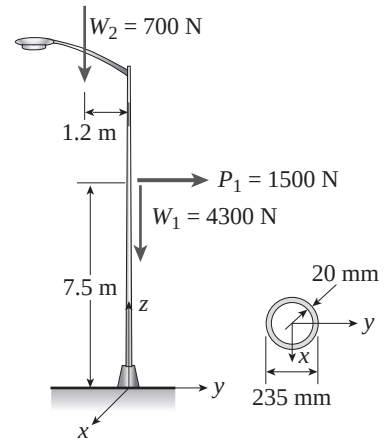
$$N = W_1 + W_2 = 5.0 \text{ kN}$$

Bending moment at base of pole:

$$M = W_2 B - P_1 H = -10.410 \text{ kN}\cdot\text{m} \quad < \text{ results in compression at right}$$

Compressive stress at right side at base of pole:

$$\sigma_c = \frac{N}{A} + \frac{|M| \left(\frac{d_2}{2} \right)}{I} = 15.9 \text{ MPa} \quad \leftarrow$$



A-5.11: Two thin cables, each having diameter $d = t/6$ and carrying tensile loads P , are bolted to the top of a rectangular steel block with cross section dimensions $b \times t$. The ratio of the maximum tensile to compressive stress in the block due to loads P is:

- (A) 1.5
- (B) 1.8
- (C) 2.0
- (D) 2.5

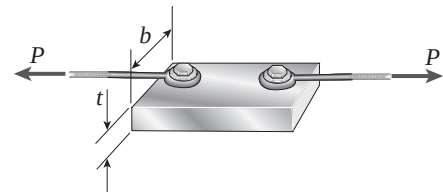
Solution

Cross section properties of block:

$$A = bt \quad I = \frac{bt^3}{12} \quad d = \frac{t}{6}$$

Tensile stress at top of block:

$$\sigma_t = \frac{P}{A} + \frac{P \left(\frac{d}{2} + \frac{t}{2} \right) \left(\frac{t}{2} \right)}{I} = \frac{9P}{2bt}$$



Compressive stress at bottom of block:

$$\sigma_c = \frac{P}{A} - \frac{P\left(\frac{d}{2} + \frac{t}{2}\right)\left(\frac{t}{2}\right)}{I} = -\frac{5P}{2bt}$$

Ratio of max. tensile to compressive stress in block:

$$\text{ratio} = \left| \frac{\sigma_t}{\sigma_c} \right| = \frac{9}{5} \quad \frac{9}{5} = 1.8 \quad \leftarrow$$

A-5.12: A rectangular beam with semicircular notches has dimensions $h = 160$ mm and $h_1 = 140$ mm. The maximum allowable bending stress in the plastic beam is $\sigma_{\max} = 6.5$ MPa, and the bending moment is $M = 185$ N·m. The minimum permissible width of the beam is:

- (A) 12 mm
- (B) 20 mm
- (C) 28 mm
- (D) 32 mm

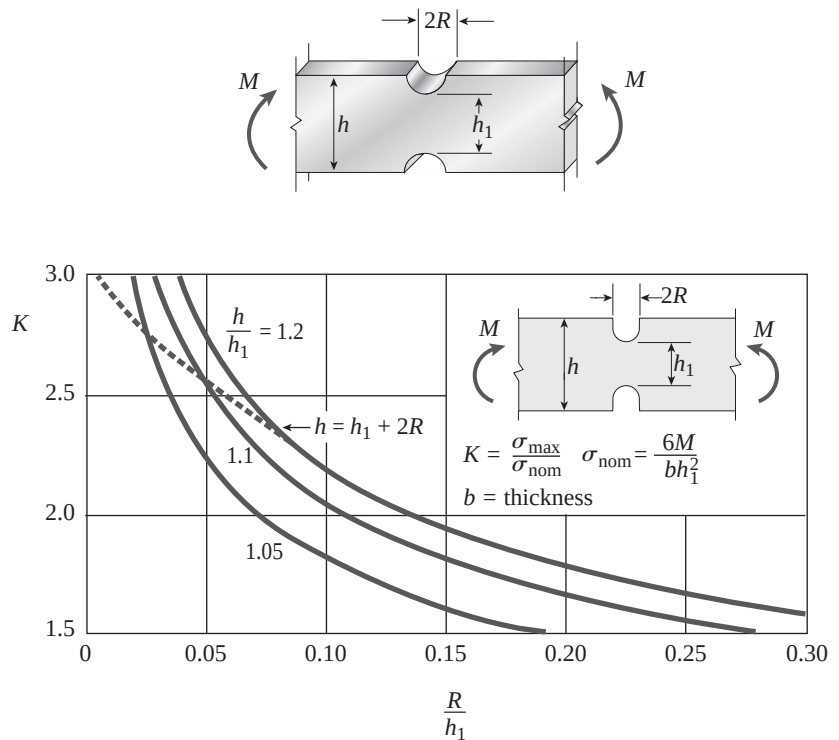


FIG. 5-50 Stress-concentration factor K for a notched beam of rectangular cross section in pure bending (h = height of beam; b = thickness of beam, perpendicular to the plane of the figure). The dashed line is for semicircular notches ($h = h_1 + 2R$)

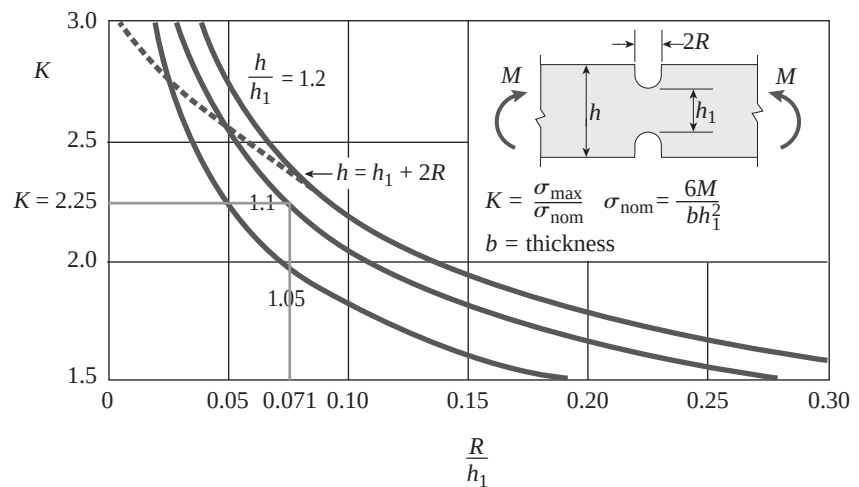
Solution

$$R = \frac{1}{2}(h - h_1) = \frac{1}{2}(160 \text{ mm} - 140 \text{ mm}) = 10.000 \text{ mm}$$

$$\frac{R}{h_1} = \frac{10}{140} = 0.071 \quad \frac{h}{h_1} = \frac{160}{140} = 1.143$$

From Fig 5-50: $K = 2.25$

$$\frac{\sigma_{allow}}{K} = \frac{6M}{bh_1^2} \quad \text{so} \quad b_{min} = \frac{6MK}{\sigma_{allow}h_1^2} = \frac{6(185 \text{ N}\cdot\text{m})(2.25)}{6.5 \text{ MPa}[(140 \text{ mm})^2]} = 19.6 \text{ mm} \quad \leftarrow$$



A-6.1: A composite beam is made up of a 200 mm $\times$ 300 mm core ($E_c = 14$ GPa) and an exterior cover sheet (300 mm $\times$ 12 mm, $E_e = 100$ GPa) on each side. Allowable stresses in core and exterior sheets are 9.5 MPa and 140 MPa, respectively. The ratio of the maximum permissible bending moment about the z-axis to that about the y-axis is most nearly:

- (A) 0.5
- (B) 0.7
- (C) 1.2
- (D) 1.5

Solution

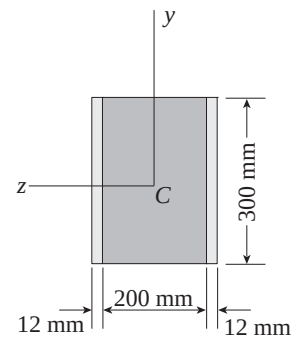
$$b = 200 \text{ mm} \quad t = 12 \text{ mm}$$

$$h = 300 \text{ mm}$$

$$E_c = 14 \text{ GPa} \quad E_e = 100 \text{ GPa}$$

$$\sigma_{ac} = 9.5 \text{ MPa}$$

$$\sigma_{ae} = 140 \text{ MPa}$$



Composite beam is symmetric about both axes so each NA is an axis of symmetry

Moments of inertia of cross section about z and y axes:

$$I_{cz} = \frac{bh^3}{12} = 4.500 \times 10^8 \text{ mm}^4 \quad I_{cy} = \frac{hb^3}{12} = 2.000 \times 10^8 \text{ mm}^4$$

$$I_{ez} = \frac{2th^3}{12} = 5.400 \times 10^7 \text{ mm}^4$$

$$I_{ey} = \frac{2ht^3}{12} = 2(th) \left(\frac{b}{2} + \frac{t}{2} \right)^2 = 8.099 \times 10^7 \text{ mm}^4$$

Bending about z axis based on allowable stress in each material (lesser value controls)

$$M_{\max_{cz}} = \sigma_{ac} \frac{(E_c I_{cz} + E_e I_{ez})}{\frac{h}{2} E_c} = 52.9 \text{ kN} \cdot \text{m}$$

$$M_{\max_{ez}} = \sigma_{ae} \frac{(E_c I_{cz} + E_e I_{ez})}{\frac{h}{2} E_e} = 109.2 \text{ kN} \cdot \text{m}$$

Bending about y axis based on allowable stress in each material (lesser value controls)

$$M_{\max_{cy}} = \sigma_{ac} \frac{(E_c I_{cy} + E_e I_{ey})}{\frac{b}{2} E_c} = 74.0 \text{ kN} \cdot \text{m}$$

$$M_{\max_{ey}} = \sigma_{ae} \frac{(E_c I_{cy} + E_e I_{ey})}{\left(\frac{b}{2} + t \right) E_e} = 136.2 \text{ kN} \cdot \text{m}$$

$$ratio_{z\_to\_y} = \frac{M_{\max_{cz}}}{M_{\max_{cy}}} = 0.72 \quad \leftarrow < \text{allowable stress in the core, not exterior cover sheet, controls moments about both axes}$$

A-6.2: A composite beam is made up of a 90 mm × 160 mm wood beam ($E_w = 11 \text{ GPa}$) and a steel bottom cover plate (90 mm × 8 mm, $E_s = 190 \text{ GPa}$). Allowable stresses in wood and steel are 6.5 MPa and 110 MPa, respectively. The allowable bending moment about the z-axis of the composite beam is most nearly:

- (A) 2.9 kN·m
- (B) 3.5 kN·m
- (C) 4.3 kN·m
- (D) 9.9 kN·m

Solution

$$b = 90 \text{ mm} \quad t = 8 \text{ mm}$$

$$h = 160 \text{ mm}$$

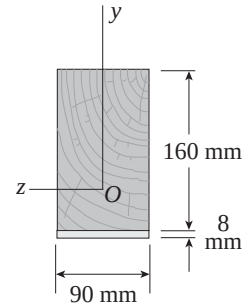
$$E_w = 11 \text{ GPa} \quad E_s = 190 \text{ GPa}$$

$$\sigma_{aw} = 6.5 \text{ MPa}$$

$$\sigma_{as} = 110 \text{ MPa}$$

$$A_w = bh = 14400 \text{ mm}^2$$

$$A_s = bt = 720 \text{ mm}^2$$



Locate NA (distance h_2 above base) by summing 1st moments of EA about base of beam; then find h_1 = dist. from NA to top of beam:

$$h_2 = \frac{E_s A_s \frac{t}{2} + E_w A_w \left(t + \frac{h}{2} \right)}{E_s A_s + E_w A_w} = 49.07 \text{ mm}$$

$$h_1 = h + t - h_2 = 118.93 \text{ mm}$$

Moments of inertia of wood and steel about NA:

$$I_s = \frac{bt^3}{12} + A_s \left(h_2 - \frac{t}{2} \right)^2 = 1.467 \times 10^6 \text{ mm}^4$$

$$I_w = \frac{bh^3}{12} + A_w \left(h_1 - \frac{h}{2} \right)^2 = 5.254 \times 10^7 \text{ mm}^4$$

Allowable moment about z axis based on allowable stress in each material (lesser value controls)

$$M_{\max_w} = \sigma_{aw} \frac{(E_w I_w + E_s I_s)}{h_1 E_w} = 4.26 \text{ kN}\cdot\text{m} \quad \leftarrow$$

$$M_{\max_s} = \sigma_{as} \frac{(E_w I_w + E_s I_s)}{h_2 E_s} = 10.11 \text{ kN}\cdot\text{m}$$

A-6.3: A steel pipe ($d_3 = 104 \text{ mm}$, $d_2 = 96 \text{ mm}$) has a plastic liner with inner diameter $d_1 = 82 \text{ mm}$. The modulus of elasticity of the steel is 75 times that of the modulus of the plastic. Allowable stresses in steel and plastic are 40 MPa and 550 kPa, respectively. The allowable bending moment for the composite pipe is approximately:

- (A) 1100 N·m
- (B) 1230 N·m
- (C) 1370 N·m
- (D) 1460 N·m

Solution

$$d_3 = 104 \text{ mm} \quad d_2 = 96 \text{ mm} \quad d_1 = 82 \text{ mm}$$

$$\sigma_{as} = 40 \text{ MPa}$$

$$\sigma_{ap} = 550 \text{ kPa}$$

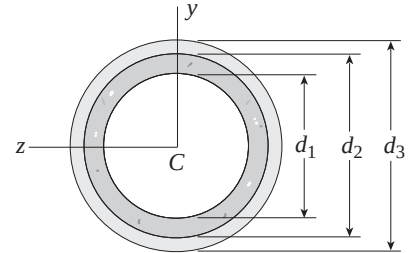
Cross section properties:

$$A_s = \frac{\pi}{4} (d_3^2 - d_2^2) = 1256.6 \text{ mm}^2$$

$$A_p = \frac{\pi}{4} (d_2^2 - d_1^2) = 1957.2 \text{ mm}^2$$

$$I_s = \frac{\pi}{64} (d_3^4 - d_2^4) = 1.573 \times 10^6 \text{ mm}^4$$

$$I_p = \frac{\pi}{64} (d_2^4 - d_1^4) = 1.950 \times 10^6 \text{ mm}^4$$



Due to symmetry, NA of composite beam is the z axis

Allowable moment about z axis based on allowable stress in each material (lesser value controls)

$$M_{\max_s} = \sigma_{as} \frac{(E_p I_p + E_s I_s)}{\left(\frac{d_3}{2}\right) E_s} \quad M_{\max_p} = \sigma_{ap} \frac{(E_p I_p + E_s I_s)}{\left(\frac{d_2}{2}\right) E_p}$$

Modular ratio: $n = \frac{E_s}{E_p} \quad n = 75$

Divide through by E_p in moment expressions above

$$M_{\max_s} = \sigma_{as} \frac{(I_p + n I_s)}{\left(\frac{d_3}{2}\right) n} = 1230 \text{ N}\cdot\text{m} \quad \leftarrow$$

$$M_{\max_p} = \sigma_{ap} \frac{(I_p + n I_s)}{\left(\frac{d_2}{2}\right)} = 1374 \text{ N}\cdot\text{m}$$

A-6.4: A bimetallic beam of aluminum ($E_a = 70 \text{ GPa}$) and copper ($E_c = 110 \text{ GPa}$) strips has width $b = 25 \text{ mm}$; each strip has thickness $t = 1.5 \text{ mm}$. A bending moment of $1.75 \text{ N}\cdot\text{m}$ is applied about the z axis. The ratio of the maximum stress in the aluminum to that in the copper is approximately:

- (A) 0.6
- (B) 0.8
- (C) 1.0
- (D) 1.5

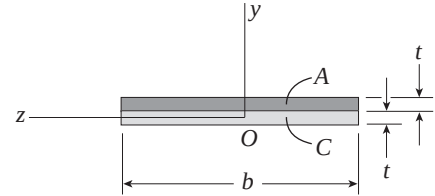
Solution

$$b = 25 \text{ mm} \quad t = 1.5 \text{ mm} \quad A_a = b t = 37.5 \text{ mm}^2 \quad A_c = A_a = 37.5 \text{ mm}^2$$

$$M = 1.75 \text{ N}\cdot\text{m}$$

$$E_a = 70 \text{ GPa} \quad E_c = 110 \text{ GPa}$$

Equate 1st moments of EA about bottom of beam to locate NA (distance h_2 above base); then find h_1 = dist. from NA to top of beam:



$$h_2 = \frac{E_c A_c \frac{t}{2} + E_a A_a \left(t + \frac{t}{2}\right)}{E_c A_c + E_a A_a} = 1.333 \text{ mm}$$

$$h_1 = 2t - h_2 = 1.667 \text{ mm} \quad h_1 + h_2 = 3.000 \text{ mm} \quad 2t = 3.000 \text{ mm}$$

Moments of inertia of aluminum and copper strips about NA:

$$I_c = \frac{bt^3}{12} + A_c \left(h_2 - \frac{t}{2}\right)^2 = 19.792 \text{ mm}^4$$

$$I_a = \frac{bt^3}{12} + A_a \left(h_1 - \frac{t}{2}\right)^2 = 38.542 \text{ mm}^4$$

Bending stresses in aluminum and copper:

$$\sigma_a = \frac{M h_1 E_a}{E_a I_a + E_c I_c} = 41.9 \text{ MPa} \quad \sigma_c = \frac{M h_2 E_c}{E_a I_a + E_c I_c} = 52.6 \text{ MPa}$$

Ratio of the stress in the aluminum to that of the copper: $\frac{\sigma_a}{\sigma_c} = 0.795$ ←

A-6.5: A composite beam of aluminum ($E_a = 72 \text{ GPa}$) and steel ($E_s = 190 \text{ GPa}$) has width $b = 25 \text{ mm}$ and heights $h_a = 42 \text{ mm}$, $h_s = 68 \text{ mm}$. A bending moment is applied about the z axis resulting in a maximum stress in the aluminum of 55 MPa. The maximum stress in the steel is approximately:

- (A) 86 MPa
- (B) 90 MPa
- (C) 94 MPa
- (D) 98 MPa

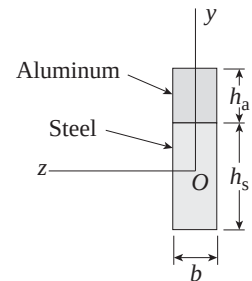
Solution

$$b = 25 \text{ mm} \quad h_a = 42 \text{ mm} \quad h_s = 68 \text{ mm}$$

$$E_a = 72 \text{ GPa} \quad E_s = 190 \text{ GPa} \quad \sigma_a = 55 \text{ MPa}$$

$$A_a = b h_a = 1050.0 \text{ mm}^2$$

$$A_s = b h_s = 1700.0 \text{ mm}^2$$



Locate NA (distance h_2 above base) by summing 1st moments of EA about base of beam; then find h_1 = dist. from NA to top of beam:

$$h_2 = \frac{E_s A_s \frac{h_s}{2} + E_a A_a \left(h_s + \frac{h_a}{2} \right)}{E_a A_a + E_s A_s} = 44.43 \text{ mm}$$

$$h_1 = h_a + h_s - h_2 = 65.57 \text{ mm}$$

$$h_1 + h_2 = 110.00 \text{ mm}$$

Moments of inertia of aluminum and steel parts about NA:

$$I_s = \frac{b h_s^3}{12} + A_s \left(h_2 - \frac{h_s}{2} \right)^2 = 8.401 \times 10^5 \text{ mm}^4$$

$$I_a = \frac{b h_a^3}{12} + A_a \left(h_1 - \frac{h_a}{2} \right)^2 = 2.240 \times 10^6 \text{ mm}^4$$

Set max. bending stress in aluminum to given value then solve for moment M :

$$M = \frac{\sigma_a (E_a I_a + E_s I_s)}{h_1 E_a} = 3.738 \text{ kN} \cdot \text{m}$$

Use M to find max. bending stress in steel: $\sigma_s = \frac{M h_2 E_s}{E_a I_a + E_s I_s} = 98.4 \text{ MPa}$ ←

A-7.1: A rectangular plate ($a = 120 \text{ mm}$, $b = 160 \text{ mm}$) is subjected to compressive stress $\sigma_x = -4.5 \text{ MPa}$ and tensile stress $\sigma_y = 15 \text{ MPa}$. The ratio of the normal stress acting perpendicular to the weld to the shear stress acting along the weld is approximately:

- (A) 0.27
- (B) 0.54
- (C) 0.85
- (D) 1.22

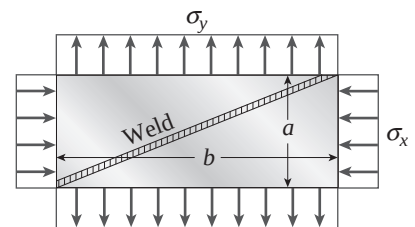
Solution

$$a = 120 \text{ mm} \quad b = 160 \text{ mm}$$

$$\theta = \arctan\left(\frac{a}{b}\right) = 36.87^\circ$$

$$\sigma_x = -4.5 \text{ MPa} \quad \sigma_y = 15 \text{ MPa}$$

$$\tau_{xy} = 0$$



Plane stress transformation: normal and shear stresses on y -face of element rotated through angle θ (perpendicular to & along weld seam):

$$\sigma_{\theta} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos\left[2\left(\theta + \frac{\pi}{2}\right)\right] + \tau_{xy} \sin\left[2\left(\theta + \frac{\pi}{2}\right)\right] = 7.98 \text{ MPa}$$

$$\tau_{\theta} = -\frac{\sigma_x - \sigma_y}{2} \sin\left[2\left(\theta + \frac{\pi}{2}\right)\right] + \tau_{xy} \cos\left[2\left(\theta + \frac{\pi}{2}\right)\right] = -9.36 \text{ MPa}$$

$$\left| \frac{\sigma_{\theta}}{\tau_{\theta}} \right| = 0.85 \quad \leftarrow$$

A-7.2: A rectangular plate in plane stress is subjected to normal stresses σ_x and σ_y and shear stress τ_{xy} . Stress σ_x is known to be 15 MPa but σ_y and τ_{xy} are unknown. However, the normal stress is known to be 33 MPa at counterclockwise angles of 35° and 75° from the x axis. Based on this, the normal stress σ_y on the element below is approximately:

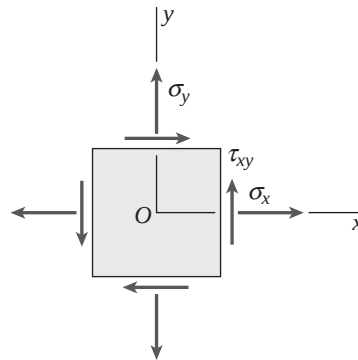
- (A) 14 MPa
- (B) 21 MPa
- (C) 26 MPa
- (D) 43 MPa

Solution

$$\sigma_x = 15 \text{ MPa} \quad \sigma_{35} = 33 \text{ MPa} \quad \sigma_{75} = \sigma_{35}$$

Plane stress transformations for 35° & 75° :

$$\sigma_{\theta} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$



For $\theta = 35^\circ$: $\theta_{35} = 35^\circ$

$$\frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos[2(\theta_{35})] + \tau_{xy} \sin[2(\theta_{35})] = \sigma_{35}$$

$$\text{Or} \quad \sigma_y + 2.8563 \tau_{xy} = 69.713$$

And for $\theta = 75^\circ$: $\theta_{75} = 75^\circ$

$$\frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos[2(\theta_{75})] + \tau_{xy} \sin[2(\theta_{75})] = \sigma_{75}$$

$$\text{Or } \sigma_y + 0.5359 \tau_{xy} = 34.292$$

Solving above two equations for σ_y and τ_{xy} gives:

$$\begin{pmatrix} \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{bmatrix} 1 & 2.8563 \\ 1 & 0.5359 \end{bmatrix}^{-1} \begin{pmatrix} 69.713 \\ 34.292 \end{pmatrix} = \begin{pmatrix} 26.1 \\ 15.3 \end{pmatrix} \text{MPa}$$

$$\text{so } \sigma_y = 26.1 \text{ MPa} \quad \leftarrow$$

A-7.3: A rectangular plate in plane stress is subjected to normal stresses $\sigma_x = 35 \text{ MPa}$, $\sigma_y = 26 \text{ MPa}$, and shear stress $\tau_{xy} = 14 \text{ MPa}$. The ratio of the magnitudes of the principal stresses (σ_1/σ_2) is approximately:

- (A) 0.8
- (B) 1.5
- (C) 2.1
- (D) 2.9

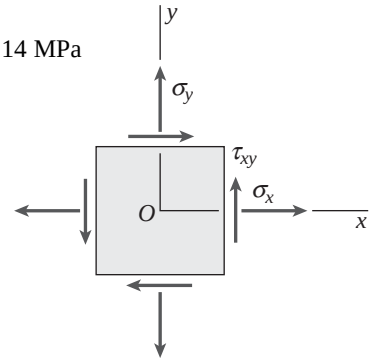
Solution

$$\sigma_x = 35 \text{ MPa} \quad \sigma_y = 26 \text{ MPa} \quad \tau_{xy} = 14 \text{ MPa}$$

Principal angles:

$$\theta_{P1} = \frac{1}{2} \arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) = 36.091^\circ$$

$$\theta_{P2} = \theta_{P1} + \frac{\pi}{2} = 126.091^\circ$$



Plane stress transformations:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{P1}) + \tau_{xy} \sin(2\theta_{P1}) = 45.21 \text{ MPa}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{P2}) + \tau_{xy} \sin(2\theta_{P2}) = 15.79 \text{ MPa}$$

Ratio of principal stresses:

$$\frac{\sigma_1}{\sigma_2} = 2.86 \quad \leftarrow$$

A-7.4: A drive shaft resists torsional shear stress of 45 MPa and axial compressive stress of 100 MPa. The ratio of the magnitudes of the principal stresses (σ_1/σ_2) is approximately:

- (A) 0.15
- (B) 0.55
- (C) 1.2
- (D) 1.9

Solution

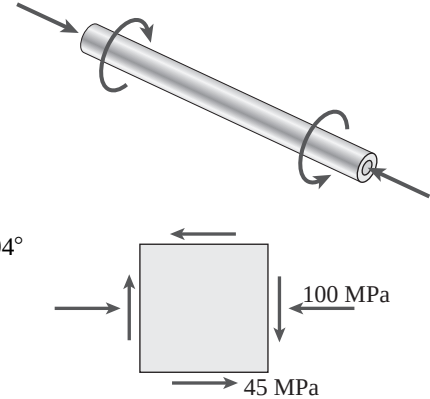
$$\sigma_x = -100 \text{ MPa} \quad \sigma_y = 0$$

$$\tau_{xy} = -45 \text{ MPa}$$

Principal angles:

$$\theta_{P1} = \frac{1}{2} \arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) = 20.994^\circ$$

$$\theta_{P2} = \theta_{P1} + \frac{\pi}{2} = 110.994^\circ$$



Plane stress transformations:

$$\sigma_{\theta P1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{P1}) + \tau_{xy} \sin(2\theta_{P1}) = -117.27 \text{ MPa}$$

< actually σ_2

$$\sigma_{\theta P2} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{P2}) + \tau_{xy} \sin(2\theta_{P2}) = 17.27 \text{ MPa}$$

< this is σ_1

So

$$\sigma_1 = \max(\sigma_{\theta P1}, \sigma_{\theta P2}) = 17.268 \text{ MPa} \quad \sigma_2 = \min(\sigma_{\theta P1}, \sigma_{\theta P2}) = -117.268 \text{ MPa}$$

Ratio of principal stresses:

$$\left| \frac{\sigma_1}{\sigma_2} \right| = 0.15 \quad \leftarrow$$

A-7.5: A drive shaft resists torsional shear stress of 45 MPa and axial compressive stress of 100 MPa. The maximum shear stress is approximately:

- (A) 42 MPa
- (B) 67 MPa
- (C) 71 MPa
- (D) 93 MPa

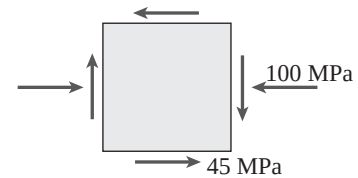
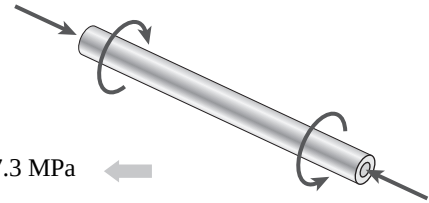
Solution

$$\sigma_x = -100 \text{ MPa} \quad \sigma_y = 0$$

$$\tau_{xy} = -45 \text{ MPa}$$

Max. shear stress:

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 67.3 \text{ MPa} \quad \leftarrow$$



A-7.6: A drive shaft resists torsional shear stress of $\tau_{xy} = 40 \text{ MPa}$ and axial compressive stress $\sigma_x = -70 \text{ MPa}$. One principal normal stress is known to be 38 MPa (tensile). The stress σ_y is approximately:

- (A) 23 MPa
- (B) 35 MPa
- (C) 62 MPa
- (D) 75 MPa

Solution

$$\sigma_x = -70 \text{ MPa} \quad \tau_{xy} = 40 \text{ MPa}$$

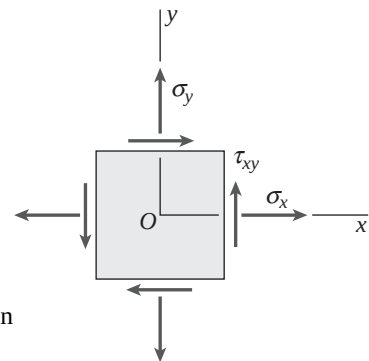
$$\sigma_y \text{ is unknown} \quad \sigma_{\text{prin}} = 38 \text{ MPa}$$

Stresses σ_x and σ_y must be smaller than the given principal stress so:

$$\sigma_1 = \sigma_{\text{prin}}$$

Substitute into stress transformation equation and **solve for σ_y** :

$$\frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_1 \text{ solve, } \sigma_y = \frac{626 \text{ MPa}}{27} = 23.2 \text{ MPa} \quad \leftarrow$$



A-7.7: A cantilever beam with rectangular cross section ($b = 95 \text{ mm}$, $h = 300 \text{ mm}$) supports load $P = 160 \text{ kN}$ at its free end. The ratio of the magnitudes of the principal stresses (σ_1/σ_2) at point A (at distance $c = 0.8 \text{ m}$ from the free end and distance $d = 200 \text{ mm}$ up from the bottom) is approximately:

- (A) 5
- (B) 12
- (C) 18
- (D) 25

Solution

$$P = 160 \text{ kN} \quad b = 95 \text{ mm} \quad h = 300 \text{ mm}$$

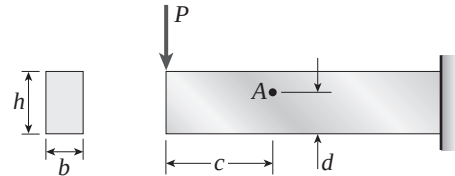
$$c = 0.8 \text{ m} \quad d = 200 \text{ mm} \quad \frac{d}{h} = 0.667$$

Cross section properties:

$$A = bh = 28500 \text{ mm}^2$$

$$I = \frac{bh^3}{12} = 2.138 \times 10^8 \text{ mm}^4$$

$$Q_A = [b(h-d)] \left[\frac{h}{2} - \frac{(h-d)}{2} \right] = 9.500 \times 10^5 \text{ mm}^3$$



Moment, shear force and normal and shear stresses at A:

$$M_A = -Pc = -1.280 \times 10^5 \text{ kN}\cdot\text{mm} \quad V_A = P$$

$$\tau_A = \frac{V_A Q_A}{Ib} = 7.485 \text{ MPa} \quad \sigma_A = \frac{-M_A \left(d - \frac{h}{2} \right)}{I} = 29.942 \text{ MPa}$$

Plane stress state at A: $\sigma_x = \sigma_A \quad \tau_{xy} = \tau_A \quad \sigma_y = 0$

Principal stresses:

$$\theta_p = \frac{1}{2} \arctan \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right) = 13.283^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 31.709 \text{ MPa}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = -1.767 \text{ MPa}$$

$$\text{Ratio of principal stresses } (\sigma_1 / \sigma_2): \quad \left| \frac{\sigma_1}{\sigma_2} \right| = 17.9 \quad \leftarrow$$

A-7.8: A simply supported beam ($L = 4.5 \text{ m}$) with rectangular cross section ($b = 95 \text{ mm}$, $h = 280 \text{ mm}$) supports uniform load $q = 25 \text{ kN/m}$. The ratio of the magnitudes of the principal stresses (σ_1/σ_2) at a point $a = 1.0 \text{ m}$ from the left support and distance $d = 100 \text{ mm}$ up from the bottom of the beam is approximately:

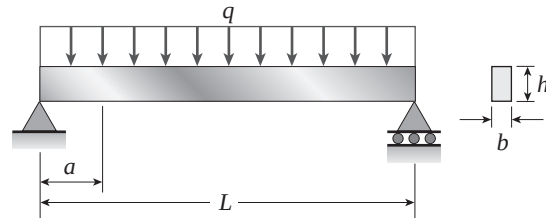
- (A) 9
- (B) 17
- (C) 31
- (D) 41

Solution

$$q = 25 \frac{\text{kN}}{\text{m}} \quad L = 4.5 \text{ m}$$

$$b = 95 \text{ mm} \quad h = 280 \text{ mm}$$

$$a = 1.0 \text{ m} \quad d = 100 \text{ mm}$$



Cross section properties:

$$A = bh = 26600 \text{ mm}^2$$

$$I = \frac{bh^3}{12} = 1.738 \times 10^8 \text{ mm}^4$$

$$Q = [b(h - d)] \left[\frac{h}{2} - \frac{(h - d)}{2} \right] = 8.550 \times 10^5 \text{ mm}^3$$

Moment, shear force and normal and shear stresses at distance a from left support:

$$V_a = \frac{qL}{2} - qa = 31.250 \text{ kN} \quad M_a = \frac{qL}{2}a - \frac{qa^2}{2} = 4.375 \times 10^4 \text{ kN} \cdot \text{mm}$$

$$\tau = \frac{V_a Q}{Ib} = 1.618 \text{ MPa} \quad \sigma = \frac{-M_a \left(d - \frac{h}{2} \right)}{I} = 10.070 \text{ MPa}$$

Plane stress state: $\sigma_x = \sigma \quad \tau_{xy} = \tau \quad \sigma_y = 0$

Principal stresses:

$$\theta_p = \frac{1}{2} \arctan \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right) = 8.909^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 10.324 \text{ MPa}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = -0.254 \text{ MPa}$$

Ratio of principal stresses (σ_1 / σ_2): $\left| \frac{\sigma_1}{\sigma_2} \right| = 40.7 \quad \leftarrow$

A-8.1: A thin wall spherical tank of diameter 1.5 m and wall thickness 65 mm has internal pressure of 20 MPa. The maximum shear stress in the wall of the tank is approximately:

- (A) 58 MPa
- (B) 67 MPa
- (C) 115 MPa
- (D) 127 MPa

Solution

$$d = 1.5 \text{ m} \quad t = 65 \text{ mm} \quad p = 20 \text{ MPa}$$

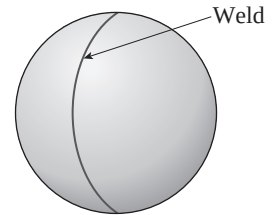
Thin wall tank since: $\frac{t}{\left(\frac{d}{2}\right)} = 0.087$

Biaxial stress:

$$\sigma = \frac{p\left(\frac{d}{2}\right)}{2t} \quad \sigma = 115.4 \text{ MPa}$$

Max. shear stress at 45 deg. rotation is 1/2 of σ

$$\tau_{\max} = \frac{\sigma}{2} = 57.7 \text{ MPa} \quad \leftarrow$$



A-8.2: A thin wall spherical tank of diameter 0.75 m has internal pressure of 20 MPa. The yield stress in tension is 920 MPa, the yield stress in shear is 475 MPa, and the factor of safety is 2.5. The modulus of elasticity is 210 GPa, Poisson's ratio is 0.28, and maximum normal strain is 1220×10^{-6} . The minimum permissible thickness of the tank is approximately:

- (A) 8.6 mm
- (B) 9.9 mm
- (C) 10.5 mm
- (D) 11.1 mm

Solution

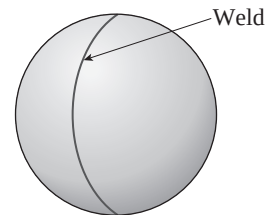
$$d = 0.75 \text{ m} \quad p = 20 \text{ MPa} \quad E = 210 \text{ GPa}$$

$$\sigma_Y = 920 \text{ MPa} \quad \tau_Y = 475 \text{ MPa} \quad FS_Y = 2.5$$

$$\nu = 0.28 \quad \epsilon_a = 1220(10^{-6})$$

Thickness based on **tensile stress**:

$$t_1 = \frac{p\left(\frac{d}{2}\right)}{2\left(\frac{\sigma_Y}{FS_Y}\right)} = 10.190 \text{ mm}$$



Thickness based on **shear stress**:

$$t_2 = \frac{p \left(\frac{d}{2} \right)}{4 \left(\frac{\tau_Y}{FS_Y} \right)} = 9.868 \text{ mm}$$

Thickness based on **normal strain**:

$$t_3 = \frac{p \left(\frac{d}{2} \right)}{2 \varepsilon_a E} (1 - \nu) \quad t_3 = 10.54 \text{ mm} \quad \leftarrow < \text{largest value controls}$$

A-8.3: A thin wall cylindrical tank of diameter 200 mm has internal pressure of 11 MPa. The yield stress in tension is 250 MPa, the yield stress in shear is 140 MPa, and the factor of safety is 2.5. The minimum permissible thickness of the tank is approximately:

- (A) 8.2 mm
- (B) 9.1 mm
- (C) 9.8 mm
- (D) 11.0 mm

Solution

$$d = 200 \text{ mm} \quad p = 11 \text{ MPa}$$

$$\sigma_Y = 250 \text{ MPa} \quad \tau_Y = 140 \text{ MPa} \quad FS_Y = 2.5$$

Wall thickness based on **tensile stress**:

$$t_1 = \frac{p \left(\frac{d}{2} \right)}{\frac{\sigma_Y}{FS_Y}} = 11.00 \text{ mm} \quad \leftarrow < \text{larger value governs} \quad \frac{t_1}{\left(\frac{d}{2} \right)} = 0.110$$

Wall thickness based on **shear stress**:

$$t_2 = \frac{p \left(\frac{d}{2} \right)}{2 \left(\frac{\tau_Y}{FS_Y} \right)} = 9.821 \text{ mm} \quad \frac{t_2}{\left(\frac{d}{2} \right)} = 0.098$$

A-8.4: A thin wall cylindrical tank of diameter 2.0 m and wall thickness 18 mm is open at the top. The height h of water (weight density = 9.81 kN/m^3) in the tank at which the circumferential stress reaches 10 MPa in the tank wall is approximately:

- (A) 14 m
- (B) 18 m
- (C) 20 m
- (D) 24 m

Solution

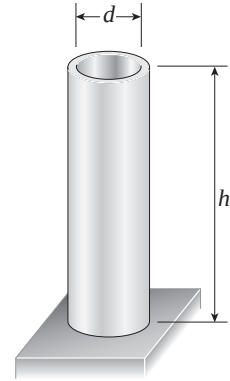
$$d = 2 \text{ m} \quad t = 18 \text{ mm} \quad \sigma_a = 10 \text{ MPa} \quad \gamma_w = 9.81 \frac{\text{kN}}{\text{m}^3}$$

Pressure at height h : $p_h = \gamma_w h$

Circumferential stress: $\sigma_c = \frac{p_h \left(\frac{d}{2}\right)}{t} \quad \sigma_c = \frac{(\gamma_w h) \left(\frac{d}{2}\right)}{t}$

Set σ_c equal to σ_a and solve for h :

$$h = \frac{\sigma_a t}{(\gamma_w) \left(\frac{d}{2}\right)} = 18.3 \text{ m} \quad \leftarrow$$



A-8.5: The pressure relief valve is opened on a thin wall cylindrical tank, with radius to wall thickness ratio of 128, thereby decreasing the longitudinal strain by 150×10^{-6} . Assume $E = 73 \text{ GPa}$ and $\nu = 0.33$. The original internal pressure in the tank was approximately:

- (A) 370 kPa
- (B) 450 kPa
- (C) 500 kPa
- (D) 590 kPa

Solution

$$r_t = \frac{r}{t} \quad r_t = 128$$

$$\varepsilon_L = 148 (10^{-6})$$

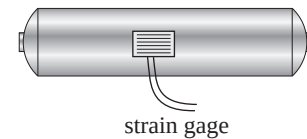
$$E = 73 \text{ GPa} \quad \nu = 0.33$$

Longitudinal strain:

$$\varepsilon = \frac{p}{2E} \left(\frac{r}{t}\right) (1 - 2\nu)$$

Set ε to ε_L and solve for pressure p :

$$p = \frac{2E\varepsilon_L}{r_t(1 - 2\nu)} = 497 \text{ kPa} \quad \leftarrow$$



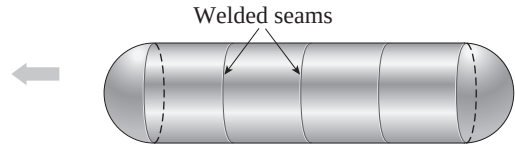
A-8.6: A cylindrical tank is assembled by welding steel sections circumferentially. Tank diameter is 1.5 m, thickness is 20 mm, and internal pressure is 2.0 MPa. The maximum stress in the heads of the tank is approximately:

- (A) 38 MPa
- (B) 45 MPa
- (C) 50 MPa
- (D) 59 MPa

Solution

$$d = 1.5 \text{ m} \quad t = 20 \text{ mm} \quad p = 2.0 \text{ MPa}$$

$$\sigma_h = \frac{p \left(\frac{d}{2} \right)}{2t} = 37.5 \text{ MPa}$$



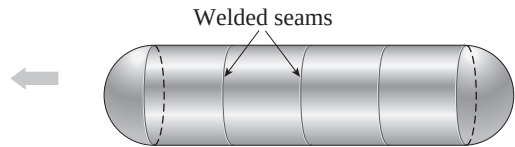
A-8.7: A cylindrical tank is assembled by welding steel sections circumferentially. Tank diameter is 1.5 m, thickness is 20 mm, and internal pressure is 2.0 MPa. The maximum tensile stress in the cylindrical part of the tank is approximately:

- (A) 45 MPa
- (B) 57 MPa
- (C) 62 MPa
- (D) 75 MPa

Solution

$$d = 1.5 \text{ m} \quad t = 20 \text{ mm} \quad p = 2.0 \text{ MPa}$$

$$\sigma_c = \frac{p \left(\frac{d}{2} \right)}{t} = 75.0 \text{ MPa}$$



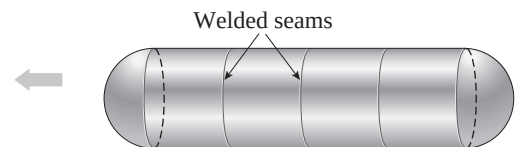
A-8.8: A cylindrical tank is assembled by welding steel sections circumferentially. Tank diameter is 1.5 m, thickness is 20 mm, and internal pressure is 2.0 MPa. The maximum tensile stress perpendicular to the welds is approximately:

- (A) 22 MPa
- (B) 29 MPa
- (C) 33 MPa
- (D) 37 MPa

Solution

$$d = 1.5 \text{ m} \quad t = 20 \text{ mm} \quad p = 2.0 \text{ MPa}$$

$$\sigma_w = \frac{p \left(\frac{d}{2} \right)}{2t} = 37.5 \text{ MPa}$$



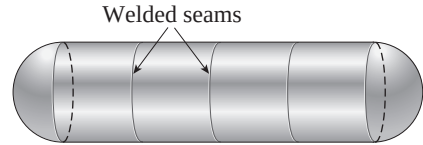
A-8.9: A cylindrical tank is assembled by welding steel sections circumferentially. Tank diameter is 1.5 m, thickness is 20 mm, and internal pressure is 2.0 MPa. The maximum shear stress in the heads is approximately:

- (A) 19 MPa
- (B) 23 MPa
- (C) 33 MPa
- (D) 35 MPa

Solution

$$d = 1.5 \text{ m} \quad t = 20 \text{ mm} \quad p = 2.0 \text{ MPa}$$

$$\tau_h = \frac{p \left(\frac{d}{2} \right)}{4t} = 18.8 \text{ MPa}$$



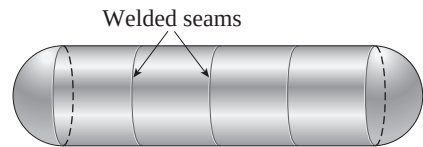
A-8.10: A cylindrical tank is assembled by welding steel sections circumferentially. Tank diameter is 1.5 m, thickness is 20 mm, and internal pressure is 2.0 MPa. The maximum shear stress in the cylindrical part of the tank is approximately:

- (A) 17 MPa
- (B) 26 MPa
- (C) 34 MPa
- (D) 38 MPa

Solution

$$d = 1.5 \text{ m} \quad t = 20 \text{ mm} \quad p = 2.0 \text{ MPa}$$

$$\tau_{\max} = \frac{p \left(\frac{d}{2} \right)}{2t} = 37.5 \text{ MPa}$$



A-8.11: A cylindrical tank is assembled by welding steel sections in a helical pattern with angle $\alpha = 50$ degrees. Tank diameter is 1.6 m, thickness is 20 mm, and internal pressure is 2.75 MPa. Modulus $E = 210$ GPa and Poisson's ratio $\nu = 0.28$. The circumferential strain in the wall of the tank is approximately:

- (A) 1.9×10^{-4}
- (B) 3.2×10^{-4}
- (C) 3.9×10^{-4}
- (D) 4.5×10^{-4}

Solution

$$d = 1.6 \text{ m} \quad t = 20 \text{ mm} \quad p = 2.75 \text{ MPa}$$

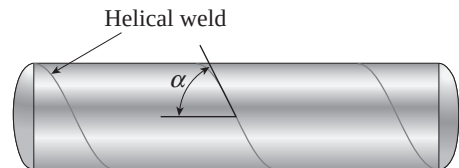
$$E = 210 \text{ GPa} \quad \nu = 0.28 \quad \alpha = 50^\circ$$

Circumferential stress:

$$\sigma_c = \frac{p \left(\frac{d}{2} \right)}{t} = 110.000 \text{ MPa}$$

Circumferential strain:

$$\varepsilon_c = \frac{\sigma_c}{2E} (2 - \nu) = 4.50 \times 10^{-4}$$



A-8.12: A cylindrical tank is assembled by welding steel sections in a helical pattern with angle $\alpha = 50$ degrees. Tank diameter is 1.6 m, thickness is 20 mm, and internal pressure is 2.75 MPa. Modulus $E = 210$ GPa and Poisson's ratio $\nu = 0.28$. The longitudinal strain in the the wall of the tank is approximately:

- (A) 1.2×10^{-4}
- (B) 2.4×10^{-4}
- (C) 3.1×10^{-4}
- (D) 4.3×10^{-4}

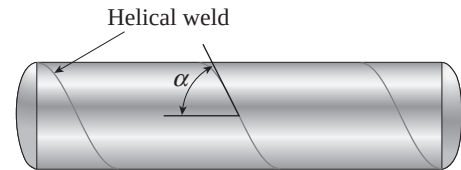
Solution

$$d = 1.6 \text{ m} \quad t = 20 \text{ mm} \quad p = 2.75 \text{ MPa}$$

$$E = 210 \text{ GPa} \quad \nu = 0.28 \quad \alpha = 50^\circ$$

Longitudinal stress:

$$\sigma_L = \frac{p \left(\frac{d}{2} \right)}{2t} = 55.000 \text{ MPa}$$



Longitudinal strain:

$$\varepsilon_L = \frac{\sigma_L}{E} (1 - 2\nu) = 1.15 \times 10^{-4} \quad \leftarrow$$

A-8.13: A cylindrical tank is assembled by welding steel sections in a helical pattern with angle $\alpha = 50$ degrees. Tank diameter is 1.6 m, thickness is 20 mm, and internal pressure is 2.75 MPa. Modulus $E = 210$ GPa and Poisson's ratio $\nu = 0.28$. The normal stress acting perpendicular to the weld is approximately:

- (A) 39 MPa
- (B) 48 MPa
- (C) 78 MPa
- (D) 84 MPa

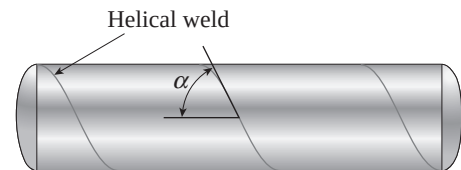
Solution

$$d = 1.6 \text{ m} \quad t = 20 \text{ mm} \quad p = 2.75 \text{ MPa} \quad E = 210 \text{ GPa}$$

$$\nu = 0.28 \quad \alpha = 50^\circ$$

Longitudinal stress:

$$\sigma_L = \frac{p \left(\frac{d}{2} \right)}{2t} = 55.000 \text{ MPa} \quad \text{So} \quad \sigma_x = \sigma_L$$



Circumferential stress:

$$\sigma_c = \frac{p \left(\frac{d}{2} \right)}{t} = 110.000 \text{ MPa} \quad \text{So} \quad \sigma_y = \sigma_c$$

Angle perpendicular to the weld: $\theta = 90^\circ - \alpha = 40.000^\circ$

Normal stress perpendicular to the weld:

$$\sigma_{40} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) = 77.7 \text{ MPa} \quad \leftarrow$$

A-8.14: A segment of a drive shaft ($d_2 = 200 \text{ mm}$, $d_1 = 160 \text{ mm}$) is subjected to a torque $T = 30 \text{ kN}\cdot\text{m}$. The allowable shear stress in the shaft is 45 MPa. The maximum permissible compressive load P is approximately:

- (A) 200 kN
- (B) 286 kN
- (C) 328 kN
- (D) 442 kN

Solution

$$d_2 = 200 \text{ mm} \quad d_1 = 160 \text{ mm} \quad \tau_a = 45 \text{ MPa} \\ T = 30 \text{ kN}\cdot\text{m}$$

Cross section properties:

$$A = \frac{\pi}{4} (d_2^2 - d_1^2) = 11310 \text{ mm}^2 \\ I_p = \frac{\pi}{32} (d_2^4 - d_1^4) = 9.274 \times 10^7 \text{ mm}^4$$

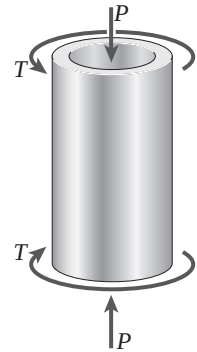
Normal and in-plane shear stresses:

$$\sigma_x = 0 \quad \sigma_y = \frac{-P}{A} \quad \tau_{xy} = \frac{T \left(\frac{d_2}{2} \right)}{I_p} = 32.349 \text{ MPa}$$

Maximum in-plane shear stress: set $\tau_{\max} = \tau_{\text{allow}}$ then solve for σ_y

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \quad \text{So} \quad \sigma_y = \sqrt{4(\tau_a - \tau_{xy})^2} = 25.303 \text{ MPa}$$

Finally solve for $P = \sigma_y A$: $P_{\max} = \sigma_y A = 286 \text{ kN} \quad \leftarrow$



A-8.15: A thin walled cylindrical tank, under internal pressure p , is compressed by a force $F = 75 \text{ kN}$. Cylinder diameter is $d = 90 \text{ mm}$ and wall thickness $t = 5.5 \text{ mm}$. Allowable normal stress is 110 MPa and allowable shear stress is 60 MPa. The maximum allowable internal pressure $p_{\max}$ is approximately:

- (A) 5 MPa
- (B) 10 MPa
- (C) 13 MPa
- (D) 17 MPa

Solution

$$d = 90 \text{ mm} \quad t = 5.5 \text{ mm} \quad \sigma_a = 110 \text{ MPa}$$

$$F = 75 \text{ kN} \quad A = 2\pi \frac{d}{2} t = 1555 \text{ mm}^2$$

Circumferential normal stress:



$$\sigma_c = \frac{p_{\max} \left(\frac{d}{2} \right)}{t} \quad \text{and setting } \sigma_c = \sigma_a \text{ and solving for } p_{\max}:$$

$$p_{\max c} = \sigma_a \left(\frac{2t}{d} \right) = 13.4 \text{ MPa} \quad \leftarrow < \text{controls}$$

Longitudinal normal stress:

$$\sigma_L = \frac{p_{\max} \left(\frac{d}{2} \right)}{2t} - \frac{F}{A} \quad \text{Or} \quad \sigma_L = \frac{p_{\max} d}{4t} - \frac{F}{A}$$

 So set $\sigma_L = \sigma_a$ and solve for $p_{\max}$:

$$p_{\max L} = \left(\sigma_a + \frac{F}{A} \right) \frac{4t}{d} = 38.7 \text{ MPa}$$

Check also in-plane & out-of-plane shear stresses: all are below allowable shear stress so circumferential normal stress controls as noted above.

A-9.1: An aluminum beam ($E = 72 \text{ GPa}$) with a square cross section and span length $L = 2.5 \text{ m}$ is subjected to uniform load $q = 1.5 \text{ kN/m}$. The allowable bending stress is 60 MPa . The maximum deflection of the beam is approximately:

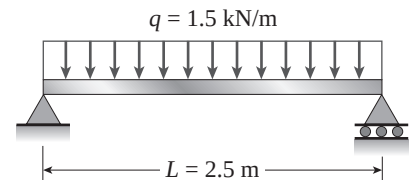
- (A) 10 mm
- (B) 16 mm
- (C) 22 mm
- (D) 26 mm

Solution

$$E = 72 (10^3) \text{ MPa} \quad \sigma_a = 60 \text{ MPa}$$

$$q = 1.5 \frac{\text{N}}{\text{mm}}$$

$$L = 2500 \text{ mm}$$


 Max. moment and deflection at $L/2$:

$$M_{\max} = \frac{qL^2}{8} \quad \delta_{\max} = \frac{5qL^4}{384EI}$$

Moment of inertia and section modulus for square cross section (height = width = b)

$$I = \frac{b^4}{12} \quad S = \frac{I}{\left(\frac{b}{2}\right)} = \frac{b^3}{6}$$

Flexure formula

$$\sigma_{\max} = \frac{M_{\max}}{S} \quad \sigma_{\max} = \frac{\frac{qL^2}{8}}{\left(\frac{b^3}{6}\right)} \quad \text{so} \quad b^3 = \frac{3}{4} \frac{qL^2}{\sigma_{\max}}$$

Max. deflection formula

$$\delta_{\max} = \frac{5qL^4}{384E\left(\frac{b^4}{12}\right)} \quad \text{so solve for } \delta_{\max} \text{ if } \sigma_{\max} = \sigma_a$$

$$\delta_{\max} = \frac{5qL^4}{384E \left[\frac{\left[\left(\frac{3}{4} \frac{qL^2}{\sigma_{\max}} \right)^{\frac{1}{3}} \right]^4}{12} \right]} = 22.2 \text{ mm} \quad \leftarrow$$

A-9.2: An aluminum cantilever beam ($E = 72 \text{ GPa}$) with a square cross section and span length $L = 2.5 \text{ m}$ is subjected to uniform load $q = 1.5 \text{ kN/m}$. The allowable bending stress is 55 MPa . The maximum deflection of the beam is approximately:

- (A) 10 mm
- (B) 20 mm
- (C) 30 mm
- (D) 40 mm

Solution

$$E = 72(10^3) \text{ MPa} \quad \sigma_a = 55 \text{ MPa}$$

$$q = 1.5 \frac{\text{N}}{\text{mm}}$$

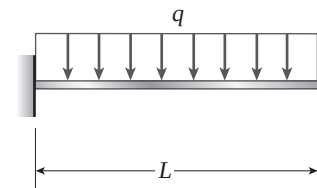
$$L = 2500 \text{ mm}$$

Max. moment at support & max. deflection at L :

$$M_{\max} = \frac{qL^2}{2} \quad \delta_{\max} = \frac{qL^4}{8EI}$$

Moment of inertia and section modulus for square cross section (height = width = b)

$$I = \frac{b^4}{12} \quad S = \frac{I}{\left(\frac{b}{2}\right)} = \frac{b^3}{6}$$



Flexure formula

$$\sigma_{\max} = \frac{M_{\max}}{S} \quad \sigma_{\max} = \frac{\frac{qL^2}{2}}{\left(\frac{b^3}{6}\right)} \quad \text{so} \quad b^3 = 3 \frac{qL^2}{\sigma_{\max}}$$

Max. deflection formula

$$\delta_{\max} = \frac{qL^4}{8E\left(\frac{b^4}{12}\right)} \quad \text{so solve for } \delta_{\max} \text{ if } \sigma_{\max} = \sigma_a$$

$$\delta_{\max} = \frac{qL^4}{8E \left[\frac{\left[\left(3 \frac{qL^2}{\sigma_{\max}} \right)^{\frac{1}{3}} \right]^4}{12} \right]} = 29.9 \text{ mm} \quad \leftarrow$$

A-9.3: A steel beam ($E = 210 \text{ GPa}$) with $I = 119 \times 10^6 \text{ mm}^4$ and span length $L = 3.5 \text{ m}$ is subjected to uniform load $q = 9.5 \text{ kN/m}$. The maximum deflection of the beam is approximately:

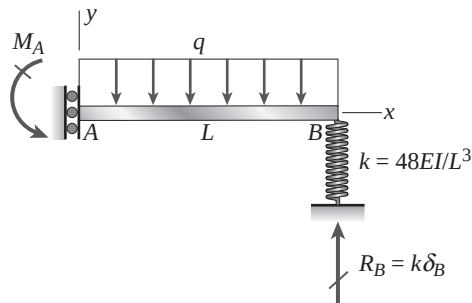
- (A) 10 mm
- (B) 13 mm
- (C) 17 mm
- (D) 19 mm

Solution

$$E = 210(10^3) \text{ MPa} \quad I = 119(10^6) \text{ mm}^4 < \text{strong axis } I \text{ for W310} \times 52$$

$$q = 9.5 \frac{\text{N}}{\text{mm}}$$

$$L = 3500 \text{ mm}$$



Max. deflection at A by superposition of SS beam mid-span deflection & R_B/k :

$$\delta_{\max} = \frac{5q(2L)^4}{384EI} + \frac{(qL)}{\left(\frac{48EI}{L^3}\right)} = 13.07 \text{ mm} \quad \leftarrow$$

A-9.4: A steel bracket ABC ($EI = 4.2 \times 10^6 \text{ N}\cdot\text{m}^2$) with span length $L = 4.5 \text{ m}$ and height $H = 2 \text{ m}$ is subjected to load $P = 15 \text{ kN}$ at C . The maximum rotation of joint B is approximately:

- (A) 0.1°
- (B) 0.3°
- (C) 0.6°
- (D) 0.9°

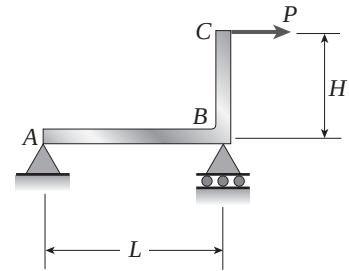
Solution

$$E = 210 \text{ GPa} \quad I = 20 \times 10^6 \text{ mm}^4 \quad < \text{strong axis } I \text{ for W200} \times 22.5$$

$$EI = 4.20 \times 10^6 \text{ N}\cdot\text{m}^2$$

$$P = 15 \text{ kN}$$

$$L = 4.5 \text{ m} \quad H = 2 \text{ m}$$



Max. rotation at B : apply statically-equivalent moment $P \times H$ at B on SS beam

$$\theta_{B\max} = \frac{(PH)L}{3EI} = 0.614^\circ \quad \leftarrow \quad \theta_{B\max} = 0.011 \text{ rad}$$

A-9.5: A steel bracket ABC ($EI = 4.2 \times 10^6 \text{ N}\cdot\text{m}^2$) with span length $L = 4.5 \text{ m}$ and height $H = 2 \text{ m}$ is subjected to load $P = 15 \text{ kN}$ at C . The maximum horizontal displacement of joint C is approximately:

- (A) 22 mm
- (B) 31 mm
- (C) 38 mm
- (D) 40 mm

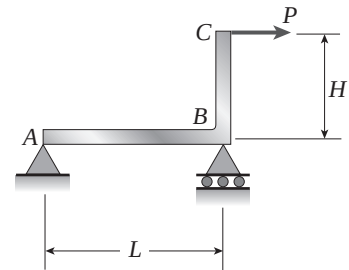
Solution

$$E = 210 \text{ GPa} \quad I = 20 \times 10^6 \text{ mm}^4 \quad < \text{strong axis } I \text{ for W200} \times 22.5$$

$$EI = 4.20 \times 10^6 \text{ N}\cdot\text{m}^2$$

$$P = 15 \text{ kN}$$

$$L = 4.5 \text{ m} \quad H = 2 \text{ m}$$



Max. rotation at B : apply statically-equivalent moment $P \times H$ at B on SS beam

$$\theta_{B\max} = \frac{(PH)L}{3EI} = 0.614^\circ \quad \theta_{B\max} = 0.011 \text{ rad}$$

Horizontal deflection of vertical cantilever BC : $\delta_{BC} = \frac{PH^3}{3EI} = 9.524 \text{ mm}$

Finally, superpose $\theta_B \times H$ and δ_{BC}

$$\delta_C = \theta_{B\max} H + \delta_{BC} = 31.0 \text{ mm} \quad \leftarrow$$

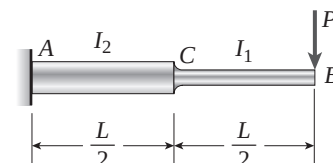
A-9.6: A nonprismatic cantilever beam of one material is subjected to load P at its free end. Moment of inertia $I_2 = 2 I_1$. The ratio r of the deflection δ_B to the deflection δ_1 at the free end of a prismatic cantilever with moment of inertia I_1 carrying the same load is approximately:

- (A) 0.25
- (B) 0.40
- (C) 0.56
- (D) 0.78

Solution

Max. deflection of prismatic cantilever (constant I_1)

$$\delta_1 = \frac{PL^3}{3EI_1}$$



Rotation at C due to both load P & moment $PL/2$ at C for nonprismatic beam:

$$\theta_C = \frac{P\left(\frac{L}{2}\right)^2}{2EI_2} + \frac{\left(P\frac{L}{2}\right)\frac{L}{2}}{EI_2} = \frac{3L^2P}{8EI_2}$$

Deflection at C due to both load P & moment $PL/2$ at C for nonprismatic beam:

$$\delta_{C1} = \frac{P\left(\frac{L}{2}\right)^3}{3EI_2} + \frac{\left(P\frac{L}{2}\right)\left(\frac{L}{2}\right)^2}{2EI_2} = \frac{5L^3P}{48EI_2}$$

Total deflection at B : $\delta_B = \delta_{C1} + \theta_C \frac{L}{2} + \frac{P\left(\frac{L}{2}\right)^3}{3EI_1}$

$$\delta_B = \frac{5L^3P}{48EI_2} + \frac{\left(\frac{3L^2P}{8EI_2}\right)\frac{L}{2}}{3EI_1} + \frac{P\left(\frac{L}{2}\right)^3}{3EI_1} = \frac{L^3P(7I_1 + I_2)}{24EI_1I_2}$$

Ratio δ_B/δ_1 :

$$r = \frac{\frac{L^3 P (7I_1 + I_2)}{24EI_1 I_2}}{\frac{PL^3}{3EI_1}} = \frac{7I_1}{8I_2} + \frac{1}{8}$$

$$\text{so } r = \frac{7}{8} \left(\frac{1}{2} \right) + \frac{1}{8} = 0.563 \quad \leftarrow$$

A-9.7: A steel bracket $ABCD$ ($EI = 4.2 \times 10^6 \text{ N}\cdot\text{m}^2$), with span length $L = 4.5 \text{ m}$ and dimension $a = 2 \text{ m}$, is subjected to load $P = 10 \text{ kN}$ at D . The maximum deflection at B is approximately:

- (A) 10 mm
- (B) 14 mm
- (C) 19 mm
- (D) 24 mm

Solution

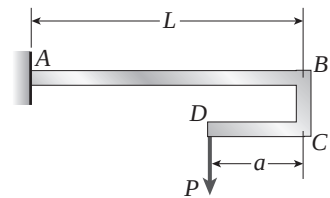
$$E = 210 \text{ GPa} \quad I = 20 \times 10^6 \text{ mm}^4 < \text{strong axis } I \text{ for W200} \times 22.5$$

$$EI = 4.20 \times 10^6 \text{ N}\cdot\text{m}^2$$

$$P = 10 \text{ kN}$$

$$L = 4.5 \text{ m} \quad a = 2 \text{ m}$$

$$h = 206 \text{ mm}$$

Statically-equivalent loads at end of cantilever AB :

- downward load P
- CCW moment $P \times a$

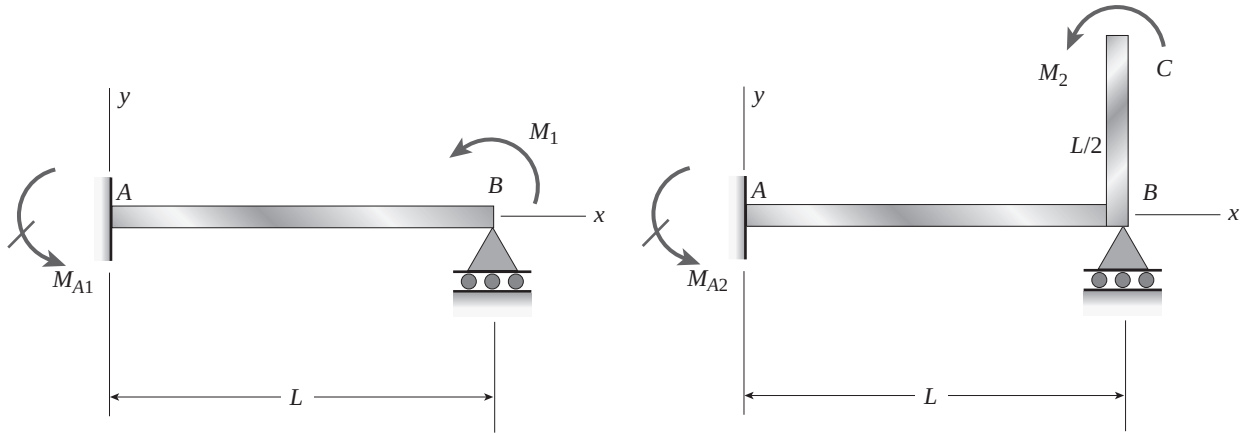
Downward deflection at B by superposition:

$$\delta_B = \frac{PL^3}{3EI} - \frac{(Pa)L^2}{2EI} = 24.1 \text{ mm} \quad \leftarrow \quad \frac{\delta_B}{L} = 0.005$$

A-10.1: Propped cantilever beam AB has moment M_1 applied at joint B . Framework ABC has moment M_2 applied at C . Both structures have constant flexural rigidity EI . If the ratio of the *applied moments* $M_1/M_2 = 3/2$, the ratio of the *reactive moments* M_{A1}/M_{A2} at clamped support A is:

- (A) 1
- (B) 3/2
- (C) 2
- (D) 5/2

Solution



From Prob. 10.3-1: $M_{A1} = \frac{M_1}{2}$

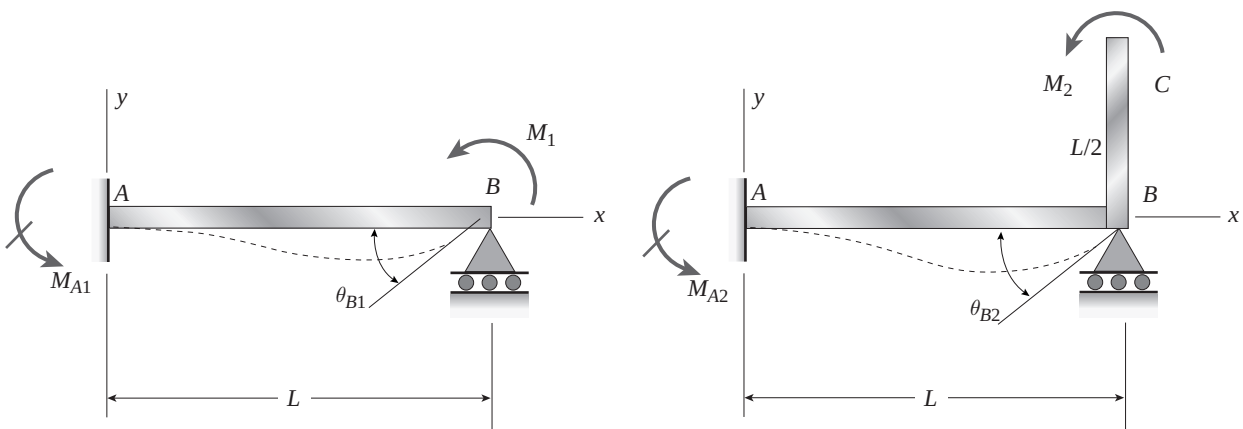
Statically-equivalent moment at B is M_2 so $M_{A2} = \frac{M_2}{2}$

Ratio of reactive moments is $\frac{M_{A1}}{M_{A2}} = \frac{M_1}{M_2}$ so $\frac{M_{A1}}{M_{A2}} = \frac{3}{2}$ ←

A-10.2: Propped cantilever beam AB has moment M_1 applied at joint B . Framework ABC has moment M_2 applied at C . Both structures have constant flexural rigidity EI . If the ratio of the applied moments $M_1/M_2 = 3/2$, the ratio of the joint rotations at B , θ_{B1}/θ_{B2} , is:

- (A) 1
- (B) 3/2
- (C) 2
- (D) 5/2

Solution



From Prob 10.3-1: $\theta_{B1} = \frac{M_1 L}{4EI}$

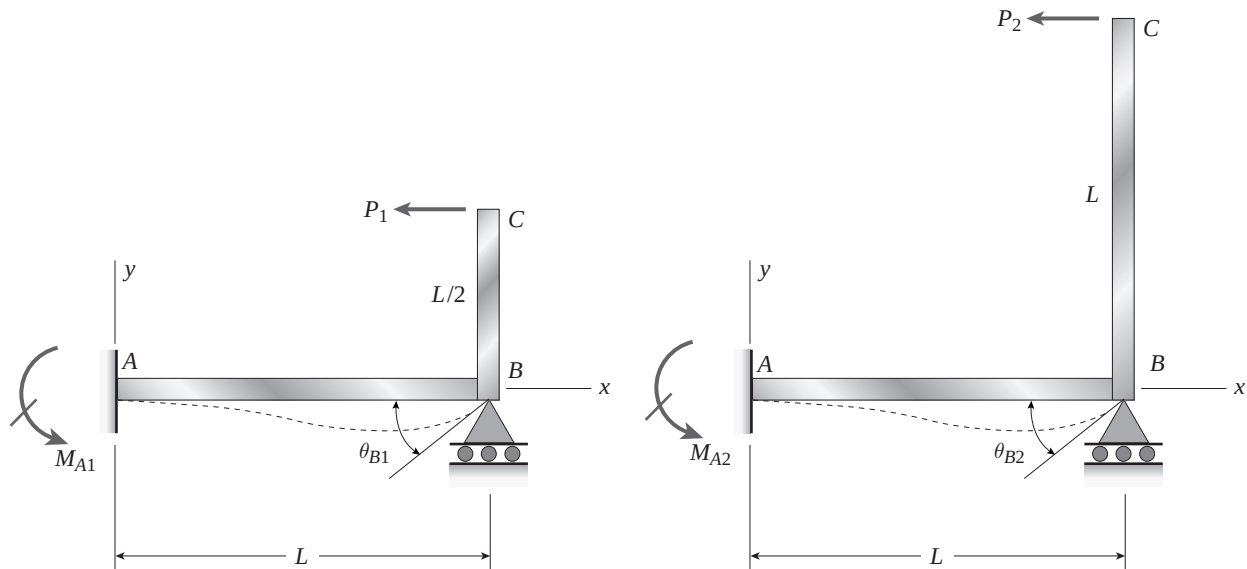
Statically-equivalent moment at B is M_2 so $\theta_{B2} = \frac{M_2 L}{4EI}$

Ratio of joint rotations is $\frac{\theta_{B1}}{\theta_{B2}} = \frac{M_1}{M_2}$ so $\frac{\theta_{B1}}{\theta_{B2}} = \frac{3}{2}$ ←

A-10.3: Structure 1 with member BC of length $L/2$ has force P_1 applied at joint C. Structure 2 with member BC of length L has force P_2 applied at C. Both structures have constant flexural rigidity EI . If the ratio of the applied forces $P_1/P_2 = 5/2$, the ratio of the joint B rotations θ_{B1}/θ_{B2} is:

- (A) 1
- (B) 5/4
- (C) 3/2
- (D) 2

Solution



Statically-equivalent moment at B is $M_1 = P_1 \times L/2$

From Prob. 10.3-1: $\theta_{B1} = \frac{\left(P_1 \frac{L}{2}\right)L}{4EI} = \frac{L^2 P_1}{8EI}$

Statically-equivalent moment at B is $M_2 = P_2 \times L$

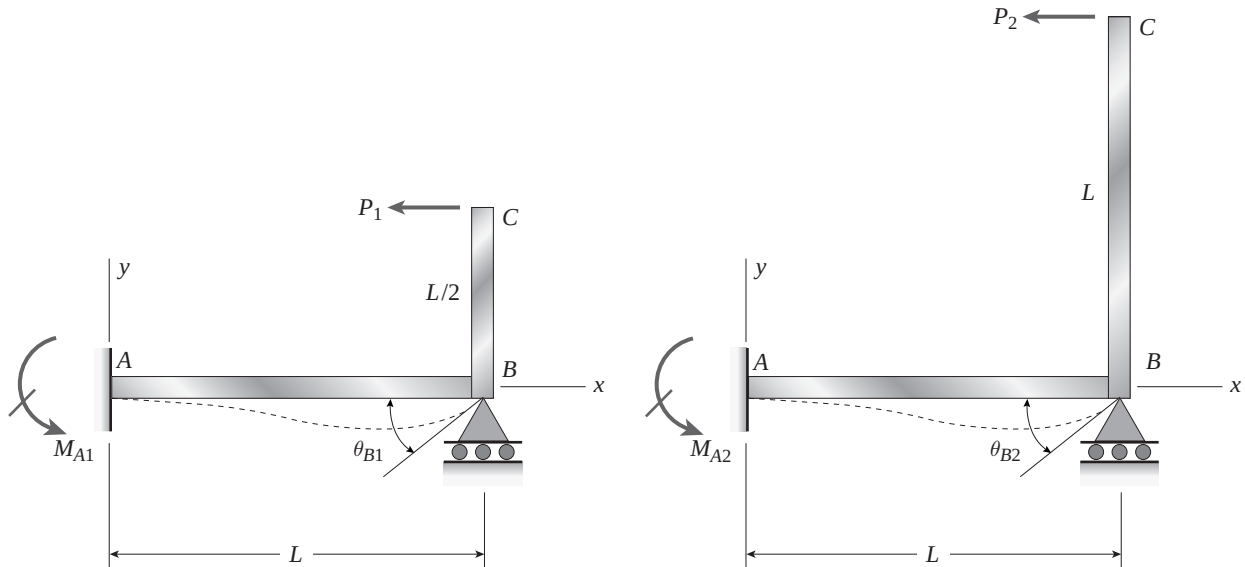
so $\theta_{B2} = \frac{(P_2 L)L}{4EI} = \frac{L^2 P_2}{4EI}$

Ratio of joint rotations is $\frac{\theta_{B1}}{\theta_{B2}} = \frac{P_1}{2P_2}$ so $\frac{\theta_{B1}}{\theta_{B2}} = \frac{5}{4}$ ←

A-10.4: Structure 1 with member BC of length $L/2$ has force P_1 applied at joint C . Structure 2 with member BC of length L has force P_2 applied at C . Both structures have constant flexural rigidity EI . The *required* ratio of the *applied* forces P_1/P_2 so that joint B rotations θ_{B1} and θ_{B2} are equal is approximately:

- (A) 1
- (B) 5/4
- (C) 3/2
- (D) 2

Solution



Statically-equivalent moment at B is $M_1 = P_1 \times L/2$

$$\text{From Prob. 10.3-1: } \theta_{B1} = \frac{\left(P_1 \frac{L}{2}\right)L}{4EI} = \frac{L^2 P_1}{8EI}$$

Statically-equivalent moment at B is $M_2 = P_2 \times L$

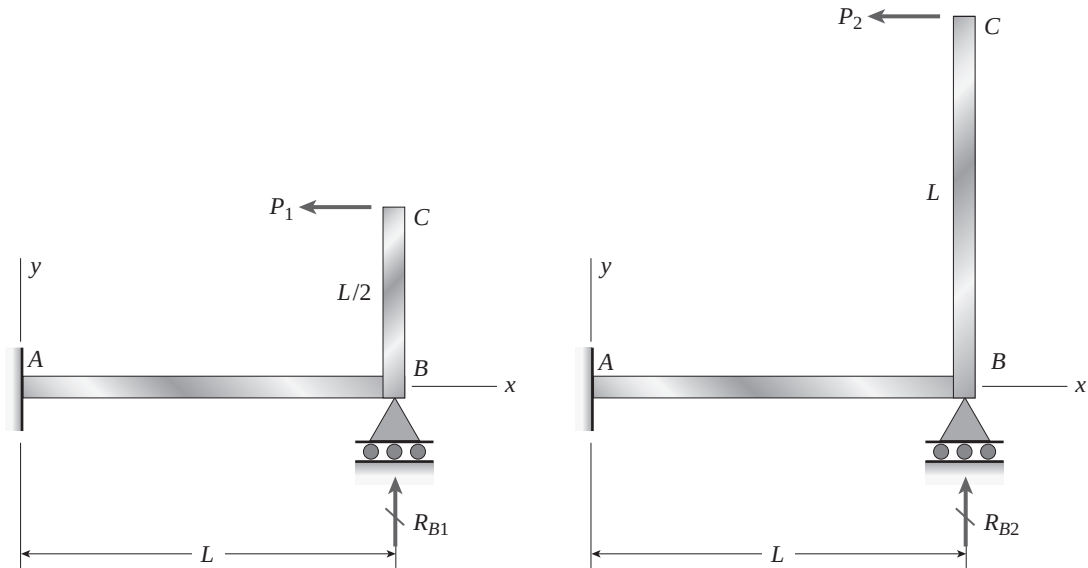
$$\text{so } \theta_{B2} = \frac{(P_2 L)L}{4EI} = \frac{L^2 P_2}{4EI}$$

$$\text{Ratio of joint rotations is } \frac{\theta_{B1}}{\theta_{B2}} = \frac{P_1}{2P_2} \text{ and } \frac{\theta_{B1}}{\theta_{B2}} = 1 \text{ so } \frac{P_1}{P_2} = 2 \quad \leftarrow$$

A-10.5: Structure 1 with member BC of length $L/2$ has force P_1 applied at joint C . Structure 2 with member BC of length L has force P_2 applied at C . Both structures have constant flexural rigidity EI . If the ratio of the *applied* forces $P_1/P_2 = 5/2$, the ratio of the *joint* B reactions R_{B1}/R_{B2} is:

- (A) 1
- (B) 5/4
- (C) 3/2
- (D) 2

Solution



Statically-equivalent moment at B is $M_1 = P_1 \times L/2$

$$\text{From Prob. 10.3-1: } R_{B1} = \frac{-3}{2} \frac{M_1}{L} \quad \text{or} \quad R_{B1} = \frac{-3}{2} \frac{\left(P_1 \frac{L}{2}\right)}{L} = -\frac{3P_1}{4}$$

Statically-equivalent moment at B is $M_2 = P_2 \times L$

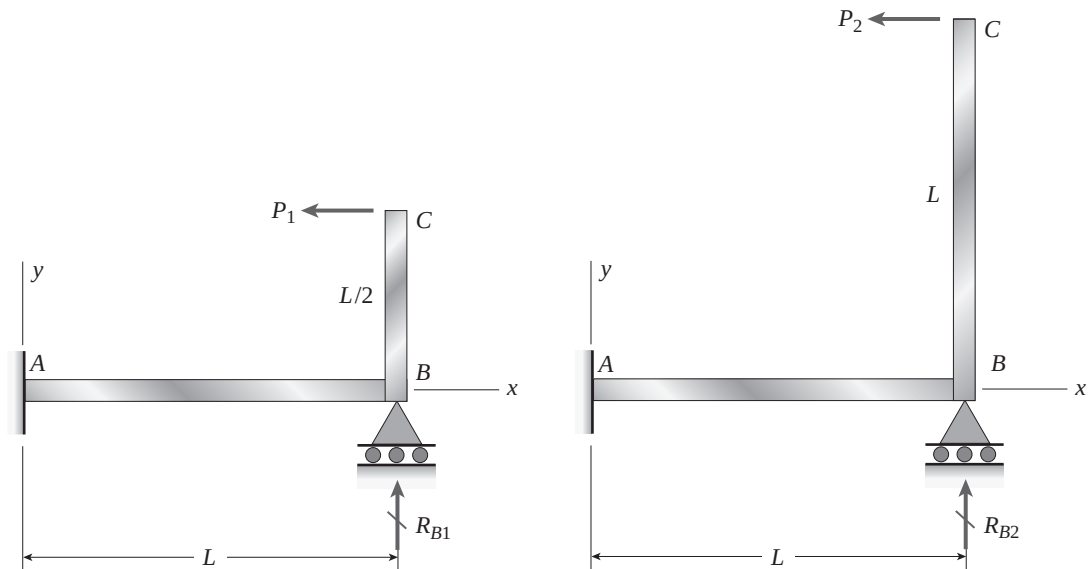
$$\text{so } R_{B2} = \frac{-3}{2} \frac{(P_2 L)}{L} = -\frac{3P_2}{2}$$

$$\text{Ratio of reactions at B is } \frac{R_{B1}}{R_{B2}} = \frac{P_1}{2P_2} \quad \text{so} \quad \frac{R_{B1}}{R_{B2}} = \frac{1}{2} \left(\frac{5}{2}\right) = \frac{5}{4} \quad \leftarrow$$

A-10.6: Structure 1 with member BC of length $L/2$ has force P_1 applied at joint C. Structure 2 with member BC of length L has force P_2 applied at C. Both structures have constant flexural rigidity EI . The *required* ratio of the *applied* forces P_1/P_2 so that joint B reactions R_{B1} and R_{B2} are equal is:

- (A) 1
- (B) 5/4
- (C) 3/2
- (D) 2

Solution



Statically-equivalent moment at B is $M_1 = P_1 \times L/2$

$$\text{From Prob. 10.3-1: } R_{B1} = \frac{-3 M_1}{2 L} \quad \text{or} \quad R_{B1} = \frac{-3 \left(P_1 \frac{L}{2} \right)}{2 L} = -\frac{3 P_1}{4}$$

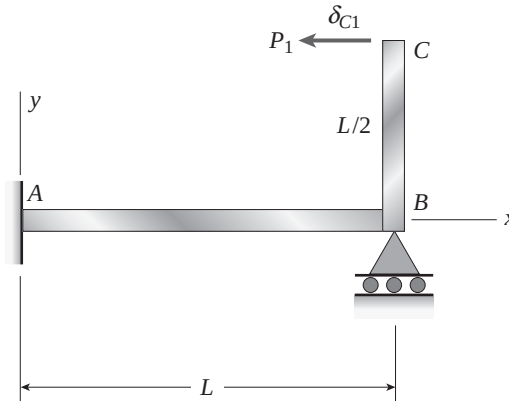
Statically-equivalent moment at B is $M_2 = P_2 \times L$

$$\text{so } R_{B2} = \frac{-3 (P_2 L)}{2 L} = -\frac{3 P_2}{2}$$

$$\text{Ratio of reactions at B is } \frac{R_{B1}}{R_{B2}} = \frac{P_1}{2 P_2} \quad \text{and} \quad \frac{R_{B1}}{R_{B2}} = 1 \quad \text{so} \quad \frac{P_1}{P_2} = 2 \quad \leftarrow$$

A-10.7: Structure 1 with member BC of length $L/2$ has force P_1 applied at joint C. Structure 2 with member BC of length L has force P_2 applied at C. Both structures have constant flexural rigidity EI . If the ratio of the *applied forces* $P_1/P_2 = 5/2$, the ratio of the *joint C lateral deflections* δ_{C1}/δ_{C2} is approximately:

- (A) 1/2
- (B) 4/5
- (C) 3/2
- (D) 2

Solution

$$\delta_{C1} = \frac{P_1 \left(\frac{L}{2}\right)^3}{3EI} + \frac{P_1 \frac{L}{2} L}{4EI} \frac{L}{2} = \frac{5L^3 P_1}{48EI}$$

$$\delta_{C2} = \frac{P_2 L^3}{3EI} + \frac{P_2 L L}{4EI} L = \frac{7L^3 P_2}{12EI}$$

$$\frac{\delta_{C1}}{\delta_{C2}} = \frac{5P_1}{28P_2} \quad \frac{5}{28} \frac{5}{2} = \frac{25}{56} \quad \frac{25}{56} = 0.446 \quad \leftarrow$$

A-11.1: Beam ACB has a sliding support at A and is supported at C by a pinned end steel column with square cross section ($E = 200 \text{ GPa}$, $b = 40 \text{ mm}$) and height $L = 3.75 \text{ m}$. The column must resist a load Q at B with a factor of safety 2.0 with respect to the critical load. The maximum permissible value of Q is approximately:

- (A) 10.5 kN
- (B) 11.8 kN
- (C) 13.2 kN
- (D) 15.0 kN

Solution

$$E = 200 \text{ GPa} \quad n = 2.0$$

$$b = 40 \text{ mm} \quad L = 3.75 \text{ m}$$

$$I = \frac{b^4}{12} = 2.133 \times 10^5 \text{ mm}^4$$

Statics: sum vertical forces to find reaction at D:

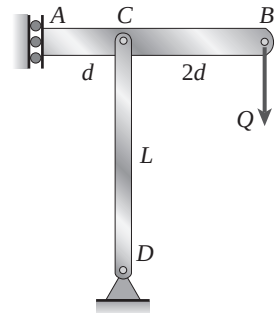
$$R_D = Q$$

So force in pin-pin column is Q

$$P_{cr} = Q_{cr} \quad Q_{cr} = \frac{\pi^2 EI}{L^2} = 29.9 \text{ kN}$$

Allowable value of Q :

$$Q_{\text{allow}} = \frac{Q_{cr}}{n} = 15.0 \text{ kN} \quad \leftarrow$$



A-11.2: Beam ACB has a pin support at A and is supported at C by a steel column with square cross section ($E = 190$ GPa, $b = 42$ mm) and height $L = 5.25$ m. The column is pinned at C and fixed at D . The column must resist a load Q at B with a factor of safety 2.0 with respect to the critical load. The maximum permissible value of Q is approximately:

- (A) 3.0 kN
- (B) 6.0 kN
- (C) 9.4 kN
- (D) 10.1 kN

Solution

$$E = 190 \text{ GPa} \quad n = 2.0$$

$$b = 42 \text{ mm} \quad L = 5.25 \text{ m}$$

Effective length of pinned-fixed column:

$$L_e = 0.699 L = 3.670 \text{ m}$$

$$I = \frac{b^4}{12} = 2.593 \times 10^5 \text{ mm}^4$$

Statics: use FBD of ACB and sum moments about A to find force in column as a multiple of Q :

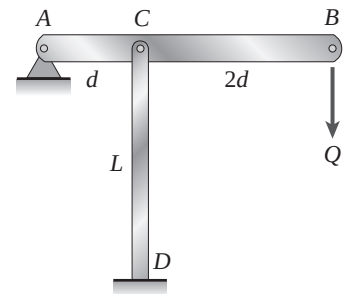
$$F_{CD} = \frac{Q(3d)}{d} = 3Q$$

So force in pin-fixed column is $3Q$

$$P_{cr} = 3Q_{cr} \quad P_{cr} = \frac{\pi^2 EI}{L_e^2} = 36.1 \text{ kN} \quad \text{So} \quad Q_{cr} = \frac{P_{cr}}{3} = 12.0 \text{ kN}$$

Allowable value of Q :

$$Q_{\text{allow}} = \frac{Q_{cr}}{n} = 6.0 \text{ kN} \quad \leftarrow$$



A-11.3: A steel pipe column ($E = 190$ GPa, $\alpha = 14 \times 10^{-6}$ per degree Celsius, $d_2 = 82$ mm, $d_1 = 70$ mm) of length $L = 4.25$ m is subjected to a temperature increase ΔT . The column is pinned at the top and fixed at the bottom. The temperature increase at which the column will buckle is approximately:

- (A) 36 °C
- (B) 42 °C
- (C) 54 °C
- (D) 58 °C

Solution

$$E = 190 \text{ GPa} \quad L = 4.25 \text{ m} \quad \alpha = [14(10^{-6})]/^\circ\text{C}$$

$$d_2 = 82 \text{ mm} \quad d_1 = 70 \text{ mm} \quad A = \frac{\pi}{4}(d_2^2 - d_1^2) = 1432.57 \text{ mm}^2$$

Effective length of pinned-fixed column:

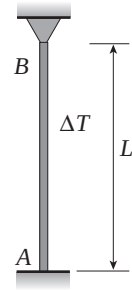
$$L_e = 0.699L = 3.0 \text{ m}$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) = 1.04076 \times 10^6 \text{ mm}^4$$

Axial compressive load in bar: $P = EA \alpha (\Delta T)$

Equate to Euler buckling load and solve for ΔT :

$$\Delta T = \frac{\frac{\pi^2 EI}{L_e^2}}{EA \alpha} \quad \text{Or} \quad \Delta T = \frac{\pi^2 I}{\alpha A L_e^2} = 58.0^\circ\text{C} \quad \leftarrow$$



A-11.4: A steel pipe ($E = 190 \text{ GPa}$, $\alpha = 14 \times 10^{-6}/^\circ\text{C}$, $d_2 = 82 \text{ mm}$, $d_1 = 70 \text{ mm}$) of length $L = 4.25 \text{ m}$ hangs from a rigid surface and is subjected to a temperature increase $\Delta T = 50^\circ\text{C}$. The column is fixed at the top and has a small gap at the bottom. To avoid buckling, the minimum clearance at the bottom should be approximately:

- (A) 2.55 mm
- (B) 3.24 mm
- (C) 4.17 mm
- (D) 5.23 mm

Solution

$$E = 190 \text{ GPa} \quad L = 4250 \text{ mm}$$

$$\alpha = [14(10^{-6})] / ^\circ\text{C} \quad \Delta T = 50^\circ\text{C}$$

$$d_2 = 82 \text{ mm} \quad d_1 = 70 \text{ mm}$$

$$A = \frac{\pi}{4} (d_2^2 - d_1^2) = 1433 \text{ mm}^2$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) = 1.041 \times 10^6 \text{ mm}^4$$

Effective length of fixed-roller support column:

$$L_e = 2.0L = 8500.0 \text{ mm}$$

Column elongation due to temperature increase:

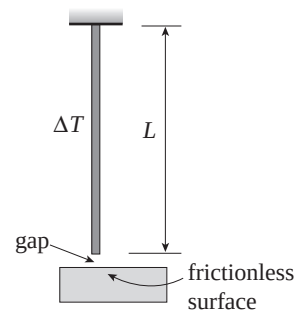
$$\delta_1 = \alpha \Delta T L = 2.975 \text{ mm}$$

$$\text{Euler buckling load for fixed-roller column: } P_{cr} = \frac{\pi^2 EI}{L_e^2} = 27.013 \text{ kN}$$

Column shortening under load of $P = P_{cr}$:

$$\delta_2 = \frac{P_{cr} L}{EA} = 0.422 \text{ mm}$$

$$\text{Minimum required gap size to avoid buckling: } \text{gap} = \delta_1 - \delta_2 = 2.55 \text{ mm} \quad \leftarrow$$



A-11.5: A pinned-end copper strut ($E = 110$ GPa) with length $L = 1.6$ m is constructed of circular tubing with outside diameter $d = 38$ mm. The strut must resist an axial load $P = 14$ kN with a factor of safety 2.0 with respect to the critical load. The required thickness t of the tube is:

- (A) 2.75 mm
- (B) 3.15 mm
- (C) 3.89 mm
- (D) 4.33 mm

Solution

$$E = 110 \text{ GPa} \quad L = 1.6 \text{ m} \quad d = 38 \text{ mm} \quad n = 2.0 \quad P = 14 \text{ kN}$$

$$P_{cr} = nP \quad P_{cr} = 28.0 \text{ kN}$$

Solve for required moment of inertia I in terms of P_{cr} then find tube thickness

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad I = \frac{P_{cr} L^2}{\pi^2 E}$$

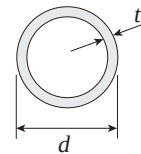
$$I = 66025 \text{ mm}^4$$

Moment of inertia

Solve numerically for min. thickness t :

$$I = \frac{\pi}{64} [d^4 - (d - 2t)^4] \quad d^4 - (d - 2t)^4 = I \frac{64}{\pi}$$

$$t_{\min} = 4.33 \text{ mm} \quad \leftarrow \quad d - 2t_{\min} = 29.3 \text{ mm} = \text{inner diameter}$$



A-11.6: A plane truss composed of two steel pipes ($E = 210$ GPa, $d = 100$ mm, wall thickness = 6.5 mm) is subjected to vertical load W at joint B . Joints A and C are $L = 7$ m apart. The critical value of load W for buckling in the plane of the truss is nearly:

- (A) 138 kN
- (B) 146 kN
- (C) 153 kN
- (D) 164 kN

Solution

$$E = 210 \text{ GPa} \quad L = 7 \text{ m}$$

$$d = 100 \text{ mm} \quad t = 6.5 \text{ mm}$$

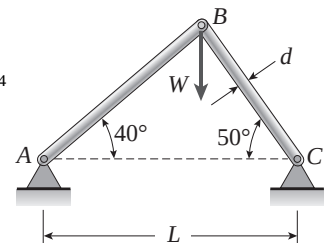
Moment of inertia

$$I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 2.097 \times 10^6 \text{ mm}^4$$

Member lengths:

$$L_{BA} = L \cos(40^\circ) = 5.362 \text{ m}$$

$$L_{BC} = L \cos(50^\circ) = 4.500 \text{ m}$$



Statics at joint B to find member forces F_{BA} and F_{BC} :

- Sum horizontal forces at joint B :

$$F_{BA} \cos(40^\circ) = F_{BC} \cos(50^\circ) \quad F_{BA} = F_{BC} \frac{\cos(50^\circ)}{\cos(40^\circ)}$$

where $\alpha = \frac{\cos(50^\circ)}{\cos(40^\circ)} = 0.839$

- Sum vertical forces at joint B :

$$W = F_{BA} \sin(40^\circ) + F_{BC} \sin(50^\circ)$$

$$W = F_{BC} \frac{\cos(50^\circ)}{\cos(40^\circ)} \sin(40^\circ) + F_{BC} \sin(50^\circ)$$

$$F_{BC} = W\beta \quad \text{where} \quad \beta = \frac{1}{\left(\frac{\cos(50^\circ)}{\cos(40^\circ)} \sin(40^\circ) + \sin(50^\circ)\right)} = 0.766$$

So member forces in terms of W are:

$$F_{BC} = W\beta \quad \text{and} \quad F_{BA} = F_{BC}\alpha \quad \text{or} \quad F_{BA} = W(\alpha\beta)$$

with $\alpha\beta = 0.643$

Euler buckling loads in BA & BC :

$$F_{BA\_cr} = \frac{\pi^2 EI}{L_{BA}^2} = 151.118 \text{ kN}$$

$$\text{so} \quad W_{BA\_cr} = \frac{\beta}{\alpha} F_{BA\_cr} = 138 \text{ kN} \quad \leftarrow < \text{lower value controls}$$

$$F_{BC\_cr} = \frac{\pi^2 EI}{L_{BC}^2} = 214.630 \text{ kN} \quad \text{so} \quad W_{BC\_cr} = \beta F_{BC\_cr} = 164 \text{ kN}$$

A-11.7: A beam is pin-connected to the tops of two identical pipe columns, each of height h , in a frame. The frame is restrained against sidesway at the top of column 1. Only buckling of columns 1 and 2 in the plane of the frame is of interest here. The ratio (a/L) defining the placement of load Q_{cr} , which causes both columns to buckle simultaneously, is approximately:

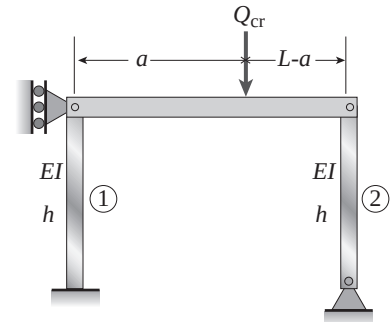
- (A) 0.25
(B) 0.33
(C) 0.67
(D) 0.75

Solution

Draw FBD of beam only; use statics to show that Q_{cr} causes forces P_1 and P_2 in columns 1 & 2 respectively:

$$P_1 = \left(\frac{L-a}{L}\right) Q_{cr}$$

$$P_2 = \frac{a}{L} Q_{cr}$$



Buckling loads for columns 1 & 2:

$$P_{cr1} = \frac{\pi^2 EI}{(0.699h)^2} = \left(\frac{L-a}{L}\right) Q_{cr}$$

$$P_{cr2} = \frac{\pi^2 EI}{h^2} = \left(\frac{a}{L}\right) Q_{cr}$$

Solve above expressions for Q_{cr} , then solve for required a/L so that columns buckle at the same time:

$$\frac{\pi^2 EI}{(0.699h)^2} \left(\frac{L-a}{L}\right) = \frac{\pi^2 EI}{h^2} \left(\frac{a}{L}\right)$$

$$\text{Or } \frac{\pi^2 EI}{(0.699h)^2} \left(\frac{L-a}{L}\right) - \frac{\pi^2 EI}{h^2} \left(\frac{a}{L}\right) = 0$$

$$\text{Or } \frac{L}{0.699^2(L-a)} - \frac{L}{a} = 0 \quad \text{Or } \frac{a}{L-a} = 0.699^2$$

$$\text{So } \frac{a}{L} = \frac{0.699^2}{(1 + 0.699^2)} = 0.328 \quad \leftarrow$$

A-11.8: A steel pipe column ($E = 210$ GPa) with length $L = 4.25$ m is constructed of circular tubing with outside diameter $d_2 = 90$ mm and inner diameter $d_1 = 64$ mm. The pipe column is fixed at the base and pinned at the top and may buckle in any direction. The Euler buckling load of the column is most nearly:

- (A) 303 kN
- (B) 560 kN
- (C) 690 kN
- (D) 720 kN

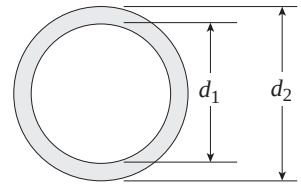
Solution

$$E = 210 \text{ GPa} \quad L = 4.25 \text{ m}$$

$$d_2 = 90 \text{ mm} \quad d_1 = 64 \text{ mm}$$

Moment of inertia

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) \quad I = 2.397 \times 10^6 \text{ mm}^4$$



Effective length of column for fixed-pinned case:

$$L_e = 0.699 L = 2.971 \text{ m}$$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = 563 \text{ kN} \quad \leftarrow$$

A-11.9: An aluminum tube ($E = 72 \text{ GPa}$) AB of circular cross section has a pinned support at the base and is pin-connected at the top to a horizontal beam supporting a load $Q = 600 \text{ kN}$. The outside diameter of the tube is 200 mm and the desired factor of safety with respect to Euler buckling is 3.0 . The required thickness t of the tube is most nearly:

- (A) 8 mm
- (B) 10 mm
- (C) 12 mm
- (D) 14 mm

Solution

$$E = 72 \text{ GPa} \quad L = 2.5 \text{ m} \quad n = 3.0 \quad Q = 600 \text{ kN} \quad d = 200 \text{ mm}$$

$$\Sigma M_c = 0 \quad P = \frac{2.5Q}{1.5} \quad P = 1000 \text{ kN}$$

Find required I based on critical buckling load
Critical load

$$P_{cr} = Pn$$

$$P_{cr} = 3000 \text{ kN}$$

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$I = \frac{P_{cr} L^2}{\pi^2 E}$$

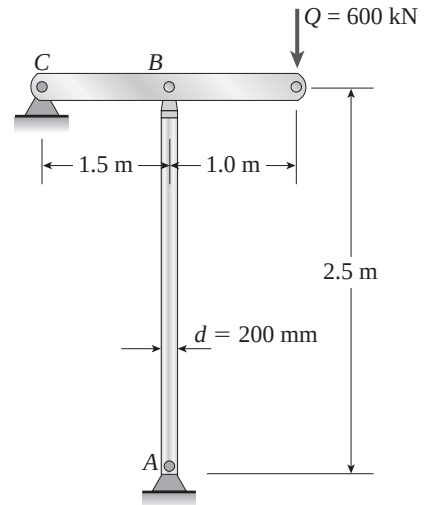
$$I = 26.386 \times 10^6 \text{ mm}^4$$

Moment of inertia

$$I = \frac{\pi}{64} [d^4 - (d - 2t)^4]$$

$$t_{\min} = \frac{d - \sqrt[4]{d^4 - I \frac{64}{\pi}}}{2}$$

$$t_{\min} = 9.73 \text{ mm} \quad \leftarrow$$



A-11.10: Two pipe columns are required to have the same Euler buckling load P_{cr} . Column 1 has flexural rigidity $E I$ and height L_1 ; column 2 has flexural rigidity $(4/3)E I$ and height L_2 . The ratio (L_2/L_1) at which both columns will buckle under the same load is approximately:

- (A) 0.55
- (B) 0.72
- (C) 0.81
- (D) 1.10

Solution

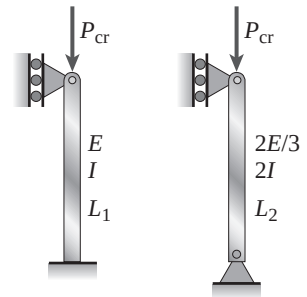
Equate Euler buckling load expressions for the two columns considering their different properties, base fixity conditions and lengths:

$$\frac{\pi^2 EI}{(0.699 L_1)^2} = \frac{\pi^2 \left(\frac{2}{3} E\right) (2I)}{L_2^2}$$

Simplify then solve for L_2/L_1 :

$$\left(\frac{L_2}{0.699 L_1}\right)^2 = \frac{4}{3}$$

$$\frac{L_2}{L_1} = \sqrt{\frac{4}{3} 0.699^2} = 0.807 \quad \leftarrow$$



A-11.11: Two pipe columns are required to have the same Euler buckling load P_{cr} . Column 1 has flexural rigidity $E I_1$ and height L ; column 2 has flexural rigidity $(2/3)E I_2$ and height L . The ratio (I_2/I_1) at which both columns will buckle under the same load is approximately:

- (A) 0.8
- (B) 1.0
- (C) 2.2
- (D) 3.1

Solution

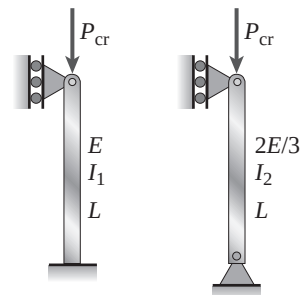
Equate Euler buckling load expressions for the two columns considering their different properties, base fixity conditions and lengths:

$$\frac{\pi^2 EI_1}{(0.699 L)^2} = \frac{\pi^2 \left(\frac{2}{3} E\right) (I_2)}{L^2}$$

Simplify then solve for I_2/I_1 :

$$\frac{I_2}{I_1} = \frac{L^2}{(0.699 L)^2} \frac{E}{\frac{2}{3} E}$$

$$\frac{I_2}{I_1} = \frac{\frac{3}{2}}{(0.699)^2} = 3.07 \quad \leftarrow$$



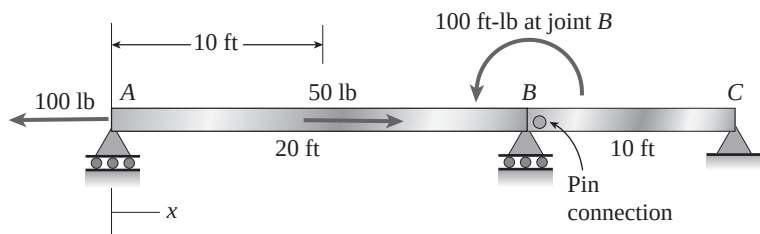
1

Tension, Compression, and Shear

Statics Review

Problem 1.2-1 Segments AB and BC of beam ABC are pin connected a small distance to the right of joint B (see figure). Axial loads act at A and at mid-span of AB . A concentrated moment is applied at joint B .

- Find reactions at supports A , B , and C .
- Find internal stress resultants N , V , and M at $x = 15$ ft.



Solution 1.2-1

(a) APPLY LAWS OF STATICS

$$\Sigma F_x = 0 \quad C_x = 100 \text{ lb} - 50 \text{ lb} = 50 \text{ lb}$$

$$\text{FBD of } BC \quad \Sigma M_B = 0 \quad C_y = \frac{1}{10 \text{ ft}}(0) = 0$$

$$\text{Entire FBD} \quad \Sigma M_A = 0 \quad B_y = \frac{1}{20 \text{ ft}}(-100 \text{ lb-ft}) = -5 \text{ lb}$$

$$\Sigma F_y = 0 \quad A_y = -B_y = 5 \text{ lb-ft}$$

$$\text{Reactions are } \boxed{A_y = 5 \text{ lb}} \quad \boxed{B_y = -5 \text{ lb}} \quad \boxed{C_x = 50 \text{ lb}} \quad \boxed{C_y = 0}$$

(b) INTERNAL STRESS RESULTANTS N , V , AND M AT $x = 15$ ft

Use FBD of segment from A to $x = 15$ ft

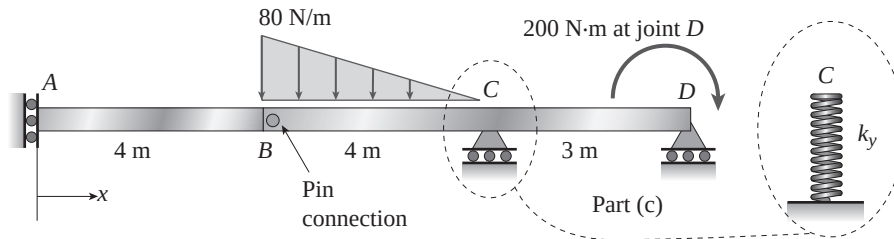
$$\Sigma F_x = 0 \quad \boxed{N = 100 \text{ lb} - 50 \text{ lb} = 50 \text{ lb}}$$

$$\Sigma F_y = 0 \quad \boxed{V = A_y = 5 \text{ lb}}$$

$$\Sigma M = 0 \quad \boxed{M = A_y 15 \text{ ft} = 75 \text{ lb-ft}}$$

Problem 1.2-2 Segments AB and BCD of beam $ABCD$ are pin connected at $x = 4$ m. The beam is supported by a sliding support at A and roller supports at C and D (see figure). A triangularly distributed load with peak intensity of 80 N/m acts on BC . A concentrated moment is applied at joint D .

- Find reactions at supports A , C , and D .
- Find internal stress resultants N , V , and M at $x = 5$ m.
- Repeat parts (a) and (b) for the case of the roller support at C replaced by a linear spring of stiffness $k_y = 200$ kN/m.



Solution 1.2-2

- (a) APPLY LAWS OF STATICS

$$\Sigma F_x = 0 \quad A_x = 0$$

$$\text{FBD of AB} \quad \Sigma M_B = 0 \quad M_A = 0$$

$$\text{Entire FBD} \quad \Sigma M_C = 0 \quad D_y = \frac{1}{3} \left[200 \text{ N}\cdot\text{m} - \frac{1}{2} (80 \text{ N/m}) 4 \text{ m} \left(\frac{2}{3} \right) 4 \text{ m} \right] = -75.556 \text{ N}$$

$$\Sigma F_y = 0 \quad C_y = \frac{1}{2} (80 \text{ N/m}) 4 \text{ m} - D_y = 235.556 \text{ N}$$

$$\text{Reactions are} \quad \boxed{M_A = 0} \quad A_x = 0 \quad \boxed{C_y = 236 \text{ N}} \quad \boxed{D_y = -75.6 \text{ N}}$$

- (b) INTERNAL STRESS RESULTANTS N , V , AND M AT $x = 5$ m

Use FBD of segment from A to $x = 5$ m; ordinate on triangular load at $x = 5$ m is $\frac{3}{4} (80 \text{ N/m}) = 60 \text{ N/m}$.

$$\Sigma F_x = 0 \quad N_x = -A_x = 0$$

$$\Sigma F_y = 0 \quad V = \frac{-1}{2} [(80 \text{ N/m} + 60 \text{ N/m}) 1 \text{ m}] = -70 \text{ N} \quad \boxed{V = -70 \text{ N}} \quad \text{Upward}$$

$$\Sigma M = 0 \quad M = -M_A - \frac{1}{2} (80 \text{ N/m}) 1 \text{ m} \left(\frac{2}{3} 1 \text{ m} \right) - \frac{1}{2} (60 \text{ N/m}) 1 \text{ m} \left(\frac{1}{3} 1 \text{ m} \right) = -36.667 \text{ N}\cdot\text{m}$$

(break trapezoidal load into two triangular loads in moment expression)

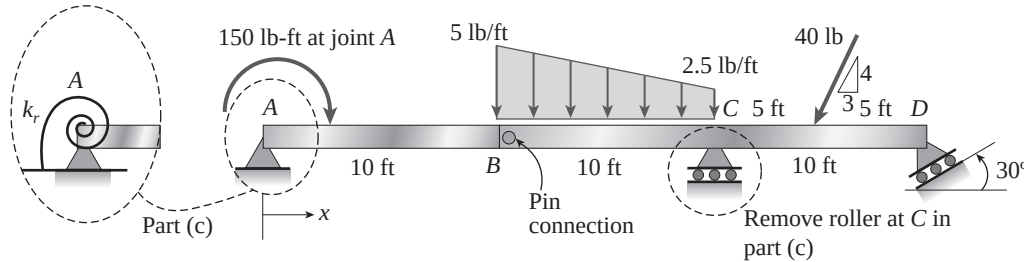
$$\boxed{M = -36.7 \text{ N}\cdot\text{m}} \quad \text{CW}$$

- (c) REPLACE ROLLER SUPPORT AT C WITH SPRING SUPPORT

Structure remains statically determinate so all results above in (a) and (b) are unchanged.

Problem 1.2-3 Segments AB and BCD of beam $ABCD$ are pin connected at $x = 10$ ft. The beam is supported by a pin support at A and roller supports at C and D ; the roller at D is rotated by 30° from the x axis (see figure). A trapezoidal distributed load on BC varies in intensity from 5 lb/ft at B to 2.5 lb/ft at C . A concentrated moment is applied at joint A and a 40 lb inclined load is applied at mid-span of CD .

- Find reactions at supports A , C , and D .
- Find the resultant force in the pin connection at B .
- Repeat parts (a) and (b) if a rotational spring ($k_r = 50$ ft-lb/rad) is added at A and the roller at C is removed.



Solution 1.2-3

(a) STATICS

FBD of AB (cut through beam at pin): $\Sigma M_B = 0 \quad A_y = \frac{1}{10 \text{ ft}}(-150 \text{ lb-ft}) = -15 \text{ lb}$

Entire FBD: $\Sigma M_D = 0$

$$C_y = \frac{1}{10 \text{ ft}} \left[\frac{4}{5} 40 \text{ lb}(5 \text{ ft}) + \frac{1}{2} (2.5 \text{ lb/ft}) 10 \text{ ft} \left(10 \text{ ft} + \frac{10 \text{ ft}}{3} \right) + \frac{1}{2} (5 \text{ lb/ft}) 10 \text{ ft} \left(10 \text{ ft} + \frac{2}{3} 10 \text{ ft} \right) - 150 \text{ lb-ft} - A_y 30 \text{ ft} \right] = 104.333 \text{ lb}$$

$$\Sigma F_y = 0 \quad D_y = \frac{4}{5} 40 \text{ lb} + \frac{1}{2} (5 \text{ lb/ft} + 2.5 \text{ lb/ft}) 10 \text{ ft} - A_y - C_y = -19.833 \text{ lb} \quad \text{so} \quad D_x = \frac{-D_y}{\tan(60^\circ)} = 11.451 \text{ lb}$$

$$\Sigma F_x = 0 \quad A_x = \frac{3}{5} 40 \text{ lb} - D_x = 12.549 \text{ lb}$$

$$\boxed{A_x = 12.55 \text{ lb}, A_y = -15 \text{ lb}, C_y = 104.3 \text{ lb}, D_x = 11.45 \text{ lb}, D_y = -19.83 \text{ lb}}$$

(b) USE FBD OF AB ONLY; MOMENT AT PIN IS ZERO

$$F_{Bx} = -A_x \quad F_{Bx} = -12.55 \text{ lb} \quad F_{By} = -A_y \quad F_{By} = 15 \text{ lb} \quad \boxed{\text{Resultant}_B = \sqrt{F_{Bx}^2 + F_{By}^2} = 19.56 \text{ lb}}$$

(c) ADD ROTATIONAL SPRING AT A AND REMOVE ROLLER AT C ; APPLY EQUATIONS OF STATICAL EQUILIBRIUM

Use FBD of BCD $\Sigma M_B = 0$

$$D_y = \frac{1}{20 \text{ ft}} \left[\frac{1}{2} (2.5 \text{ lb/ft}) 10 \text{ ft} \left(\frac{2}{3} 10 \text{ ft} \right) + \frac{1}{2} (5 \text{ lb/ft}) 10 \text{ ft} \left(\frac{1}{3} 10 \text{ ft} \right) + \frac{4}{5} 40 \text{ lb} (15 \text{ ft}) \right] = 32.333 \text{ lb}$$

$$\text{so} \quad D_x = \frac{-D_y}{\tan(60^\circ)} = -18.668 \text{ lb}$$

$$\text{Use entire FBD} \quad \Sigma F_y = 0 \quad A_y = \frac{1}{2} (5 \text{ lb/ft} + 2.5 \text{ lb/ft}) 10 \text{ ft} + \frac{4}{5} (40 \text{ lb}) - D_y = 37.167 \text{ lb}$$

$$\Sigma F_x = 0 \quad A_x = \frac{3}{5} (40 \text{ lb}) - D_x = 42.668 \text{ lb}$$

Use FBD of AB $\Sigma M_B = 0$ $M_A = 150 \text{ lb-ft} + A_y 10 \text{ ft} = 521.667 \text{ lb-ft}$

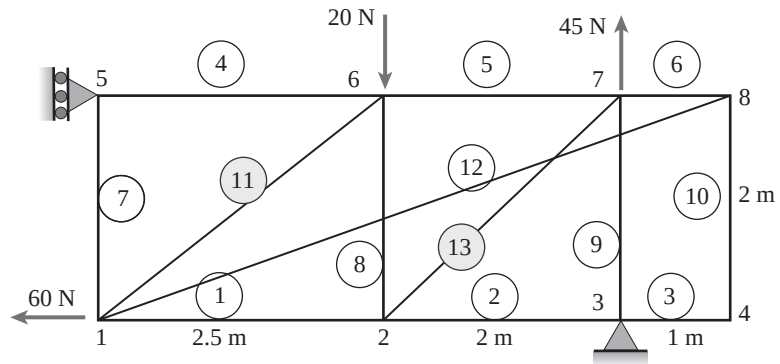
SO REACTIONS ARE $A_x = 42.7 \text{ lb}$ $A_y = 37.2 \text{ lb}$ $M_A = 522 \text{ lb-ft}$ $D_x = -18.67 \text{ lb}$ $D_y = 32.3 \text{ lb}$

RESULTANT FORCE IN PIN CONNECTION AT B

$$F_{Bx} = -A_x \quad F_{By} = -A_y \quad \text{Resultant}_B = \sqrt{F_{Bx}^2 + F_{By}^2} = 56.6 \text{ lb}$$

Problem 1.2-4 Consider the plane truss with a pin support at joint 3 and a vertical roller support at joint 5 (see figure).

- Find reactions at support joints 3 and 5.
- Find axial forces in truss members 11 and 13.



Solution 1.2-4

(a) STATICS

$$\Sigma F_y = 0 \quad R_{3y} = 20 \text{ N} - 45 \text{ N} = -25 \text{ N}$$

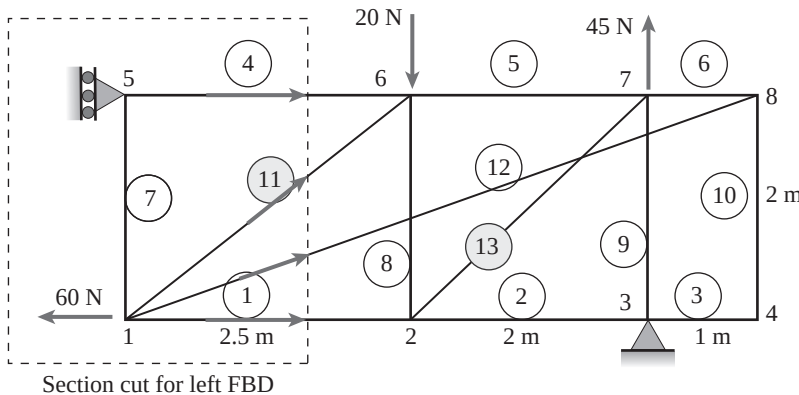
$$\Sigma M_3 = 0 \quad R_{5x} = \frac{1}{2 \text{ m}} (20 \text{ N} \times 2 \text{ m}) = 20 \text{ N}$$

$$\Sigma F_x = 0 \quad R_{3x} = -R_{5x} + 60 \text{ N} = 40 \text{ N}$$

(b) MEMBER FORCES IN MEMBERS 11 and 13

$$\text{Number of unknowns: } m = 13 \quad r = 3 \quad m + r = 16$$

$$\text{Number of equations: } j = 8 \quad 2j = 16 \quad \text{So statically determinate}$$



Section cut for left FBD

TRUSS ANALYSIS

$$(1) \Sigma F_V = 0 \quad \text{at joint 4 so } F_{10} = 0$$

$$(2) \Sigma F_V = 0 \quad \text{at joint 8 so } F_{12} = 0$$

$$(3) \Sigma F_H = 0 \quad \text{at joint 5 so } F_4 = -R_{5x} = -20 \text{ N}$$

$$(4) \text{ Cut vertically through 4, 11, 12, and 1; use left FBD; sum moments about joint 2}$$

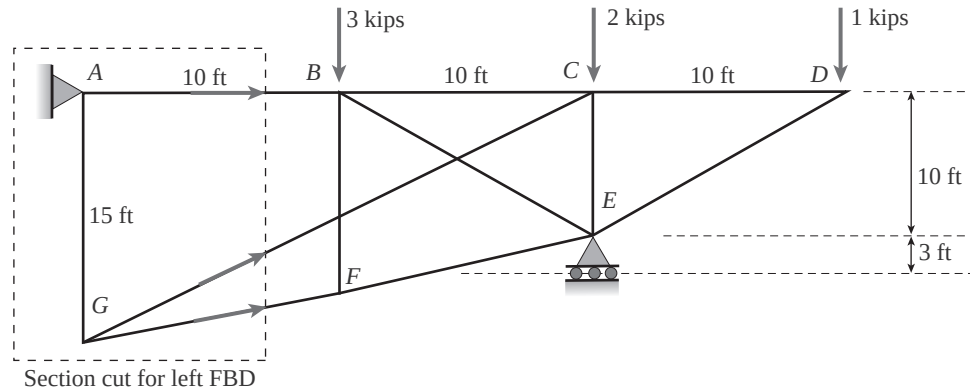
$$F_{11V} = \frac{1}{2.5 \text{ m}} (R_{5x} - F_4) \quad \text{so } F_{11} = 0$$

$$(5) \text{ Sum vertical forces at joint 3; } F_9 = R_{3y} \quad F_9 = 25 \text{ N}$$

$$(6) \text{ Sum vertical forces at joint 7} \quad F_{13V} = 45 \text{ N} - F_9 = 20 \text{ N} \quad F_{13} = \sqrt{2} F_{13V} = 28.3 \text{ N}$$

Problem 1.2-5 A plane truss has a pin support at A and a roller support at E (see figure).

- Find reactions at all supports.
- Find the axial force in truss member FE.



Solution 1.2-5

(a) STATICS

$$\Sigma F_x = 0 \quad A_x = 0$$

$$\Sigma M_A = 0 \quad E_y = \frac{1}{20 \text{ ft}}(3 \text{ k} \times 10 \text{ ft} + 2 \text{ k} \times 20 \text{ ft} + 1 \text{ k} \times 30 \text{ ft}) = 5 \text{ k}$$

$$\Sigma F_y = 0 \quad A_y = 3 \text{ k} + 2 \text{ k} + 1 \text{ k} - E_y = 1 \text{ k}$$

(b) MEMBER FORCE IN MEMBER FE

$$\text{Number of unknowns:} \quad m = 11 \quad r = 3 \quad m + r = 14$$

$$\text{Number of equations:} \quad j = 7 \quad 2j = 14 \quad \text{So statically determinate}$$

TRUSS ANALYSIS

(1) Cut vertically through AB, GC, and GF; use left FBD; sum moments about C

$$F_{GFx}(15 \text{ ft}) - F_{GFy}(20 \text{ ft}) = A_y(20 \text{ ft}) = 20 \text{ ft-k} \quad F_{GFx} = F_{GF} \frac{10}{\sqrt{2^2 + 10^2}} \quad F_{GFy} = F_{GF} \frac{2}{\sqrt{2^2 + 10^2}}$$

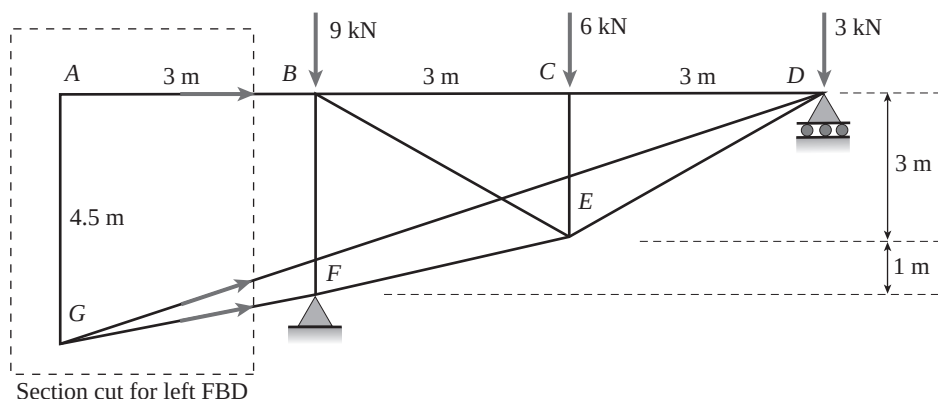
$$\text{so } F_{GF} = \frac{A_y(20 \text{ ft})}{15 \text{ ft} \frac{10}{\sqrt{2^2 + 10^2}} - 20 \text{ ft} \frac{2}{\sqrt{2^2 + 10^2}}} = 1.854 \text{ k} \quad \text{and} \quad F_{GFx} = F_{GF} \frac{10}{\sqrt{2^2 + 10^2}} = 1.818 \text{ k}$$

$$(2) \text{ Sum horizontal forces at joint F} \quad F_{FE} = F_{GFx} = 1.818 \text{ k} \quad F_{FE} = \frac{\sqrt{10^2 + 3^2}}{10} F_{FE} = 1.898 \text{ k}$$

$$\boxed{F_{FE} = 1.898 \text{ k}}$$

Problem 1.2-6 A plane truss has a pin support at F and a roller support at D (see figure).

- Find reactions at both supports.
- Find the axial force in truss member FE .



Solution 1.2-6

(a) STATICS

$$\Sigma F_x = 0 \quad F_x = 0$$

$$\Sigma M_F = 0 \quad D_y = \frac{1}{6 \text{ m}} [3 \text{ kN}(6 \text{ m}) + 6 \text{ kN}(3 \text{ m})] = 6 \text{ kN}$$

$$\Sigma F_y = 0 \quad F_y = 9 \text{ kN} + 6 \text{ kN} + 3 \text{ kN} - D_y = 12 \text{ kN}$$

(b) MEMBER FORCE IN MEMBER FE

$$\text{Number of unknowns: } m = 11 \quad r = 3 \quad m + r = 14$$

$$\text{Number of equations: } j = 7 \quad 2j = 14 \quad \text{So statically determinate}$$

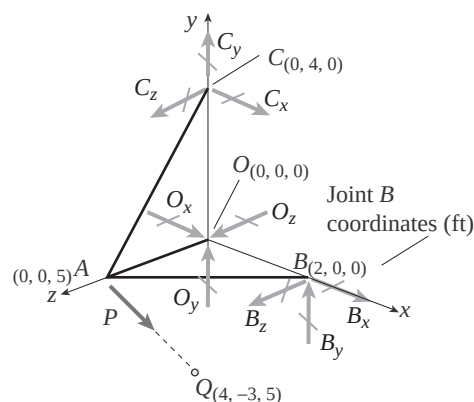
TRUSS ANALYSIS

(1) Cut vertically through AB , GD , and GF ; use left FBD; sum moments about D to get $F_{GF} = 0$

(2) Sum horizontal forces at joint F $F_{FE} = -F_x = 0$ so $F_{FE} = 0$

Problem 1.2-7 A space truss has three-dimensional pin supports at joints O , B , and C . Load P is applied at joint A and acts toward point Q . Coordinates of all joints are given in feet (see figure).

- Find reaction force components B_x , B_z , and O_z .
- Find the axial force in truss member AC .



Solution 1.2-7

(a) FIND REACTIONS USING STATICS $m = 3$ $r = 9$ $m + r = 12$ $j = 4$ $3j = 12$
 $m + r = 3j$ So truss is statically determinate

$$r_{AQ} = \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix} \quad r_{OA} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} \quad e_{AQ} = \frac{r_{AQ}}{|r_{AQ}|} = \begin{pmatrix} 0.8 \\ -0.6 \\ 0 \end{pmatrix} \quad P_A = P e_{AQ} = \begin{pmatrix} 0.8P \\ -0.6P \\ 0 \end{pmatrix} \quad r_{OC} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \quad r_{OB} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\Sigma M = 0$$

$$M_O = r_{OA} \times P_A + r_{OC} \times \begin{pmatrix} C_x \\ C_y \\ C_z \end{pmatrix} + r_{OB} \times \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 4C_z + 3.0P \\ 4.0P - 2B_z \\ 2B_y - 4C_x \end{pmatrix} \quad \text{so} \quad \Sigma M_x = 0 \quad \text{gives} \quad C_z = \frac{-3}{4}P$$

$$\Sigma M_y = 0 \quad \text{gives} \quad \boxed{B_z = 2P}$$

$$\Sigma F = 0$$

$$R_O = P_A + \begin{pmatrix} O_x \\ O_y \\ O_z \end{pmatrix} + \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} + \begin{pmatrix} C_x \\ C_y \\ C_z \end{pmatrix} = \begin{pmatrix} B_x + C_x + O_x + 0.8P \\ B_y + C_y + O_y - 0.6P \\ O_z + \frac{5P}{4} \end{pmatrix} \quad \text{so} \quad \Sigma M_z = 0 \quad \text{gives} \quad \boxed{O_z = \frac{-5}{4}P}$$

METHOD OF JOINTS Joint O $\Sigma F_x = 0$ $O_x = 0$ $\Sigma F_y = 0$ $O_y = 0$

Joint B $\Sigma F_y = 0$ $B_y = 0$

Joint C $\Sigma F_x = 0$ $C_x = 0$

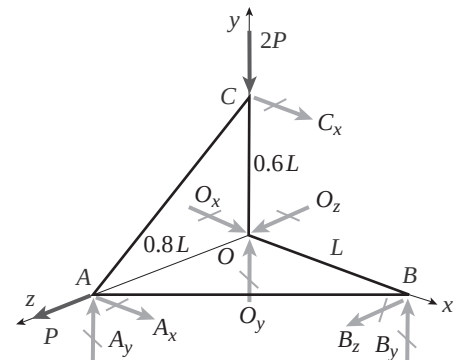
For entire structure $\Sigma F_x = 0$ gives $\boxed{B_x = -0.8P}$ $\Sigma F_y = 0$ $C_y = 0.6P - B_y = O_y$ $C_y = 0.6P$

(b) FORCE IN MEMBER AC

$$\Sigma F_z = 0 \quad \text{at joint C} \quad F_{AC} = \frac{\sqrt{4^2 + 5^2}}{5} |C_z| = \frac{3\sqrt{41}}{20} |P| \quad \boxed{F_{AC} = \frac{3\sqrt{41}}{20} P} \quad \text{tension} \quad \frac{3\sqrt{41}}{20} = 0.96$$

Problem 1.2-8 A space truss is restrained at joints O, A, B, and C, as shown in the figure. Load P is applied at joint A and load $2P$ acts downward at joint C.

- (a) Find reaction force components A_x , B_y , and B_z in terms of load variable P .
 (b) Find the axial force in truss member AB in terms of load variable P .



Solution 1.2-8(a) FIND REACTIONS USING STATICS $m = 4$ $r = 8$ $m + r = 12$ $j = 4$ $3j = 12$ $m + r = 3j$ so truss is statically determinate

$$r_{OA} = \begin{pmatrix} 0 \\ 0 \\ 0.8L \end{pmatrix} \quad r_{OB} = \begin{pmatrix} L \\ 0 \\ 0 \end{pmatrix} \quad r_{OC} = \begin{pmatrix} 0 \\ 0.6L \\ 0 \end{pmatrix} \quad F_A = \begin{pmatrix} A_x \\ A_y \\ P \end{pmatrix} \quad F_B = \begin{pmatrix} 0 \\ B_y \\ B_z \end{pmatrix} \quad F_C = \begin{pmatrix} C_x \\ -2P \\ 0 \end{pmatrix} \quad F_O = \begin{pmatrix} O_x \\ O_y \\ O_z \end{pmatrix}$$

$$\Sigma M = 0$$

Resultant moment at O

$$M_O = r_{OA} \times F_A + r_{OB} \times F_B + r_{OC} \times F_C = \begin{pmatrix} -0.8A_yL \\ 0.8A_xL - B_zL \\ B_yL - 0.6C_xL \end{pmatrix} \quad \text{so} \quad \Sigma M_x = 0 \quad \text{gives} \quad A_y = 0$$

$$\Sigma F = 0$$

Resultant force at O

$$R_O = F_O + F_A + F_B + F_C = \begin{pmatrix} A_x + C_x + O_x \\ A_y + B_y + O_y - 2P \\ B_z + O_z + P \end{pmatrix}$$

METHOD OF JOINTS Joint O $\Sigma F_z = 0$ $O_z = 0$

$$\text{so from} \quad \Sigma F_z = 0 \quad \boxed{B_z = -P} \quad \text{and} \quad \Sigma M_y = 0 \quad \boxed{A_x = \frac{B_z}{0.8} = -1.25P}$$

$$\text{Joint B} \quad \Sigma F_y = 0 \quad \boxed{B_y = 0}$$

$$\text{Joint C} \quad \Sigma F_x = 0 \quad C_x = 0$$

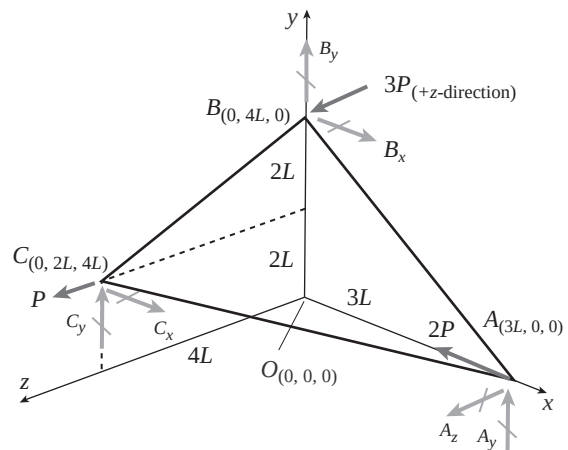
(b) FORCE IN MEMBER AB

$$\Sigma F_z = 0 \quad \text{at joint B} \quad F_{AB} = \frac{\sqrt{(0.8L)^2 + L^2}}{0.8L} |B_z| \quad |B_z| = |P| \quad \frac{\sqrt{(0.8L)^2 + L^2}}{0.8L} = 1.601$$

$$\boxed{F_{AB} = 1.601P} \quad \text{tension}$$

Problem 1.2-9 A space truss is restrained at joints A, B, and C, as shown in the figure. Load $2P$ is applied at in the $-x$ direction at joint A, load $3P$ acts in the $+z$ direction at joint B and load P is applied in the $+z$ direction at joint C. Coordinates of all joints are given in terms of dimension variable L (see figure).

- Find reaction force components A_y and A_z in terms of load variable P .
- Find the axial force in truss member AB in terms of load variable P .



Solution 1.2-9(a) FIND REACTIONS USING STATICS $m = 3$ $r = 6$ $m + r = 9$ $j = 3$ $3j = 9$ $m + r = 3j$ So truss is statically determinate

$$r_{OA} = \begin{pmatrix} 3L \\ 0 \\ 0 \end{pmatrix} \quad r_{OB} = \begin{pmatrix} 0 \\ 4L \\ 0 \end{pmatrix} \quad r_{OC} = \begin{pmatrix} 0 \\ 2L \\ 4L \end{pmatrix} \quad F_A = \begin{pmatrix} -2P \\ A_y \\ A_z \end{pmatrix} \quad F_B = \begin{pmatrix} B_x \\ B_y \\ 3P \end{pmatrix} \quad F_C = \begin{pmatrix} C_x \\ C_y \\ P \end{pmatrix}$$

$$\Sigma M = 0$$

Resultant moment at O

$$M_O = r_{OA} \times F_A + r_{OB} \times F_B + r_{OC} \times F_C = \begin{pmatrix} 14LP - 4C_yL \\ 4C_xL - 3A_zL \\ 3A_yL - 4B_xL - 2C_xL \end{pmatrix} \quad \text{so} \quad \Sigma M_x = 0 \quad \text{gives} \quad C_y = \frac{14}{4}P$$

$$\Sigma F = 0$$

Resultant force at O

$$R_O = F_A + F_B + F_C = \begin{pmatrix} B_x + C_x - 2P \\ A_y + B_y + C_y \\ A_z + 4P \end{pmatrix} \quad \text{so} \quad \Sigma F_z = 0 \quad \text{gives} \quad \boxed{A_z = -4.0P}$$

METHOD OF JOINTS

$$\text{Joint A} \quad \Sigma F_z = 0 \quad F_{ACz} = -A_z = 4.0P \quad \text{so} \quad F_{ACy} = \frac{2}{4}F_{ACz} = 2.0P \quad F_{ACx} = \frac{3}{4}F_{ACz} = 3.0P$$

$$\Sigma F_x = 0 \quad F_{ABx} = -2P - F_{ACx} = -3.0P - 2P \quad \text{so} \quad F_{ABx} = -5.0P$$

$$\Sigma F_y = 0 \quad A_y = -(F_{ABx} + F_{ACy}) = \frac{8P}{3} + 4.0P + -2.0P \quad \boxed{A_y = 4.67P}$$

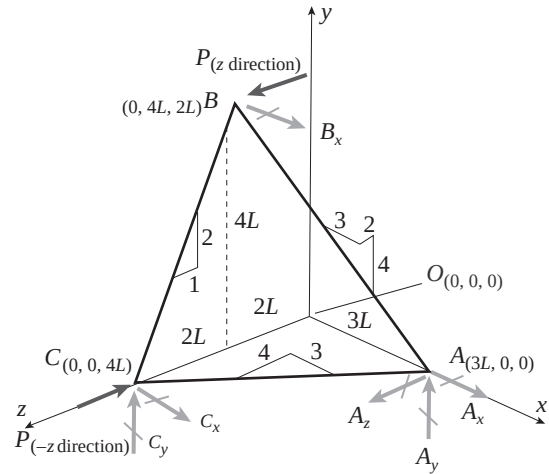
(b) FORCE IN MEMBER AB

$$F_{AB} = \sqrt{F_{ABx}^2 + F_{ABy}^2} \quad F_{AB} = -\sqrt{5^2 + \left(\frac{20}{3}\right)^2}P = -\frac{25P}{3} \quad \frac{25}{3} = 8.33$$

$$\boxed{F_{AB} = -8.33P} \quad \text{compression}$$

Problem 1.2-10 A space truss is restrained at joints A , B , and C , as shown in the figure. Load P acts in the $+z$ direction at joint B and in the $-z$ direction at joint C . Coordinates of all joints are given in terms of dimension variable L (see figure). Let $P = 5$ kN and $L = 2$ m.

- (a) Find the reaction force components A_z and B_x .
 (b) Find the axial force in truss member AB .



Solution 1.2-10

- (a) FIND REACTIONS USING STATICS $m = 3$ $r = 6$ $m + r = 9$ $j = 3$ $3j = 9$
 $m + r = 3j$ so truss is statically determinate

$$L = 2 \text{ m} \quad P = 5 \text{ kN}$$

$$r_{OA} = \begin{pmatrix} 3L \\ 0 \\ 0 \end{pmatrix} \quad r_{OB} = \begin{pmatrix} 0 \\ 4L \\ 2L \end{pmatrix} \quad r_{OC} = \begin{pmatrix} 0 \\ 0 \\ 4L \end{pmatrix} \quad F_A = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \quad F_B = \begin{pmatrix} B_x \\ 0 \\ P \end{pmatrix} \quad F_C = \begin{pmatrix} C_x \\ C_y \\ -P \end{pmatrix}$$

$$\Sigma F = 0$$

$$\text{Resultant force at } O \quad R_O = F_A + F_B + F_C = \begin{pmatrix} A_x + B_x + C_x \\ A_y + C_y \\ A_z \end{pmatrix} \quad \text{so} \quad \Sigma F_z = 0 \quad \text{gives} \quad A_z = 0$$

RESULTANT MOMENT AT A

$$r_{AC} = \begin{pmatrix} -3L \\ 0 \\ 4L \end{pmatrix} \quad e_{AC} = \frac{r_{AC}}{|r_{AC}|} = \begin{pmatrix} -0.6 \\ 0 \\ 0.8 \end{pmatrix} \quad r_{AB} = \begin{pmatrix} -3L \\ 4L \\ 2L \end{pmatrix}$$

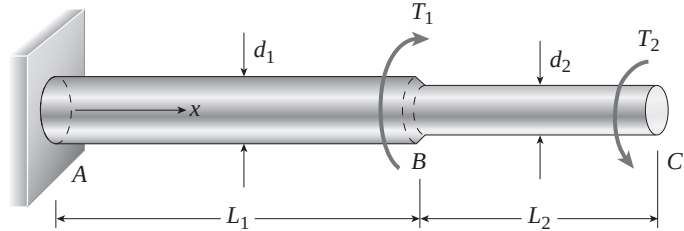
$$M_A = r_{AB} \times F_B + r_{AC} \times F_C = \begin{pmatrix} 120 \text{ kN} - 24C_y \\ 12B_x + 24C_x \\ -24B_x - 18C_y \end{pmatrix} \quad M_A e_{AC} = -19.2B_x - 72.0 \text{ kN} \quad \text{so} \quad B_x = \frac{-72}{19.2} \text{ kN} = -3.75 \text{ kN}$$

- (b) FORCE IN MEMBER AB

$$\text{Method of joints at } B \quad \Sigma F_x = 0 \quad F_{ABx} = -B_x \quad F_{AB} = \frac{\sqrt{29}}{3} F_{ABx} = 6.73 \text{ kN}$$

Problem 1.2-11 A stepped shaft ABC consisting of two solid, circular segments is subjected to torques T_1 and T_2 acting in opposite directions, as shown in the figure. The larger segment of the shaft has a diameter of $d_1 = 2.25$ in. and a length of $L_1 = 30$ in.; the smaller segment has a diameter $d_2 = 1.75$ in. and a length $L_2 = 20$ in. The torques are $T_1 = 21,000$ lb-in. and $T_2 = 10,000$ lb-in.

- Find reaction torque T_A at support A .
- Find the internal torque $T(x)$ at two locations: $x = L_1/2$ and $x = L_1 + L_2/2$. Show these internal torques on properly drawn free-body diagrams (FBDs).



Solution 1.2-11

- (a) APPLY LAWS OF STATICS $L_1 = 30$ in. $L_2 = 20$ in. $T_1 = 21000$ lb-in. $T_2 = 10000$ lb-in.

$$\Sigma M_x = 0 \quad \boxed{T_A = T_1 - T_2 = 11,000 \text{ lb-in.}}$$

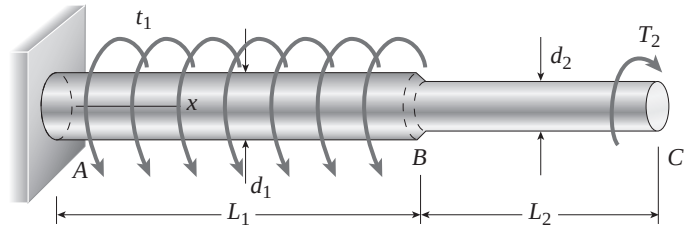
- (b) INTERNAL STRESS RESULTANT T AT TWO LOCATIONS

Cut shaft at midpoint between A and B at $x = L_1/2$ $\Sigma M_x = 0$ $\boxed{T_{AB} = -T_A = -11,000 \text{ lb-in.}}$
(use left FBD)

Cut shaft at midpoint between B and C at $x = L_1 + L_2/2$ $\Sigma M_x = 0$ $\boxed{T_{BC} = T_2 = 10,000 \text{ lb-in.}}$
(use right FBD)

Problem 1.2-12 A stepped shaft ABC consisting of two solid, circular segments is subjected to uniformly distributed torque t_1 acting over segment 1 and concentrated torque T_2 applied at C , as shown in the figure. Segment 1 of the shaft has a diameter of $d_1 = 57$ mm and length of $L_1 = 0.75$ m; segment 2 has a diameter $d_2 = 44$ mm and length $L_2 = 0.5$ m. Torque intensity $t_1 = 3100$ N·m/m and $T_2 = 1100$ N·m.

- Find reaction torque T_A at support A .
- Find the internal torque $T(x)$ at two locations: $x = L_1/2$ and $x = L_1 + L_2/2$. Show these internal torques on properly drawn free-body diagrams (FBDs).



Solution 1.2-12

- (a) REACTION TORQUE AT A $L_1 = 0.75$ m $L_2 = 0.75$ m $t_1 = 3100$ N·m/m $T_2 = 1100$ N·m

Statics $\Sigma M_x = 0$ $T_A = -t_1 L_1 + T_2 = -1225$ N·m $\boxed{T_A = -1225 \text{ N·m}}$

- (b) INTERNAL TORSIONAL MOMENTS AT TWO LOCATIONS

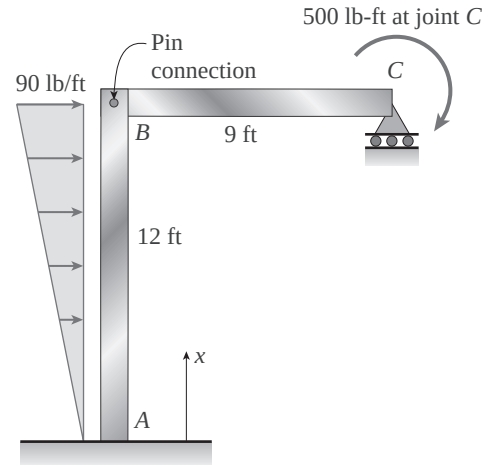
Cut shaft between A and B $T_1(x) = -T_A - t_1 x$ $\boxed{T_1\left(\frac{L_1}{2}\right) = 62.5 \text{ N·m}}$
(use left FBD)

Cut shaft between B and C $T_2(x) = -T_A - t_1 L_1$ $\boxed{T_2\left(L_1 + \frac{L_2}{2}\right) = -1100 \text{ N·m}}$
(use left FBD)

Problem 1.2-13 A plane frame is restrained at joints A and C , as shown in the figure. Members AB and BC are pin connected at B . A triangularly distributed lateral load with peak intensity of 90 lb/ft acts on AB . A concentrated moment is applied at joint C .

A concentrated moment is applied at joint C .

- Find reactions at supports A and C .
- Find internal stress resultants N , V , and M at $x = 3$ ft on column AB .



Solution 1.2-13

(a) STATICS

$$\Sigma F_H = 0 \quad A_x = \frac{-1}{2}(90 \text{ lb/ft}) 12 \text{ ft} = -540 \text{ lb}$$

$$\Sigma F_V = 0 \quad A_y + C_y = 0$$

$$\Sigma M_{FBD BC} = 0 \quad C_y = \frac{500 \text{ lb-ft}}{9 \text{ ft}} = 55.6 \text{ lb} \quad A_y = -C_y = -55.6 \text{ lb}$$

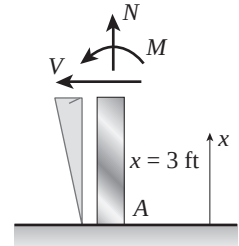
$$\Sigma M_A = 0 \quad M_A = 500 \text{ lb-ft} + \frac{1}{2}(90 \text{ lb/ft}) 12 \text{ ft} \left(\frac{2}{3} 12 \text{ ft} \right) - C_y 9 \text{ ft} = 4320 \text{ lb-ft}$$

(b) INTERNAL STRESS RESULTANTS

$$N = -A_y = 55.6 \text{ lb}$$

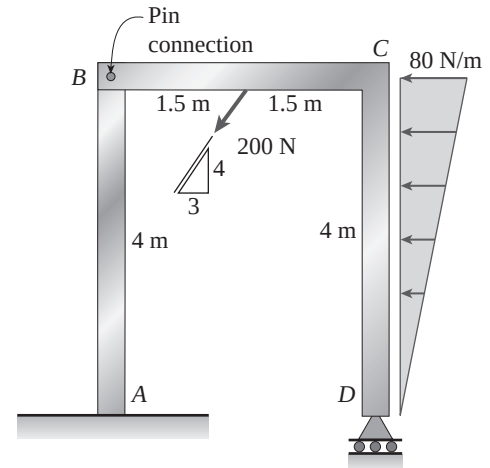
$$V = -A_x - \frac{1}{2} \left(\frac{3}{12} 90 \text{ lb/ft} \right) 3 \text{ ft} = 506 \text{ lb}$$

$$M = -M_A - A_x 3 \text{ ft} - \frac{1}{2} \left(\frac{3}{12} 90 \text{ lb/ft} \right) 3 \text{ ft} \left(\frac{1}{3} 3 \text{ ft} \right) = -2734 \text{ lb-ft}$$



Problem 1.2-14 A plane frame is restrained at joints A and D , as shown in the figure. Members AB and BCD are pin connected at B . A triangularly distributed lateral load with peak intensity of 80 N/m acts on CD . An inclined concentrated force of 200 N acts at the mid-span of BC .

- Find reactions at supports A and D .
- Find resultant forces in the pins at B and C .



Solution 1.2-14

(a) STATICS

$$\Sigma F_x = 0 \quad A_x = \frac{3}{5} (200 \text{ N}) + \frac{1}{2} (80 \text{ N/m}) 4 \text{ m} = 280 \text{ N}$$

$$\Sigma M_{BRHFB} = 0 \quad D_y = \frac{1}{3 \text{ m}} \left[\frac{4}{5} (200 \text{ N}) (1.5 \text{ m}) + \frac{1}{2} (80 \text{ N/m}) 4 \text{ m} \left(\frac{1}{3} 4 \text{ m} \right) \right]$$

$$= 151.1 \text{ N} < \text{use right hand FBD (BCD only)}$$

$$\Sigma F_y = 0 \quad A_y = -D_y + \frac{4}{5} (200 \text{ N}) = 8.89 \text{ N}$$

$$\Sigma M_A = 0 \quad M_A = \frac{4}{5} (200 \text{ N}) (1.5 \text{ m}) - \frac{3}{5} (200 \text{ N}) (4 \text{ m}) - D_y 3 \text{ m} - \frac{1}{2} (80 \text{ N/m}) 4 \text{ m} \left(\frac{2}{3} 4 \text{ m} \right) = -1120 \text{ N}\cdot\text{m}$$

(b) RESULTANT FORCE IN PIN AT B

LEFT HAND FBD (SEE FIGURE)

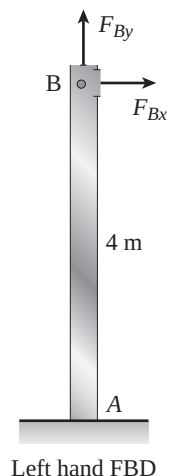
$$F_{Bx} = -A_x = -280 \text{ N} \quad F_{By} = -A_y = -8.89 \text{ N}$$

RIGHT HAND FBD

$$F_{Bx} = \frac{3}{5} (200 \text{ N}) + \frac{1}{2} (80 \text{ N/m}) 4 \text{ m} = 280 \text{ N}$$

$$F_{By} = \frac{4}{5} (200 \text{ N}) - D_y = 8.89 \text{ N}$$

$$\text{Resultant}_B = \sqrt{F_{Bx}^2 + F_{By}^2} = 280 \text{ N}$$

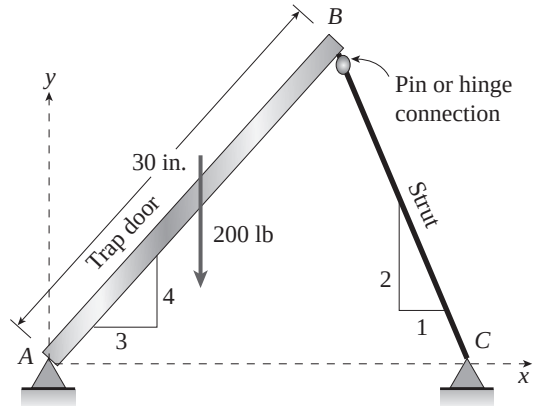


Problem 1.2-15 A 200 lb trap door (AB) is supported by a strut (BC) which is pin connected to the door at B (see figure).

- Find reactions at supports A and C .
- Find internal stress resultants N , V , and M on the trap door at 20 in. from A .

$$L_{BC} = \frac{\frac{4}{5} 30 \text{ in.}}{\frac{2}{\sqrt{5}}} = 26.833 \text{ in.}$$

$$L_{AC} = \frac{3}{5} (30 \text{ in.}) + \frac{1}{\sqrt{5}} L_{BC} = 30 \text{ in.}$$



Solution 1.2-15

(a) STATICS

$$\begin{aligned} \Sigma M_A = 0 \quad C_y &= \frac{1}{L_{AC}} \left[200 \text{ lb} \left(\frac{1}{2} \right) \left(\frac{3}{5} \right) 30 \text{ in.} \right] = 60 \text{ lb} \quad C_x = \frac{-1}{2} C_y = -30 \text{ lb} \\ \Sigma F_x = 0 \quad A_x &= -C_x = 30 \text{ lb} \quad (\text{resultant of } C_x \text{ and } C_y \text{ acts along line of strut}) \\ \Sigma F_y = 0 \quad A_y &= 200 \text{ lb} - C_y = 140 \text{ lb} \end{aligned}$$

(b) INTERNAL STRESS RESULTANTS N , V , M (SEE FIGURE)

$$\text{Distributed weight of door in } -y \text{ direction} \quad w = \frac{200 \text{ lb}}{30 \text{ in.}} = 6.667 \text{ lb/in.}$$

Components of w along and perpendicular to door

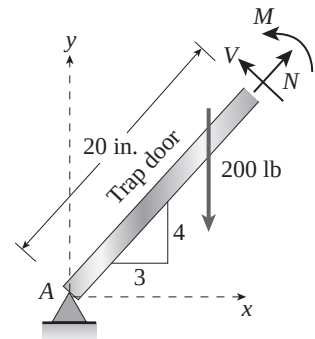
$$w_a = \frac{4}{5} w = 5.333 \text{ lb/in.} \quad w_p = \frac{3}{5} w = 4 \text{ lb/in.}$$

$$N = w_a (20 \text{ in.}) - \frac{3}{5} A_x - \frac{4}{5} A_y = -23.333 \text{ lb}$$

$$V = -w_p (20 \text{ in.}) - \frac{4}{5} A_x + \frac{3}{5} A_y = -20 \text{ lb}$$

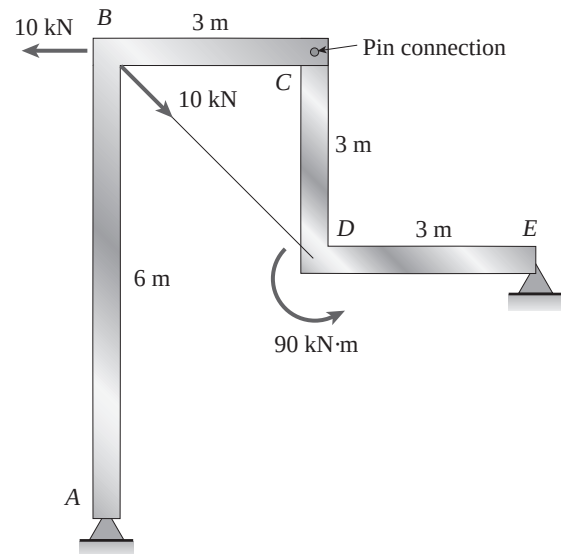
$$M = -w_p (20 \text{ in.}) \frac{20 \text{ in.}}{2} - \frac{4}{5} A_x (20 \text{ in.}) + \frac{3}{5} A_y (20 \text{ in.}) = 33.333 \text{ lb-ft}$$

$$\boxed{N = -23.3 \text{ lb}} \quad \boxed{V = -20 \text{ lb}} \quad \boxed{M = 33.3 \text{ lb-ft}}$$



Problem 1.2-16 A plane frame is constructed by using a pin connection between segments ABC and CDE . The frame has pin supports at A and E and has joint loads at B and D (see figure).

- Find reactions at supports A and E .
- Find resultant force in the pin at C .



Solution 1.2-16

(a) STATICS

$$\Sigma M_A = 0$$

$$10 \text{ kN}(6 \text{ m}) - 10 \text{ kN}\left(\frac{1}{\sqrt{2}}\right)(6 \text{ m}) + 90 \text{ kN}\cdot\text{m} + E_y(6 \text{ m}) - E_x(3 \text{ m}) = 6E_y \text{ m} - 3E_x \text{ m} + 150 \text{ kN}\cdot\text{m} - 30\sqrt{2} \text{ kN}\cdot\text{m}$$

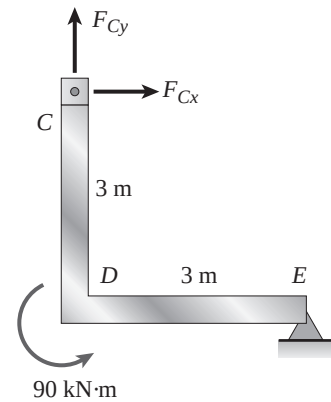
$$\text{so } 6E_y \text{ m} - 3E_x \text{ m} + 150 \text{ kN}\cdot\text{m} - 30\sqrt{2} \text{ kN}\cdot\text{m} = 0$$

$$\text{or } -E_x + 2E_y = \frac{-(150 \text{ kN}\cdot\text{m} - 30\sqrt{2} \text{ kN}\cdot\text{m})}{3 \text{ m}} = -35.858 \text{ kN}$$

$$\Sigma M_{\text{CRHFB}} = 0 < \text{right hand FBD (CDE) - see figure.}$$

$$(E_x + E_y)3 \text{ m} = -90 \text{ kN}\cdot\text{m} \quad E_x + E_y = \frac{-90 \text{ kN}\cdot\text{m}}{3 \text{ m}} = -30 \text{ kN}$$

$$\text{Solving } \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -35.858 \text{ kN} \\ -30 \text{ kN} \end{pmatrix} = \begin{pmatrix} -8.05 \\ -21.95 \end{pmatrix} \text{ kN}$$



$$E_x = -8.05 \text{ kN}$$

$$E_y = -22 \text{ kN}$$

$$\Sigma F_x = 0 \quad A_x = -E_x + 10 \text{ kN} - 10 \text{ kN}\left(\frac{1}{\sqrt{2}}\right) = 10.98 \text{ kN}$$

$$A_x = 10.98 \text{ kN}$$

$$\Sigma F_y = 0 \quad A_y = -E_y + 10 \text{ kN}\left(\frac{1}{\sqrt{2}}\right) = 29.07 \text{ kN}$$

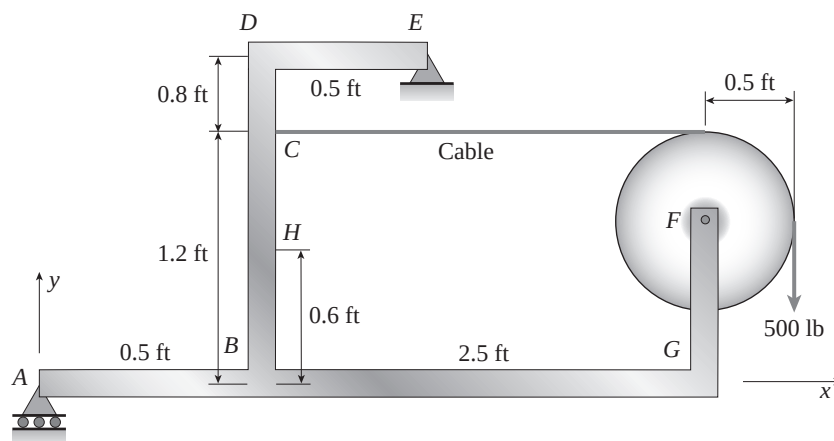
$$A_y = 29.1 \text{ kN}$$

(b) RIGHT HAND FBD $C_x = -E_x = 8.05 \text{ kN} \quad C_y = -E_y = 22 \text{ kN}$

$$\text{Resultant}_C = \sqrt{C_x^2 + C_y^2} = 23.4 \text{ kN}$$

Problem 1.2-17 A plane frame with pin supports at A and E has a cable attached at C , which runs over a frictionless pulley at F (see figure). The cable force is known to be 500 lb.

- (a) Find reactions at supports A and E .
 (b) Find internal stress resultants, N , V , and M at point H .



Solution 1.2-17

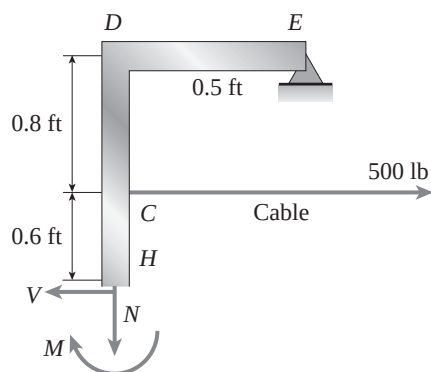
(a) STATICS

$$\Sigma F_x = 0 \quad E_x = 0$$

$$\Sigma M_E = 0 \quad A_y = \frac{1}{1 \text{ ft}}(-500 \text{ lb} \times 2.5 \text{ ft}) = -1250 \text{ lb}$$

$$\Sigma F_y = 0 \quad E_y = 500 \text{ lb} - A_y = 1750 \text{ lb}$$

(b) USE UPPER (SEE FIGURE BELOW) OR LOWER FBD TO FIND STRESS RESULTANTS N , V , AND M AT H



$$\Sigma F_x = 0 \quad V = E_x + 500 \text{ lb} = 500 \text{ lb}$$

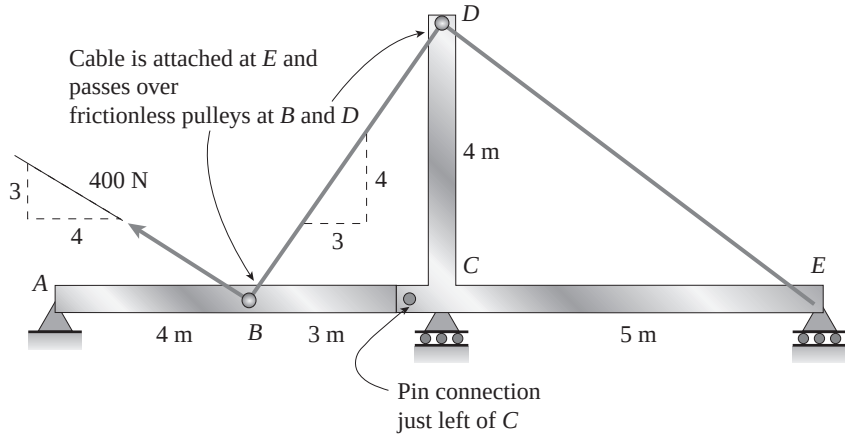
$$\Sigma F_y = 0 \quad N = E_y = 1750 \text{ lb}$$

$$\Sigma M_H = 0$$

$$M = -0.6 \text{ ft}(500 \text{ lb}) - E_x 1.4 \text{ ft} + E_y 0.5 \text{ ft} = 575 \text{ lb-ft}$$

Problem 1.2-18 A plane frame with a pin support at A and roller supports at C and E has a cable attached at E, which runs over frictionless pulleys at D and B (see figure). The cable force is known to be 400 N. There is a pin connection just to the left of joint C.

- Find reactions at supports A, C, and E.
- Find internal stress resultants N , V , and M just to the right of joint C.
- Find resultant force in the pin near C.



Solution 1.2-18

(a) STATICS

$$\Sigma F_x = 0 \quad A_x = \frac{4}{5}(400 \text{ N}) = 320 \text{ N} \quad \boxed{A_x = 320 \text{ N}}$$

Use left hand FBD (cut through pin just left of C)

$$\Sigma M_C = 0 \quad A_y = \frac{1}{7 \text{ m}} \left[\left[\frac{-3}{5}(400 \text{ N}) - \frac{4}{5}(400 \text{ N}) \right] (3 \text{ m}) \right] = -240 \text{ N} \quad \boxed{A_y = -240 \text{ N}}$$

$$\text{Use entire FBD} \quad \Sigma M_C = 0 \quad E_y = \frac{1}{5 \text{ m}} \left[A_y (7 \text{ m}) + \left(\frac{3}{5} 400 \text{ N} \right) (3 \text{ m}) \right] = -192 \text{ N} \quad \boxed{E_y = -192 \text{ N}}$$

$$\Sigma F_y = 0 \quad C_y = -A_y - E_y - \frac{3}{5}(400 \text{ N}) = 192 \text{ N} \quad \boxed{C_y = 192 \text{ N}}$$

(b) N , V , AND M JUST RIGHT OF C; USE RIGHT HAND FBD

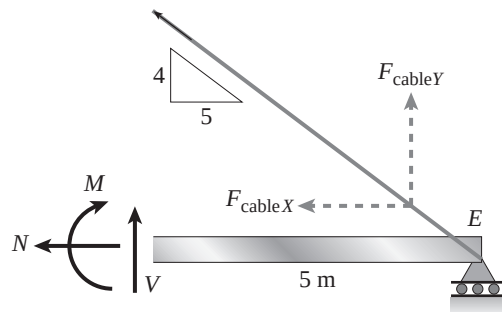
$$F_{\text{cable}X} = 400 \text{ N} \left(\frac{5}{\sqrt{4^2 + 5^2}} \right) = 312.348 \text{ N}$$

$$F_{\text{cable}Y} = \frac{4}{5} F_{\text{cable}X} = 249.878 \text{ N}$$

$$\Sigma F_x = 0 \quad \boxed{N_x = -F_{\text{cable}X} = -312 \text{ N}}$$

$$\Sigma F_y = 0 \quad \boxed{V = -F_{\text{cable}Y} - E_y = -57.9 \text{ N}}$$

$$\Sigma M_C = 0 \quad M = (F_{\text{cable}Y} + E_y)(5 \text{ m}) = \boxed{289 \text{ N}\cdot\text{m}}$$



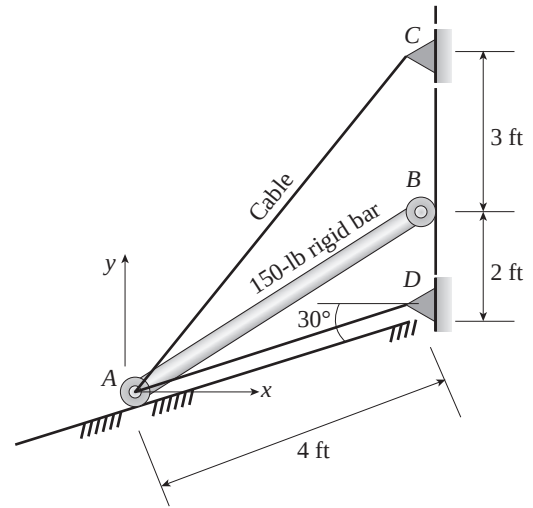
(c) RESULTANT FORCE IN PIN JUST LEFT OF C; USE LEFT HAND FBD $A_x = 320 \text{ N}$

$$F_{Cx} = -A_x + \left(\frac{4}{5} - \frac{3}{5} \right) 400 \text{ N} = -240 \text{ N} \quad F_{Cy} = -A_y - \left(\frac{3}{5} + \frac{4}{5} \right) 400 \text{ N} = -320 \text{ N}$$

$$\boxed{\text{Res}_C = \sqrt{F_{Cx}^2 + F_{Cy}^2} = 400 \text{ N}}$$

Problem 1.2-19 A 150-lb rigid bar AB , with frictionless rollers at each end, is held in the position shown in the figure by a continuous cable CAD . The cable is pinned at C and D and runs over a pulley at A .

- (a) Find reactions at supports A and B .
 (b) Find the force in the cable.



Solution 1.2-19

(a) STATICS $W = 150$ lb

$$\Sigma M_A = 0 \quad B_x(4) + W\left(\frac{2\sqrt{3}}{2}\right) = 0 \text{ solve, } B_x = -\frac{75\sqrt{3}}{2}$$

$$\text{so } B_x = -\frac{75\sqrt{3}}{2} = -64.952$$

$$\Sigma F_x = 0 \quad -A\sin(30^\circ) + B_x + T\cos(30^\circ) + T\cos\left(\arctan\left(\frac{7}{2\sqrt{3}}\right)\right) = 0$$

$$\Sigma F_y = 0 \quad A\cos(30^\circ) + T\sin(30^\circ) + T\sin\left(\arctan\left(\frac{7}{2\sqrt{3}}\right)\right) = W$$

$$\begin{pmatrix} A \\ T \end{pmatrix} = \begin{pmatrix} -\sin(30^\circ) & \cos(30^\circ) + \cos\left(\arctan\left(\frac{7}{2\sqrt{3}}\right)\right) \\ \cos(30^\circ) & \sin(30^\circ) + \sin\left(\arctan\left(\frac{7}{2\sqrt{3}}\right)\right) \end{pmatrix}^{-1} \begin{pmatrix} -B_x \\ W \end{pmatrix} \quad \begin{pmatrix} A \\ T \end{pmatrix} = \begin{pmatrix} 57.713 \\ 71.634 \end{pmatrix} \text{ lb}$$

SUPPORT REACTIONS

$$\boxed{B_x = -65} \quad \boxed{A = 57.7} \quad \text{Units} = \text{lbs}$$

$$A_x = -A\sin(30^\circ) = -28.9 \text{ lb} \quad A_y = A\cos(30^\circ) = 50 \text{ lb}$$

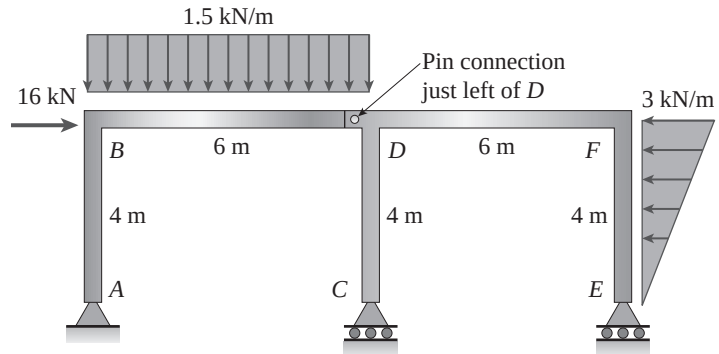
$$\sqrt{A_x^2 + A_y^2} = 57.713$$

(b) CABLE FORCE IS T (LBS) FROM ABOVE SOLUTION

$$\boxed{T = 71.6 \text{ lb}}$$

Problem 1.2-20 A plane frame has a pin support at A and roller supports at C and E (see figure). Frame segments ABD and $CDEF$ are joined just left of joint D by a pin connection.

- Find reactions at supports A , C , and E .
- Find the resultant force in the pin just left of D .



Solution 1.2-20

(a) STATICS

RIGHT-HAND FBD

$$\Sigma M_{\text{pin}} = 0 \quad E_y = \frac{1}{6 \text{ m}} \left[\frac{1}{2} (3 \text{ kN/m}) 4 \text{ m} \left(\frac{1}{3} 4 \text{ m} \right) \right] = 1.333 \text{ kN} \quad \boxed{E_y = 1.333 \text{ kN}}$$

ENTIRE FBD

$$\Sigma M_A = 0 \quad C_y = \frac{1}{6 \text{ m}} \left[-E_y 12 \text{ m} + (16 \text{ kN}) 4 \text{ m} + (1.5 \text{ kN/m}) 6 \text{ m} (3 \text{ m}) - \frac{1}{2} (3 \text{ kN/m}) 4 \text{ m} \left(\frac{2}{3} 4 \text{ m} \right) \right] = 9.833 \text{ kN}$$

$$\boxed{C_y = 9.83 \text{ kN}}$$

$$\Sigma F_y = 0 \quad A_y = -C_y - E_y + (1.5 \text{ kN/m}) 6 \text{ m} = -2.167 \text{ kN} \quad \boxed{A_y = -2.17 \text{ kN}}$$

$$\Sigma F_x = 0 \quad A_x = -16 \text{ kN} + \frac{1}{2} (3 \text{ kN/m}) 4 \text{ m} = -10 \text{ kN} \quad \boxed{A_x = -10 \text{ kN}}$$

(b) RESULTANT FORCE IN PIN; USE EITHER RIGHT HAND OR LEFT HAND FBD (CUT THROUGH PIN EXPOSING PIN FORCES F_{Dx} AND F_{Dy}) THEN SUM FORCES IN x AND y DIRECTIONS FOR EITHER FBD

LHFB:

$$F_{Dx} = -16 \text{ kN} - A_x = -6 \text{ kN}$$

$$F_{Dy} = -A_y + (1.5 \text{ kN/m}) 6 \text{ m} = 11.167 \text{ kN}$$

$$\text{Resultant}_D = \sqrt{F_{Dx}^2 + F_{Dy}^2} = 12.68 \text{ kN}$$

RHFB:

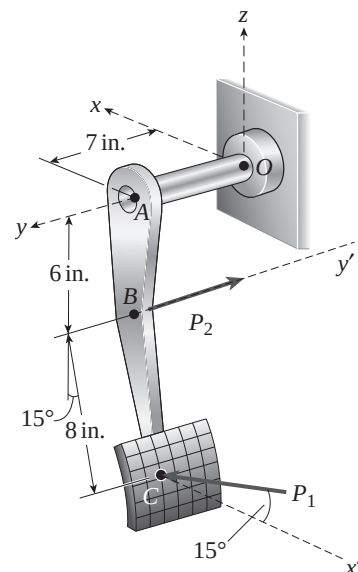
$$F_{Dx} = \frac{1}{2} (3 \text{ kN/m}) 4 \text{ m} = 6 \text{ kN}$$

$$F_{Dy} = -C_y - E_y = -11.167 \text{ kN}$$

$$\boxed{\text{Resultant}_D = 12.68 \text{ kN}}$$

Problem 1.2-21 A special vehicle brake is clamped at O , (when the brake force P_1 is applied—see figure). Force $P_1 = 50$ lb and lies in a plane which is parallel to the xz plane and is applied at C normal to line BC . Force $P_2 = 40$ lb and is applied at B in the $-y$ direction.

- (a) Find reactions at support O .
 (b) Find internal stress resultants N , V , T , and M at the midpoint of segment OA .



Solution 1.2-21

(a) STATICS $P_1 = 50$ lb $P_2 = 40$ lb

$$\Sigma F_x = 0 \quad O_x = -P_1 \cos(15^\circ) = -48.3 \text{ lb} \quad \Sigma F_y = 0 \quad O_y = P_2 = 40 \text{ lb}$$

$$\Sigma F_z = 0 \quad O_z = P_1 \sin(15^\circ) = 12.94 \text{ lb}$$

$$\Sigma M_x = 0 \quad M_{Ox} = P_2(6 \text{ in.}) + P_1 \sin(15^\circ)(7 \text{ in.}) = 331 \text{ lb-in.}$$

$$\Sigma M_y = 0 \quad M_{Oy} = P_1 \sin(15^\circ)(8 \text{ in.} \sin(15^\circ)) + P_1 \cos(15^\circ)(6 \text{ in.} + 8 \text{ in.} \cos(15^\circ))$$

$$M_{Oy} = 690 \text{ lb-in.}$$

$$\Sigma M_z = 0 \quad M_{Oz} = -P_1 \cos(15^\circ)(7 \text{ in.}) = -338 \text{ lb-in.}$$

- (b) INTERNAL STRESS RESULTANTS AT MIDPOINT OF OA

$$N = -O_y = -40 \text{ lb}$$

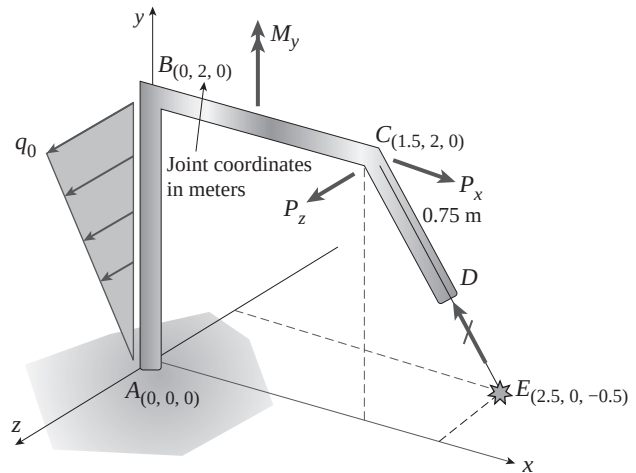
$$V_x = -O_x = 48.3 \text{ lb} \quad V_z = -O_z = -12.94 \text{ lb} \quad V = \sqrt{V_x^2 + V_z^2} = 50 \text{ lb}$$

$$T = -M_{Oy} = -690 \text{ lb-in.}$$

$$M_x = -M_{Ox} = -330.59 \text{ lb-in.} \quad M_z = -M_{Oz} = 338.07 \text{ lb-in.} \quad M = \sqrt{M_x^2 + M_z^2} = 473 \text{ lb-in.}$$

Problem 1.2-22 Space frame $ABCD$ is clamped at A , except it is free to translate in the x -direction. There is also a roller support at D , which is normal to line CDE . A triangularly distributed force with peak intensity $q_0 = 75 \text{ N/m}$ acts along AB in the positive z direction. Forces $P_x = 60 \text{ N}$ and $P_z = -45 \text{ N}$ are applied at joint C and a concentrated moment $M_y = 120 \text{ N}\cdot\text{m}$ acts at the mid-span of member BC .

- Find reactions at supports A and D .
- Find internal stress resultants N , V , T , and M at the mid-height of segment AB .



Solution 1.2-22

FORCES

$$P_x = 60 \text{ N} \quad P_z = -45 \text{ N} \quad M_y = 120 \text{ N}\cdot\text{m} \quad q_0 = 75 \text{ N/m}$$

$$F_C = \begin{pmatrix} P_x \\ 0 \\ P_z \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \\ -45 \end{pmatrix} \text{ N} \quad R_A = \begin{pmatrix} 0 \\ A_y \\ A_z \end{pmatrix}$$

VECTOR ALONG MEMBER CD

$$r_{EC} = \begin{bmatrix} 1.5 - 2.5 \\ 2 - 0 \\ 0 - (-0.5) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0.5 \end{bmatrix} \quad |r_{EC}| = 2.291 \quad e_{EC} = \frac{r_{EC}}{|r_{EC}|} = \begin{pmatrix} -0.436 \\ 0.873 \\ 0.218 \end{pmatrix}$$

(a) STATICS (FORCE AND MOMENT EQUILIBRIUM)

$$\Sigma F = 0 \quad \begin{pmatrix} 0 \\ A_y \\ A_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R_T \end{pmatrix} + \begin{pmatrix} P_x \\ 0 \\ P_z \end{pmatrix} + \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = 0 \quad \text{resultant of triangular load: } R_T = \frac{1}{2} q_0 (2 \text{ m}) = 75 \text{ N}$$

$$\text{where} \quad \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = D e_{EC}$$

SOLVING ABOVE THREE EQUATIONS:

$$\Sigma F_x = 0 \quad D_x = -P_x \quad \text{so} \quad D = \frac{-P_x}{e_{EC1}}$$

$$D = 137.477 \text{ N} \quad \boxed{D_x = -60 \text{ N}}$$

$$\Sigma F_y = 0 \quad D_y = e_{EC2} D$$

$$\boxed{D_y = 120 \text{ N}}$$

$$\boxed{A_y = -D_y = -120 \text{ N}}$$

$$\Sigma F_z = 0 \quad D_z = e_{EC3} D$$

$$\boxed{D_z = 30 \text{ N}}$$

$$\sqrt{D_x^2 + D_y^2 + D_z^2} = 137.477 \text{ N}$$

$$\text{so } A_z = -D_z - R_T - P_z$$

$$\boxed{A_z = -60 \text{ N}}$$

$$\Sigma M_A = 0$$

$$\begin{pmatrix} M_{Ax} \\ M_{Ay} \\ M_{Az} \end{pmatrix} + r_{AE} \times D + r_{AC} \times \begin{pmatrix} P_x \\ 0 \\ P_z \end{pmatrix} + \begin{pmatrix} 0 \\ M_y \\ 0 \end{pmatrix} + r_{cg} \times \begin{pmatrix} 0 \\ 0 \\ R_T \end{pmatrix} = 0$$

$$r_{AE} = \begin{pmatrix} 2.5 - 0 \\ 0 - 0 \\ -0.5 - 0 \end{pmatrix} \text{ m} \quad D = \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} \quad D = \begin{pmatrix} -60 \\ 120 \\ 30 \end{pmatrix} \text{ N} \quad |D| = 137.477 \text{ N} \quad r_{AE} \times D = \begin{pmatrix} 60 \\ -45 \\ 300 \end{pmatrix} \text{ N}\cdot\text{m}$$

$$r_{AC} = \begin{pmatrix} 1.5 - 0 \\ 2 - 0 \\ 0 - 0 \end{pmatrix} \text{ m} \quad r_{AC} \times \begin{pmatrix} P_x \\ 0 \\ P_z \end{pmatrix} = \begin{pmatrix} -90 \\ 67.5 \\ -120 \end{pmatrix} \text{ J} \quad r_{cg} = \begin{pmatrix} 0 \\ \frac{2}{3}(2 \text{ m}) \\ 0 \end{pmatrix} \quad r_{cg} \times \begin{pmatrix} 0 \\ 0 \\ R_T \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} \text{ N}\cdot\text{m}$$

$$\begin{pmatrix} M_{Ax} \\ M_{Ay} \\ M_{Az} \end{pmatrix} = - \left[r_{AE} \times D + r_{AC} \times \begin{pmatrix} P_x \\ 0 \\ P_z \end{pmatrix} + \begin{pmatrix} 0 \\ M_y \\ 0 \end{pmatrix} + r_{cg} \times \begin{pmatrix} 0 \\ 0 \\ R_T \end{pmatrix} \right] = \begin{pmatrix} -70 \\ -142.5 \\ -180 \end{pmatrix} \text{ N}\cdot\text{m} \quad \boxed{\begin{pmatrix} M_{Ax} \\ M_{Ay} \\ M_{Az} \end{pmatrix} = \begin{pmatrix} -70 \\ -142.5 \\ -80 \end{pmatrix} \text{ N}\cdot\text{m}}$$

(b) RESULTANTS AT MID-HEIGHT OF AB (SEE FBD IN FIGURE BELOW)

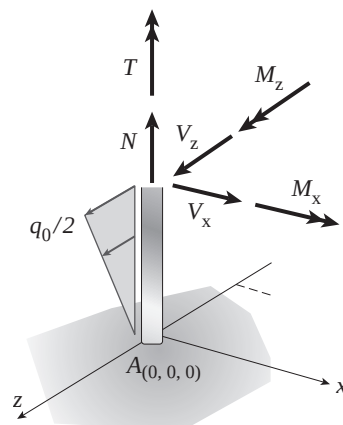
$$\boxed{N = -A_y = 120 \text{ N}} \quad V_x = -D_x - P_x = 0 \text{ N} \quad V_z = -A_z - \frac{1}{2} \frac{q_0}{2} (2 \text{ m})/2 = 41.25 \text{ N} \quad \boxed{V = V_z = 41.3 \text{ N}}$$

$$\boxed{T = -M_{Ay} = 142.5 \text{ N}\cdot\text{m}} \quad M_x = -M_{Ax} + A_z(1 \text{ m}) + \frac{1}{2} \frac{q_0}{2} 1 \text{ m} \left(\frac{1}{3} 1 \text{ m} \right) = 16.25 \text{ N}\cdot\text{m}$$

$$M_z = -M_{Az} = 180 \text{ N}\cdot\text{m}$$

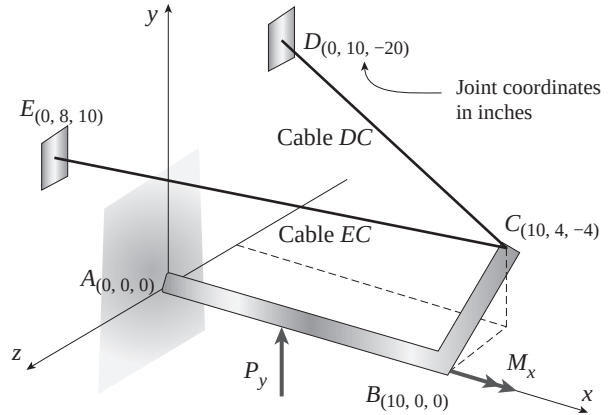
$$M_{\text{resultant}} = \sqrt{M_x^2 + M_z^2} = 180.732 \text{ N}\cdot\text{m}$$

$$\boxed{M_{\text{resultant}} = 180.7 \text{ N}\cdot\text{m}}$$



Problem 1.2-23 Space frame ABC is clamped at A except it is free to rotate at A about the x and y axes. Cables DC and EC support the frame at C . Forces $P_y = -50$ lb is applied at mid-span of AB and a concentrated moment $M_x = -20$ in-lb acts at joint B .

- (a) Find reactions at supports A .
 (b) Find cable tension forces.



Solution 1.2-23

POSITION AND UNIT VECTORS

$$r_{AB} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} \quad r_{AP} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \quad r_{AC} = \begin{pmatrix} 10 \\ 4 \\ -4 \end{pmatrix} \quad r_{CD} = \begin{bmatrix} 0 - 10 \\ 10 - 4 \\ -20 - (-4) \end{bmatrix} = \begin{pmatrix} -10 \\ 6 \\ -16 \end{pmatrix} \quad e_{CD} = \frac{r_{CD}}{|r_{CD}|} = \begin{pmatrix} -0.505 \\ 0.303 \\ -0.808 \end{pmatrix}$$

APPLIED FORCE AND MOMENT

$$r_{CE} = \begin{bmatrix} 0 - 10 \\ 8 - 4 \\ 10 - (-4) \end{bmatrix} = \begin{pmatrix} -10 \\ 4 \\ 14 \end{pmatrix} \quad e_{CE} = \frac{r_{CE}}{|r_{CE}|} = \begin{pmatrix} -0.566 \\ 0.226 \\ 0.793 \end{pmatrix}$$

$$P_y = -50 \text{ lb} \quad M_x = -20 \text{ lb-in.}$$

STATICS FORCE AND MOMENT EQUILIBRIUM

First sum moment about point A

$$\Sigma M_A = 0$$

$$M_A = \begin{pmatrix} 0 \\ 0 \\ M_{Az} \end{pmatrix} + r_{AP} \times \begin{pmatrix} 0 \\ P_y \\ 0 \end{pmatrix} + \begin{pmatrix} M_x \\ 0 \\ 0 \end{pmatrix} + r_{AC} \times (T_D e_{CD} + T_E e_{CE}) = \begin{pmatrix} -2.0203 T_D + 4.0762 T_E - 20.0 \\ 10.102 T_D + -5.6614 T_E \\ M_{Az} + 5.0508 T_D + 4.5291 T_E - 250.0 \end{pmatrix}$$

Solve moment equilibrium equations for moments about x and y axes to get cable tension forces

$$\begin{pmatrix} T_D \\ T_E \end{pmatrix} = \begin{pmatrix} -2.0203 & 4.0762 \\ 10.102 & -5.6614 \end{pmatrix}^{-1} \begin{pmatrix} 20 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.81 \\ 6.79 \end{pmatrix} \text{ lb} \quad (b)$$

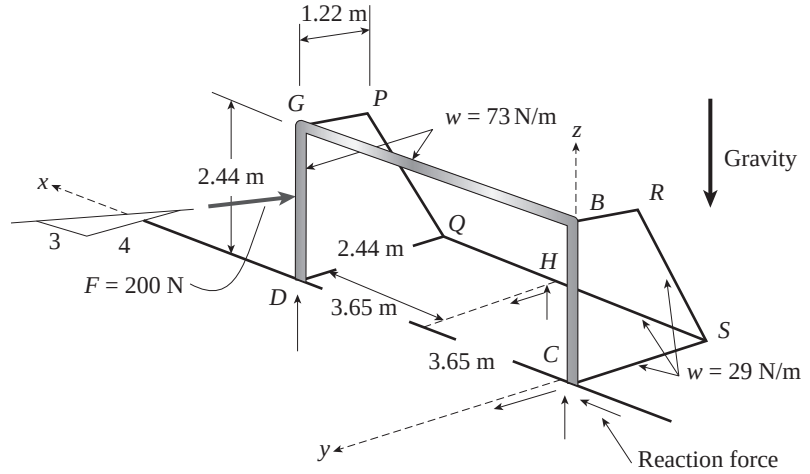
Next, solve moment equilibrium equation about z axis now that cable forces are known

$$M_{Az} = -(5.0508 T_D + 4.5291 T_E - 250.0) = 200 \text{ lb-in.} \quad (a)$$

Finally, use force equilibrium to find reaction forces at point A

$$\Sigma F = 0 \quad \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = - \begin{pmatrix} 0 \\ P_y \\ 0 \end{pmatrix} - (T_D e_{CD} + T_E e_{CE}) = \begin{pmatrix} 5.77 \\ 47.31 \\ -2.31 \end{pmatrix} \text{ lb}$$

Problem 1.2-24 A soccer goal is subjected to gravity loads (in the $-z$ direction, $w = 73 \text{ N/m}$ for DG , BG , and BC ; $w = 29 \text{ N/m}$ for all other members; see figure) and a force $F = 200 \text{ N}$ applied eccentrically at mid-height of member DG . Find reactions at supports C , D , and H .

**Solution 1.2-24**

FIND MEMBER LENGTHS

$$L_{QS} = 2(3.65 \text{ m}) = 7.3 \text{ m} \quad L_{RS} = \sqrt{(2.44 \text{ m})^2 + (2.44 \text{ m} - 1.22 \text{ m})^2} = 2.728 \text{ m} \quad L_{PQ} = L_{RS}$$

Assume that soccer goal is supported only at points C , H , and D (see reaction force components at each location in figure)

STATICS SUM MOMENT ABOUT EACH AXIS AND FORCES IN EACH AXIS DIRECTION $F = 200 \text{ N}$ $\Sigma M_x = 0$ TO FIND REACTION COMPONENT H_y :Find moments about x due to for component F_y and also for distributed weight of each frame component

$$M_{xGP} = \frac{(1.22 \text{ m})^2}{2} (29 \text{ N/m}) \quad M_{xBR} = M_{xGP} \quad M_{xDQ} = \frac{(2.44 \text{ m})^2}{2} (29 \text{ N/m}) \quad M_{xCS} = M_{xDQ}$$

$$M_{xRS} = L_{RS} (29 \text{ N/m}) \left(1.22 \text{ m} + \frac{1.22 \text{ m}}{2} \right) \quad M_{xPQ} = M_{xRS} \quad M_{xQS} = L_{QS} (29 \text{ N/m}) (2.44 \text{ m})$$

$$H_z = \frac{1}{2.44 \text{ m}} \left[\frac{4}{5} F \left(\frac{2.44 \text{ m}}{2} \right) + 2M_{xGP} + 2M_{xDQ} + 2M_{xPQ} + M_{xQS} \right] = 498.818 \text{ N} \quad \boxed{H_z = 499 \text{ N}}$$

 $\Sigma M_y = 0$ TO FIND REACTION FORCE D_z :

$$M_{yGD} = 2.44 \text{ m} (73 \text{ N/m}) L_{QS} \quad M_{yGP} = 1.22 \text{ m} (29 \text{ N/m}) L_{QS} \quad M_{yDQ} = 2.44 \text{ m} (29 \text{ N/m}) L_{QS}$$

$$M_{yPQ} = L_{RS} (29 \text{ N/m}) L_{QS} \quad M_{yBG} = L_{QS} (73 \text{ N/m}) \frac{L_{QS}}{2} \quad M_{yQS} = L_{QS} (29 \text{ N/m}) \frac{L_{QS}}{2}$$

$$D_z = \frac{1}{L_{QS}} \left[M_{yGD} + M_{yGP} + M_{yDQ} + M_{yPQ} + M_{yBG} + M_{yQS} - H_z \frac{L_{QS}}{2} - \frac{3}{5} F \left(\frac{2.44 \text{ m}}{2} \right) \right] = 466.208 \text{ N}$$

$$\boxed{D_z = 466 \text{ N}}$$

 $\Sigma M_z = 0$ TO FIND REACTION FORCE H_y :

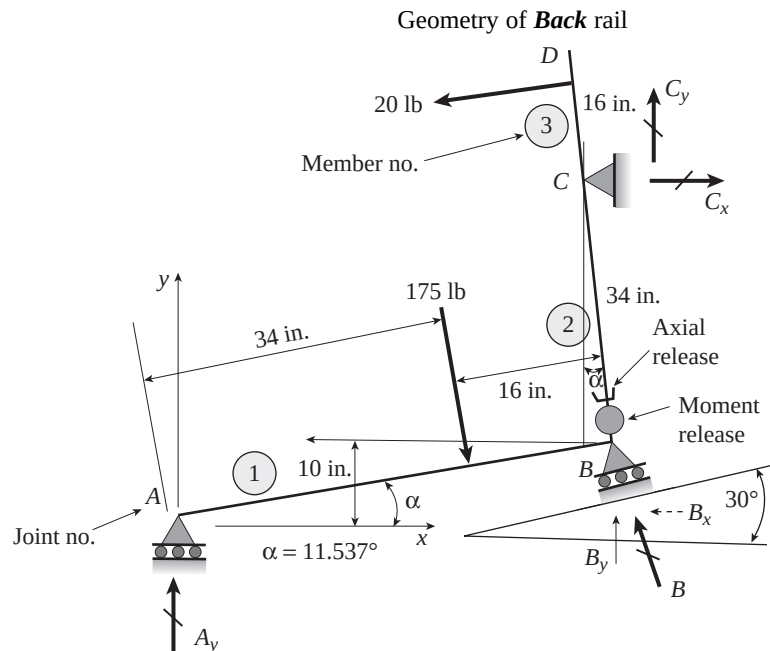
$$H_y = \frac{1}{3.65 \text{ m}} \left(\frac{4}{5} F L_{QS} \right) = 320 \text{ N} \quad \boxed{H_y = 320 \text{ N}}$$

$$\Sigma F_x = 0 \text{ TO FIND REACTION FORCE } C_x: \quad C_x = \frac{3}{5}F = 120 \text{ N}$$

$$\Sigma F_y = 0 \text{ TO FIND REACTION FORCE } C_y: \quad C_y = -H_y + \frac{4}{5}F = -160 \text{ N} \quad C_y = -160 \text{ N}$$

$$\Sigma F_z = 0 \text{ TO FIND REACTION FORCE } C_z: \\ C_z = -D_z - H_z + (29 \text{ N/m})(21.22 \text{ m} + 22.44 \text{ m} + 2L_{RS} + L_{QS}) \\ + (73 \text{ N/m})(22.44 \text{ m} + L_{QS}) = 506.318 \text{ N} \quad C_z = 506 \text{ N}$$

Problem 1.2-25 An elliptical exerciser machine (see figure part a) is composed of front and back rails. A simplified plane frame model of the back rail is shown in figure part b. Analyze the plane frame model to find reaction forces at supports A, B, and C for the position and applied loads given in figure part b. Note that there are axial and moment releases at the base of member 2 so that member 2 can lengthen and shorten as the roller support at B moves along the 30° incline. (These releases indicate that the internal axial force N and moment M must be zero at this location.)



Solution 1.2-25

$$\alpha = \arcsin\left(\frac{10}{50}\right) = 11.537^\circ \quad \text{Analysis pertains to this position of exerciser only}$$

STATICS UFB (CUT AT AXIAL AND MOMENT RELEASES JUST ABOVE B)

Inclined vertical component of reaction at C = 0 (due to axial release)

Sum moments about moment release to get inclined normal reaction at C

$$C = \frac{20 \text{ lb}(34 \text{ in.} + 16 \text{ in.})}{34 \text{ in.}} = 29.412 \text{ lb} \quad C_x = C \cos(\alpha) = 28.8 \text{ lb}$$

$$C_y = C \sin(\alpha) = 5.88 \text{ lb} \quad \sqrt{C_x^2 + C_y^2} = 29.412 \text{ lb}$$

STATICS LFB (CUT THROUGH AXIAL AND MOMENT RELEASES)

Sum moments to find reaction A_y

$$A_y = \frac{175 \text{ lb}(16 \text{ in.})}{(34 \text{ in.} + 16 \text{ in.}) \cos(\alpha)} = 57.2 \text{ lb}$$

STATICS SUM FORCES FOR ENTIRE FBD TO FIND REACTION AT B

$$\text{Sum forces in } x\text{-direction: } B_x = C_x + 175 \text{ lb}(\sin(\alpha)) - 20 \text{ lb}(\cos(\alpha)) = 44.2 \text{ lb} \quad < \text{ acts leftward}$$

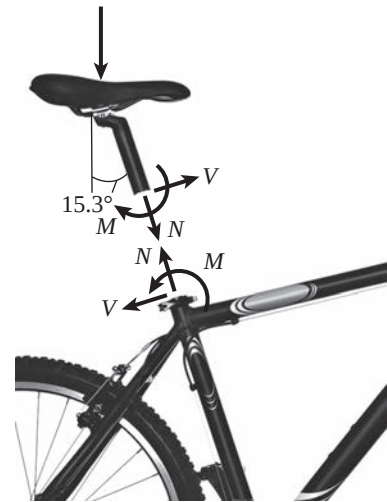
$$\text{Sum forces in } y\text{-direction: } B_y = -A_y - C_y + 175 \text{ lb}(\cos(\alpha)) + 20 \text{ lb}(\sin(\alpha)) = 112.4 \text{ lb}$$

$$\boxed{B_x = 44.2 \text{ lb}} \quad \boxed{B_y = 112.4 \text{ lb}}$$

$$\text{Resultant reaction force at } B: \quad B = \sqrt{B_x^2 + B_y^2} = 120.8 \text{ lb}$$

Problem 1.2-26 A mountain bike is moving along a flat path at constant velocity. At some instant, the rider (weight = 670 N) applies pedal and hand forces, as shown in the figure part a.

- Find reactions forces at the front and rear hubs. (Assume that the bike is pin supported at the rear hub and roller supported at the front hub).
- Find internal stress resultants N , V , and M in the inclined seat post (see figure part b).



Solution 1.2-26

- (a) REACTIONS: SUM MOMENTS ABOUT REAR HUB TO FIND VERTICAL REACTION AT FRONT HUB (FIG. 1)

$$\Sigma M_B = 0$$

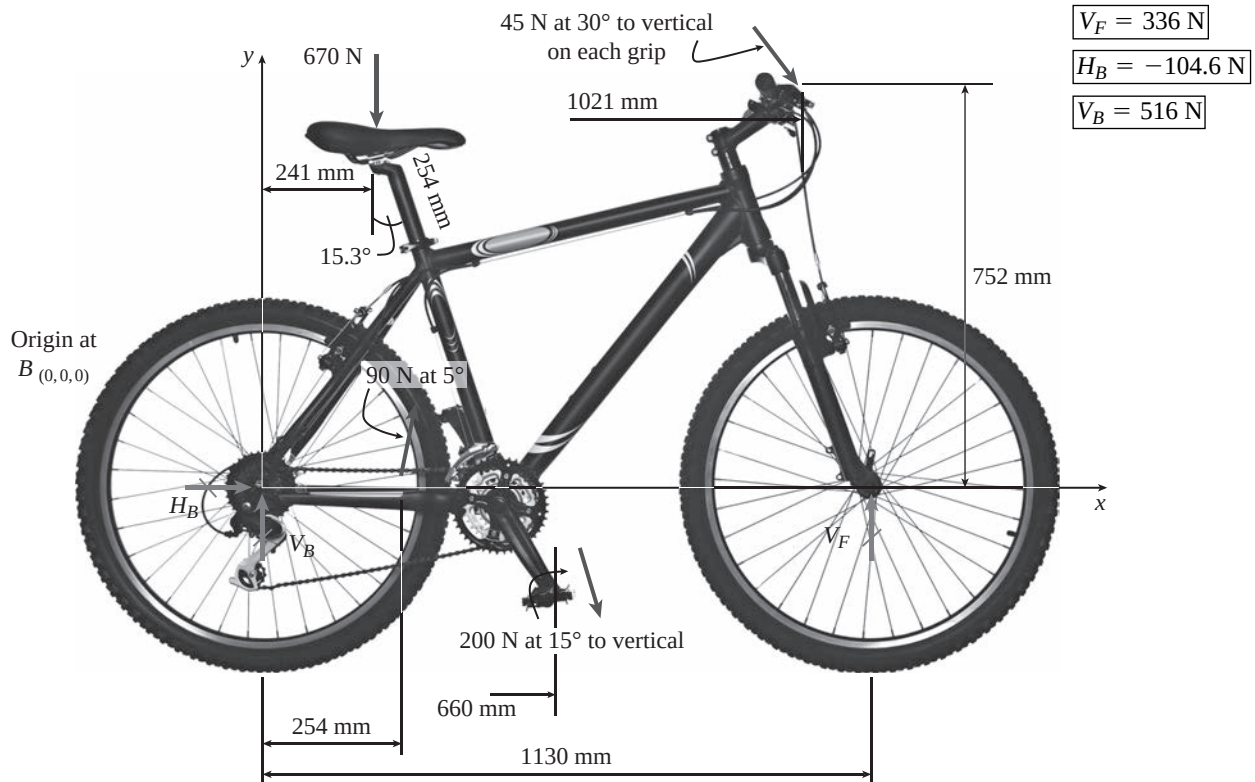
$$V_F = \frac{1}{1130} [670(241) - 90(\cos(5^\circ))254 + 200\cos(15^\circ)660 + 2(45)\cos(30^\circ)1021 + 2(45)\sin(30^\circ)752]$$

$$V_F = 335.945 \text{ N}$$

Sum forces to get force components at rear hub

$$\Sigma F_{\text{vert}} = 0 \quad V_B = 670 - 90 \cos(5^\circ) + 200 \cos(15^\circ) + 2(45) \cos(30^\circ) - V_F = 515.525 \text{ N}$$

$$\Sigma F_{\text{horiz}} = 0 \quad H_B = -90 \sin(5^\circ) - 200 \sin(15^\circ) - 2(45) \sin(30^\circ) = -104.608 \text{ N}$$



(b) STRESS RESULTANTS N , V , AND M IN SEAT POST (Fig. 2)

SEAT POST RESULTANTS (FIG. 2)

$$N = -670 \cos(15.3^\circ) = -646.253 \text{ N}$$

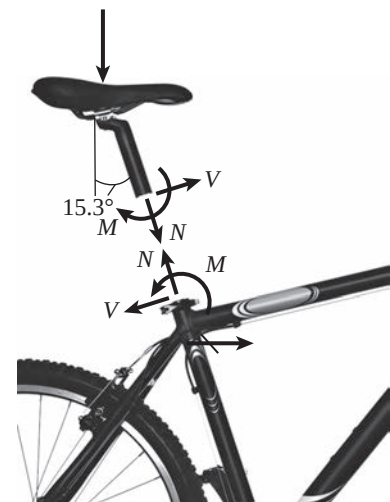
$$N = -646 \text{ N}$$

$$V = 670 \sin(15.3^\circ) = 176.795 \text{ N}$$

$$V = 176.8 \text{ N}$$

$$M = 670 \sin(15.3^\circ) 254 = 44,905.916 \text{ N}\cdot\text{mm}$$

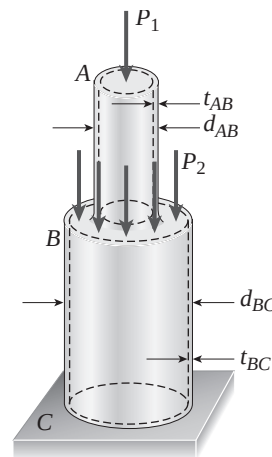
$$M = 44.9 \text{ N}\cdot\text{m}$$



Normal Stress and Strain

Problem 1.3-1 A hollow circular post ABC (see figure) supports a load $P_1 = 1700$ lb acting at the top. A second load P_2 is uniformly distributed around the cap plate at B . The diameters and thicknesses of the upper and lower parts of the post are $d_{AB} = 1.25$ in., $t_{AB} = 0.5$ in., $d_{BC} = 2.25$ in., and $t_{BC} = 0.375$ in., respectively.

- Calculate the normal stress σ_{AB} in the upper part of the post.
- If it is desired that the lower part of the post have the same compressive stress as the upper part, what should be the magnitude of the load P_2 ?
- If P_1 remains at 1700 lb and P_2 is now set at 2260 lb, what new thickness of BC will result in the same compressive stress in both parts?



Solution 1.3-1

PART (a)

$$P_1 = 1700 \text{ lb} \quad d_{AB} = 1.25 \text{ in.} \quad t_{AB} = 0.5 \text{ in.}$$

$$d_{BC} = 2.25 \text{ in.} \quad t_{BC} = 0.375 \text{ in.}$$

$$A_{AB} = \frac{\pi[d_{AB}^2 - (d_{AB} - 2t_{AB})^2]}{4}$$

$$A_{AB} = 1.178 \text{ in.}^2 \quad \sigma_{AB} = \frac{P_1}{A_{AB}}$$

$$\sigma_{AB} = 1443 \text{ psi} \quad \leftarrow$$

PART (c)

$$P_2 = 2260 \quad \frac{P_1 + P_2}{\sigma_{AB}} = A_{BC}$$

$$\frac{P_1 + P_2}{\sigma_{AB}} = 2.744$$

$$(d_{BC} - 2t_{BC})^2 = d_{BC}^2 - \frac{4}{\pi} \left(\frac{P_1 + P_2}{\sigma_{AB}} \right)$$

PART (b)

$$A_{BC} = \frac{\pi[d_{BC}^2 - (d_{BC} - 2t_{BC})^2]}{4}$$

$$A_{BC} = 2.209 \text{ in.}^2 \quad P_2 = \sigma_{AB} A_{BC} - P_1$$

$$P_2 = 1488 \text{ lbs} \quad \leftarrow$$

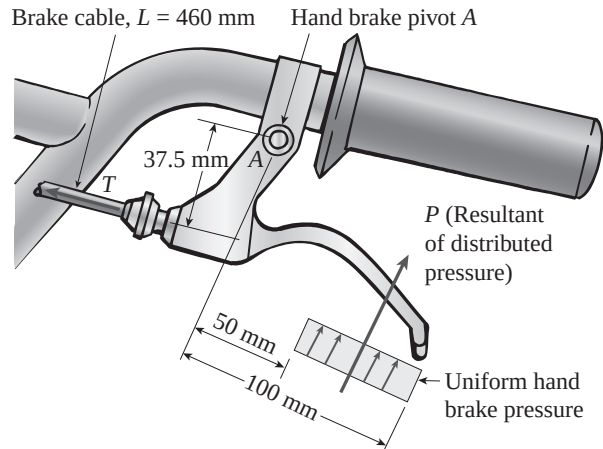
$$\text{CHECK:} \quad \frac{P_1 + P_2}{A_{BC}} = 1443 \text{ psi}$$

$$d_{BC} - 2t_{BC} = \sqrt{d_{BC}^2 - \frac{4}{\pi} \left(\frac{P_1 + P_2}{\sigma_{AB}} \right)}$$

$$t_{BC} = \frac{d_{BC} - \sqrt{d_{BC}^2 - \frac{4}{\pi} \left(\frac{P_1 + P_2}{\sigma_{AB}} \right)}}{2}$$

$$t_{BC} = 0.499 \text{ in.} \quad \leftarrow$$

Problem 1.3-2 A force P of 70 N is applied by a rider to the front hand brake of a bicycle (P is the resultant of an evenly distributed pressure). As the hand brake pivots at A , a tension T develops in the 460-mm long brake cable ($A_e = 1.075 \text{ mm}^2$) which elongates by $\delta = 0.214 \text{ mm}$. Find normal stress σ and strain ε in the brake cable.



Solution 1.3-2

$$P = 70 \text{ N} \quad A_e = 1.075 \text{ mm}^2$$

$$L = 460 \text{ mm} \quad \delta = 0.214 \text{ mm}$$

Statics: sum moments about A to get $T = 2P$

$$\sigma = \frac{T}{A_e} \quad \sigma = 103.2 \text{ MPa} \quad \leftarrow$$

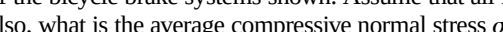
$$\varepsilon = \frac{\delta}{L} \quad \varepsilon = 4.65 \times 10^{-4} \quad \leftarrow$$

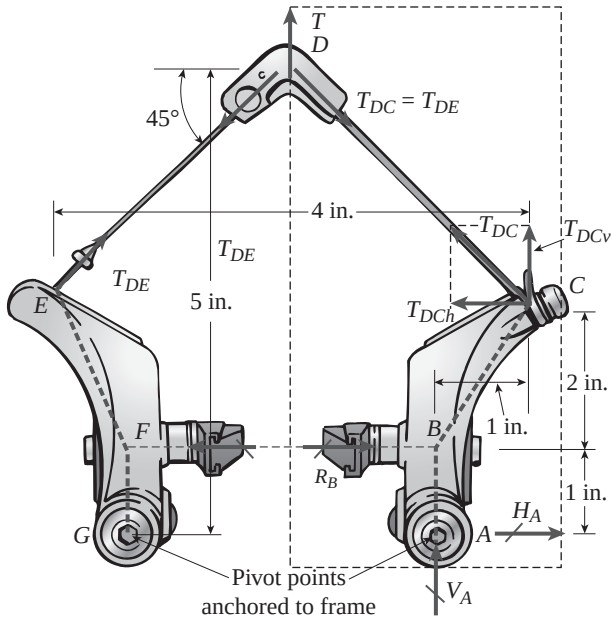
$$E = \frac{\sigma}{\varepsilon} = 1.4 \times 10^5 \text{ MPa}$$

NOTE: (E for cables is approximately 140 GPa)

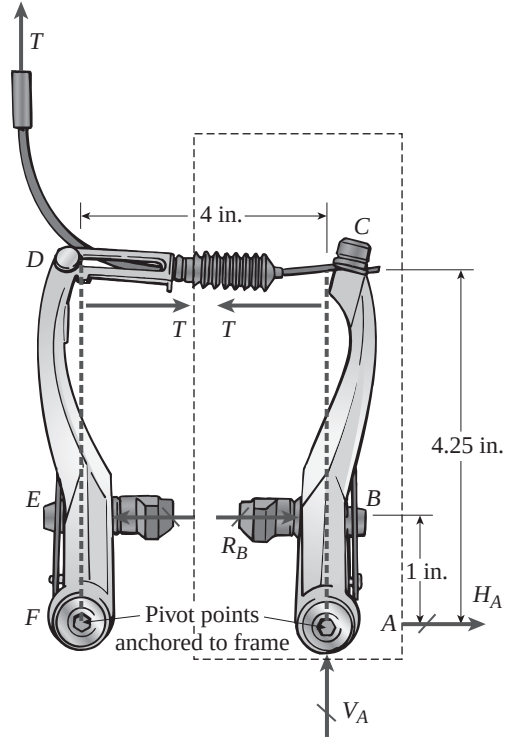
30

Problem 1.3-3 A bicycle rider would like to compare the effectiveness of cantilever hand brakes [see figure part (a)] versus V-brakes [figure part (b)].

- (a) Calculate the braking force R_B at the wheel rims for each of the bicycle brake systems shown. Assume that all forces act in the plane of the figure and that cable tension $T = 45$ lbs. Also, what is the average compressive normal stress σ_c on the brake pad ($A = 0.625$ in.²)?
- (b) For each braking system, what is the stress in the brake cable (assume effective cross-sectional area of 0.00167 in.²)?
- (*HINT: Because of symmetry, you only need to use the right half of each figure in your analysis.*)
- 



(a) Cantilever brakes



(b) V-brakes

Solution 1.3-3

$$T = 45 \text{ lbs} \quad A_{\text{pad}} = 0.625 \text{ in.}^2$$

$$A_{\text{cable}} = 0.00167 \text{ in.}^2$$

- (a) CANTILEVER BRAKES—BRAKING FORCE

R_B and PAD PRESSURE

STATICS SUM FORCES AT D TO GET $T_{DCV} = T/2$

$$\sum M_A = 0$$

$$R_B(1) = T_{DCh}(3) + T_{DCv}(1)s$$

$$T_{DCh} = T_{DCv} \quad T_{DCh} = T/2$$

$$R_B = 2T \quad R_B = 90 \text{ lbs} \quad \leftarrow$$

so $R_B = 2T$ versus $4.25T$ for V-brakes (next)

$$\sigma_{\text{pad}} = \frac{R_B}{A_{\text{pad}}} \quad \sigma_{\text{pad}} = 144 \text{ psi} \quad \leftarrow \quad \frac{4.25}{2} = 2.125$$

$$\sigma_{\text{cable}} = \frac{T}{A_{\text{cable}}} \quad \sigma_{\text{cable}} = 26,946 \text{ psi} \quad \leftarrow \quad (\text{same for V-brakes (below)})$$

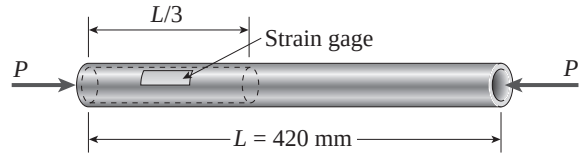
(b) V-BRAKES—BRAKING FORCE R_B AND PAD PRESSURE

$$\sum M_A = 0 \quad R_B = 4.25T \quad R_B = 191.3 \text{ lbs} \quad \leftarrow$$

$$\sigma_{\text{pad}} = \frac{R_B}{A_{\text{pad}}} \quad \sigma_{\text{pad}} = 306 \text{ psi} \quad \leftarrow$$

Problem 1.3-4 A circular aluminum tube of length $L = 420 \text{ mm}$ is loaded in compression by forces P (see figure). The hollow segment of length $L/3$ has outside and inside diameters of 60 mm and 35 mm, respectively. The solid segment of length $2L/3$ has diameter of 60 mm. A strain gage is placed on the outside of the hollow segment of the bar to measure normal strains in the longitudinal direction.

- (a) If the measured strain in the hollow segment is $\epsilon_h = 470 \times 10^{-6}$, what is the strain ϵ_s in the solid part? (*Hint: The strain in the solid segment is equal to that in the hollow segment multiplied by the ratio of the area of the hollow to that of the solid segment.*)
- (b) What is the overall shortening δ of the bar?
- (c) If the compressive stress in the bar cannot exceed 48 MPa, what is the maximum permissible value of load P ?



Solution 1.3-4

$$L = 420 \text{ mm} \quad d_2 = 60 \text{ mm} \quad d_1 = 35 \text{ mm} \quad \epsilon_h = 470 (10^{-6}) \quad \sigma_a = 48 \text{ MPa}$$

PART (a)

$$A_s = \frac{\pi}{4} d_2^2 = 2.827 \times 10^{-3} \text{ m}^2 \quad A_h = \frac{\pi}{4} (d_2^2 - d_1^2) = 1.865 \times 10^{-3} \text{ m}^2$$

$$\epsilon_h = \frac{A_h}{A_s} \epsilon_s = 3.101 \times 10^{-4}$$

PART (b)

$$\delta = \epsilon_h \frac{L}{3} + \epsilon_s \left(\frac{2L}{3} \right) = 0.1526 \text{ mm} \quad \epsilon_h \frac{L}{3} = 0.066 \text{ mm} \quad \epsilon_s \left(\frac{2L}{3} \right) = 0.087 \text{ mm}$$

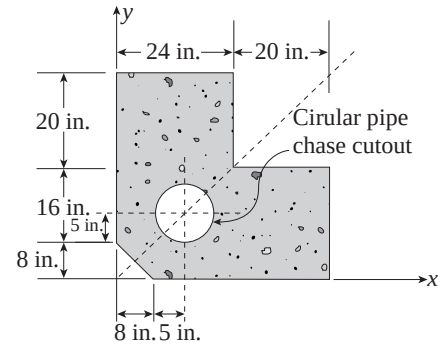
PART (c)

$$P_{\text{maxh}} = \sigma_a A_h = 89.535 \text{ kN} \quad P_{\text{maxs}} = \sigma_a A_s = 135.717 \text{ kN} \quad < \text{lesser value controls}$$

$$\boxed{P_{\text{max}} = P_{\text{maxh}} = 89.5 \text{ kN}}$$

Problem 1.3-5 The cross section of a concrete corner column that is loaded uniformly in compression is shown in the figure. A circular pipe chase cut-out of 10 in. in diameter runs the height of the column (see figure).

- Determine the average compressive stress σ_c in the concrete if the load is equal to 3500 kips.
- Determine the coordinates x_c and y_c of the point where the resultant load must act in order to produce uniform normal stress in the column.



Solution 1.3-5

$$P = 3500 \text{ kips}$$

$$A = (24 + 20)(20 + 16 + 8) - \left(\frac{1}{2} 8^2\right) - 20^2 - \frac{\pi}{4} 10^2$$

$$A = 1425.46 \text{ in.}^2$$

(a) AVERAGE COMPRESSIVE STRESS

$$\sigma_c = \frac{P}{A} \quad \boxed{\sigma_c = 2.46 \text{ ksi}}$$

(b) CENTROID

$$x_c = \frac{(24 + 20)^2 \frac{(24 + 20)}{2} - (20^2)(24 + 10) - \frac{1}{2} 8^2 \left(\frac{8}{3}\right) - \left(\frac{\pi}{4} 10^2\right)(8 + 5)}{A}$$

$$\boxed{x_c = 19.56 \text{ in.}}$$

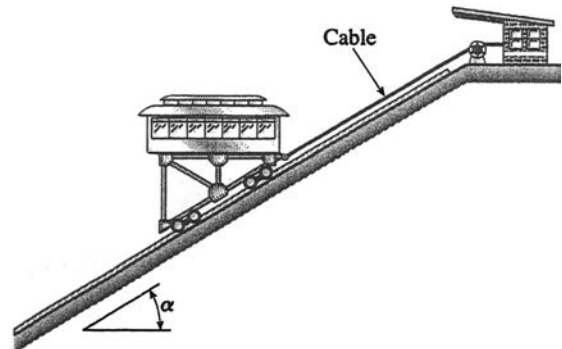
$$y_c = \frac{(24 + 20)^2 \frac{(24 + 20)}{2} - (20^2)(24 + 10) - \frac{1}{2} 8^2 \left(\frac{8}{3}\right) - \left(\frac{\pi}{4} 10^2\right)(8 + 5)}{A}$$

$$\boxed{y_c = 19.56 \text{ in.}}$$

x_c and y_c are the same as expected due to symmetry about a diagonal

Problem 1.3-6 A car weighing 130 kN when fully loaded is pulled slowly up a steep inclined track by a steel cable (see figure). The cable has an effective cross-sectional area of 490 mm², and the angle α of the incline is 30°.

- Calculate the tensile stress σ_t in the cable.
- If the allowable stress in the cable is 150 MPa, what is the maximum acceptable angle of the incline for a fully loaded car?



Solution 1.3-6

$$W = 130 \text{ kN} \quad \alpha = 30^\circ \quad A = 490 \text{ mm}^2 \quad \sigma_a = 150 \text{ MPa}$$

PART (a)

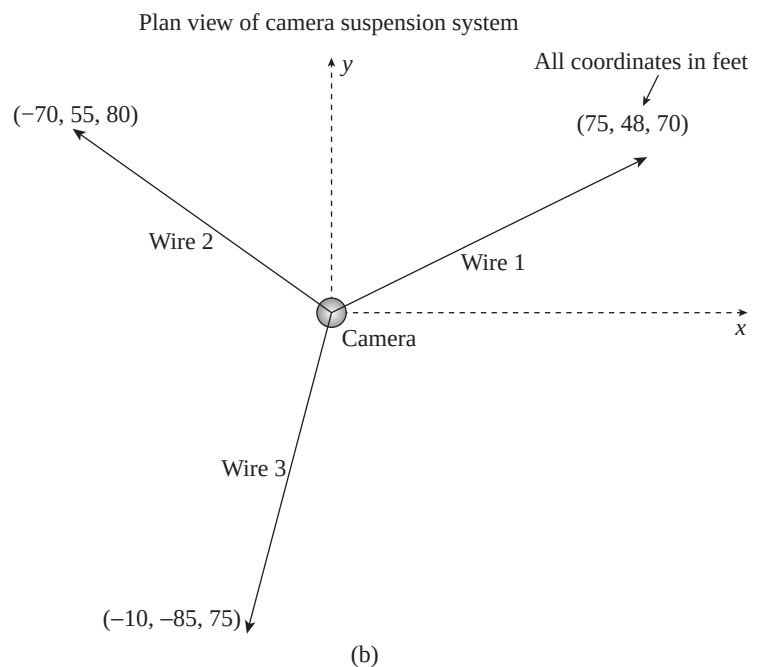
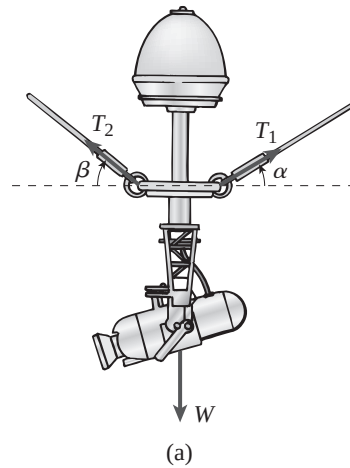
$$\sigma_t = \frac{W \sin(\alpha)}{A} = 132.7 \text{ MPa}$$

PART (b)

$$\alpha_{\max} = \arcsin\left(\frac{\sigma_a A}{W}\right) = 34.4^\circ$$

Problem 1.3-7 Two steel wires support a moveable overhead camera weighing $W = 28 \text{ lb}$ (see figure part a) used for close-up viewing of field action at sporting events. At some instant, wire 1 is at an angle $\alpha = 22^\circ$ to the horizontal and wire 2 is at an angle $\beta = 40^\circ$. Wires 1 and 2 have diameters of 30 and 35 mils, respectively. (Wire diameters are often expressed in mils; one mil equals 0.001 in.)

- Determine the tensile stresses σ_1 and σ_2 in the two wires.
- If the stresses in wires 1 and 2 must be the same, what is the required diameter of wire 1?
- Now, to stabilize the camera for windy outdoor conditions, a third wire is added (see figure part b). Assume the three wires meet at a common point (coordinates (0, 0, 0) above the camera at the instant shown in figure part b). Wire 1 is attached to a support at coordinates (75 ft, 48 ft, 70 ft). Wire 2 is supported at (-70 ft, 55 ft, 80 ft). Wire 3 is supported at (-10 ft, -85 ft, 75 ft). Assume that all three wires have a diameter of 30 mils. Find the tensile stresses in wires 1 to 3.



Solution 1.3-7

$$d_1 = 30(10^{-3}) \text{ in.} \quad d_2 = 35(10^{-3}) \text{ in.} \quad A_1 = \frac{\pi}{4}d_1^2 = 7.069 \times 10^{-4} \text{ in.}^2$$

$$W = 28 \text{ lb} \quad A_2 = \frac{\pi}{4}d_2^2 = 9.621 \times 10^{-4} \text{ in.}^2$$

$$\alpha = 22^\circ \quad \beta = 40^\circ$$

(a) FIND NORMAL STRESS IN WIRES

$$T_2 = \frac{W}{\frac{\cos(\beta)}{\cos(\alpha)} \sin(\alpha) + \sin(\beta)} = 29.403 \text{ lb} \quad \sigma_2 = \frac{T_2}{A_2} = 30.6 \text{ ksi}$$

$$T_1 = T_2 \frac{\cos(\beta)}{\cos(\alpha)} = 24.293 \text{ lb} \quad \sigma_1 = \frac{T_1}{A_1} = 34.4 \text{ ksi}$$

(b) FIND NEW d_1 S.T. NORMAL STRESSES IN WIRES IS THE SAME

$$A_{1\text{new}} = \frac{T_1}{\sigma_2} = 7.949 \times 10^{-4} \text{ in.}^2 \quad d_{1\text{new}} = \sqrt{\frac{4}{\pi}A_{1\text{new}}} = 3.18 \times 10^{-2} \text{ in.} \quad \text{or} \quad 31.8 \text{ mils}$$

$$\sigma_{1\text{new}} = \frac{T_1}{\frac{\pi}{4}d_{1\text{new}}^2} = 30.6 \text{ ksi}$$

(c) Now, to stabilize the camera for windy outdoor conditions, a third wire is added (see figure b); assume the 3 wires meet at a common point (coordinates = (0, 0, 0) above the camera at the instant shown in figure b); wire 1 is attached to a support at coordinates (75', 48', 70'); wire 2 is supported at (-70', 55', 80'); and wire 3 is supported at (-10', -85', 75'); assume that all three wires have diameter of 30 mils. Find tensile stresses in wires 1 to 3.

$$d = 30(10^{-3}) \text{ in.} \quad A = \frac{\pi}{4}d^2 = 7.069 \times 10^{-4} \text{ in.}^2$$

$$\text{Position vectors from camera to each support} \quad r_1 = \begin{pmatrix} 75 \\ 48 \\ 70 \end{pmatrix} \text{ ft} \quad r_2 = \begin{pmatrix} -70 \\ 55 \\ 80 \end{pmatrix} \text{ ft} \quad r_3 = \begin{pmatrix} -10 \\ -85 \\ 75 \end{pmatrix} \text{ ft} \quad W = 28 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ lb}$$

$$L_1 = |r_1| = 113.265 \quad L_2 = |r_2| = 119.687 \quad L_3 = |r_3| = 113.798$$

$$\text{Unit vectors along wires 1 to 3} \quad e_1 = \frac{r_1}{|r_1|} = \begin{pmatrix} 0.662 \\ 0.424 \\ 0.618 \end{pmatrix} \quad e_2 = \frac{r_2}{|r_2|} = \begin{pmatrix} -0.585 \\ 0.46 \\ 0.668 \end{pmatrix} \quad e_3 = \frac{r_3}{|r_3|} = \begin{pmatrix} -0.088 \\ -0.747 \\ 0.659 \end{pmatrix}$$

$$T_1 = F_1 e_1 \quad T_2 = F_2 e_2 \quad T_3 = F_3 e_3 \quad i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Equilibrium of forces $T_1 + T_2 + T_3 = W$

$$T^{(1)} = e_1 \quad T^{(2)} = e_2 \quad T^{(3)} = e_3 \quad T = \begin{pmatrix} 0.662 & -0.585 & -0.088 \\ 0.424 & 0.46 & -0.747 \\ 0.618 & 0.668 & 0.659 \end{pmatrix}$$

$$F = T^{-1} W = \begin{pmatrix} 13.854 \\ 13.277 \\ 16.028 \end{pmatrix} \text{ lb} \quad \sigma_1 = \frac{F_1}{A} = 19.6 \text{ ksi} \quad \sigma_2 = \frac{F_2}{A} = 18.78 \text{ ksi} \quad \sigma_3 = \frac{F_3}{A} = 22.7 \text{ ksi}$$

$$\sigma_1 = 19.6 \text{ ksi}$$

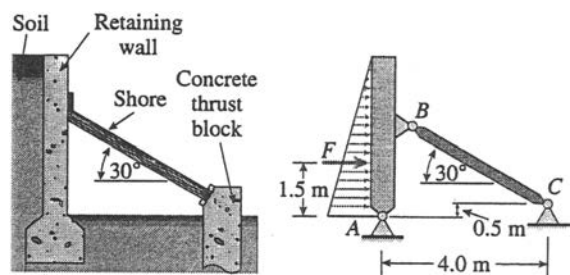
$$\sigma_2 = 18.78 \text{ ksi}$$

$$\sigma_3 = 22.7 \text{ ksi}$$

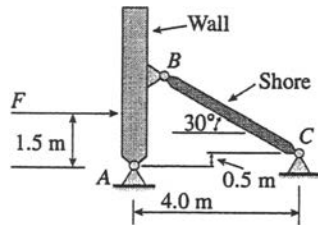
Problem 1.3-8 A long retaining wall is braced by wood shores set at an angle of 30° and supported by concrete thrust blocks, as shown in the first part of the figure. The shores are evenly spaced, 3 m apart.

For analysis purposes, the wall and shores are idealized as shown in the second part of the figure. Note that the base of the wall and both ends of the shores are assumed to be pinned. The pressure of the soil against the wall is assumed to be triangularly distributed, and the resultant force acting on a 3-meter length of the wall is $F = 190 \text{ kN}$.

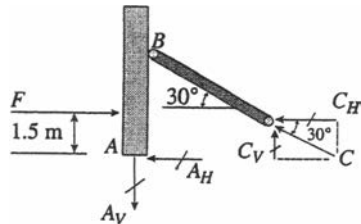
If each shore has a $150 \text{ mm} \times 150 \text{ mm}$ square cross section, what is the compressive stress σ_c in the shores?



Solution 1.3-8 Retaining wall braced by wood shores



FREE-BODY DIAGRAM OF WALL AND SHORE



C = compressive force in wood shore
 C_H = horizontal component of C
 C_V = vertical component of C
 $C_H = C \cos 30^\circ$
 $C_V = C \sin 30^\circ$

$$F = 190 \text{ kN}$$

A = area of one shore

$$\begin{aligned} A &= (150 \text{ mm})(150 \text{ mm}) \\ &= 22,500 \text{ mm}^2 \\ &= 0.0225 \text{ m}^2 \end{aligned}$$

SUMMATION OF MOMENTS ABOUT POINT A

$$\sum M_A = 0 \quad \curvearrowright \quad \curvearrowleft$$

$$-F(1.5 \text{ m}) + C_V(4.0 \text{ m}) + C_H(0.5 \text{ m}) = 0$$

or

$$\begin{aligned} -(190 \text{ kN})(1.5 \text{ m}) + C(\sin 30^\circ)(4.0 \text{ m}) \\ + C(\cos 30^\circ)(0.5 \text{ m}) = 0 \end{aligned}$$

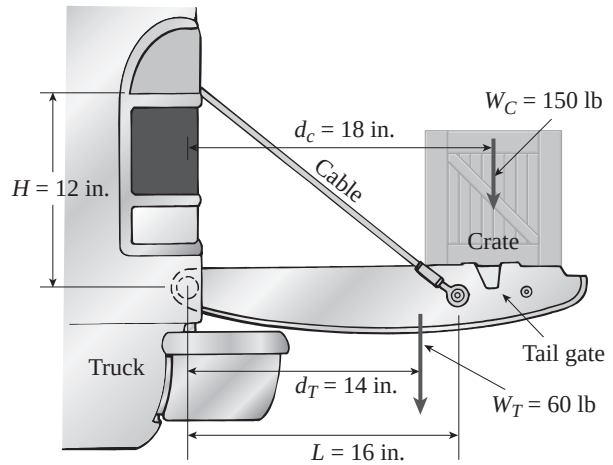
$$\therefore C = 117.14 \text{ kN}$$

COMPRESSIVE STRESS IN THE SHORES

$$\begin{aligned} \sigma_c &= \frac{C}{A} = \frac{117.14 \text{ kN}}{0.0225 \text{ m}^2} \\ &= 5.21 \text{ MPa} \quad \leftarrow \end{aligned}$$

Problem 1.3-9 A pickup truck tailgate supports a crate ($W_C = 150$ lb), as shown in the figure. The tailgate weighs $W_T = 60$ lb and is supported by two cables (only one is shown in the figure). Each cable has an effective cross-sectional area $A_e = 0.017$ in².

- Find the tensile force T and normal stress σ in each cable.
- If each cable elongates $\delta = 0.01$ in. due to the weight of both the crate and the tailgate, what is the average strain in the cable?



Solution 1.3-9

$$W_C = 150 \text{ lb}$$

$$A_e = 0.017 \text{ in.}^2$$

$$W_T = 60$$

$$\delta = 0.01$$

$$d_c = 18$$

$$d_T = 14$$

$$H = 12$$

$$L = 16$$

$$L_c = \sqrt{L^2 + H^2} \quad L_c = 20$$

$$\sum M_{\text{hinge}} = 0 \quad 2T_v L = W_C d_c + W_T d_T$$

$$T_v = \frac{W_C d_c + W_T d_T}{2L} \quad T_v = 110.625 \text{ lb}$$

$$T_h = \frac{L}{H} T_v \quad T_h = 147.5$$

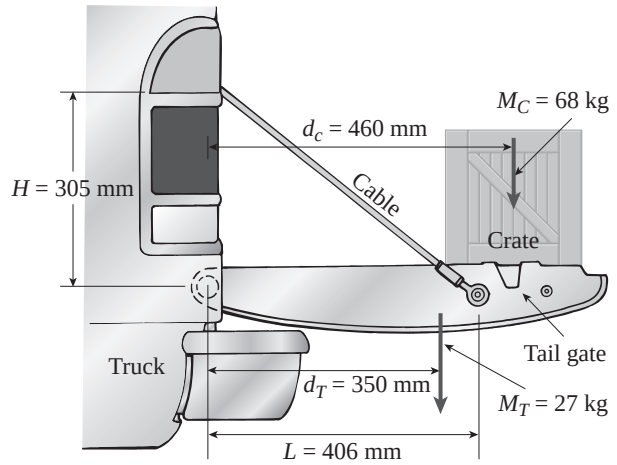
$$(a) \quad T = \sqrt{T_v^2 + T_h^2} \quad T = 184.4 \text{ lb} \quad \leftarrow$$

$$\sigma_{\text{cable}} = \frac{T}{A_e} \quad \sigma_{\text{cable}} = 10.8 \text{ ksi} \quad \leftarrow$$

$$(b) \quad \epsilon_{\text{cable}} = \frac{\delta}{L_c} \quad \epsilon_{\text{cable}} = 5 \times 10^{-4} \quad \leftarrow$$

Problem 1.3-10 Solve the preceding problem if the mass of the tail gate is $M_T = 27$ kg and that of the crate is $M_C = 68$ kg. Use dimensions $H = 305$ mm, $L = 406$ mm, $d_C = 460$ mm, and $d_T = 350$ mm. The cable cross-sectional area is $A_e = 11.0$ mm².

- Find the tensile force T and normal stress σ in each cable.
- If each cable elongates $\delta = 0.25$ mm due to the weight of both the crate and the tailgate, what is the average strain in the cable?



Solution 1.3-10

$$M_C = 68$$

$$M_T = 27 \text{ kg} \quad g = 9.81 \text{ m/s}^2$$

$$W_C = M_C g \quad W_T = M_T g$$

$$W_C = 667.08 \quad W_T = 264.87$$

$$N = \text{kg} \cdot \text{m/s}^2$$

$$A_e = 11.0 \text{ mm}^2 \quad \delta = 0.25$$

$$d_C = 460 \quad d_T = 350$$

$$H = 305 \quad L = 406$$

$$L_c = \sqrt{L^2 + H^2} \quad L_c = 507.8 \text{ mm}$$

$$\sum M_{\text{hinge}} = 0 \quad 2T_v L = W_C d_C + W_T d_T$$

$$T_v = \frac{W_C d_C + W_T d_T}{2L} \quad T_v = 492.071 \text{ N}$$

$$T_h = \frac{L}{H} T_v \quad T_h = 655.019 \text{ N}$$

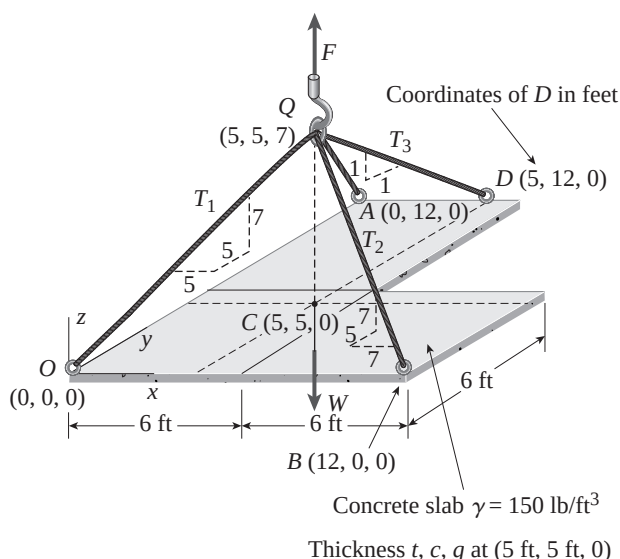
$$(a) \quad T = \sqrt{T_v^2 + T_h^2} \quad T = 819 \text{ N} \quad \leftarrow$$

$$\sigma_{\text{cable}} = \frac{T}{A_e} \quad \sigma_{\text{cable}} = 74.5 \text{ MPa} \quad \leftarrow$$

$$(b) \quad \varepsilon_{\text{cable}} = \frac{\delta}{L_c} \quad \varepsilon_{\text{cable}} = 4.92 \times 10^{-4} \quad \leftarrow$$

Problem 1.3-11 An L-shaped reinforced concrete slab $12\text{ ft} \times 12\text{ ft}$ (but with a $6\text{ ft} \times 6\text{ ft}$ cutout) and thickness $t = 9.0\text{ in.}$ is lifted by three cables attached at O , B and D , as shown in the figure. The cables are combined at point Q , which is 7.0 ft above the top of the slab and directly above the center of mass at C . Each cable has an effective cross-sectional area of $A_e = 0.12\text{ in.}^2$.

- Find the tensile force T_i ($i = 1, 2, 3$) in each cable due to the weight W of the concrete slab (ignore weight of cables).
- Find the average stress σ_i in each cable. (see Table I-1 in Appendix I for the weight density of reinforced concrete.)
- Add cable AQ so that OQA is one continuous cable, with each segment having force T_1 , which is connected to cables BQ and DQ at point Q . Repeat parts (a) and (b). (Hint: There are now three force equilibrium equations and one constraint equation, $T_1 = T_4$)



Solution 1.3-11

CABLE LENGTHS (FT)

$$L_1 = \sqrt{5^2 + 5^2 + 7^2} \quad L_1 = 9.95 \quad L_2 = \sqrt{5^2 + 7^2 + 7^2} \quad L_2 = 11.091 \quad L_3 = \sqrt{7^2 + 7^2} \quad L_3 = 9.899$$

(a) SOLUTION FOR CABLE FORCES USING STATICS (THREE EQUATIONS, THREE UNKNOWN); UNITS = lb, ft

$$r_{OQ} = \begin{pmatrix} 5 \\ 5 \\ 7 \end{pmatrix} \quad r_{BQ} = \begin{pmatrix} -7 \\ 5 \\ 7 \end{pmatrix} \quad r_{DQ} = \begin{pmatrix} 0 \\ -7 \\ 7 \end{pmatrix}$$

$$e_{OQ} = \frac{r_{OQ}}{|r_{OQ}|} = \begin{pmatrix} 0.503 \\ 0.503 \\ 0.704 \end{pmatrix} \quad e_{BQ} = \frac{r_{BQ}}{|r_{BQ}|} = \begin{pmatrix} -0.631 \\ 0.451 \\ 0.631 \end{pmatrix} \quad e_{DQ} = \frac{r_{DQ}}{|r_{DQ}|} = \begin{pmatrix} 0 \\ -0.707 \\ 0.707 \end{pmatrix}$$

$$W = 150(12^2 - 6^2) \frac{9}{12} = 12,150 \text{ lbs}$$

$$\text{STATICS} \quad \Sigma F = 0 \quad T_1 e_{OQ} + T_2 e_{BQ} + T_3 e_{DQ} - \begin{pmatrix} 0 \\ 0 \\ W \end{pmatrix} = \begin{pmatrix} 0.50252 T_1 - 0.63117 T_2 \\ 0.50252 T_1 + 0.45083 T_2 - 0.70711 T_3 \\ 0.70353 T_1 + 0.63117 T_2 + 0.70711 T_3 - 12,150 \end{pmatrix}$$

or in matrix form; solve simultaneous equations to get cable tension forces

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} e_{OQ,1} & e_{BQ,1} & e_{DQ,1} \\ e_{OQ,2} & e_{BQ,2} & e_{DQ,2} \\ e_{OQ,3} & e_{BQ,3} & e_{DQ,3} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ W \end{pmatrix} = \begin{pmatrix} 5877 \\ 4679 \\ 7159 \end{pmatrix} \text{ lb} \quad T = \begin{pmatrix} 5877 \\ 4679 \\ 7159 \end{pmatrix} \text{ lb}$$

(b) AVERAGE NORMAL STRESS IN EACH CABLE

$$i = 1 \dots 3 \quad \sigma_i = \frac{T_i}{A_e} \quad \sigma = \begin{pmatrix} 48975 \\ 38992 \\ 59658 \end{pmatrix} \text{ psi} \quad A_e = 0.12 \text{ in.}^2$$

(c) ADD CONTINUOUS CABLE OQA

$$r_{OQ} = \begin{pmatrix} 5 \\ 5 \\ 7 \end{pmatrix} \quad r_{AQ} = \begin{pmatrix} 5 \\ -7 \\ 7 \end{pmatrix} \quad r_{BQ} = \begin{pmatrix} -7 \\ 5 \\ 7 \end{pmatrix} \quad r_{DQ} = \begin{pmatrix} 0 \\ -7 \\ 7 \end{pmatrix} \quad e_{AQ} = \frac{r_{AQ}}{|r_{AQ}|} = \begin{pmatrix} 0.451 \\ -0.631 \\ 0.631 \end{pmatrix}$$

$$e_{OQ} = \frac{r_{OQ}}{|r_{OQ}|} = \begin{pmatrix} 0.503 \\ 0.503 \\ 0.704 \end{pmatrix} \quad e_{AQ} = \frac{r_{AQ}}{|r_{AQ}|} = \begin{pmatrix} 0.451 \\ -0.631 \\ 0.631 \end{pmatrix} \quad e_{BQ} = \frac{r_{BQ}}{|r_{BQ}|} = \begin{pmatrix} -0.631 \\ 0.451 \\ 0.631 \end{pmatrix}$$

STATICS Solve simultaneous equations to get cable tension forces

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} e_{OQ_{1,1}} & e_{BQ_{1,1}} & e_{DQ_{1,1}} & e_{AQ_{1,1}} \\ e_{OQ_{2,1}} & e_{BQ_{2,1}} & e_{DQ_{2,1}} & e_{AQ_{2,1}} \\ e_{OQ_{3,1}} & e_{BQ_{3,1}} & e_{DQ_{3,1}} & e_{AQ_{3,1}} \\ 1 & 0 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ W \\ 0 \end{pmatrix} = \begin{pmatrix} 4278 \\ 6461 \\ 3341 \\ 4278 \end{pmatrix} \text{ lbs} \quad T = \begin{pmatrix} 4278 \\ 6461 \\ 3341 \\ 4278 \end{pmatrix} \text{ lb}$$

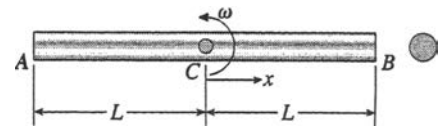
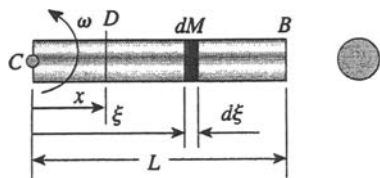
< for case of $T_1 = T_4$

Normal stresses in cables

$$i = 1 \dots 4 \quad \sigma_i = \frac{T_i}{A_e} \quad \sigma = \begin{pmatrix} 35650 \\ 53842 \\ 27842 \\ 35650 \end{pmatrix} \text{ psi}$$

Problem 1.3-12 A round bar ACB of length $2L$ (see figure) rotates about an axis through the midpoint C with constant angular speed ω (radians per second). The material of the bar has weight density γ .

- Derive a formula for the tensile stress σ_x in the bar as a function of the distance x from the midpoint C .
- What is the maximum tensile stress $\sigma_{\max}$?

**Solution 1.3-12 Rotating Bar** ω = angular speed (rad/s) A = cross-sectional area γ = weight density $\frac{\gamma}{g}$ = mass density

We wish to find the axial force F_x in the bar at Section D , distance x from the midpoint C .

The force F_x equals the inertia force of the part of the rotating bar from D to B .

Consider an element of mass dM at distance ξ from the midpoint C . The variable ξ ranges from x to L .

$$dM = \frac{\gamma}{g} A d\xi$$

dF = Inertia force (centrifugal force) of element of mass dM

$$dF = (dM)(j\omega^2) = \frac{\gamma}{g} A \omega^2 j d\xi$$

$$F_x = \int_D^B dF = \int_x^L \frac{\gamma}{g} A \omega^2 j d\xi = \frac{\gamma A \omega^2}{2g} (L^2 - x^2)$$

(a) TENSILE STRESS IN BAR AT DISTANCE x

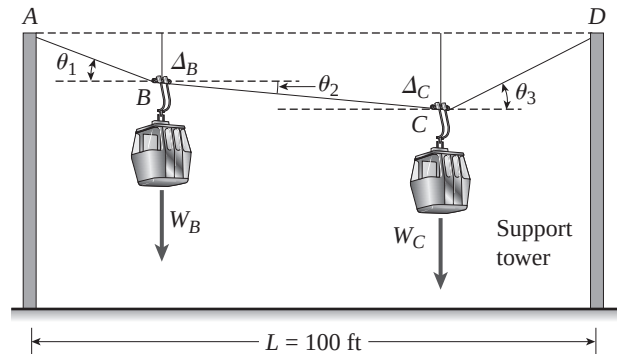
$$\sigma_x = \frac{F_x}{A} = \frac{\gamma \omega^2}{2g} (L^2 - x^2) \quad \leftarrow$$

(b) MAXIMUM TENSILE STRESS

$$x = 0 \quad \sigma_{\max} = \frac{\gamma \omega^2 L^2}{2g} \quad \leftarrow$$

Problem 1.3-13 Two gondolas on a ski lift are locked in the position shown in the figure while repairs are being made elsewhere. The distance between support towers is $L = 100$ ft. The length of each cable segment under gondola weights $W_B = 450$ lb and $W_C = 650$ lb are $D_{AB} = 12$ ft, $D_{BC} = 70$ ft, and $D_{CD} = 20$ ft. The cable sag at B is $\Delta_B = 3.9$ ft and that at C (Δ_C) is 7.1 ft. The effective cross-sectional area of the cable is $A_e = 0.12$ in².

- Find the tension force in each cable segment; neglect the mass of the cable.
- Find the average stress (σ) in each cable segment.



Solution 1.3-13

$$W_B = 450$$

$$W_C = 650 \text{ lb}$$

$$\Delta_B = 3.9 \text{ ft}$$

$$\Delta_C = 7.1 \text{ ft}$$

$$L = 100 \text{ ft}$$

$$D_{AB} = 12 \text{ ft}$$

$$D_{BC} = 70 \text{ ft}$$

$$D_{CD} = 20 \text{ ft}$$

$$D_{AB} + D_{BC} + D_{CD} = 102 \text{ ft}$$

$$A_e = 0.12 \text{ in}^2$$

COMPUTE INITIAL VALUES OF THETA ANGLES (RADIANs)

$$\theta_1 = \arcsin\left(\frac{\Delta_B}{D_{AB}}\right) \quad \theta_1 = 0.331$$

$$\theta_2 = \arcsin\left(\frac{\Delta_C - \Delta_B}{D_{BC}}\right) \quad \theta_2 = 0.046$$

$$\theta_3 = \arcsin\left(\frac{\Delta_C}{D_{CD}}\right) \quad \theta_3 = 0.363$$

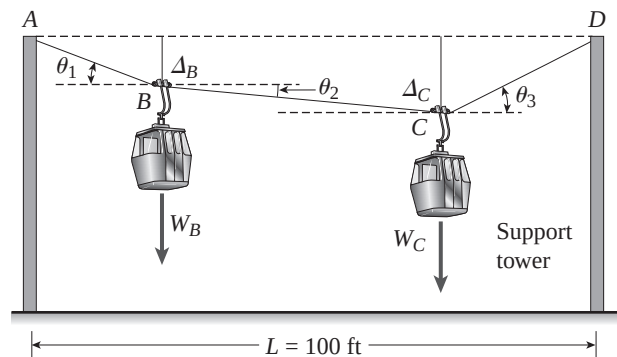
(a) STATICS AT B AND C

$$-T_{AB} \cos(\theta_1) + T_{BC} \cos(\theta_2) = 0$$

$$T_{AB} \sin(\theta_1) - T_{BC} \sin(\theta_2) = W_B$$

$$-T_{BC} \cos(\theta_2) + T_{CD} \cos(\theta_3) = 0$$

$$T_{BC} \sin(\theta_2) + T_{CD} \sin(\theta_3) = W_C$$



CONSTRAINT EQUATIONS

$$D_{AB} \cos(\theta_1) + D_{BC} \cos(\theta_2) + D_{CD} \cos(\theta_3) = L$$

$$D_{AB} \sin(\theta_1) + D_{BC} \sin(\theta_2) = D_{CD} \sin(\theta_3)$$

SOLVE SIMULTANEOUS EQUATIONS NUMERICALLY FOR TENSION FORCE IN EACH CABLE SEGMENT

$$T_{AB} = 1620 \text{ lb} \quad T_{CB} = 1536 \text{ lb} \quad T_{CD} = 1640 \text{ lb} \quad \leftarrow$$

CHECK EQUILIBRIUM AT B AND C

$$T_{AB} \sin(\theta_1) - T_{BC} \sin(\theta_2) = 450$$

$$T_{BC} \sin(\theta_2) + T_{CD} \sin(\theta_3) = 650$$

(b) COMPUTE STRESSES IN CABLE SEGMENTS

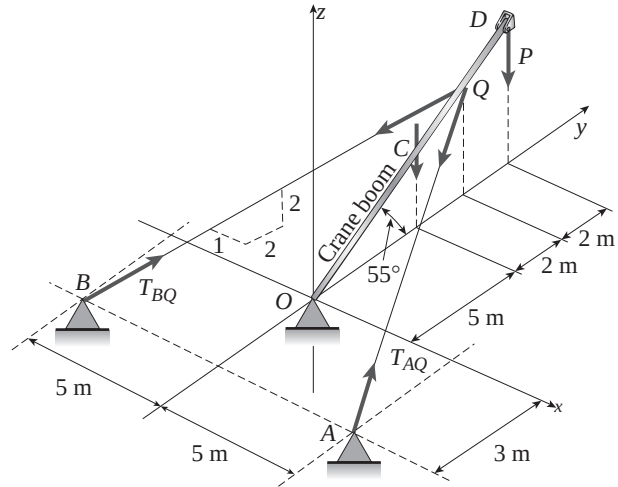
$$\sigma_{AB} = \frac{T_{AB}}{A_e} \quad \sigma_{BC} = \frac{T_{BC}}{A_e} \quad \sigma_{CD} = \frac{T_{CD}}{A_e}$$

$$\sigma_{AB} = 13.5 \text{ ksi} \quad \sigma_{BC} = 12.8 \text{ ksi}$$

$$\sigma_{CD} = 13.67 \text{ ksi} \quad \leftarrow$$

Problem 1.3-14 A crane boom of mass 450 kg with its center of mass at C is stabilized by two cables AQ and BQ ($A_e = 304 \text{ mm}^2$ for each cable) as shown in the figure. A load $P = 20 \text{ kN}$ is supported at point D . The crane boom lies in the y - z plane.

- Find the tension forces in each cable: T_{AQ} and T_{BQ} (kN). Neglect the mass of the cables, but include the mass of the boom in addition to load P .
- Find the average stress (σ) in each cable.



Solution 1.3-14

Data $M_{\text{boom}} = 450 \text{ kg}$

$$g = 9.81 \text{ m/s}^2 \quad W_{\text{boom}} = M_{\text{boom}} g$$

$$W_{\text{boom}} = 4415 \text{ N}$$

$$P = 20 \text{ kN}$$

$$A_e = 304 \text{ mm}^2$$

$$(a) \text{ Symmetry: } T_{AQ} = T_{BQ}$$

$$\sum M_x = 0$$

$$2T_{AQZ}(3000) = W_{\text{boom}}(5000) + P(9000)$$

$$T_{AQZ} = \frac{W_{\text{boom}}(5000) + P(9000)}{2(3000)}$$

$$T_{AQ} = \frac{\sqrt{2^2 + 2^2 + 1^2}}{2} T_{AQZ}$$

$$T_{AQ} = 50.5 \text{ kN} = T_{BQ} \quad \leftarrow$$

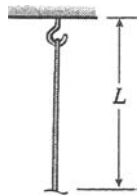
$$(b) \sigma = \frac{T_{AQ}}{A_e} \quad \sigma = 166.2 \text{ MPa} \quad \leftarrow$$

Mechanical Properties of Materials

Problem 1.4-1 Imagine that a long steel wire hangs vertically from a high-altitude balloon.

- (a) What is the greatest length (feet) it can have without yielding if the steel yields at 40 ksi?
 (b) If the same wire hangs from a ship at sea, what is the greatest length? (Obtain the weight densities of steel and sea water from Appendix I, Table I-1.)
-

Solution 1.4-1 Hanging wire of length L



W = total weight of steel wire

γ_S = weight density of steel
 $= 490 \text{ lb/ft}^3$

γ_w = weight density of sea water
 $= 63.8 \text{ lb/ft}^3$

A = cross-sectional area of wire

$\sigma_{\max} = 40 \text{ ksi}$ (yield strength)

(a) WIRE HANGING IN AIR

$$W = \gamma_S AL$$

$$\sigma_{\max} = \frac{W}{A} = \gamma_S L$$

$$L_{\max} = \frac{\sigma_{\max}}{\gamma_S} = \frac{40,000 \text{ psi}}{490 \text{ lb/ft}^3} (144 \text{ in.}^2/\text{ft}^2)$$

$$= 11,800 \text{ ft} \quad \leftarrow$$

(b) WIRE HANGING IN SEA WATER

F = tensile force at top of wire

$$F = (\gamma_S - \gamma_w)AL \quad \sigma_{\max} = \frac{F}{A} = (\gamma_S - \gamma_w)L$$

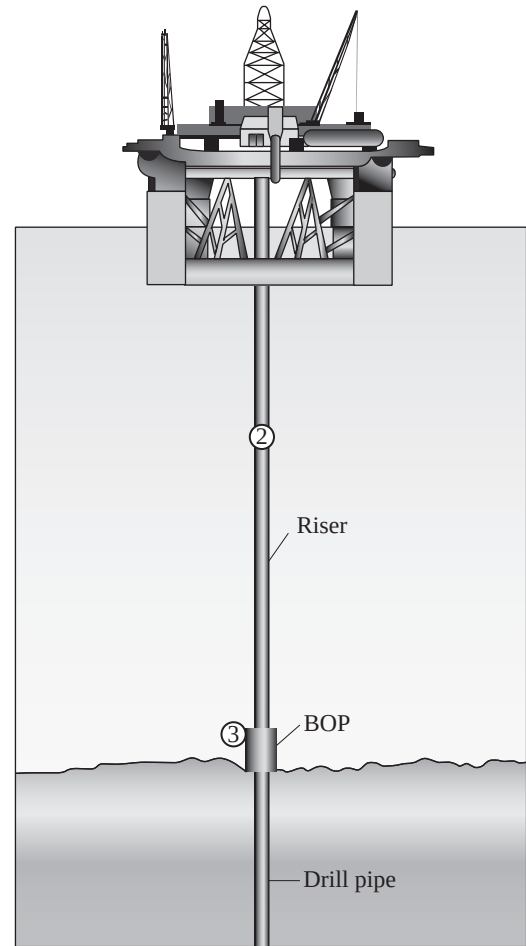
$$L_{\max} = \frac{\sigma_{\max}}{\gamma_S - \gamma_w}$$

$$= \frac{40,000 \text{ psi}}{(490 - 63.8) \text{ lb/ft}^3} (144 \text{ in.}^2/\text{ft}^2)$$

$$= 13,500 \text{ ft} \quad \leftarrow$$

Problem 1.4-2 Steel riser pipe hangs from a drill rig located offshore in deep water (see figure).

- What is the greatest length (meters) it can have without breaking if the pipe is suspended in air and the ultimate strength (or breaking strength) is 550 MPa?
- If the same riser pipe hangs from a drill rig at sea, what is the greatest length? (Obtain the weight densities of steel and sea water from Appendix I, Table I-1. Neglect the effect of buoyant foam casings on the pipe).



Solution 1.4-2

(a) PIPE SUSPENDED IN AIR

$$\sigma_U = 550 \text{ MPa}$$

$$\gamma_s = 77 \text{ kN/m}^3$$

$$W = \gamma_s AL$$

$$L_{\max} = \frac{\sigma_U}{\gamma_s} = 7143 \text{ m}$$

(b) PIPE SUSPENDED IN SEA WATER

$$\gamma_w = 10 \text{ kN/m}^3$$

$$\text{Force at top of pipe: } F = (\gamma_s - \gamma_w)AL$$

Stress at top of pipe:

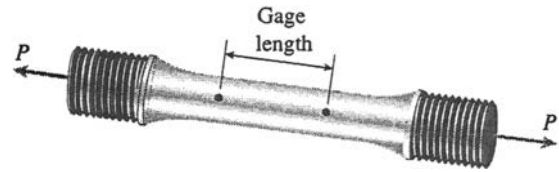
$$\sigma_{\max} = \frac{F}{A} \quad \sigma_{\max} = (\gamma_s - \gamma_w)L$$

Set max stress equal to ultimate and then solve for $L_{\max}$

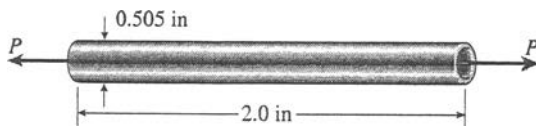
$$L_{\max} = \frac{\sigma_U}{(\gamma_s - \gamma_w)} = 8209 \text{ m}$$

Problem 1.4-3 Three different materials, designated A, B, and C, are tested in tension using test specimens having diameters of 0.505 in. and gage lengths of 2.0 in. (see figure). At failure, the distances between the gage marks are found to be 2.13, 2.48, and 2.78 in., respectively. Also, at the failure cross sections the diameters are found to be 0.484, 0.398, and 0.253 in., respectively.

Determine the percent elongation and percent reduction in area of each specimen, and then, using your own judgment, classify each material as brittle or ductile.



Solution 1.4-3 Tensile tests of three materials



$$\text{Percent elongation} = \frac{L_1 - L_0}{L_0}(100) = \left(\frac{L_1}{L_0} - 1\right)100$$

$$L_0 = 2.0 \text{ in.}$$

$$\text{Percent elongation} = \left(\frac{L_1}{2.0} - 1\right)(100) \quad (\text{Eq. 1})$$

$$\frac{A_1}{A_0} = \left(\frac{d_1}{d_0}\right)^2 \quad d_0 = 0.505 \text{ in.}$$

Percent reduction in area

$$= \left[1 - \left(\frac{d_1}{0.505}\right)^2\right](100) \quad (\text{Eq. 2})$$

where d_1 is in inches.

where L_1 is in inches.

$$\begin{aligned} \text{Percent reduction in area} &= \frac{A_0 - A_1}{A_0}(100) \\ &= \left(1 - \frac{A_1}{A_0}\right)(100) \end{aligned}$$

d_0 = initial diameter d_1 = final diameter

Material	L_1 (in.)	d_1 (in.)	% Elongation (Eq. 1)	% Reduction (Eq. 2)	Brittle or Ductile?
A	2.13	0.484	6.5%	8.1%	Brittle
B	2.48	0.398	24.0%	37.9%	Ductile
C	2.78	0.253	39.0%	74.9%	Ductile

Problem 1.4-4 The *strength-to-weight ratio* of a structural material is defined as its load-carrying capacity divided by its weight. For materials in tension, we may use a characteristic tensile stress (as obtained from a stress-strain curve) as a measure of strength. For instance, either the yield stress or the ultimate stress could be used, depending upon the particular application. Thus, the strength-to-weight ratio $R_{S/W}$ for a material in tension is defined as

$$R_{S/W} = \frac{\sigma}{\gamma}$$

in which σ is the characteristic stress and γ is the weight density. Note that the ratio has units of length.

Using the ultimate stress σ_U as the strength parameter, calculate the strength-to-weight ratio (in units of meters) for each of the following materials: aluminum alloy 6061-T6, Douglas fir (in bending), nylon, structural steel ASTM-A572, and a titanium alloy. (Obtain the material properties from Appendix I, Tables I-1 and I-3. When a range of values is given in a table, use the average value.)

Solution 1.4-4 Strength-to-weight ratio

The ultimate stress σ_U for each material is obtained from Appendix I, Tables I-3, and the weight density γ is obtained from Table I-1.

The strength-to-weight ratio (meters) is

$$R_{S/W} = \frac{\sigma_U (\text{MPa})}{\gamma (\text{kN/m}^3)} (10^3)$$

Values of σ_U , γ , and $R_{S/W}$ are listed in the table.

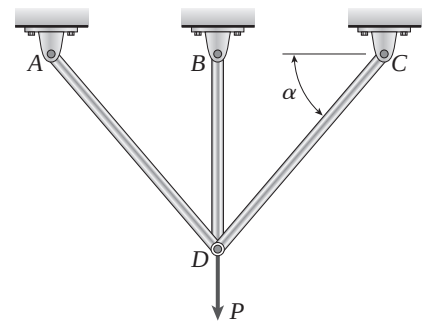
	σ_U (MPa)	γ (kN/m ³)	$R_{S/W}$ (m)
Aluminum alloy 6061-T6	310	26.0	11.9×10^3
Douglas fir	65	5.1	12.7×10^3
Nylon	60	9.8	6.1×10^3
Structural steel ASTM-A572	500	77.0	6.5×10^3
Titanium alloy	1050	44.0	23.9×10^3

Titanium has a high strength-to-weight ratio, which is why it is used in space vehicles and high-performance airplanes. Aluminum is higher than steel, which makes it desirable for commercial aircraft. Some woods are also higher than steel, and nylon is about the same as steel.

Problem 1.4-5 A symmetrical framework consisting of three pin-connected bars is loaded by a force P (see figure). The angle between the inclined bars and the horizontal is $\alpha = 52^\circ$. The axial strain in the middle bar is measured as 0.036.

Determine the tensile stress in the outer bars if they are constructed of a copper alloy having the following stress-strain relationship:

$$\sigma = \frac{18,000\varepsilon}{1 + 300\varepsilon} \quad 0 \leq \varepsilon \leq 0.03 \quad (\sigma = \text{ksi})$$

**Solution 1.4-5**

DATA

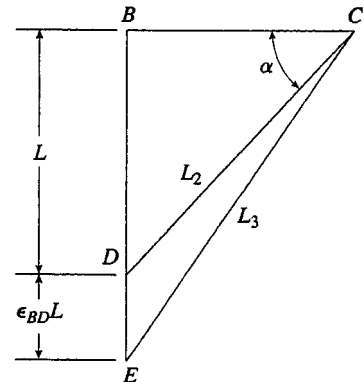
$$\varepsilon_{BD} = 0.036 \quad \alpha = 52^\circ \quad L_{BD} = 1$$

< assume unit length to facilitate numerical calculations below

Strain in CE

$$\varepsilon_{CE} = \frac{L_3 - L_2}{L_2}$$

$$L_2 = \frac{L_{BD}}{\sin(\alpha)} \quad L_{BC} = \frac{L_{BD}}{\tan(\alpha)}$$



Increased length of CE (see figure)

$$L_3 = \sqrt{L_{BC}^2 + (L_{BD} + \varepsilon_{BD}L_{BD})^2} = \sqrt{\frac{1}{\tan(52^\circ)^2} + 1.073296}$$

Compute strain in CE then substitute strain value into stress-strain relationship to find tensile stress in outer bars:

$$\varepsilon_{CE} = \frac{L_3 - L_2}{L_2} = 0.023 \quad \sigma = \frac{18000 \varepsilon_{CE}}{1 + 300 \varepsilon_{CE}} \quad \boxed{\sigma = 52.3 \text{ ksi}}$$

Problem 1.4-6 A specimen of a methacrylate plastic is tested in tension at room temperature (see figure), producing the stress-strain data listed in the accompanying table.

Plot the stress-strain curve and determine the proportional limit, modulus of elasticity (i.e., the slope of the initial part of the stress-strain curve), and yield stress at 0.2% offset. Is the material ductile or brittle?

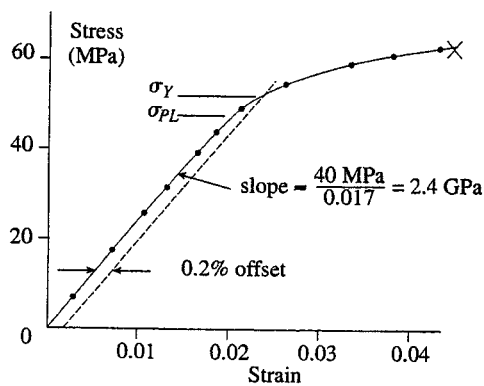


STRESS-STRAIN DATA FOR PROBLEM 1.4-6

Stress (MPa)	Strain
8.0	0.0032
17.5	0.0073
25.6	0.0111
31.1	0.0129
39.8	0.0163
44.0	0.0184
48.2	0.0209
53.9	0.0260
58.1	0.0331
62.0	0.0429
62.1	Fracture

Solution 1.4-6 Tensile test of a plastic

Using the stress-strain data given in the problem statement, plot the stress-strain curve:



σ_{PL} = proportional limit $\sigma_{PL} \approx 47 \text{ MPa}$ ←

Modulus of elasticity (slope) $\approx 2.4 \text{ GPa}$ ←

σ_Y = yield stress at 0.2% offset

$\sigma_Y \approx 53 \text{ MPa}$ ←

Material is *brittle*, because the strain after the proportional limit is exceeded is relatively small. ←

Problem 1.4-7 The data shown in the accompanying table were obtained from a tensile test of high-strength steel. The test specimen had a diameter of 0.505 in. and a gage length of 2.00 in. (see figure for Prob. 1.4-3). At fracture, the elongation between the gage marks was 0.12 in. and the minimum diameter was 0.42 in.

Plot the conventional stress-strain curve for the steel and determine the proportional limit, modulus of elasticity (i.e., the slope of the initial part of the stress-strain curve), yield stress at 0.1% offset, ultimate stress, percent elongation in 2.00 in., and percent reduction in area.

TENSILE-TEST DATA FOR PROBLEM 1.4-7

Load (lb)	Elongation (in.)
1,000	0.0002
2,000	0.0006
6,000	0.0019
10,000	0.0033
12,000	0.0039
12,900	0.0043
13,400	0.0047
13,600	0.0054
13,800	0.0063
14,000	0.0090
14,400	0.0102
15,200	0.0130
16,800	0.0230
18,400	0.0336
20,000	0.0507
22,400	0.1108
22,600	Fracture

Solution 1.4-7 Tensile test of high-strength steel

$$d_0 = 0.505 \text{ in.} \quad L_0 = 2.00 \text{ in.}$$

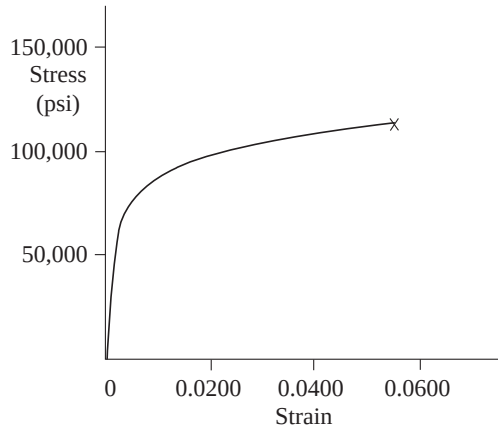
$$A_0 = \frac{\pi d_0^2}{4} = 0.200 \text{ in.}^2$$

CONVENTIONAL STRESS AND STRAIN

$$\sigma = \frac{P}{A_0} \quad \varepsilon = \frac{\delta}{L_0}$$

Load P (lb)	Elongation δ (in.)	Stress σ (psi)	Strain ε
1,000	0.0002	5,000	0.00010
2,000	0.0006	10,000	0.00030
6,000	0.0019	30,000	0.00100
10,000	0.0033	50,000	0.00165
12,000	0.0039	60,000	0.00195
12,900	0.0043	64,500	0.00215
13,400	0.0047	67,000	0.00235
13,600	0.0054	68,000	0.00270
13,800	0.0063	69,000	0.00315
14,000	0.0090	70,000	0.00450
14,400	0.0102	72,000	0.00510
15,200	0.0130	76,000	0.00650
16,800	0.0230	84,000	0.01150
18,400	0.0336	92,000	0.01680
20,000	0.0507	100,000	0.02535
22,400	0.1108	112,000	0.05540
22,600	Fracture	113,000	

STRESS-STRAIN DIAGRAM



RESULTS

Proportional limit $\approx 65,000$ psi ←

Modulus of elasticity (slope) $\approx 30 \times 10^6$ psi ←

Yield stress at 0.1% offset $\approx 69,000$ psi ←

Ultimate stress (maximum stress)

$\approx 113,000$ psi ←

Percent elongation in 2.00 in.

$$= \frac{L_1 - L_0}{L_0} (100)$$

$$= \frac{0.12 \text{ in.}}{2.00 \text{ in.}} (100) = 6\% \quad \leftarrow$$

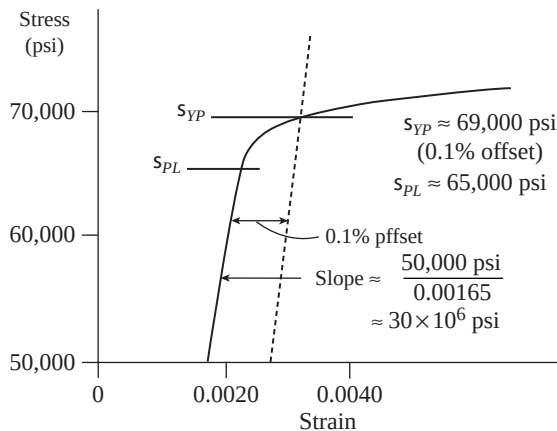
Percent reduction in area

$$= \frac{A_0 - A_1}{A_0} (100)$$

$$= \frac{0.200 \text{ in.}^2 - \frac{\pi}{4} (0.42 \text{ in.})^2}{0.200 \text{ in.}^2} (100)$$

$$= 31\% \quad \leftarrow$$

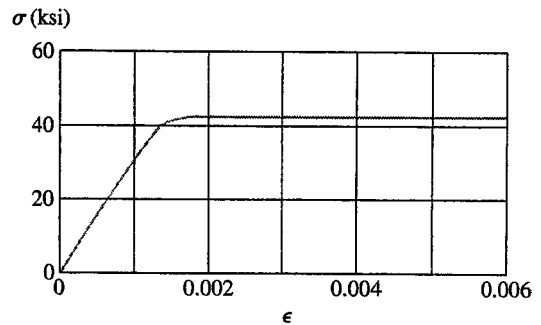
ENLARGEMENT OF PART OF THE STRESS-STRAIN CURVE



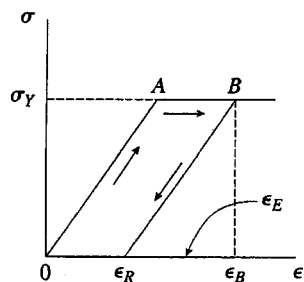
Elasticity, Plasticity, and Creep

Problem 1.5-1 A bar made of structural steel having the stress-strain diagram shown in the figure has a length of 48 in. The yield stress of the steel is 42 ksi and the slope of the initial linear part of the stress-strain curve (modulus of elasticity) is 30×10^3 ksi. The bar is loaded axially until it elongates 0.20 in., and then the load is removed.

How does the final length of the bar compare with its original length? (*Hint:* Use the concepts illustrated in Fig. 1-36b.)



Solution 1.5-1 Steel bar in tension


$$L = 48 \text{ in.}$$

Yield stress $\sigma_Y = 42$ ksi

$$\text{Slope} = 30 \times 10^3 \text{ ksi}$$
$$\delta = 0.20 \text{ in.}$$

STRESS AND STRAIN AT POINT B

$$\sigma_B = \sigma_Y = 42 \text{ ksi}$$

$$\epsilon_B = \frac{\delta}{L} = \frac{0.20 \text{ in.}}{48 \text{ in.}} = 0.00417$$

ELASTIC RECOVERY ε_F

$$\epsilon_E = \frac{\sigma_B}{\text{Slope}} = \frac{42 \text{ ksi}}{30 \times 10^3 \text{ ksi}} = 0.00140$$

RESIDUAL STRAIN ε_R

$$\begin{aligned}\varepsilon_R &= \varepsilon_B - \varepsilon_E = 0.00417 - 0.00140 \\ &= 0.00277\end{aligned}$$

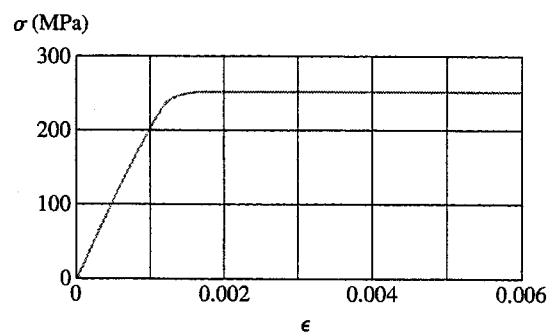
PERMANENT SET

$$\begin{aligned}\varepsilon_R L &= (0.00277)(48 \text{ in.}) \\ &= 0.13 \text{ in.}\end{aligned}$$

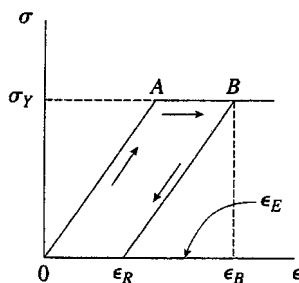
Final length of bar is 0.13 in. greater than its original length. ←

Problem 1.5-2 A bar of length 2.0 m is made of a structural steel having the stress-strain diagram shown in the figure. The yield stress of the steel is 250 MPa and the slope of the initial linear part of the stress-strain curve (modulus of elasticity) is 200 GPa. The bar is loaded axially until it elongates 6.5 mm, and then the load is removed.

How does the final length of the bar compare with its original length? (*Hint:* Use the concepts illustrated in Fig. 1-36b.)



Solution 1.5-2 Steel bar in tension


$$L = 2.0 \text{ m} = 2000 \text{ mm}$$

Yield stress $\sigma_v = 250$ MPa

Slope = 200 GPa

$$\delta = 6.5 \text{ mm}$$
ELASTIC RECOVERY ε_F

$$\varepsilon_E = \frac{\sigma_B}{\text{Slope}} = \frac{250 \text{ MPa}}{200 \text{ GPa}} = 0.00125$$

RESIDUAL STRAIN ε_R

$$\begin{aligned}\varepsilon_R &= \varepsilon_B - \varepsilon_E = 0.00325 - 0.00125 \\ &= 0.00200\end{aligned}$$

$$\begin{aligned}\text{Permanent set} &= \varepsilon_R L = (0.00200)(2000 \text{ mm}) \\ &= 4.0 \text{ mm}\end{aligned}$$

Final length of bar is 4.0 mm greater than its original length. ←

STRESS AND STRAIN AT POINT B

$$\sigma_B = \sigma_Y = 250 \text{ MPa}$$

$$\varepsilon_B = \frac{\delta}{L} = \frac{6.5 \text{ mm}}{2000 \text{ mm}} = 0.00325$$

Problem 1.5-3 An aluminum bar has length $L = 6$ ft and diameter $d = 1.375$ in.

The stress-strain curve for the aluminum is shown in Fig. 1-31 of Section 1.4.

The initial straight-line part of the curve has a slope (modulus of elasticity) of 10.6×10^6 psi. The bar is loaded by tensile forces $P = 44.6$ k and then unloaded.

- (a) What is the permanent set of the bar?
(b) If the bar is reloaded, what is the proportional limit? (*Hint:* Use the concepts illustrated in Figs. 1-36b and 1-37.)

Solution 1.5-3

DATA

$$P = 44.6 \text{ kip} \quad L = 6 \text{ ft}$$

$$d = 1.375 \text{ in.} \quad E = 10.6(10^6) \text{ psi}$$

NORMAL STRESS IN BAR

$$\sigma_B = \frac{P}{\frac{\pi}{4}d^2} = 30036 \text{ psi}$$

from curve, say that $\varepsilon_B = 0.025$

ELASTIC RECOVERY unloading parallel to initial straight line

$$\varepsilon_E = \frac{\sigma_B}{E} = 2.834 \times 10^{-3}$$

RESIDUAL STRAIN

$$\varepsilon_R = \varepsilon_B - \varepsilon_E = 0.022$$

- (a) PERMANENT SET

$$\varepsilon_R L = 1.596 \text{ in.}$$

- (b) PROPORTIONAL LIMIT WHEN RELOADED IS $\sigma_R = 30$ ksi

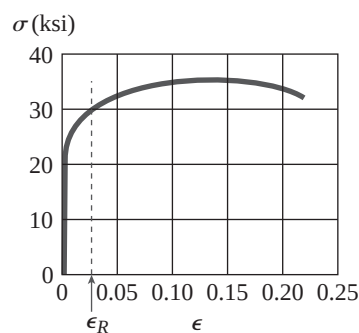


FIG 1-31 Typical stress-strain diagram for an aluminum alloy

Problem 1.5-4 A circular bar of magnesium alloy is 750 mm long. The stress-strain diagram for the material is shown in the figure. The bar is loaded in tension to an elongation of 6.0 mm, and then the load is removed.

- (a) What is the permanent set of the bar?
 (b) If the bar is reloaded, what is the proportional limit? (*Hint:* Use the concepts illustrated in Figs. 1-36b and 1-37.)

Solution 1.5-4

numerical data $L = 750 \text{ mm}$ $\delta = 6 \text{ mm}$

$$E_t(\epsilon) = \frac{d}{d\epsilon} \frac{\alpha \epsilon}{1 + \beta \epsilon} \quad E_t(0) = 41000 \text{ MPa}$$

(or 41 GPa > magnesium alloy)

$$\epsilon_B = \frac{\delta}{L} = 8 \times 10^{-3} \quad \sigma_B 65.6 \text{ MPa} < \text{from curve}$$

(see figure)

$$\epsilon_E = 0.0023 < \text{elastic recovery (see figure)}$$

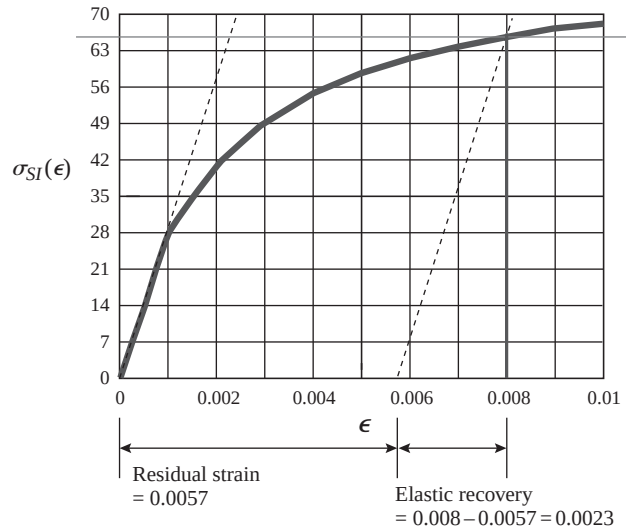
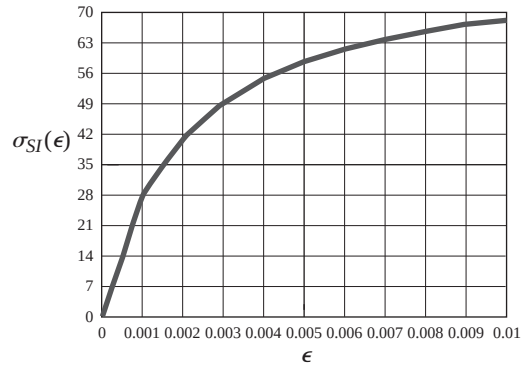
$$\epsilon_R = \epsilon_B - \epsilon_E = 5.7 \times 10^{-3} < \text{residual strain}$$

(a) PERMANENT SET

$$\delta_{pset} = \epsilon_R L = 4.275 \quad \boxed{\delta_{pset} = 4.28 \text{ mm}}$$

(b) PROPORTIONAL LIMIT WHEN RELOADED

$$\boxed{\sigma_B = 65.6 \text{ MPa}}$$



Problem 1.5-5 A wire of length $L = 4 \text{ ft}$ and diameter $d = 0.125 \text{ in.}$ is stretched by tensile forces $P = 600 \text{ lb.}$ The wire is made of a copper alloy having a stress-strain relationship that may be described mathematically by the following equation:

$$\sigma = \frac{18,000\epsilon}{1 + 300\epsilon} \quad 0 \leq \epsilon \leq 0.03 \quad (\sigma = \text{ksi})$$

in which ϵ is nondimensional and σ has units of kips per square inch (ksi).

- (a) Construct a stress-strain diagram for the material.
 (b) Determine the elongation of the wire due to the forces P .
 (c) If the forces are removed, what is the permanent set of the bar?
 (d) If the forces are applied again, what is the proportional limit?

Solution 1.5-5 Wire stretched by forces P

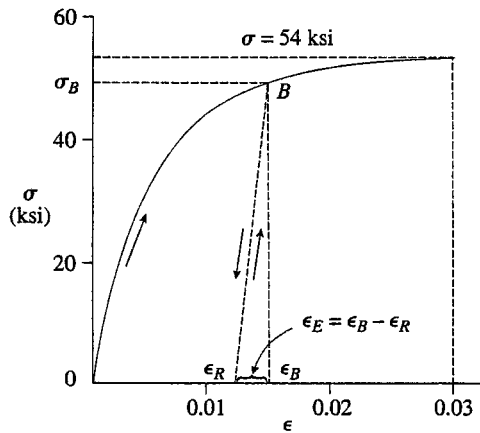
$$L = 4 \text{ ft} = 48 \text{ in.} \quad d = 0.125 \text{ in.}$$

$$P = 600 \text{ lb}$$

COPPER ALLOY

$$\sigma = \frac{18,000\varepsilon}{1 + 300\varepsilon} \quad 0 \leq \varepsilon \leq 0.03 \quad (\sigma = \text{ksi}) \quad (\text{Eq. 1})$$

(a) STRESS-STRAIN DIAGRAM (From Eq. 1)



INITIAL SLOPE OF STRESS-STRAIN CURVE

Take the derivative of σ with respect to ε :

$$\begin{aligned} \frac{d\sigma}{d\varepsilon} &= \frac{(1 + 300\varepsilon)(18,000) - (18,000)(300)\varepsilon}{(1 + 300\varepsilon)^2} \\ &= \frac{18,000}{(1 + 300\varepsilon)^2} \end{aligned}$$

$$\text{At } \varepsilon = 0, \quad \frac{d\sigma}{d\varepsilon} = 18,000 \text{ ksi}$$

$$\therefore \text{Initial slope} = 18,000 \text{ ksi}$$

ALTERNATIVE FORM OF THE STRESS-STRAIN RELATIONSHIP

Solve Eq. (1) for ε in terms of σ :

$$\varepsilon = \frac{\sigma}{18,000 - 300\sigma} \quad 0 \leq \sigma \leq 54 \text{ ksi} \quad (\sigma = \text{ksi}) \quad (\text{Eq. 2})$$

This equation may also be used when plotting the stress-strain diagram.

(b) ELONGATION δ OF THE WIRE

$$\sigma = \frac{P}{A} = \frac{600 \text{ lb}}{\frac{\pi}{4}(0.125 \text{ in.})^2} = 48,900 \text{ psi} = 48.9 \text{ ksi}$$

From Eq. (2) or from the stress-strain diagram:

$$\varepsilon = 0.0147$$

$$\delta = \varepsilon L = (0.0147)(48 \text{ in.}) = 0.71 \text{ in.} \quad \leftarrow$$

STRESS AND STRAIN AT POINT B (see diagram)

$$\sigma_B = 48.9 \text{ ksi} \quad \varepsilon_B = 0.0147$$

ELASTIC RECOVERY ε_E

$$\varepsilon_E = \frac{\sigma_B}{\text{Slope}} = \frac{48.9 \text{ ksi}}{18,000 \text{ ksi}} = 0.00272$$

RESIDUAL STRAIN ε_R

$$\varepsilon_R = \varepsilon_B - \varepsilon_E = 0.0147 - 0.0027 = 0.0120$$

$$\begin{aligned} \text{(c) Permanent set} &= \varepsilon_R L = (0.0120)(48 \text{ in.}) \\ &= 0.58 \text{ in.} \quad \leftarrow \end{aligned}$$

(d) Proportional limit when reloaded = σ_B

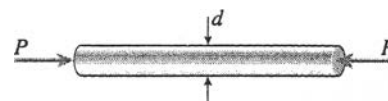
$$\sigma_B = 49 \text{ ksi} \quad \leftarrow$$

Linear Elasticity, Hooke's Law, and Poisson's Ratio

When solving the problems for Section 1.5, assume that the material behaves linearly elastically.

Problem 1.6-1 A high-strength steel bar used in a large crane has diameter $d = 2.00$ in. (see figure). The steel has modulus of elasticity $E = 29 \times 10^6$ psi and Poisson's ratio $\nu = 0.29$. Because of clearance requirements, the diameter of the bar is limited to 2.001 in. when it is compressed by axial forces.

What is the largest compressive load $P_{\max}$ that is permitted?



Solution 1.6-1 Steel bar in compression

STEEL BAR

$$d = 2.00 \text{ in.} \quad \text{Maximum } \Delta d = 0.001 \text{ in.}$$

$$E = 29 \times 10^6 \text{ psi} \quad \nu = 0.29$$

LATERAL STRAIN

$$\epsilon' = \frac{\Delta d}{d} = \frac{0.001 \text{ in.}}{2.00 \text{ in.}} = 0.0005$$

AXIAL STRAIN

$$\epsilon = -\frac{\epsilon'}{\nu} = -\frac{0.0005}{0.29} = -0.001724$$

(shortening)

AXIAL STRESS

$$\begin{aligned} \sigma &= E\epsilon = (29 \times 10^6 \text{ psi})(-0.001724) \\ &= -50.00 \text{ ksi (compression)} \end{aligned}$$

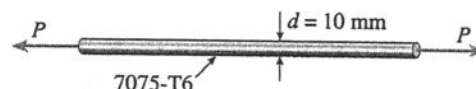
Assume that the yield stress for the high-strength steel is greater than 50 ksi. Therefore, Hooke's law is valid.

MAXIMUM COMPRESSIVE LOAD

$$\begin{aligned} P_{\max} &= \sigma A = (50.00 \text{ ksi})\left(\frac{\pi}{4}\right)(2.00 \text{ in.})^2 \\ &= 157 \text{ k} \quad \leftarrow \end{aligned}$$

Problem 1.6-2 A round bar of 10 mm diameter is made of aluminum alloy 7075-T6 (see figure). When the bar is stretched by axial forces P , its diameter decreases by 0.016 mm.

Find the magnitude of the load P . (Obtain the material properties from Appendix I.)



Solution 1.6-2 Aluminum bar in tension

$$d = 10 \text{ mm} \quad \Delta d = 0.016 \text{ mm}$$

(Decrease in diameter)

7075-T6

From Table I-2: $E = 72 \text{ GPa}$ $\nu = 0.33$

From Table I-3: Yield stress $\sigma_Y = 480 \text{ MPa}$

LATERAL STRAIN

$$\epsilon' = \frac{\Delta d}{d} = \frac{-0.016 \text{ mm}}{10 \text{ mm}} = -0.0016$$

AXIAL STRAIN

$$\begin{aligned} \epsilon &= -\frac{\epsilon'}{\nu} = \frac{0.0016}{0.33} \\ &= 0.004848 \text{ (Elongation)} \end{aligned}$$

AXIAL STRESS

$$\begin{aligned} \sigma &= E\epsilon = (72 \text{ GPa})(0.004848) \\ &= 349.1 \text{ MPa (Tension)} \end{aligned}$$

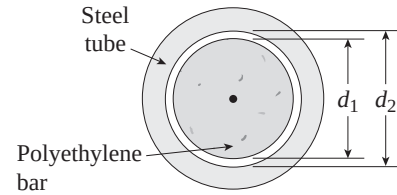
Because $\sigma < \sigma_Y$, Hooke's law is valid.

LOAD P (TENSILE FORCE)

$$\begin{aligned} P &= \sigma A = (349.1 \text{ MPa})\left(\frac{\pi}{4}\right)(10 \text{ mm})^2 \\ &= 27.4 \text{ kN} \quad \leftarrow \end{aligned}$$

Problem 1.6-3 A polyethylene bar having diameter $d_1 = 4.0$ in. is placed inside a steel tube having inner diameter $d_2 = 4.01$ in. (see figure). The polyethylene bar is then compressed by an axial force P .

At what value of the force P will the space between the nylon bar and the steel tube be closed? (For nylon, assume $E = 400$ ksi and $\nu = 0.4$.)



Solution 1.6-3

NUMERICAL DATA

$$\begin{aligned} d_1 &= 4 \text{ in.} & d_2 &= 4.01 \text{ in.} & E &= 200 \text{ ksi} \\ \nu &= 0.4 & \Delta d_1 &= 0.01 \text{ in.} \\ A_1 &= \frac{\pi}{4} d_1^2 & A_2 &= \frac{\pi}{4} d_2^2 & A_1 &= 12.566 \text{ in.}^2 \\ A_2 &= 12.629 \text{ in.}^2 \end{aligned}$$

LATERAL STRAIN

$$\varepsilon_p = \frac{\Delta d_1}{d_1} \quad \varepsilon_p = \frac{0.01}{4} \quad \varepsilon_p = 2.5 \times 10^{-3}$$

NORMAL STRAIN

$$\varepsilon_1 = \frac{-\varepsilon_p}{\nu} \quad \varepsilon_1 = -6.25 \times 10^{-3}$$

AXIAL STRESS

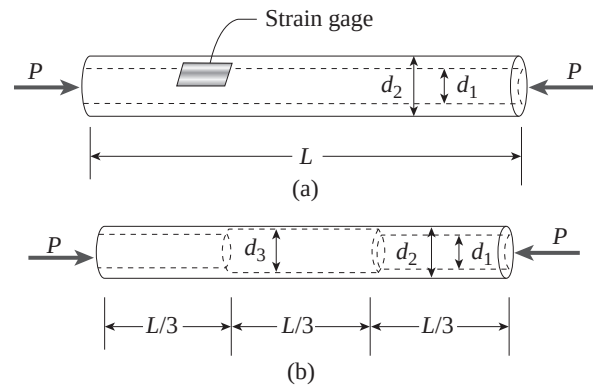
$$\sigma_1 = E \varepsilon_1 \quad \sigma_1 = -1.25 \text{ ksi}$$

COMPRESSION FORCE

$$\begin{aligned} P &= EA_1 \varepsilon_1 \\ P &= -15.71 \text{ kips} \quad \leftarrow \end{aligned}$$

Problem 1.6-4 A circular aluminum tube of length $L = 600$ mm is loaded in compression by forces P (see figure). The outside and inside diameters are $d_2 = 75$ mm and $d_1 = 63$ mm, respectively. A strain gage is placed on the outside of the tube to measure normal strains in the longitudinal direction. Assume that $E = 73$ GPa and Poisson's ratio $\nu = 0.33$.

- If the compressive stress in the tube is 57 MPa, what is the load P ?
- If the measured strain is $\varepsilon = 781 \times 10^{-6}$, what is the shortening δ of the tube? What is the percent change in its cross-sectional area? What is the volume change of the tube?
- If the tube has a constant outer diameter of $d_2 = 75$ mm along its entire length L but now has increased inner diameter d_3 with a normal stress of 70 MPa over the middle third, while the rest of the tube remains at normal stress of 57 MPa, what is the diameter d_3 ?



Solution 1.6-4(a) GIVEN STRESS, FIND FORCE P IN BAR FIGURE (A)

$$\sigma = 57 \text{ MPa} \quad E = 73 \text{ GPa} \quad \nu = 0.33$$

$$L = 600 \text{ mm} \quad d_2 = 75 \text{ mm} \quad d_1 = 63 \text{ mm}$$

$$A = \frac{\pi}{4} (d_2^2 - d_1^2) = 1301 \text{ mm}^2$$

$$P = \sigma A = 74.1 \text{ kN}$$

(b) GIVEN STRAIN, FIND CHANGE IN LENGTH IN BAR FIGURE (A) AND ALSO VOLUME CHANGE

$$\varepsilon = -781 (10^{-6}) \quad t = \frac{d_2 - d_1}{2} = 6 \text{ mm}$$

$$\delta = \varepsilon L = -0.469 \text{ mm} \quad \text{shortening}$$

$$\text{Vol}_1 = L(A) = 7.804 \times 10^5 \text{ mm}^3$$

$$\varepsilon_{\text{lat}} = -\nu \varepsilon = 2.577 \times 10^{-4} \quad \Delta t = \varepsilon_{\text{lat}} t = 1.546 \times 10^{-3} \text{ mm}$$

$$\Delta d_2 = \varepsilon_{\text{lat}} d_2 = 0.019 \text{ mm} \quad \Delta d_1 = \varepsilon_{\text{lat}} d_1 = 0.016 \text{ mm}$$

$$A_f = \frac{\pi}{4} \left[(d_2 + \Delta d_2)^2 - (d_1 + \Delta d_1)^2 \right]$$

$$A_f = 1301.29 \text{ mm}^2$$

$$\frac{A_f - A}{A} = \frac{1301.29 - 1300.62}{1300.62} = 0.052\%$$

$$V_{1f} = (L + \delta) (A_f) = 7.802 \times 10^5 \text{ mm}^3$$

$$\Delta V_1 = V_{1f} - \text{Vol}_1 = -207.482 \text{ mm}^3$$

$$\Delta V_1 = -207 \text{ mm}^3 \quad \text{change}$$

(c) IF THE TUBE HAS CONSTANT OUTER DIAMETER OF $d_2 = 75 \text{ mm}$ ALONG ITS ENTIRE LENGTH L BUT NOW HAS INCREASED INNER DIAMETER d_3 OVER THE MIDDLE THIRD WITH NORMAL STRESS OF 70 MPa , WHILE THE REST OF THE BAR REMAINS AT NORMAL STRESS OF 57 MPa , WHAT IS THE DIAMETER d_3 ?

$$\sigma_{M3} = 70 \text{ MPa} \quad P = 74.135 \text{ kN} \quad A_{M3} = \frac{P}{\sigma_{M3}} = 1059.076 \text{ mm}^2 \quad d_2 = 75 \text{ mm} \quad d_1 = 63 \text{ mm}$$

$$d_2^2 - d_3^2 = \frac{4}{\pi} A_{M3} \quad \text{SO} \quad d_3 = \sqrt{d_2^2 - \frac{4}{\pi} A_{M3}} = 65.4 \text{ mm} \quad t_{M3} = \frac{d_2 - d_3}{2} = 4.802 \text{ mm}$$

$$d_3 = 65.4 \text{ mm}$$

Problem 1.6-5 A bar of monel metal as in the figure (length $L = 9 \text{ in.}$, diameter $d = 0.225 \text{ in.}$) is loaded axially by a tensile force P . If the bar elongates by 0.0195 in. , what is the decrease in diameter d ? What is the magnitude of the load P ? Use the data in Table I-2, Appendix I.

Solution 1.6-5

NUMERICAL DATA

$$E = 25,000 \text{ ksi}$$

$$\nu = 0.32$$

$$L = 9 \text{ in.}$$

$$\delta = 0.0195 \text{ in.}$$

$$d = 0.225 \text{ in.}$$

NORMAL STRAIN

$$\epsilon = \frac{\delta}{L} \quad \epsilon = 2.167 \times 10^{-3}$$

LATERAL STRAIN

$$\epsilon_p = -\nu\epsilon \quad \epsilon_p = -6.933 \times 10^{-4}$$

DECREASE IN DIAMETER

$$\Delta d = \epsilon_p d$$

$$\Delta d = -1.56 \times 10^{-4} \text{ in.} \quad \leftarrow$$

INITIAL CROSS SECTIONAL AREA

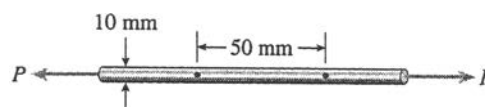
$$A_i = \frac{\pi}{4} d^2 \quad A_i = 0.04 \text{ in.}^2$$

MAGNITUDE OF LOAD P

$$P = EA_i \epsilon$$

$$P = 2.15 \text{ kips} \quad \leftarrow$$

Problem 1.6-6 A tensile test is performed on a brass specimen 10 mm in diameter using a gage length of 50 mm (see figure). When the tensile load P reaches a value of 20 kN, the distance between the gage marks has increased by 0.122 mm.



- (a) What is the modulus of elasticity E of the brass?
 (b) If the diameter decreases by 0.00830 mm, what is Poisson's ratio?

Solution 1.6-6 Brass specimen in tension

$$d = 10 \text{ mm} \quad \text{Gage length } L = 50 \text{ mm}$$

$$P = 20 \text{ kN} \quad \delta = 0.122 \text{ mm} \quad \Delta d = 0.00830 \text{ mm}$$

AXIAL STRESS

$$\sigma = \frac{P}{A} = \frac{20 \text{ k}}{\frac{\pi}{4} (10 \text{ mm})^2} = 254.6 \text{ MPa}$$

Assume σ is below the proportional limit so that Hooke's law is valid.

AXIAL STRAIN

$$\epsilon = \frac{\delta}{L} = \frac{0.122 \text{ mm}}{50 \text{ mm}} = 0.002440$$

(a) MODULUS OF ELASTICITY

$$E = \frac{\sigma}{\epsilon} = \frac{254.6 \text{ MPa}}{0.002440} = 104 \text{ GPa} \quad \leftarrow$$

(b) POISSON'S RATIO

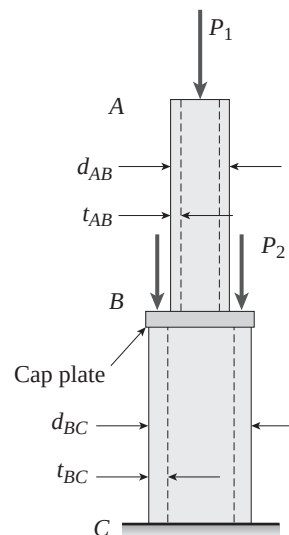
$$\epsilon' = \nu\epsilon$$

$$\Delta d = \epsilon' d = \nu\epsilon d$$

$$\nu = \frac{\Delta d}{\epsilon d} = \frac{0.00830 \text{ mm}}{(0.002440)(10 \text{ mm})} = 0.34 \quad \leftarrow$$

Problem 1.6-7 A hollow, brass circular pipe ABC (see figure) supports a load $P_1 = 26.5$ kips acting at the top. A second load $P_2 = 22.0$ kips is uniformly distributed around the cap plate at B. The diameters and thicknesses of the upper and lower parts of the pipe are $d_{AB} = 1.25$ in., $t_{AB} = 0.5$ in., $d_{BC} = 2.25$ in., and $t_{BC} = 0.375$ in., respectively. The modulus of elasticity is 14,000 ksi. When both loads are fully applied, the wall thickness of pipe BC increases by 200×10^{-6} in.

- Find the increase in the inner diameter of pipe segment BC.
- Find Poisson's ratio for the brass.
- Find the increase in the wall thickness of pipe segment AB and the increase in the inner diameter of AB.



Solution 1.6-7

NUMERICAL DATA

$$\begin{aligned} P_1 &= 26.5 \text{ k} \\ P_2 &= 22 \text{ k} \\ d_{AB} &= 1.25 \text{ in.} \\ t_{AB} &= 0.5 \text{ in.} \\ d_{BC} &= 2.25 \text{ in.} \\ t_{BC} &= 0.375 \text{ in.} \\ E &= 14000 \text{ ksi} \\ \Delta t_{BC} &= 200 \times 10^{-6} \end{aligned}$$

- (a) INCREASE IN THE INNER DIAMETER OF PIPE SEGMENT BC

$$\epsilon_{pBC} = \frac{\Delta t_{BC}}{t_{BC}} \quad \epsilon_{pBC} = 5.333 \times 10^{-4}$$

$$\begin{aligned} \Delta d_{BC\text{inner}} &= \epsilon_{pBC}(d_{BC} - 2t_{BC}) \\ \Delta d_{BC\text{inner}} &= 8 \times 10^{-4} \text{ in.} \quad \leftarrow \end{aligned}$$

- (b) POISSON'S RATIO FOR THE BRASS

$$A_{BC} = \frac{\pi}{4} [d_{BC}^2 - (d_{BC} - 2t_{BC})^2]$$

$$A_{BC} = 2.209 \text{ in.}^2$$

$$\epsilon_{BC} = \frac{-(P_1 + P_2)}{EA_{BC}} \quad \epsilon_{BC} = -1.568 \times 10^{-3}$$

$$\nu_{\text{brass}} = \frac{-\epsilon_{pBC}}{\epsilon_{BC}} \quad \nu_{\text{brass}} = 0.34$$

(agrees with App. I (Table I-2))

- (c) INCREASE IN THE WALL THICKNESS OF PIPE SEGMENT AB AND THE INCREASE IN THE INNER DIAMETER OF AB

$$A_{AB} = \frac{\pi}{4} [d_{AB}^2 - (d_{AB} - 2t_{AB})^2]$$

$$\epsilon_{AB} = \frac{-P_1}{EA_{AB}} \quad \epsilon_{AB} = -1.607 \times 10^{-3}$$

$$\epsilon_{pAB} = -\nu_{\text{brass}} \epsilon_{AB} \quad \epsilon_{pAB} = 5.464 \times 10^{-4}$$

$$\Delta t_{AB} = \epsilon_{pAB} t_{AB} \quad \Delta t_{AB} = 2.73 \times 10^{-4} \text{ in.} \quad \leftarrow$$

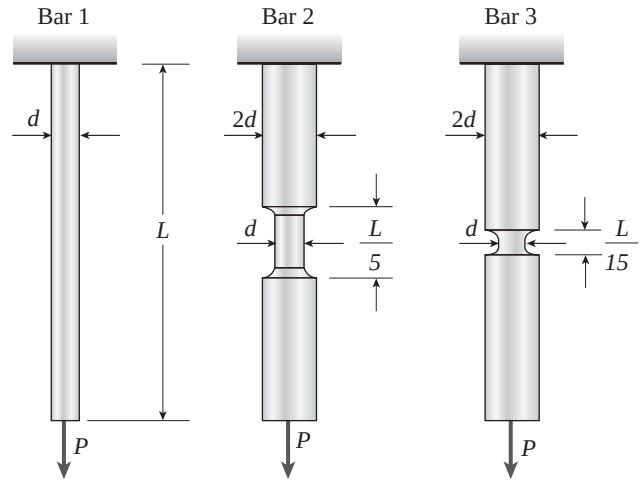
$$\Delta d_{AB\text{inner}} = \epsilon_{pAB}(d_{AB} - 2t_{AB})$$

$$\Delta d_{AB\text{inner}} = 1.366 \times 10^{-4} \text{ in.}$$

Problem 1.6-8 Three round, copper alloy bars having the same length L but different shapes are shown in the figure. The first bar has a diameter d over its entire length, the second has a diameter d over one-fifth of its length, and the third has a diameter d over one-fifteenth of its length. Elsewhere, the second and third bars have diameter $2d$. All three bars are subjected to the same axial load P .

Use the following numerical data: $P = 1400$ kN, $L = 5$ m, $d = 80$ mm, $E = 110$ GPa, and $\nu = 0.33$.

- Find the change in length of each bar.
- Find the change in volume of each bar.



Solution 1.6-8

$$P = 1400 \text{ kN} \quad L = 5 \text{ m} \quad d = 80 \text{ mm} \quad E = 110 \text{ GPa} \quad \nu = 0.33$$

$$A_d = \frac{\pi}{4} d^2 = 5026.5 \text{ mm}^2 \quad A_{2d} = \frac{\pi}{4} (2d)^2 = 20,106.2 \text{ mm}^2$$

(a) FIND CHANGE IN LENGTH OF EACH BAR

BAR #1 $\epsilon_1 = \frac{P}{EA_d} = 2.532 \times 10^{-3}$ $\sigma_1 = E\epsilon_1 = 279 \text{ MPa}$ $< \text{Appendix I, Table I-3: copper alloys can have yield stress in range 55–760 MPa so assume this is below proportional limit so that Hooke's Law applies}$

$$\Delta L_1 = \epsilon_1 L = 12.66 \text{ mm} \quad L_{f1} = L + \Delta L_1 = 5012.66 \text{ mm}$$

BAR #2 $\epsilon_{2a} = \frac{P}{EA_d} = 2.532 \times 10^{-3}$ $\epsilon_{2b} = \frac{P}{EA_{2d}} = 6.33 \times 10^{-4}$ $\frac{\epsilon_{2a}}{4} = 6.33 \times 10^{-4}$

$$\Delta L_{2a} = \epsilon_{2a} \frac{L}{5} = 2.532 \text{ mm} \quad \Delta L_{2b} = \epsilon_{2b} \left(\frac{4L}{5} \right) = 2.532 \text{ mm}$$

$$\Delta L_2 = \Delta L_{2a} + \Delta L_{2b} = 5.06 \text{ mm} \quad L_{f2} = L + \Delta L_2 = 5005.06 \text{ mm} \quad \frac{\Delta L_2}{\Delta L_1} = 0.4$$

BAR #3

$$\Delta L_{3a} = \epsilon_{2a} \frac{L}{15} = 0.844 \text{ mm} \quad \Delta L_{3b} = \epsilon_{2b} \left(\frac{14L}{15} \right) = 2.954 \text{ mm}$$

$$\Delta L_3 = \Delta L_{3a} + \Delta L_{3b} = 3.8 \text{ mm} \quad L_{f3} = L + \Delta L_3 = 5003.08 \text{ mm} \quad \frac{\Delta L_3}{\Delta L_1} = 0.3$$

(b) FIND CHANGE IN VOLUME OF EACH BAR

Use lateral strain (ϵ_p) in each segment to find change in diameter Δd , then find change in cross sectional area, then volume

BAR #1

$$\epsilon_{p1} = -\nu\epsilon_1 = -8.356 \times 10^{-4} \quad \Delta d_1 = \epsilon_{p1}d = -0.067 \text{ mm} \quad A_1 = \frac{\pi}{4}(d + \Delta d_1)^2 = 5018.152 \text{ mm}^2$$

$$\Delta \text{Vol}_1 = A_1 L_{f1} - A_d L = 21548 \text{ mm}^3 \quad \frac{\Delta \text{Vol}_1}{A_d L} = 8.574 \times 10^{-4}$$

BAR #2

$$\varepsilon_{p2a} = \varepsilon_{p1} \quad \varepsilon_{p2b} = -\nu \varepsilon_{2b} = -2.089 \times 10^{-4} \quad \frac{\varepsilon_{p1}}{4} = -2.089 \times 10^{-4}$$

$$\Delta d_{2b} = \varepsilon_{p2b}(2d) = -0.33 \text{ mm} \quad A_{2a} = A_1 \quad A_{2b} = \frac{\pi}{4}(2d + \Delta d_{2b})^2 = 20097.794 \text{ mm}^2$$

$$\Delta L_{2a} = \varepsilon_{2a} \frac{L}{5} = 2.532 \text{ mm} \quad \Delta L_{2b} = \varepsilon_{2b} \left(\frac{4L}{5} \right) = 2.532 \text{ mm}$$

$$\Delta \text{Vol}_2 = \left[A_1 \left(\frac{L}{5} + \Delta L_{2a} \right) + A_{2b} \left(\frac{4L}{5} + \Delta L_{2b} \right) \right] - \left[A_{2d} \left(\frac{4L}{5} \right) + A_d \left(\frac{L}{5} \right) \right]$$

$$= 21601 \text{ mm}^3 \quad \frac{\Delta \text{Vol}_2}{\Delta \text{Vol}_1} = 1.002$$

BAR #3

$$\Delta L_{2a} = \varepsilon_{2a} \frac{L}{15} = 0.844 \text{ mm} \quad \Delta L_{2b} = \varepsilon_{2b} \left(\frac{14L}{15} \right) = 2.954 \text{ mm}$$

$$\Delta \text{Vol}_3 = \left[A_1 \left(\frac{L}{15} + \Delta L_{2a} \right) + A_{2b} \left(\frac{14L}{15} + \Delta L_{2b} \right) \right] - \left[A_{2d} \left(\frac{14L}{15} \right) + A_d \left(\frac{L}{15} \right) \right]$$

$$= 21610 \text{ mm}^3 \quad \frac{\Delta \text{Vol}_3}{\Delta \text{Vol}_2} = 1.003$$

$$\Delta \text{Vol}_1 = 21548 \text{ mm}^3$$

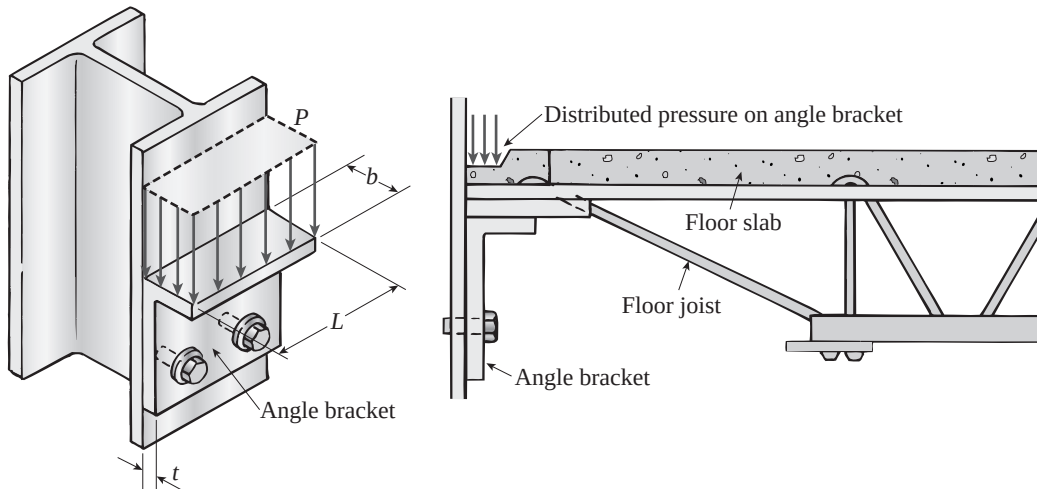
$$\Delta \text{Vol}_2 = 21601 \text{ mm}^3$$

$$\Delta \text{Vol}_3 = 21610 \text{ mm}^3$$

Shear Stress and Strain

Problem 1.7-1 An angle bracket having thickness $t = 0.75$ in. is attached to the flange of a column by two 5/8-inch diameter bolts (see figure). A uniformly distributed load from a floor joist acts on the top face of the bracket with a pressure $p = 275$ psi. The top face of the bracket has length $L = 8$ in. and width $b = 3.0$ in.

Determine the average bearing pressure σ_b between the angle bracket and the bolts and the average shear stress τ_{aver} in the bolts. (Disregard friction between the bracket and the column.)



Solution 1.7-1

NUMERICAL DATA

$$t = 0.75 \text{ in.} \quad L = 8 \text{ in.}$$

$$b = 3 \text{ in.} \quad p = \frac{275}{1000} \text{ ksi} \quad d = \frac{5}{8} \text{ in.}$$

BEARING FORCE

$$F = p b L \quad F = 6.6 \text{ k}$$

SHEAR AND BEARING AREAS

$$A_s = \frac{\pi}{4} d^2 \quad A_s = 0.307 \text{ in.}^2$$

$$A_b = d t \quad A_b = 0.469 \text{ in.}^2$$

BEARING STRESS

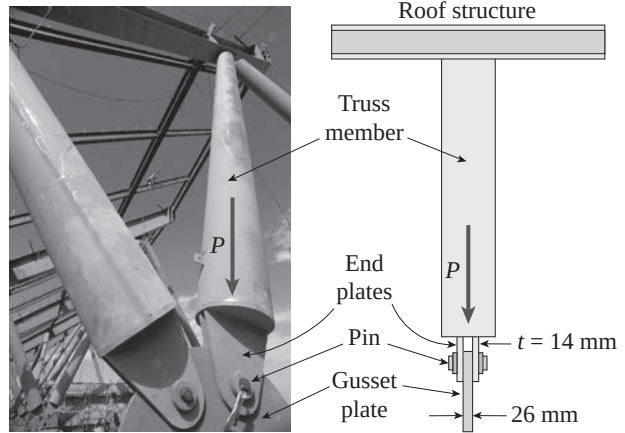
$$\sigma_b = \frac{F}{2A_b} \quad \sigma_b = 7.04 \text{ ksi} \quad \leftarrow$$

SHEAR STRESS

$$\tau_{\text{ave}} = \frac{F}{2A_s} \quad \tau_{\text{ave}} = 10.76 \text{ ksi} \quad \leftarrow$$

Problem 1.7-2 Truss members supporting a roof are connected to a 26-mm-thick gusset plate by a 22-mm diameter pin as shown in the figure and photo. The two end plates on the truss members are each 14 mm thick.

- (a) If the load $P = 80 \text{ kN}$, what is the largest bearing stress acting on the pin?
 (b) If the ultimate shear stress for the pin is 190 MPa , what force P_{ult} is required to cause the pin to fail in shear?
 (Disregard friction between the plates.)

**Solution 1.7-2**

NUMERICAL DATA

$$t_{ep} = 14 \text{ mm}$$

$$t_{gp} = 26 \text{ mm}$$

$$P = 80 \text{ kN}$$

$$d_p = 22 \text{ mm}$$

$$\tau_{\text{ult}} = 190 \text{ MPa}$$

(a) BEARING STRESS ON PIN

$$\sigma_b = \frac{P}{d_p t_{gp}} \quad \text{gusset plate is thinner than} \\ (2 t_{ep}) \text{ so gusset plate controls}$$

$$\sigma_b = 139.9 \text{ MPa} \quad \leftarrow$$

(b) ULTIMATE FORCE IN SHEAR

Cross sectional area of pin

$$A_p = \frac{\pi d_p^2}{4}$$

$$A_p = 380.133 \text{ mm}^2$$

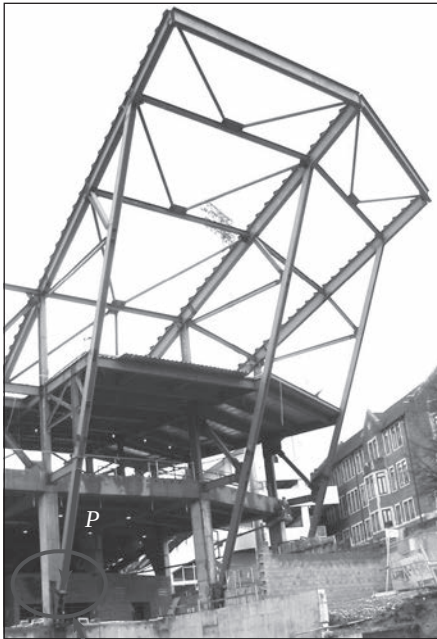
$$P_{\text{ult}} = 2\tau_{\text{ult}}A_p \quad P_{\text{ult}} = 144.4 \text{ kN} \quad \leftarrow$$

Problem 1.7-3 The upper deck of a football stadium is supported by braces each of which transfers a load $P = 160$ kips to the base of a column [see figure part (a)]. A cap plate at the bottom of the brace distributes the load P to four flange plates ($t_f = 1$ in.) through a pin ($d_p = 2$ in.) to two gusset plates ($t_g = 1.5$ in.) [see figure parts (b) and (c)].

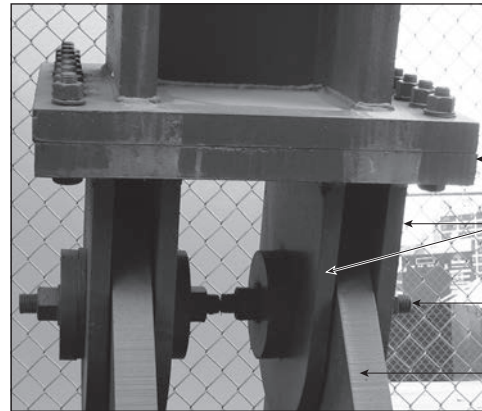
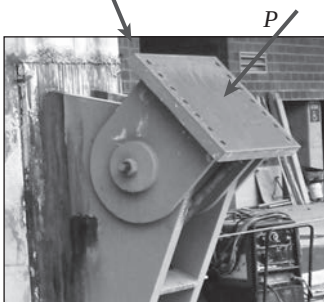
Determine the following quantities.

- The average shear stress τ_{aver} in the pin.
- The average bearing stress between the flange plates and the pin (σ_{bf}), and also between the gusset plates and the pin (σ_{bg}).

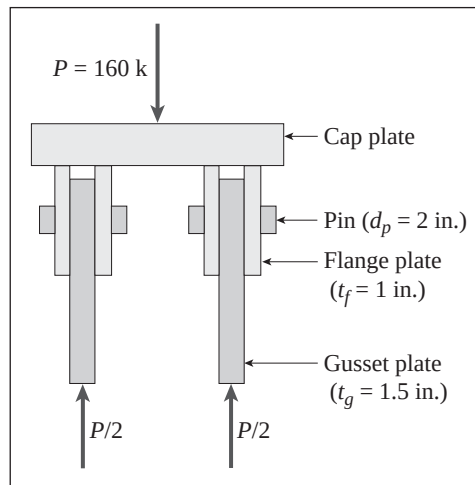
(Disregard friction between the plates.)



(a) Stadium brace



(b) Detail at bottom of brace



(c) Section through bottom of brace

Solution 1.7-3

NUMERICAL DATA

$$P = 160 \text{ kips} \quad d_p = 2 \text{ in.}$$

$$t_g = 1.5 \text{ in.} \quad t_f = 1 \text{ in.}$$

(a) SHEAR STRESS ON PIN

$$\tau = \frac{V}{\left(\frac{\pi d_p^2}{4}\right)} \quad \tau = \frac{\frac{P}{4}}{\left(\frac{\pi d_p^2}{4}\right)}$$

$$\tau = 12.73 \text{ ksi} \quad \leftarrow$$

(b) BEARING STRESS ON PIN FROM FLANGE PLATE

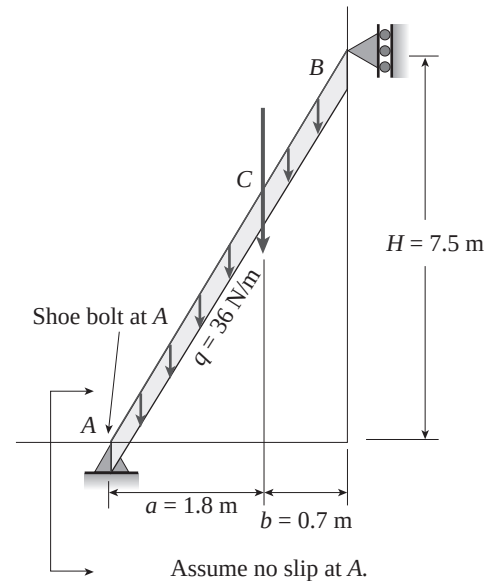
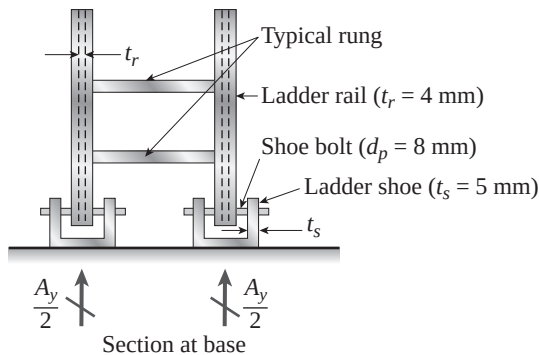
$$\sigma_{bf} = \frac{\frac{P}{4}}{d_p t_f} \quad \sigma_{bf} = 20 \text{ ksi} \quad \leftarrow$$

BEARING STRESS ON PIN FROM GUSSET PLATE

$$\sigma_{bg} = \frac{\frac{P}{2}}{d_p t_g} \quad \sigma_{bg} = 26.7 \text{ ksi} \quad \leftarrow$$

Problem 1.7-4 The inclined ladder AB supports a house painter (85 kg) at C and the self weight ($q = 40 \text{ N/m}$) of the ladder itself. Each ladder rail ($t_r = 4 \text{ mm}$) is supported by a shoe ($t_s = 5 \text{ mm}$) which is attached to the ladder rail by a bolt of diameter $d_p = 8 \text{ mm}$.

- Find support reactions at A and B .
- Find the resultant force in the shoe bolt at A .
- Find maximum average shear (τ) and bearing (σ_b) stresses in the shoe bolt at A .

**Solution 1.7-4**

NUMERICAL DATA

$$t_r = 4 \text{ mm} \quad t_s = 5 \text{ mm} \quad d_p = 8 \text{ mm} \quad P = 85(9.81) \quad P = 833.85 \text{ N}$$

$$a = 1.8 \text{ m} \quad b = 0.7 \text{ m} \quad H = 7.5 \text{ m} \quad q = 40 \text{ N/m}$$

(a) SUPPORT REACTIONS

$$L = \sqrt{(a + b)^2 + H^2} \quad L = 7.906 \text{ m} \quad L_{AC} = \frac{a}{a + b} L \quad L_{AC} = 5.692$$

$$L_{CB} = \frac{a}{a + b} L \quad L_{CB} = 2.214 \quad L_{AC} + L_{CB} = 7.906$$

SUM MOMENTS ABOUT A

$$B_x = \frac{Pa + qL\left(\frac{a+b}{2}\right)}{-H} \quad B_x = -252.829 \text{ N (left) and } A_x = -B_x (A_x \text{ acts to right}) \quad A_x = -B_x$$

$$A_y = P + qL \quad A_y = 1150.078 \text{ N} \quad \boxed{B_x = -252.8 \text{ N}} \quad \boxed{A_x = -B_x} \quad \boxed{A_y = 1150.1 \text{ N}}$$

(b) RESULTANT FORCE IN SHOE BOLT AT A $A_{\text{resultant}} = \sqrt{A_x^2 + A_y^2}$

$$A_{\text{resultant}} = 1177.54 \text{ N}$$

$$\boxed{A_{\text{resultant}} = 1178 \text{ N}}$$

(c) MAXIMUM SHEAR AND BEARING STRESSES IN SHOE BOLT AT A

$$d_p = 8 \text{ mm} \quad t_s = 5 \text{ mm} \quad t_r = 4 \text{ mm}$$

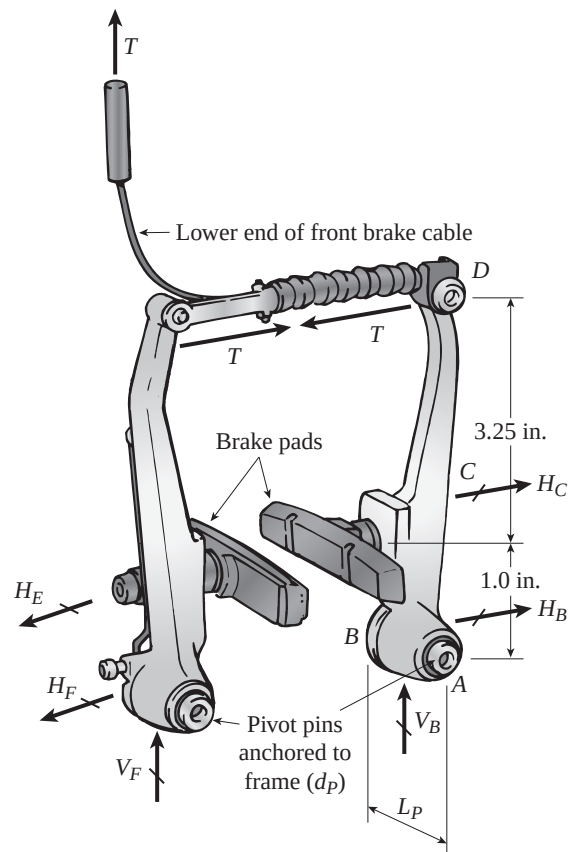
$$\text{Shear area: } A_s = \frac{\pi}{4} d_p^2 \quad A_s = 50.256 \text{ mm}^2 \quad \text{Shear stress: } \tau = \frac{\frac{A_{\text{resultant}}}{2}}{2A_s} \quad \boxed{\tau = 5.86 \text{ MPa}}$$

$$\text{Bearing area: } A_b = 2d_p t_s \quad A_b = 80 \text{ mm}^2 \quad \text{Bearing stress: } \sigma_{b\text{shoe}} = \frac{\frac{A_{\text{resultant}}}{2}}{A_b} \quad \boxed{\sigma_{b\text{shoe}} = 7.36 \text{ MPa}}$$

Problem 1.7-5 The force in the brake cable of the V-brake system shown in the figure is $T = 45 \text{ lb}$. The pivot pin at A has diameter $d_p = 0.25 \text{ in.}$ and length $L_p = 5/8 \text{ in.}$

Use dimensions shown in the figure. Neglect the weight of the brake system.

- Find the average shear stress τ_{aver} in the pivot pin where it is anchored to the bicycle frame at B.
- Find the average bearing stress $\sigma_{b,\text{aver}}$ in the pivot pin over segment AB.



Solution 1.7-5

NUMERICAL DATA

$$d_p = 0.25 \text{ in.} \quad L = \frac{5}{8} \text{ in.} \quad CD = 3.25 \text{ in.}$$

$$BC = 1 \text{ in.} \quad T = 45 \text{ lb}$$

EQUILIBRIUM - FIND HORIZONTAL FORCES
AT B AND C [VERTICAL REACTION $V_B = 0$]

$$\sum M_B = 0 \quad H_C = \frac{T(BC + CD)}{BC}$$

$$H_C = 191.25 \text{ lb} \quad \sum F_H = 0$$

$$H_B = T - H_C \quad H_B = -146.25 \text{ lb}$$

- (a) FIND THE AVE SHEAR STRESS τ_{ave} IN THE PIVOT PIN WHERE IT IS ANCHORED TO THE BICYCLE FRAME AT B:

$$A_S = \frac{\pi d_p^2}{4} \quad A_S = 0.049 \text{ in.}^2$$

$$\tau_{ave} = \frac{|H_B|}{A_S} \quad \tau_{ave} = 2979 \text{ psi} \quad \leftarrow$$

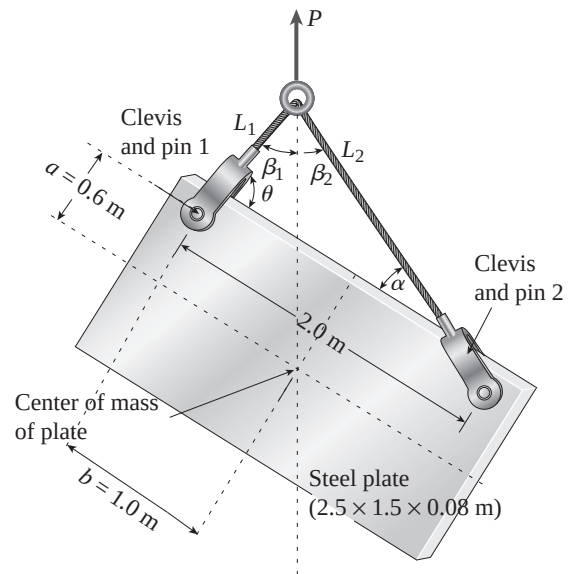
- (b) FIND THE AVE BEARING STRESS $\sigma_{b,ave}$ IN THE PIVOT PIN OVER SEGMENT AB.

$$A_b = d_p L \quad A_b = 0.156 \text{ in.}^2$$

$$\sigma_{b,ave} = \frac{|H_B|}{A_b} \quad \sigma_{b,ave} = 936 \text{ psi} \quad \leftarrow$$

Problem 1.7-6 A steel plate of dimensions $2.5 \times 1.5 \times 0.08 \text{ m}$ and weighing 23.1 kN is hoisted by steel cables with lengths $L_1 = 3.2 \text{ m}$ and $L_2 = 3.9 \text{ m}$ that are each attached to the plate by a clevis and pin (see figure). The pins through the clevises are 18 mm in diameter and are located 2.0 m apart. The orientation angles are measured to be $\theta = 94.4^\circ$ and $\alpha = 54.9^\circ$.

For these conditions, first determine the cable forces T_1 and T_2 , then find the average shear stress τ_{aver} in both pin 1 and pin 2, and then the average bearing stress σ_b between the steel plate and each pin. Ignore the mass of the cables.



Solution 1.7-6

NUMERICAL DATA

$$L_1 = 3.2 \text{ m} \quad L_2 = 3.9 \text{ m} \quad \alpha = 54.9 \left(\frac{\pi}{180} \right) \text{ rad}$$

$$\theta = 94.4 \left(\frac{\pi}{180} \right) \text{ rad}$$

$$a = 0.6 \text{ m} \quad b = 1.0 \text{ m}$$

$$W = 77.0(2.5 \times 1.5 \times 0.08) \quad W = 23.1 \text{ kN}$$

$$(77 = \text{wt density of steel, kN/m}^3)$$

SOLUTION APPROACH

$$\text{STEP (1)} \quad d = \sqrt{a^2 + b^2} \quad d = 1.166 \text{ m}$$

$$\text{STEP (2)} \quad \theta_1 = \arctan \left(\frac{a}{b} \right) \quad \theta_1 \frac{180}{\pi} = 30.964^\circ$$

STEP (3)-Law of cosines

$$H = \sqrt{d^2 + L_1^2 - 2dL_1 \cos(\theta + \theta_1)}$$

$$H = 3.99 \text{ m}$$

$$\text{STEP (4)} \quad \beta_1 = \arccos\left(\frac{L_2^2 + H^2 - d^2}{2L_1H}\right)$$

$$\beta_1 \frac{180}{\pi} = 13.789^\circ$$

$$\text{STEP (5)} \quad \beta_2 = \arccos\left(\frac{L_2^2 + H^2 - d^2}{2L_2H}\right)$$

$$\beta_2 \frac{180}{\pi} = 16.95^\circ$$

STEP (6)

$$\text{Check } (\beta_1 + \beta_2 + \theta + \alpha) \frac{180}{\pi}$$

$$= 180.039^\circ$$

STATICS

$$T_1 \sin(\beta_1) = T_2 \sin(\beta_2)$$

$$T_1 = T_2 \left(\frac{\sin(\beta_2)}{\sin(\beta_1)} \right)$$

$$T_1 \cos(\beta_1) + T_2 \cos(\beta_2) = W$$

$$T_2 = \frac{W}{\cos(\beta_1) \frac{\sin(\beta_2)}{\sin(\beta_1)} + \cos(\beta_2)}$$

$$T_2 = 10.77 \text{ kN} \quad \leftarrow$$

$$T_1 = T_2 \left(\frac{\sin(\beta_2)}{\sin(\beta_1)} \right) \quad T_1 = 13.18 \text{ kN} \quad \leftarrow$$

$$T_1 \cos(\beta_1) + T_2 \cos(\beta_2) = 23.1 < \text{checks}$$

SHEAR & BEARING STRESSES

$$d_p = 18 \text{ mm} \quad t = 80 \text{ mm}$$

$$A_S = \frac{\pi}{4} d_p^2 \quad A_b = t d_p$$

$$\tau_{1\text{ave}} = \frac{T_1}{A_S} \quad \tau_{1\text{ave}} = 25.9 \text{ MPa} \quad \leftarrow$$

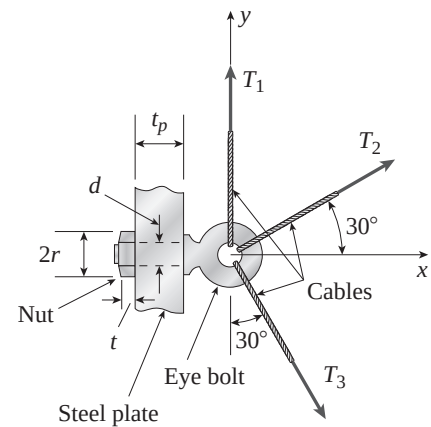
$$\tau_{2\text{ave}} = \frac{T_2}{A_S} \quad \tau_{2\text{ave}} = 21.2 \text{ MPa} \quad \leftarrow$$

$$\sigma_{b1} = \frac{T_1}{A_b} \quad \sigma_{b1} = 9.15 \text{ MPa} \quad \leftarrow$$

$$\sigma_{b2} = \frac{T_2}{A_b} \quad \sigma_{b2} = 7.48 \text{ MPa} \quad \leftarrow$$

Problem 1.7-7 A special-purpose eye bolt of shank diameter $d = 0.50$ in. passes through a hole in a steel plate of thickness $t_p = 0.75$ in. (see figure) and is secured by a nut with thickness $t = 0.25$ in. The hexagonal nut bears directly against the steel plate. The radius of the circumscribed circle for the hexagon is $r = 0.40$ in. (which means that each side of the hexagon has length 0.40 in.). The tensile forces in three cables attached to the eye bolt are $T_1 = 800$ lb., $T_2 = 550$ lb., and $T_3 = 1241$ lb.

- Find the resultant force acting on the eye bolt.
- Determine the average bearing stress σ_b between the hexagonal nut on the eye bolt and the plate.
- Determine the average shear stress τ_{aver} in the nut and also in the steel plate.



Solution 1.7-7

CABLE FORCES

$$T_1 = 800 \text{ lb} \quad T_2 = 550 \text{ lb} \quad T_3 = 1241 \text{ lb}$$

(a) RESULTANT

$$P = T_2 \frac{\sqrt{3}}{2} + T_3 0.5 \quad P = 1097 \text{ lb} \quad \leftarrow$$

(b) AVERAGE BEARING STRESS

$$A_b = 0.2194 \text{ in.}^2 \quad \text{hexagon (Case 25, Appendix E)}$$

$$\sigma_b = \frac{P}{A_b} \quad \sigma_b = 4999 \text{ psi} \quad \leftarrow$$

(c) AVERAGE SHEAR THROUGH NUT

$$d = 0.5 \text{ in.} \quad t = 0.25 \text{ in.}$$

$$A_{\text{sn}} = \pi dt \quad A_{\text{sn}} = 0 \quad \tau_{\text{nut}} = \frac{P}{A_{\text{sn}}}$$

$$\tau_{\text{nut}} = 2793 \text{ psi} \quad \leftarrow$$

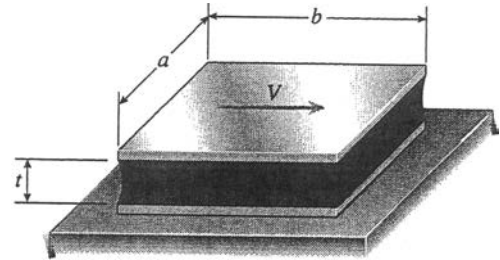
SHEAR THROUGH PLATE $t_p = 0.75 \quad r = 0.40$

$$A_{\text{spl}} = 6rt_p \quad A_{\text{spl}} = 2$$

$$\tau_{\text{pl}} = \frac{P}{A_{\text{spl}}} \quad \tau_{\text{pl}} = 609 \text{ psi} \quad \leftarrow$$

Problem 1.7-8 An elastomeric bearing pad consisting of two steel plates bonded to a chloroprene elastomer (an artificial rubber) is subjected to a shear force V during a static loading test (see figure). The pad has dimensions $a = 125 \text{ mm}$ and $b = 240 \text{ mm}$, and the elastomer has thickness $t = 50 \text{ mm}$. When the force V equals 12 kN , the top plate is found to have displaced laterally 8.0 mm with respect to the bottom plate.

What is the shear modulus of elasticity G of the chloroprene?

**Solution 1.7-8**

NUMERICAL DATA

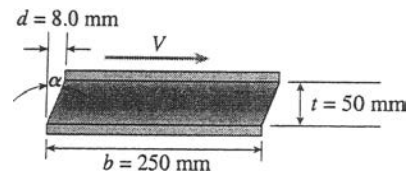
$$V = 12 \text{ kN} \quad a = 125 \text{ mm}$$

$$b = 240 \text{ mm} \quad t = 50 \text{ mm} \quad d = 8 \text{ mm}$$

AVERAGE SHEAR STRESS

$$\tau_{\text{ave}} = \frac{V}{ab} \quad \tau_{\text{ave}} = 0.4 \text{ MPa}$$

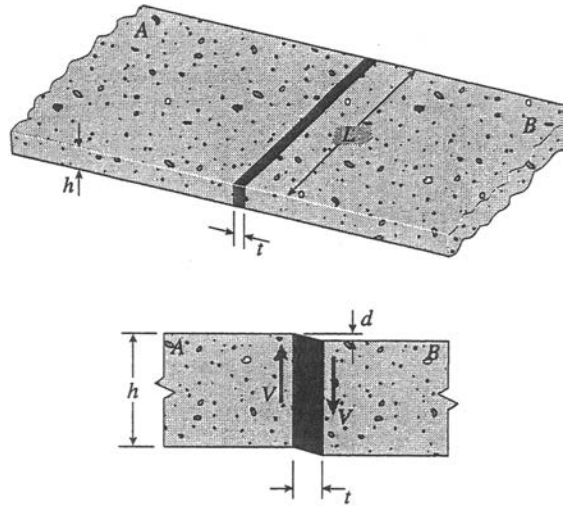
$$\text{AVERAGE SHEAR STRAIN} \quad \gamma_{\text{ave}} = \frac{d}{t} \quad \gamma_{\text{ave}} = 0.16$$



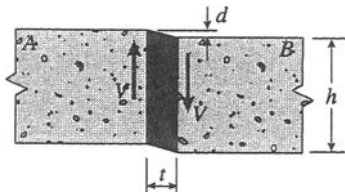
$$\text{SHEAR MODULUS } G \quad G = \frac{\tau_{\text{ave}}}{\gamma_{\text{ave}}} \\ G = 2.5 \text{ MPa} \quad \leftarrow$$

Problem 1.7-9 A joint between two concrete slabs *A* and *B* is filled with a flexible epoxy that bonds securely to the concrete (see figure). The height of the joint is $h = 4.0$ in., its length is $L = 40$ in., and its thickness is $t = 0.5$ in. Under the action of shear forces V , the slabs displace vertically through the distance $d = 0.002$ in. relative to each other.

- What is the average shear strain γ_{aver} in the epoxy?
- What is the magnitude of the forces V if the shear modulus of elasticity G for the epoxy is 140 ksi?



Solution 1.7-9 Epoxy joint between concrete slabs



$$\begin{aligned} h &= 4.0 \text{ in.} & t &= 0.5 \text{ in.} \\ L &= 40 \text{ in.} & d &= 0.002 \text{ in.} \\ G &= 140 \text{ ksi} \end{aligned}$$

- (a) AVERAGE SHEAR STRAIN

$$\gamma_{\text{aver}} = \frac{d}{t} = 0.004 \quad \leftarrow$$

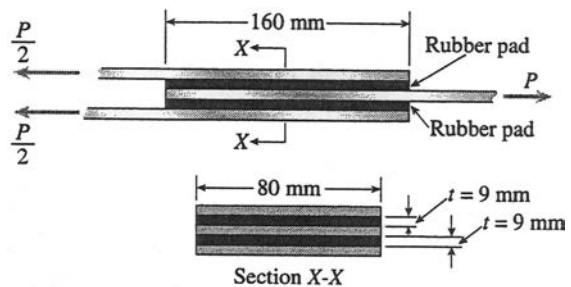
- (b) SHEAR FORCES V

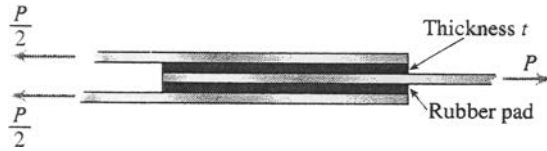
$$\text{Average shear stress: } \tau_{\text{aver}} = G\gamma_{\text{aver}}$$

$$\begin{aligned} V &= \tau_{\text{aver}}(hL) = G\gamma_{\text{aver}}(hL) \\ &= (140 \text{ ksi})(0.004)(4.0 \text{ in.})(40 \text{ in.}) \\ &= 89.6 \text{ k} \quad \leftarrow \end{aligned}$$

Problem 1.7-10 A flexible connection consisting of rubber pads (thickness $t = 9$ mm) bonded to steel plates is shown in the figure. The pads are 160 mm long and 80 mm wide.

- Find the average shear strain γ_{aver} in the rubber if the force $P = 16$ kN and the shear modulus for the rubber is $G = 1250$ kPa.
- Find the relative horizontal displacement δ between the interior plate and the outer plates.



Solution 1.7-10 Rubber pads bonded to steel plates

Rubber pads: $t = 9 \text{ mm}$

Length $L = 160 \text{ mm}$

Width $b = 80 \text{ mm}$

$G = 1250 \text{ kPa}$

$P = 16 \text{ kN}$

(a) SHEAR STRESS AND STRAIN IN THE RUBBER PADS

$$\tau_{\text{aver}} = \frac{P/2}{bL} = \frac{8 \text{ kN}}{(80 \text{ mm})(160 \text{ mm})} = 625 \text{ kPa}$$

$$\gamma_{\text{aver}} = \frac{\tau_{\text{aver}}}{G} = \frac{625 \text{ kPa}}{1250 \text{ kPa}} = 0.50 \quad \leftarrow$$

(b) HORIZONTAL DISPLACEMENT

$$\delta = t \times \tan(\gamma_{\text{ave}}) = 4.92 \text{ mm}$$

Problem 1.7-11 Steel riser pipe hangs from a drill rig located offshore in deep water (see figure). Separate segments are joined using bolted flange plates (see figure and photo). Assume that there are six bolts at each pipe segment connection. Assume that the total length of riser pipe is $L = 5000 \text{ ft}$; outer and inner diameters are $d_2 = 16 \text{ in.}$, $d_1 = 15 \text{ in.}$; flange plate thickness is $t_f = 1.75 \text{ in.}$; and bolt and washer diameters are $d_b = 1.125 \text{ in.}$, $d_w = 1.875 \text{ in.}$

- If the entire length of the riser pipe were suspended in air, find the average normal stress σ in each bolt, the average bearing stress σ_b beneath each washer, and the average shear stress τ through the flange plate at each bolt location for the topmost bolted connection.
- If the same riser pipe hangs from a drill rig at sea, what are the normal, bearing, and shear stresses in the connection? (Obtain the weight densities of steel and sea water from Table I-1, Appendix I. Neglect the effect of buoyant foam casings on the riser pipe).

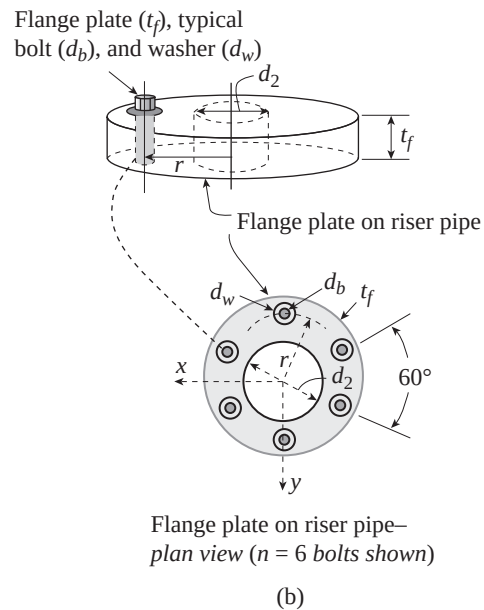
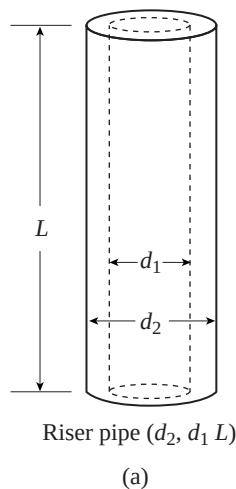




Photo Courtesy of
Transocean

Solution 1.7-11

(a) PIPE SUSPENDED IN AIR

$$L = 5000 \text{ ft} \quad \gamma_s = 490 \text{ lb/ft}^3 \quad \gamma_w = 63.8 \text{ lb/ft}^3$$

$$d_2 = 16 \text{ in.} \quad d_1 = 15 \text{ in.} \quad t = \frac{d_2 - d_1}{2} = 0.5 \text{ in.} \quad t_f = 1.75 \text{ in.} \quad A_{\text{pipe}} = \frac{\pi}{4} (d_2^2 - d_1^2) = 24.347 \text{ in.}^2$$

$$W_{\text{pipe}} = \gamma_s A_{\text{pipe}} L = 414.243 \text{ k}$$

$$n = 6 \quad d_b = 1.125 \text{ in.} \quad d_w = 1.875 \text{ in.} \quad A_b = \frac{\pi}{4} d_b^2 = 0.994 \text{ in.}^2 \quad A_w = \frac{\pi}{4} (d_w^2 - d_b^2) = 1.8 \text{ in.}^2$$

$$\sigma_b = \frac{W_{\text{pipe}}}{n A_b} = 69.5 \text{ ksi}$$

$$\sigma_{\text{brg}} = \frac{W_{\text{pipe}}}{n A_w} = 39.1 \text{ ksi}$$

$$\tau_f = \frac{W_{\text{pipe}}}{n d_w t_f} = 21 \text{ ksi}$$

(b) PIPE SUSPENDED IN SEA WATER $W_{\text{inwater}} = (\gamma_s - \gamma_w) A_{\text{pipe}} L = 360.307 \text{ kip}$

$$\sigma_b = \frac{W_{\text{inwater}}}{n A_b} = 60.4 \text{ ksi}$$

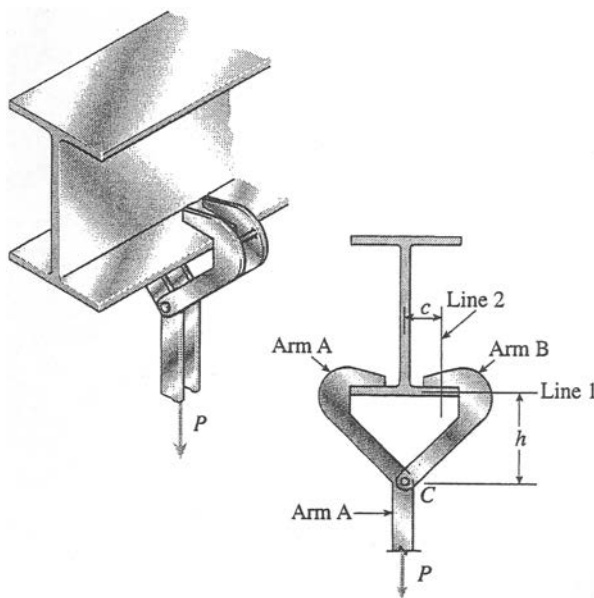
$$\sigma_{\text{brg}} = \frac{W_{\text{inwater}}}{n A_w} = 34 \text{ ksi}$$

$$\tau_f = \frac{W_{\text{inwater}}}{n d_w t_f} = 18.3 \text{ ksi}$$

Problem 1.7-12 The clamp shown in the figure is used to support a load hanging from the lower flange of a steel beam. The clamp consists of two arms (A and B) joined by a pin at C. The pin has diameter $d = 12$ mm. Because arm B straddles arm A, the pin is in double shear.

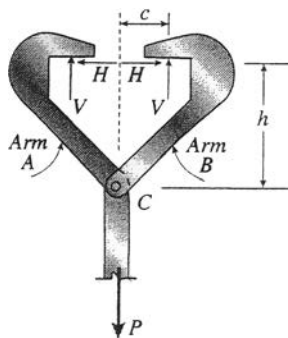
Line 1 in the figure defines the line of action of the resultant horizontal force H acting between the lower flange of the beam and arm B. The vertical distance from this line to the pin is $h = 250$ mm. Line 2 defines the line of action of the resultant vertical force V acting between the flange and arm B. The horizontal distance from this line to the centerline of the beam is $c = 100$ mm. The force conditions between arm A and the lower flange are symmetrical with those given for arm B.

Determine the average shear stress in the pin at C when the load $P = 18$ kN.



Solution 1.7-12 Clamp supporting a load P

FREE-BODY DIAGRAM OF CLAMP



$$h = 250 \text{ mm}$$

$$c = 100 \text{ mm}$$

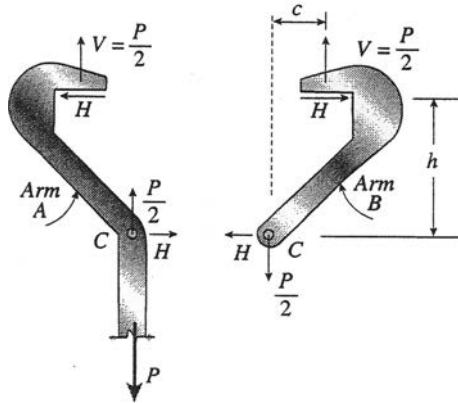
$$P = 18 \text{ kN}$$

From vertical equilibrium:

$$V = \frac{P}{2} = 9 \text{ kN}$$

$$d = \text{diameter of pin at } C = 12 \text{ mm}$$

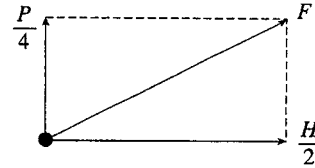
FREE-BODY DIAGRAMS OF ARMS A AND B



$$\Sigma M_C = 0 \quad \curvearrowright$$

$$V_C - Hh = 0$$

$$H = \frac{V_C}{h} = \frac{P_C}{2h} = 3.6 \text{ kN}$$

SHEAR FORCE F IN PIN

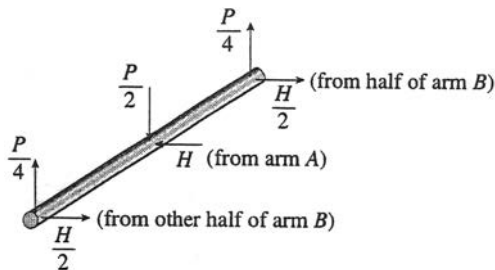
$$F = \sqrt{\left(\frac{P}{4}\right)^2 + \left(\frac{H}{2}\right)^2}$$

$$= 4.847 \text{ kN}$$

AVERAGE SHEAR STRESS IN THE PIN

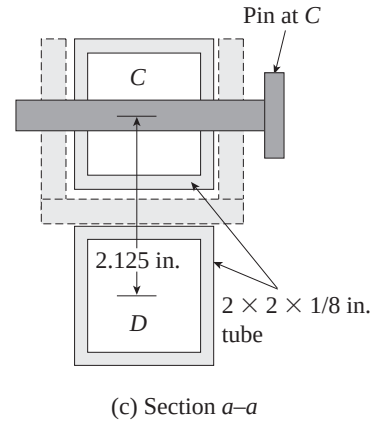
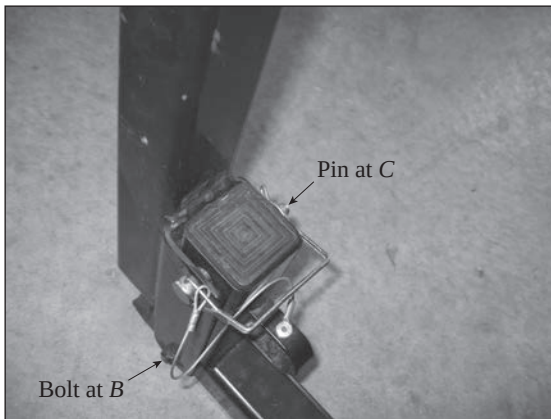
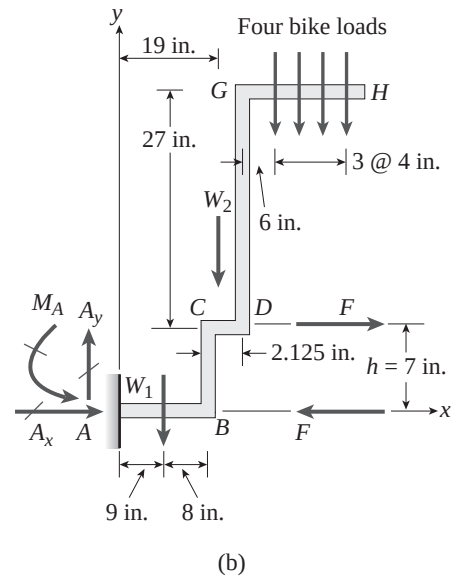
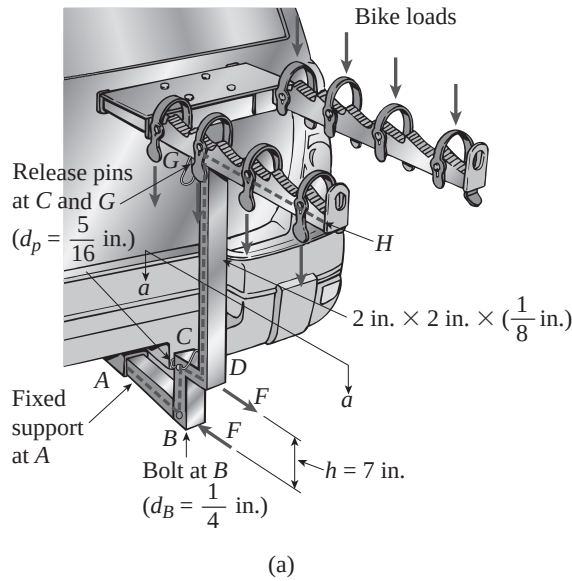
$$\tau_{\text{aver}} = \frac{F}{A_{\text{pin}}} = \frac{F}{\frac{\pi a^2}{4}} = 42.9 \text{ MPa} \quad \leftarrow$$

FREE-BODY DIAGRAM OF PIN



Problem 1.7-13 A hitch-mounted bicycle rack is designed to carry up to four 30-lb. bikes mounted on and strapped to two arms GH [see bike loads in the figure part (a)]. The rack is attached to the vehicle at A and is assumed to be like a cantilever beam $ABCDGH$ [figure part (b)]. The weight of fixed segment AB is $W_1 = 10$ lb, centered 9 in. from A [see the figure part (b)] and the rest of the rack weighs $W_2 = 40$ lb, centered 19 in. from A . Segment $ABCDGH$ is a steel tube, 2×2 in., of thickness $t = 1/8$ in. Segment $BCDGH$ pivots about a bolt at B of diameter $d_B = 0.25$ in. to allow access to the rear of the vehicle without removing the hitch rack. When in use, the rack is secured in an upright position by a pin at C (diameter of pin $d_p = 5/16$ in.) [see photo and figure part (c)]. The overturning effect of the bikes on the rack is resisted by a force couple Fh at BC .

- Find the support reactions at A for the fully loaded rack.
- Find forces in the bolt at B and the pin at C .
- Find average shear stresses τ_{aver} in both the bolt at B and the pin at C .
- Find average bearing stresses σ_b in the bolt at B and the pin at C .



Solution 1.7-13

NUMERICAL DATA

$$t = \frac{1}{8} \text{ in.} \quad b = 2 \text{ in.}$$

$$h = 7 \text{ in.} \quad W_1 = 10 \text{ lb} \quad W_2 = 40 \text{ lb}$$

$$P = 30 \text{ lb} \quad d_B = 0.25 \text{ in.} \quad d_p = \frac{5}{16} \text{ in.}$$

(a) REACTIONS AT A

$$A_x = 0 \quad \leftarrow$$

$$A_y = W_1 + W_2 + 4P \quad \leftarrow$$

$$A_y = 170 \text{ lb} \quad \leftarrow$$

$$L_1 = 17 + 2.125 + 6 \quad L_1 = 25 \text{ in.}$$

(dist from A to first bike)

$$M_A = W_1(9) + W_2(19) + P(4L_1 + 4 + 8 + 12)$$

$$M_A = 4585 \text{ in.-lb}$$

(b) FORCES IN BOLT AT B AND PIN AT C

$$\Sigma F_y = 0 \quad B_y = W_2 + 4P \quad B_y = 160 \text{ lb} \quad \leftarrow$$

$$\Sigma M_B = 0$$

Right hand FBD

$$B_x = \frac{[W_2(19 - 17) + P(6 + 2.125) + P(8.125 + 4) + P(8.125 + 8) + P(8.125 + 12)]}{h}$$

$$B_x = 254 \text{ lb} \quad \leftarrow \quad C_x = -B_x$$

$$B_{\text{res}} = \sqrt{B_x^2 + B_y^2} \quad B_{\text{res}} = 300 \text{ lb} \quad \leftarrow$$

(c) AVERAGE SHEAR STRESSES τ_{ave} IN BOTH THE BOLT AT *B* AND THE PIN AT *C*

$$A_{sB} = 2 \frac{\pi d_B^2}{4} \quad A_{sB} = 0.098 \text{ in.}^2$$

$$\tau_B = \frac{B_{\text{res}}}{A_{sB}} \quad \tau_B = 3054 \text{ psi} \quad \leftarrow$$

$$A_{sC} = 2 \frac{\pi d_p^2}{4} \quad A_{sC} = 0.153 \text{ in.}^2$$

$$\tau_C = \frac{B_x}{A_{sC}} \quad \tau_C = 1653 \text{ psi} \quad \leftarrow$$

(d) BEARING STRESSES σ_B IN THE BOLT AT *B* AND THE PIN AT *C*

$$t = 0.125 \text{ in.}$$

$$A_{bB} = 2td_B \quad A_{bB} = 0.063 \text{ in.}^2$$

$$\sigma_{bB} = \frac{B_{\text{res}}}{A_{bB}} \quad \sigma_{bB} = 4797 \text{ psi} \quad \leftarrow$$

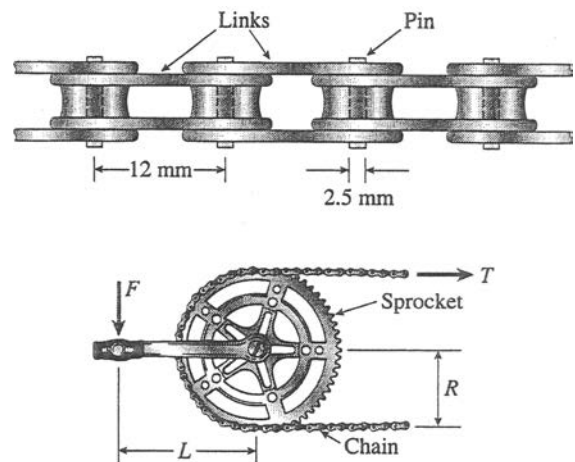
$$A_{bC} = 2td_p \quad A_{bC} = 0.078 \text{ in.}^2$$

$$\sigma_{bC} = \frac{C_x}{A_{bC}} \quad \sigma_{bC} = 3246 \text{ psi} \quad \leftarrow$$

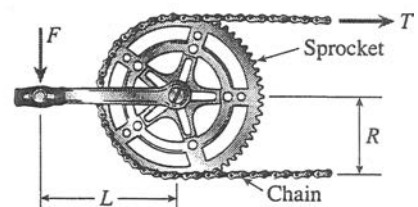
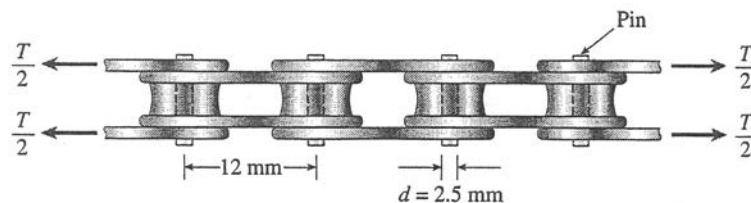
Problem 1.7-14 A bicycle chain consists of a series of small links, each 12 mm long between the centers of the pins (see figure). You might wish to examine a bicycle chain and observe its construction. Note particularly the pins, which we will assume to have a diameter of 2.5 mm.

In order to solve this problem, you must now make two measurements on a bicycle (see figure): (1) the length *L* of the crank arm from main axle to pedal axle, and (2) the radius *R* of the sprocket (the toothed wheel, sometimes called the chainring).

- Using your measured dimensions, calculate the tensile force *T* in the chain due to a force *F* = 800 N applied to one of the pedals.
- Calculate the average shear stress τ_{aver} in the pins.



Solution 1.7-14 Bicycle chain



F = force applied to pedal = 800 N

L = length of crank arm

R = radius of sprocket

MEASUREMENTS (FOR AUTHOR'S BICYCLE)

(1) $L = 162 \text{ mm}$ (2) $R = 90 \text{ mm}$

(a) TENSILE FORCE T IN CHAIN

$$\Sigma M_{\text{axle}} = 0 \quad FL = TR \quad T = \frac{FL}{R}$$

Substitute numerical values:

$$T = \frac{(800 \text{ N})(162 \text{ mm})}{90 \text{ mm}} = 1440 \text{ N} \quad \leftarrow$$

(b) SHEAR STRESS IN PINS

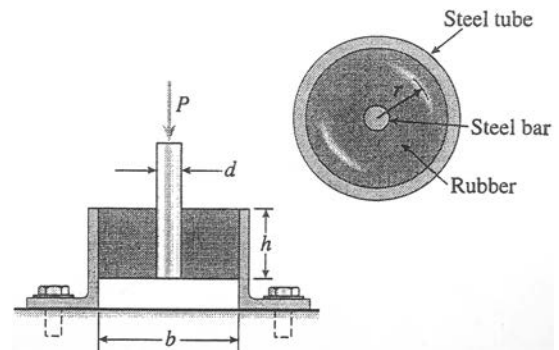
$$\begin{aligned} \tau_{\text{aver}} &= \frac{T/2}{A_{\text{pin}}} = \frac{T}{2 \frac{\pi d^2}{4}} = \frac{2T}{\pi d^2} \\ &= \frac{2FL}{\pi d^2 R} \end{aligned}$$

Substitute numerical values:

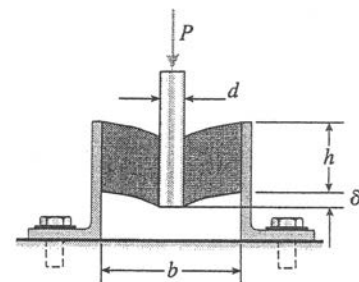
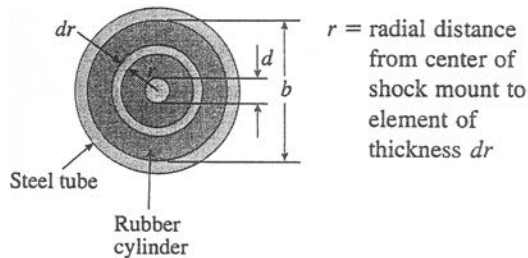
$$\tau_{\text{aver}} = \frac{2(800 \text{ N})(162 \text{ mm})}{\pi(2.5 \text{ mm})^2(90 \text{ mm})} = 147 \text{ MPa} \quad \leftarrow$$

Problem 1.7-15 A shock mount constructed as shown in the figure is used to support a delicate instrument. The mount consists of an outer steel tube with inside diameter b , a central steel bar of diameter d that supports the load P , and a hollow rubber cylinder (height h) bonded to the tube and bar.

- Obtain a formula for the shear τ in the rubber at a radial distance r from the center of the shock mount.
- Obtain a formula for the downward displacement δ of the central bar due to the load P , assuming that G is the shear modulus of elasticity of the rubber and that the steel tube and bar are rigid.



Solution 1.7-15 Shock mount



r = radial distance from center of shock mount to element of thickness dr

(a) SHEAR STRESS τ AT RADIAL DISTANCE r

A_s = shear area at distance $r = 2\pi rh$

$$\tau = \frac{P}{A_s} = \frac{P}{2\pi rh} \quad \leftarrow$$

(b) DOWNWARD DISPLACEMENT δ

γ = shear strain at distance r

$$\gamma = \frac{\tau}{G} = \frac{P}{2\pi rhG}$$

$d\delta$ = downward displacement for element dr

$$d\delta = \gamma dr = \frac{Pdr}{2\pi rhG}$$

$$\delta = \int d\delta = \int_{d/2}^{b/2} \frac{Pdr}{2\pi rhG}$$

$$\delta = \frac{P}{2\pi hG} \int_{d/2}^{b/2} \frac{dr}{r} = \frac{P}{2\pi hG} [\ln r]_{d/2}^{b/2}$$

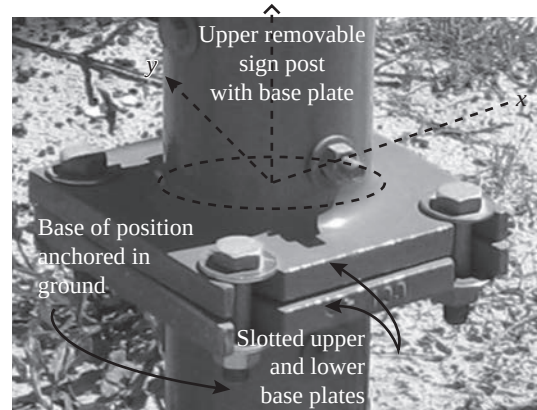
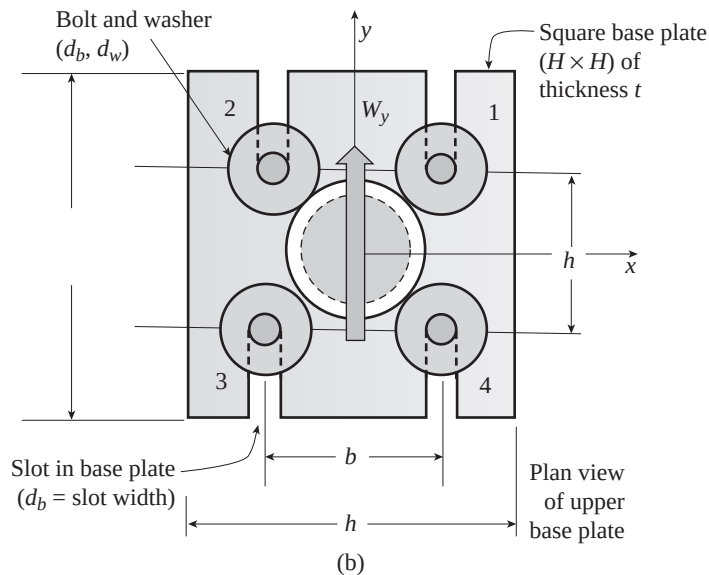
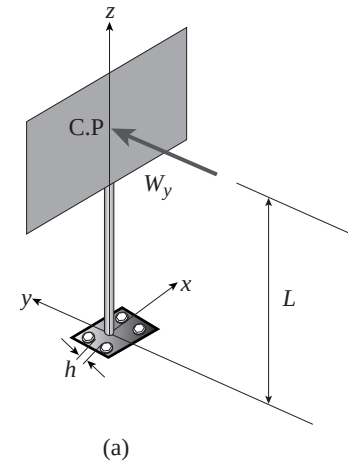
$$\delta = \frac{P}{2\pi hG} \ln \frac{b}{d} \quad \leftarrow$$

Problem 1.7-16 A removable sign post on a hurricane evacuation route has a square base plate with four slots (or cut outs) at bolts 1 through 4 (see drawing and photo) for ease of installation and removal. The upper portion of the post has a separate base plate which is bolted to an anchored base (see photo). Each of the four bolts has diameter d_b and a washer of diameter d_w . The bolts are arranged in a rectangular pattern ($b \times h$). Consider **wind force** W_y applied in the y -direction at the center of pressure of the sign structure at a height $z = L$ above the base. Neglect the weight of the sign and post, and also neglect friction between the upper and lower base plates. Assume that the lower base plate and short anchored post are rigid.

- Find average **shear stress** τ , (MPa) at **bolt #1** due to the wind force W_y ; repeat for **bolt #4**.
- Find average **bearing stress** σ_b , (MPa) between the bolt and the base plate (thickness t) at **bolt #1**; repeat for **bolt #4**.
- Find average **bearing stress** σ_b , (MPa) between base plate and washer at **bolt #4** due to the wind force W_y (assume initial bolt pretension is zero)
- Find average **shear stress** τ , (MPa) through the base plate at **bolt #4** due to the wind force W_y .
- Find an expression for **normal stress** σ in **bolt #3** due to the wind force W_y .

See Prob. 1.8-15 for additional discussion of wind on a sign, and the resulting forces acting on a *conventional* base plate.

NUMERICAL DATA $L = 2.75 \text{ m}$ $W_y = 667 \text{ N}$ $H = 150 \text{ mm}$ $h = 108 \text{ mm}$
 $b = 96 \text{ mm}$ $t = 14 \text{ mm}$ $d_b = 12 \text{ mm}$ $d_w = 22 \text{ mm}$



Solution 1.7-16

- (a) FIND AVERAGE
- SHEAR STRESS**
- (
- τ
- , MPa) AT
- BOLT #1**
- DUE TO THE WIND FORCE
- W_y
- ; REPEAT FOR
- BOLT #4**

$$A_b = \frac{\pi}{4} d_b^2 = 113.097 \text{ mm}^2 \quad \tau_1 = \frac{\frac{W_y}{2}}{A_b} = 2.95 \text{ MPa} \quad \boxed{\tau_1 = 2.95 \text{ MPa}} \quad \boxed{\tau_4 = 0} \quad \frac{W_y}{2} = 333.5 \text{ N}$$

(only bolts 1 and 2 resist wind force shear in +y-direction)

- (b) FIND AVERAGE
- BEARING STRESS**
- (
- σ_b
- , MPa) BETWEEN THE BOLT AND THE BASE PLATE (THICKNESS
- t
-) AT
- BOLT #1**
- ; REPEAT FOR
- BOLT #4**

$$A_{\text{brg}} = d_b t = 168 \text{ mm}^2 \quad \sigma_{b1} = \frac{\frac{W_y}{2}}{A_{\text{brg}}} = 1.985 \text{ MPa} \quad \boxed{\sigma_{b1} = 1.985 \text{ MPa}} \quad \boxed{\sigma_{b4} = 0}$$

(only bolts 1 and 2 resist wind force bearing in +y-direction)

- (c) FIND AVERAGE
- BEARING STRESS**
- (
- σ_b
- , MPa) BETWEEN BASE PLATE AND WASHER AT
- BOLT #4**
- DUE TO THE WIND FORCE
- W_y
- (ASSUME INITIAL BOLT PRETENSION IS ZERO)

Assume wind force creates overturning moment about x axis = OTM_x $\text{OTM}_x = W_y L = 1834.25 \text{ N}\cdot\text{m}$

OTM is resisted by force couples pairs at bolts 1 to 4 and 2 to 3; so force in bolt 4 is

$$F_4 = \frac{\text{OTM}_x}{2h} = 8491.898 \text{ N}$$

Bearing area is donut shaped area of washer in contact with the plate minus approximate rectangular cutout for slot

$$A_{\text{brg}} = \frac{\pi}{4} (d_w^2 - d_b^2) - d_b \left(\frac{d_w - d_b}{2} \right) = 207.035 \text{ mm}^2 \quad \sigma_{b4} = \frac{F_4}{A_{\text{brg}}} = 41 \text{ MPa} \quad \boxed{\sigma_{b4} = 41 \text{ MPa}}$$

- (d) FIND AVERAGE
- SHEAR STRESS**
- (
- τ
- , MPa) THROUGH THE BASE PLATE AT
- BOLT #4**
- DUE TO THE WIND FORCE
- W_y
- ;

Use force F_4 above; shear stress is on cylindrical surface at perimeter of washer; must deduct approximate rectangular area due to slot

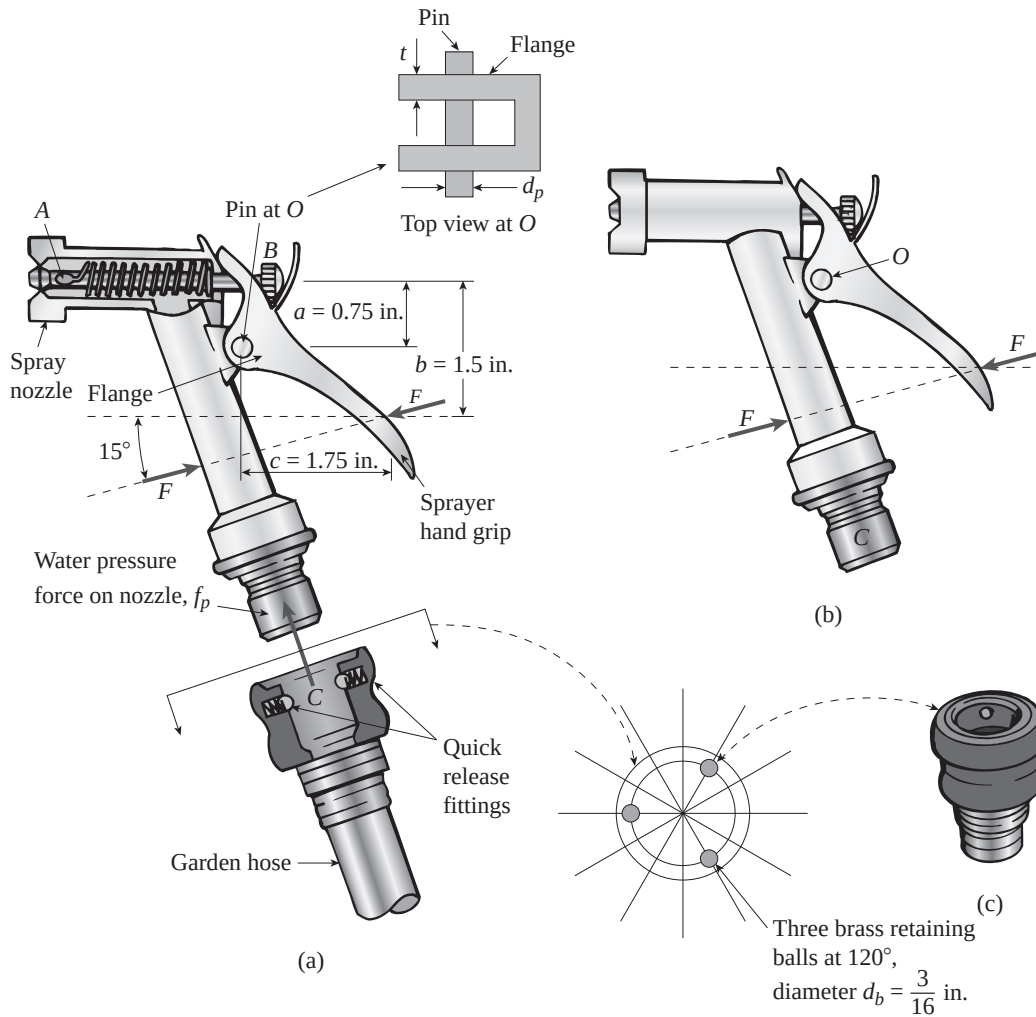
$$A_{\text{sh}} = (\pi d_w - d_b) t = 799.611 \text{ mm}^2 \quad \tau = \frac{F_4}{A_{\text{sh}}} = 10.62 \text{ MPa} \quad \boxed{\tau = 10.62 \text{ MPa}}$$

- (e) FIND AN EXPRESSION FOR
- NORMAL STRESS**
- (
- σ
-) IN
- BOLT #3**
- DUE TO THE WIND FORCE
- W_y
- .

$$\text{Force in bolt \#3 due to } \text{OTM}_x \text{ is same as that in bolt \#4} \quad \sigma_3 = \frac{F_4}{A_b} = 75.1 \text{ MPa} \quad \boxed{\sigma_3 = 75.1 \text{ MPa}}$$

Problem 1.7-17 A spray nozzle for a garden hose requires a force $F = 5$ lb. to open the spring-loaded spray chamber AB . The nozzle hand grip pivots about a pin through a flange at O . Each of the two flanges has thickness $t = 1/16$ in., and the pin has diameter $d_p = 1/8$ in. [see figure part (a)]. The spray nozzle is attached to the garden hose with a quick release fitting at B [see figure part (b)]. Three brass balls (diameter $d_b = 3/16$ in.) hold the spray head in place under water pressure force $f_p = 30$ lb at C [see figure part (c)]. Use dimensions given in figure part (a).

- Find the force in the pin at O due to applied force F .
- Find average shear stress τ_{aver} and bearing stress σ_b in the pin at O .



Solution 1.7-17

NUMERICAL DATA

$$F = 5 \text{ lb} \quad t = \frac{1}{16} \text{ in.} \quad d_p = \frac{1}{8} \text{ in.} \quad d_b = \frac{3}{16} \text{ in.}$$

$$f_p = 30 \text{ lb} \quad d_N = \frac{5}{8} \text{ in.} \quad \theta = 15 \frac{\pi}{180} \text{ rad}$$

$$a = 0.75 \text{ in.} \quad b = 1.5 \text{ in.} \quad c = 1.75 \text{ in.}$$

(a) FIND THE FORCE IN THE PIN AT O DUE TO APPLIED FORCE F

$$\sum M_o = 0$$

$$F_{AB} = \frac{[F \cos(\theta)(b - a)] + F \sin(\theta)(c)}{a}$$

$$F_{AB} = 7.849 \text{ lb}$$

$$\sum F_H = 0 \quad O_x = F_{AB} + F \cos(\theta)$$

$$O_y = F \sin(\theta)$$

$$O_x = 12.68 \text{ lb} \quad O_y = 1.294 \text{ lb}$$

$$O_{\text{res}} = \sqrt{O_x^2 + O_y^2} \quad O_{\text{res}} = 12.74 \text{ lb} \quad \leftarrow$$

(b) FIND AVERAGE SHEAR STRESS τ_{ave} AND BEARING STRESS σ_b IN THE PIN AT O

$$A_s = 2 \frac{\pi d_p^2}{4} \quad \tau_O = \frac{O_{\text{res}}}{A_s} \quad \tau_O = 519 \text{ psi} \quad \leftarrow$$

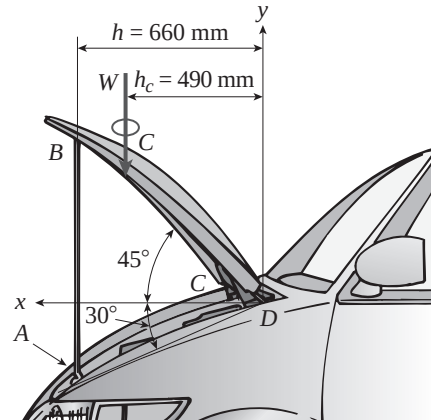
$$A_b = 2td_p \quad \sigma_{bO} = \frac{O_{\text{res}}}{A_b} \quad \sigma_{bO} = 816 \text{ psi} \quad \leftarrow$$

(c) FIND THE AVERAGE SHEAR STRESS τ_{ave} IN THE BRASS RETAINING BALLS AT B DUE TO WATER PRESSURE FORCE F_p

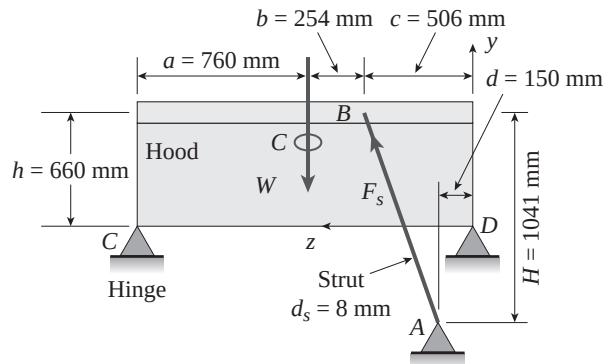
$$A_s = 3 \frac{\pi d_b^2}{4} \quad \tau_{\text{ave}} = \frac{f_p}{A_s} \quad \tau_{\text{ave}} = 362 \text{ psi} \quad \leftarrow$$

Problem 1.7-18 A single steel strut AB with diameter $d_s = 8 \text{ mm}$, supports the vehicle engine hood of mass 20 kg which pivots about hinges at C and D [see figures (a) and (b)]. The strut is bent into a loop at its end and then attached to a bolt at A with diameter $d_b = 10 \text{ mm}$. Strut AB lies in a vertical plane.

- Find the strut force F_s and average normal stress σ in the strut.
- Find the average shear stress τ_{aver} in the bolt at A .
- Find the average bearing stress σ_b on the bolt at A .



(a)



(b)

Solution 1.7-18

NUMERICAL DATA

$$d_s = 8 \text{ mm} \quad d_b = 10 \text{ mm} \quad m = 20 \text{ kg}$$

$$a = 760 \text{ mm} \quad b = 254 \text{ mm}$$

$$c = 506 \text{ mm} \quad d = 150 \text{ mm}$$

$$h = 660 \text{ mm} \quad h_c = 490 \text{ mm}$$

$$H = h \left(\tan \left(30 \frac{\pi}{180} \right) + \tan \left(45 \frac{\pi}{180} \right) \right)$$

$$H = 1041 \text{ mm}$$

$$W = m (9.81 \text{ m/s}^2) \quad W = 196.2 \text{ N}$$

$$\frac{a + b + c}{2} = 760 \text{ mm}$$

VECTOR r_{AB}

$$r_{AB} = \begin{pmatrix} 0 \\ H \\ c - d \end{pmatrix} \quad r_{AB} = \begin{pmatrix} 0 \\ 1.041 \times 10^3 \\ 356 \end{pmatrix}$$

UNIT VECTOR e_{AB}

$$e_{AB} = \frac{r_{AB}}{|r_{AB}|} \quad e_{AB} = \begin{pmatrix} 0 \\ 0.946 \\ 0.324 \end{pmatrix} \quad |e_{AB}| = 1$$

$$W = \begin{pmatrix} 0 \\ -W \\ 0 \end{pmatrix} \quad W = \begin{pmatrix} 0 \\ -196.2 \\ 0 \end{pmatrix}$$

$$r_{DC} = \begin{pmatrix} h_c \\ h_c \\ b + c \end{pmatrix} \quad r_{DC} = \begin{pmatrix} 490 \\ 490 \\ 760 \end{pmatrix}$$

$$\sum M_D \quad M_D = r_{DB} \times F_s e_{AB} + W \times r_{DC}$$

(ignore force at hinge C since it will vanish with moment about line DC)

$$F_{sx} = 0 \quad F_{sy} = \frac{H}{\sqrt{H^2 + (c - d)^2}} F_s$$

$$F_{sz} = \frac{c - d}{\sqrt{H^2 + (c - d)^2}} F_s$$

where

$$\frac{H}{\sqrt{H^2 + (c - d)^2}} = 0.946$$

$$\frac{c - d}{\sqrt{H^2 + (c - d)^2}} = 0.324$$

(a) FIND THE STRUT FORCE F_s AND AVERAGE NORMAL STRESS σ IN THE STRUT

$$\sum M_{\text{lineDC}} = 0 \quad F_{sy} = \frac{|W| h_c}{h}$$

$$F_{sy} = 145.664$$

$$F_s = \frac{F_{sy}}{\frac{H}{\sqrt{H^2 + (c - d)^2}}} \quad F_s = 153.9 \text{ N} \quad \leftarrow$$

$$A_{\text{strut}} = \frac{\pi}{4} d_s^2 \quad A_{\text{strut}} = 50.265 \text{ mm}^2$$

$$\sigma = \frac{F_s}{A_{\text{strut}}} \quad \sigma = 3.06 \text{ MPa} \quad \leftarrow$$

(b) FIND THE AVERAGE SHEAR STRESS τ_{ave} IN THE BOLT AT A

$$d_b = 10 \text{ mm}$$

$$A_s = \frac{\pi}{4} d_b^2 \quad A_s = 78.54 \text{ mm}^2$$

$$\tau_{\text{ave}} = \frac{F_s}{A_s} \quad \tau_{\text{ave}} = 1.96 \text{ MPa} \quad \leftarrow$$

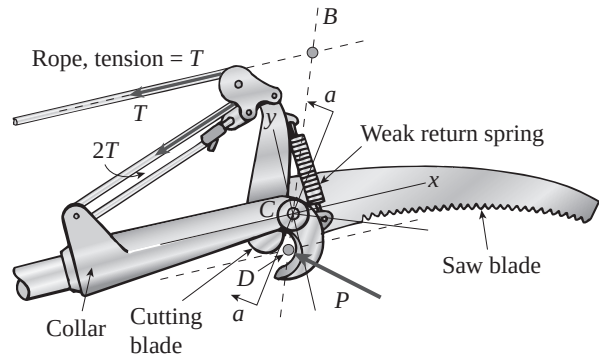
(c) FIND THE BEARING STRESS σ_b ON THE BOLT AT A

$$A_b = d_s d_b \quad A_b = 80 \text{ mm}^2$$

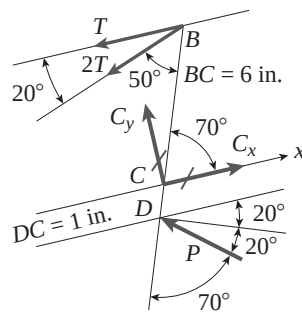
$$\sigma_b = \frac{F_s}{A_b} \quad \sigma_b = 1.924 \text{ MPa} \quad \leftarrow$$

Problem 1.7-19 The top portion of a pole saw used to trim small branches from trees is shown in the figure part (a). The cutting blade BCD [see figure parts (a) and (c)] applies a force P at point D . Ignore the effect of the weak return spring attached to the cutting blade below B . Use properties and dimensions given in the figure.

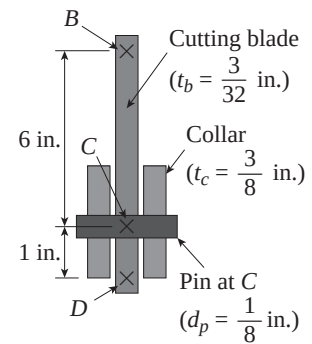
- Find the force P on the cutting blade at D if the tension force in the rope is $T = 25$ lb (see free body diagram in part (b)).
- Find force in the pin at C .
- Find average shear stress τ_{ave} and bearing stress σ_b in the support pin at C [see Section $a-a$ through cutting blade in figure part (c)].



(a) Top part of pole saw



(b) Free-body diagram

(c) Section $a-a$ **Solution 1.7-19**

NUMERICAL PROPERTIES

$$d_p = \frac{1}{8} \text{ in.} \quad t_b = \frac{3}{32} \text{ in.} \quad t_c = \frac{3}{8} \text{ in.}$$

$$T = 25 \text{ lb} \quad d_{BC} = 6 \text{ in.}$$

$$d_{CD} = 1 \text{ in.}$$

- (a) FIND THE CUTTING FORCE P ON THE CUTTING BLADE AT D IF THE TENSION FORCE IN THE ROPE IS $T = 25$ lb:

$$\sum M_C = 0$$

$$\begin{aligned} M_C &= T(6 \sin(70^\circ)) \\ &\quad + 2T \cos(20^\circ)(6 \sin(70^\circ)) \\ &\quad - 2T \sin(20^\circ)(6 \cos(70^\circ)) \\ &\quad - P \cos(20^\circ)(1) \end{aligned}$$

SOLVE EQUATION FOR P

$$\begin{aligned} &[T(6 \sin(70^\circ)) + 2T \cos(20^\circ) \\ &P = \frac{6 \sin(70^\circ) - 2T \sin(20^\circ)(6 \cos(70^\circ))}{\cos(20^\circ)} \end{aligned}$$

$$P = 395 \text{ lbs} \quad \leftarrow$$

- (b) SOLVE FOR FORCES ON PIN AT C

$$\sum F_x = 0 \quad C_x = T + 2T \cos(20^\circ) + P \cos(40^\circ)$$

$$C_x = 374 \text{ lbs} \quad \leftarrow$$

$$\sum F_y = 0 \quad C_y = 2T \sin(20^\circ) - P \sin(40^\circ)$$

$$C_y = -237 \text{ lbs} \quad \leftarrow$$

RESULTANT AT C

$$C_{\text{res}} = \sqrt{C_x^2 + C_y^2} \quad C_{\text{res}} = 443 \text{ lbs} \quad \leftarrow$$

- (c) FIND MAXIMUM SHEAR AND BEARING STRESSES IN THE SUPPORT PIN AT C (SEE SECTION A-A THROUGH SAW).

SHEAR STRESS—PIN IN DOUBLE SHEAR

$$A_s = \frac{\pi}{4} d_p^2 \quad A_s = 0.012 \text{ in.}^2$$

$$\tau_{\text{ave}} = \frac{C_{\text{res}}}{2A_s} \quad \tau_{\text{ave}} = 18.04 \text{ ksi}$$

BEARING STRESSES ON PIN ON EACH SIDE OF COLLAR

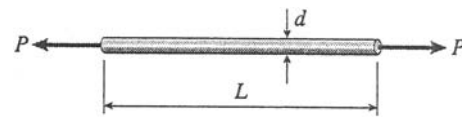
$$\sigma_{bC} = \frac{C_{\text{res}}}{d_p t_c} \quad \sigma_{bC} = 4.72 \text{ ksi} \quad \leftarrow$$

BEARING STRESS ON PIN AT CUTTING BLADE

$$\sigma_{bcb} = \frac{C_{\text{res}}}{d_p t_b} \quad \sigma_{bcb} = 37.8 \text{ ksi} \quad \leftarrow$$

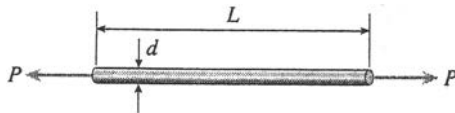
Allowable Stresses and Allowable Loads

Problem 1.8-1 A bar of solid circular cross section is loaded in tension by forces P (see figure). The bar has length $L = 16.0$ in. and diameter $d = 0.50$ in. The material is a magnesium alloy having modulus of elasticity $E = 6.4 \times 10^6$ psi. The allowable stress in tension is $\sigma_{\text{allow}} = 17,000$ psi, and the elongation of the bar must not exceed 0.04 in.



What is the allowable value of the forces P ?

Solution 1.8-1 Magnesium bar in tension



$$L = 16.0 \text{ in.} \quad d = 0.50 \text{ in.}$$

$$E = 6.4 \times 10^6 \text{ psi}$$

$$\sigma_{\text{allow}} = 17,000 \text{ psi} \quad \delta_{\text{max}} = 0.04 \text{ in.}$$

MAXIMUM LOAD BASED UPON ELONGATION

$$\epsilon_{\text{max}} = \frac{\delta_{\text{max}}}{L} = \frac{0.04 \text{ in.}}{16 \text{ in.}} = 0.00250$$

$$\sigma_{\text{max}} = E \epsilon_{\text{max}} = (6.4 \times 10^6 \text{ psi})(0.00250)$$

$$= 16,000 \text{ psi}$$

$$P_{\text{max}} = \sigma_{\text{max}} A = (16,000 \text{ psi}) \left(\frac{\pi}{4} \right) (0.50 \text{ in.})^2$$

$$= 3140 \text{ lb}$$

MAXIMUM LOAD BASED UPON TENSILE STRESS

$$P_{\text{max}} = \sigma_{\text{allow}} A = (17,000 \text{ psi}) \left(\frac{\pi}{4} \right) (0.50 \text{ in.})^2$$

$$= 3340 \text{ lb}$$

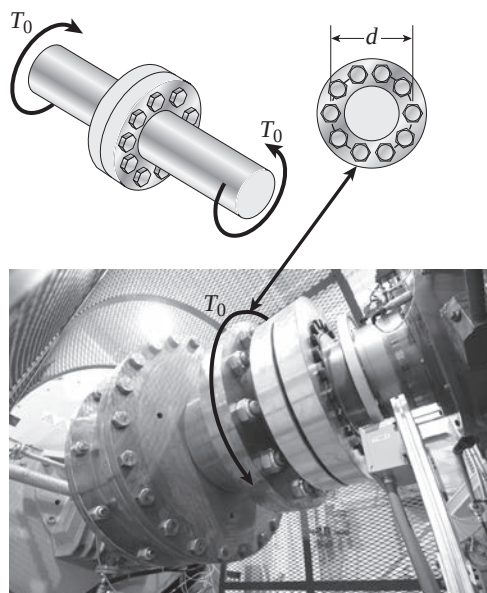
ALLOWABLE LOAD

Elongation governs.

$$P_{\text{allow}} = 3140 \text{ lb} \quad \leftarrow$$

Problem 1.8-2 A torque T_0 is transmitted between two flanged shafts by means of ten 20-mm bolts (see figure and photo). The diameter of the bolt circle is $d = 250$ mm.

If the allowable shear stress in the bolts is 90 MPa, what is the maximum permissible torque? (Disregard friction between the flanges.)



Solution 1.8-2 Shafts with flanges

NUMERICAL DATA

$$r = 10 \quad d = 250 \text{ mm}$$

$$\wedge \text{ bolts} \quad \wedge \text{ flange}$$

$$A_s = \pi r^2$$

$$A_s = 314.159 \text{ m}^2$$

$$\tau_a = 85 \text{ MPa}$$

MAXIMUM PERMISSIBLE TORQUE

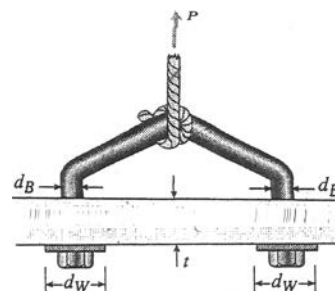
$$T_{\max} = \tau_a A_s \left(r \frac{d}{2} \right)$$

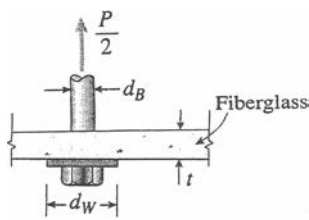
$$T_{\max} = 3.338 \times 10^7 \text{ N}\cdot\text{mm}$$

$$T_{\max} = 33.4 \text{ kN}\cdot\text{m} \quad \leftarrow$$

Problem 1.8-3 A tie-down on the deck of a sailboat consists of a bent bar bolted at both ends, as shown in the figure. The diameter d_B of the bar is $\frac{1}{4}$ in., the diameter d_W of the washers is $\frac{7}{8}$ in., and the thickness t of the fiberglass deck is $\frac{3}{8}$ in.

If the allowable shear stress in the fiberglass is 300 psi, and the allowable bearing pressure between the washer and the fiberglass is 550 psi, what is the allowable load P_{allow} on the tie-down?



Solution 1.8-3 Bolts through fiberglass

$$d_B = \frac{1}{4} \text{ in.}$$

$$d_W = \frac{7}{8} \text{ in.}$$

$$t = \frac{3}{8} \text{ in.}$$

ALLOWABLE LOAD BASED UPON SHEAR STRESS IN FIBERGLASS

$$\tau_{\text{allow}} = 300 \text{ psi}$$

$$\text{Shear area } A_s = \pi d_W t$$

$$\begin{aligned} \frac{P_1}{2} &= \tau_{\text{allow}} A_s = \tau_{\text{allow}} (\pi d_W t) \\ &= (300 \text{ psi})(\pi) \left(\frac{7}{8} \text{ in.} \right) \left(\frac{3}{8} \text{ in.} \right) \end{aligned}$$

$$\frac{P_1}{2} = 309.3 \text{ lb}$$

$$P_1 = 619 \text{ lb}$$

ALLOWABLE LOAD BASED UPON BEARING PRESSURE

$$\sigma_b = 550 \text{ psi}$$

$$\text{Bearing area } A_b = \frac{\pi}{4} (d_W^2 - d_B^2)$$

$$\begin{aligned} \frac{P_2}{2} &= \sigma_b A_b = (550 \text{ psi}) \left(\frac{\pi}{4} \right) \left[\left(\frac{7}{8} \text{ in.} \right)^2 - \left(\frac{1}{4} \text{ in.} \right)^2 \right] \\ &= 303.7 \text{ lb} \end{aligned}$$

$$P_2 = 607 \text{ lb}$$

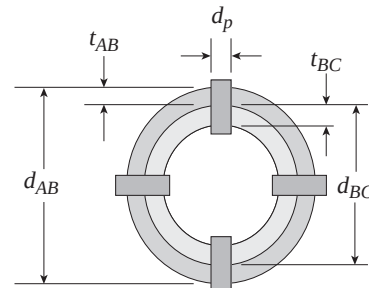
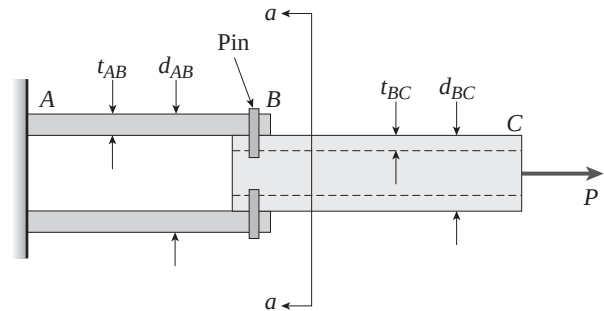
ALLOWABLE LOAD

Bearing pressure governs.

$$P_{\text{allow}} = 607 \text{ lb} \quad \leftarrow$$

Problem 1.8-4 Two steel tubes are joined at *B* by four pins ($d_p = 11 \text{ mm}$), as shown in the cross section *a-a* in the figure. The outer diameters of the tubes are $d_{AB} = 41 \text{ mm}$ and $d_{BC} = 28 \text{ mm}$. The wall thicknesses are $t_{AB} = 6.5 \text{ mm}$ and $t_{BC} = 7.5 \text{ mm}$. The yield stress in tension for the steel is $\sigma_Y = 200 \text{ MPa}$ and the ultimate stress in tension is $\sigma_U = 340 \text{ MPa}$. The corresponding yield and ultimate values in shear for the pin are 80 MPa and 140 MPa, respectively. Finally, the yield and ultimate values in bearing between the pins and the tubes are 260 MPa and 450 MPa, respectively. Assume that the factors of safety with respect to yield stress and ultimate stress are 3.5 and 4.5, respectively.

- Calculate the allowable tensile force P_{allow} considering tension in the tubes.
- Recompute P_{allow} for shear in the pins.
- Finally, recompute P_{allow} for bearing between the pins and the tubes. Which is the controlling value of P ?



Section *a-a*

Solution 1.8-4

Yield and ultimate stresses (all in MPa)

TUBES: $\sigma_Y = 200$ $\sigma_u = 340$ $FS_Y = 3.5$

PIN (SHEAR): $\tau_Y = 8$ $\tau_u = 140$ $FS_u = 4.5$

PIN (BEARING): $\sigma_{bY} = 260$ $\sigma_{bu} = 450$

TUBES AND PIN DIMENSIONS (MM)

$d_{AB} = 41$ $t_{AB} = 6.5$

$d_{BC} = d_{AB} - 2 t_{AB}$ $d_{BC} = 28$

$t_{BC} = 7.5$ $d_p = 11$

(a) P_{ALLOW} CONSIDERING TENSION IN THE TUBES

$$A_{netAB} = \frac{\pi}{4} [d_{AB}^2 - (d_{AB} - 2 t_{AB})^2] - 4 d_p t_{AB} \quad A_{netAB} = 418.502 \text{ mm}^2$$

$$A_{netBC} = \frac{\pi}{4} [d_{BC}^2 - (d_{BC} - 2 t_{BC})^2] - 4 d_p t_{BC} \quad A_{netBC} = 153.02 < \text{use smaller}$$

$$P_{aT1} = \frac{\sigma_Y}{FS_Y} A_{netBC} \quad P_{aT1} = 8743.993 \text{ N} < \text{controls} \quad \boxed{P_{allow} = 8.74 \text{ kN}}$$

$$P_{aT2} = \frac{\sigma_u}{FS_u} A_{netBC} \quad P_{aT2} = 11,561.501 \text{ N}$$

(b) P_{allow} CONSIDERING SHEAR IN THE PINS $A_s = \frac{\pi}{4} d_p^2$ $A_s = 95.033 \text{ mm}^2$

$$P_{aS1} = (4 A_s) \frac{\tau_Y}{FS_Y} \quad P_{aS1} = 8688.748 \text{ N} < \text{controls} \quad \boxed{P_{allow} = 8.69 \text{ kN}}$$

$$P_{aS2} = (4 A_s) \frac{\tau_u}{FS_u} \quad P_{aS2} = 11,826.351 \text{ N}$$

(c) P_{allow} CONSIDERING BEARING IN THE PINS

$$A_{bAB} = 4 d_p t_{AB} \quad A_{bAB} = 286 \text{ mm}^2 < \text{smaller controls}$$

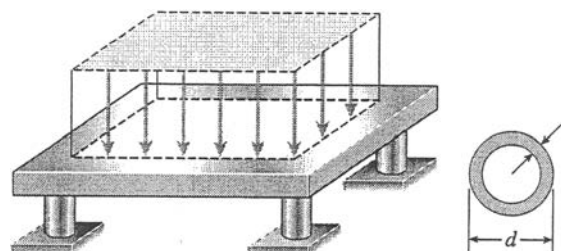
$$A_{bBC} = 4 d_p t_{BC} \quad A_{bBC} = 330$$

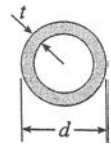
$$P_{ab1} = A_{bAB} \left(\frac{\sigma_{bY}}{FS_Y} \right) \quad P_{ab1} = 21,245.714 \text{ N} < \text{controls} \quad \boxed{P_{allow} = 21.2 \text{ kN}}$$

$$P_{ab2} = A_{bAB} \left(\frac{\sigma_{bu}}{FS_u} \right) \quad P_{ab2} = 28,600 \text{ N} \quad \boxed{\text{Overall, shear controls (Part (b))}}$$

Problem 1.8-5 A steel pad supporting heavy machinery rests on four short, hollow, cast iron piers (see figure). The ultimate strength of the cast iron in compression is 50 ksi. The outer diameter of the piers is $d = 4.5$ in. and the wall thickness is $t = 0.40$ in.

Using a factor of safety of 3.5 with respect to the ultimate strength, determine the total load P that may be supported by the pad.



Solution 1.8-5 Cast iron piers in compression

Four piers

$$\sigma_U = 50 \text{ ksi}$$

$$n = 3.5$$

$$\sigma_{\text{allow}} = \frac{\sigma_U}{n} = \frac{50 \text{ ksi}}{3.5} = 14.29 \text{ ksi}$$

$$d = 4.5 \text{ in.}$$

$$t = 0.4 \text{ in.}$$

$$d_0 = d - 2t = 3.7 \text{ in.}$$

$$A = \frac{\pi}{4} (d^2 - d_0^2) = \frac{\pi}{4} [(4.5 \text{ in.})^2 - (3.7 \text{ in.})^2]$$

$$= 5.152 \text{ in.}^2$$

$$P_1 = \text{allowable load on one pier}$$

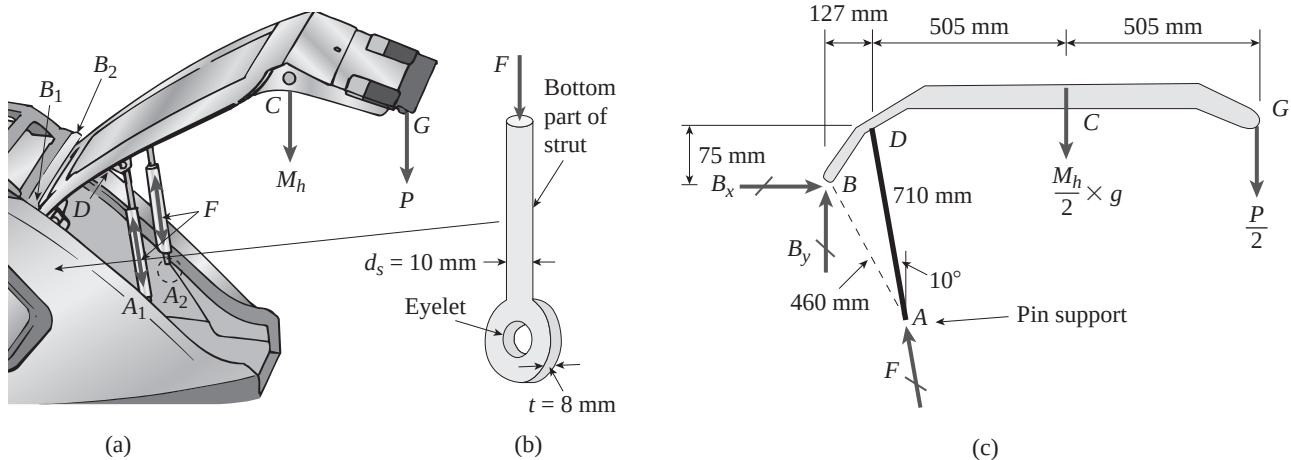
$$= \sigma_{\text{allow}} A = (14.29 \text{ ksi})(5.152 \text{ in.}^2)$$

$$= 73.62 \text{ k}$$

$$\text{Total load } P = 4P_1 = 294 \text{ k} \quad \leftarrow$$

Problem 1.8-6 The rear hatch of a van [BDCF in figure part (a)] is supported by two hinges at B_1 and B_2 and by two struts A_1B_1 and A_2B_2 (diameter $d_s = 10 \text{ mm}$) as shown in figure part (b). The struts are supported at A_1 and A_2 by pins, each with diameter $d_p = 9 \text{ mm}$ and passing through an eyelet of thickness $t = 8 \text{ mm}$ at the end of the strut [figure part (b)]. If a closing force $P = 50 \text{ N}$ is applied at G and the mass of the hatch $M_h = 43 \text{ kg}$ is concentrated at C :

- What is the force F in each strut? [Use the free-body diagram of one half of the hatch in the figure part (c)]
- What is the maximum permissible force in the strut, F_{allow} , if the allowable stresses are as follows: compressive stress in the strut, 70 MPa ; shear stress in the pin, 45 MPa ; and bearing stress between the pin and the end of the strut, 110 MPa .

**Solution 1.8-6**

NUMERICAL DATA

$$M_h = 43 \text{ kg} \quad \sigma_a = 70 \text{ MPa}$$

$$\tau_a = 45 \text{ MPa} \quad \sigma_{ba} = 110 \text{ MPa}$$

$$d_s = 10 \text{ mm} \quad d_p = 9 \text{ mm} \quad t = 8 \text{ mm}$$

$$P = 50 \text{ N} \quad g = 9.81 \text{ m/s}^2$$

- FORCE F IN EACH STRUT FROM STATICS (SUM MOMENTS ABOUT B)

$$F_V = F \cos(10^\circ) \quad F_H = F \sin(10^\circ)$$

$$\sum M_B = 0$$

$$\begin{aligned}
 F_V(127) + F_H(75) &= \frac{M_h}{2}g(127 + 505) \\
 &\quad + \frac{P}{2}[127 + 2(505)] \\
 F(127\cos(10^\circ) + 75\sin(10^\circ)) \\
 &= \frac{M_h}{2}g(127 + 505) + \frac{P}{2} + 2(505)] \\
 F &= \frac{\frac{M_h}{2}g(127 + 505) + \frac{P}{2}[127 + 2(505)]}{(127\cos(10^\circ) + 75\sin(10^\circ))} \\
 F &= 1.171 \text{ kN} \quad \leftarrow
 \end{aligned}$$

(b) MAXIMUM PERMISSIBLE FORCE F IN EACH STRUT
 $F_{\max}$ IS SMALLEST OF THE FOLLOWING

$$F_{a1} = \sigma_a \frac{\pi}{4} d_s^2 \quad F_{a1} = 5.50 \text{ kN}$$

$$F_{a2} = \tau_a \frac{\pi}{4} d_p^2$$

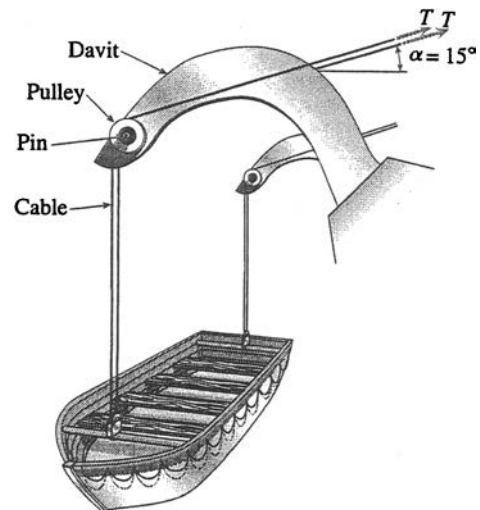
$$F_{a2} = 2.86 \text{ kN} \quad \leftarrow \quad \frac{F_{a2}}{F} = 2.445$$

$$F_{a3} = \sigma_{ba} d_p t \quad F_{a3} = 7.92 \text{ kN}$$

Problem 1.8-7 A lifeboat hangs from two ship's davits, as shown in the figure. A pin of diameter $d = 0.80$ in. passes through each davit and supports two pulleys, one on each side of the davit.

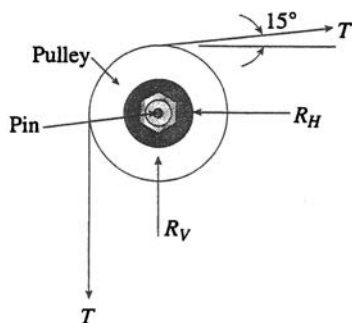
Cables attached to the lifeboat pass over the pulleys and wind around winches that raise and lower the lifeboat. The lower parts of the cables are vertical and the upper parts make an angle $\alpha = 15^\circ$ with the horizontal. The allowable tensile force in each cable is 1800 lb, and the allowable shear stress in the pins is 4000 psi.

If the lifeboat weighs 1500 lb, what is the maximum weight that should be carried in the lifeboat?



Solution 1.8-7 Lifeboat supported by four cables

FREE-BODY DIAGRAM OF ONE PULLEY



Pin diameter $d = 0.80$ in.

T = tensile force in one cable

$$T_{\text{allow}} = 1800 \text{ lb}$$

$$\tau_{\text{allow}} = 4000 \text{ psi}$$

W = weight of lifeboat

$$= 1500 \text{ lb}$$

$$\Sigma F_{\text{horiz}} = 0 \quad R_H = T \cos 15^\circ = 0.9659T$$

$$\Sigma F_{\text{vert}} = 0 \quad R_V = T - T \sin 15^\circ = 0.7412T$$

V = shear force in pin

$$V = \sqrt{(R_H)^2 + (R_V)^2} = 1.2175T$$

ALLOWABLE TENSILE FORCE IN ONE CABLE BASED
UPON SHEAR IN THE PINS

$$V_{\text{allow}} = \tau_{\text{allow}} A_{\text{pin}} = (4000 \text{ psi}) \left(\frac{\pi}{4} \right) (0.80 \text{ in.})^2$$

$$= 2011 \text{ lb}$$

$$V = 1.2175T \quad T_1 = \frac{V_{\text{allow}}}{1.2175} = 1652 \text{ lb}$$

ALLOWABLE FORCE IN ONE CABLE BASED UPON
TENSION IN THE CABLE

$$T_2 = T_{\text{allow}} = 1800 \text{ lb}$$

MAXIMUM WEIGHT

Shear in the pins governs.

$$T_{\text{max}} = T_1 = 1652 \text{ lb}$$

Total tensile force in four cables

$$= 4T_{\text{max}} = 6608 \text{ lb}$$

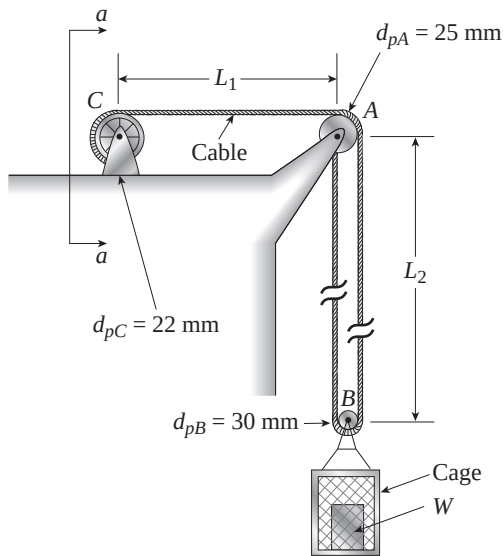
$$W_{\text{max}} = 4T_{\text{max}} - W$$

$$= 6608 \text{ lb} - 1500 \text{ lb}$$

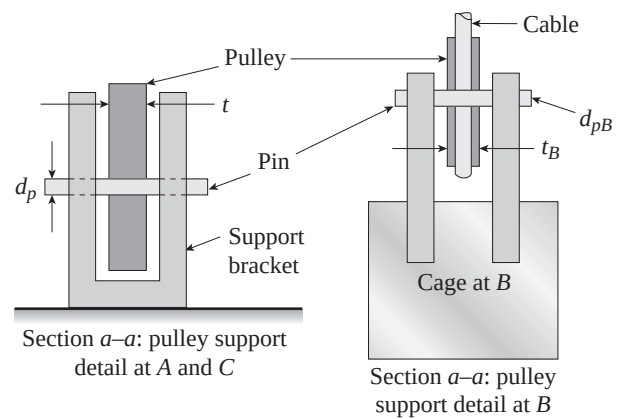
$$= 5110 \text{ lb} \quad \leftarrow$$

Problem 1.8-8 A cable and pulley system in figure part (a) supports a cage of mass 300 kg at B. Assume that this includes the mass of the cables as well. The thickness of each of the three steel pulleys is $t = 40 \text{ mm}$. The pin diameters are $d_{pA} = 25 \text{ mm}$, $d_{pB} = 30 \text{ mm}$ and $d_{pC} = 22 \text{ mm}$ [see figure, parts (a) and (b)].

- Find expressions for the resultant forces acting on the pulleys at A, B, and C in terms of cable tension T .
- What is the maximum weight W that can be added to the cage at B based on the following allowable stresses? Shear stress in the pins is 50 MPa; bearing stress between the pin and the pulley is 110 MPa.



(a)



(b)

Solution 1.8-8

NUMERICAL DATA

$$M = 300 \text{ kg} \quad g = 9.81 \text{ m/s}^2$$

$$\tau_a = 50 \text{ MPa} \quad \sigma_{ba} = 110 \text{ MPa}$$

$$t_A = 40 \text{ mm} \quad t_B = 40 \text{ mm}$$

$$t_C = 50 \quad d_{pA} = 25 \text{ mm}$$

$$d_{pB} = 30 \quad d_{pC} = 22 \text{ mm}$$

(a) RESULTANT FORCES F ACTING ON PULLEYS A , B , AND C

$$F_A = \sqrt{2} T \quad F_B = 2T$$

$$F_C = T \quad T = \frac{Mg}{2} + \frac{W_{\max}}{2}$$

$$W_{\max} = 2T - Mg$$

From statics at B (b) MAXIMUM LOAD W THAT CAN BE ADDED AT B DUE TO τ_a AND σ_{ba} IN PINS AT A , B , AND C PULLEY AT A

$$\tau_A = \frac{F_A}{A_s}$$

DOUBLE SHEAR

$$F_A = \tau_a A_s \quad \sqrt{2} T = \tau_a A_s$$

$$\frac{Mg}{2} + \frac{W_{\max}}{2} = \frac{\tau_a A_s}{\sqrt{2}}$$

$$W_{\max 1} = \frac{2}{\sqrt{2}} \left(\tau_a A_s \right) - Mg$$

$$W_{\max 1} = \frac{2}{\sqrt{2}} \left(\tau_a 2 \frac{\pi}{4} d_{pA}^2 \right) - Mg$$

$$\frac{W_{\max 1}}{Mg} = 22.6$$

$$W_{\max 1} = 66.5 \text{ kN} \quad \leftarrow$$

(shear at A controls)

OR check bearing stress

$$W_{\max 2} = \frac{2}{\sqrt{2}} \left(\sigma_{ba} A_b \right) - Mg$$

$$W_{\max 2} = \frac{2}{\sqrt{2}} \left(\sigma_{ba} t_A d_{pA} \right) - Mg$$

$$W_{\max 2} = 152.6 \text{ kN} \quad (\text{bearing at } A)$$

PULLEY AT B $2T = \tau_a A_s$

$$W_{\max 3} = \frac{2}{2} (\tau_a A_s) - Mg$$

$$W_{\max 3} = \left[\tau_a \left(2 \frac{\pi}{4} d_{pB}^2 \right) \right] - Mg$$

$$W_{\max 3} = 67.7 \text{ kN} \quad (\text{shear at } B)$$

$$W_{\max 4} = \frac{2}{2} (\sigma_{ba} A_b) - Mg$$

$$W_{\max 4} = \sigma_{ba} t_B d_{pB} - Mg$$

$$W_{\max 4} = 129.1 \text{ kN} \quad (\text{bearing at } B)$$

PULLEY AT C $T = \tau_a A_s$

$$W_{\max 5} = 2(\tau_a A_s) - Mg$$

$$W_{\max 5} = \left[2\tau_a \left(2 \frac{\pi}{4} d_{pC}^2 \right) \right] - Mg$$

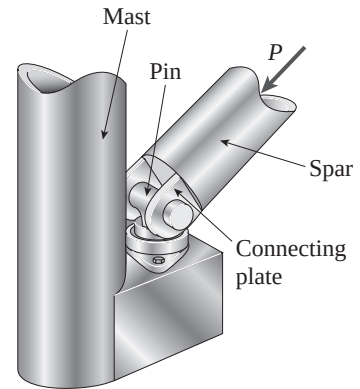
$$W_{\max 5} = 7.3 \times 10^4 \quad W_{\max 5} = 73.1 \text{ kN} \quad (\text{shear at } C)$$

$$W_{\max 6} = 2\sigma_{ba} t_C d_{pC} - Mg$$

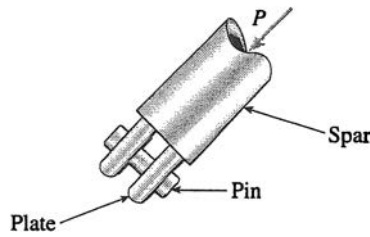
$$W_{\max 6} = 239.1 \text{ kN} \quad (\text{bearing at } C)$$

Problem 1.8-9 A ship's spar is attached at the base of a mast by a pin connection (see figure). The spar is a steel tube of outer diameter $d_2 = 3.5$ in. and inner diameter $d_1 = 2.8$ in. The steel pin has diameter $d = 1$ in., and the two plates connecting the spar to the pin have thickness $t = 0.5$ in. The allowable stresses are as follows: compressive stress in the spar, 10 ksi; shear stress in the pin, 6.5 ksi; and bearing stress between the pin and the connecting plates, 16 ksi.

Determine the allowable compressive force P_{allow} in the spar.



Solution 1.8-9



NUMERICAL DATA

$$d_2 = 3.5 \text{ in.} \quad d_1 = 2.8 \text{ in.}$$

$$d_p = 1 \text{ in.} \quad t = 0.5 \text{ in.}$$

$$\sigma_a = 10 \text{ ksi} \quad \tau_a = 6.5 \text{ ksi} \quad \sigma_{ba} = 16 \text{ ksi}$$

COMPRESSIVE STRESS IN SPAR

$$P_{a1} = \sigma_a \frac{\pi}{4} (d_2^2 - d_1^2) \quad P_{a1} = 34.636 \text{ k}$$

SHEAR STRESS IN PIN

$$P_{a2} = \tau_a \left(2 \frac{\pi}{4} d_p^2 \right)$$

$$P_{a2} = 10.21 \text{ kips} < \text{controls} \quad \leftarrow$$

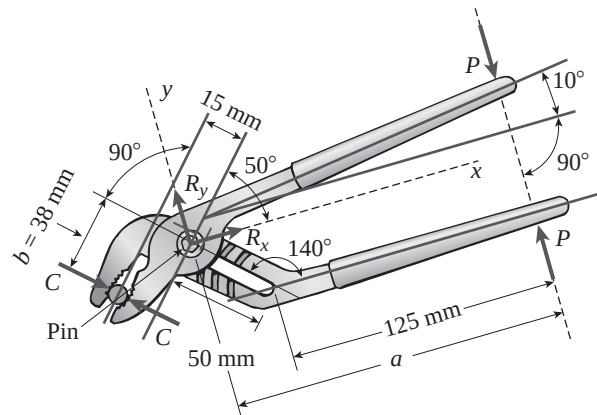
^double shear

BEARING STRESS BETWEEN PIN AND CONNECTING PLATES

$$P_{a3} = \sigma_{ba} (2 d_p t) \quad P_{a3} = 16 \text{ k}$$

Problem 1.8-10 What is the maximum possible value of the clamping force C in the jaws of the pliers shown in the figure if the ultimate shear stress in the 5-mm diameter pin is 340 MPa?

What is the maximum permissible value of the applied load P if a factor of safety of 3.0 with respect to failure of the pin is to be maintained?



Solution 1.8-10

NUMERICAL DATA

$$FS = 3 \quad \tau_u = 340 \text{ MPa} \quad \tau_a = \frac{\tau_u}{FS}$$

$$d = 5 \text{ mm}$$

$$\tau_a = \frac{\sqrt{R_x^2 + R_y^2}}{A_s} < \text{pin at C in single shear}$$

$$R_x = -C \cos(40^\circ) \quad R_y = P + C \sin(40^\circ)$$

$$a = 50 \cos(40^\circ) + 125 \quad a = 163.302 \text{ mm}$$

$$b = 38 \text{ mm}$$

$$\text{STATICS} \quad \sum M_{\text{pin}} = 0 \quad C = \frac{P(a)}{b}$$

$$R_x = -\frac{P(a)}{b} \cos(40^\circ) \quad R_y = P \left[1 + \frac{a}{b} \sin(40^\circ) \right]$$

$$P \sqrt{\left[-\frac{a}{b} \cos(40^\circ) \right]^2 + \left[1 + \frac{a}{b} \sin(40^\circ) \right]^2} = \tau_a A_s$$

$$A_s = \frac{\pi}{4} d^2$$

$$\tau_a = \frac{\tau_u}{FS} \quad \tau_a = 113.333 \text{ MPa}$$

Find $P_{\max}$

$$P_{\max} = \frac{\tau_a A_s}{\sqrt{\left[-\frac{a}{b} \cos(40^\circ) \right]^2 + \left[1 + \frac{a}{b} \sin(40^\circ) \right]^2}}$$

$$P_{\max} = 445 \text{ N} \quad \leftarrow$$

$$\text{here } \frac{a}{b} = 4.297 < a/b = \text{mechanical advantage}$$

FIND MAXIMUM CLAMPING FORCE

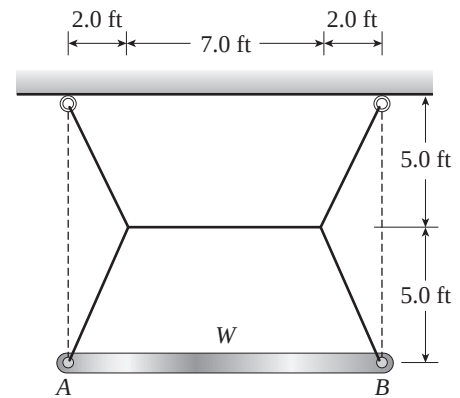
$$C_{\text{ult}} = P_{\max} FS \left(\frac{a}{b} \right) \quad C_{\text{ult}} = 5739 \text{ N} \quad \leftarrow$$

$$P_{\text{ult}} = P_{\max} FS \quad P_{\text{ult}} = 1335$$

$$\frac{C_{\text{ult}}}{P_{\text{ult}}} = 4.297$$

Problem 1.8-11 A metal bar AB of weight W is suspended by a system of steel wires arranged as shown in the figure. The diameter of the wires is $5/64$ in., and the yield stress of the steel is 65 ksi.

Determine the maximum permissible weight $W_{\max}$ for a factor of safety of 1.9 with respect to yielding.

**Solution 1.8-11**

NUMERICAL DATA

$$d = \frac{5}{64} \text{ in.} \quad \sigma_Y = 65 \text{ ksi} \quad FS_y = 1.9$$

$$\sigma_a = \frac{\sigma_Y}{FS_y} \quad \sigma_a = 34.211 \text{ ksi}$$

FORCES IN WIRES AC , EC , BD , AND FD

$$\sum F_V = 0 \quad \text{at } A, B, E, \text{ or } F$$

$$F_W = \frac{\sqrt{2^2 + 5^2}}{5} \times \frac{W}{2} \quad \frac{\sqrt{2^2 + 5^2}}{10} = 0.539$$

$$W_{\max} = 0.539 \sigma_a \times A$$

$$W_{\max} = 0.539 \left(\frac{\sigma_Y}{FS_Y} \right) \left(\frac{\pi}{4} d^2 \right)$$

$$W_{\max} = 0.305 \text{ kips} \quad \leftarrow$$

CHECK ALSO FORCE IN WIRE CD

$$\sum F_H = 0 \quad \text{at } C \text{ or } D$$

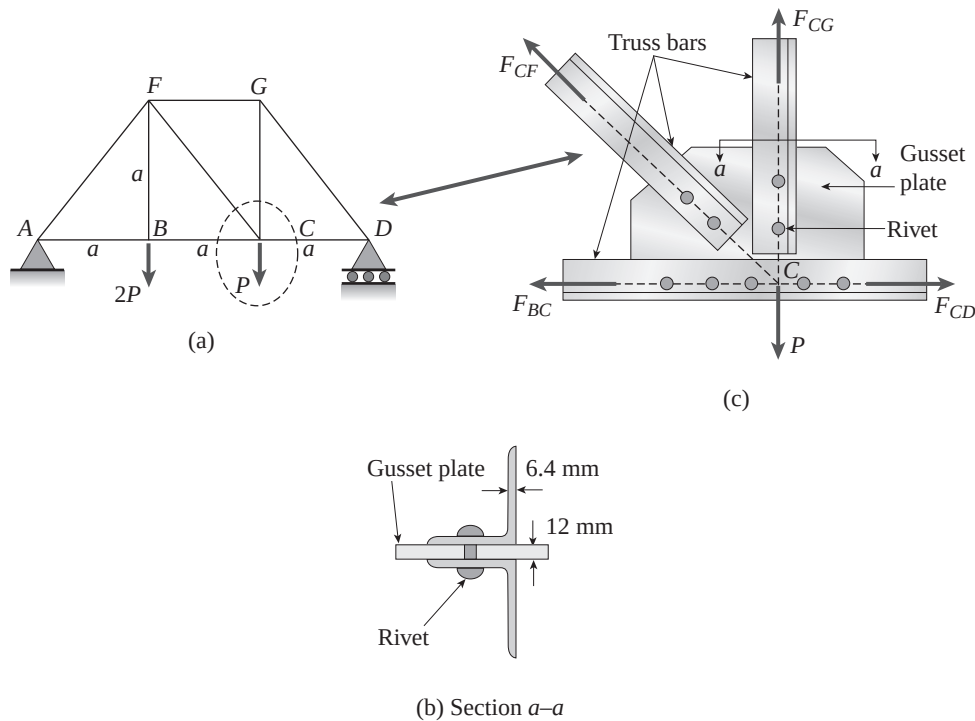
$$F_{CD} = 2 \left(\frac{2}{\sqrt{2^2 + 5^2}} F_w \right)$$

$$F_{CD} = 2 \left[\frac{2}{\sqrt{2^2 + 5^2}} \left(\frac{\sqrt{2^2 + 5^2}}{5} \times \frac{W}{2} \right) \right]$$

$$F_{CD} = \frac{2}{5} W \quad \text{less than } F_{AC} \text{ so } AC \text{ controls}$$

Problem 1.8-12 A plane truss is subjected to loads $2P$ and P at joints B and C , respectively, as shown in the figure part (a). The truss bars are made of two L102 $\times$ 76 $\times$ 6.4 steel angles [see Table F-5(b): cross sectional area of the two angles, $A = 2180 \text{ mm}^2$, figure part (b)] having an ultimate stress in tension equal to 390 MPa. The angles are connected to an 12 mm-thick gusset plate at C [figure part (c)] with 16-mm diameter rivets; assume each rivet transfers an equal share of the member force to the gusset plate. The ultimate stresses in shear and bearing for the rivet steel are 190 MPa and 550 MPa, respectively.

Determine the allowable load P_{allow} if a safety factor of 2.5 is desired with respect to the ultimate load that can be carried. (Consider tension in the bars, shear in the rivets, bearing between the rivets and the bars, and also bearing between the rivets and the gusset plate. Disregard friction between the plates and the weight of the truss itself.)



Solution 1.8-12

NUMERICAL DATA

$$A = 2180 \text{ mm}^2$$

$$t_g = 12 \text{ mm} \quad d_r = 16 \text{ mm} \quad t_{\text{ang}} = 6.4 \text{ mm}$$

$$\sigma_u = 390 \text{ MPa} \quad \tau_u = 190 \text{ MPa}$$

$$\sigma_{bu} = 550 \text{ MPa} \quad \text{FS} = 2.5$$

$$\sigma_a = \frac{\sigma_u}{\text{FS}} \quad \tau_a = \frac{\tau_u}{\text{FS}} \quad \sigma_{ba} = \frac{\sigma_{bu}}{\text{FS}}$$

MEMBER FORCES FROM TRUSS ANALYSIS

$$F_{BC} = \frac{5}{3}P \quad F_{CD} = \frac{4}{3}P \quad F_{CF} = \frac{\sqrt{2}}{3}P$$

$$\frac{\sqrt{2}}{3} = 0.471 \quad F_{CG} = \frac{4}{3}P$$

 P_{allow} FOR TENSION ON NET SECTION IN TRUSS BARS

$$A_{\text{net}} = A - 2d_r t_{\text{ang}} \quad A_{\text{net}} = 1975 \text{ mm}^2$$

$$\frac{A_{\text{net}}}{A} = 0.906$$

$$F_{\text{allow}} = \sigma_a A_{\text{net}} < \text{allowable force in a member}$$

so BC controls since it has the largest
member force for this loading

$$P_{\text{allow}} = \frac{3}{5}F_{BC\text{max}} \quad P_{\text{allow}} = \frac{3}{5}(\sigma_a A_{\text{net}})$$

$$P_{\text{allow}} = 184.879 \text{ kN}$$

Next, P_{allow} for shear in rivets (all are in double shear)

$$A_s = 2\frac{\pi}{4}d_r^2 < \text{for one rivet in DOUBLE shear}$$

$$\frac{F_{\text{max}}}{N} = \tau_a A_s \quad N = \text{number of rivets in a particular member (see drawing of connection detail)}$$

$$P_{BC} = 3\left(\frac{3}{5}\right)(\tau_a A_s) \quad P_{BC} = 55.0 \text{ kN}$$

$$P_{CF} = 2\left(\frac{3}{\sqrt{2}}\right)(\tau_a A_s) \quad P_{CF} = 129.7 \text{ kN}$$

$$P_{CG} = 2\left(\frac{3}{4}\right)(\tau_a A_s)$$

$$P_{CG} = 45.8 \text{ kN} \leftarrow < \text{so shear in rivets in CG and CD controls } P_{\text{allow}} \text{ here}$$

$$P_{CD} = 2\left(\frac{3}{4}\right)(\tau_a A_s) \quad P_{CD} = 45.8 \text{ kN} \leftarrow$$

Next, P_{allow} for bearing of rivets on truss bars
 $A_b = 2d_r t_{\text{ang}} < \text{rivet bears on each angle in two angle pairs}$

$$\frac{F_{\text{max}}}{N} = \sigma_{ba} A_b$$

$$P_{BC} = 3\left(\frac{3}{5}\right)(\sigma_{ba} A_b) \quad P_{BC} = 81.101 \text{ kN}$$

$$P_{CF} = 2\left(\frac{3}{\sqrt{2}}\right)(\sigma_{ba} A_b) \quad P_{CF} = 191.156 \text{ kN}$$

$$P_{CG} = 2\left(\frac{3}{4}\right)(\sigma_{ba} A_b) \quad P_{CG} = 67.584 \text{ kN}$$

$$P_{CD} = 2\left(\frac{3}{4}\right)(\sigma_{ba} A_b) \quad P_{CD} = 67.584 \text{ kN}$$

Finally, P_{allow} for bearing of rivets on gusset plate

$$A_b = d_r t_g$$

(bearing area for each rivet on gusset plate)

$$t_g = 12 \text{ mm} < 2t_{\text{ang}} = 12.8 \text{ mm}$$

so gusset will control over angles

$$P_{BC} = 3\left(\frac{3}{5}\right)(\sigma_{ba} A_b) \quad P_{BC} = 76.032 \text{ kN}$$

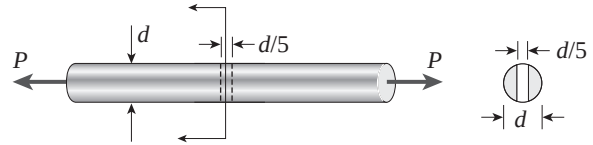
$$P_{CF} = 2\left(\frac{3}{\sqrt{2}}\right)(\sigma_{ba} A_b) \quad P_{CF} = 179.209 \text{ kN}$$

$$P_{CG} = 2\left(\frac{3}{4}\right)(\sigma_{ba} A_b) \quad P_{CG} = 63.36 \text{ kN}$$

$$P_{CD} = 2\left(\frac{3}{4}\right)(\sigma_{ba} A_b) \quad P_{CD} = 63.36 \text{ kN}$$

$$\text{So, shear in rivets controls: } P_{\text{allow}} = 45.8 \text{ kN} \leftarrow$$

Problem 1.8-13 A solid bar of circular cross section (diameter d) has a hole of diameter $d/5$ drilled laterally through the center of the bar (see figure). The allowable average tensile stress on the net cross section of the bar is σ_{allow} .



- (a) Obtain a formula for the allowable load P_{allow} that the bar can carry in tension.
 (b) Calculate the value of P_{allow} if the bar is made of brass with diameter $d = 1.75$ in. and $\sigma_{\text{allow}} = 12$ ksi.
 (Hint: Use the formulas of Case 15 Appendix E.)

Solution 1.8-13

NUMERICAL DATA

$$d = 1.75 \text{ in.} \quad \sigma_a = 12 \text{ ksi}$$

(a) FORMULA FOR P_{ALLOW} IN TENSION

From Case 15, Appendix E:

$$A = 2r^2 \left(\alpha - \frac{ab}{r^2} \right) \quad r = \frac{d}{2} \quad a = \frac{d}{10}$$

$$\alpha = \arccos\left(\frac{a}{r}\right) \quad r = 0.875 \text{ in.} \quad a = 0.175 \text{ in.}$$

$$\alpha \frac{180}{\pi} = 78.463^\circ$$

$$b = \sqrt{r^2 - a^2}$$

$$b = \sqrt{\left[\left(\frac{d}{2}\right)^2 - \left(\frac{d}{10}\right)^2\right]}$$

$$b = \sqrt{\left(\frac{6}{25}d^2\right)} \quad b = \frac{d}{5}\sqrt{6}$$

$$P_a = \sigma_a A$$

$$P_a = \sigma_a \left[\frac{1}{2} d^2 \left(\arccos\left(\frac{1}{5}\right) - \frac{2}{25} \sqrt{6} \right) \right]$$

$$\frac{\arccos\left(\frac{1}{5}\right) - \frac{2}{25} \sqrt{6}}{2} = 0.587 \quad \frac{\pi}{4} = 0.785$$

$$P_a = \sigma_a (0.587 d^2) \quad \leftarrow$$

$$\frac{0.587}{0.785} = 0.748$$

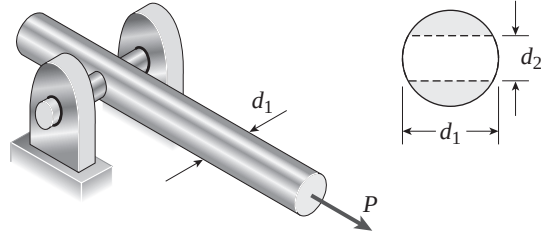
(b) EVALUATE NUMERICAL RESULT

$$d = 1.75 \text{ in.} \quad \sigma_a = 12 \text{ ksi}$$

$$P_a = 21.6 \text{ k} \quad \leftarrow$$

Problem 1.8-14 A solid steel bar of diameter $d_1 = 60$ mm has a hole of diameter $d_2 = 32$ mm drilled through it (see figure). A steel pin of diameter d_2 passes through the hole and is attached to supports.

Determine the maximum permissible tensile load P_{allow} in the bar if the yield stress for shear in the pin is $\tau_Y = 120$ MPa, the yield stress for tension in the bar is $\sigma_Y = 250$ MPa and a factor of safety of 2.0 with respect to yielding is required. (Hint: Use the formulas of Case 15, Appendix E.)



Solution 1.8-14

NUMERICAL DATA

$$\begin{aligned} d_1 &= 60 \text{ mm} & d_2 &= 32 \text{ mm} \\ \tau_Y &= 120 \text{ MPa} & \sigma_Y &= 250 \text{ MPa} \\ \text{FS}_y &= 2 \end{aligned}$$

ALLOWABLE STRESSES

$$\begin{aligned} \tau_a &= \frac{\tau_Y}{\text{FS}_y} & \tau_a &= 60 \text{ MPa} \\ \sigma_a &= \frac{\sigma_Y}{\text{FS}_y} & \sigma_a &= 125 \text{ MPa} \end{aligned}$$

From Case 15, Appendix E: $r = \frac{d_1}{2}$

$$A = 2r^2 \left(\alpha - \frac{ab}{r^2} \right) \quad \alpha = \arccos \frac{d_2/2}{d_1/2} = \arccos \frac{d_2}{d_1}$$

$$a = \frac{d_2}{2} \quad b = \sqrt{r^2 - a^2}$$

SHEAR AREA (DOUBLE SHEAR)

$$A_s = 2 \left(\frac{\pi}{4} d_2^2 \right) \quad A_s = 1608 \text{ mm}^2$$

NET AREA IN TENSION (FROM APPENDIX E)

$$A_{\text{net}} = 2 \left(\frac{d_1}{2} \right)^2 \left[\arccos \left(\frac{d_2}{d_1} \right) - \frac{\frac{d_2}{2} \left[\sqrt{\left(\frac{d_1}{2} \right)^2 - \left(\frac{d_2}{2} \right)^2} \right]}{\left(\frac{d_1}{2} \right)^2} \right]$$

$$A_{\text{net}} = 1003 \text{ mm}^2$$

P_{allow} in tension: smaller of values based on either shear or tension allowable stress $\times$ appropriate area

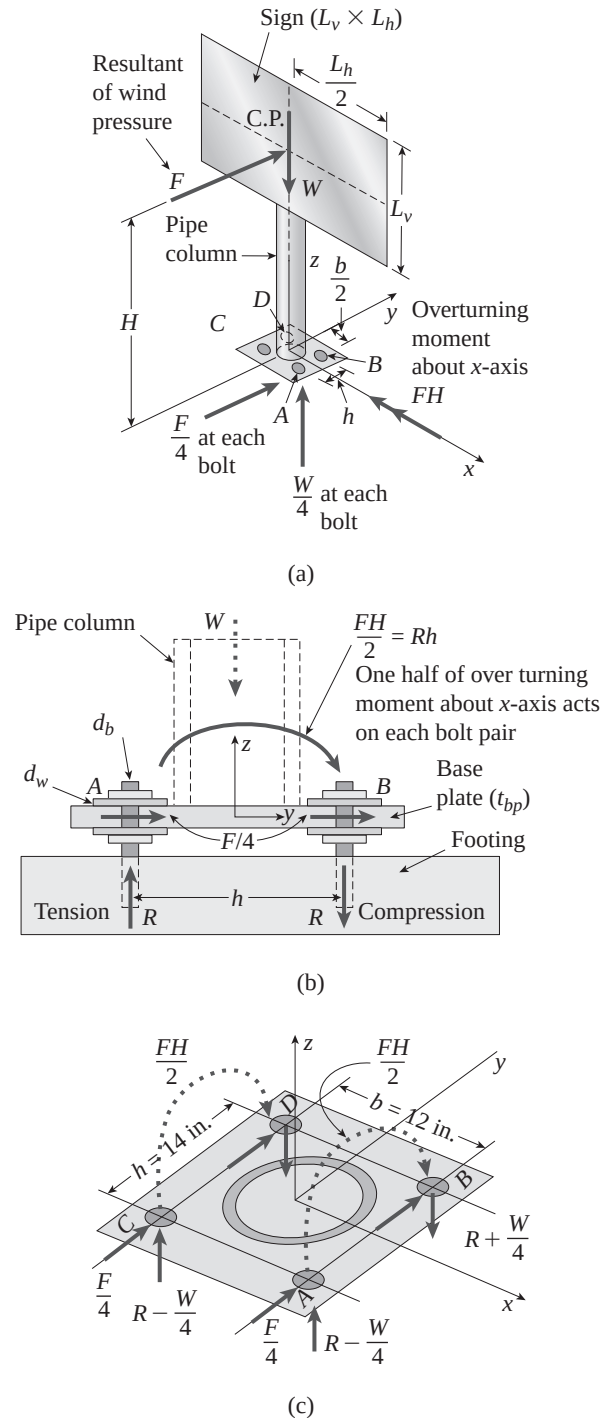
$$P_{a1} = \tau_a A_s \quad P_{a1} = 96.5 \text{ kN} < \text{shear governs} \quad \leftarrow$$

$$P_{a2} = \sigma_a A_{\text{net}} \quad P_{a2} = 125.4 \text{ kN}$$

Problem 1.8-15 A sign of weight W is supported at its base by four bolts anchored in a concrete footing. Wind pressure p acts normal to the surface of the sign; the resultant of the uniform wind pressure is force F at the center of pressure. The wind force is assumed to create equal shear forces $F/4$ in the y -direction at each bolt [see figure parts (a) and (c)]. The overturning effect of the wind force also causes an uplift force R at bolts A and C and a downward force ($-R$) at bolts B and D [see figure part (b)]. The resulting effects of the wind, and the associated ultimate stresses for each stress condition, are: normal stress in each bolt ($\sigma_u = 60$ ksi); shear through the base plate ($\tau_u = 17$ ksi); horizontal shear and bearing on each bolt ($\tau_{hu} = 25$ ksi and $\sigma_{bu} = 75$ ksi); and bearing on the bottom washer at B (or D) ($\sigma_{bw} = 50$ ksi).

Find the maximum wind pressure $p_{\max}$ (psf) that can be carried by the bolted support system for the sign if a safety factor of 2.5 is desired with respect to the ultimate wind load that can be carried.

Use the following numerical data: bolt $d_b = 3/4$ in.; washer $d_w = 1.5$ in.; base plate $t_{bp} = 1$ in.; base plate dimensions $h = 14$ in. and $b = 12$ in.; $W = 500$ lb; $H = 17$ ft; sign dimensions ($L_v = 10$ ft. $\times$ $L_h = 12$ ft.); pipe column diameter $d = 6$ in., and pipe column thickness $t = 3/8$ in.



Solution 1.8-15

NUMERICAL DATA

$$\begin{aligned}
 \sigma_u &= 60 \text{ ksi} & \tau_u &= 17 \text{ ksi} & \tau_{hu} &= 25 \text{ ksi} \\
 \sigma_{bu} &= 75 \text{ ksi} & \sigma_{bw} &= 50 \text{ ksi} & \text{FS}_u &= 2.5 \\
 d_b &= \frac{3}{4} \text{ in.} & d_w &= 1.5 \text{ in.} & t_{bp} &= 1 \text{ in.} \\
 h &= 14 \text{ in.} & b &= 12 \text{ in.} & d &= 6 \text{ in.} & t &= \frac{3}{8} \text{ in.} \\
 W &= 0.500 \text{ kips} & H &= 17(12) & H &= 204 \text{ in.} \\
 L_v &= 10(12) & L_h &= 12(12) & L_v &= 120 \text{ in.} \\
 L_h &= 144 \text{ in.}
 \end{aligned}$$

ALLOWABLE STRESSES (ksi)

$$\begin{aligned}
 \sigma_a &= \frac{\sigma_u}{\text{FS}_u} & \sigma_a &= 24 & \tau_a &= \frac{\tau_u}{\text{FS}_u} \\
 \tau_a &= 6.8 & \tau_{ha} &= \frac{\tau_{hu}}{\text{FS}_u} & \tau_{ha} &= 10 \\
 \sigma_{ba} &= \frac{\sigma_{bu}}{\text{FS}_u} & \sigma_{ba} &= 30 & \sigma_{bwa} &= \frac{\sigma_{bw}}{\text{FS}_u} \\
 \sigma_{bwa} &= 20
 \end{aligned}$$

FORCES F AND R IN TERMS OF $p_{\max}$

$$\begin{aligned}
 F &= p_{\max} L_v L_h & R &= \frac{FH}{2h} \\
 R &= p_{\max} \frac{L_v L_h H}{2h}
 \end{aligned}$$

- (1) COMPUTE
- $p_{\max}$
- BASED ON NORMAL STRESS IN EACH BOLT (GREATER AT
- B
- AND
- D
-)

$$\begin{aligned}
 \sigma &= \frac{R + \frac{W}{4}}{\frac{\pi}{4} d_b^2} & R_{\max} &= \sigma_a \left(\frac{\pi}{4} d_b^2 \right) - \frac{W}{4} \\
 p_{\max 1} &= \frac{\sigma_a \left(\frac{\pi}{4} d_b^2 \right) - \frac{W}{4}}{\frac{L_v L_h H}{2h}} \\
 p_{\max 1} &= 11.98 \text{ psf} \quad \leftarrow \quad \text{controls}
 \end{aligned}$$

- (2) COMPUTE
- $p_{\max}$
- BASED ON SHEAR THROUGH BASE PLATE (GREATER AT
- B
- AND
- D
-)

$$\begin{aligned}
 \tau &= \frac{R + \frac{W}{4}}{\pi d_w t_{bp}} \\
 R_{\max} &= \tau_a (\pi d_w t_{bp}) - \frac{W}{4} \\
 p_{\max 2} &= \frac{\tau_a (\pi d_w t_{bp}) - \frac{W}{4}}{\frac{L_v L_h H}{2h}} \\
 p_{\max 2} &= 36.5 \text{ psf}
 \end{aligned}$$

- (3) COMPUTE $p_{\max}$ BASED ON HORIZONTAL SHEAR ON EACH BOLT

$$\tau_h = \frac{\frac{F}{4}}{\left(\frac{\pi}{4} d_b^2\right)} \quad F_{\max} = 4\tau_{ha} \left(\frac{\pi}{4} d_b^2\right)$$

$$p_{\max 3} = \frac{\tau_{ha}(\pi d_b^2)}{L_v L_h}$$

$$p_{\max 3} = 147.3 \text{ psf}$$

- (4) COMPUTE $p_{\max}$ BASED ON HORIZONTAL BEARING ON EACH BOLT

$$\sigma_b = \frac{\frac{F}{4}}{(t_{bp} d_b)} \quad F_{\max} = 4\sigma_{ba}(t_{bp} d_b)$$

$$p_{\max 4} = \frac{4\sigma_{ba}(t_{bp} d_b)}{L_v L_h}$$

$$p_{\max 4} = 750 \text{ psf}$$

- (5) COMPUTE $p_{\max}$ BASED ON BEARING UNDER THE TOP WASHER AT A (OR C) AND THE BOTTOM WASHER AT B (OR D)

$$\sigma_{bw} = \frac{R + \frac{W}{4}}{\frac{\pi}{4}(d_w^2 - d_b^2)}$$

$$R_{\max} = \sigma_{bwa} \left[\frac{\pi}{4}(d_w^2 - d_b^2) \right] - \frac{W}{4}$$

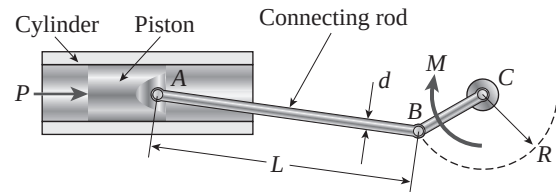
$$p_{\max 5} = \frac{\sigma_{bwa} \left[\frac{\pi}{4}(d_w^2 - d_b^2) \right] - \frac{W}{4}}{\frac{L_v L_h H}{2h}}$$

$$p_{\max 5} = 30.2 \text{ psf}$$

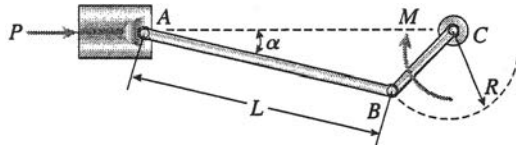
So, normal/stress in bolts controls; $p_{\max} = 11.98 \text{ psf}$

Problem 1.8-16 The piston in an engine is attached to a connecting rod AB , which in turn is connected to a crank arm BC (see figure). The piston slides without friction in a cylinder and is subjected to a force P (assumed to be constant) while moving to the right in the figure. The connecting rod, which has diameter d and length L , is attached at both ends by pins. The crank arm rotates about the axle at C with the pin at B moving in a circle of radius R . The axle at C , which is supported by bearings, exerts a resisting moment M against the crank arm.

- Obtain a formula for the maximum permissible force P_{allow} based upon an allowable compressive stress σ_c in the connecting rod.
- Calculate the force P_{allow} for the following data:
 $\sigma_c = 160 \text{ MPa}$, $d = 9.00 \text{ mm}$, and $R = 0.28L$.

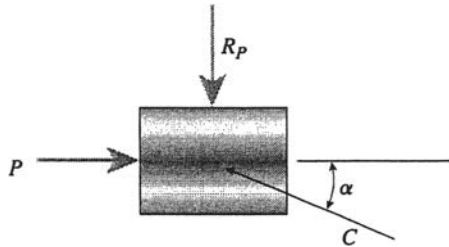


Solution 1.8-16



d = diameter of rod AB

FREE-BODY DIAGRAM OF PISTON



P = applied force (constant)

C = compressive force in connecting rod

RP = resultant of reaction forces between cylinder and piston (no friction)

$$\sum F_{\text{horiz}} = 0 \rightarrow \leftarrow$$

$$P - C \cos \alpha = 0$$

$$P = C \cos \alpha$$

MAXIMUM COMPRESSIVE FORCE C IN CONNECTING ROD

$$C_{\text{max}} = \sigma_c A_c$$

in which A_c = area of connecting rod

$$A_c = \frac{\pi d^2}{4}$$

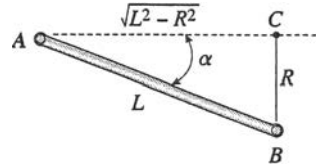
MAXIMUM ALLOWABLE FORCE P

$$P = C_{\text{max}} \cos \alpha$$

$$= \sigma_c A_c \cos \alpha$$

The maximum allowable force P occurs when $\cos \alpha$ has its smallest value, which means that α has its largest value.

LARGEST VALUE OF α



The largest value of α occurs when point B is the farthest distance from line AC. The farthest distance is the radius R of the crank arm.

Therefore,

$$\overline{BC} = R$$

$$\text{Also, } \overline{AC} = \sqrt{L^2 - R^2}$$

$$\cos \alpha = \frac{\sqrt{L^2 - R^2}}{L} = \sqrt{1 - \left(\frac{R}{L}\right)^2}$$

(a) MAXIMUM ALLOWABLE FORCE P

$$P_{\text{allow}} = \sigma_c A_c \cos \alpha$$

$$= \sigma_c \left(\frac{\pi d^2}{4} \right) \sqrt{1 - \left(\frac{R}{L}\right)^2} \quad \leftarrow$$

(b) SUBSTITUTE NUMERICAL VALUES

$$\sigma_c = 160 \text{ MPa} \quad d = 9.00 \text{ mm}$$

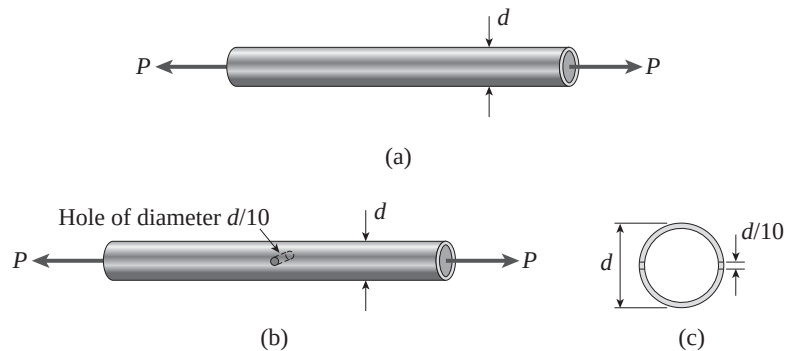
$$R = 0.28L \quad R/L = 0.28$$

$$P_{\text{allow}} = 9.77 \text{ kN} \quad \leftarrow$$

Design for Axial Loads and Direct Shear

Problem 1.9-1 An aluminum tube is required to transmit an axial tensile force $P = 33$ k [see figure part (a)]. The thickness of the wall of the tube is to be 0.25 in.

- What is the minimum required outer diameter $d_{\min}$ if the allowable tensile stress is 12,000 psi?
- Repeat part (a) if the tube will have a hole of diameter $d/10$ at mid-length [see figure parts (b) and (c)].



Solution 1.9-1

NUMERICAL DATA

$$P = 33 \text{ kips} \quad t = 0.25 \text{ in.} \quad \sigma_a = 12 \text{ ksi}$$

- (a) MINIMUM DIAMETER OF TUBE (NO HOLES)

$$A_1 = \frac{\pi}{4} [d^2 - (d-2t)^2] \quad A_2 = \frac{P}{\sigma_a}$$

$$A_2 = 2.75 \text{ in.}^2$$

Equating A_1 and A_2 and solving for d :

$$d = \frac{P}{\pi \sigma_a t} + t \quad d = 3.75 \text{ in.} \quad \leftarrow$$

- (b) MINIMUM DIAMETER OF TUBE (WITH HOLES)

$$A_1 = \left[\frac{\pi}{4} [d^2 - (d-2t)^2] - 2 \left(\frac{d}{10} \right) t \right]$$

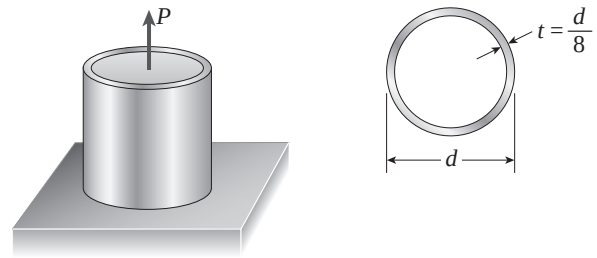
$$A_1 = d \left(\pi t - \frac{t}{5} \right) - \pi t^2$$

Equating A_1 and A_2 and solving for d :

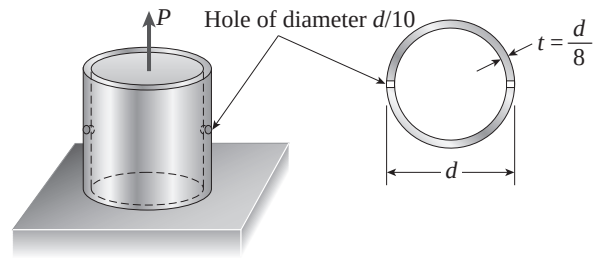
$$d = \frac{\frac{P}{\sigma_a} + \pi t^2}{\pi t - \frac{t}{5}} \quad d = 4.01 \text{ in.} \quad \leftarrow$$

Problem 1.9-2 A copper alloy pipe having yield stress $\sigma_Y = 290$ MPa is to carry an axial tensile load $P = 1500$ kN [see figure part (a)]. A factor of safety of 1.8 against yielding is to be used.

- If the thickness t of the pipe is to be one-eighth of its outer diameter, what is the minimum required outer diameter $d_{\min}$?
- Repeat part (a) if the tube has a hole of diameter $d/10$ drilled through the entire tube as shown in the figure [part (b)].



(a)



(b)

Solution 1.9-2

NUMERICAL DATA

$$\sigma_Y = 290 \text{ MPa}$$

$$P = 1500 \text{ kN}$$

$$FS_y = 1.8$$

- (a) MINIMUM DIAMETER (NO HOLES)

$$A_1 = \frac{\pi}{4} \left[d^2 - \left(d - \frac{d}{4} \right)^2 \right]$$

$$A_1 = \frac{7}{64} \pi d^2$$

$$A_2 = \frac{P}{\frac{\sigma_Y}{FS_y}} \quad A_2 = 9.31 \times 10^3 \text{ mm}^2$$

Equate A_1 and A_2 and solve for d :

$$d^2 = \frac{7}{64\pi} \left(\frac{P}{\frac{\sigma_Y}{FS_y}} \right)$$

$$d_{\min} = \sqrt{\frac{7}{64\pi} \left(\frac{P}{\frac{\sigma_Y}{FS_y}} \right)}$$

$$d_{\min} = 164.6 \text{ mm} \quad \leftarrow$$

(b) MINIMUM DIAMETER (WITH HOLES)

Redefine A_1 —subtract area for two holes—then equate to A_2

$$A_1 = \left[\frac{\pi}{4} \left[d^2 - \left(d - \frac{d}{4} \right)^2 \right] - 2 \left(\frac{d}{10} \right) \left(\frac{d}{8} \right) \right]$$

$$A_1 = \frac{7}{64} \pi d^2 - \frac{1}{40} d^2$$

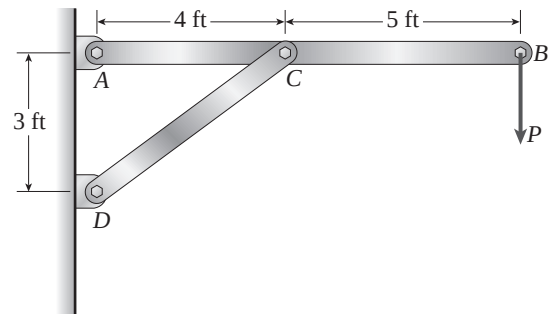
$$A_1 = d^2 \left(\frac{7}{64} \pi - \frac{1}{40} \right) \quad \frac{7}{64} \pi - \frac{1}{40} = 0.319$$

$$d^2 = \frac{\left(\frac{P}{\sigma_y} \right)}{\left(\frac{7}{64} \pi - \frac{1}{40} \right)}$$

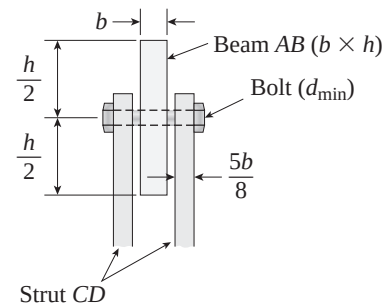
$$d_{\min} = \sqrt{\frac{\left(\frac{P}{\sigma_y} \right)}{\left(\frac{7}{64} \pi - \frac{1}{40} \right)}} \quad d_{\min} = 170.9 \text{ mm} \quad \leftarrow$$

Problem 1.9-3 A horizontal beam AB with cross-sectional dimensions ($b = 0.75 \text{ in.}$) $\times$ ($h = 8.0 \text{ in.}$) is supported by an inclined strut CD and carries a load $P = 2700 \text{ lb}$ at joint B [see figure part (a)]. The strut, which consists of two bars each of thickness $5b/8$, is connected to the beam by a bolt passing through the three bars meeting at joint C [see figure part (b)].

- If the allowable shear stress in the bolt is 13,000 psi, what is the minimum required diameter $d_{\min}$ of the bolt at C ?
- If the allowable bearing stress in the bolt is 19,000 psi, what is the minimum required diameter $d_{\min}$ of the bolt at C ?



(a)



(b)

Solution 1.9-3

NUMERICAL DATA

$$P = 2.7 \text{ k} \quad b = 0.75 \text{ in.} \quad h = 8 \text{ in.}$$

$$\tau_a = 13 \text{ ksi} \quad \sigma_{ba} = 19 \text{ ksi}$$

(a) $d_{\min}$ BASED ON ALLOWABLE SHEAR—DOUBLE SHEAR
IN STRUT

$$\tau_a = \frac{F_{DC}}{A_s} \quad F_{DC} = \frac{15}{4}P$$

$$A_s = 2\left(\frac{\pi}{4}d^2\right)$$

$$d_{\min} = \sqrt{\frac{\frac{15}{4}P}{\tau_a\left(\frac{\pi}{2}\right)}} \quad d_{\min} = 0.704 \text{ in.} \quad \leftarrow$$

(b) $d_{\min}$ BASED ON ALLOWABLE BEARING AT JT C

$$\text{Bearing from beam } ACB \quad \sigma_b = \frac{15P/4}{bd}$$

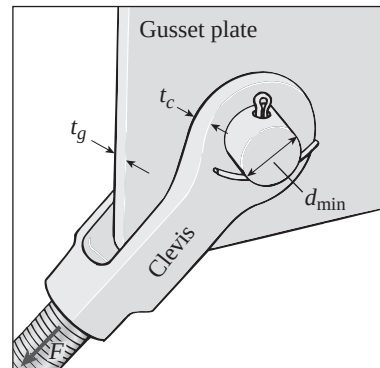
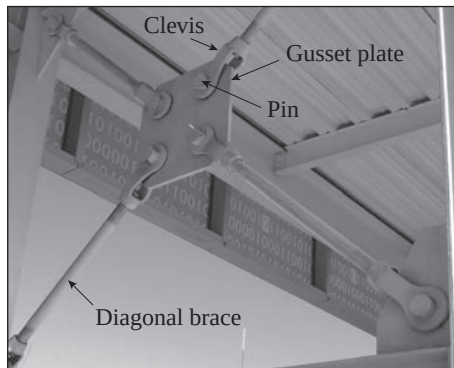
$$d_{\min} = \frac{15P/4}{b\sigma_{ba}} \quad d_{\min} = 0.711 \text{ in.} \quad \leftarrow$$

$$\text{Bearing from strut } DC \quad \sigma_b = \frac{\frac{15}{4}P}{2\frac{5}{8}bd}$$

$$\sigma_b = 3\frac{P}{bd} \quad (\text{lower than } ACB)$$

Problem 1.9-4 Lateral bracing for an elevated pedestrian walkway is shown in the figure part (a). The thickness of the clevis plate $t_c = 16 \text{ mm}$ and the thickness of the gusset plate $t_g = 20 \text{ mm}$ [see figure part (b)]. The maximum force in the diagonal bracing is expected to be $F = 190 \text{ kN}$.

If the allowable shear stress in the pin is 90 MPa and the allowable bearing stress between the pin and both the clevis and gusset plates is 150 MPa , what is the minimum required diameter $d_{\min}$ of the pin?



Solution 1.9-4

NUMERICAL DATA

$$F = 190 \text{ kN} \quad \tau_a = 90 \text{ MPa} \quad \sigma_{ba} = 150 \text{ MPa}$$

$$t_g = 20 \text{ mm} \quad t_c = 16 \text{ mm}$$

(1) $d_{\min}$ BASED ON ALLOW SHEAR—DOUBLE SHEAR
IN PIN

$$\tau = \frac{F}{A_s} \quad A_s = 2 \left(\frac{\pi}{4} d^2 \right)$$

$$d_{\min} = \sqrt{\frac{F}{\tau_a \left(\frac{\pi}{2} \right)}} \quad d_{\min} = 36.7 \text{ mm}$$

(2) $d_{\min}$ BASED ON ALLOW BEARING IN GUSSET AND CLEVIS
PLATES

Bearing on gusset plate

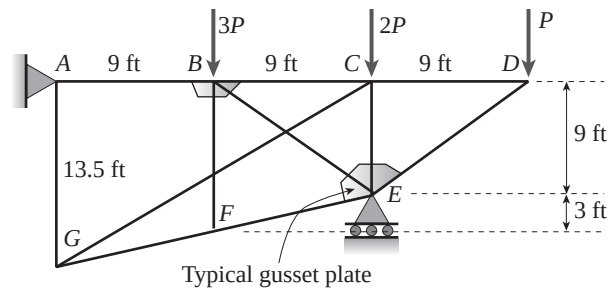
$$\sigma_b = \frac{F}{A_b} \quad A_b = t_g d \quad d_{\min} = \frac{F}{t_g \sigma_{ba}}$$

$$d_{\min} = 63.3 \text{ mm} \quad < \text{controls} \quad \leftarrow$$

Bearing on clevis $A_b = d(2t_c)$

$$d_{\min} = \frac{F}{2t_c \sigma_{ba}} \quad d_{\min} = 39.6 \text{ mm}$$

Problem 1.9-5 A plane truss has joint loads P , $2P$, and $3P$ at joints D , C , and B , respectively (see figure) where load variable $P = 5200 \text{ lb}$. All members have two end plates (see figure for Prob. 1.7-2) which are pin connected to gusset plates (see also figure for Prob. 1.8-12). Each end plate has thickness $t_p = 0.625 \text{ in.}$, and all gusset plates have thickness $t_g = 1.125 \text{ in.}$ If the allowable shear stress in each pin is $12,000 \text{ psi}$ and the allowable bearing stress in each pin is $18,000 \text{ psi}$, what is the minimum required diameter $d_{\min}$ of the pins used at either end of member BE ?

**Solution 1.9-5**

$$P = 5200 \text{ lb}$$

$$F_{BE} = 3.83858 P = 19,960.616 \text{ lb} < \text{from plane truss analysis (see Probs. 1.2-4 to 1.2-6)}$$

$$\tau_a = 12 \text{ ksi}$$

$$t_p = \frac{5}{8} \text{ in.}$$

$$t_g = 1.125 \text{ in.}$$

$$t_p = 0.625 \text{ in.}$$

$$2 t_p = 1.25 \text{ in.}$$

$$\sigma_{ba} = 18 \text{ ksi}$$

PIN DIAMETER BASED ON ALLOWABLE SHEAR STRESS (PINS IN DOUBLE SHEAR)

$$d_{p1} = \sqrt{\frac{\frac{F_{BE}}{2}}{\frac{\pi}{4} \tau_a}} = 1.029 \text{ in.} \quad < \text{controls} \quad \boxed{d_{\text{pin}} = 1.029 \text{ in.}}$$

PIN DIAMETER BASED ON BEARING BETWEEN PIN AND EACH OF TWO END PLATES

< $2t_p$ is greater than t_g so gusset will control

$$d_{p2} = \frac{F_{BE}}{2 t_p \sigma_{ba}} = 0.887 \text{ in.}$$

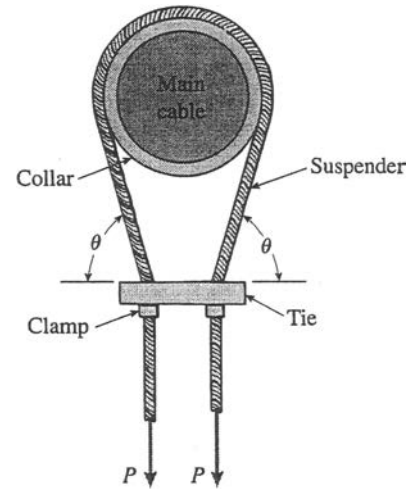
PIN DIAMETER BASED ON BEARING BETWEEN PIN AND GUSSET PLATE

$$d_{p3} = \frac{F_{BE}}{t_g \sigma_{ba}} = 0.986 \text{ in.}$$

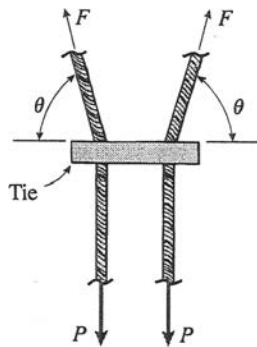
Problem 1.9-6 A suspender on a suspension bridge consists of a cable that passes over the main cable (see figure) and supports the bridge deck, which is far below. The suspender is held in position by a metal tie that is prevented from sliding downward by clamps around the suspender cable.

Let P represent the load in each part of the suspender cable, and let θ represent the angle of the suspender cable just above the tie. Finally, let σ_{allow} represent the allowable tensile stress in the metal tie.

- Obtain a formula for the minimum required cross-sectional area of the tie.
- Calculate the minimum area if $P = 130 \text{ kN}$, $\theta = 75^\circ$, and $\sigma_{\text{allow}} = 80 \text{ MPa}$.



Solution 1.9-6 Suspender tie on a suspension bridge



F = tensile force in cable above tie

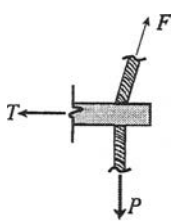
P = tensile force in cable below tie

σ_{allow} = allowable tensile stress in the tie

FREE-BODY DIAGRAM OF HALF THE TIE

Note: Include a small amount of the cable in the free-body diagram

T = tensile force in the tie



FORCE TRIANGLE

$$\cot \theta = \frac{T}{P}$$

$$T = P \cot \theta$$

- MINIMUM REQUIRED AREA OF TIE

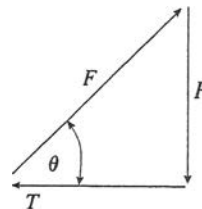
$$A_{\text{min}} = \frac{T}{\sigma_{\text{allow}}} = \frac{P \cot \theta}{\sigma_{\text{allow}}} \quad \leftarrow$$

- SUBSTITUTE NUMERICAL VALUES:

$$P = 130 \text{ kN} \quad \theta = 75^\circ$$

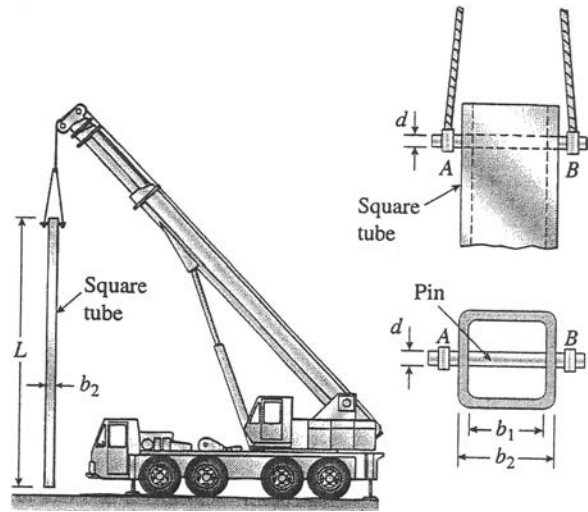
$$\sigma_{\text{allow}} = 80 \text{ MPa}$$

$$A_{\text{min}} = 435 \text{ mm}^2 \quad \leftarrow$$

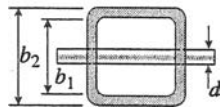
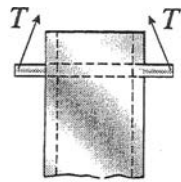


Problem 1.9-7 A square steel tube of length $L = 20$ ft and width $b_2 = 10.0$ in. is hoisted by a crane (see figure). The tube hangs from a pin of diameter d that is held by the cables at points A and B. The cross section is a hollow square with inner dimension $b_1 = 8.5$ in. and outer dimension $b_2 = 10.0$ in. The allowable shear stress in the pin is 8,700 psi, and the allowable bearing stress between the pin and the tube is 13,000 psi.

Determine the minimum diameter of the pin in order to support the weight of the tube. (Note: Disregard the rounded corners of the tube when calculating its weight.)



Solution 1.9-7 Tube hoisted by a crane



T = tensile force in cable

W = weight of steel tube

d = diameter of pin

b_1 = inner dimension of tube
= 8.5 in.

b_2 = outer dimension of tube
= 10.0 in.

L = length of tube = 20 ft

$\tau_{\text{allow}} = 8,700$ psi

$\sigma_b = 13,000$ psi

WEIGHT OF TUBE

γ_s = weight density of steel

$$= 490 \text{ lb/ft}^3$$

A = area of tube

$$= b_2^2 - b_1^2 = (10.0 \text{ in.})^2 - (8.5 \text{ in.})^2 \\ = 27.75 \text{ in.}^2$$

$$W = \gamma_s AL$$

$$= (490 \text{ lb/ft}^3)(27.75 \text{ in.}^2) \left(\frac{1 \text{ ft}^2}{144 \text{ in.}^2} \right) (20 \text{ ft}) \\ = 1,889 \text{ lb}$$

DIAMETER OF PIN BASED UPON SHEAR

Double shear. $2\tau_{\text{allow}} A_{\text{pin}} = W$

$$2(8,700 \text{ psi}) \left(\frac{\pi d^2}{4} \right) = 1,889 \text{ lb}$$

$$d^2 = 0.1382 \text{ in.}^2 \quad d_1 = 0.372 \text{ in.}$$

DIAMETER OF PIN BASED UPON BEARING

$$\sigma_b(b_2 - b_1)d = W$$

$$(13,000 \text{ psi})(10.0 \text{ in.} - 8.5 \text{ in.}) d = 1,889 \text{ lb}$$

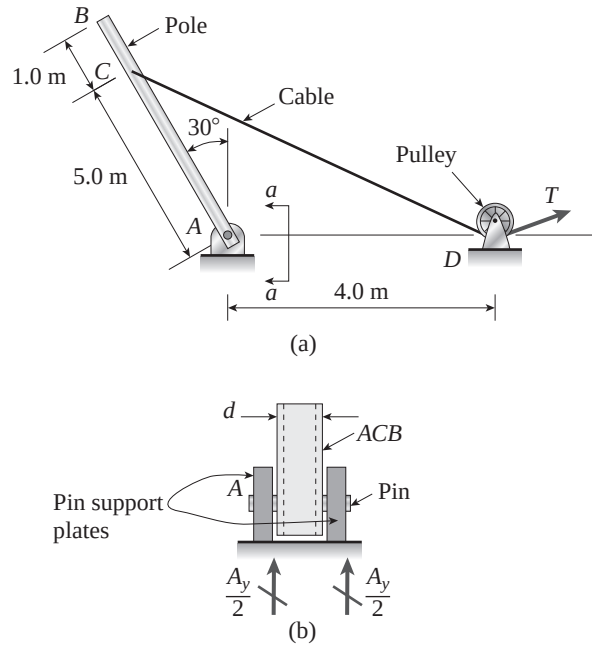
$$d_2 = 0.097 \text{ in.}$$

MINIMUM DIAMETER OF PIN

Shear governs. $d_{\text{min}} = 0.372 \text{ in.}$

Problem 1.9-8 A cable and pulley system at D is used to bring a 230-kg pole (ACB) to a vertical position as shown in the figure part (a). The cable has tensile force T and is attached at C . The length L of the pole is 6.0 m, the outer diameter is $d = 140$ mm, and the wall thickness $t = 12$ mm. The pole pivots about a pin at A in figure part (b). The allowable shear stress in the pin is 60 MPa and the allowable bearing stress is 90 MPa.

Find the minimum diameter of the pin at A in order to support the weight of the pole in the position shown in the figure part (a).



Solution 1.9-8

ALLOWABLE SHEAR AND BEARING STRESSES

$$\tau_a = 60 \text{ MPa} \quad \sigma_{ba} = 90 \text{ MPa}$$

FIND INCLINATION OF AND FORCE IN CABLE, T

let α = angle between pole and cable at C ; use law of cosines

$$DC = \sqrt{5^2 + 4^2 - 2(5)(4)\cos\left(120 \frac{\pi}{180}\right)}$$

$$DC = 7.81 \text{ m} \quad \alpha = \arccos\left[\frac{5^2 + DC^2 - 4^2}{2DC(5)}\right]$$

$$\alpha = 26.33^\circ \quad \theta = 60 - \alpha$$

$$\theta = 33.67^\circ < \text{angle between cable and horizontal at } D$$

$$W = 230 \text{ kg}(9.81 \text{ m/s}^2) \quad W = 2.256 \times 10^3 \text{ N}$$

STATICS TO FIND CABLE FORCE T

$$\sum M_A = 0 \quad W(3 \sin(30^\circ)) - T_x(5 \cos(30^\circ)) + T_y(5 \sin(30^\circ)) = 0$$

substitute for T_x and T_y in terms of T and solve for T :

$$T = \frac{\frac{3}{2}W}{\frac{-5}{2}\sin(\theta) + \frac{5\sqrt{3}}{2}\cos(\theta)}$$

$$T = 1.53 \times 10^3 \text{ N} \quad T_x = T \cos(\theta)$$

$$T_y = T \sin(\theta) \quad T_x = 1.27 \times 10^3 \text{ N} \quad T_y = 846.11 \text{ N}$$

(1) $d_{\min}$ BASED ON ALLOWABLE SHEAR—DOUBLE SHEAR AT A

$$A_x = -T_x \quad A_y = T_y + W$$

CHECK SHEAR DUE TO RESULTANT FORCE ON PIN AT A

$$R_A = \sqrt{A_x^2 + A_y^2} \quad R_A = 3.35 \times 10^3 \text{ N}$$

$$d_{\min} = \sqrt{\frac{\frac{R_A}{2}}{\tau_a \left(\frac{\pi}{4} \right)}}$$

$$d_{\min} = 5.96 \text{ mm} < \text{controls} \quad \leftarrow$$

(2) $d_{\min}$ BASED ON ALLOWABLE BEARING ON PIN

$$d_{\text{pole}} = 140 \text{ mm} \quad t_{\text{pole}} = 12 \text{ mm}$$

$$L_{\text{pole}} = 6000 \text{ mm}$$

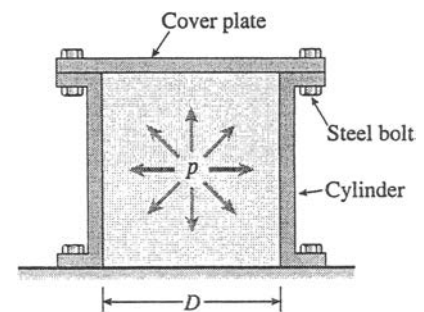
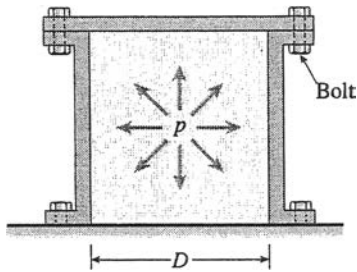
MEMBER AB BEARING ON PIN

$$\sigma_b = \frac{R_A}{A_b} \quad A_b = 2t_{\text{pole}}d$$

$$d_{\min} = \frac{R_A}{2t_{\text{pole}}\sigma_{ba}} \quad d_{\min} = 1.55 \text{ mm}$$

Problem 1.9-9 A pressurized circular cylinder has a sealed cover plate fastened with steel bolts (see figure). The pressure p of the gas in the cylinder is 290 psi, the inside diameter D of the cylinder is 10.0 in., and the diameter d_b of the bolts is 0.50 in.

If the allowable tensile stress in the bolts is 10,000 psi, find the number n of bolts needed to fasten the cover.

**Solution 1.9-9 Pressurized cylinder**

$$p = 290 \text{ psi} \quad D = 10.0 \text{ in.} \quad d_b = 0.50 \text{ in.}$$

$$\sigma_{\text{allow}} = 10,000 \text{ psi} \quad n = \text{number of bolts}$$

F = total force acting on the cover plate from the internal pressure

$$F = p \left(\frac{\pi D^2}{4} \right)$$

NUMBER OF BOLTS

P = tensile force in one bolt

$$P = \frac{F}{n} = \frac{\pi p D^2}{4n}$$

$$A_b = \text{area of one bolt} = \frac{\pi}{4} d_b^2$$

$$P = \sigma_{\text{allow}} A_b$$

$$\sigma_{\text{allow}} = \frac{P}{A_b} = \frac{\pi p D^2}{(4n)(\frac{\pi}{4} d_b^2)} = \frac{p D^2}{n d_b^2}$$

$$n = \frac{p D^2}{d_b^2 \sigma_{\text{allow}}}$$

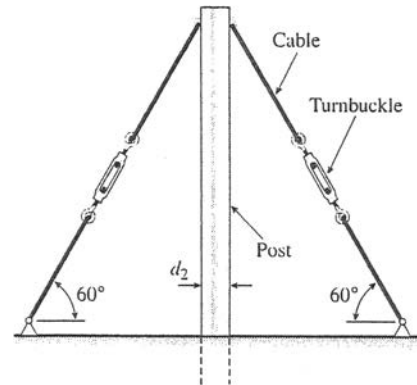
SUBSTITUTE NUMERICAL VALUES:

$$n = \frac{(290 \text{ psi})(10 \text{ in.})^2}{(0.5 \text{ in.})^2(10,000 \text{ psi})} = 11.6$$

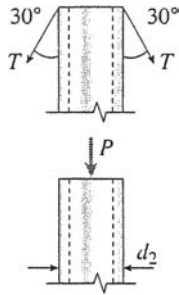
Use 12 bolts $\leftarrow$

Problem 1.9-10 A tubular post of outer diameter d_2 is guyed by two cables fitted with turnbuckles (see figure). The cables are tightened by rotating the turnbuckles, thus producing tension in the cables and compression in the post. Both cables are tightened to a tensile force of 110 kN. Also, the angle between the cables and the ground is 60° , and the allowable compressive stress in the post is $\sigma_c = 35$ MPa.

If the wall thickness of the post is 15 mm, what is the minimum permissible value of the outer diameter d_2 ?



Solution 1.9-10 Tubular post with guy cables



d_2 = outer diameter

d_1 = inner diameter

t = wall thickness

= 15 mm

T = tensile force in a cable

= 110 kN

$\sigma_{\text{allow}} = 35$ MPa

P = compressive force in post

= $2T \cos 30^\circ$

REQUIRED AREA OF POST

$$A = \frac{P}{\sigma_{\text{allow}}} = \frac{2T \cos 30^\circ}{\sigma_{\text{allow}}}$$

AREA OF POST

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}[d_2^2 - (d_2 - 2t)^2] \\ = \pi t(d_2 - t)$$

EQUATE AREAS AND SOLVE FOR d_2 :

$$\frac{2T \cos 30^\circ}{\sigma_{\text{allow}}} = \pi t(d_2 - t)$$

$$d_2 = \frac{2T \cos 30^\circ}{\pi t \sigma_{\text{allow}}} + t \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

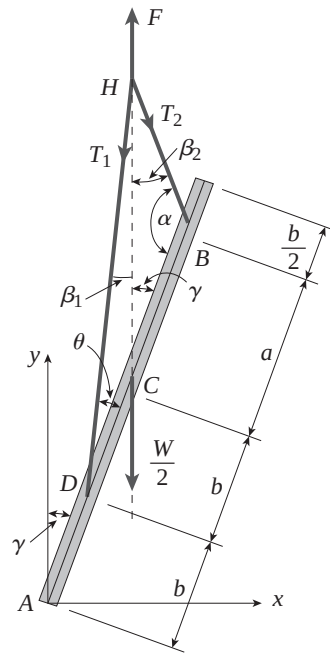
$$(d_2)_{\text{min}} = 131 \text{ mm} \quad \leftarrow$$

Problem 1.9-11 A large precast concrete panel for a warehouse is being raised to a vertical position using two sets of cables at two lift lines as shown in the figure part (a). Cable 1 has length $L_1 = 22$ ft and distances along the panel (see figure part (b)) are $a = L_1/2$ and $b = L_1/4$. The cables are attached at lift points B and D and the panel is rotated about its base at A . However, as a worst case, assume that the panel is momentarily lifted off the ground and its total weight must be supported by the cables. Assuming the cable lift forces F at each lift line are about equal, use the simplified model of one half of the panel in figure part (b) to perform your analysis for the lift position shown. The total weight of the panel is $W = 85$ kips. The orientation of the panel is defined by the following angles: $\gamma = 20^\circ$ and $\theta = 10^\circ$.

Find the required cross-sectional area A_C of the cable if its breaking stress is 91 ksi and a factor of safety of 4 with respect to failure is desired.



(a)



(b)

Solution 1.9-11

GEOMETRY

$$L_1 = 22 \text{ ft} \quad a = \frac{1}{2}L_1 \quad b = \frac{1}{4}L_1$$

$$\theta = 10^\circ \quad a + 2.5b = 24.75 \text{ ft}$$

$$\gamma = 20^\circ$$

Using law of cosines

$$L_2 = \sqrt{(a + b)^2 + L_1^2 - 2(a + b)L_1 \cos(\theta)}$$

$$L_2 = 6.425 \text{ ft}$$

$$\beta = \arccos \left[\frac{L_1^2 + L_2^2 - (a + b)^2}{2L_1L_2} \right]$$

$$\beta = 26.484^\circ$$

$$\beta_1 = \pi - (\theta + \pi - \gamma) \quad \beta_1 = 10^\circ$$

$$\beta_2 = \beta - \beta_1 \quad \beta_2 = 16.484^\circ$$

SOLUTION APPROACH: FIND T THEN $A_C = T/(\sigma_u/\text{FS})$

STATICS at point H

$$\sum F_x = 0 \quad T_1 \sin(\beta_1) = T_2 \sin(\beta_2)$$

$$\text{So} \quad T_2 = T_1 \frac{\sin(\beta_1)}{\sin(\beta_2)}$$

$$\sum F_y = 0 \quad T_1 \cos(\beta_1) + T_2 \cos(\beta_2) = F$$

$$\text{and} \quad F = W/2, \quad W = 85 \text{ k}$$

$$\text{So} \quad T_1 \left(\cos(\beta_1) + \frac{\sin(\beta_1)}{\sin(\beta_2)} \cos(\beta_2) \right) = F$$

$$T_1 = \frac{\frac{W}{2}}{\left(\cos(\beta_1) + \frac{\sin(\beta_1)}{\sin(\beta_2)} \cos(\beta_2) \right)}$$

$$T_1 = 27.042 \text{ k}$$

$$T_2 = T_1 \frac{\sin(\beta_1)}{\sin(\beta_2)} \quad T_2 = 16.549 \text{ k}$$

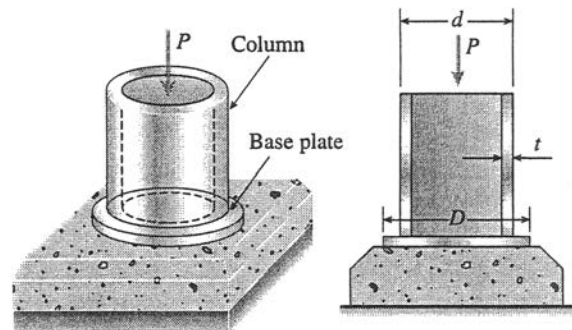
COMPUTE REQUIRED CROSS-SECTIONAL AREA

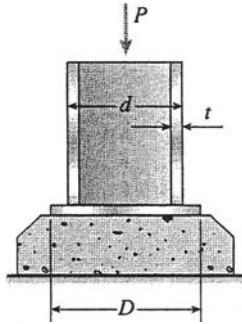
$$\sigma_u = 91 \text{ ksi} \quad \text{FS} = 4 \quad \frac{\sigma_u}{\text{FS}} = 22.75 \text{ ksi}$$

$$A_c = \frac{T_1}{\frac{\sigma_u}{\text{FS}}} \quad A_c = 1.189 \text{ in.}^2 \quad \leftarrow$$

Problem 1.9-12 A steel column of hollow circular cross section is supported on a circular steel base plate and a concrete pedestal (see figure). The column has outside diameter $d = 250 \text{ mm}$ and supports a load $P = 750 \text{ kN}$.

- If the allowable stress in the column is 55 MPa , what is the minimum required thickness t ? Based upon your result, select a thickness for the column. (Select a thickness that is an even integer, such as 10, 12, 14, . . . , in units of millimeters.)
- If the allowable bearing stress on the concrete pedestal is 11.5 MPa , what is the minimum required diameter D of the base plate if it is designed for the allowable load P_{allow} that the column with the selected thickness can support?



Solution 1.9-12 Hollow circular column

$$d = 250 \text{ mm} \quad P = 750 \text{ kN}$$

$$\sigma_{\text{allow}} = 55 \text{ MPa (compression in column)}$$

t = thickness of column

D = diameter of base plate

$$\sigma_b = 11.5 \text{ MPa (allowable pressure on concrete)}$$

(a) THICKNESS t OF THE COLUMN

$$A = \frac{P}{\sigma_{\text{allow}}} \quad A = \frac{\pi d^2}{4} - \frac{\pi}{4}(d - 2t)^2$$

$$= \frac{\pi}{4}(4t)(d - t) = \pi t(d - t)$$

$$\pi t(d - t) = \frac{P}{\sigma_{\text{allow}}}$$

$$\pi t^2 - \pi t d + \frac{P}{\sigma_{\text{allow}}} = 0$$

$$t^2 - t d + \frac{P}{\pi \sigma_{\text{allow}}} = 0 \quad (\text{Eq. 1})$$

SUBSTITUTE NUMERICAL VALUES IN EQ. (1):

$$t^2 - 250t + \frac{(750 \times 10^3 \text{ N})}{\pi(55 \text{ N/mm}^2)} = 0$$

(Note: In this eq., t has units of mm.)

$$t^2 - 250t + 4,340.6 = 0$$

Solve the quadratic eq. for t :

$$t = 18.77 \text{ mm} \quad t_{\min} = 18.8 \text{ mm} \quad \leftarrow$$

$$\text{Use } t = 20 \text{ mm} \quad \leftarrow$$

(b) DIAMETER D OF THE BASE PLATE

$$\text{For the column,} \quad P_{\text{allow}} = \sigma_{\text{allow}} A$$

where A is the area of the column with $t = 20 \text{ mm}$.

$$A = \pi t(d - t) \quad P_{\text{allow}} = \sigma_{\text{allow}} \pi t(d - t)$$

$$\text{Area of base plate} = \frac{\pi D^2}{4} = \frac{P_{\text{allow}}}{\sigma_b}$$

$$\frac{\pi D^2}{4} = \frac{\sigma_{\text{allow}} \pi t(d - t)}{\sigma_b}$$

$$D^2 = \frac{4\sigma_{\text{allow}} t(d - t)}{\sigma_b}$$

$$= \frac{4(55 \text{ MPa})(20 \text{ mm})(230 \text{ mm})}{11.5 \text{ MPa}}$$

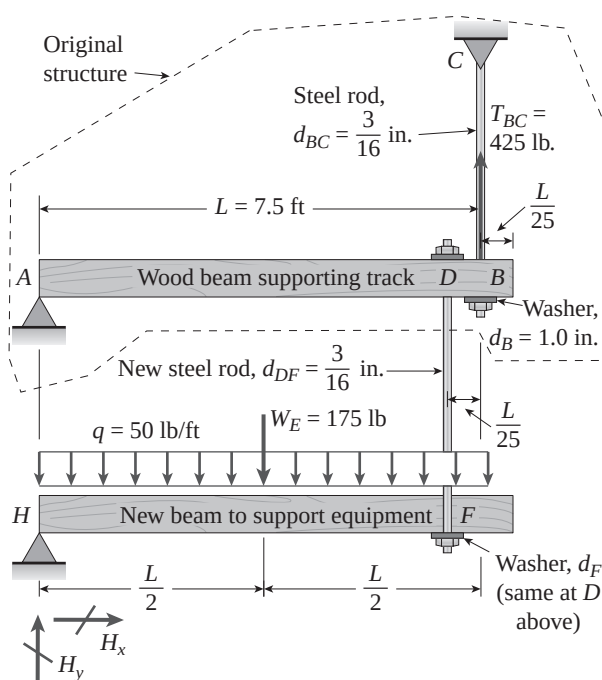
$$D^2 = 88,000 \text{ mm}^2 \quad D = 296.6 \text{ mm}$$

$$D_{\min} = 297 \text{ mm} \quad \leftarrow$$

Problem 1.9-13 An elevated jogging track is supported at intervals by a wood beam AB ($L = 7.5$ ft) which is pinned at A and supported by steel rod BC and a steel washer at B . Both the rod ($d_{BC} = 3/16$ in.) and the washer ($d_B = 1.0$ in.) were designed using a rod tension force of $T_{BC} = 425$ lb. The rod was sized using a factor of safety of 3 against reaching the ultimate stress $\sigma_u = 60$ ksi. An allowable bearing stress $\sigma_{ba} = 565$ psi was used to size the washer at B .

Now, a small platform HF is to be suspended below a section of the elevated track to support some mechanical and electrical equipment. The equipment load is uniform load $q = 50$ lb/ft and concentrated load $W_E = 175$ lb at mid-span of beam HF . The plan is to drill a hole through beam AB at D and install the same rod (d_{BC}) and washer (d_B) at both D and F to support beam HF .

- Use σ_u and σ_{ba} to check the proposed design for rod DF and washer d_F ; are they acceptable?
- Also re-check the normal tensile stress in rod BC and bearing stress at B ; if either is inadequate under the additional load from platform HF , redesign them to meet the original design criteria.



Solution 1.9-13

NUMERICAL DATA

$$L = 7.5(12) \quad L = 90 \text{ in.} \quad T_{BC} = 425 \text{ lb}$$

$$\sigma_u = 60 \text{ ksi} \quad \text{FS}_u = 3 \quad \sigma_{ba} = 0.565 \text{ ksi}$$

$$q = \frac{50}{12} \quad q = 4.167 \text{ lb/in.} \quad W_E = 175 \text{ lb}$$

$$d_{BC} = \frac{3}{16} \text{ in.} \quad d_B = 1.0 \text{ in.}$$

- (a) FIND FORCE IN ROD DF AND FORCE ON WASHER AT F

$$\sum M_H = 0 \quad T_{DF} = \frac{W_E \frac{L}{2} + qL \frac{L}{2}}{\left(L - \frac{L}{25}\right)}$$

$$T_{DF} = 286.458 \text{ lb}$$

NORMAL STRESS IN ROD DF :

$$\sigma_{DF} = \frac{T_{DF}}{\frac{\pi}{4} d_{BC}^2}$$

$$\sigma_{DF} = 10.38 \text{ ksi} \quad \text{OK—less than } \sigma_u; \text{ rod is acceptable} \quad \leftarrow$$

$$\sigma_a = \frac{\sigma_u}{\text{FS}_u} \quad \sigma_a = 20 \text{ ksi}$$

BEARING STRESS ON WASHER AT F :

$$\sigma_{bF} = \frac{T_{DF}}{\frac{\pi}{4} (d_B^2 - d_{BC}^2)}$$

$$\sigma_{bF} = 378 \text{ psi} \quad \text{OK—less than } \sigma_{ba}; \text{ washer is acceptable} \quad \leftarrow$$

- (b) FIND NEW FORCE IN ROD BC —SUM MOMENT ABOUT A FOR UPPER FBD—THEN CHECK NORMAL STRESS IN BC AND BEARING STRESS AT B

$$\sum M_A = 0$$

$$T_{BC2} = \frac{T_{BC}L + T_{DF} \left(L - \frac{L}{25}\right)}{L}$$

$$T_{BC2} = 700 \text{ lb}$$

REVISED NORMAL STRESS IN ROD BC :

$$\sigma_{BC2} = \frac{T_{BC2}}{\left(\frac{\pi}{4} d_{BC}^2\right)}$$

$$\sigma_{BC2} = 25.352 \text{ ksi} \quad \text{exceeds } \sigma_a = 20 \text{ ksi}$$

SO RE-DESIGN ROD BC :

$$d_{BC\text{reqd}} = \sqrt{\frac{T_{BC2}}{\frac{\pi}{4} \sigma_a}}$$

$$d_{BC\text{reqd}} = 0.211 \text{ in.} \quad d_{BC\text{reqd}} \times 16 = 3.38$$

^say 4/16 = 1/4 in. $d_{BC2} = \frac{1}{4} \text{ in.}$

RE-CHECK BEARING STRESS IN WASHER AT B :

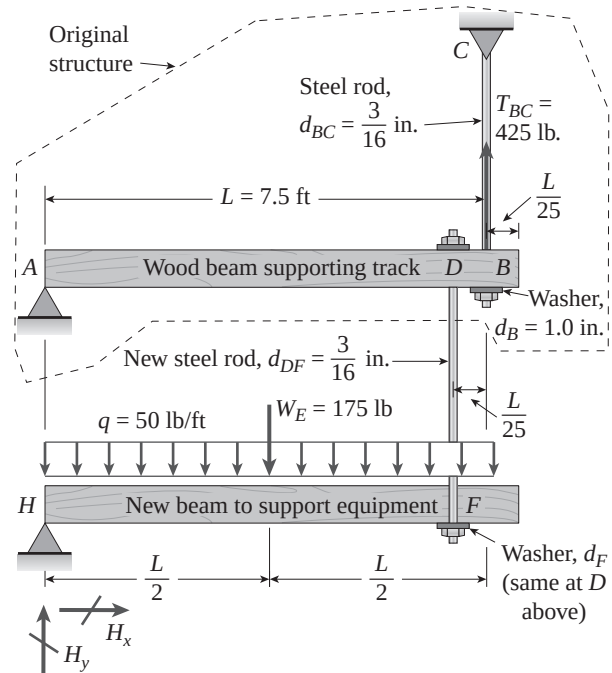
$$\sigma_{bB2} = \frac{T_{BC2}}{\left[\frac{\pi}{4} (d_B^2 - d_{BC}^2)\right]} \quad \sigma_{bB2} = 924 \text{ psi}$$

^ exceeds $\sigma_{ba} = 565 \text{ psi}$

SO RE-DESIGN WASHER AT B :

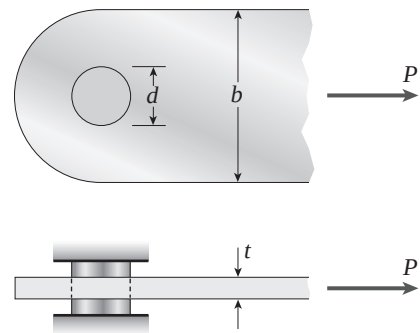
$$d_{B\text{reqd}} = \sqrt{\frac{T_{BC2}}{\frac{\pi}{4} \sigma_{ba}} + d_{BC}^2} \quad d_{B\text{reqd}} = 1.281 \text{ in.}$$

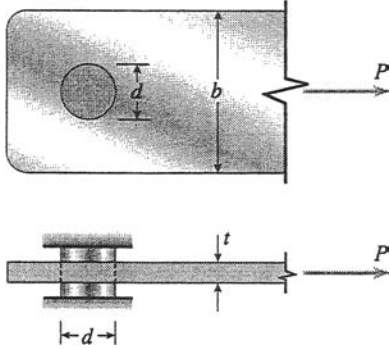
use 1 - 5/16 in washer at B : 1 + 5/16 = 1.312 in. ←



Problem 1.9-14 A flat bar of width $b = 60 \text{ mm}$ and thickness $t = 10 \text{ mm}$ is loaded in tension by a force P (see figure). The bar is attached to a support by a pin of diameter d that passes through a hole of the same size in the bar. The allowable tensile stress on the net cross section of the bar is $\sigma_T = 140 \text{ MPa}$, the allowable shear stress in the pin is $\tau_S = 80 \text{ MPa}$, and the allowable bearing stress between the pin and the bar is $\sigma_B = 200 \text{ MPa}$.

- Determine the pin diameter d_m for which the load P will be a maximum.
- Determine the corresponding value $P_{\max}$ of the load.



Solution 1.9-14 Bar with a pin connection

$$b = 60 \text{ mm}$$

$$t = 10 \text{ mm}$$

d = diameter of hole and pin

$$\sigma_T = 140 \text{ MPa}$$

$$\tau_S = 80 \text{ MPa}$$

$$\sigma_B = 200 \text{ MPa}$$

UNITS USED IN THE FOLLOWING CALCULATIONS:

P is in kN

σ and τ are in N/mm^2 (same as MPa)

b , t , and d are in mm

TENSION IN THE BAR

$$\begin{aligned} P_T &= \sigma_T (\text{Net area}) = \sigma_t(t)(b - d) \\ &= (140 \text{ MPa})(10 \text{ mm})(60 \text{ mm} - d) \left(\frac{1}{1000} \right) \\ &= 1.40(60 - d) \end{aligned} \quad (\text{Eq. 1})$$

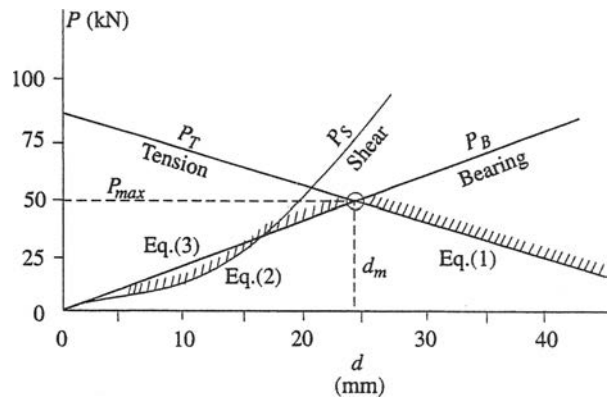
SHEAR IN THE PIN

$$\begin{aligned} P_S &= 2\tau_S A_{\text{pin}} = 2\tau_S \left(\frac{\pi d^2}{4} \right) \\ &= 2(80 \text{ MPa}) \left(\frac{\pi}{4} \right) (d^2) \left(\frac{1}{1000} \right) \\ &= 0.040 \pi d^2 = 0.12566d^2 \end{aligned} \quad (\text{Eq. 2})$$

BEARING BETWEEN PIN AND BAR

$$\begin{aligned} P_B &= \sigma_B td \\ &= (200 \text{ MPa})(10 \text{ mm})(d) \left(\frac{1}{1000} \right) \\ &= 2.0 d \end{aligned} \quad (\text{Eq. 3})$$

GRAPH OF EQS. (1), (2), AND (3)



(a) PIN DIAMETER d_m

$$\begin{aligned} P_T &= P_B \text{ or } 1.40(60 - d) = 2.0 d \\ \text{Solving, } d_m &= \frac{84.0}{3.4} \text{ mm} = 24.7 \text{ mm} \quad \leftarrow \end{aligned}$$

(b) LOAD $P_{\max}$

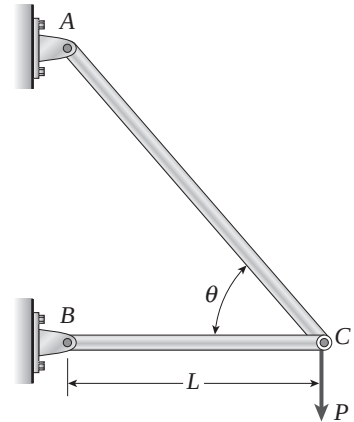
Substitute d_m into Eq. (1) or Eq. (3):

$$P_{\max} = 49.4 \text{ kN} \quad \leftarrow$$

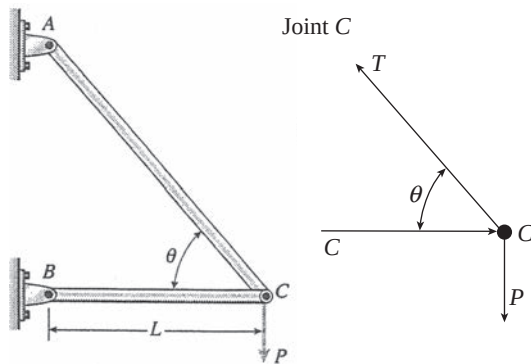
Problem 1.9-15 Two bars AC and BC of the same material support a vertical load P (see figure). The length L of the horizontal bar is fixed, but the angle θ can be varied by moving support A vertically and changing the length of bar AC to correspond with the new position of support A . The allowable stresses in the bars are the same in tension and compression.

We observe that when the angle θ is reduced, bar AC becomes shorter but the cross-sectional areas of both bars increase (because the axial forces are larger). The opposite effects occur if the angle θ is increased. Thus, we see that the weight of the structure (which is proportional to the volume) depends upon the angle θ .

Determine the angle θ so that the structure has minimum weight without exceeding the allowable stresses in the bars. (Note: The weights of the bars are very small compared to the force P and may be disregarded.)



Solution 1.9-15 Two bars supporting a load P



T = tensile force in bar AC

C = compressive force in bar BC

$$\sum F_{\text{vert}} = 0 \quad T = \frac{P}{\sin \theta}$$

$$\sum F_{\text{horiz}} = 0 \quad C = \frac{P}{\tan \theta}$$

AREAS OF BARS

$$A_{AC} = \frac{T}{\sigma_{\text{allow}}} = \frac{P}{\sigma_{\text{allow}} \sin \theta}$$

$$A_{BC} = \frac{C}{\sigma_{\text{allow}}} = \frac{P}{\sigma_{\text{allow}} \tan \theta}$$

LENGTHS OF BARS

$$L_{AC} = \frac{L}{\cos \theta} \quad L_{BC} = L$$

WEIGHT OF TRUSS

γ = weight density of material

$$\begin{aligned} W &= \gamma(A_{AC}L_{AC} + A_{BC}L_{BC}) \\ &= \frac{\gamma PL}{\sigma_{\text{allow}}} \left(\frac{1}{\sin \theta \cos \theta} + \frac{1}{\tan \theta} \right) \\ &= \frac{\gamma PL}{\sigma_{\text{allow}}} \left(\frac{1 + \cos^2 \theta}{\sin \theta \cos \theta} \right) \end{aligned} \quad \text{Eq. (1)}$$

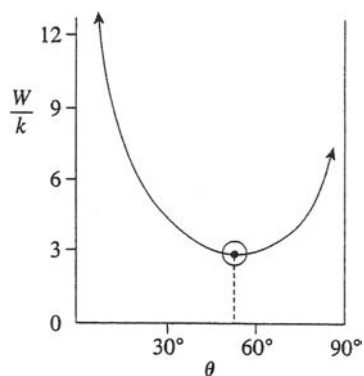
γ , P , L , and σ_{allow} are constants

W varies only with θ

$$\text{Let } k = \frac{\gamma PL}{\sigma_{\text{allow}}} \quad (k \text{ has units of force})$$

$$\frac{W}{k} = \frac{1 + \cos^2 \theta}{\sin \theta \cos \theta} \quad (\text{Nondimensional}) \quad \text{Eq. (2)}$$

GRAPH OF EQ. (2):

ANGLE θ THAT MAKES W_A MINIMUM

Use Eq. (2)

$$\text{Let } f = \frac{1 + \cos^2 \theta}{\sin \theta \cos \theta}$$

$$\frac{df}{d\theta} = 0$$

$$\begin{aligned} \frac{df}{d\theta} &= \frac{(\sin \theta \cos \theta)(2)(\cos \theta)(-\sin \theta) - (1 + \cos^2 \theta)(-\sin^2 \theta + \cos^2 \theta)}{\sin^2 \theta \cos^2 \theta} \\ &= \frac{-\sin^2 \theta \cos^2 \theta + \sin^2 \theta - \cos^2 \theta - \cos^4 \theta}{\sin^2 \theta \cos^2 \theta} \end{aligned}$$

SET THE NUMERATOR = 0 AND SOLVE FOR θ :

$$-\sin^2 \theta \cos^2 \theta + \sin^2 \theta - \cos^2 \theta - \cos^4 \theta = 0$$

Replace $\sin^2 \theta$ by $1 - \cos^2 \theta$:

$$-(1 - \cos^2 \theta)(\cos^2 \theta) + 1 - \cos^2 \theta - \cos^2 \theta - \cos^4 \theta = 0$$

Combine terms to simplify the equation:

$$1 - 3 \cos^2 \theta = 0 \quad \cos \theta = \frac{1}{\sqrt{3}}$$

$$\theta = 54.7^\circ \quad \leftarrow$$

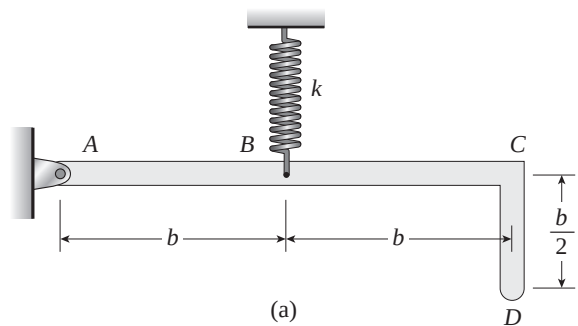
2

Axially Loaded Members

Changes in Lengths of Axially Loaded Members

Problem 2.2-1 The L-shaped arm $ABCD$ shown in the figure lies in a vertical plane and pivots about a horizontal pin at A . The arm has constant cross-sectional area and total weight W . A vertical spring of stiffness k supports the arm at point B .

- Obtain a formula for the elongation of the spring due to the weight of the arm.
- Repeat part (a) if the pin support at A is moved to D .

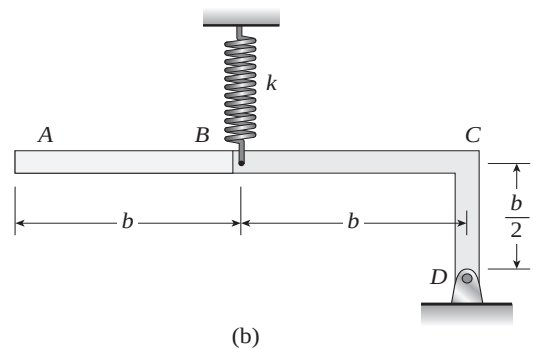


Solution 2.2-1

- SUM MOMENTS ABOUT A

$$\Sigma M_A = 0 \quad \frac{2b}{5}Wb + \frac{b}{5}W(2b)k\delta b$$

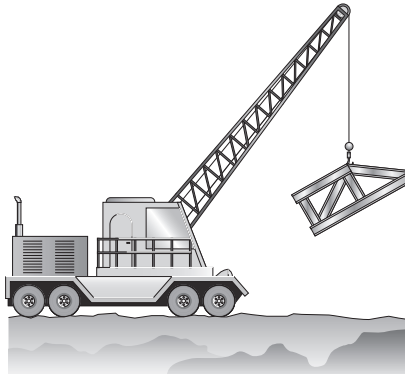
$$\delta = \frac{\frac{2b}{5}Wb + \frac{b}{5}W(2b)}{kb} = \frac{6W}{5k}$$



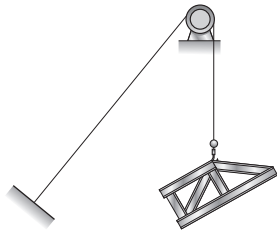
$$(b) \Sigma M_D = 0 \quad kb\delta = \frac{2b}{5}Wb = \frac{4Wb}{5} \quad \text{so} \quad \delta = \frac{\frac{2b}{5}Wb}{kb} = \frac{4W}{5k}$$

Problem 2.2-2 A steel cable with nominal diameter 25 mm (see Table 2-1) is used in a construction yard to lift a bridge section weighing 38 kN, as shown in the figure. The cable has an effective modulus of elasticity $E = 140$ GPa.

- If the cable is 14 m long, how much will it stretch when the load is picked up?
- If the cable is rated for a maximum load of 70 kN, what is the factor of safety with respect to failure of the cable?



Solution 2.2-2 Bridge section lifted by a cable



$A = 304 \text{ mm}^2$ (from Table 2-1)
 $W = 38 \text{ kN}$
 $E = 140 \text{ GPa}$
 $L = 14 \text{ m}$

(b) FACTOR OF SAFETY

$P_{ULT} = 406 \text{ kN}$ (from Table 2-1)

$P_{max} = 70 \text{ kN}$

$$n = \frac{P_{ULT}}{P_{max}} = \frac{406 \text{ kN}}{70 \text{ kN}} = 5.8 \quad \leftarrow$$

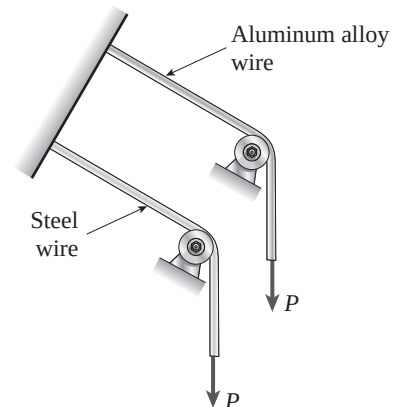
(a) STRETCH OF CABLE

$$\delta = \frac{WL}{EA} = \frac{(38 \text{ kN})(14 \text{ m})}{(140 \text{ GPa})(304 \text{ mm}^2)}$$

$$= 12.5 \text{ mm} \quad \leftarrow$$

Problem 2.2-3 A steel wire and an aluminum alloy wire have equal lengths and support equal loads P (see figure). The moduli of elasticity for the steel and aluminum alloy are $E_s = 30,000$ ksi and $E_a = 11,000$ ksi, respectively.

- If the wires have the same diameters, what is the ratio of the elongation of the aluminum alloy wire to the elongation of the steel wire?
- If the wires stretch the same amount, what is the ratio of the diameter of the aluminum alloy wire to the diameter of the steel wire?
- If the wires have the same diameters and same load P , what is the ratio of the initial length of the aluminum alloy wire to that of the steel wire if the aluminum alloy wire stretches 1.5 times that of the steel wire?
- If the wires have the same diameters, same initial length, and same load P , what is the material of the upper wire if it elongates 1.7 times that of the steel wire?



Solution 2.2-3

$$(a) \frac{\delta_a}{\delta_s} = \frac{\frac{PL}{E_a A}}{\left(\frac{PL}{E_s A}\right)} \rightarrow \frac{E_s}{E_a}$$

$$E_s = 30,000 \text{ ksi} \quad E_a = 11,000 \text{ ksi}$$

$$\boxed{\frac{E_s}{E_a} = 2.727} \quad \frac{30}{11} = 2.727$$

$$(b) \delta_a = \delta_s \quad \text{so} \quad \frac{PL}{E_a A_a} = \frac{PL}{E_s A_s} \quad \text{so} \quad \frac{A_a}{A_s} = \frac{E_s}{E_a} \quad \text{and} \quad \boxed{\frac{d_a}{d_s} = \sqrt{\frac{E_s}{E_a}} = 1.651}$$

- (c) SAME DIAMETER, SAME LOAD, FIND RATIO OF LENGTH OF ALUMINUM TO STEEL WIRE IF ELONGATION OF ALUMINUM IS 1.5 TIMES THAT OF STEEL WIRE

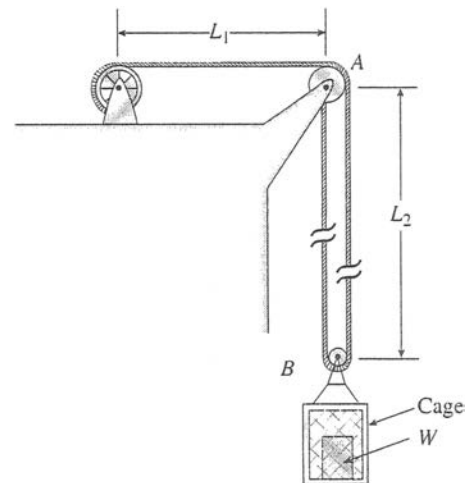
$$\frac{\delta_a}{\delta_s} = \frac{\frac{PL_a}{E_a A}}{\left(\frac{PL_s}{E_s A}\right)} = \frac{\frac{PL_a}{E_a A}}{\left(\frac{PL_s}{E_s A}\right)} = 1.5 \quad \boxed{\frac{L_a}{L_s} = 1.5 \frac{E_a}{E_s} = 0.55}$$

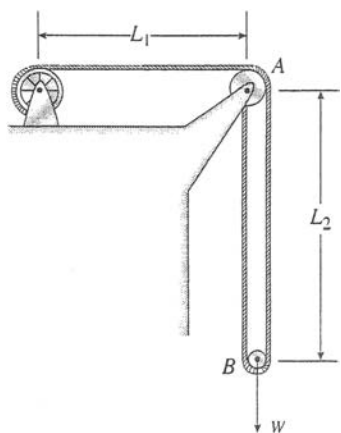
- (d) SAME DIAMETER, SAME LENGTH, SAME LOAD—BUT WIRE 1 ELONGATION 1.7 TIMES THE STEEL WIRE > WHAT IS WIRE 1 MATERIAL?

$$\frac{\delta_1}{\delta_s} = \frac{\frac{PL}{E_1 A}}{\left(\frac{PL}{E_s A}\right)} = \frac{\frac{PL}{E_1 A}}{\left(\frac{PL}{E_s A}\right)} = 1.7 \quad E_1 = \frac{E_s}{1.7} = 17,647 \text{ ksi} \quad \boxed{< \text{cast iron or copper alloy (see App. I)}}$$

Problem 2.2-4 By what distance h does the cage shown in the figure move downward when the weight W is placed inside it?

Consider only the effects of the stretching of the cable, which has axial rigidity $EA = 10,700 \text{ kN}$. The pulley at A has diameter $d_A = 300 \text{ mm}$ and the pulley at B has diameter $d_B = 150 \text{ mm}$. Also, the distance $L_1 = 4.6 \text{ m}$, the distance $L_2 = 10.5 \text{ m}$, and the weight $W = 22 \text{ kN}$. (Note: When calculating the length of the cable, include the parts of the cable that go around the pulleys at A and B .)



Solution 2.2-4 Cage supported by a cable

$$\begin{aligned}d_A &= 300 \text{ mm} \\d_B &= 150 \text{ mm} \\L_1 &= 4.6 \text{ m} \\L_2 &= 10.5 \text{ m} \\EA &= 10,700 \text{ kN} \\W &= 22 \text{ kN}\end{aligned}$$

TENSILE FORCE IN CABLE

$$T = \frac{W}{2} = 11 \text{ kN}$$

LENGTH OF CABLE

$$\begin{aligned}L &= L_1 + 2L_2 + \frac{1}{4}(\pi d_A) + \frac{1}{2}(\pi d_B) \\&= 4,600 \text{ mm} + 21,000 \text{ mm} + 236 \text{ mm} + 236 \text{ mm} \\&= 26,072 \text{ mm}\end{aligned}$$

ELONGATION OF CABLE

$$\delta = \frac{TL}{EA} = \frac{(11 \text{ kN})(26,072 \text{ mm})}{(10,700 \text{ kN})} = 26.8 \text{ mm}$$

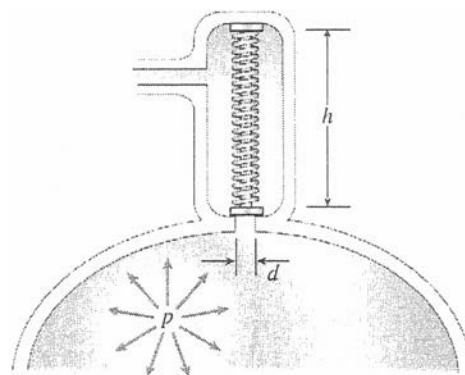
LOWERING OF THE CAGE

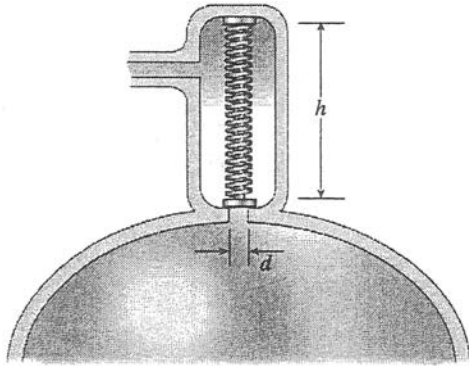
 h = distance the cage moves downward

$$h = \frac{1}{2} \delta = 13.4 \text{ mm} \quad \leftarrow$$

Problem 2.2-5 A safety valve on the top of a tank containing steam under pressure p has a discharge hole of diameter d (see figure). The valve is designed to release the steam when the pressure reaches the value $p_{\max}$.

If the natural length of the spring is L and its stiffness is k , what should be the dimension h of the valve? (Express your result as a formula for h .)



Solution 2.2-5 Safety valve

h = height of valve (compressed length of the spring)

d = diameter of discharge hole

p = pressure in tank

$p_{\max}$ = pressure when valve opens

L = natural length of spring ($L > h$)

k = stiffness of spring

FORCE IN COMPRESSED SPRING

$$F = k(L - h) \text{ (From Eq. 2-1a)}$$

PRESSURE FORCE ON SPRING

$$P = p_{\max} \left(\frac{\pi d^2}{4} \right)$$

EQUATE FORCES AND SOLVE FOR h :

$$F = P \quad k(L - h) = \frac{\pi p_{\max} d^2}{4}$$

$$h = L - \frac{\pi p_{\max} d^2}{4k} \quad \leftarrow$$

Problem 2.2-6 The device shown in the figure consists of a prismatic rigid pointer ABC supported by a uniform translational spring of stiffness $k = 950 \text{ N/m}$. The spring is positioned at distance $b = 165 \text{ mm}$ from the pinned end A of the pointer. The device is adjusted so that when there is no load P , the pointer reads zero on the angular scale.

- If the load $P = 11 \text{ N}$, at what distance x should the load be placed so that the pointer will read $\theta = 2.5^\circ$ on the scale (see figure part a)?
- Repeat part (a) if a rotational spring $k_r = kb^2$ is added at A (see figure part b).
- Let $x = 7b/8$. What is $P_{\max}$ (N) if θ cannot exceed 2° ? Include spring k_r in your analysis.
- Now, if the weight of the pointer ABC is known to be $W_p = 3 \text{ N}$ and the weight of the spring is $W_s = 2.75 \text{ N}$, what initial angular position (i.e., θ in degrees) of the pointer will result in a zero reading on the angular scale once the pointer is released from rest? Assume $P = k_r = 0$.
- If the pointer is rotated to a vertical position (see figure part c), find the required load P , applied at mid-height of the pointer, that will result in a pointer reading of $\theta = 2.5^\circ$ on the scale. Consider the weight of the pointer W_p in your analysis.

Solution 2.2-6

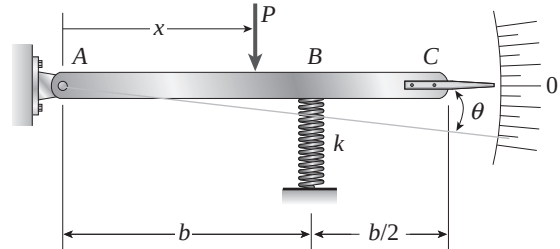
NUMERICAL DATA $k = 950 \text{ N/m}$ $b = 165 \text{ mm}$ $P = 11 \text{ N}$ $\theta = 2.5^\circ$ $\theta_{\max} = 2^\circ$

$$W_p = 3 \text{ N} \quad W_s = 2.75 \text{ N}$$

- (a) If the load $P = 11 \text{ N}$, at what distance x should the load be placed so that the pointer will read $\theta = 2.5^\circ$ on the scale (see Fig. a)?

Sum moments about A, then solve for x :

$$x = \frac{k\theta b^2}{P} = 102.6 \text{ mm} \quad \boxed{x = 102.6 \text{ mm}}$$



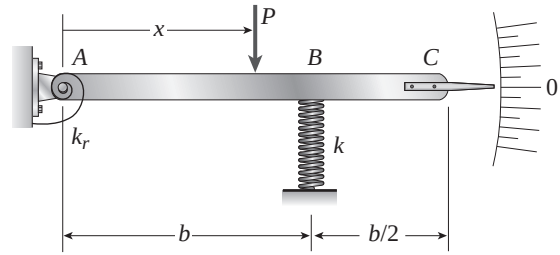
(a)

- (b) Repeat (a) if a rotational spring $k_r = kb^2$ is added at A (see Fig. b).

$$k_r = k b^2 = 25864 \text{ N}\cdot\text{mm}$$

Sum moments about A, then solve for x :

$$x = \frac{k\theta b^2 + k_r\theta}{P} = 205 \text{ mm} \quad \frac{x}{b} = 1.244 \quad \boxed{x = 205 \text{ mm}}$$



(b)

- (c) Now if $x = 7b/8$, what is $P_{\max}$ (N) if θ cannot exceed 2° ? $x = \frac{7}{8}b = 144.375 \text{ mm}$

Sum moments about A, then solve for P :
$$P_{\max} = \frac{k\theta_{\max}b^2 + k_r\theta_{\max}}{\frac{7}{8}b} = 12.51 \text{ N} \quad \boxed{P_{\max} = 12.51 \text{ N}}$$

- (d) Now, if the weight of the pointer ABC is known to be $W_p = 3 \text{ N}$ and the weight of the spring is $W_s = 2.75 \text{ N}$, what initial angular position (i.e., θ in degrees) of the pointer will result in a zero reading on the angular scale once the pointer is released from rest? Assume $P = k_r = 0$.

Deflection at spring due to W_p :

$$\delta_{Bp} = \frac{W_p \left(\frac{3}{4}b \right)}{kb} = 2.368 \text{ mm}$$

Deflection at B due to self weight of spring:

$$\delta_{Bk} = \frac{W_s}{2k} = 1.447 \text{ mm}$$

$$\delta_B = \delta_{Bp} + \delta_{Bk} = 3.816 \text{ mm} \quad \theta_{\text{init}} = \frac{\delta_B}{b} = 1.325^\circ$$

$$\text{OR } \theta_{\text{init}} = \arctan \left(\frac{\delta_B}{b} \right) = 1.325^\circ \quad \boxed{\theta_{\text{init}} = 1.325^\circ}$$

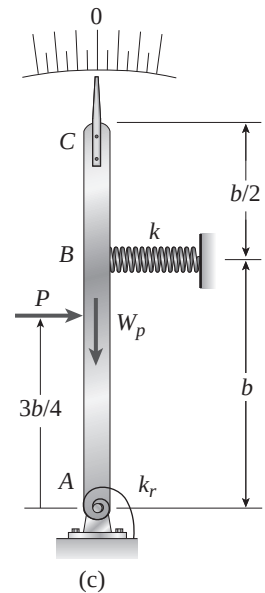
- (e) If the pointer is rotated to a vertical position (figure part c), find the required load P , applied at mid-height of the pointer that will result in a pointer reading of $\theta = 2.5^\circ$ on the scale. Consider the weight of the pointer, W_p , in your analysis.

$$k = 950 \text{ N/m} \quad b = 165 \text{ mm} \quad W_p = 3 \text{ N}$$

$$k_r = kb^2 = 25.864 \text{ N}\cdot\text{m} \quad \theta = 2.5^\circ$$

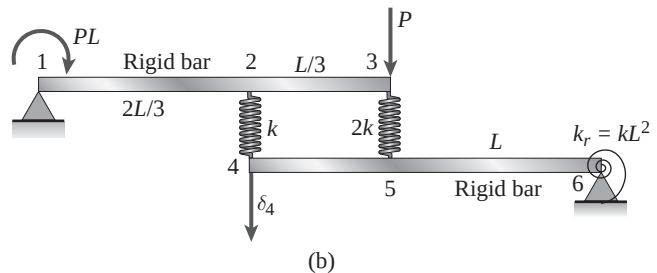
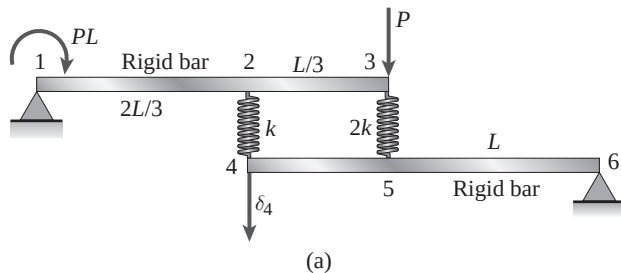
Sum moments about A to get P :

$$P = \frac{\theta}{\left(\frac{3b}{4}\right)} \left[k_r + k \left(\frac{5}{4}b^2 \right) - W_p \left(\frac{3b}{4} \right) \right] = 20.388 \text{ N} \quad \boxed{P = 20.4 \text{ N}}$$



Problem 2.2-7 Two rigid bars are connected to each other by two linearly elastic springs. Before loads are applied, the lengths of the springs are such that the bars are parallel and the springs are without stress.

- (a) Derive a formula for the displacement δ_4 at point 4 when the load P is applied at joint 3 and moment PL is applied at joint 1, as shown in the figure part a. (Assume that the bars rotate through very small angles under the action of the load P .)
- (b) Repeat part (a) if a rotational spring, $k_r = kL^2$, is now added at joint 6. What is the ratio of the deflection δ_4 in the figure part a to that in the figure part b?



Solution 2.2-7

- (a) Derive a formula for the displacement δ_4 at point 4 when the load P is applied at joint 3 and moment PL is applied at joint 1, as shown.

Cut horizontally through both springs to create upper and lower FBD's. Sum moments about joint 1 for upper FBD and also sum moments about joint 6 for lower FBD to get two equations of equilibrium; assume both springs are in tension.

Note that $\delta_2 = \frac{2}{3} \delta_3$ and $\delta_5 = \frac{3}{4} \delta_4$

Force in left spring: $k\left(\delta_4 - \frac{2}{3} \delta_3\right)$

Force in right spring: $2k\left(\frac{3}{4} \delta_4 - \delta_3\right)$

Summing moments about joint 1 (upper FBD) and about joint 6 (lower FBD) then dividing through by k gives

$$\begin{pmatrix} \frac{-22}{9} & \frac{13}{6} \\ \frac{-26}{9} & \frac{17}{6} \end{pmatrix} \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} \frac{-2P}{k} \\ 0 \end{pmatrix} \quad \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} \frac{-22}{9} & \frac{13}{6} \\ \frac{-26}{9} & \frac{17}{6} \end{pmatrix}^{-1} \begin{pmatrix} \frac{-2P}{k} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{17P}{2k} \\ \frac{26P}{3k} \end{pmatrix} \quad \begin{matrix} \frac{17}{2} = 8.5 \\ \frac{26}{3} = 8.667 \end{matrix} \quad \boxed{\delta_4 = \frac{26P}{3k}}$$

^ deltas are **positive downward**

- (b) Repeat part (a) if a rotational spring $k_r = kL^2$ is now added at joint 6. What is the ratio of the deflection δ_4 in part (a) to that in (b)?

Upper FBD—sum moments about joint 1:

$$k\left(\delta_4 - \frac{2}{3} \delta_3\right) \frac{2L}{3} + 2k\left(\frac{3}{4} \delta_4 - \delta_3\right)L = -2PL \quad \text{OR} \quad \left(\frac{22Lk}{9}\right)\delta_3 + \frac{13Lk}{6}\delta_4 = -2PL$$

Lower FBD—sum moments about joint 6:

$$k\left(\delta_4 - \frac{2}{3} \delta_3\right) \frac{4L}{3} + 2k\left(\frac{3}{4} \delta_4 - \delta_3\right)L - k_r \theta_6 = 0$$

$$\left[k\left(\delta_4 - \frac{2}{3} \delta_3\right) \frac{4L}{3} + 2k\left(\frac{3}{4} \delta_4 - \delta_3\right)L \right] + (kL^2) \left(\frac{\delta_4}{\frac{4}{3}L} \right) = 0 \quad \text{OR} \quad \left(\frac{26Lk}{9} \right) \delta_3 + \frac{43Lk}{12} \delta_4 = 0$$

Divide matrix equilibrium equations through by k to get the following displacement equations:

$$\begin{pmatrix} \frac{-22}{9} & \frac{13}{6} \\ \frac{-26}{9} & \frac{43}{12} \end{pmatrix} \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} \frac{-2P}{k} \\ 0 \end{pmatrix} \quad \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} \frac{-22}{9} & \frac{13}{6} \\ \frac{-26}{9} & \frac{43}{12} \end{pmatrix}^{-1} \begin{pmatrix} \frac{-2P}{k} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{43P}{15k} \\ \frac{104P}{45k} \end{pmatrix} \quad \begin{matrix} \frac{43}{15} = 2.867 \\ \frac{104}{45} = 2.311 \end{matrix} \quad \boxed{\delta_4 = \frac{104P}{45k}}$$

^ deltas are positive downward

$$\text{Ratio of the deflection } \delta_4 \text{ in part (a) to that in (b): } \frac{\frac{26}{104}}{\frac{3}{45}} = \frac{15}{4} \quad \boxed{\text{Ratio} = \frac{15}{4} = 3.75}$$

Problem 2.2-8 The three-bar truss ABC shown in figure part a has a span $L = 3$ m and is constructed of steel pipes having cross-sectional area $A = 3900 \text{ mm}^2$ and modulus of elasticity $E = 200 \text{ GPa}$. Identical loads P act both vertically and horizontally at joint C , as shown.

- If $P = 475 \text{ kN}$, what is the horizontal displacement of joint B ?
- What is the maximum permissible load value $P_{\max}$ if the displacement of joint B is limited to 1.5 mm ?
- Repeat parts (a) and (b) if the plane truss is replaced by a space truss (see figure part b).

Solution 2.2-8

NUMERICAL DATA

$$A = 3900 \text{ mm}^2 \quad E = 200 \text{ GPa}$$

$$P = 475 \text{ kN} \quad L = 3000 \text{ mm}$$

$$\delta_{B\max} = 1.5 \text{ mm}$$

- (a) FIND HORIZONTAL DISPLACEMENT OF JOINT B

STATICS TO FIND SUPPORT REACTIONS AND THEN MEMBER FORCES:

$$\sum M_A = 0 \quad B_y = \frac{1}{L} \left(2P \frac{L}{2} \right)$$

$$B_y = P$$

$$\sum F_H = 0 \quad A_x = -P$$

$$\sum F_V = 0 \quad A_y = P - B_y \quad A_y = 0$$

METHOD OF JOINTS: $AC_V = A_Y \quad AC_V = 0 \quad \text{Force in } AC = 0$

$$AB = A_X$$

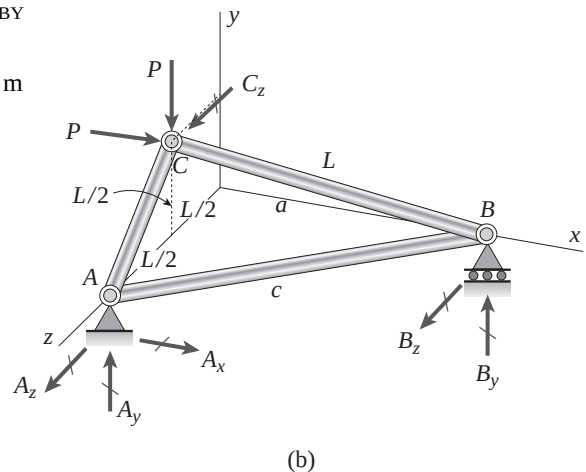
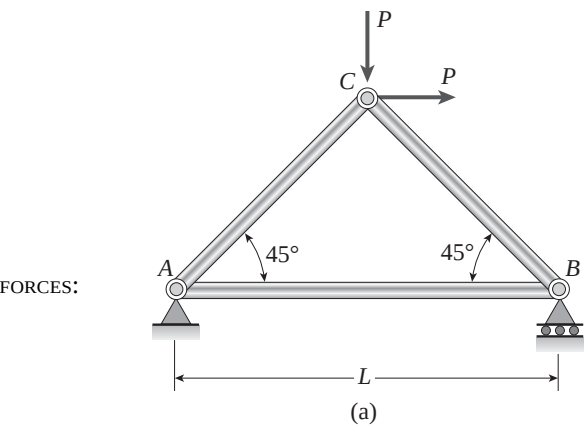
Force in AB is P (tension) so elongation of AB is the horizontal displacement of joint B .

$$\delta_B = \frac{F_{AB}L}{EA} \quad \delta_B = \frac{PL}{EA} \quad \delta_B = 1.82692 \text{ mm} \quad \boxed{\delta_B = 1.827 \text{ mm}}$$

- (b) FIND $P_{\max}$ IF DISPLACEMENT OF JOINT $B = \delta_{B\max} = 1.5 \text{ mm}$ $P_{\max} = \frac{EA}{L} \delta_{B\max} \quad \boxed{P_{\max} = 390 \text{ kN}}$

- (c) REPEAT PARTS (a) AND (b) IF THE PLANE TRUSS IS REPLACED BY A SPACE TRUSS (SEE FIGURE PART b).

FIND MISSING DIMENSIONS a AND c : $P = 475 \text{ kN} \quad L = 3 \text{ m}$



$$a = \sqrt{L^2 - 2\left(\frac{L}{2}\right)^2} = 2.12132 \text{ m} \quad \frac{a}{L} = 0.707 \quad a = \frac{L}{\sqrt{2}} = 2.12132 \text{ m}$$

$$c = \sqrt{L^2 + a^2} = 3.67423 \text{ m} \quad c = \sqrt{L^2 + \left(\frac{L}{\sqrt{2}}\right)^2} = 3.67423 \text{ m} \quad c = L\sqrt{\frac{3}{2}} = 3.67423 \text{ m}$$

- (1) SUM MOMENTS ABOUT A LINE THRU
- A
- WHICH IS PARALLEL TO THE
- y
- AXIS

$$B_z = -P \frac{L}{a} = -671.751 \text{ kN}$$

- (2) SUM MOMENTS ABOUT THE
- z
- AXIS

$$B_y = \frac{P\left(\frac{L}{2}\right)}{a} = 335.876 \text{ kN} \quad \text{SO} \quad A_y = P - B_y = 139.124 \text{ kN}$$

- (3) SUM MOMENTS ABOUT THE
- x
- AXIS

$$C_z = \frac{A_y L - P \frac{L}{2}}{\frac{L}{2}} = -196.751 \text{ kN}$$

- (4) SUM FORCES IN THE
- x
- AND
- z
- DIRECTIONS
- $A_x = -P = -475 \text{ kN}$
- $A_z = -C_z - B_z = 868.503 \text{ kN}$

- (5) USE METHOD OF JOINTS TO FIND MEMBER FORCES

$$\text{Sum forces in } x\text{-direction at joint } A: \frac{a}{c}F_{AB} + A_x = 0 \quad F_{AB} = \frac{-c}{a}A_x = 823 \text{ kN}$$

$$\text{Sum forces in } y\text{-direction at joint } A: \frac{\frac{L}{2}}{\sqrt{2} \frac{L}{2}}F_{AC} + A_y = 0 \quad F_{AC} = \sqrt{2}(-A_y) = -196.8 \text{ kN}$$

$$\text{Sum forces in } y\text{-direction at joint } B: \frac{\frac{L}{2}}{L}F_{BC} + B_y = 0 \quad F_{BC} = -2B_y = -672 \text{ kN}$$

- (6) FIND DISPLACEMENT ALONG
- x
- AXIS AT JOINT
- B

Find change in length of member AB then find its projection along x axis:

$$\delta_{AB} = \frac{F_{AB}c}{EA} = 3.875 \text{ mm} \quad \beta = \arctan\left(\frac{L}{a}\right) = 54.736^\circ \quad \delta_{Bx} = \frac{\delta_{AB}}{\cos(\beta)} = 6.713 \text{ mm} \quad \boxed{\delta_{Bx} = 6.71 \text{ mm}}$$

- (7) FIND
- $P_{\max}$
- FOR SPACE TRUSS IF
- δ_{Bx}
- MUST BE LIMITED TO 1.5 mm

Displacements are linearly related to the loads for this linear elastic small displacement problem, so reduce load variable P from 475 kN to

$$\frac{1.5}{6.71254} 475 = 106.145 \text{ kN} \quad \boxed{P_{\max} = 106.1 \text{ kN}}$$

Repeat space truss analysis using vector operations $a = 2.121 \text{ m}$ $L = 3 \text{ m}$ $P = 475 \text{ kN}$

POSITION AND UNIT VECTORS:

$$r_{AB} = \begin{pmatrix} a \\ 0 \\ -L \end{pmatrix} \quad e_{AB} = \frac{r_{AB}}{|r_{AB}|} = \begin{pmatrix} 0.577 \\ 0 \\ -0.816 \end{pmatrix} \quad r_{AC} = \begin{pmatrix} 0 \\ \frac{L}{2} \\ -\frac{L}{2} \end{pmatrix} \quad e_{AC} = \frac{r_{AC}}{|r_{AC}|} = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \end{pmatrix}$$

FIND MOMENT AT A:

$$M_A = r_{AB} \times R_B + r_{AC} \times R_C$$

$$M_A = r_{AB} \times \begin{pmatrix} 0 \\ RB_y \\ RB_z \end{pmatrix} + r_{AC} \times \begin{pmatrix} 2P \\ -P \\ RC_z \end{pmatrix} = \begin{pmatrix} 3.0 \text{ m } RB_y + 1.5 \text{ m } RC_z - 712.5 \text{ kN}\cdot\text{m} \\ -2.1213 \text{ m } RB_z - 1425.0 \text{ kN}\cdot\text{m} \\ 2.1213 \text{ m } RB_y - 1425.0 \text{ kN}\cdot\text{m} \end{pmatrix}$$

FIND MOMENTS ABOUT LINES OR AXES:

$$M_A e_{AB} = -1.732 \text{ m } RB_y + 1.7321 \text{ m } RB_z + 0.86603 \text{ m } RC_z + 752.15 \text{ kN}\cdot\text{m}$$

$$RC_z = \frac{-244.12}{0.72169} = -338.262 \quad C_z = -196.751 \text{ kN}$$

$$M_A e_{AC} = -1.5 \text{ m } RB_y + -1.5 \text{ m } RB_z \quad \text{So} \quad RB_y = -RB_z$$

$$M_A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -2.1213 \text{ m } RB_z + -1425.0 \text{ kN}\cdot\text{m} \quad \text{So} \quad RB_z = \frac{462.5}{-1.7678} = -261.625 \quad B_z = -671.75 \text{ kN}$$

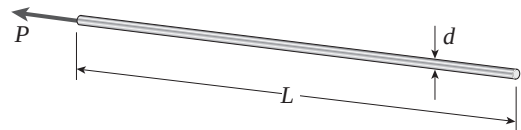
$$M_A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2.1213 \text{ m } RB_y + -1425.0 \text{ kN}\cdot\text{m} \quad \text{So} \quad RB_y = -RB_z = 261.625 \quad B_y = -335.876 \text{ kN}$$

$$\sum F_y = 0 \quad A_y = P - B_y = 139.124 \text{ kN}$$

Reactions obtained using vector operations agree with those based on scalar operations.

Problem 2.2-9 An aluminum wire having a diameter $d = 1/10$ in. and length $L = 12$ ft is subjected to a tensile load P (see figure). The aluminum has modulus of elasticity $E = 10,600$ ksi

If the maximum permissible elongation of the wire is $1/8$ in. and the allowable stress in tension is 10 ksi, what is the allowable load $P_{\max}$?



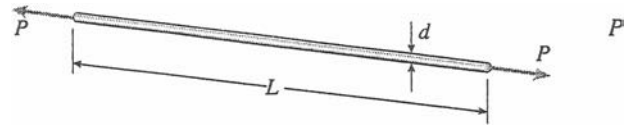
Solution 2.2-9

$$d = \frac{1}{10} \text{ in.} \quad L = 12(12) \text{ in.} \quad E = 10,600 \times (10^3) \text{ psi}$$

$$\delta_a = \frac{1}{8} \text{ in.} \quad \sigma_a = 10 \times (10^3) \text{ psi}$$

$$A = \frac{\pi d^2}{4} \quad A = 7.854 \times 10^{-3} \text{ in.}^2$$

$$EA = 8.325 \times 10^4 \text{ lb}$$



Maximum load based on elongation:

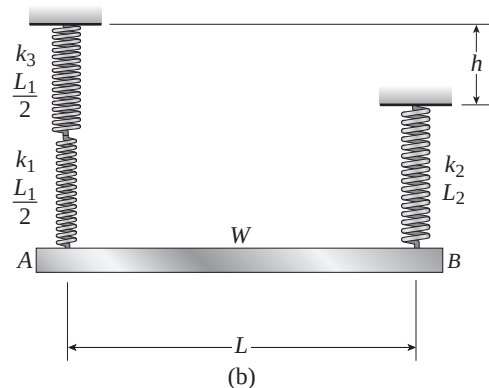
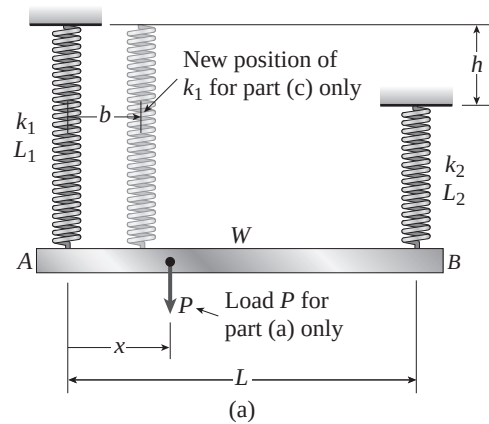
$$P_{\max 1} = \frac{EA}{L} \delta_a \quad P_{\max 1} = 72.3 \text{ lb} \quad \leftarrow \text{controls}$$

Maximum load based on stress:

$$P_{\max 2} = \sigma_a A \quad P_{\max 2} = 78.5 \text{ lb}$$

Problem 2.2-10 A uniform bar AB of weight $W = 25 \text{ N}$ is supported by two springs, as shown in the figure. The spring on the left has stiffness $k_1 = 300 \text{ N/m}$ and natural length $L_1 = 250 \text{ mm}$. The corresponding quantities for the spring on the right are $k_2 = 400 \text{ N/m}$ and $L_2 = 200 \text{ mm}$. The distance between the springs is $L = 350 \text{ mm}$, and the spring on the right is suspended from a support that is distance $h = 80 \text{ mm}$ below the point of support for the spring on the left. Neglect the weight of the springs.

- At what distance x from the left-hand spring (figure part a) should a load $P = 18 \text{ N}$ be placed in order to bring the bar to a horizontal position?
- If P is now removed, what new value of k_1 is required so that the bar (figure part a) will hang in a horizontal position under weight W ?
- If P is removed and $k_1 = 300 \text{ N/m}$, what distance b should spring k_1 be moved to the right so that the bar (figure part a) will hang in a horizontal position under weight W ?
- If the spring on the left is now replaced by two springs in series ($k_1 = 300 \text{ N/m}$, k_3) with overall natural length $L_1 = 250 \text{ mm}$ (see figure part b), what value of k_3 is required so that the bar will hang in a horizontal position under weight W ?



Solution 2.2-10

NUMERICAL DATA

$$W = 25 \text{ N} \quad k_1 = 0.300 \text{ N/mm} \quad L_1 = 250 \text{ mm}$$

$$k_2 = 0.400 \text{ N/mm} \quad L_2 = 200 \text{ mm}$$

$$L = 350 \text{ mm} \quad h = 80 \text{ mm} \quad P = 18 \text{ N}$$

- (a) LOCATION OF LOAD P TO BRING BAR TO HORIZONTAL POSITION

Use statics to get forces in both springs:

$$\sum M_A = 0 \quad F_2 = \frac{1}{L} \left(W \frac{L}{2} + Px \right)$$

$$F_2 = \frac{W}{2} + P \frac{x}{L}$$

$$\sum F_V = 0 \quad F_1 = W + P - F_2$$

$$F_1 = \frac{W}{2} + P \left(1 - \frac{x}{L} \right)$$

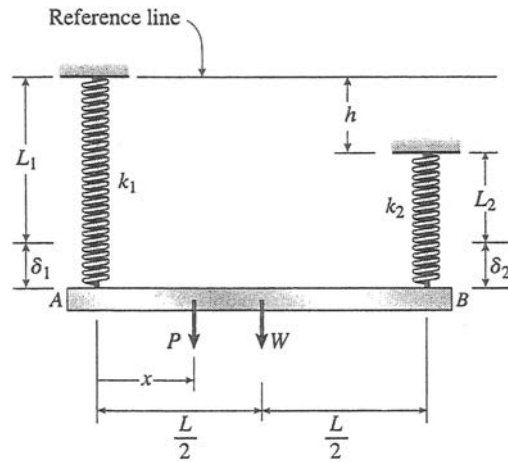
Use constraint equation to define horizontal position, then solve for location x :

$$L_1 + \frac{F_1}{k_1} = L_2 + h + \frac{F_2}{k_2}$$

Substitute expressions for F_1 and F_2 above into constraint equilibrium and solve for x :

$$x = \frac{-2L_1 L k_1 k_2 - k_2 W L - 2k_2 P L + 2L_2 L k_1 k_2 + 2h L k_1 k_2 + k_1 W L}{-2P(k_1 + k_2)}$$

$$x = 134.7 \text{ mm} \quad \leftarrow$$



- (b) NEXT REMOVE P AND FIND NEW VALUE OF SPRING CONSTANT k_1 SO THAT BAR IS HORIZONTAL UNDER WEIGHT W

$$\text{Now, } F_1 = \frac{W}{2} \quad F_2 = \frac{W}{2} \quad \text{since } P = 0$$

Same constraint equation as above but now $P = 0$:

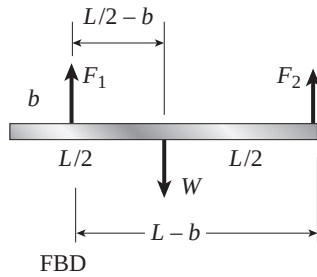
$$L_1 + \frac{W}{k_1} - (L_2 + h) - \frac{\left(\frac{W}{2}\right)}{k_2} = 0$$

Solve for k_1 :

$$k_1 = \frac{-Wk_2}{[2k_2[L_1 - (L_2 + h)]] - W}$$

$$k_1 = 0.204 \text{ N/mm} \quad \leftarrow$$

- (c) USE $k_1 = 0.300 \text{ N/mm}$ BUT RELOCATE SPRING k_1 ($x = b$) SO THAT BAR ENDS UP IN HORIZONTAL POSITION UNDER WEIGHT W



$$b = \frac{2L_1k_1k_2L + Wk_2 - 2L_2k_1k_2L - 2hk_1k_2L - Wk_1L}{(2L_1k_1k_2) - 2L_2k_1k_2 - 2hk_1k_2 - 2Wk_1}$$

- (d) REPLACE SPRING k_1 WITH SPRINGS IN SERIES: $k_1 = 0.3 \text{ N/mm}$, $L_1/2$, AND k_3 , $L_1/2$. FIND k_3 SO THAT BAR HANGS IN HORIZONTAL POSITION

$$\text{STATICS} \quad F_1 = \frac{W}{2} \quad F_2 = \frac{W}{2}$$

$$k_3 = \frac{Wk_1k_2}{-2L_1k_1k_2 - Wk_2 + 2L_2k_1k_2 + 2hk_1k_2 + Wk_1}$$

NOTE—equivalent spring constant for series springs:

$$k_e = \frac{k_1k_3}{k_1 + k_3}$$

PART (c)—CONTINUED

STATICS

$$\sum M_{k_1} = 0 \quad F_2 = \frac{W\left(\frac{L}{2} - b\right)}{L - b}$$

$$\sum F_V = 0$$

$$F_1 = W - F_2$$

$$F_1 = W - \frac{W\left(\frac{L}{2} - b\right)}{L - b}$$

$$F_1 = \frac{WL}{2(L - b)}$$

Constraint equation—substitute above expressions for F_1 and F_2 and solve for b :

$$L_1 + \frac{F_1}{k_1} - (L_2 + h) - \frac{F_2}{k_2} = 0$$

Use the following data:

$$k_1 = 0.300 \text{ N/mm} \quad k_2 = 0.4 \text{ N/mm} \quad L_1 = 250 \text{ mm}$$

$$L_2 = 200 \text{ mm} \quad L = 350 \text{ mm}$$

$$b = 74.1 \text{ mm} \quad \leftarrow$$

New constraint equation; solve for k_3 :

$$L_1 + \frac{F_1}{k_1} + \frac{F_1}{k_3} - (L_2 + h) - \frac{F_2}{k_2} = 0$$

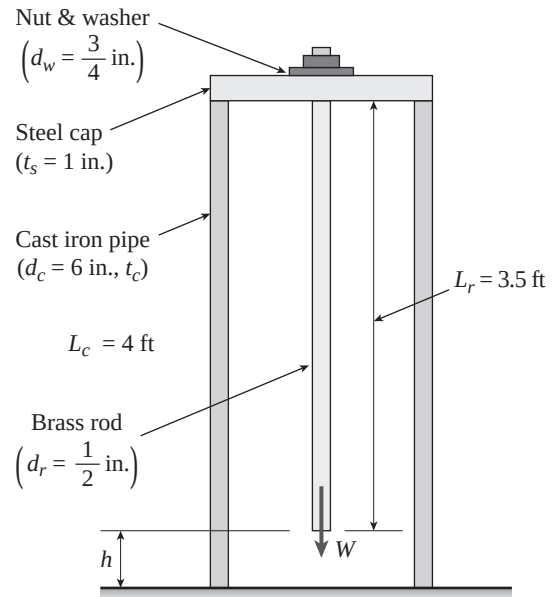
$$L_1 + \frac{W/2}{k_1} + \frac{W/2}{k_3} - (L_2 + h) - \frac{W/2}{k_2} = 0$$

$$k_3 = 0.638 \text{ N/mm} \quad \leftarrow$$

$$k_e = 0.204 \text{ N/mm} \quad \leftarrow \quad \text{checks—same as (b) above}$$

Problem 2.2-11 A hollow, circular, cast-iron pipe ($E_c = 12,000$ ksi) supports a brass rod ($E_b = 14,000$ ksi) and weight $W = 2$ kips, as shown. The outside diameter of the pipe is $d_c = 6$ in.

- If the allowable compressive stress in the pipe is 5000 psi and the allowable shortening of the pipe is 0.02 in., what is the minimum required wall thickness $t_{c,min}$? (Include the weights of the rod and steel cap in your calculations.)
- What is the elongation of the brass rod δ_r due to both load W and its own weight?
- What is the minimum required clearance h ?



Solution 2.2-11

The figure shows a section cut through the pipe, cap, and rod.

NUMERICAL DATA

$$E_c = 12000 \text{ ksi} \quad E_b = 14,000 \text{ ksi}$$

$$W = 2 \text{ k} \quad d_c = 6 \text{ in.} \quad d_r = \frac{1}{2} \text{ in.}$$

$$\sigma_a = 5 \text{ ksi} \quad \delta_a = 0.02 \text{ in.}$$

$$\text{Unit weights (see Table I-1): } \gamma_s = 2.836 \times 10^{-4} \text{ k/in.}^3$$

$$\gamma_b = 3.009 \times 10^{-4} \text{ k/in.}^3$$

$$L_c = 48 \text{ in.} \quad L_r = 42 \text{ in.}$$

$$t_s = 1 \text{ in.}$$

- (a) MINIMUM REQUIRED WALL THICKNESS OF CAST IRON PIPE, $t_{c,min}$

First check allowable stress then allowable shortening:

$$W_{cap} = \gamma_s \left(\frac{\pi}{4} d_c^2 t_s \right)$$

$$W_{cap} = 8.018 \times 10^{-3} \text{ k}$$

$$W_{rod} = \gamma_b \left(\frac{\pi}{4} d_r^2 L_r \right)$$

$$W_{rod} = 2.482 \times 10^{-3} \text{ k}$$

$$W_t = W + W_{cap} + W_{rod} \quad W_t = 2.01 \text{ k}$$

$$A_{min} = \frac{W_t}{\sigma_a} \quad A_{min} = 0.402 \text{ in.}^2$$

$$A_{pipe} = \frac{\pi}{4} [d_c^2 - (d_c - 2t_c)^2]$$

$$A_{\text{pipe}} = \pi t_c (d_c - t_c)$$

$$t_c (d_c - t_c) = \frac{W_t}{\pi \sigma_a}$$

$$\text{Let } \alpha = \frac{W_t}{\pi \sigma_a} \quad \alpha = 0.128;$$

$$t_c^2 - d_c t_c + \alpha = 0$$

$$t_c = \frac{d_c - \sqrt{d_c^2 - 4\alpha}}{2} \quad t_c = 0.021 \text{ in.}$$

^ minimum based
on σ_a

Now check allowable shortening requirement:

$$\delta_{\text{pipe}} = \frac{W_t L_c}{E_c A_{\text{min}}} \quad A_{\text{min}} = \frac{W_t L_c}{E_c \delta_a}$$

$$A_{\text{min}} = 0.447 \text{ in.}^2 < \text{larger than value based on } \sigma_a \text{ above}$$

$$\pi t_c (d_c - t_c) = \frac{W_t L_c}{E_c \delta_a}$$

$$t_c^2 - d_c t_c + \beta = 0 \quad \beta = \frac{W_t L_c}{\pi E_c \delta_a}$$

$$\beta = 0.142$$

$$t_c = \frac{d_c - \sqrt{d_c^2 - 4\beta}}{2}$$

$$t_c = 0.021 \text{ in.} \quad \leftarrow \text{minimum based on } \delta_a \text{ and } \sigma_a \text{ controls}$$

(b) ELONGATION OF ROD DUE TO SELF WEIGHT AND ALSO WEIGHT W

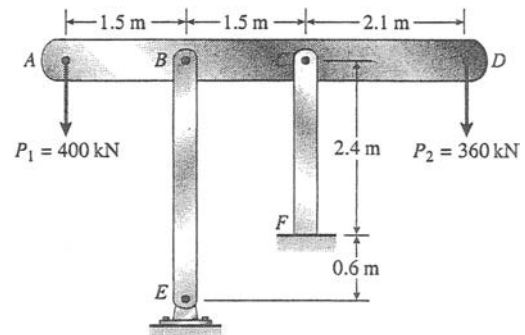
$$\delta_r = \frac{\left(W + \frac{W_{\text{rod}}}{2}\right) L_r}{E_b \left(\frac{\pi}{4} d_r^2\right)} \quad \delta_r = 0.031 \text{ in.} \quad \leftarrow$$

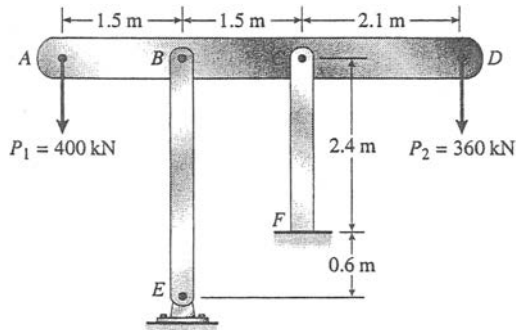
(c) MINIMUM CLEARANCE h

$$h_{\text{min}} = \delta_a + \delta_r \quad h_{\text{min}} = 0.051 \text{ in.} \quad \leftarrow$$

Problem 2.2-12 The horizontal rigid beam $ABCD$ is supported by vertical bars BE and CF and is loaded by vertical forces $P_1 = 400 \text{ kN}$ and $P_2 = 360 \text{ kN}$ acting at points A and D , respectively (see figure). Bars BE and CF are made of steel ($E = 200 \text{ GPa}$) and have cross-sectional areas $A_{BE} = 11,100 \text{ mm}^2$ and $A_{CF} = 9,280 \text{ mm}^2$. The distances between various points on the bars are shown in the figure.

Determine the vertical displacements δ_A and δ_D of points A and D , respectively.



Solution 2.2-12 Rigid beam supported by vertical bars

$$A_{BE} = 11,100 \text{ mm}^2$$

$$A_{CF} = 9,280 \text{ mm}^2$$

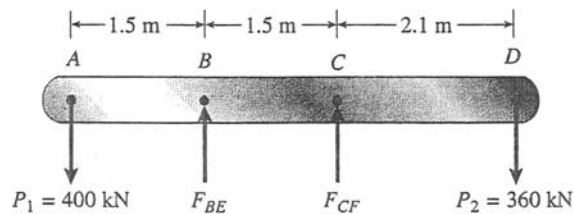
$$E = 200 \text{ GPa}$$

$$L_{BE} = 3.0 \text{ m}$$

$$L_{CF} = 2.4 \text{ m}$$

$$P_1 = 400 \text{ kN}; P_2 = 360 \text{ kN}$$

FREE-BODY DIAGRAM OF BAR *ABCD*



$$\Sigma M_B = 0 \quad \curvearrowright$$

$$(400 \text{ kN})(1.5 \text{ m}) + F_{CF}(1.5 \text{ m}) - (360 \text{ kN})(3.6 \text{ m}) = 0$$

$$F_{CF} = 464 \text{ kN}$$

$$\Sigma M_C = 0$$

$$(400 \text{ kN})(3.0 \text{ m}) - F_{BE}(1.5 \text{ m}) - (360 \text{ kN})(2.1 \text{ m}) = 0$$

$$F_{BE} = 296 \text{ kN}$$

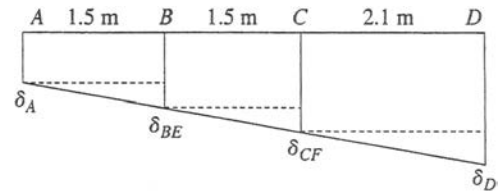
SHORTENING OF BAR *BE*

$$\delta_{BE} = \frac{F_{BE}L_{BE}}{EA_{BE}} = \frac{(296 \text{ kN})(3.0 \text{ m})}{(200 \text{ GPa})(11,100 \text{ mm}^2)} = 0.400 \text{ mm}$$

SHORTENING OF BAR *CF*

$$\delta_{CF} = \frac{F_{CF}L_{CF}}{EA_{CF}} = \frac{(464 \text{ kN})(2.4 \text{ m})}{(200 \text{ GPa})(9,280 \text{ mm}^2)} = 0.600 \text{ mm}$$

DISPLACEMENT DIAGRAM



$$\delta_{BE} - \delta_A = \delta_{CF} - \delta_{BE} \text{ or } \delta_A = 2\delta_{BE} - \delta_{CF}$$

$$\delta_A = 2(0.400 \text{ mm}) - 0.600 \text{ mm}$$

$$= 0.200 \text{ mm} \quad \leftarrow$$

(Downward)

$$\delta_D - \delta_{CF} = \frac{2.1}{1.5}(\delta_{CF} - \delta_{BE})$$

$$\text{or } \delta_D = \frac{12}{5}\delta_{CF} - \frac{7}{5}\delta_{BE}$$

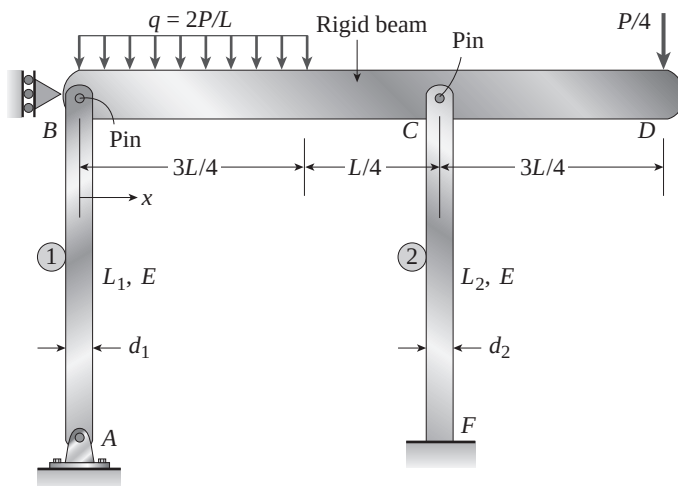
$$= \frac{12}{5}(0.600 \text{ mm}) - \frac{7}{5}(0.400 \text{ mm})$$

$$= 0.880 \text{ mm} \quad \leftarrow$$

(Downward)

Problem 2.2-13 Two pipe columns (AB , FC) are pin-connected to a rigid beam (BCD) as shown in the figure. Each pipe column has modulus E , but heights (L_1 or L_2) and outer diameters (d_1 or d_2) are different for each column. Assume the inner diameter of each column is $3/4$ of outer diameter. Uniformly distributed downward load $q = 2P/L$ is applied over a distance of $3L/4$ along BC , and concentrated load $P/4$ is applied downward at D .

- Derive a formula for the displacement δ_D at point D in terms of P and column flexibilities f_1 and f_2 .
- If $d_1 = (9/8)d_2$, find the L_1/L_2 ratio so that beam BCD displaces downward to a horizontal position under the load system in (a).
- If $L_1 = 2L_2$, find the d_1/d_2 ratio so that beam BCD displaces downward to a horizontal position under the load system in (a).
- If $d_1 = (9/8)d_2$ and $L_1/L_2 = 1.5$, at what horizontal distance x from B should load $P/4$ be placed so that beam BCD displaces downward to a horizontal position under the load system in part (a)?



Solution 2.2-13

(a) DISPLACEMENT δ_D

Use FBD of beam BCD $\sum M_B = 0$ $R_C = \frac{1}{L} \left[\left(2 \frac{P}{L} \right) \left(\frac{3}{4} L \right) \left(\frac{3}{8} L \right) + \frac{P}{4} \left(L + \frac{3}{4} L \right) \right] = P$ <compression force in column CF

$$\sum F_V = 0 \quad R_B = \left(2 \frac{P}{L} \right) \left(\frac{3}{4} L \right) + \frac{P}{4} - R_C = \frac{3P}{4} \quad \text{<compression force in column } BA$$

Downward displacements at B and C : $\delta_B = R_B f_1 = \frac{3P f_1}{4}$ $\delta_C = R_C f_2 = P f_2$

Geometry: $\delta_D = \delta_B + (\delta_C - \delta_B) \left(\frac{L + \frac{3}{4}L}{L} \right) = \frac{7P f_2}{4} - \frac{9P f_1}{16}$ $\delta_D = \frac{7P f_2}{4} - \frac{9P f_1}{16} = \boxed{\frac{P}{16} (28 f_2 - 9 f_1)}$

(b) DISPLACEMENT TO HORIZONTAL POSITION, SO $\delta_C = \delta_B$ and $\frac{3P f_1}{4} = P f_2$ or $\frac{f_1}{f_2} = \frac{4}{3}$

$$\frac{\frac{L_1}{EA_1}}{\frac{L_2}{EA_2}} = \frac{4}{3} \quad \text{or} \quad \frac{L_1}{L_2} = \frac{4}{3} \left(\frac{A_1}{A_2} \right) \quad \frac{L_1}{L_2} = \frac{4}{3} \left(\frac{\frac{\pi}{4} d_1^2}{\frac{\pi}{4} d_2^2} \right) = \frac{4 d_1^2}{3 d_2^2} \quad \frac{L_1}{L_2} = \frac{4}{3} \left(\frac{d_1}{d_2} \right)^2 \quad \text{with} \quad \frac{d_1}{d_2} = \frac{9}{8}$$

$$\frac{L_1}{L_2} = \frac{4}{3} \left(\frac{9}{8} \right)^2 = \frac{27}{16} \quad \boxed{\frac{L_1}{L_2} = \frac{27}{16}}$$

(c) If $L_1 = 2 L_2$, FIND THE d_1/d_2 RATIO SO THAT BEAM BCD DISPLACES DOWNWARD TO A HORIZONTAL POSITION

$$\frac{L_1}{L_2} = 2 \quad \text{and} \quad \delta_C = \delta_B \quad \text{from part (b).} \quad \left(\frac{d_1}{d_2}\right)^2 = \frac{3}{4} \left(\frac{L_1}{L_2}\right) \quad \text{so} \quad \boxed{\frac{d_1}{d_2} = \sqrt{\frac{3}{4}}(2) = 1.225}$$

(d) If $d_1 = (9/8) d_2$ AND $L_1/L_2 = 1.5$, AT WHAT HORIZONTAL DISTANCE x FROM B SHOULD LOAD $P/4$ AT D BE PLACED?

$$\text{Given} \quad \frac{d_1}{d_2} = \frac{9}{8} \quad \text{and} \quad \frac{L_1}{L_2} = 1.5 \quad \text{or} \quad \frac{f_1}{f_2} = \frac{L_1}{L_2} \left(\frac{A_2}{A_1}\right) \quad \frac{f_1}{f_2} = \frac{L_1}{L_2} \left(\frac{d_2}{d_1}\right)^2 = \frac{3}{2} \left(\frac{8}{9}\right)^2 = \frac{32}{27}$$

Recompute column forces R_B and R_C but now with load $P/4$ positioned at distance x from B .

$$\text{Use FBD of beam } BCD: \quad \sum M_B = 0 \quad R_C = \frac{1}{L} \left[\left(2\frac{P}{L}\right) \left(\frac{3}{4}L\right) \left(\frac{3}{8}L\right) + \frac{P}{4}(x) \right] = \frac{\frac{9LP}{16} + \frac{Px}{4}}{L}$$

$$\sum F_V = 0 \quad R_B = \left(2\frac{P}{L}\right) \left(\frac{3}{4}L\right) + \frac{P}{4} - R_C = \frac{7P}{4} - \frac{\frac{9LP}{16} + \frac{Px}{4}}{L}$$

Horizontal displaced position under load q and load $P/4$ so $\delta_C = \delta_B$ or $R_C f_2 = R_B f_1$.

$$\left(\frac{\frac{9LP}{16} + \frac{Px}{4}}{L}\right) f_2 = \left(\frac{7P}{4} - \frac{\frac{9LP}{16} + \frac{Px}{4}}{L}\right) f_1 \quad \text{solve, } x = -\frac{9Lf_2 - 19Lf_1}{4f_1 + 4f_2} = -\frac{L(9f_2 - 19f_1)}{4(f_1 + f_2)}$$

$$x = -\frac{L(9f_2 - 19f_1)}{4(f_1 + f_2)} \quad \text{or} \quad x = L \left[\frac{19\frac{f_1}{f_2} - 9}{4\left(\frac{f_1}{f_2} + 1\right)} \right]$$

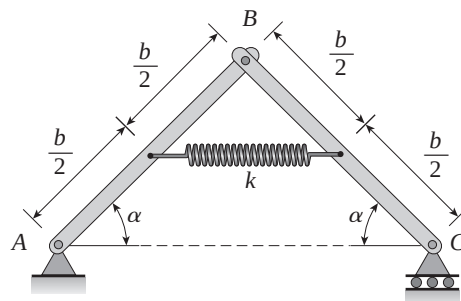
Now substitute f_1/f_2 ratio from above:

$$\boxed{x = L \left[\frac{19\frac{32}{27} - 9}{4\left(\frac{32}{27} + 1\right)} \right] = \frac{365L}{236}} \quad \frac{365}{236} = 1.547$$

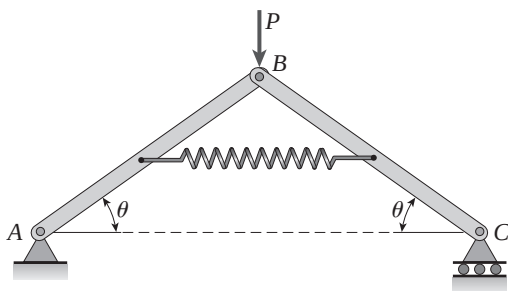
Problem 2.2-14 A framework ABC consists of two rigid bars AB and BC , each having a length b (see the first part of the figure part a). The bars have pin connections at A, B , and C and are joined by a spring of stiffness k . The spring is attached at the midpoints of the bars. The framework has a pin support at A and a roller support at C , and the bars are at an angle α to the horizontal.

When a vertical load P is applied at joint B (see the second part of the figure part a) the roller support C moves to the right, the spring is stretched, and the angle of the bars decreases from α to the angle θ .

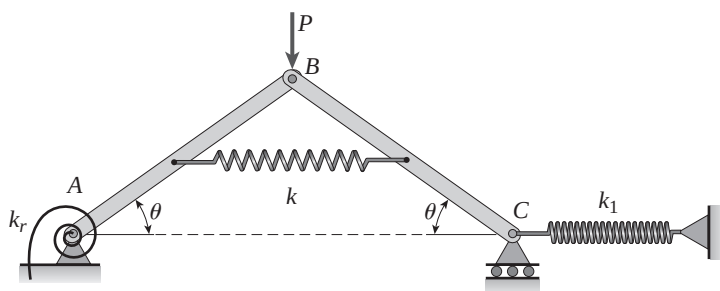
- Determine the angle θ and the increase δ in the distance between points A and C . Also find reactions at A and C . (Use the following data: $b = 200$ mm, $k = 3.2$ kN/m, $\alpha = 45^\circ$, and $P = 50$ N.)
- Repeat part (a) if a translational spring $k_1 = k/2$ is added at C and a rotational spring $k_r = kb^2/2$ is added at A (see figure part b).



(a) Initial position of structure



(a) - cont'd: displaced position of structure



(b) Displaced structure

Solution 2.2-14

Apply the laws of statics to the structure in its displaced position; also use FBD's of the left and right bars alone (referred to as LHFB and RHFB below).

$$\text{OVERALL FBD: } \sum F_H = 0 \quad H_A - k_1 \delta = 0 \quad \text{so} \quad H_A = k_1 \delta$$

$$\sum F_V = 0 \quad R_A + R_C = P$$

$$\sum M_A = 0 \quad k_r(\alpha - \theta) - P \frac{L_2}{2} + R_C L_2 = 0 \quad R_C = \frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right]$$

$$\text{LHFB: } \sum M_B = 0 \quad H_A h + k \frac{\delta}{2} \left(\frac{h}{2} \right) - R_A \left(\frac{L_2}{2} \right) + k_r(\alpha - \theta) = 0$$

$$R_A = \frac{2}{L_2} \left[k_1 \delta h + k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_r(\alpha - \theta) \right]$$

$$\text{RHFB: } \sum M_B = 0 \quad -k \frac{\delta}{2} \left(\frac{h}{2} \right) - k_1 \delta h + R_C \frac{L_2}{2} = 0 \quad R_C = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right]$$

Equate the two expressions for R_C then substitute expressions for L_2 , k_r , k_1 , h and δ

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right] \quad \text{OR}$$

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] - \left[\frac{2}{L_2} \left[k \frac{2b(\cos(\theta) - \cos(\alpha))}{2} \frac{b \sin(\theta)}{2} + k_1 [2b(\cos(\theta) - \cos(\alpha))] (b \sin(\theta)) \right] \right] = 0$$

(a) SUBSTITUTE NUMERICAL VALUES, THEN SOLVE NUMERICALLY FOR ANGLE θ AND DISTANCE INCREASE δ

$$b = 200 \text{ mm} \quad k = 3.2 \text{ kN/m} \quad \alpha = 45^\circ \quad P = 50 \text{ N} \quad k_1 = 0 \quad k_r = 0$$

$$L_2 = 2b \cos(\theta) \quad L_1 = 2b \cos(\alpha) \quad \delta = L_2 - L_1 \quad \delta = 2b(\cos(\theta) - \cos(\alpha)) \quad h = b \sin(\theta)$$

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] - \left[\frac{1}{L_2} \left[k \frac{2b(\cos(\theta) - \cos(\alpha))}{2} \frac{b \sin(\theta)}{2} + k_1 [2b(\cos(\theta) - \cos(\alpha))] (b \sin(\theta)) \right] \right] = 0$$

$$\text{Solving above equation numerically gives } \boxed{\theta = 35.1^\circ} \quad \boxed{\delta = 44.6 \text{ mm}}$$

COMPUTE REACTIONS

$$R_C = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right] = 25 \text{ N} \quad R_C = \frac{1}{L_C} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] = 25 \text{ N}$$

$$R_A = \frac{2}{L_2} \left[k_1 \delta h + k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_r(\alpha - \theta) \right] = 25 \text{ N} \quad M_A = k_r(\alpha - \theta) = 0$$

$$R_A + R_C = 50 \text{ N} < \text{check}$$

$$\boxed{R_A = 25 \text{ N}}$$

$$\boxed{R_C = 25 \text{ N}}$$

(b) SUBSTITUTE NUMERICAL VALUES, THEN SOLVE NUMERICALLY FOR ANGLE θ AND DISTANCE INCREASE δ

$$b = 200 \text{ mm} \quad k = 3.2 \text{ kN/m} \quad \alpha = 45^\circ \quad P = 50 \text{ N} \quad k_1 = \frac{k}{2} \quad k_r = \frac{k}{2} b^2$$

$$L_2 = 2b \cos(\theta) \quad L_1 = 2b \cos(\alpha) \quad \delta = L_2 - L_1 \quad \delta = 2b(\cos(\theta) - \cos(\alpha)) \quad h = b \sin(\theta)$$

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] - \left[\frac{2}{L_2} \left[k \frac{2b(\cos(\theta) - \cos(\alpha))}{2} \frac{b \sin(\theta)}{2} + k_1 [2b(\cos(\theta) - \cos(\alpha))] (b \sin(\theta)) \right] \right] = 0$$

$$\text{Solving above equation numerically gives } \boxed{\theta = 43.3^\circ} \quad \boxed{\delta = 8.19 \text{ mm}}$$

COMPUTE REACTIONS

$$R_C = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right] = 18.5 \text{ N} \quad R_2 = \frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] = 18.5 \text{ N}$$

$$R_A = \frac{2}{L_2} \left[k_1 \delta h + k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_r(\alpha - \theta) \right] = 31.5 \text{ N} \quad M_A = k_r(\alpha - \theta) = 1.882 \text{ N}\cdot\text{m}$$

$$R_A + R_C = 50 \text{ N} < \text{check}$$

$$\boxed{R_A = 31.5 \text{ N}}$$

$$\boxed{R_C = 18.5 \text{ N}}$$

$$\boxed{M_A = 1.882 \text{ N}\cdot\text{m}}$$

Problem 2.2-15 Solve the preceding problem for the following data:

$$b = 8.0 \text{ in.}, k = 16 \text{ lb/in.}, \alpha = 45^\circ, \text{ and } P = 10 \text{ lb.}$$

Solution 2.2-15

Apply the laws of statics to the structure in its displaced position; also use FBD's of the left and right bars alone (referred to as LHFB and RHFB below)

$$\text{OVERALL FBD} \quad \sum F_H = 0 \quad H_A - k_1 \delta = 0 \quad \text{so} \quad H_A = k_1 \delta$$

$$\sum F_V = 0 \quad R_A + R_C = P$$

$$\sum M_A = 0 \quad k_r(\alpha - \theta) - P \frac{L_2}{2} + R_C L_2 = 0 \quad R_C = \frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right]$$

$$\text{LHFB} \quad \sum M_B = 0 \quad H_A h + k \frac{\delta}{2} \left(\frac{h}{2} \right) - R_A \frac{L_2}{2} + k_r(\alpha - \theta) = 0$$

$$R_A = \frac{2}{L_2} \left[k_1 \delta h + k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_r(\alpha - \theta) \right]$$

$$\text{RHFB} \quad \sum M_B = 0 \quad -k \frac{\delta}{2} \left(\frac{h}{2} \right) - k_1 \delta h + R_C \frac{L_2}{2} = 0 \quad R_C = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right]$$

Equate the two expressions above for R_C then substitute expressions for L_2 , k_r , k_1 , h , and δ

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right] \quad \text{OR}$$

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] - \left[\frac{2}{L_2} \left[k \frac{2b(\cos(\theta) - \cos(\alpha))}{2} \frac{b \sin(\theta)}{2} + k_1 [2b(\cos(\theta) - \cos(\alpha))] (b \sin(\theta)) \right] \right] = 0$$

(a) SUBSTITUTE NUMERICAL VALUES, THEN SOLVE NUMERICALLY FOR ANGLE θ AND DISTANCE INCREASE δ

$$b = 8 \text{ in.} \quad k = 16 \text{ lb/in.} \quad \alpha = 45^\circ \quad P = 10 \text{ lb} \quad k_1 = 0 \quad k_r = 0$$

$$L_2 = 2b \cos(\theta) \quad L_1 = 2b \cos(\alpha) \quad \delta = L_2 - L_1 \quad \delta = 2b(\cos(\theta) - \cos(\alpha)) \quad h = b \sin(\theta)$$

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] - \left[\frac{2}{L_2} \left[k \frac{2b(\cos(\theta) - \cos(\alpha))}{2} \frac{b \sin(\theta)}{2} + k_1 [2b(\cos(\theta) - \cos(\alpha))] (b \sin(\theta)) \right] \right] = 0$$

$$\text{Solving above equation numerically gives } \boxed{\theta = 35.1^\circ} \quad \boxed{\delta = 1.782 \text{ in.}}$$

COMPUTE REACTIONS

$$R_C = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right] = 5 \text{ lb} \quad R_C = \frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] = 5 \text{ lb}$$

$$R_A = \frac{2}{L_2} \left[k_1 \delta h + k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_r(\alpha - \theta) \right] = 5 \text{ lb} \quad M_A = k_r(\alpha - \theta) = 0$$

$$R_A + R_C = 10 \text{ lb} \quad < \text{check} \quad \boxed{R_A = 5 \text{ lb}} \quad \boxed{R_C = 5 \text{ lb}}$$

- (b) SUBSTITUTE NUMERICAL VALUES, THEN SOLVE NUMERICALLY FOR ANGLE
- θ
- AND DISTANCE INCREASE
- δ

$$b = 8 \text{ in.} \quad k = 16 \text{ lb/in.} \quad \alpha = 45^\circ \quad P = 101b \quad k_1 = \frac{k}{2} \quad k_r = \frac{k}{2} b^2$$

$$L_2 = 2b \cos(\theta) \quad L_1 = 2b \cos(\alpha) \quad \delta = L_2 - L_1 \quad \delta = 2b(\cos(\theta) - \cos(\alpha)) \quad h = b \sin(\theta)$$

$$\frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] - \left[\frac{2}{L_2} \left[k \frac{2b(\cos(\theta) - \cos(\alpha))}{2} \frac{b \sin(\theta)}{2} + k_1 [2b(\cos(\theta) - \cos(\alpha))] (b \sin(\theta)) \right] \right] = 0$$

$$\text{Solving above equation numerically gives } \boxed{\theta = 43.3^\circ} \quad \boxed{\delta = 0.327 \text{ in.}}$$

COMPUTE REACTIONS

$$R_C = \frac{2}{L_2} \left[k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_1 \delta h \right] = 3.71 \text{ lb} \quad R_C = \frac{1}{L_2} \left[P \frac{L_2}{2} - k_r(\alpha - \theta) \right] = 3.71 \text{ lb}$$

$$R_A = \frac{2}{L_2} \left[k_1 \delta h + k \frac{\delta}{2} \left(\frac{h}{2} \right) + k_r(\alpha - \theta) \right] = 6.3 \text{ lb} \quad M_A = k_r(\alpha - \theta) = 1.252 \text{ ft}\cdot\text{lb}$$

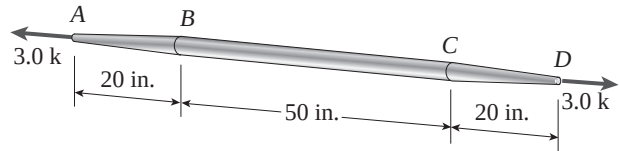
$$R_A + R_C = 10.01 \text{ lb} < \text{check} \quad \boxed{R_A = 6.3 \text{ lb}} \quad \boxed{R_C = 3.71 \text{ lb}} \quad \boxed{M_A = 1.252 \text{ lb}\cdot\text{ft}}$$

Changes in Lengths under Nonuniform Conditions

Problem 2.3-1

- (a) Calculate the elongation of a copper bar of solid circular cross section with tapered ends when it is stretched by axial loads of magnitude 3.0 k (see figure).

(The length of the end segments is 20 in. and the length of the prismatic middle segment is 50 in. Also, the diameters at cross sections A, B, C, and D are 0.5, 1.0, 1.0, and 0.5 in., respectively, and the modulus of elasticity is 18,000 ksi. (Hint: Use the result of Example 2-4.)



- (b) If the total elongation of the bar cannot exceed 0.025 in., what are the required diameters at B and C? Assume that diameters at A and D remain at 0.5 in.

Solution 2.3-1

NUMERICAL DATA

$$P = 3 \text{ k} \quad L_1 = 20 \text{ in.} \quad L_2 = 50 \text{ in.} \quad d_A = 0.5 \text{ in.} \quad d_B = 1 \text{ in.} \quad E = 18000 \text{ ksi}$$

- (a) TOTAL ELONGATION

$$\delta_1 = \frac{4PL_1}{\pi E d_A d_B} = 0.00849 \text{ in.} \quad \delta_2 = \frac{PL_2}{E \frac{\pi}{4} d_B^2} = 0.01061 \text{ in.}$$

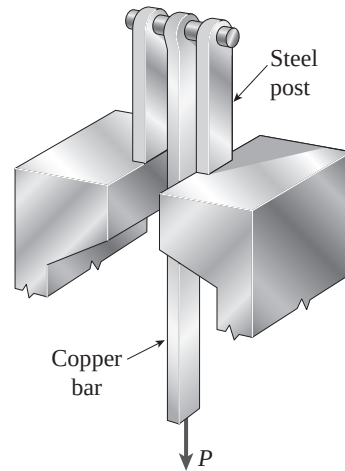
$$\delta = 2\delta_1 + \delta_2 = 0.0276 \text{ in.} \quad \boxed{\delta = 0.0276 \text{ in.}}$$

- (b) FIND NEW DIAMETERS AT B AND C IF TOTAL ELONGATION CANNOT EXCEED 0.025 in.

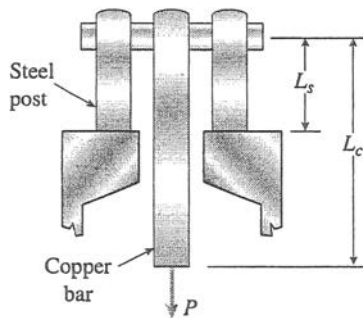
$$2 \left(\frac{4PL_1}{\pi E d_A d_B} \right) + \frac{PL_2}{E \frac{\pi}{4} d_B^2} = 0.025 \text{ in.} \quad \text{Solving for } d_B: \quad \boxed{d_B = 1.074 \text{ in.}}$$

Problem 2.3-2 A long, rectangular copper bar under a tensile load P hangs from a pin that is supported by two steel posts (see figure). The copper bar has a length of 2.0 m, a cross-sectional area of 4800 mm^2 , and a modulus of elasticity $E_c = 120 \text{ GPa}$. Each steel post has a height of 0.5 m, a cross-sectional area of 4500 mm^2 , and a modulus of elasticity $E_s = 200 \text{ GPa}$.

- Determine the downward displacement δ of the lower end of the copper bar due to a load $P = 180 \text{ kN}$.
- What is the maximum permissible load $P_{\max}$ if the displacement δ is limited to 1.0 mm?



Solution 2.3-2 Copper bar with a tensile load



$$\begin{aligned} L_c &= 2.0 \text{ m} \\ A_c &= 4800 \text{ mm}^2 \\ E_c &= 120 \text{ GPa} \\ L_s &= 0.5 \text{ m} \\ A_s &= 4500 \text{ mm}^2 \\ E_s &= 200 \text{ GPa} \end{aligned}$$

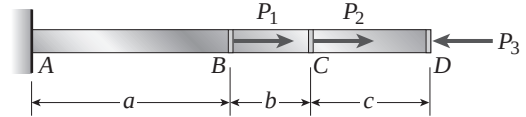
- (a) DOWNWARD DISPLACEMENT δ ($P = 180 \text{ kN}$)

$$\begin{aligned} \delta_c &= \frac{PL_c}{E_c A_c} = \frac{(180 \text{ kN})(2.0 \text{ m})}{(120 \text{ GPa})(4800 \text{ mm}^2)} \\ &= 0.625 \text{ mm} \\ \delta_s &= \frac{(P/2)L_s}{E_s A_s} = \frac{(90 \text{ kN})(0.5 \text{ m})}{(200 \text{ GPa})(4500 \text{ mm}^2)} \\ &= 0.050 \text{ mm} \\ \delta &= \delta_c + \delta_s = 0.625 \text{ mm} + 0.050 \text{ mm} \\ &= 0.675 \text{ mm} \quad \leftarrow \end{aligned}$$

- (b) MAXIMUM LOAD $P_{\max}$ ($\delta_{\max} = 1.0 \text{ mm}$)

$$\begin{aligned} \frac{P_{\max}}{P} &= \frac{\delta_{\max}}{\delta} \quad P_{\max} = P \left(\frac{\delta_{\max}}{\delta} \right) \\ P_{\max} &= (180 \text{ kN}) \left(\frac{1.0 \text{ mm}}{0.675 \text{ mm}} \right) = 267 \text{ kN} \quad \leftarrow \end{aligned}$$

Problem 2.3-3 An aluminum bar AD (see figure) has a cross-sectional area of 0.40 in.^2 and is loaded by forces $P_1 = 1700 \text{ lb}$, $P_2 = 1200 \text{ lb}$, and $P_3 = 1300 \text{ lb}$. The lengths of the segments of the bar are $a = 60 \text{ in.}$, $b = 24 \text{ in.}$, and $c = 36 \text{ in.}$



- Assuming that the modulus of elasticity $E = 10.4 \times 10^6 \text{ psi}$, calculate the change in length of the bar. Does the bar elongate or shorten?
- By what amount P should the load P_3 be increased so that the bar does not change in length when the three loads are applied?
- If P_3 remains at 1300 lb , what revised cross-sectional area for segment AB will result in no change of length when all three loads are applied?

Solution 2.3-3

NUMERICAL DATA

$$A = 0.40 \text{ in.}^2 \quad P_1 = 1700 \text{ lb}$$

$$P_2 = 1200 \text{ lb} \quad P_3 = 1300 \text{ lb}$$

$$E = 10.4 (10^6) \text{ psi}$$

$$a = 60 \text{ in.} \quad b = 24 \text{ in.} \quad c = 36 \text{ in.}$$

(a) TOTAL ELONGATION

$$\delta = \frac{1}{EA} [(P_1 + P_2 - P_3)a + (P_2 - P_3)b + (-P_3)c] = 0.01125 \text{ in.} \quad \boxed{\delta = 0.01125 \text{ in.}} \quad (\text{elongation})$$

(b) INCREASE P_3 SO THAT BAR DOES NOT CHANGE LENGTH

$$\frac{1}{EA} [(P_1 + P_2 - P_3)a + (P_2 - P_3)b + (-P_3)c] = 0 \text{ solve, } P_3 = 1690 \text{ lb}$$

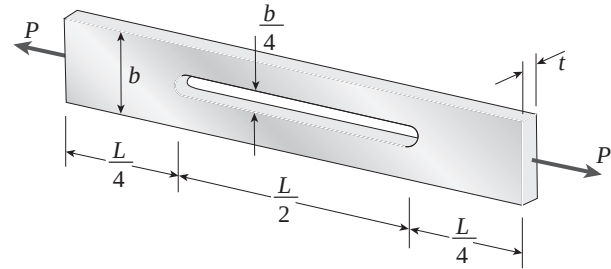
So new value of P_3 is 1690 lb ,
 an increase of 390 lb .

(c) NOW CHANGE CROSS-SECTIONAL AREA OF AB SO THAT BAR DOES NOT CHANGE LENGTH $P_3 = 1300 \text{ lb}$

$$\frac{1}{E} \left[(P_1 + P_2 - P_3) \frac{a}{A_{AB}} + (P_2 - P_3) \frac{b}{A} + (-P_3) \frac{c}{A} \right] = 0$$

Solving for A_{AB} : $\boxed{A_{AB} = 0.78 \text{ in.}^2}$ $\frac{A_{AB}}{A} = 1.951$

Problem 2.3-4 A rectangular bar of length L has a slot in the middle half of its length (see figure). The bar has width b , thickness t , and modulus of elasticity E . The slot has width $b/4$.



- Obtain a formula for the elongation δ of the bar due to the axial loads P .
- Calculate the elongation of the bar if the material is high-strength steel, the axial stress in the middle region is 160 MPa, the length is 750 mm, and the modulus of elasticity is 210 GPa.
- If the total elongation of the bar is limited to $\delta_{\max} = 0.475$ mm, what is the maximum length of the slotted region? Assume that the axial stress in the middle region remains at 160 MPa.

Solution 2.3-4

$$(a) \quad \delta = \frac{P}{E} \left(\frac{2 \frac{L}{4}}{bt} + \frac{\frac{L}{2}}{\frac{3}{4}bt} \right) = \frac{7LP}{6Ebt} \quad \boxed{\delta = \frac{7PL}{6Ebt}}$$

$$(b) \quad \text{NUMERICAL DATA} \quad E = 210 \text{ GPa} \quad L = 750 \text{ mm} \quad \sigma_{\text{mid}} = 160 \text{ MPa}$$

$$\text{so} \quad \sigma_{\text{mid}} = \frac{P}{\frac{3}{4}bt} \quad \text{and} \quad \frac{P}{bt} = \frac{3}{4}\sigma_{\text{mid}}$$

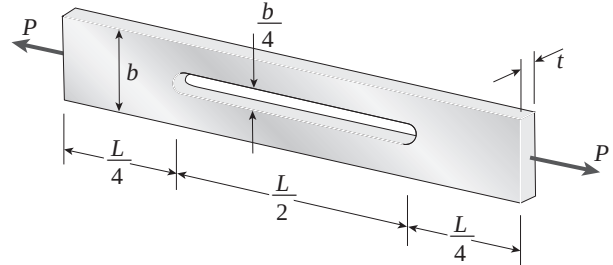
$$\delta = \frac{7LP}{6Ebt} \quad \text{or} \quad \delta = \frac{7L}{6E} \left(\frac{3}{4}\sigma_{\text{mid}} \right) = 0.5 \text{ mm} \quad \boxed{\delta = 0.5 \text{ mm}}$$

$$(c) \quad \delta_{\max} = \frac{P}{E} \left(\frac{L - L_{\text{slot}}}{bt} + \frac{L_{\text{slot}}}{\frac{3}{4}bt} \right) \quad \text{or} \quad \delta_{\max} = \left(\frac{P}{bt} \right) \left(\frac{1}{E} \right) \left(L - L_{\text{slot}} + \frac{4}{3}L_{\text{slot}} \right)$$

$$\text{or} \quad \delta_{\max} = \left(\frac{3}{4}\sigma_{\text{mid}} \right) \left(\frac{1}{E} \right) \left(L + \frac{L_{\text{slot}}}{3} \right) \quad \text{Solving for } L_{\text{slot}} \text{ with } \delta_{\max} = 0.475 \text{ mm}$$

$$L_{\text{slot}} = \frac{4E\delta_{\max} - 3L\sigma_{\text{mid}}}{\sigma_{\text{mid}}} = 244 \text{ mm} \quad \boxed{L_{\text{slot}} = 244 \text{ mm}} \quad \frac{L_{\text{slot}}}{L} = 0.325$$

Problem 2.3-5 Solve the preceding problem if the axial stress in the middle region is 24,000 psi, the length is 30 in., and the modulus of elasticity is 30×10^6 psi. In part (c), assume that $\delta_{\max} = 0.02$ in.



Solution 2.3-5

$$(a) \quad \delta = \frac{P}{E} \left(\frac{2 \frac{L}{4}}{\frac{b}{4}t} + \frac{\frac{L}{2}}{\frac{3}{4}bt} \right) = \frac{7LP}{6Ebt}$$

$$(b) \quad E = 30,000 \text{ ksi} \quad L = 30 \text{ in.} \quad \sigma_{\text{mid}} = 24 \text{ ksi}$$

$$\text{So} \quad \sigma_{\text{mid}} = \frac{P}{\frac{3}{4}bt} \quad \text{and} \quad \frac{P}{bt} = \frac{3}{4} \sigma_{\text{mid}}$$

$$\delta = \frac{7LP}{6Ebt} \quad \text{or} \quad \delta = \frac{7L}{6E} \left(\frac{3}{4} \sigma_{\text{mid}} \right) = 0.021 \text{ in.} \quad \boxed{\delta = 0.021 \text{ in.}}$$

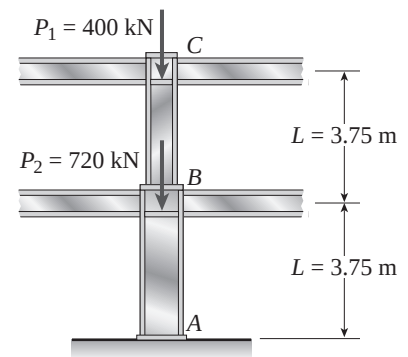
$$(c) \quad \delta_{\max} = \frac{P}{E} \left(\frac{L - L_{\text{slot}}}{bt} + \frac{L_{\text{slot}}}{\frac{3}{4}bt} \right) \quad \text{or} \quad \delta_{\max} = \left(\frac{P}{bt} \right) \left(\frac{1}{E} \right) \left(L - L_{\text{slot}} + \frac{4}{3} L_{\text{slot}} \right)$$

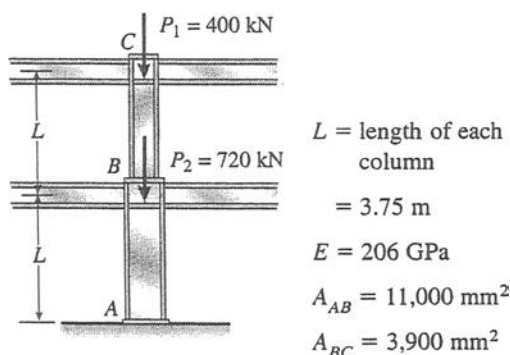
$$\text{or} \quad \delta_{\max} = \left(\frac{3}{4} \sigma_{\text{mid}} \right) \left(\frac{1}{E} \right) \left(L + \frac{L_{\text{slot}}}{3} \right) \quad \text{Solving for } L_{\text{slot}} \text{ with } \delta_{\max} = 0.02 \text{ in.:$$

$$L_{\text{slot}} = \frac{4E\delta_{\max} - 3L\sigma_{\text{mid}}}{\sigma_{\text{mid}}} = 10 \text{ in.} \quad \boxed{L_{\text{slot}} = 10 \text{ in.}} \quad \frac{L_{\text{slot}}}{L} = 0.333$$

Problem 2.3-6 A two-story building has steel columns AB in the first floor and BC in the second floor, as shown in the figure. The roof load P_1 equals 400 kN and the second-floor load P_2 equals 720 kN. Each column has length $L = 3.75$ m. The cross-sectional areas of the first- and second-floor columns are 11,000 mm² and 3,900 mm², respectively.

- Assuming that $E = 206$ GPa, determine the total shortening δ_{AC} of the two columns due to the combined action of the loads P_1 and P_2 .
- How much additional load P_0 can be placed at the top of the column (point C) if the total shortening δ_{AC} is not to exceed 4.0 mm?



Solution 2.3-6 Steel columns in a building(a) SHORTENING δ_{AC} OF THE TWO COLUMNS

$$\begin{aligned}
 \delta_{AC} &= \sum \frac{N_i L_i}{E_i A_i} = \frac{N_{AB} L}{E A_{AB}} + \frac{N_{BC} L}{E A_{BC}} \\
 &= \frac{(1120 \text{ kN})(3.75 \text{ m})}{(206 \text{ GPa})(11,000 \text{ mm}^2)} + \frac{(400 \text{ kN})(3.75 \text{ m})}{(206 \text{ GPa})(3,900 \text{ mm}^2)} \\
 &= 1.8535 \text{ mm} + 1.8671 \text{ mm} = 3.7206 \text{ mm} \\
 \delta_{AC} &= 3.72 \text{ mm} \quad \leftarrow
 \end{aligned}$$

(b) ADDITIONAL LOAD P_0 AT POINT C

$$(\delta_{AC})_{\max} = 4.0 \text{ mm}$$

 δ_0 = additional shortening of the two columns due to the load P_0

$$\begin{aligned}
 \delta_0 &= (\delta_{AC})_{\max} - \delta_{AC} = 4.0 \text{ mm} - 3.7206 \text{ mm} \\
 &= 0.2794 \text{ mm}
 \end{aligned}$$

$$\text{Also, } \delta_0 = \frac{P_0 L}{E A_{AB}} + \frac{P_0 L}{E A_{BC}} = \frac{P_0 L}{E} \left(\frac{1}{A_{AB}} + \frac{1}{A_{BC}} \right)$$

Solve for P_0 :

$$P_0 = \frac{E \delta_0}{L} \left(\frac{A_{AB} A_{BC}}{A_{AB} + A_{BC}} \right)$$

SUBSTITUTE NUMERICAL VALUES:

$$E = 206 \times 10^9 \text{ N/m}^2 \quad \delta_0 = 0.2794 \times 10^{-3} \text{ m}$$

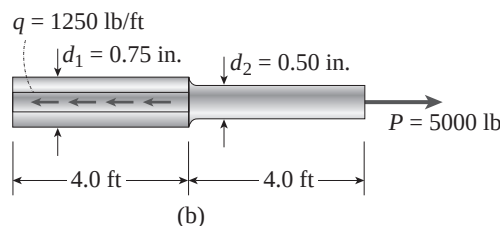
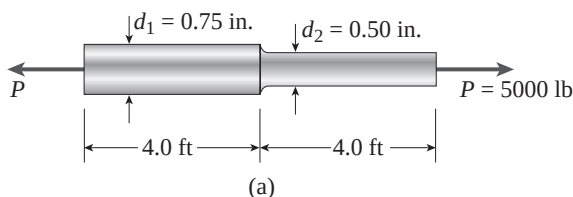
$$L = 3.75 \text{ m} \quad A_{AB} = 11,000 \times 10^{-6} \text{ m}^2$$

$$A_{BC} = 3,900 \times 10^{-6} \text{ m}^2$$

$$P_0 = 44,200 \text{ N} = 44.2 \text{ kN} \quad \leftarrow$$

Problem 2.3-7 A steel bar 8.0 ft long has a circular cross section of diameter $d_1 = 0.75$ in. over one-half of its length and diameter $d_2 = 0.5$ in. over the other half (see figure part a). The modulus of elasticity $E = 30 \times 10^6$ psi.

- How much will the bar elongate under a tensile load $P = 5000$ lb?
- If the same volume of material is made into a bar of constant diameter d and length 8.0 ft, what will be the elongation under the same load P ?
- If the uniform axial centroidal load $q = 1250$ lb/ft is applied to the left over segment 1 (see figure part b), find the ratio of the total elongation of the bar to that in parts (a) and (b).



Solution 2.3-7

NUMERICAL DATA $E = 30(10^6)$ psi $P = 5000$ lb $L = 4$ ft $d_1 = 0.75$ in. $d_2 = 0.5$ in.

$$(a) \delta_a = \frac{PL}{E} \left(\frac{1}{\frac{\pi}{4}d_1^2} + \frac{1}{\frac{\pi}{4}d_2^2} \right) = 0.0589 \text{ in.} \quad \boxed{\delta_a = 0.0589 \text{ in.}}$$

$$(b) V_a = \left(\frac{\pi}{4}d_1^2 + \frac{\pi}{4}d_2^2 \right)L = 30.631 \text{ in.}^3 \quad d = \sqrt{\frac{V_a}{\frac{\pi}{4}(2L)}} = 0.637 \text{ in.} \quad A = \frac{\pi}{4}d^2 = 0.31907 \text{ in.}^2$$

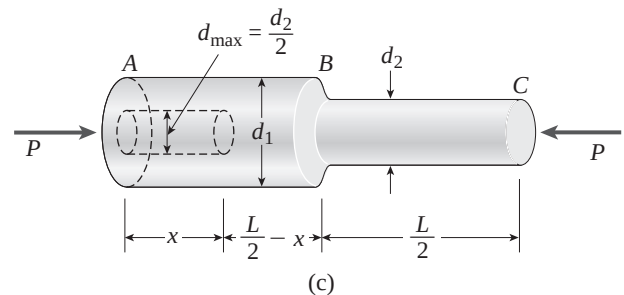
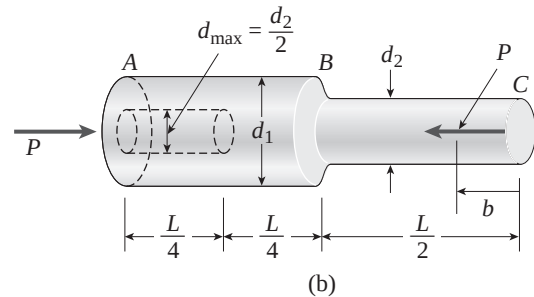
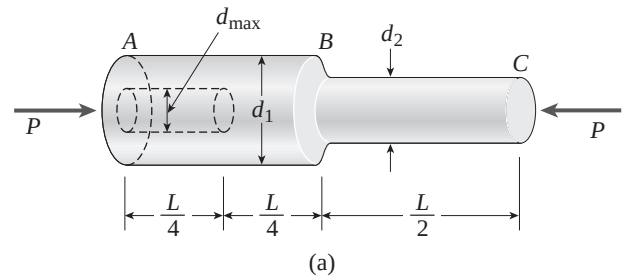
$$\delta_b = \frac{P(2L)}{EA} = 0.0501 \text{ in.} \quad \boxed{\delta_b = 0.0501 \text{ in.}}$$

$$(c) q = 1250 \text{ lb/ft} \quad L = 4 \text{ ft}$$

$$\delta_c = \frac{qL^2}{2E\left(\frac{\pi}{4}d_1^2\right)} + \frac{PL}{EA} = 0.0341 \text{ in.} \quad \boxed{\frac{\delta_c}{\delta_a} = 0.58} \quad \boxed{\frac{\delta_c}{\delta_b} = 0.681}$$

Problem 2.3-8 A bar ABC of length L consists of two parts of equal lengths but different diameters. Segment AB has diameter $d_1 = 100$ mm, and segment BC has diameter $d_2 = 60$ mm. Both segments have length $L/2 = 0.6$ m. A longitudinal hole of diameter d is drilled through segment AB for one-half of its length (distance $L/4 = 0.3$ m). The bar is made of plastic having modulus of elasticity $E = 4.0$ GPa. Compressive loads $P = 110$ kN act at the ends of the bar.

- If the shortening of the bar is limited to 8.0 mm, what is the maximum allowable diameter $d_{\max}$ of the hole? (See figure part a.)
- Now, if $d_{\max}$ is instead set at $d_2/2$, at what distance b from end C should load P be applied to limit the bar shortening to 8.0 mm? (See figure part b.)
- Finally, if loads P are applied at the ends and $d_{\max} = d_2/2$, what is the permissible length x of the hole if shortening is to be limited to 8.0 mm? (See figure part c.)



Solution 2.3-8

NUMERICAL DATA

$$d_1 = 100 \text{ mm} \quad d_2 = 60 \text{ mm}$$

$$L = 1200 \text{ mm} \quad E = 4.0 \text{ GPa} \quad P = 110 \text{ kN}$$

$$\delta_a = 8.0 \text{ mm}$$

(a) FIND $d_{\max}$ IF SHORTENING IS LIMITED TO δ_a

$$A_1 = \frac{\pi}{4} d_1^2 \quad A_2 = \frac{\pi}{4} d_2^2$$

$$\delta = \frac{P}{E} \left[\frac{\frac{L}{4}}{\frac{\pi}{4} (d_1^2 - d_{\max}^2)} + \frac{\frac{L}{4}}{A_1} + \frac{\frac{L}{2}}{A_2} \right]$$

Set δ to δ_a , and solve for $d_{\max}$:

$$d_{\max} = d_1 \sqrt{\frac{E\delta_a \pi d_1^2 d_2^2 - 2PLd_2^2 - 2PLd_1^2}{E\delta_a \pi d_1^2 d_2^2 - PLd_2^2 - 2PLd_1^2}}$$

$$d_{\max} = 23.9 \text{ mm} \quad \leftarrow$$

(b) NOW, IF $d_{\max}$ IS INSTEAD SET AT $d_2/2$, AT WHAT DISTANCE b FROM END C SHOULD LOAD P BE APPLIED TO LIMIT THE BAR SHORTENING TO $\delta_a = 8.0 \text{ mm}$?

$$A_0 = \frac{\pi}{4} \left[d_1^2 - \left(\frac{d_2}{2} \right)^2 \right]$$

$$A_1 = \frac{\pi}{4} d_1^2 \quad A_2 = \frac{\pi}{4} d_2^2$$

$$\delta = \frac{P}{E} \left[\frac{L}{4A_0} + \frac{L}{4A_1} + \frac{\left(\frac{L}{2} - b \right)}{A_2} \right]$$

No axial force in segment at end of length b ; set $\delta = \delta_a$ and solve for b :

$$b = \left[\frac{L}{2} - A_2 \left[\frac{E\delta_a}{P} - \left(\frac{L}{4A_0} + \frac{L}{4A_1} \right) \right] \right]$$

$$b = 4.16 \text{ mm} \quad \leftarrow$$

(c) FINALLY IF LOADS P ARE APPLIED AT THE ENDS AND $d_{\max} = d_2/2$, WHAT IS THE PERMISSIBLE LENGTH x OF THE HOLE IF SHORTENING IS TO BE LIMITED TO $\delta_a = 8.0 \text{ mm}$?

$$\delta = \frac{P}{E} \left[\frac{x}{A_0} + \frac{\left(\frac{L}{2} - x \right)}{A_1} + \frac{\left(\frac{L}{2} \right)}{A_2} \right]$$

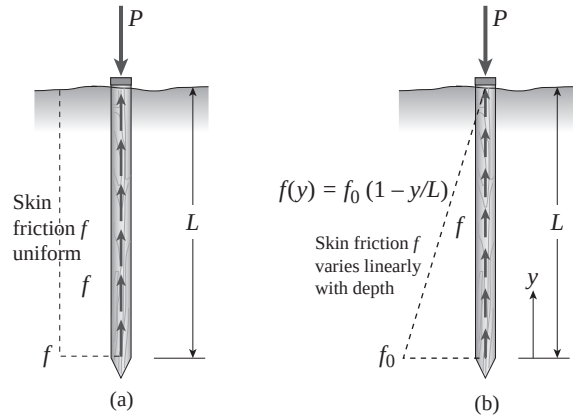
Set $\delta = \delta_a$ and solve for x :

$$x = \frac{\left[A_0 A_1 \left(\frac{E\delta_a}{P} - \frac{L}{2A_2} \right) \right] - \frac{1}{2} A_0 L}{A_1 - A_0}$$

$$x = 183.3 \text{ mm} \quad \leftarrow$$

Problem 2.3-9 A wood pile, driven into the earth, supports a load P entirely by friction along its sides (see figure part a). The friction force f per unit length of pile is assumed to be uniformly distributed over the surface of the pile. The pile has length L , cross-sectional area A , and modulus of elasticity E .

- Derive a formula for the shortening δ of the pile in terms of P , L , E , and A .
- Draw a diagram showing how the compressive stress σ_c varies throughout the length of the pile.
- Repeat parts (a) and (b) if skin friction f varies linearly with depth (see figure part b).



Solution 2.3-9

AFD LINEAR

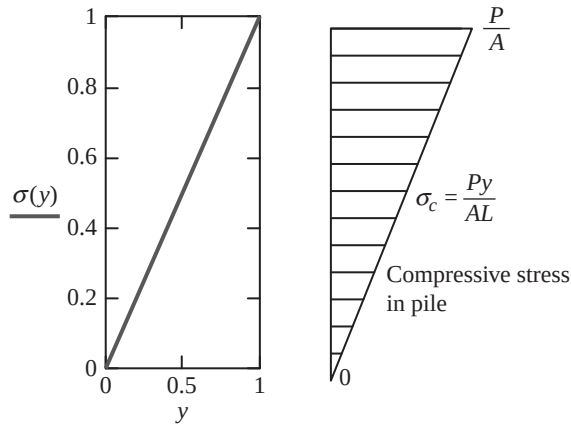
$$(a) \quad N(y) = fy \quad \delta = \int_0^L \frac{(fy)}{EA} dy = \frac{L^2 f}{2AE} \quad \boxed{\delta = \frac{PL}{2EA}}$$

$$(b) \quad \sigma(y) = \frac{N(y)}{A} \quad \sigma(y) = \frac{fy}{A} \quad \sigma(L) = \frac{fL}{A} = \frac{P}{A}$$

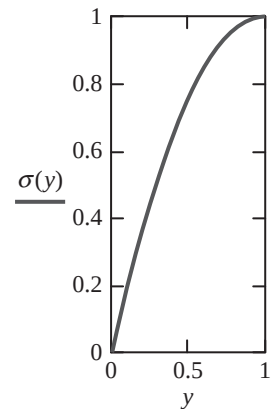
$$\sigma(0) = 0$$

So linear variation, zero at bottom, P/A at top (i.e., at ground surface)

$$N(L) = f \quad \boxed{\sigma(y) = \frac{P}{A} \left(\frac{y}{L} \right)}$$



$f(y)$ is constant
and AFD is linear



$f(y)$ is linear and
AFD quadratic

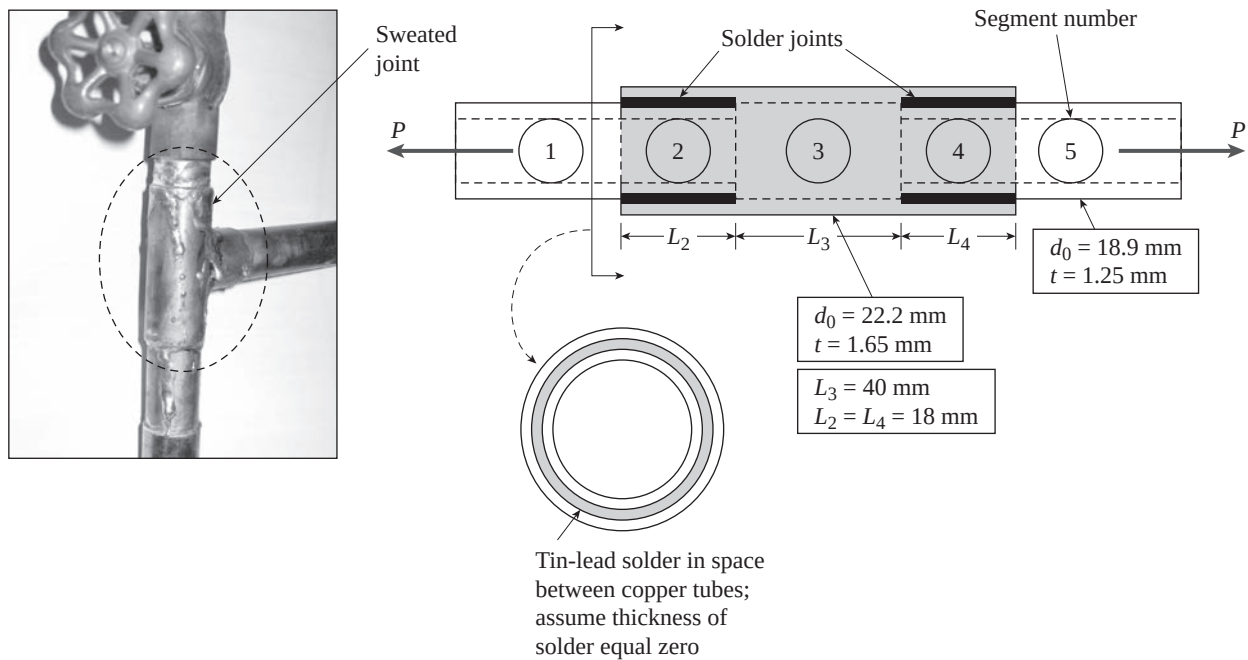
(c) $N(y) = f(y)y$

$$N(y) = \int_0^y f_0 \left(1 - \frac{\zeta}{L}\right) d\zeta = \frac{f_0 y (y - 2)}{2} \quad N(L) = \frac{f_0}{2} \quad N(0) = 0$$

$$\delta = \frac{\left(\frac{f_0 L}{2}\right)}{\frac{3}{2}EA} \quad P = \frac{1}{2} f_0 L \quad \boxed{\delta = \frac{PL}{EA} \left(\frac{2}{3}\right)} \quad \boxed{\sigma(y) = \frac{P}{A} \left[\frac{y}{L} \left(2 - \frac{y}{L}\right)\right]} \quad \sigma(0) = 0 \quad \sigma(L) = \frac{f_0}{2} = P/A$$

Problem 2.3-10 Consider the copper tubes joined below using a “sweated” joint. Use the properties and dimensions given.

- Find the total elongation of segment 2-3-4 (δ_{2-4}) for an applied tensile force of $P = 5$ kN. Use $E_c = 120$ GPa.
- If the yield strength in shear of the tin-lead solder is $\tau_y = 30$ MPa and the tensile yield strength of the copper is $\sigma_y = 200$ MPa, what is the maximum load $P_{\max}$ that can be applied to the joint if the desired factor of safety in shear is $FS_\tau = 2$ and in tension is $FS_\sigma = 1.7$?
- Find the value of L_2 at which tube and solder capacities are equal.



Solution 2.3-10

NUMERICAL DATA

$$\begin{aligned}
 P &= 5 \text{ kN} & E_c &= 120 \text{ GPa} \\
 L_2 &= 18 \text{ mm} & L_4 &= L_2 \\
 L_3 &= 40 \text{ mm} \\
 d_{o3} &= 22.2 \text{ mm} & t_3 &= 1.65 \text{ mm} \\
 d_{o5} &= 18.9 \text{ mm} & t_5 &= 1.25 \text{ mm} \\
 \tau_Y &= 30 \text{ MPa} & \sigma_Y &= 200 \text{ MPa} \\
 \text{FS}_\tau &= 2 & \text{FS}_\sigma &= 1.7
 \end{aligned}$$

$$\tau_a = \frac{\tau_Y}{\text{FS}_\tau} \quad \tau_a = 15 \text{ MPa}$$

$$\sigma_a = \frac{\sigma_Y}{\text{FS}_\sigma} \quad \sigma_a = 117.6 \text{ MPa}$$

(a) ELONGATION OF SEGMENT 2-3-4

$$A_2 = \frac{\pi}{4} [d_{o3}^2 - (d_{o5} - 2t_3)^2]$$

$$A_3 = \frac{\pi}{4} [d_{o3}^2 - (d_{o3} - 2t_3)^2]$$

$$A_2 = 175.835 \text{ mm}^2 \quad A_3 = 106.524 \text{ mm}^2$$

$$\delta_{24} = \frac{P}{E_c} \left(\frac{L_2 + L_4}{A_2} + \frac{L_3}{A_3} \right)$$

$$\delta_{24} = 0.024 \text{ mm} \quad \leftarrow$$

(b) MAXIMUM LOAD $P_{\max}$ THAT CAN BE APPLIED TO THE JOINT

First check normal stress:

$$A_1 = \frac{\pi}{4} [d_{o5}^2 - (d_{o5} - 2t_5)^2]$$

$$A_1 = 69.311 \text{ mm}^2 < \text{smallest cross-sectional area controls normal stress}$$

$$P_{\max\sigma} = \sigma_a A_1 \quad P_{\max\sigma} = 8.15 \text{ kN} \quad \leftarrow \text{smaller than } P_{\max\tau} \text{ based on shear below so normal stress controls}$$

Next check shear stress in solder joint:

$$A_{\text{sh}} = \pi d_{o5} L_2 \quad A_{\text{sh}} = 1.069 \times 10^3 \text{ mm}^2$$

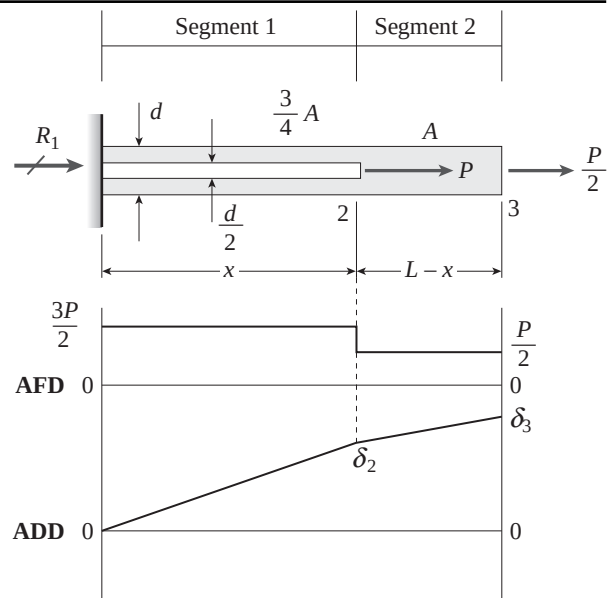
$$P_{\max\tau} = \tau_a A_{\text{sh}} \quad P_{\max\tau} = 16.03 \text{ kN}$$

(c) FIND THE VALUE OF L_2 AT WHICH TUBE AND SOLDER CAPACITIES ARE EQUALSet $P_{\max}$ based on shear strength equal to $P_{\max}$ based on tensile strength and solve for L_2 :

$$L_2 = \frac{\sigma_a A_1}{\tau_a (\pi d_{o5})} \quad L_2 = 9.16 \text{ mm} \quad \leftarrow$$

Problem 2.3-11 The nonprismatic cantilever circular bar shown has an internal cylindrical hole of diameter $d/2$ from 0 to x , so the net area of the cross section for Segment 1 is $(3/4)A$. Load P is applied at x , and load $P/2$ is applied at $x = L$. Assume that E is constant.

- Find reaction force R_1 .
- Find internal axial forces N_i in segments 1 and 2.
- Find x required to obtain axial displacement at joint 3 of $\delta_3 = PL/EA$.
- In (c), what is the displacement at joint 2, δ_2 ?
- If P acts at $x = 2L/3$ and $P/2$ at joint 3 is replaced by βP , find β so that $\delta_3 = PL/EA$.
- Draw the axial force (AFD: $N(x)$, $0 \leq x \leq L$) and axial displacement (ADD: $\delta(x)$, $0 \leq x \leq L$) diagrams using results from (b) through (d) above.



Solution 2.3-11

(a) STATICS $\sum F_H = 0 \quad R_1 = -P - \frac{P}{2}$

$$R_1 = \frac{-3}{2}P \quad \leftarrow$$

- (b) DRAW FBD'S CUTTING THROUGH SEGMENT 1 AND AGAIN THROUGH SEGMENT 2

$$N_1 = \frac{3P}{2} < \text{tension} \quad N_2 = \frac{P}{2} < \text{tension}$$

- (c) FIND x REQUIRED TO OBTAIN AXIAL DISPLACEMENT AT JOINT 3 OF $\delta_3 = PL/EA$

Add axial deformations of segments 1 and 2, then set to δ_3 ; solve for x :

$$\frac{N_1 x}{E \frac{3}{4}A} + \frac{N_2 (L - x)}{EA} = \frac{PL}{EA}$$

$$\frac{\frac{3P}{2}x}{E \frac{3}{4}A} + \frac{\frac{P}{2}(L - x)}{EA} = \frac{PL}{EA}$$

$$\frac{3}{2}x = \frac{L}{2} \quad x = \frac{L}{3} \quad \leftarrow$$

- (d) WHAT IS THE DISPLACEMENT AT JOINT 2, δ_2 ?

$$\delta_2 = \frac{N_1 x}{E \frac{3}{4}A} \quad \delta_2 = \frac{\left(\frac{3P}{2}\right)\frac{L}{3}}{E \frac{3}{4}A}$$

$$\delta_2 = \frac{2}{3} \frac{PL}{EA}$$

- (e) IF $x = 2L/3$ AND $P/2$ AT JOINT 3 IS REPLACED BY βP , FIND β SO THAT $\delta_3 = PL/EA$

$$N_1 = (1 + \beta)P \quad N_2 = \beta P \quad x = \frac{2L}{3}$$

substitute in axial deformation expression above and solve for β

$$\frac{[(1 + \beta)P] \frac{2L}{3}}{E \frac{3}{4}A} + \frac{\beta P \left(L - \frac{2L}{3}\right)}{EA} = \frac{PL}{EA}$$

$$\frac{1}{9} PL \frac{8 + 11\beta}{EA} = \frac{PL}{EA}$$

$$(8 + 11\beta) = 9$$

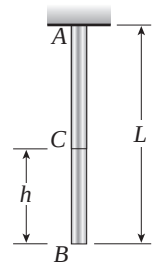
$$\beta = \frac{1}{11} \quad \leftarrow$$

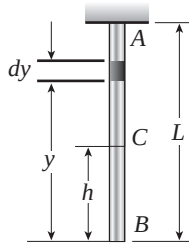
$$\beta = 0.091$$

- (f) Draw AFD, ADD—see plots for $x = \frac{L}{3}$

Problem 2.3-12 A prismatic bar AB of length L , cross-sectional area A , modulus of elasticity E , and weight W hangs vertically under its own weight (see figure).

- Derive a formula for the downward displacement δ_C of point C , located at distance h from the lower end of the bar.
- What is the elongation δ_B of the entire bar?
- What is the ratio β of the elongation of the upper half of the bar to the elongation of the lower half of the bar?
- If bar AB is a riser pipe hanging from a drill rig at sea, what is the total elongation of the pipe? Let $L = 1500$ m, $A = 0.0157$ m², $E = 210$ GPa. See Appendix I for weight densities of steel and sea water. (See Problems 1.4-2 and 1.7-11 for additional figures).



Solution 2.3-12 Prismatic bar hanging vertically W = Weight of bar(a) DOWNWARD DISPLACEMENT δ_C
Consider an element at distance y from the lower end.

$$N(y) = \frac{Wy}{L} \quad d\delta = \frac{N(y)dy}{EA} = \frac{Wydy}{EAL}$$

$$\delta_C = \int_h^L d\delta = \int_h^L \frac{Wydy}{EAL} = \frac{W}{2EAL} (L^2 - h^2)$$

$$\delta_C = \frac{W}{2EAL} (L^2 - h^2) \quad \leftarrow$$

(b) ELONGATION OF BAR ($h = 0$)

$$\delta_B = \frac{WL}{2EA} \quad \leftarrow$$

(c) RATIO OF ELONGATIONS

Elongation of upper half of bar ($h = \frac{L}{2}$):

$$\delta_{\text{upper}} = \frac{3WL}{8EA}$$

Elongation of lower half of bar:

$$\delta_{\text{lower}} = \delta_B - \delta_{\text{upper}} = \frac{WL}{2EA} - \frac{3WL}{8EA} = \frac{WL}{8EA}$$

$$\beta = \frac{\delta_{\text{upper}}}{\delta_{\text{lower}}} = \frac{3/8}{1/8} = 3 \quad \leftarrow$$

(d) NUMERICAL DATA

$$\gamma_s = 77 \text{ kN/m}^3$$

$$\gamma_w = 10 \text{ kN/m}^3$$

$$L = 1500 \text{ m}$$

$$A = 0.0157 \text{ m}^2$$

$$E = 210 \text{ GPa}$$

In sea water:

$$W = (\gamma_s - \gamma_w)AL = 1577.85 \text{ kN}$$

$$\delta = \frac{WL}{2EA} = 359 \text{ mm}$$

$$\frac{\delta}{L} = 2.393 \times 10^{-4}$$

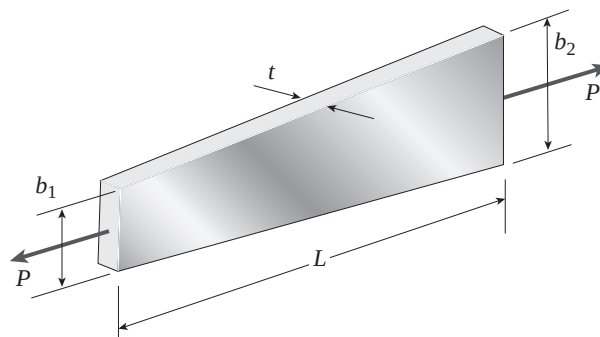
In air:

$$W = (\gamma_s)AL = 1813.35 \text{ kN}$$

$$\delta = \frac{WL}{2EA} = 412 \text{ mm}$$

$$\frac{\delta}{L} = 2.75 \times 10^{-4}$$

Problem 2.3-13 A flat bar of rectangular cross section, length L , and constant thickness t is subjected to tension by forces P (see figure). The width of the bar varies linearly from b_1 at the smaller end to b_2 at the larger end. Assume that the angle of taper is small.

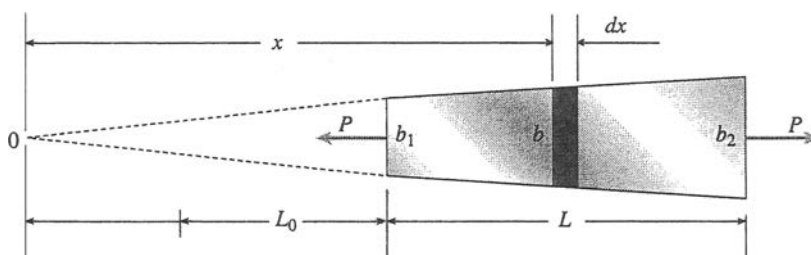


- (a) Derive the following formula for the elongation of the bar:

$$\delta = \frac{PL}{Et(b_2 - b_1)} \ln \frac{b_2}{b_1}$$

- (b) Calculate the elongation, assuming $L = 5$ ft, $t = 1.0$ in., $P = 25$ k, $b_1 = 4.0$ in., $b_2 = 6.0$ in., and $E = 30 \times 10^6$ psi.

Solution 2.3-13 Tapered bar (rectangular cross section)



t = thickness (constant)

$$b = b_1 \left(\frac{x}{L_0} \right) \quad b_2 = b_1 \left(\frac{L_0 + L}{L_0} \right) \quad (\text{Eq. 1})$$

$$A(x) = bt = b_1 t \left(\frac{x}{L_0} \right)$$

(a) ELONGATION OF THE BAR

$$d\delta = \frac{Pdx}{EA(x)} = \frac{PL_0 dx}{Eb_1 tx}$$

$$\begin{aligned} \delta &= \int_{L_0}^{L_0+L} d\delta = \frac{PL_0}{Eb_1 t} \int_{L_0}^{L_0+L} \frac{dx}{x} \\ &= \frac{PL_0}{Eb_1 t} \ln x \Big|_{L_0}^{L_0+L} = \frac{PL_0}{Eb_1 t} \ln \frac{L_0 + L}{L_0} \quad (\text{Eq. 2}) \end{aligned}$$

$$\text{From Eq. (1): } \frac{L_0 + L}{L_0} = \frac{b_2}{b_1} \quad (\text{Eq. 3})$$

$$\text{Solve Eq. (3) for } L_0: L_0 = L \left(\frac{b_1}{b_2 - b_1} \right) \quad (\text{Eq. 4})$$

Substitute Eqs. (3) and (4) into Eq. (2):

$$\delta = \frac{PL}{Et(b_2 - b_1)} \ln \frac{b_2}{b_1} \quad (\text{Eq. 5})$$

(b) SUBSTITUTE NUMERICAL VALUES:

$$L = 5 \text{ ft} = 60 \text{ in.} \quad t = 1.0 \text{ in.}$$

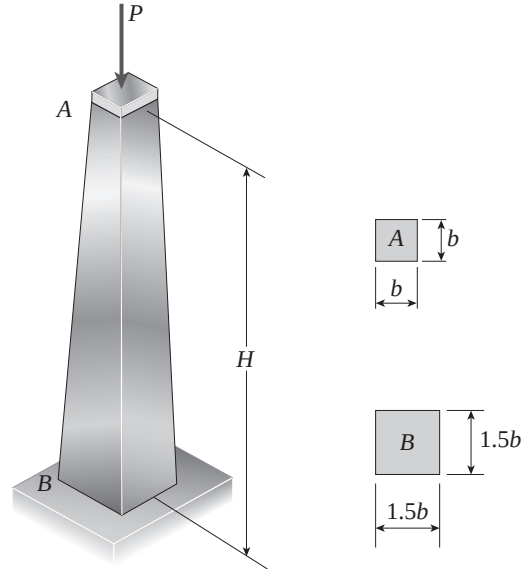
$$P = 25 \text{ k} \quad b_1 = 4.0 \text{ in.}$$

$$b_2 = 6.0 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$$

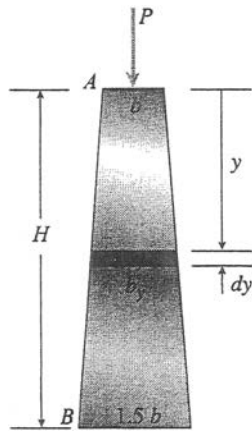
$$\text{From Eq. (5): } \delta = 0.010 \text{ in.} \quad \leftarrow$$

Problem 2.3-14 A post AB supporting equipment in a laboratory is tapered uniformly throughout its height H (see figure). The cross sections of the post are square, with dimensions $b \times b$ at the top and $1.5b \times 1.5b$ at the base.

Derive a formula for the shortening δ of the post due to the compressive load P acting at the top. (Assume that the angle of taper is small and disregard the weight of the post itself.)



Solution 2.3-14 Tapered post



Square cross sections:

$$b = \text{width at } A$$

$$1.5b = \text{width at } B$$

$$b_y = \text{width at distance } y$$

$$= b + (1.5b - b) \frac{y}{H}$$

$$= \frac{b}{H}(H + 0.5y)$$

$A_y =$ cross-sectional area at distance y

$$= (b_y)^2 = \frac{b^2}{H^2}(H + 0.5y)^2$$

SHORTENING OF ELEMENT dy

$$d\delta = \frac{Pdy}{EA_y} = \frac{Pdy}{E\left(\frac{b^2}{H^2}\right)(H + 0.5y)^2}$$

SHORTENING OF ENTIRE POST

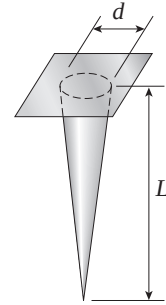
$$\delta = \int d\delta = \frac{PH^2}{Eb^2} \int_0^H \frac{dy}{(H + 0.5y)^2}$$

From Appendix D: $\int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)}$

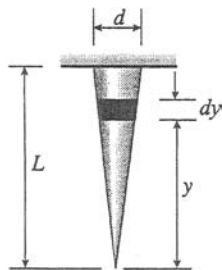
$$\begin{aligned} \delta &= \frac{PH^2}{Eb^2} \left[-\frac{1}{(0.5)(H + 0.5y)} \right]_0^H \\ &= \frac{PH^2}{Eb^2} \left[-\frac{1}{(0.5)(1.5H)} + \frac{1}{0.5H} \right] \\ &= \frac{2PH}{3Eb^2} \quad \leftarrow \end{aligned}$$

Problem 2.3-15 A long, slender bar in the shape of a right circular cone with length L and base diameter d hangs vertically under the action of its own weight (see figure). The weight of the cone is W and the modulus of elasticity of the material is E .

Derive a formula for the increase δ in the length of the bar due to its own weight. (Assume that the angle of taper of the cone is small.)



Solution 2.3-15 Conical bar hanging vertically



TERMINOLOGY

N_y = axial force acting on element dy

A_y = cross-sectional area at element dy

A_B = cross-sectional area at base of cone

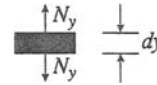
$$= \frac{\pi d^2}{4} \quad V = \text{volume of cone}$$

$$= \frac{1}{3} A_B L \quad V_y = \text{volume of cone below element } dy$$

$$= \frac{1}{3} A_y y \quad W_y = \text{weight of cone below element } dy$$

$$= \frac{V_y}{V} (W) = \frac{A_y y W}{A_B L} \quad N_y = W_y$$

ELEMENT OF BAR



W = weight of cone

ELONGATION OF ELEMENT dy

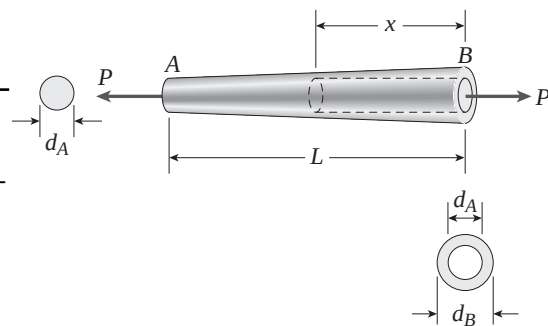
$$d\delta = \frac{N_y dy}{E A_y} = \frac{W_y dy}{E A_B L} = \frac{4W}{\pi d^2 E L} y dy$$

ELONGATION OF CONICAL BAR

$$\delta = \int d\delta = \frac{4W}{\pi d^2 E L} \int_0^L y dy = \frac{2WL}{\pi d^2 E} \quad \leftarrow$$

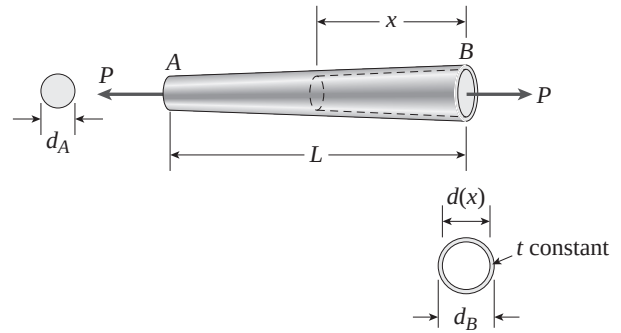
Problem 2.3-16 A uniformly tapered plastic tube AB of circular cross section and length L is shown in the figure. The average diameters at the ends are d_A and $d_B = 2d_A$. Assume E is constant. Find the elongation δ of the tube when it is subjected to loads P acting at the ends. Use the following numerical data: $d_A = 35$ mm, $L = 300$ mm, $E = 2.1$ GPa, $P = 25$ kN. Consider two cases as follows:

- (a) A hole of constant diameter d_A is drilled from B toward A to form a hollow section of length $x = L/2$ (see figure part a).



(a)

- (b) A hole of *variable* diameter $d(x)$ is drilled from B toward A to form a hollow section of length $x = L/2$ and constant thickness t (see figure part b). (Assume that $t = d_A/20$.)



(b)

Solution 2.3-16

- (a) ELONGATION δ FOR CASE OF CONSTANT DIAMETER HOLE

$$d(\zeta) = d_A \left(1 + \frac{\zeta}{L} \right) \quad A(\zeta) = \frac{\pi}{4} d(\zeta)^2 \quad < \text{solid portion of length } L - x$$

$$A(\zeta) = \frac{\pi}{4} (d(\zeta)^2 - d_A^2) \quad < \text{hollow portion of length } x$$

$$\delta = \frac{P}{E} \left(\int \frac{1}{A(\zeta)} d\zeta \right) \quad \delta = \frac{P}{E} \left[\int_0^{L-x} \frac{4}{\pi d(\zeta)^2} d\zeta + \int_{L-x}^L \frac{4}{\pi (d(\zeta)^2 - d_A^2)} d\zeta \right]$$

$$\delta = \frac{P}{E} \left[\int_0^{L-x} \frac{1}{\left[\frac{\pi}{4} \left[d_A \left(1 + \frac{\zeta}{L} \right) \right]^2 \right]} d\zeta + \int_{L-x}^L \frac{1}{\left[\frac{\pi}{4} \left[d_A \left(1 + \frac{\zeta}{L} \right) \right]^2 - d_A^2 \right]} d\zeta \right]$$

$$\delta = \frac{P}{E} \left[4 \frac{L^2}{(-2+x)\pi d_A^2} + \left[4 \frac{L}{\pi d_A^2} + \int_{L-x}^L \frac{1}{\left[\frac{\pi}{4} \left[d_A \left(1 + \frac{\zeta}{L} \right) \right]^2 - d_A^2 \right]} d\zeta \right] \right]$$

$$\delta = \frac{P}{E} \left[4 \frac{L^2}{(-2+x)\pi d_A^2} + \left(4 \frac{L}{\pi d_A^2} - 2L \frac{\ln(3)}{\pi d_A^2} + 2L \frac{-\ln(L-x) + \ln(3L-x)}{\pi d_A^2} \right) \right]$$

$$\text{if } x = L/2 \quad \delta = \frac{P}{E} \left(\frac{4}{3} \frac{L}{\pi d_A^2} - 2L \frac{\ln(3)}{\pi d_A^2} + 2L \frac{-\ln\left(\frac{1}{2}L\right) + \ln\left(\frac{5}{2}L\right)}{\pi d_A^2} \right)$$

Substitute numerical data:

$$\delta = 2.18 \text{ mm} \quad \leftarrow$$

(b) ELONGATION δ FOR CASE OF VARIABLE DIAMETER HOLE BUT CONSTANT WALL THICKNESS $t = d_A/20$ OVER SEGMENT x

$$d(\zeta) = d_A \left(1 + \frac{\zeta}{L} \right) \quad A(\zeta) = \frac{\pi}{4} d(\zeta)^2 < \text{solid portion of length } L - x$$

$$A(\zeta) = \frac{\pi}{4} \left[d(\zeta)^2 - \left(d(\zeta) - 2 \frac{d_A}{20} \right)^2 \right] < \text{hollow portion of length } x$$

$$\delta = \frac{P}{E} \left(\int \frac{1}{A(\zeta)} d\zeta \right) \quad \delta = \frac{P}{E} \left[\int_0^{L-x} \frac{4}{\pi d(\zeta)^2} d\zeta + \int_{L-x}^L \frac{4}{\pi \left[d(\zeta)^2 - \left(d(\zeta) - 2 \frac{d_A}{20} \right)^2 \right]} d\zeta \right]$$

$$\delta = \frac{P}{E} \left[\int_0^{L-x} \frac{4}{\pi \left[d_A \left(1 + \frac{\zeta}{L} \right) \right]^2} d\zeta + \int_{L-x}^L \frac{4}{\pi \left[\left[d_A \left(1 + \frac{\zeta}{L} \right) \right]^2 - \left[d_A \left(1 + \frac{\zeta}{L} \right) - 2 \frac{d_A}{20} \right]^2 \right]} d\zeta \right]$$

$$\delta = \frac{P}{E} \left[4 \frac{L^2}{(-2L + x) \pi d_A^2} + 4 \frac{L}{\pi d_A^2} + 20L \frac{\ln(3) + \ln(13) + 2\ln(d_A) + \ln(L)}{\pi d_A^2} - 20L \frac{2\ln(d_A) + \ln(39L - 20x)}{\pi d_A^2} \right]$$

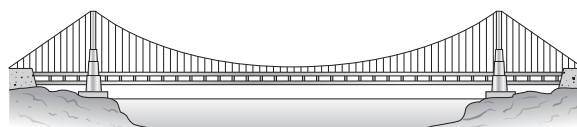
if $x = L/2$

$$\delta = \frac{P}{E} \left(\frac{4}{3} \frac{L}{\pi d_A^2} + 20L \frac{\ln(3) + \ln(13) + 2\ln(d_A) + \ln(L)}{\pi d_A^2} - 20L \frac{2\ln(d_A) + \ln(29L)}{\pi d_A^2} \right)$$

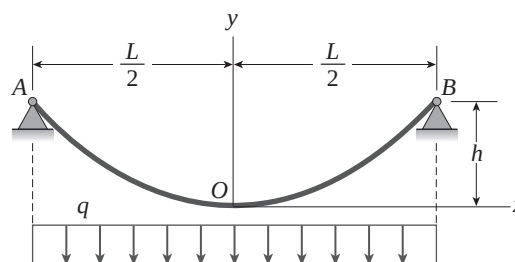
Substitute numerical data:

$$\delta = 6.74 \text{ mm} \quad \leftarrow$$

Problem 2.3-17 The main cables of a suspension bridge [see part (a) of the figure] follow a curve that is nearly parabolic because the primary load on the cables is the weight of the bridge deck, which is uniform in intensity along the horizontal. Therefore, let us represent the central region AOB of one of the main cables [see part (b) of the figure] as a parabolic cable supported at points A and B and carrying a uniform load of intensity q along the horizontal. The span of the cable is L , the sag is h , the axial rigidity is EA , and the origin of coordinates is at midspan.



(a)



(b)

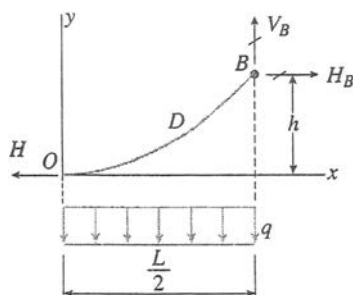
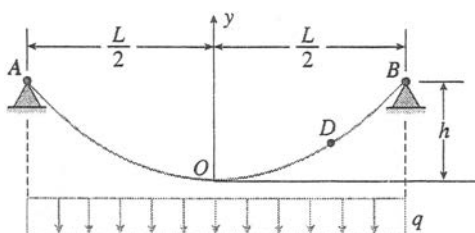
- (a) Derive the following formula for the elongation of cable AOB shown in part (b) of the figure:

$$\delta = \frac{qL^3}{8hEA} \left(1 + \frac{16h^2}{3L^2} \right)$$

- (b) Calculate the elongation δ of the central span of one of the main cables of the Golden Gate Bridge, for which the dimensions and properties are $L = 4200$ ft, $h = 470$ ft, $q = 12,700$ lb/ft, and $E = 28,800,000$ psi. The cable consists of 27,572 parallel wires of diameter 0.196 in.

Hint: Determine the tensile force T at any point in the cable from a free-body diagram of part of the cable; then determine the elongation of an element of the cable of length ds ; finally, integrate along the curve of the cable to obtain an equation for the elongation δ .

Solution 2.3-17 Cable of a suspension bridge



Equation of parabolic curve:

$$y = \frac{4hx^2}{L^2}$$

$$\frac{dy}{dx} = \frac{8hx}{L^2}$$

FREE-BODY DIAGRAM OF HALF OF CABLE

$$\Sigma M_B = 0 \quad (\curvearrowright)$$

$$-Hh + \frac{qL}{2} \left(\frac{L}{4} \right) = 0$$

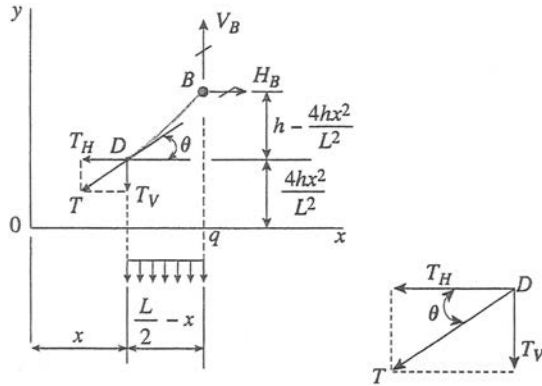
$$H = \frac{qL^2}{8h}$$

$$\Sigma F_{\text{horizontal}} = 0$$

$$H_B = H = \frac{qL^2}{8h} \quad (\text{Eq. 1})$$

$$\Sigma F_{\text{vertical}} = 0$$

$$V_B = \frac{qL}{2} \quad (\text{Eq. 2})$$

FREE-BODY DIAGRAM OF SEGMENT *DB* OF CABLE

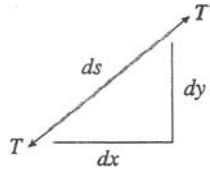
$$\Sigma F_{\text{horiz}} = 0 \quad T_H = H_B = \frac{qL^2}{8h} \quad (\text{Eq. 3})$$

$$\Sigma F_{\text{vert}} = 0 \quad V_B - T_V - q\left(\frac{L}{2} - x\right) = 0$$

$$T_V = V_B - q\left(\frac{L}{2} - x\right) = \frac{qL}{2} - \frac{qL}{2} + qx = qx \quad (\text{Eq. 4})$$

TENSILE FORCE *T* IN CABLE

$$T = \sqrt{T_H^2 + T_V^2} = \sqrt{\left(\frac{qL^2}{8h}\right)^2 + (qx)^2} = \frac{qL^2}{8h} \sqrt{1 + \frac{64h^2x^2}{L^4}} \quad (\text{Eq. 5})$$

ELONGATION $d\delta$ OF AN ELEMENT OF LENGTH ds 

$$\begin{aligned} d\delta &= \frac{Tds}{EA} \\ ds &= \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \\ &= dx\sqrt{1 + \left(\frac{8hx}{L^2}\right)^2} \\ &= dx\sqrt{1 + \frac{64h^2x^2}{L^4}} \end{aligned} \quad (\text{Eq. 6})$$

(a) ELONGATION δ OF CABLE *AOB*

$$\delta = \int d\delta = \int \frac{T ds}{EA}$$

Substitute for *T* from Eq. (5) and for *ds* from Eq. (6):

$$\delta = \frac{1}{EA} \int \frac{qL^2}{8h} \left(1 + \frac{64h^2x^2}{L^4}\right) dx$$

For both halves of cable:

$$\begin{aligned} \delta &= \frac{2}{EA} \int_0^{L/2} \frac{qL^2}{8h} \left(1 + \frac{64h^2x^2}{L^4}\right) dx \\ \delta &= \frac{qL^3}{8hEA} \left(1 + \frac{16h^2}{3L^4}\right) \leftarrow \end{aligned} \quad (\text{Eq. 7})$$

(b) GOLDEN GATE BRIDGE CABLE

$$L = 4200 \text{ ft} \quad h = 470 \text{ ft}$$

$$q = 12,700 \text{ lb/ft} \quad E = 28,800,000 \text{ psi}$$

27,572 wires of diameter $d = 0.196 \text{ in.}$

$$A = (27,572) \left(\frac{\pi}{4}\right) (0.196 \text{ in.})^2 = 831.90 \text{ in.}^2$$

Substitute into Eq. (7):

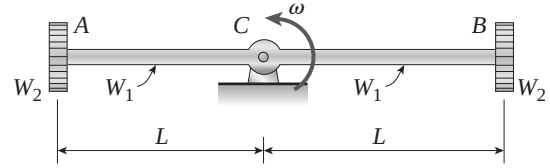
$$\delta = 133.7 \text{ in} = 11.14 \text{ ft} \leftarrow$$

Problem 2.3-18 A bar ABC revolves in a horizontal plane about a vertical axis at the midpoint C (see figure). The bar, which has length $2L$ and cross-sectional area A , revolves at constant angular speed ω . Each half of the bar (AC and BC) has weight W_1 and supports a weight W_2 at its end.

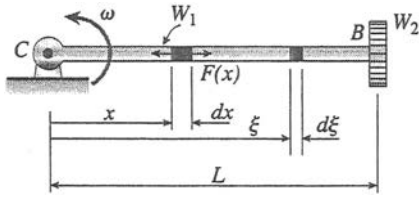
Derive the following formula for the elongation of one-half of the bar (that is, the elongation of either AC or BC):

$$\delta = \frac{L^2 \omega^2}{3gEA} (W_1 + 3W_2)$$

in which E is the modulus of elasticity of the material of the bar and g is the acceleration of gravity.



Solution 2.3-18 Rotating bar



ω = angular speed

A = cross-sectional area

E = modulus of elasticity

g = acceleration of gravity

$F(x)$ = axial force in bar at distance x from point C

Consider an element of length dx at distance x from point C .

To find the force $F(x)$ acting on this element, we must find the inertia force of the part of the bar from distance x to distance L , plus the inertia force of the weight W_2 .

Since the inertia force varies with distance from point C , we now must consider an element of length $d\xi$ at distance ξ , where ξ varies from x to L .

$$\text{Mass of element } d = \frac{d}{L} \left(\frac{W_1}{g} \right)$$

$$\text{Acceleration of element} = \xi \omega^2$$

Centrifugal force produced by element

$$= (\text{mass})(\text{acceleration}) = \frac{W_1 \omega^2}{gL} d$$

Centrifugal force produced by weight W_2

$$= \left(\frac{W_2}{g} \right) (L \omega^2)$$

AXIAL FORCE $F(x)$

$$\begin{aligned} F(x) &= \int_x^L \frac{W_1 \omega^2}{gL} d + \frac{W_2 L \omega^2}{g} \\ &= \frac{W_1 \omega^2}{2gL} (L^2 - x^2) + \frac{W_2 L \omega^2}{g} \end{aligned}$$

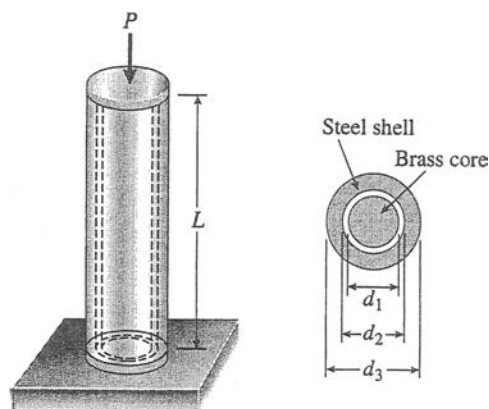
ELONGATION OF BAR BC

$$\begin{aligned} \delta &= \int_0^L \frac{F(x) dx}{EA} \\ &= \int_0^L \frac{W_1 \omega^2}{2gL} (L^2 - x^2) dx + \int_0^L \frac{W_2 L \omega^2 dx}{gEA} \\ &= \frac{W_1 L \omega^2}{2gLEA} \left[\int_0^L L^2 dx - \int_0^L x^2 dx \right] + \frac{W_2 L \omega^2}{gEA} \int_0^L dx \\ &= \frac{W_1 L^2 \omega^2}{3gEA} + \frac{W_2 L^2 \omega^2}{gEA} \\ &= \frac{L^2 \omega^2}{3gEA} + (W_1 + 3W_2) \quad \leftarrow \end{aligned}$$

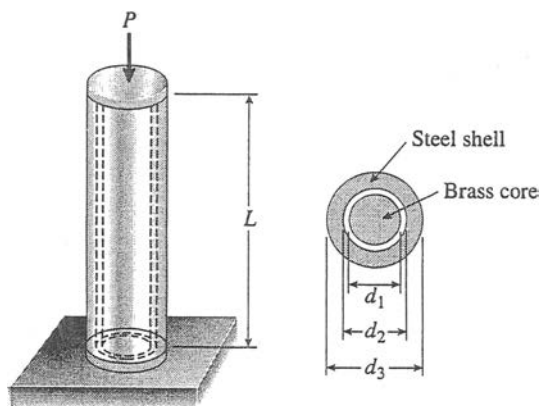
Statically Indeterminate Structures

Problem 2.4-1 The assembly shown in the figure consists of a brass core (diameter $d_1 = 0.25$ in.) surrounded by a steel shell (inner diameter $d_2 = 0.28$ in., outer diameter $d_3 = 0.35$ in.). A load P compresses the core and shell, which have length $L = 4.0$ in. The moduli of elasticity of the brass and steel are $E_b = 15 \times 10^6$ psi and $E_s = 30 \times 10^6$ psi, respectively.

- What load P will compress the assembly by 0.003 in.?
- If the allowable stress in the steel is 22 ksi and the allowable stress in the brass is 16 ksi, what is the allowable compressive load P_{allow} ? (Suggestion: Use the equations derived in Example 2-6.)



Solution 2.4-1 Cylindrical assembly in compression



$$d_1 = 0.25 \text{ in.} \quad E_b = 15 \times 10^6 \text{ psi}$$

$$d_2 = 0.28 \text{ in.} \quad E_s = 30 \times 10^6 \text{ psi}$$

$$d_3 = 0.35 \text{ in.} \quad A_s = \frac{\pi}{4}(d_3^2 - d_2^2) = 0.03464 \text{ in.}^2$$

$$L = 4.0 \text{ in.} \quad A_b = \frac{\pi}{4}d_1^2 = 0.04909 \text{ in.}^2$$

- (a) DECREASE IN LENGTH ($\delta = 0.003$ in.)

Use Eq. (2-18) of Example 2-6.

$$\delta = \frac{PL}{E_s A_s + E_b A_b} \quad \text{or}$$

$$P = (E_s A_s + E_b A_b) \left(\frac{\delta}{L} \right)$$

Substitute numerical values:

$$\begin{aligned} E_s A_s + E_b A_b &= (30 \times 10^6 \text{ psi})(0.03464 \text{ in.}^2) \\ &\quad + (15 \times 10^6 \text{ psi})(0.04909 \text{ in.}^2) \\ &= 1.776 \times 10^6 \text{ lb} \end{aligned}$$

$$\begin{aligned} P &= (1.776 \times 10^6 \text{ lb}) \left(\frac{0.003 \text{ in.}}{4.0 \text{ in.}} \right) \\ &= 1330 \text{ lb} \quad \leftarrow \end{aligned}$$

- (b) ALLOWABLE LOAD

$$\sigma_s = 22 \text{ ksi} \quad \sigma_b = 16 \text{ ksi}$$

Use Eqs. (2-17a and b) of Example 2-6.

For steel:

$$\begin{aligned} \sigma_s &= \frac{PE_s}{E_s A_s + E_b A_b} \quad P_s = (E_s A_s + E_b A_b) \frac{\sigma_s}{E_s} \\ P_s &= (1.776 \times 10^6 \text{ lb}) \left(\frac{22 \text{ ksi}}{30 \times 10^6 \text{ psi}} \right) = 1300 \text{ lb} \end{aligned}$$

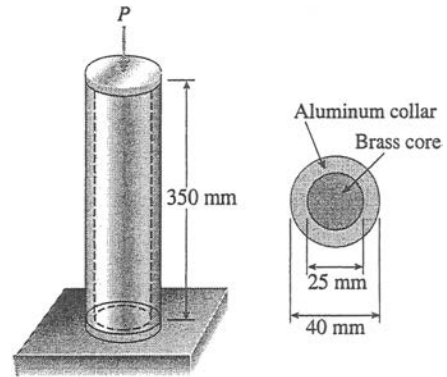
For brass:

$$\begin{aligned} \sigma_b &= \frac{PE_b}{E_s A_s + E_b A_b} \quad P_b = (E_s A_s + E_b A_b) \frac{\sigma_b}{E_b} \\ P_b &= (1.776 \times 10^6 \text{ lb}) \left(\frac{16 \text{ ksi}}{15 \times 10^6 \text{ psi}} \right) = 1890 \text{ lb} \end{aligned}$$

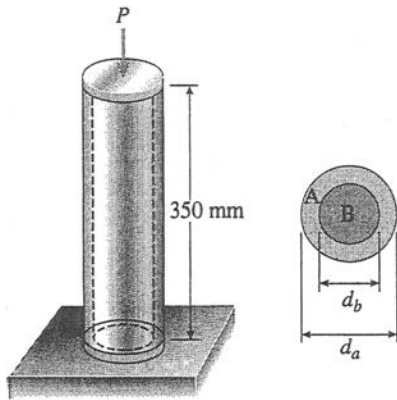
Steel governs. $P_{\text{allow}} = 1300 \text{ lb} \quad \leftarrow$

Problem 2.4-2 A cylindrical assembly consisting of a brass core and an aluminum collar is compressed by a load P (see figure). The length of the aluminum collar and brass core is 350 mm, the diameter of the core is 25 mm, and the outside diameter of the collar is 40 mm. Also, the moduli of elasticity of the aluminum and brass are 72 GPa and 100 GPa, respectively.

- If the length of the assembly decreases by 0.1% when the load P is applied, what is the magnitude of the load?
- What is the maximum permissible load $P_{\max}$ if the allowable stresses in the aluminum and brass are 80 MPa and 120 MPa, respectively? (*Suggestion:* Use the equations derived in Example 2-6.)



Solution 2.4-2 Cylindrical assembly in compression



A = aluminum

B = brass

$L = 350 \text{ mm}$

$d_a = 40 \text{ mm}$

$d_b = 25 \text{ mm}$

$$A_a = \frac{\pi}{4} (d_a^2 - d_b^2)$$

$$= 765.8 \text{ mm}^2$$

$$E_a = 72 \text{ GPa} \quad E_b = 100 \text{ GPa} \quad A_b = \frac{\pi}{4} d_b^2$$

$$= 490.9 \text{ mm}^2$$

(a) DECREASE IN LENGTH

$$(\delta = 0.1\% \text{ of } L = 0.350 \text{ mm})$$

Use Eq. (2-18) of Example 2-6.

$$\delta = \frac{PL}{E_a A_a + E_b A_b} \quad \text{or}$$

$$P = (E_a A_a + E_b A_b) \left(\frac{\delta}{L} \right)$$

Substitute numerical values:

$$\begin{aligned} E_a A_a + E_b A_b &= (72 \text{ GPa})(765.8 \text{ mm}^2) \\ &\quad + (100 \text{ GPa})(490.9 \text{ mm}^2) \\ &= 55.135 \text{ MN} + 49.090 \text{ MN} \\ &= 104.23 \text{ MN} \end{aligned}$$

$$\begin{aligned} P &= (104.23 \text{ MN}) \left(\frac{0.350 \text{ mm}}{350 \text{ mm}} \right) \\ &= 104.2 \text{ kN} \quad \leftarrow \end{aligned}$$

(b) ALLOWABLE LOAD

$$\sigma_a = 80 \text{ MPa} \quad \sigma_b = 120 \text{ MPa}$$

Use Eqs. (2-17a and b) of Example 2-6.

For aluminum:

$$\sigma_a = \frac{P E_a}{E_a A_a + E_b A_b} \quad P_a = (E_a A_a + E_b A_b) \left(\frac{\sigma_a}{E_a} \right)$$

$$P_a = (104.23 \text{ MN}) \left(\frac{80 \text{ MPa}}{72 \text{ GPa}} \right) = 115.8 \text{ kN}$$

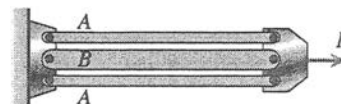
For brass:

$$\sigma_b = \frac{P E_b}{E_a A_a + E_b A_b} \quad P_b = (E_a A_a + E_b A_b) \left(\frac{\sigma_b}{E_b} \right)$$

$$P_b = (104.23 \text{ MN}) \left(\frac{120 \text{ MPa}}{100 \text{ GPa}} \right) = 125.1 \text{ kN}$$

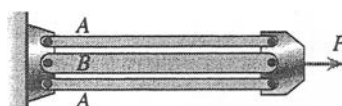
Aluminum governs. $P_{\max} = 116 \text{ kN} \quad \leftarrow$

Problem 2.4-3 Three prismatic bars, two of material A and one of material B, transmit a tensile load P (see figure). The two outer bars (material A) are identical. The cross-sectional area of the middle bar (material B) is 50% larger than the cross-sectional area of one of the outer bars. Also, the modulus of elasticity of material A is twice that of material B.

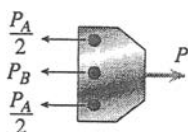


- What fraction of the load P is transmitted by the middle bar?
- What is the ratio of the stress in the middle bar to the stress in the outer bars?
- What is the ratio of the strain in the middle bar to the strain in the outer bars?

Solution 2.4-3 Prismatic bars in tension



FREE-BODY DIAGRAM OF END PLATE



EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{horiz}} = 0 \quad P_A + P_B - P = 0 \quad (1)$$

EQUATION OF COMPATIBILITY

$$\delta_A = \delta_B \quad (2)$$

FORCE-DISPLACEMENT RELATIONS

A_A = total area of both outer bars

$$\delta_A = \frac{P_A L}{E_A A_A} \quad \delta_B = \frac{P_B L}{E_B A_B} \quad (3)$$

Substitute into Eq. (2):

$$\frac{P_A L}{E_A A_A} = \frac{P_B L}{E_B A_B} \quad (4)$$

SOLUTION OF THE EQUATIONS

Solve simultaneously Eqs. (1) and (4):

$$P_A = \frac{E_A A_A P}{E_A A_A + E_B A_B} \quad P_B = \frac{E_B A_B P}{E_A A_A + E_B A_B} \quad (5)$$

Substitute into Eq. (3):

$$\delta = \delta_A = \delta_B = \frac{PL}{E_A A_A + E_B A_B} \quad (6)$$

STRESSES:

$$\sigma_A = \frac{P_A}{A_A} = \frac{E_A P}{E_A A_A + E_B A_B}$$

$$\sigma_B = \frac{P_B}{A_B} = \frac{E_B P}{E_A A_A + E_B A_B} \quad (7)$$

(a) LOAD IN MIDDLE BAR

$$\frac{P_B}{P} = \frac{E_B A_B}{E_A A_A + E_B A_B} = \frac{1}{\frac{E_A A_A}{E_B A_B} + 1}$$

$$\text{Given: } \frac{E_A}{E_B} = 2 \quad \frac{A_A}{A_B} = \frac{1 + 1}{1.5} = \frac{4}{3}$$

$$\therefore \frac{P_B}{P} = \frac{1}{\left(\frac{E_A}{E_B}\right)\left(\frac{A_A}{A_B}\right) + 1} = \frac{1}{\frac{8}{3} + 1} = \frac{3}{11} \quad \leftarrow$$

(b) RATIO OF STRESSES

$$\frac{\sigma_B}{\sigma_A} = \frac{E_B}{E_A} = \frac{1}{2} \quad \leftarrow$$

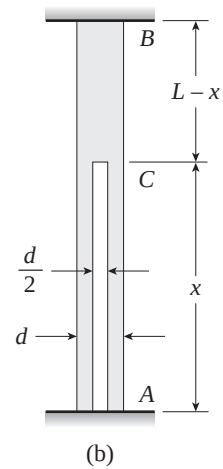
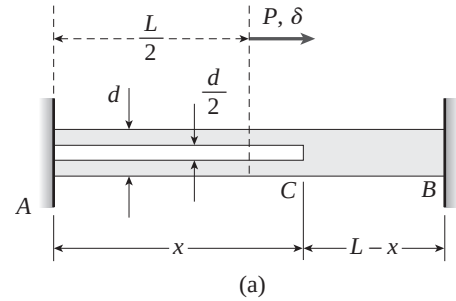
(c) RATIO OF STRAINS

All bars have the same strain

$$\text{Ratio} = 1 \quad \leftarrow$$

Problem 2.4-4 A circular bar ACB of diameter d having a cylindrical hole of length x and diameter $d/2$ from A to C is held between rigid supports at A and B . A load P acts at $L/2$ from ends A and B . Assume E is constant.

- Obtain formulas for the reactions R_A and R_B at supports A and B , respectively, due to the load P (see figure part a).
- Obtain a formula for the displacement δ at the point of load application (see figure part a).
- For what value of x is $R_B = (6/5) R_A$? (See figure part a.)
- Repeat part (a) if the bar is now rotated to a vertical position, load P is removed, and the bar is hanging under its own weight (assume mass density $= \rho$). (See figure part b.) Assume that $x = L/2$.



Solution 2.4-4

(a) REACTIONS AT A AND B DUE TO LOAD P AT $L/2$

$$A_{AC} = \frac{\pi}{4} \left[d^2 - \left(\frac{d}{2} \right)^2 \right] \quad A_{AC} = \frac{3}{16} \pi d^2$$

$$A_{CB} = \frac{\pi}{4} d^2$$

Select R_B as the redundant; use superposition and a compatibility equation at B :

$$\text{if } x \leq L/2 \quad \delta_{B1a} = \frac{Px}{EA_{AC}} + \frac{P\left(\frac{L}{2} - x\right)}{EA_{CB}} \quad \delta_{B1a} = \frac{P}{E} \left(\frac{x}{\frac{3}{16} \pi d^2} + \frac{\frac{L}{2} - x}{\frac{\pi}{4} d^2} \right)$$

$$\delta_{B1a} = \frac{2}{3} P \frac{2x + 3L}{E\pi d^2}$$

$$\text{if } x \geq L/2 \quad \delta_{B1b} = \frac{P \frac{L}{2}}{EA_{AC}} \quad \delta_{B1b} = \frac{P \frac{L}{2}}{E \left(\frac{3}{16} \pi d^2 \right)} \quad \delta_{B1b} = \frac{8}{3} \frac{PL}{E\pi d^2}$$

The following expression for δ_{B2} is good for all x :

$$\delta_{B2} = \frac{R_B}{E} \left(\frac{x}{A_{AC}} + \frac{L-x}{A_{CB}} \right) \quad \delta_{B2} = \frac{R_B}{E} \left(\frac{x}{\frac{3}{16}\pi d^2} + \frac{L-x}{\frac{\pi}{4}d^2} \right)$$

$$\delta_{B2} = \frac{R_B}{E} \left(\frac{16}{3} \frac{x}{\pi d^2} + 4 \frac{L-x}{\pi d^2} \right)$$

Solve for R_B and R_A assuming that $x \leq L/2$:

$$\text{Compatibility:} \quad \delta_{B1a} + \delta_{B2} = 0 \quad R_{Ba} = \frac{-\left(\frac{2}{3}P \frac{2x+3L}{\pi d^2}\right)}{\left(\frac{16}{3} \frac{x}{\pi d^2} + 4 \frac{L-x}{\pi d^2}\right)} \quad R_{Ba} = \frac{-1}{2} P \frac{2x+3L}{x+3L} \quad \leftarrow$$

^ check—if $x = 0$, $R_B = -P/2$

$$\text{Statics:} \quad R_{Aa} = -P - R_{Ba} \quad R_{Aa} = -P - \frac{-1}{2} P \frac{2x+3L}{x+3L} \quad R_{Aa} = \frac{-3}{2} P \frac{L}{x+3L} \quad \leftarrow$$

^ check—if $x = 0$, $R_{Aa} = -P/2$

Solve for R_B and R_A assuming that $x \geq L/2$:

$$\text{Compatibility:} \quad \delta_{B1b} + \delta_{B2} = 0 \quad R_{Bb} = \frac{\frac{-8}{3} \frac{PL}{\pi d^2}}{\left(\frac{16}{3} \frac{x}{\pi d^2} + 4 \frac{L-x}{\pi d^2}\right)} \quad R_{Bb} = \frac{-2PL}{x+3L} \quad \leftarrow$$

^ check—if $x = L$, $R_B = -P/2$

$$\text{Statics:} \quad R_{Ab} = -P - R_{Bb} \quad R_{Ab} = -P - \left(\frac{-2PL}{x+3L}\right) \quad R_{Ab} = -P \frac{x+L}{x+3L} \quad \leftarrow$$

(b) FIND δ AT POINT OF LOAD APPLICATION; AXIAL FORCE FOR SEGMENT 0 TO $L/2 = -R_A$ AND $\delta =$ ELONGATION OF THIS SEGMENT

Assume that $x \leq L/2$:

$$\delta_a = \frac{-R_{Aa}}{E} \left(\frac{x}{A_{AC}} + \frac{L-x}{A_{CB}} \right) \quad \delta_a = \frac{-\left(\frac{-3}{2} P \frac{L}{x+3L}\right)}{E} \left(\frac{x}{\frac{3}{16}\pi d^2} + \frac{L-x}{\frac{\pi}{4}d^2} \right)$$

$$\delta_a = PL \frac{2x+3L}{(x+3L)E\pi d^2}$$

$$\text{For } x = L/2, \quad \delta_a = \frac{8}{7} L \frac{P}{E\pi d^2} \quad \leftarrow$$

ASSUME THAT $x \geq L/2$:

$$\delta_b = \frac{(-R_{Ab})\frac{L}{2}}{EA_{AC}} \quad \delta_b = \frac{\left(P \frac{x+L}{x+3L}\right)\frac{L}{2}}{E\left(\frac{3}{16}\pi d^2\right)} \quad \delta_b = \frac{8}{3}P\left(\frac{x+L}{x+3L}\right)\frac{L}{E\pi d^2} \quad \leftarrow$$

$$\text{for } x = L/2 \quad \delta_b = \frac{8}{7}P\frac{L}{E\pi d^2} < \text{same as } \delta_a \text{ above (OK)}$$

(c) FOR WHAT VALUE OF x IS $R_B = (6/5)R_A$?

Guess that $x < L/2$ here and use R_{Ba} expression above to find x :

$$\frac{-1}{2}P\frac{2x+3L}{x+3L} - \frac{6}{5}\left(\frac{-3}{2}P\frac{L}{x+3L}\right) = 0 \quad \frac{-1}{10}P\frac{10x-3L}{x+3L} = 0 \quad x = \frac{3L}{10} \quad \leftarrow$$

Now try $R_{Bb} = (6/5)R_{Ab}$, assuming that $x > L/2$

$$\frac{-2PL}{x+3L} - \frac{6}{5}\left(-P\frac{x+L}{x+3L}\right) = 0 \quad \frac{2}{5}P\frac{-2L+3x}{x+3L} = 0 \quad x = \frac{2}{3}L \quad \leftarrow$$

So, there are two solutions for x .

(d) FIND REACTIONS IF THE BAR IS NOW ROTATED TO A VERTICAL POSITION, LOAD P IS REMOVED, AND THE BAR IS HANGING UNDER ITS OWN WEIGHT (ASSUME MASS DENSITY = ρ). ASSUME THAT $x = L/2$.

$$A_{AC} = \frac{3}{16}\pi d^2 \quad A_{CB} = \frac{\pi}{4}d^2$$

Select R_B as the redundant; use superposition and a compatibility equation at B

from (a) above. compatibility: $\delta_{B1} + \delta_{B2} = 0$

$$\delta_{B2} = \frac{R_B}{E}\left(\frac{x}{A_{AC}} + \frac{L-x}{A_{CB}}\right) \quad \text{For } x = L/2, \delta_{B2} = \frac{R_B}{E}\left(\frac{14}{3}\frac{L}{\pi d^2}\right)$$

$$\delta_{B1} = \int_0^{\frac{L}{2}} \frac{N_{AC}}{EA_{AC}} d\zeta + \int_{\frac{L}{2}}^L \frac{N_{CB}}{EA_{CB}} d\zeta$$

Where axial forces in bar due to self weight are $W_{AC} = \rho g A_{AC} \frac{L}{2}$ $W_{CB} = \rho g A_{CB} \frac{L}{2}$
(assume ζ is measured upward from A):

$$N_{AC} = -\left[\rho g A_{CB} \frac{L}{2} + \rho g A_{AC} \left(\frac{L}{2} - \zeta\right)\right] \quad A_{AC} = \frac{3}{16}\pi d^2 \quad A_{CB} = \frac{\pi}{4}d^2$$

$$N_{CB} = -[\rho g A_{CB}(L - \zeta)]$$

$$N_{AC} = \frac{-1}{8}\rho g \pi d^2 L - \frac{3}{16}\rho g \pi d^2 \left(\frac{1}{2}L - \zeta\right) \quad N_{CB} = -\left[\frac{1}{4}\rho g \pi d^2 (L - \zeta)\right]$$

$$\delta_{B1} = \int_0^{\frac{L}{2}} \frac{-\frac{1}{8} \rho g \pi d^2 L - \frac{3}{16} \rho g \pi d^2 \left(\frac{1}{2} L - \zeta \right)}{E \left(\frac{3}{16} \pi d^2 \right)} d\zeta + \int_{\frac{L}{2}}^L \frac{-\left[\frac{1}{4} \rho g \pi d^2 (L - \zeta) \right]}{E \left(\frac{\pi}{4} d^2 \right)} d\zeta$$

$$\delta_{B1} = \left(-\frac{11}{24} \rho g \frac{L^2}{E} + \frac{-1}{8} \rho g \frac{L^2}{E} \right) \quad \delta_{B1} = \frac{-7}{12} \rho g \frac{L^2}{E} \quad \frac{7}{12} = 0.583$$

Compatibility: $\delta_{B1} + \delta_{B2} = 0$

$$R_B = \frac{-\left(\frac{-7}{12} \rho g \frac{L^2}{E} \right)}{\left(\frac{14}{3} \frac{L}{E \pi d^2} \right)} \quad R_B = \frac{1}{8} \rho g \pi d^2 L \quad \leftarrow$$

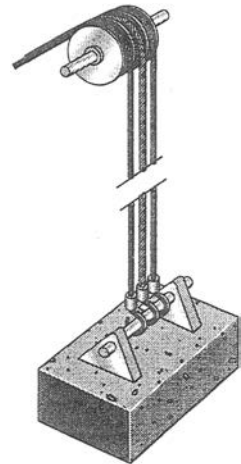
Statics: $R_A = (W_{AC} + W_{CB}) - R_B$

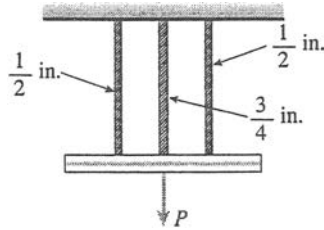
$$R_A = \left[\left[\rho g \left(\frac{3}{16} \pi d^2 \right) \frac{L}{2} + \rho g \left(\frac{\pi}{4} d^2 \right) \frac{L}{2} \right] - \frac{1}{8} \rho g \pi d^2 L \right]$$

$$R_A = \frac{3}{32} \rho g \pi d^2 L \quad \leftarrow$$

Problem 2.4-5 Three steel cables jointly support a load of 12 k (see figure). The diameter of the middle cable is $\frac{3}{4}$ in. and the diameter of each outer cable is $\frac{1}{2}$ in. The tensions in the cables are adjusted so that each cable carries one-third of the load (i.e., 4 k). Later, the load is increased by 9 k to a total load of 21 k.

- What percent of the total load is now carried by the middle cable?
 - What are the stresses σ_M and σ_O in the middle and outer cables, respectively?
- (NOTE: See Table 2-1 in Section 2.2 for properties of cables.)



Solution 2.4-5 Three cables in tension

AREAS OF CABLES (from Table 2-1)

Middle cable: $A_M = 0.268 \text{ in.}^2$

Outer cables: $A_O = 0.119 \text{ in.}^2$

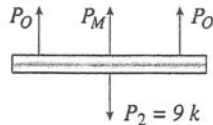
(for each cable)

FIRST LOADING

$$P_1 = 12 \text{ k} \left(\text{Each cable carries } \frac{P_1}{3} \text{ or } 4 \text{ k.} \right)$$

SECOND LOADING

$P_2 = 9 \text{ k}$ (additional load)



EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{vert}} = 0 \quad 2P_O + P_M - P_2 = 0 \quad (1)$$

EQUATION OF COMPATIBILITY

$$\delta_M = \delta_O \quad (2)$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_M = \frac{P_M L}{EA_M} \quad \delta_O = \frac{P_O L}{EA_O} \quad (3, 4)$$

SUBSTITUTE INTO COMPATIBILITY EQUATION:

$$\frac{P_M L}{EA_M} = \frac{P_O L}{EA_O} \quad \frac{P_M}{A_M} = \frac{P_O}{A_O} \quad (5)$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (5):

$$P_M = P_2 \left(\frac{A_M}{A_M + 2A_O} \right) = (9 \text{ k}) \left(\frac{0.268 \text{ in.}^2}{0.506 \text{ in.}^2} \right) = 4.767 \text{ k}$$

$$P_O = P_2 \left(\frac{A_O}{A_M + 2A_O} \right) = (9 \text{ k}) \left(\frac{0.119 \text{ in.}^2}{0.506 \text{ in.}^2} \right) = 2.117 \text{ k}$$

FORCES IN CABLES

Middle cable: Force = $4 \text{ k} + 4.767 \text{ k} = 8.767 \text{ k}$

Outer cables: Force = $4 \text{ k} + 2.117 \text{ k} = 6.117 \text{ k}$

(for each cable)

(a) PERCENT OF TOTAL LOAD CARRIED BY MIDDLE CABLE

$$\text{Percent} = \frac{8.767 \text{ k}}{21 \text{ k}} (100\%) = 41.7\% \quad \leftarrow$$

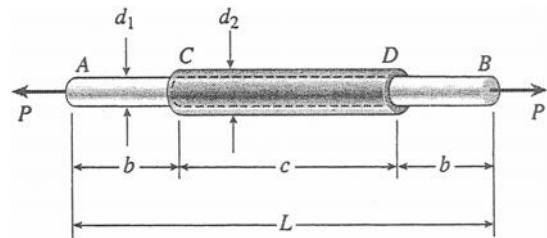
(b) STRESSES IN CABLES ($\sigma = P/A$)

$$\text{Middle cable: } \sigma_M = \frac{8.767 \text{ k}}{0.268 \text{ in.}^2} = 32.7 \text{ ksi} \quad \leftarrow$$

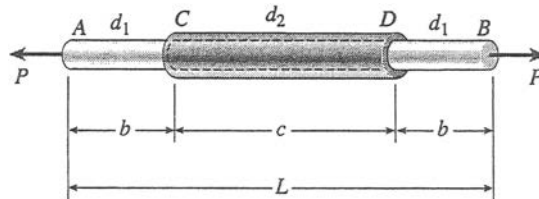
$$\text{Outer cables: } \sigma_O = \frac{6.117 \text{ k}}{0.119 \text{ in.}^2} = 51.4 \text{ ksi} \quad \leftarrow$$

Problem 2.4-6 A plastic rod AB of length $L = 0.5$ m has a diameter $d_1 = 30$ mm (see figure). A plastic sleeve CD of length $c = 0.3$ m and outer diameter $d_2 = 45$ mm is securely bonded to the rod so that no slippage can occur between the rod and the sleeve. The rod is made of an acrylic with modulus of elasticity $E_1 = 3.1$ GPa and the sleeve is made of a polyamide with $E_2 = 2.5$ GPa.

- Calculate the elongation δ of the rod when it is pulled by axial forces $P = 12$ kN.
- If the sleeve is extended for the full length of the rod, what is the elongation?
- If the sleeve is removed, what is the elongation?



Solution 2.4-6 Plastic rod with sleeve



$$P = 12 \text{ kN} \quad d_1 = 30 \text{ mm} \quad b = 100 \text{ mm}$$

$$L = 500 \text{ mm} \quad d_2 = 45 \text{ mm} \quad c = 300 \text{ mm}$$

$$\text{Rod: } E_1 = 3.1 \text{ GPa}$$

$$\text{Sleeve: } E_2 = 2.5 \text{ GPa}$$

$$\text{Rod: } A_1 = \frac{\pi d_1^2}{4} = 706.86 \text{ mm}^2$$

$$\text{Sleeve: } A_2 = \frac{\pi}{4}(d_2^2 - d_1^2) = 883.57 \text{ mm}^2$$

$$E_1 A_1 + E_2 A_2 = 4.400 \text{ MN}$$

(a) ELONGATION OF ROD

$$\text{Part AC: } \delta_{AC} = \frac{Pb}{E_1 A_1} = 0.5476 \text{ mm}$$

$$\begin{aligned} \text{Part CD: } \delta_{CD} &= \frac{Pc}{E_1 A_1 + E_2 A_2} \\ &= 0.81815 \text{ mm} \end{aligned}$$

(From Eq. 2-18 of Example 2-6)

$$\delta = 2\delta_{AC} + \delta_{CD} = 1.91 \text{ mm} \quad \leftarrow$$

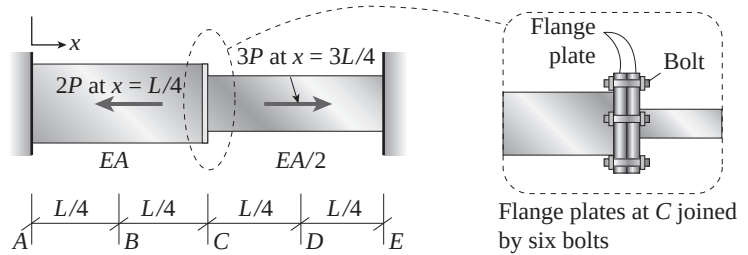
(b) SLEEVE AT FULL LENGTH

$$\begin{aligned} \delta &= \delta_{CD} \left(\frac{L}{c} \right) = (0.81815 \text{ mm}) \left(\frac{500 \text{ mm}}{300 \text{ mm}} \right) \\ &= 1.36 \text{ mm} \quad \leftarrow \end{aligned}$$

(c) SLEEVE REMOVED

$$\delta = \frac{PL}{E_1 A_1} = 2.74 \text{ mm} \quad \leftarrow$$

Problem 2.4-7 A tube structure is acted on by loads at B and D , as shown in the figure. The tubes are joined using two flange plates at C , which are bolted together using six 0.5 in. diameter bolts.



- Derive formulas for the reactions R_A and R_E at the ends of the bar.
- Determine the axial displacement δ_B , δ_C , and δ_D at points B , C , and D , respectively.
- Draw an axial-displacement diagram (ADD) in which the abscissa is the distance x from support A to any point on the bar and the ordinate is the horizontal displacement δ at that point.
- Find the maximum value of the load variable P if allowable normal stress in the bolts is 14 ksi.

Solution 2.4-7

NUMERICAL DATA

$$n = 6 \quad d_b = 0.5 \text{ in.} \quad \sigma_a = 14 \text{ ksi} \quad A_b = \frac{\pi}{4} d_b^2 = 0.196 \text{ in.}^2$$

- (a) FORMULAS FOR REACTIONS F

$$\text{Segment } ABC \text{ flexibility: } f_1 = \frac{2\left(\frac{L}{4}\right)}{EA} = \frac{L}{2EA}$$

$$\text{Segment } CDE \text{ flexibility: } f_2 = \frac{2\left(\frac{L}{4}\right)}{\frac{1}{2}EA} = \frac{L}{EA}$$

Loads at points B and D :

$$P_B = -2P \quad P_D = 3P$$

- (1) Select R_E as the redundant; find axial displacement δ_1 = displacement at E due to loads P_B and P_D :

$$\delta_1 = \frac{(P_B + P_D)\frac{L}{4}}{EA} + \frac{P_D\frac{L}{4}}{EA} + \frac{P_D\frac{L}{4}}{\frac{1}{2}EA} = \frac{5LP}{2EA}$$

- (2) Next apply redundant R_E and find axial displacement δ_2 = displacement at E due to redundant R_E :

$$\delta_2 = R_E(f_1 + f_2) = \frac{3LR_E}{2EA}$$

- (3) Use compatibility equation to find redundant R_E then use statics to find R_A :

$$\delta_1 + \delta_2 = 0 \text{ solve, } R_E = -\frac{5P}{3} \quad R_E = -\frac{5}{3}P$$

$$R_A = -R_E - P_B - P_D = \frac{2P}{3} \quad R_A = \frac{2P}{3} \quad \boxed{R_A = \frac{2P}{3}} \quad \boxed{R_E = \frac{5P}{3}}$$

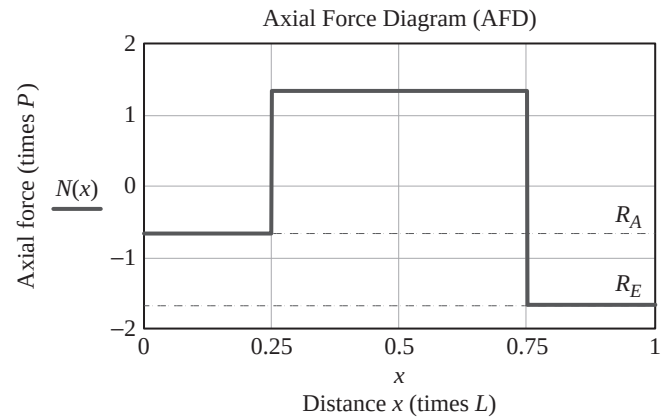
- (b) DETERMINE THE AXIAL DISPLACEMENTS δ_B , δ_C , AND δ_D AT POINTS B , C , AND D , RESPECTIVELY.

$$\delta_B = \frac{\left(\frac{-2P}{3}\right)\left(\frac{L}{4}\right)}{EA} = \frac{LP}{6EA} \quad \text{leftward} \quad \delta_C = \delta_B + \frac{\left(2P - \frac{2P}{3}\right)\left(\frac{L}{4}\right)}{EA} = \frac{LP}{6EA} \quad \text{to the right} \quad \delta_D = \frac{\left(\frac{5P}{3}\right)\left(\frac{L}{4}\right)}{\frac{EA}{2}} = \frac{5LP}{6EA} \quad \text{to the right}$$

- (c) DRAW AN AXIAL-DISPLACEMENT DIAGRAM (ADD) IN WHICH THE ABSCISSA IS THE DISTANCE x FROM SUPPORT A TO ANY POINT ON THE BAR AND THE ORDINATE IS THE HORIZONTAL DISPLACEMENT δ AT THAT POINT.

AFD for use below in Part (d)

AFD is composed of 4 constant segments, so ADD is linear with zero displacements at supports A and E .

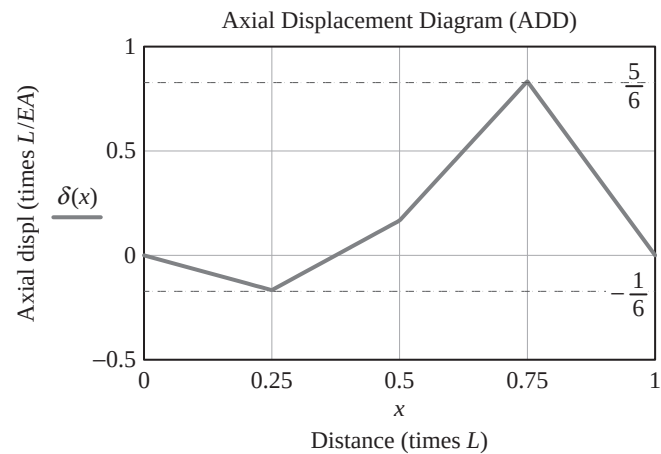


Plot displacements δ_B , δ_C , and δ_D from part (b) above, then connect points using straight lines showing linear variation of axial displacement between points

$$\delta_{\max} = \delta_D \quad \delta_{\max} = \frac{5LP}{6EA} \quad \text{to the right}$$

Boundary conditions at supports:

$$\delta_A = \delta_E = 0$$



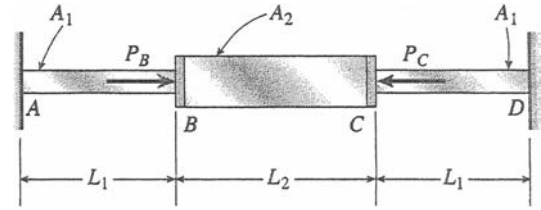
- (d) MAXIMUM PERMISSIBLE VALUE OF LOAD VARIABLE P BASED ON ALLOWABLE NORMAL STRESS IN FLANGE BOLTS
FROM AFD, FORCE AT $L/2$:

$$F_{\max} = \frac{4}{3}P \quad \text{and} \quad F_{\max} = n\sigma_a A_b = 16.493 \text{ k}$$

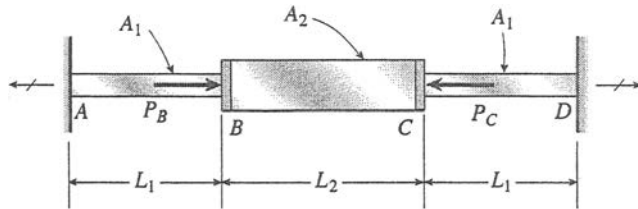
$$P_{\max} = \frac{3}{4}F_{\max} = 12.37 \text{ k} \quad \boxed{P_{\max} = 12.37 \text{ k}}$$

Problem 2.4-8 The fixed-end bar $ABCD$ consists of three prismatic segments, as shown in the figure. The end segments have cross-sectional area $A_1 = 840 \text{ mm}^2$ and length $L_1 = 200 \text{ mm}$. The middle segment has cross-sectional area $A_2 = 1260 \text{ mm}^2$ and length $L_2 = 250 \text{ mm}$. Loads P_B and P_C are equal to 25.5 kN and 17.0 kN , respectively.

- Determine the reactions R_A and R_D at the fixed supports.
- Determine the compressive axial force F_{BC} in the middle segment of the bar.



Solution 2.4-8 Bar with three segments



FREE-BODY DIAGRAM

EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{horiz}} = 0 \rightarrow \leftarrow$$

$$P_B + R_D - P_C - R_A = 0 \text{ or}$$

$$R_A - R_D = P_B - P_C = 8.5 \text{ kN} \quad (\text{Eq. 1})$$

EQUATION OF COMPATIBILITY

$$\delta_{AD} = \text{elongation of entire bar}$$

$$\delta_{AD} = \delta_{AB} + \delta_{BC} + \delta_{CD} = 0 \quad (\text{Eq. 2})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AB} = \frac{R_A L_1}{EA_1} = \frac{R_A}{E} \left(238.05 \frac{1}{\text{m}} \right) \quad (\text{Eq. 3})$$

$$\begin{aligned} \delta_{BC} &= \frac{(R_A - P_B)L_2}{EA_2} \\ &= \frac{R_A}{E} \left(198.413 \frac{1}{\text{m}} \right) - \frac{P_B}{E} \left(198.413 \frac{1}{\text{m}} \right) \end{aligned} \quad (\text{Eq. 4})$$

$$\delta_{CD} = \frac{R_D L_1}{EA_1} = \frac{R_D}{E} \left(238.095 \frac{1}{\text{m}} \right) \quad (\text{Eq. 5})$$

$$\begin{aligned} P_B &= 25.5 \text{ kN} & P_C &= 17.0 \text{ kN} \\ L_1 &= 200 \text{ mm} & L_2 &= 250 \text{ mm} \\ A_1 &= 840 \text{ mm}^2 & A_2 &= 1260 \text{ mm}^2 \\ m &= \text{meter} \end{aligned}$$

SOLUTION OF EQUATIONS

Substitute Eqs. (3), (4), and (5) into Eq. (2):

$$\begin{aligned} \frac{R_A}{E} \left(238.095 \frac{1}{\text{m}} \right) + \frac{R_A}{E} \left(198.413 \frac{1}{\text{m}} \right) \\ - \frac{P_B}{E} \left(198.413 \frac{1}{\text{m}} \right) + \frac{R_D}{E} \left(238.095 \frac{1}{\text{m}} \right) = 0 \end{aligned}$$

Simplify and substitute $P_B = 25.5 \text{ kN}$:

$$\begin{aligned} R_A \left(436.508 \frac{1}{\text{m}} \right) + R_D \left(238.095 \frac{1}{\text{m}} \right) \\ = 5,059.53 \text{ kN/m} \end{aligned} \quad (\text{Eq. 6})$$

- (a) REACTIONS R_A AND R_D

Solve simultaneously Eqs. (1) and (6).

$$\text{From (1): } R_D = R_A - 8.5 \text{ kN}$$

Substitute into (6) and solve for R_A :

$$R_A \left(674.603 \frac{1}{\text{m}} \right) = 7083.34 \text{ kN/m}$$

$$R_A = 10.5 \text{ kN} \quad \leftarrow$$

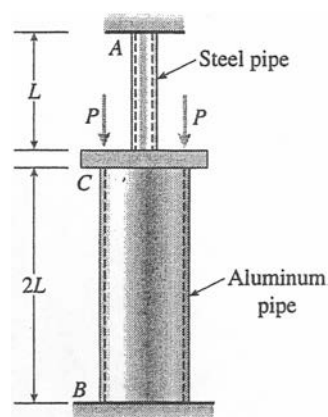
$$R_D = R_A - 8.5 \text{ kN} = 2.0 \text{ kN} \quad \leftarrow$$

- (b) COMPRESSIVE AXIAL FORCE F_{BC}

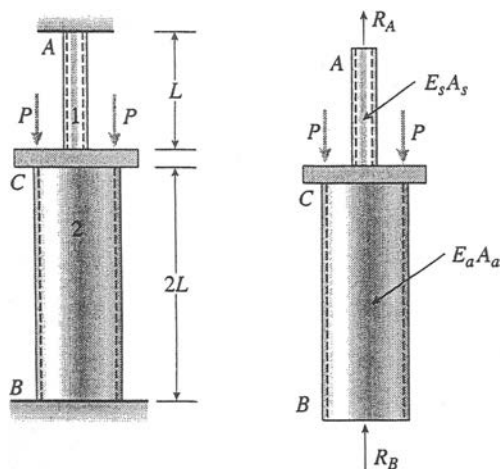
$$F_{BC} = P_B - R_A = P_C - R_D = 15.0 \text{ kN} \quad \leftarrow$$

Problem 2.4-9 The aluminum and steel pipes shown in the figure are fastened to rigid supports at ends A and B and to a rigid plate C at their junction. The aluminum pipe is twice as long as the steel pipe. Two equal and symmetrically placed loads P act on the plate at C.

- Obtain formulas for the axial stresses σ_a and σ_s in the aluminum and steel pipes, respectively.
- Calculate the stresses for the following data: $P = 12$ k, cross-sectional area of aluminum pipe $A_a = 8.92$ in.², cross-sectional area of steel pipe $A_s = 1.03$ in.², modulus of elasticity of aluminum $E_a = 10 \times 10^6$ psi, and modulus of elasticity of steel $E_s = 29 \times 10^6$ psi.



Solution 2.4-9 Pipes with intermediate loads



Pipe 1 is steel.
Pipe 2 is aluminum.

EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{vert}} = 0 \quad R_A + R_B = 2P \quad (\text{Eq. 1})$$

EQUATION OF COMPATIBILITY

$$\delta_{AB} = \delta_{AC} + \delta_{CB} = 0 \quad (\text{Eq. 2})$$

(A positive value of δ means elongation.)

FORCE-DISPLACEMENT RELATIONS

$$\delta_{AC} = \frac{R_A L}{E_s A_s} \quad \delta_{BC} = -\frac{R_B (2L)}{E_a A_a} \quad (\text{Eqs. 3, 4})$$

SOLUTION OF EQUATIONS

Substitute Eqs. (3) and (4) into Eq. (2):

$$\frac{R_A L}{E_s A_s} - \frac{R_B (2L)}{E_a A_a} = 0 \quad (\text{Eq. 5})$$

Solve simultaneously Eqs. (1) and (5):

$$R_A = \frac{4E_s A_s P}{E_a A_a + 2E_s A_s} \quad R_B = \frac{2E_a A_a P}{E_a A_a + 2E_s A_s} \quad (\text{Eqs. 6, 7})$$

(a) AXIAL STRESSES

$$\text{Aluminum: } \sigma_a = \frac{R_B}{A_a} = \frac{2E_a P}{E_a A_a + 2E_s A_s} \quad \leftarrow$$

(compression) (Eq. 8)

$$\text{Steel: } \sigma_s = \frac{R_A}{A_s} = \frac{4E_s P}{E_a A_a + 2E_s A_s} \quad \leftarrow$$

(tension) (Eq. 9)

(b) NUMERICAL RESULTS

$$P = 12 \text{ k} \quad A_a = 8.92 \text{ in.}^2 \quad A_s = 1.03 \text{ in.}^2$$

$$E_a = 10 \times 10^6 \text{ psi} \quad E_s = 29 \times 10^6 \text{ psi}$$

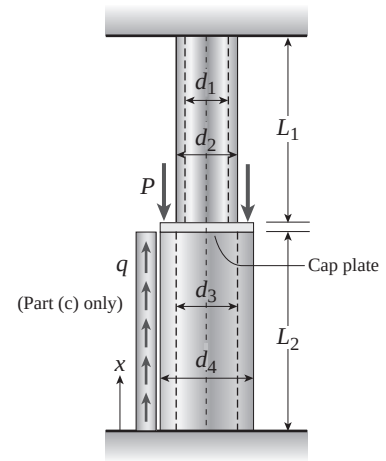
Substitute into Eqs. (8) and (9):

$$\sigma_a = 1,610 \text{ psi (compression)} \quad \leftarrow$$

$$\sigma_s = 9,350 \text{ psi (tension)} \quad \leftarrow$$

Problem 2.4-10 A hollow circular pipe (see figure) supports a load P which is uniformly distributed around a cap plate at the top of the lower pipe. The inner and outer diameters of the upper and lower parts of the pipe are $d_1 = 50$ mm, $d_2 = 60$ mm, $d_3 = 57$ mm, and $d_4 = 64$ mm, respectively. Pipe lengths are $L_1 = 2$ m and $L_2 = 3$ m. Neglect the self-weight of the pipes. Assume that cap plate thickness is small compared to L_1 and L_2 . Let $E = 110$ MPa.

- If the tensile stress in the upper part is $\sigma_1 = 10.5$ MPa, what is load P ? Also, what are reactions R_1 at the upper support and R_2 at the lower support. What is the stress σ_2 MPa in the lower part?
- Find displacement δ (mm) at the cap plate. Plot the Axial Force Diagram, AFD [$N(x)$] and Axial Displacement Diagram, ADD [$\delta(x)$].
- Add the uniformly distributed load q along the centroidal axis of pipe segment 2. Find q (kN/m) so that $R_2 = 0$. Assume that load P from part (a) is also applied.



Solution 2.4-10

- (a) STRESSES AND REACTIONS: SELECT R_1 AS REDUNDANT AND DO SUPERPOSITION ANALYSIS (HERE $q = 0$; DEFLECTION POSITION UPWARD)

$$d_1 = 50 \text{ mm} \quad d_2 = 60 \text{ mm} \quad d_3 = 57 \text{ mm} \quad d_4 = 64 \text{ mm} \quad A_1 = \frac{\pi}{4} (d_2^2 - d_1^2) = 863.938 \text{ mm}^2$$

$$E = 110 \text{ MPa}$$

$$A_2 = \frac{\pi}{4} (d_4^2 - d_3^2) = 665.232 \text{ mm}^2$$

$$\text{SEGMENT FLEXIBILITIES} \quad L_1 = 2 \text{ m} \quad L_2 = 3 \text{ m}$$

$$f_1 = \frac{L_1}{EA_1} = 0.02105 \text{ mm/N} \quad f_2 = \frac{L_2}{EA_2} = 0.041 \text{ mm/N} \quad \frac{f_1}{f_2} = 0.513$$

$$\text{TENSILE stress } (\sigma_1) \text{ is known in upper segment so } R_1 = \sigma_1 \times A_1 \quad \sigma_1 = 10.5 \text{ MPa} \quad R_1 = \sigma_1 A_1 = 9.07 \text{ kN}$$

$$\delta_{1a} = -Pf_2 \quad \delta_{1b} = R_1(f_1 + f_2) \quad \text{Compatibility: } \delta_{1a} + \delta_{1b} = 0$$

$$\text{Solve for } P: \quad P = R_1 \left(\frac{f_1 + f_2}{f_2} \right) = 13.73 \text{ kN}$$

$$\text{Finally, use statics to find } R_2: \quad R_2 = P - R_1 = 4.66 \text{ kN} \quad \sigma_2 = \frac{R_2}{A_2} = 7 \text{ MPa} < \text{compressive since } R_2 \text{ is positive (upward)}$$

$$P = 13.73 \text{ kN}$$

$$R_1 = 9.07 \text{ kN}$$

$$R_2 = 4.66 \text{ kN}$$

$$\sigma_2 = 7 \text{ MPa}$$

- (b) DISPLACEMENT AT CAP PLATE

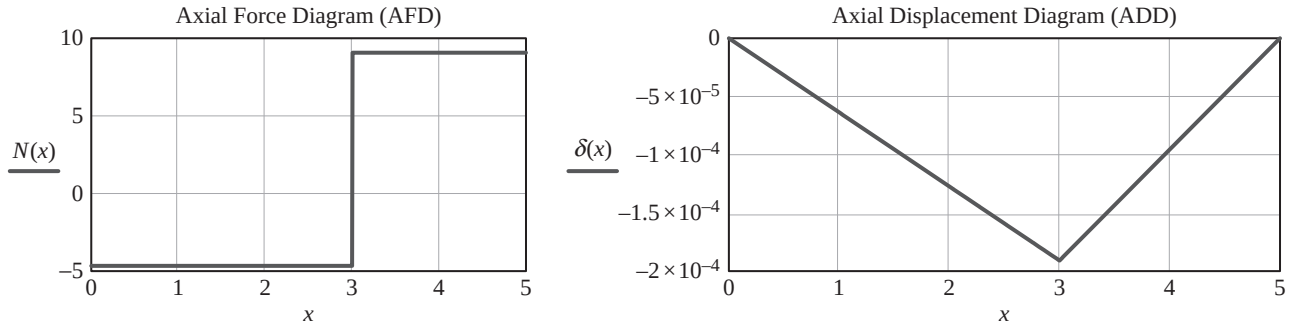
$$\delta_c = R_1 f_1 = 190.909 \text{ mm} < \text{downward} \quad \text{OR} \quad \delta_c = (R_2) f_2 = 190.909 \text{ mm} < \text{downward (neg. } x\text{-direction)}$$

$$\delta_{\text{cap}} = \delta_c = 0.191 \text{ m} \quad \boxed{\delta_{\text{cap}} = 190.9 \text{ mm}}$$

$$\text{AFD and ADD:} \quad R_1 = 9.071 \quad R_2 = 4.657 \quad L_1 = 2 \quad A_1 = 863.938 \quad A_2 = 665.232 \quad E = 110$$

$$L_2 = 3$$

NOTE: x is measured up from lower support.



(c) UNIFORM LOAD Q ON SEGMENT 2 SUCH THAT $R_2 = 0$

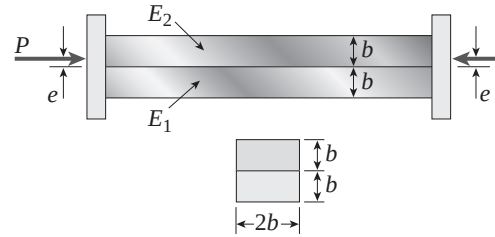
$$P = 13.728 \text{ kN} \quad R_1 = \sigma_1 A_1 = 9.071 \text{ kN} \quad L_2 = 3 \text{ m}$$

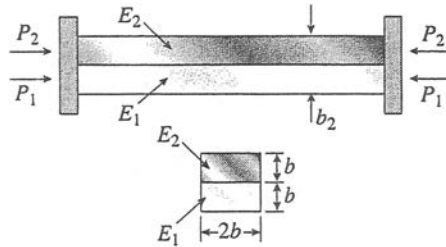
$$\text{Equilibrium: } R_1 + R_2 = P - qL_2 < \text{set } R_2 = 0, \text{ solve for req'd } q \quad q = \frac{P - R_1}{L_2} = 1.552 \text{ kN/m}$$

$$q = 1.552 \text{ kN/m}$$

Problem 2.4-11 A bimetallic bar (or composite bar) of square cross section with dimensions $2b \times 2b$ is constructed of two different metals having moduli of elasticity E_1 and E_2 (see figure). The two parts of the bar have the same cross-sectional dimensions. The bar is compressed by forces P acting through rigid end plates. The line of action of the loads has an eccentricity e of such magnitude that each part of the bar is stressed uniformly in compression.

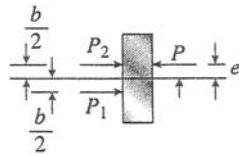
- Determine the axial forces P_1 and P_2 in the two parts of the bar.
- Determine the eccentricity e of the loads.
- Determine the ratio σ_1/σ_2 of the stresses in the two parts of the bar.



Solution 2.4-11 Bimetallic bar in compression

FREE-BODY DIAGRAM

(Plate at right-hand end)



EQUATIONS OF EQUILIBRIUM

$$\Sigma F = 0 \quad P_1 + P_2 = P \quad (\text{Eq. 1})$$

$$\Sigma M = 0 \quad \curvearrowright \quad Pe + P_1\left(\frac{b}{2}\right) - P_2\left(\frac{b}{2}\right) = 0 \quad (\text{Eq. 2})$$

EQUATION OF COMPATIBILITY

$$\delta_2 = \delta_1$$

$$\frac{P_2 L}{E_2 A} = \frac{P_1 L}{E_1 A} \quad \text{or} \quad \frac{P_2}{E_2} = \frac{P_1}{E_1} \quad (\text{Eq. 3})$$

(a) AXIAL FORCES

Solve simultaneously Eqs. (1) and (3):

$$P_1 = \frac{PE_1}{E_1 + E_2} \quad P_2 = \frac{PE_2}{E_1 + E_2} \quad \leftarrow$$

(b) ECCENTRICITY OF LOAD P Substitute P_1 and P_2 into Eq. (2) and solve for e :

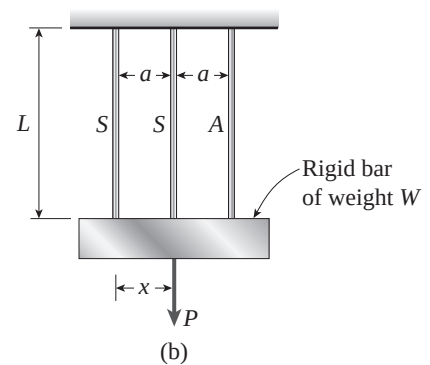
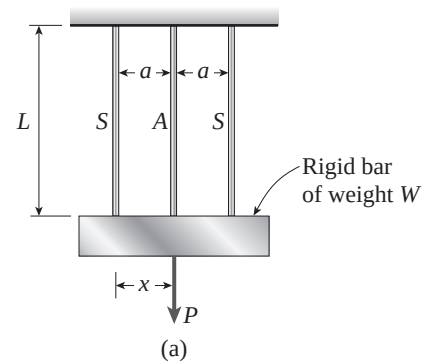
$$e = \frac{b(E_2 - E_1)}{2(E_2 + E_1)} \quad \leftarrow$$

(c) RATIO OF STRESSES

$$\sigma_1 = \frac{P_1}{A} \quad \sigma_2 = \frac{P_2}{A} \quad \frac{\sigma_1}{\sigma_2} = \frac{P_1}{P_2} = \frac{E_1}{E_2} \quad \leftarrow$$

Problem 2.4-12 A rigid bar of weight $W = 800$ N hangs from three equally spaced vertical wires (length $L = 150$ mm, spacing $a = 50$ mm): two of steel and one of aluminum. The wires also support a load P acting on the bar. The diameter of the steel wires is $d_s = 2$ mm, and the diameter of the aluminum wire is $d_a = 4$ mm. Assume $E_s = 210$ GPa and $E_a = 70$ GPa.

- What load P_{allow} can be supported at the midpoint of the bar ($x = a$) if the allowable stress in the steel wires is 220 MPa and in the aluminum wire is 80 MPa? (See figure part a.)
- What is P_{allow} if the load is positioned at $x = a/2$? (See figure part a.)
- Repeat (b) above if the second and third wires are switched as shown in figure part b.



Solution 2.4-12

Numerical data:

$$\begin{aligned} W &= 800 \text{ N} & L &= 150 \text{ mm} \\ a &= 50 \text{ mm} & d_s &= 2 \text{ mm} \\ d_a &= 4 \text{ mm} & E_s &= 210 \text{ GPa} \\ E_a &= 70 \text{ GPa} \\ \sigma_{s\text{a}} &= 220 \text{ MPa} & \sigma_{a\text{a}} &= 80 \text{ MPa} \end{aligned}$$

$$A_A = \frac{\pi}{4} d_a^2 \quad A_S = \frac{\pi}{4} d_s^2$$

$$A_A = 13 \text{ mm}^2 \quad A_S = 3 \text{ mm}^2$$

(a) P_{allow} AT CENTER OF BAR

One-degree statically indeterminate - use reaction (R_A) at top of aluminum bar as the redundant

compatibility: $\delta_1 - \delta_2 = 0$ Statics: $2R_S + R_A = P + W$

$$\delta_1 = \frac{P + W}{2} \left(\frac{L}{E_s A_S} \right) \quad \begin{array}{l} < \text{downward displacement due to elongation of each steel wire under } P + W \\ \text{aluminum wire is cut at top} \end{array}$$

$$\delta_2 = R_A \left(\frac{L}{2E_S A_S} + \frac{L}{E_A A_A} \right) \quad < \text{upward displ. due to shortening of steel wires and elongation of aluminum wire under redundant } R_A$$

Enforce compatibility and then solve for R_A :

$$\delta_1 = \delta_2 \quad \text{so} \quad R_A = \frac{\frac{P+W}{2} \left(\frac{L}{E_S A_S} \right)}{\frac{L}{2E_S A_S} + \frac{L}{E_A A_A}} \quad R_A = (P+W) \frac{E_A A_A}{E_A A_A + 2E_S A_S} \quad \text{and} \quad \sigma_{Aa} = \frac{R_A}{A_A}$$

Now use statics to find R_S :

$$R_S = \frac{P+W-R_A}{2} \quad R_S = \frac{P+W - (P+W) \frac{E_A A_A}{E_A A_A + 2E_S A_S}}{2} \quad R_S = (P+W) \frac{E_S A_S}{E_A A_A + 2E_S A_S}$$

$$\text{and} \quad \sigma_{Sa} = \frac{R_S}{A_S}$$

Compute stresses and apply allowable stress values:

$$\sigma_{Aa} = (P+W) \frac{E_A}{E_A A_A + 2E_S A_S} \quad \sigma_{Sa} = (P+W) \frac{E_S}{E_A A_A + 2E_S A_S}$$

Solve for allowable load P :

$$P_{Aa} = \sigma_{Aa} \left(\frac{E_A A_A + 2E_S A_S}{E_A} \right) - W \quad P_{Sa} = \sigma_{Sa} \left(\frac{E_A A_A + 2E_S A_S}{E_S} \right) - W \quad (\text{lower value of } P \text{ controls})$$

$$P_{Aa} = 1713 \text{ N}$$

$$P_{Sa} = 1504 \text{ N} \quad \leftarrow P_{\text{allow}} \text{ is controlled by steel wires}$$

(b) P_{allow} IF LOAD P AT $x = a/2$

Again, cut aluminum wire at top, then compute elongations of left and right steel wires:

$$\delta_{1L} = \left(\frac{3P}{4} + \frac{W}{2} \right) \left(\frac{L}{E_S A_S} \right) \quad \delta_{1R} = \left(\frac{P}{4} + \frac{W}{2} \right) \left(\frac{L}{E_S A_S} \right)$$

$$\delta_1 = \frac{\delta_{1L} + \delta_{1R}}{2} \quad \delta_1 = \frac{P+W}{2} \left(\frac{L}{E_S A_S} \right) \quad \text{where } \delta_1 = \text{displacement at } x = a$$

Use δ_2 from part (a):

$$\delta_2 = R_A \left(\frac{L}{2E_S A_S} + \frac{L}{E_A A_A} \right)$$

$$\text{So equating } \delta_1 \text{ and } \delta_2, \text{ solve for } R_A: \quad R_A = (P+W) \frac{E_A A_A}{E_A A_A + 2E_S A_S}$$

^ same as in part (a)

$$R_{SL} = \frac{3P}{4} + \frac{W}{2} - \frac{R_A}{2} \quad < \text{stress in left steel wire exceeds that in right steel wire}$$

$$R_{SL} = \frac{3P}{4} + \frac{W}{2} - \frac{(P+W) \frac{E_A A_A}{E_A A_A + 2E_S A_S}}{2}$$

$$R_{SL} = \frac{PE_A A_A + 6PE_S A_S + 4WE_S A_S}{4E_A A_A + 8E_S A_S} \quad \sigma_{Sa} = \frac{PE_A A_A + 6PE_S A_S + 4WE_S A_S}{4E_A A_A + 8E_S A_S} \left(\frac{1}{A_S} \right)$$

Solve for P_{allow} based on allowable stresses in steel and aluminum:

$$P_{Sa} = \frac{\sigma_{Sa}(4A_S E_A A_A + 8E_S A_S^2) - (4WE_S A_S)}{E_A A_A + 6E_S A_S} \quad P_{Aa} = 1713 \text{ N} \quad < \text{same as in part(a)}$$

$$P_{Sa} = 820 \text{ N} \quad \leftarrow \text{steel controls}$$

(c) P_{allow} IF WIRES ARE SWITCHED AS SHOWN AND $x = a/2$

Select R_A as the redundant; statics on the two released structures:

(1) Cut aluminum wire—apply P and W , compute forces in left and right steel wires, then compute displacements at each steel wire:

$$R_{SL} = \frac{P}{2} \quad R_{SR} = \frac{P}{2} + W$$

$$\delta_{1L} = \frac{P}{2} \left(\frac{L}{E_S A_S} \right) \quad \delta_{1R} = \left(\frac{P}{2} + W \right) \left(\frac{L}{E_S A_S} \right)$$

By geometry, δ at aluminum wire location at far right is $\delta_1 = \left(\frac{P}{2} + 2W \right) \left(\frac{L}{E_S A_S} \right)$

(2) Next apply redundant R_A at right wire, compute wire force and displacement at aluminum wire:

$$R_{SL} = -R_A \quad R_{SR} = 2R_A \quad \delta_2 = R_A \left(\frac{5L}{E_S A_S} + \frac{L}{E_A A_A} \right)$$

(3) Compatibility equate δ_1 , δ_2 and solve for R_A , then P_{allow} for aluminum wire:

$$R_A = \frac{\left(\frac{P}{2} + 2W \right) \left(\frac{L}{E_S A_S} \right)}{\frac{5L}{E_S A_S} + \frac{L}{E_A A_A}} \quad R_A = \frac{E_A A_A P + 4E_A A_A W}{10E_A A_A + 2E_S A_S} \quad \sigma_{Aa} = \frac{R_A}{A_A}$$

$$\sigma_{Aa} = \frac{E_A P + 4E_A W}{10E_A A_A + 2E_S A_S}$$

$$P_{Aa} = \frac{\sigma_{Aa}(10E_A A_A + 2E_S A_S) - 4E_A W}{E_A} \quad P_{Aa} = 1713 \text{ N}$$

(4) Statics or superposition—find forces in steel wires, then P_{allow} for steel wires:

$$R_{SL} = \frac{P}{2} + R_A \quad R_{SL} = \frac{P}{2} + \frac{E_A A_A P + 4E_A A_A W}{10E_A A_A + 2E_S A_S}$$

$$R_{SL} = \frac{6E_A A_A P + PE_S A_S + 4E_A A_A W}{10E_A A_A + 2E_S A_S} \quad < \text{larger than } R_{SR}, \text{ so use in allowable stress calculations}$$

$$R_{SR} = \frac{P}{2} + W - 2R_A \quad R_{SR} = \frac{P}{2} + W - \frac{E_A A_A P + 4E_A A_A W}{5E_A A_A + E_S A_S}$$

$$R_{SR} = \frac{3E_A A_A P + PE_S A_S + 2E_A A_A W + 2WE_S A_S}{10E_A A_A + 2E_S A_S}$$

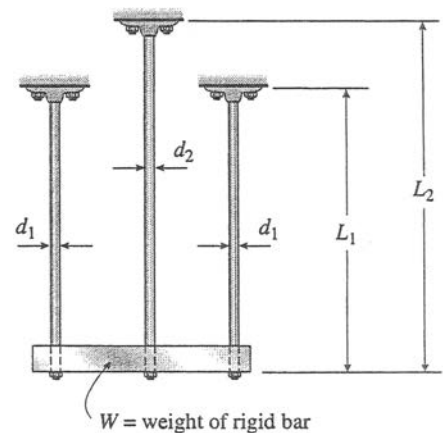
$$\sigma_{Sa} = \frac{R_{SL}}{A_S} \quad P_{Sa} = \sigma_{Sa} A_S \left(\frac{10E_A A_A + 2E_S A_S}{6E_A A_A + E_S A_S} \right) - \frac{4E_A A_A W}{6E_A A_A + E_S A_S}$$

$$P_{Sa} = \frac{10\sigma_{Sa} A_S E_A A_A + 2\sigma_{Sa} A_S^2 E_S - 4E_A A_A W}{6E_A A_A + E_S A_S} \quad P_{Sa} = 703 \text{ N} \quad \leftarrow$$

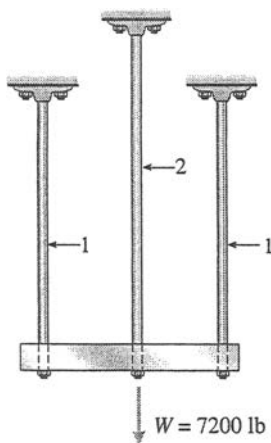
^ steel controls

Problem 2.4-13 A horizontal rigid bar of weight $W = 7200 \text{ lb}$ is supported by three slender circular rods that are equally spaced (see figure). The two outer rods are made of aluminum ($E_1 = 10 \times 10^6 \text{ psi}$) with diameter $d_1 = 0.4 \text{ in.}$ and length $L_1 = 40 \text{ in.}$ The inner rod is magnesium ($E_2 = 6.5 \times 10^6 \text{ psi}$) with diameter d_2 and length L_2 . The allowable stresses in the aluminum and magnesium are 24,000 psi and 13,000 psi, respectively.

If it is desired to have all three rods loaded to their maximum allowable values, what should be the diameter d_2 and length L_2 of the middle rod?



Solution 2.4-13 Bar supported by three rods



BAR 1 ALUMINUM

$$E_1 = 10 \times 10^6 \text{ psi}$$

$$d_1 = 0.4 \text{ in.}$$

$$L_1 = 40 \text{ in.}$$

$$\sigma_1 = 24,000 \text{ psi}$$

BAR 2 MAGNESIUM

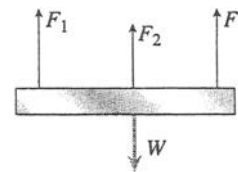
$$E_2 = 6.5 \times 10^6 \text{ psi}$$

$$d_2 = ? \quad L_2 = ?$$

$$\sigma_2 = 13,000 \text{ psi}$$

FREE-BODY DIAGRAM OF RIGID BAR

EQUATION OF EQUILIBRIUM



$$\Sigma F_{\text{vert}} = 0$$

$$2F_1 + F_2 - W = 0 \quad (\text{Eq. 1})$$

FULLY STRESSED RODS

$$F_1 = \sigma_1 A_1 \quad F_2 = \sigma_2 A_2$$

$$A_1 = \frac{\pi d_1^2}{4} \quad A_2 = \frac{\pi d_2^2}{4}$$

Substitute into Eq. (1):

$$2\sigma_1 \left(\frac{\pi d_1^2}{4} \right) + \sigma_2 \left(\frac{\pi d_2^2}{4} \right) = W$$

Diameter d_1 is known; solve for d_2 :

$$d_2^2 = \frac{4W}{\pi\sigma_2} - \frac{2\sigma_1 d_1^2}{\sigma_2} \quad \leftarrow \quad (\text{Eq. 2})$$

SUBSTITUTE NUMERICAL VALUES:

$$\begin{aligned} d_2^2 &= \frac{4(7200 \text{ lb})}{\pi(13,000 \text{ psi})} - \frac{2(24,000 \text{ psi})(0.4 \text{ in.})^2}{13,000 \text{ psi}} \\ &= 0.70518 \text{ in.}^2 - 0.59077 \text{ in.}^2 = 0.11441 \text{ in.}^2 \\ d_2 &= 0.338 \text{ in.} \quad \leftarrow \end{aligned}$$

EQUATION OF COMPATIBILITY

$$\delta_1 = \delta_2 \quad (\text{Eq. 3})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_1 = \frac{F_1 L_1}{E_1 A_1} = \sigma_1 \left(\frac{L_1}{E_1} \right) \quad (\text{Eq. 4})$$

$$\delta_2 = \frac{F_2 L_2}{E_2 A_2} = \sigma_2 \left(\frac{L_2}{E_2} \right) \quad (\text{Eq. 5})$$

Substitute (4) and (5) into Eq. (3):

$$\sigma_1 \left(\frac{L_1}{E_1} \right) = \sigma_2 \left(\frac{L_2}{E_2} \right)$$

Length L_1 is known; solve for L_2 :

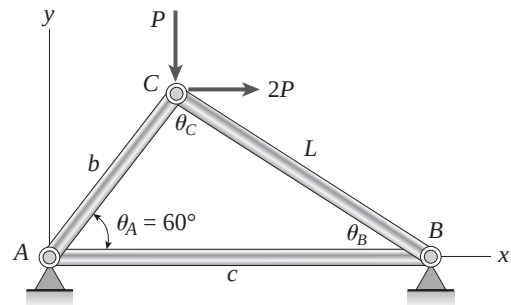
$$L_2 = L_1 \left(\frac{\sigma_1 E_2}{\sigma_2 E_1} \right) \quad \leftarrow \quad (\text{Eq. 6})$$

SUBSTITUTE NUMERICAL VALUES:

$$\begin{aligned} L_2 &= (40 \text{ in.}) \left(\frac{24,000 \text{ psi}}{13,000 \text{ psi}} \right) \left(\frac{6.5 \times 10^6 \text{ psi}}{10 \times 10^6 \text{ psi}} \right) \\ &= 48.0 \text{ in.} \end{aligned}$$

Problem 2.4-14 Three-bar truss ABC (see figure) is constructed of steel pipes having a cross-sectional area $A = 3500 \text{ mm}^2$ and a modulus of elasticity $E = 210 \text{ GPa}$. Member BC is of length $L = 2.5 \text{ m}$, and the angle between members AC and AB is known to be 60° . Member AC length is $b = 0.71L$. Loads $P = 185 \text{ kN}$ and $2P = 370 \text{ kN}$ act vertically and horizontally at joint C, as shown. Joints A and B are pinned supports. (Use the law of sines and law of cosines to find missing dimensions and angles in the figure.)

- Find the support reactions at joints A and B. Use horizontal reaction B_x as the redundant.
- What is the maximum permissible value of load variable P if the allowable normal stress in each truss member is 150 MPa ?



Solution 2.4-14

NUMERICAL DATA

$$L = 2.5 \text{ m} \quad b = 0.71 \quad L = 1.775 \text{ m} \quad E = 210 \text{ GPa} \quad A = 3500 \text{ mm}^2 \quad P = 185 \text{ kN} \quad \theta_A = 60^\circ$$

$$\sigma_a = 150 \text{ MPa}$$

FIND MISSING DIMENSIONS AND ANGLES IN PLANE TRUSS FIGURE

$$x_c = b \cos(\theta_A) = 0.8875 \text{ m} \quad y_c = b \sin(\theta_A) = 1.5372 \text{ m}$$

$$\frac{b}{\sin(\theta_B)} = \frac{L}{\sin(\theta_A)} \quad \text{so} \quad \theta_B = \sin^{-1} \left(\frac{b \sin(\theta_A)}{L} \right) = 37.94306^\circ$$

$$\theta_C = 180^\circ - (\theta_A + \theta_B) = 82.05694^\circ$$

$$c = \frac{L}{\sin(\theta_A)} \sin(\theta_C) = 2.85906 \text{ m} \quad \text{or} \quad c = \sqrt{b^2 + L^2 - 2bL \cos(\theta_C)} = 2.85906 \text{ m}$$

- (a) SELECT B_x AS THE REDUNDANT; PERFORM SUPERPOSITION ANALYSIS TO FIND B_x THEN USE STATICS TO FIND REMAINING REACTIONS. FINALLY USE METHOD OF JOINTS TO FIND MEMBER FORCES (SEE EXAMPLE 1-1)

δ_{Bx1} = displacement in x -direction in released structure acted upon by loads P and $2P$ at joint C :

$$\delta_{Bx1} = 1.2789911 \text{ mm} < \text{this displacement equals force in } AB \text{ divided by flexibility of } AB$$

$$\delta_{Bx2} = \text{displacement in } x\text{-direction in released structure acted upon by redundant } B_x: \quad \delta_{Bx2} = B_x \frac{c}{EA}$$

$$\text{COMPATIBILITY EQUATION: } \delta_{Bx1} + \delta_{Bx2} = 0 \quad \text{so} \quad B_x = \frac{-EA}{c} \delta_{Bx1} = -328.8 \text{ kN}$$

$$\text{STATICS: } \Sigma F_x = 0 \quad A_x = -B_x - 2P = -41.2 \text{ kN}$$

$$\Sigma M_A = 0 \quad B_y = \frac{1}{c} [2P(b \sin(\theta_A)) + P(b \cos(\theta_A))] = 256.361 \text{ kN}$$

$$\Sigma F_y = 0 \quad A_y = P - B_y = -71.361 \text{ kN}$$

REACTIONS:

$$\boxed{A_x = -41.2 \text{ kN}} \quad \boxed{A_y = -71.4 \text{ kN}} \quad \boxed{B_x = -329 \text{ kN}} \quad \boxed{B_y = 256 \text{ kN}}$$

- (b) FIND MAXIMUM PERMISSIBLE VALUE OF LOAD VARIABLE P IF ALLOWABLE NORMAL STRESS IS 150 MPa

(1) Use reactions and Method of Joints to find member forces in each member for above loading.

$$\text{Results: } F_{AB} = 0 \quad F_{BC} = -416.929 \text{ kN} \quad F_{AC} = 82.40 \text{ kN}$$

(2) Compute member stresses:

$$\sigma_{AB} = 0 \quad \sigma_{BC} = \frac{-416.93 \text{ kN}}{A} = -119.123 \text{ MPa} \quad \sigma_{AC} = \frac{82.4 \text{ kN}}{A} = 23.543 \text{ MPa}$$

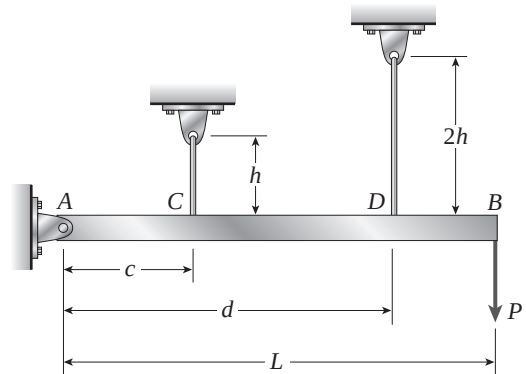
(3) Maximum stress occurs in member BC . For linear analysis, the stress is proportional to the load so

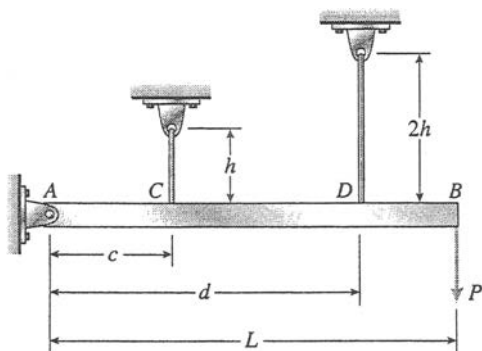
$$P_{\max} = \frac{\sigma_a}{\sigma_{BC}} P = 233 \text{ kN}$$

So when downward load $P = 233 \text{ kN}$ is applied at C and horizontal load $2P = 466 \text{ kN}$ is applied to the right at C , the stress in BC is 150 MPa

Problem 2.4-15 A rigid bar AB of length $L = 66 \text{ in.}$ is hinged to a support at A and supported by two vertical wires attached at points C and D (see figure). Both wires have the same cross-sectional area ($A = 0.0272 \text{ in.}^2$) and are made of the same material (modulus $E = 30 \times 10^6 \text{ psi}$). The wire at C has length $h = 18 \text{ in.}$ and the wire at D has length twice that amount. The horizontal distances are $c = 20 \text{ in.}$ and $d = 50 \text{ in.}$

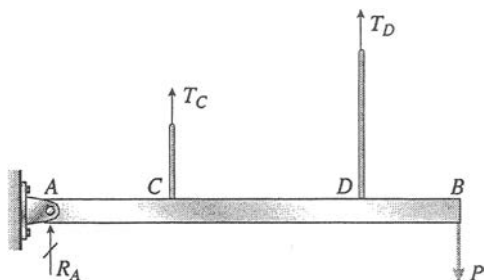
- (a) Determine the tensile stresses σ_C and σ_D in the wires due to the load $P = 340 \text{ lb}$ acting at end B of the bar.
 (b) Find the downward displacement δ_B at end B of the bar.



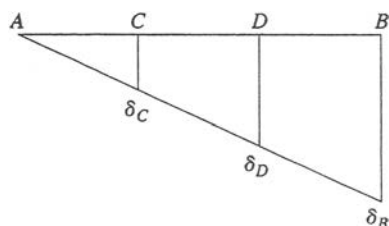
Solution 2.4-15 Bar supported by two wires

$$\begin{aligned} h &= 18 \text{ in.} \\ 2h &= 36 \text{ in.} \\ c &= 20 \text{ in.} \\ d &= 50 \text{ in.} \\ L &= 66 \text{ in.} \\ E &= 30 \times 10^6 \text{ psi} \\ A &= 0.0272 \text{ in.}^2 \\ P &= 340 \text{ lb} \end{aligned}$$

FREE-BODY DIAGRAM



DISPLACEMENT DIAGRAM



EQUATION OF EQUILIBRIUM

$$\Sigma M_A = 0 \quad \curvearrowright \quad T_C(c) + T_D(d) = PL \quad (\text{Eq. 1})$$

EQUATION OF COMPATIBILITY

$$\frac{\delta_C}{c} = \frac{\delta_D}{d} \quad (\text{Eq. 2})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_C = \frac{T_C h}{EA} \quad \delta_D = \frac{T_D (2h)}{EA} \quad (\text{Eqs. 3, 4})$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{T_C h}{cEA} = \frac{T_D (2h)}{dEA} \quad \text{or} \quad \frac{T_C}{c} = \frac{2T_D}{d} \quad (\text{Eq. 5})$$

TENSILE FORCES IN THE WIRES

Solve simultaneously Eqs. (1) and (5):

$$T_C = \frac{2cPL}{2c^2 + d^2} \quad T_D = \frac{dPL}{2c^2 + d^2} \quad (\text{Eqs. 6, 7})$$

TENSILE STRESSES IN THE WIRES

$$\sigma_C = \frac{T_C}{A} = \frac{2cPL}{A(2c^2 + d^2)} \quad (\text{Eq. 8})$$

$$\sigma_D = \frac{T_D}{A} = \frac{dPL}{A(2c^2 + d^2)} \quad (\text{Eq. 9})$$

DISPLACEMENT AT END OF BAR

$$\delta_B = \delta_D \left(\frac{L}{d} \right) = \frac{2hT_D}{EA} \left(\frac{L}{d} \right) = \frac{2hPL^2}{EA(2c^2 + d^2)} \quad (\text{Eq. 10})$$

SUBSTITUTE NUMERICAL VALUES

$$2c^2 + d^2 = 2(20 \text{ in.})^2 + (50 \text{ in.})^2 = 3300 \text{ in.}^2$$

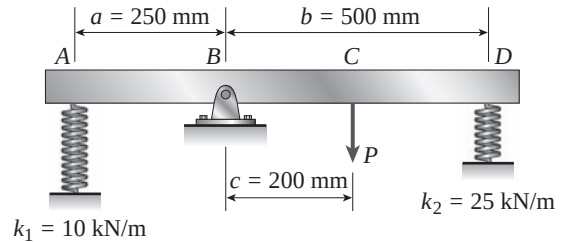
$$\begin{aligned} \text{(a) } \sigma_C &= \frac{2cPL}{A(2c^2 + d^2)} = \frac{2(20 \text{ in.})(340 \text{ lb})(66 \text{ in.})}{(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)} \\ &= 10,000 \text{ psi} \quad \leftarrow \end{aligned}$$

$$\begin{aligned} \sigma_D &= \frac{dPL}{A(2c^2 + d^2)} = \frac{(50 \text{ in.})(340 \text{ lb})(66 \text{ in.})}{(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)} \\ &= 12,500 \text{ psi} \quad \leftarrow \end{aligned}$$

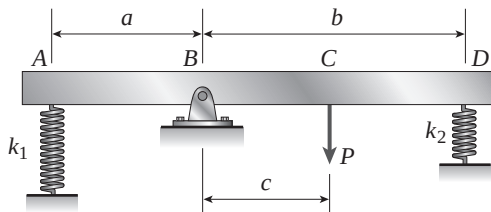
$$\begin{aligned} \text{(b) } \delta_B &= \frac{2hPL^2}{EA(2c^2 + d^2)} \\ &= \frac{2(18 \text{ in.})(340 \text{ lb})(66 \text{ in.})^2}{(30 \times 10^6 \text{ psi})(0.0272 \text{ in.}^2)(3300 \text{ in.}^2)} \\ &= 0.0198 \text{ in.} \quad \leftarrow \end{aligned}$$

Problem 2.4-16 A rigid bar $ABCD$ is pinned at point B and supported by springs at A and D (see figure). The springs at A and D have stiffnesses $k_1 = 10 \text{ kN/m}$ and $k_2 = 25 \text{ kN/m}$, respectively, and the dimensions a , b , and c are 250 mm, 500 mm, and 200 mm, respectively. A load P acts at point C .

If the angle of rotation of the bar due to the action of the load P is limited to 3° , what is the maximum permissible load $P_{\max}$?



Solution 2.4-16 Rigid bar supported by springs



NUMERICAL DATA

$$a = 250 \text{ mm}$$

$$b = 500 \text{ mm}$$

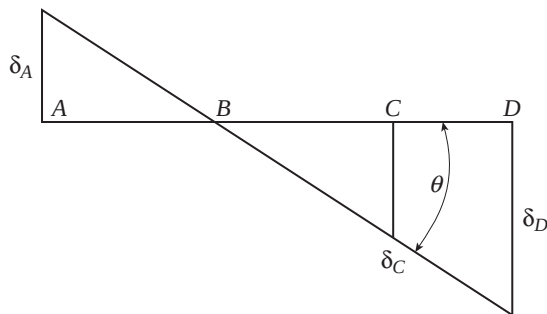
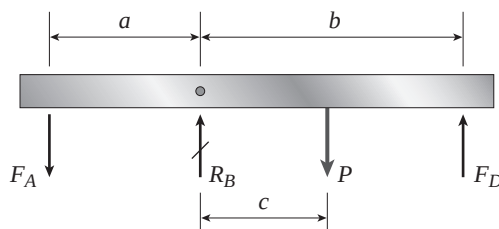
$$c = 200 \text{ mm}$$

$$k_1 = 10 \text{ kN/m}$$

$$k_2 = 25 \text{ kN/m}$$

$$\theta_{\max} = 3^\circ = \frac{\pi}{60} \text{ rad}$$

FREE-BODY DIAGRAM AND DISPLACEMENT DIAGRAM



EQUATION OF EQUILIBRIUM

$$\Sigma M_B = 0 + -F_A(a) - P(c) + F_D(b) = 0 \quad (\text{Eq. 1})$$

EQUATION OF COMPATIBILITY

$$\frac{\delta_A}{a} = \frac{\delta_D}{b} \quad (\text{Eq. 2})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_A = \frac{F_A}{k_1} \quad \delta_D = \frac{F_D}{k_2} \quad (\text{Eqs. 3, 4})$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{F_A}{ak_1} = \frac{F_D}{bk_2} \quad (\text{Eq. 5})$$

SOLVE SIMULTANEOUSLY EQS. (1) AND (5):

$$F_A = \frac{ack_1P}{a^2k_1 + b^2k_2} \quad F_D = \frac{bck_2P}{a^2k_1 + b^2k_2}$$

ANGLE OF ROTATION

$$\delta_D = \frac{F_D}{k_2} = \frac{bcP}{a^2k_1 + b^2k_2} \quad \theta = \frac{\delta_D}{b} = \frac{cP}{a^2k_1 + b^2k_2}$$

MAXIMUM LOAD

$$P = \frac{\theta}{c} (a^2k_1 + b^2k_2)$$

$$P_{\max} = \frac{\theta_{\max}}{c} (a^2k_1 + b^2k_2) \quad \leftarrow$$

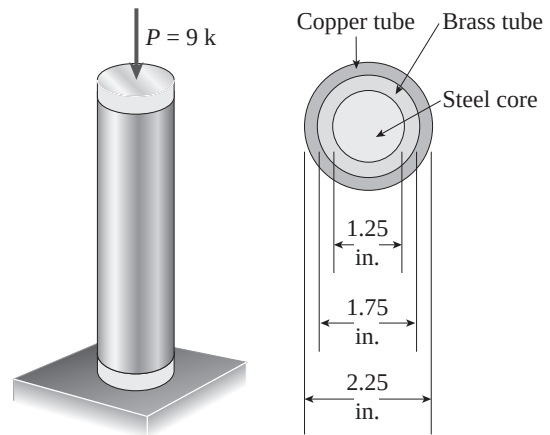
SUBSTITUTE NUMERICAL VALUES:

$$P_{\max} = \frac{\pi/60 \text{ rad}}{200 \text{ mm}} [(250 \text{ mm})^2(10 \text{ kN/m}) + (500 \text{ mm})^2(25 \text{ kN/m})]$$

$$= 1800 \text{ N} \quad \leftarrow$$

Problem 2.4-17 A trimetallic bar is uniformly compressed by an axial force $P = 9$ kips applied through a rigid end plate (see figure). The bar consists of a circular steel core surrounded by brass and copper tubes. The steel core has diameter 1.25 in., the brass tube has outer diameter 1.75 in., and the copper tube has outer diameter 2.25 in. The corresponding moduli of elasticity are $E_s = 30,000$ ksi, $E_b = 16,000$ ksi, and $E_c = 18,000$ ksi.

Calculate the compressive stresses σ_s , σ_b , and σ_c in the steel, brass, and copper, respectively, due to the force P .



Solution 2.4-17

Numerical properties (kips, inches):

$$d_c = 2.25 \text{ in.} \quad d_b = 1.75 \text{ in.} \quad d_s = 1.25 \text{ in.}$$

$$E_c = 18,000 \text{ ksi} \quad E_b = 16,000 \text{ ksi}$$

$$E_s = 30,000 \text{ ksi}$$

$$P = 9 \text{ k}$$

EQUATION OF EQUILIBRIUM

$$\Sigma F_{\text{vert}} = 0 \quad P_s + P_b + P_c = P \quad (\text{Eq. 1})$$

EQUATIONS OF COMPATIBILITY

$$\delta_s = \delta_b \quad \delta_c = \delta_s \quad (\text{Eqs. 2})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \quad \delta_b = \frac{P_b L}{E_b A_b} \quad \delta_c = \frac{P_c L}{E_c A_c} \quad (\text{Eqs. 3, 4, 5})$$

SOLUTION OF EQUATIONS

Substitute (3), (4), and (5) into Eqs. (2):

$$P_b = P_s \frac{E_b A_b}{E_s A_s} \quad P_c = P_s \frac{E_c A_c}{E_s A_s} \quad (\text{Eqs. 6, 7})$$

$$A_s = \frac{\pi}{4} d_s^2$$

$$A_b = \frac{\pi}{4} (d_b^2 - d_s^2)$$

$$A_c = \frac{\pi}{4} (d_c^2 - d_b^2)$$

SOLVE SIMULTANEOUSLY EQS. (1), (6), AND (7):

$$P_s = P \frac{E_s A_s}{E_s A_s + E_b A_b + E_c A_c} = 3.95 \text{ k}$$

$$P_b = P \frac{E_b A_b}{E_s A_s + E_b A_b + E_c A_c} = 2.02 \text{ k}$$

$$P_c = P \frac{E_c A_c}{E_s A_s + E_b A_b + E_c A_c} = 3.03 \text{ k}$$

$$P_s + P_b + P_c = 9 \quad \text{statics check}$$

COMPRESSIVE STRESSES

$$\text{Let } \Sigma EA = E_s A_s + E_b A_b + E_c A_c$$

$$\sigma_s = \frac{P_s}{A_s} = \frac{PE_s}{\Sigma EA} \quad \sigma_s = 3.22 \text{ ksi} \quad \leftarrow$$

$$\sigma_b = \frac{P_b}{A_b} = \frac{PE_b}{\Sigma EA} \quad \sigma_b = 1.716 \text{ ksi} \quad \leftarrow$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{PE_c}{\Sigma EA} \quad \sigma_c = 1.93 \text{ ksi} \quad \leftarrow$$

Thermal Effects

Problem 2.5-1 The rails of a railroad track are welded together at their ends (to form continuous rails and thus eliminate the clacking sound of the wheels) when the temperature is 60°F.

What compressive stress σ is produced in the rails when they are heated by the sun to 120°F if the coefficient of thermal expansion $\alpha = 6.5 \times 10^{-6}/^\circ\text{F}$ and the modulus of elasticity $E = 30 \times 10^6$ psi?

Solution 2.5-1 Expansion of railroad rails

The rails are prevented from expanding because of their great length and lack of expansion joints.

Therefore, each rail is in the same condition as a bar with fixed ends (see Example 2-7).

The compressive stress in the rails may be calculated as follows.

$$\Delta T = 120^\circ\text{F} - 60^\circ\text{F} = 60^\circ\text{F}$$

$$\sigma = E\alpha(\Delta T)$$

$$= (30 \times 10^6 \text{ psi})(6.5 \times 10^{-6}/^\circ\text{F})(60^\circ\text{F})$$

$$\sigma = 11,700 \text{ psi} \quad \leftarrow$$

Problem 2.5-2 An aluminum pipe has a length of 60 m at a temperature of 10°C. An adjacent steel pipe at the same temperature is 5 mm longer than the aluminum pipe.

At what temperature (degrees Celsius) will the aluminum pipe be 15 mm longer than the steel pipe? (Assume that the coefficients of thermal expansion of aluminum and steel are $\alpha_a = 23 \times 10^{-6}/^\circ\text{C}$ and $\alpha_s = 12 \times 10^{-6}/^\circ\text{C}$, respectively.)

Solution 2.5-2 Aluminum and steel pipes

INITIAL CONDITIONS

$$L_a = 60 \text{ m} \quad T_0 = 10^\circ\text{C}$$

$$L_s = 60.005 \text{ m} \quad T_0 = 10^\circ\text{C}$$

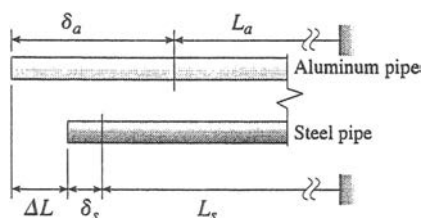
$$\alpha_a = 23 \times 10^{-6}/^\circ\text{C} \quad \alpha_s = 12 \times 10^{-6}/^\circ\text{C}$$

FINAL CONDITIONS

Aluminum pipe is longer than the steel pipe by the amount $\Delta L = 15$ mm.

ΔT = increase in temperature

$$\delta_a = \alpha_a(\Delta T)L_a \quad \delta_s = \alpha_s(\Delta T)L_s$$



From the figure above:

$$\delta_a + L_a = \Delta L + \delta_s + L_s$$

$$\text{or, } \alpha_a(\Delta T)L_a + L_a = \Delta L + \alpha_s(\Delta T)L_s + L_s$$

Solve for ΔT :

$$\Delta T = \frac{\Delta L + (L_s - L_a)}{\alpha_a L_a - \alpha_s L_s} \quad \leftarrow$$

Substitute numerical values:

$$\alpha_a L_a - \alpha_s L_s = 659.9 \times 10^{-6} \text{ m}/^\circ\text{C}$$

$$\Delta T = \frac{15 \text{ mm} + 5 \text{ mm}}{659.9 \times 10^{-6} \text{ m}/^\circ\text{C}} = 30.31^\circ\text{C}$$

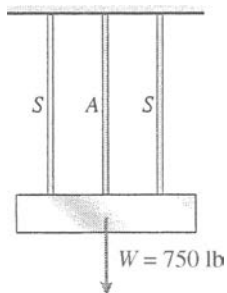
$$T = T_0 + \Delta T = 10^\circ\text{C} + 30.31^\circ\text{C}$$

$$= 40.3^\circ\text{C} \quad \leftarrow$$

Problem 2.5-3 A rigid bar of weight $W = 750$ lb hangs from three equally spaced wires, two of steel and one of aluminum (see figure). The diameter of the wires is $\frac{1}{8}$ in. Before they were loaded, all three wires had the same length.

What temperature increase ΔT in all three wires will result in the entire load being carried by the steel wires? (Assume $E_s = 30 \times 10^6$ psi, $\alpha_s = 6.5 \times 10^{-6}/^\circ\text{F}$, and $\alpha_a = 12 \times 10^{-6}/^\circ\text{F}$.)

Solution 2.5-3 Bar supported by three wires



$S = \text{steel}$ $A = \text{aluminum}$

$W = 750$ lb

$d = \frac{1}{8}$ in.

$A_s = \frac{\pi d^2}{4} = 0.012272 \text{ in.}^2$

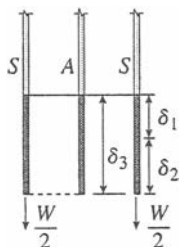
$E_s = 30 \times 10^6$ psi

$E_s A_s = 368,155$ lb

$\alpha_s = 6.5 \times 10^{-6}/^\circ\text{F}$

$\alpha_a = 12 \times 10^{-6}/^\circ\text{F}$

$L = \text{Initial length of wires}$



$\delta_1 = \text{increase in length of a steel wire due to temperature increase } \Delta T$

$$= \alpha_s (\Delta T) L$$

$\delta_2 = \text{increase in length of a steel wire due to load } W/2$

$$= \frac{WL}{2E_s A_s}$$

$\delta_3 = \text{increase in length of aluminum wire due to temperature increase } \Delta T$

$$= \alpha_a (\Delta T) L$$

For no load in the aluminum wire:

$$\delta_1 + \delta_2 = \delta_3$$

$$\alpha_s (\Delta T) L + \frac{WL}{2E_s A_s} = \alpha_a (\Delta T) L$$

or

$$\Delta T = \frac{W}{2E_s A_s (\alpha_a - \alpha_s)} \quad \leftarrow$$

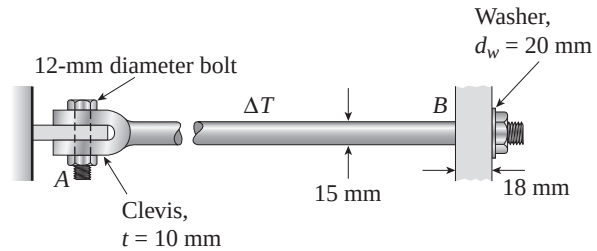
Substitute numerical values:

$$\begin{aligned} \Delta T &= \frac{750 \text{ lb}}{(2)(368,155 \text{ lb})(5.5 \times 10^{-6}/^\circ\text{F})} \\ &= 185^\circ\text{F} \quad \leftarrow \end{aligned}$$

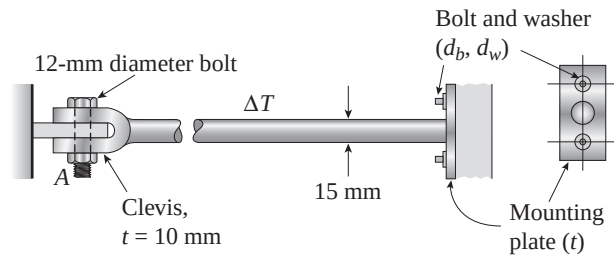
NOTE: If the temperature increase is larger than ΔT , the aluminum wire would be in compression, which is not possible. Therefore, the steel wires continue to carry all of the load. If the temperature increase is less than ΔT , the aluminum wire will be in tension and carry part of the load.

Problem 2.5-4 A steel rod of 15-mm diameter is held snugly (but without any initial stresses) between rigid walls by the arrangement shown in figure part a. (For the steel rod, use $\alpha = 12 \times 10^{-6}/^\circ\text{C}$ and $E = 200 \text{ GPa}$.)

- Calculate the temperature drop ΔT (degrees Celsius) at which the average shear stress in the 12-mm diameter bolt becomes 45 MPa. Also, what is the normal stress in the rod?
- What are the average bearing stresses in the bolt and clevis at A and between the washer ($d_w = 20 \text{ mm}$) and wall ($t = 18 \text{ mm}$) at B?
- If the connection to the wall at B is changed to an end plate with two bolts (see figure part b), what is the required diameter d_b of each bolt if the temperature drop is $\Delta T = 38^\circ\text{C}$ and the allowable bolt stress is 90 MPa?



(a)



(b)

Solution 2.5-4

NUMERICAL PROPERTIES

$$\begin{array}{llllll} d_r = 15 \text{ mm} & d_b = 12 \text{ mm} & d_w = 20 \text{ mm} & t_c = 10 \text{ mm} & t_{\text{wall}} = 18 \text{ mm} \\ \tau_b = 45 \text{ MPa} & \alpha = 12 (10^{-6}) & E = 200 \text{ GPa} \end{array}$$

(a) TEMPERATURE DROP RESULTING IN BOLT SHEAR STRESS $\epsilon = \alpha \Delta T$ $\sigma = E \alpha \Delta T$

$$\text{Rod force} = P = (E \alpha \Delta T) \frac{\pi}{4} d_r^2 \quad \text{and bolt in double shear with shear stress} \quad \tau = \frac{P}{A_s} \quad \tau = \frac{P}{2 \frac{\pi}{4} d_b^2}$$

$$\tau_b = \frac{2}{\pi d_b^2} \left[(E \alpha \Delta T) \frac{\pi}{4} d_r^2 \right] \quad \tau_b = \frac{E \alpha \Delta T}{2} \left(\frac{d_r}{d_b} \right)^2$$

$$\tau_b = 45 \text{ MPa}$$

$$\Delta T = \frac{2 \tau_b}{E (1000) \alpha} \left(\frac{d_b}{d_r} \right)^2 \quad \Delta T = 24^\circ\text{C} \quad P = (E \alpha \Delta T) \frac{\pi}{4} d_r^2 \quad P = 10 \text{ kN}$$

$$\sigma_{\text{rod}} = \frac{P 1000}{\frac{\pi}{4} d_r^2} \quad \boxed{\sigma_{\text{rod}} = 57.6 \text{ MPa}}$$

(b) BEARING STRESSES

$$\text{BOLT AND CLEVIS} \quad \sigma_{bc} = \frac{\frac{P}{2}}{d_b t_c} \quad \boxed{\sigma_{bc} = 42.4 \text{ MPa}}$$

$$\text{WASHER AT WALL} \quad \sigma_{bw} = \frac{P}{\frac{\pi}{4}(d_w^2 - d_r^2)} \quad \boxed{\sigma_{bw} = 74.1 \text{ MPa}}$$

- (c) If the connection to the wall at B is changed to an end plate with two bolts (see Fig. b), what is the required diameter d_b of each bolt if temperature drop $\Delta T = 38^\circ\text{C}$ and the allowable bolt stress is 90 MPa?

Find force in rod due to temperature drop.

$$\Delta T = 38^\circ\text{C} \quad P = (E \alpha \Delta T) \frac{\pi}{4} d_r^2$$

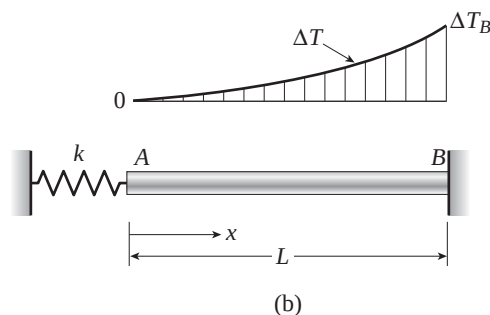
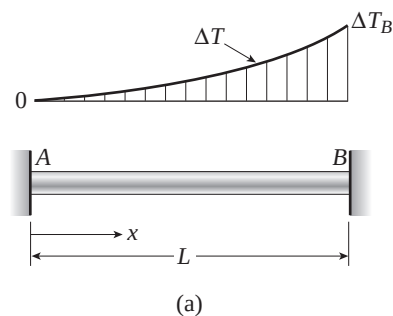
$$P = 200 \text{ GPa} \frac{\pi}{4} (15 \text{ mm})^2 [12(10^{-6})](38) = 16116 \text{ N} \quad P = 16.12 \text{ kN}$$

Each bolt carries one half of the force P :

$$d_b = \sqrt{\frac{\frac{16.12 \text{ kN}}{2}}{\frac{\pi}{4}(90 \text{ MPa})}} = 10.68 \text{ mm} \quad \boxed{d_b = 10.68 \text{ mm}}$$

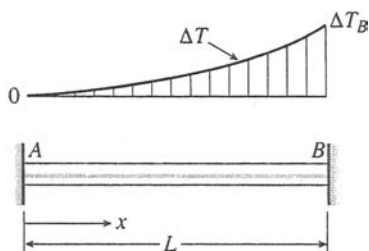
Problem 2.5-5 A bar AB of length L is held between rigid supports and heated nonuniformly in such a manner that the temperature increase ΔT at distance x from end A is given by the expression $\Delta T = \Delta T_B x^3/L^3$, where ΔT_B is the increase in temperature at end B of the bar (see figure part a).

- Derive a formula for the compressive stress σ_c in the bar. (Assume that the material has modulus of elasticity E and coefficient of thermal expansion α).
- Now modify the formula in (a) if the rigid support at A is replaced by an elastic support at A having a spring constant k (see figure part b). Assume that only bar AB is subject to the temperature increase.



Solution 2.5-5

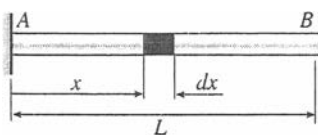
- (a) ONE DEGREE STATICALLY INDETERMINATE—USE SUPERPOSITION SELECT REACTION R_B AS THE REDUNDANT; FOLLOW PROCEDURE
Bar with nonuniform temperature change.



At distance x :

$$\Delta T = \Delta T_B \left(\frac{x^3}{L^3} \right)$$

REMOVE THE SUPPORT AT THE END B OF THE BAR:



Consider an element dx at a distance x from end A .

$d\delta$ = Elongation of element dx

$$d\delta = \alpha(\Delta T)dx = \alpha(\Delta T_B) \left(\frac{x^3}{L^3} \right) dx$$

$d\delta$ = elongation of bar

$$\delta = \int_0^L d\delta = \int_0^L \alpha(\Delta T_B) \left(\frac{x^3}{L^3} \right) dx = \frac{1}{4} \alpha(\Delta T_B) L$$

COMPRESSIVE FORCE P REQUIRED TO SHORTEN THE BAR BY THE AMOUNT δ

$$P = \frac{EA\delta}{L} = \frac{1}{4} EA\alpha(\Delta T_B)$$

COMPRESSIVE STRESS IN THE BAR

$$\sigma_c = \frac{P}{A} = \frac{E\alpha(\Delta T_B)}{4} \leftarrow$$

- (b) ONE DEGREE STATICALLY INDETERMINATE—USE SUPERPOSITION.

Select reaction R_B as the redundant then compute bar elongations due to ΔT and due to R_B

$$\delta_{B1} = \alpha\Delta T_B \frac{L}{4} \quad \text{due to temperature from above}$$

$$\delta_{B2} = R_B \left(\frac{1}{k} + \frac{L}{EA} \right)$$

Compatibility: solve for R_B : $\delta_{B1} + \delta_{B2} = 0$

$$R_B = \frac{-\left(\alpha\Delta T_B \frac{L}{4} \right)}{\left(\frac{1}{k} + \frac{L}{EA} \right)}$$

$$R_B = -\alpha\Delta T_B \left[\frac{EA}{4 \left(\frac{EA}{kL} + 1 \right)} \right]$$

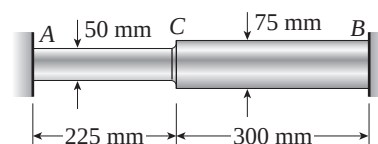
So compressive stress in bar is

$$\sigma_c = \frac{R_B}{A} \quad \sigma_c = \frac{E\alpha(\Delta T_B)}{4 \left(\frac{EA}{kL} + 1 \right)} \leftarrow$$

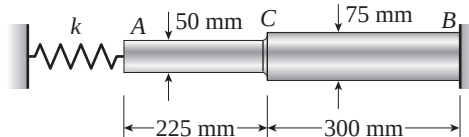
NOTE: σ_c in part (b) is the same as in part (a) if spring constant k goes to infinity.

Problem 2.5-6 A plastic bar ACB having two different solid circular cross sections is held between rigid supports as shown in the figure. The diameters in the left- and right-hand parts are 50 mm and 75 mm, respectively. The corresponding lengths are 225 mm and 300 mm. Also, the modulus of elasticity E is 6.0 GPa, and the coefficient of thermal expansion α is $100 \times 10^{-6}/^\circ\text{C}$. The bar is subjected to a uniform temperature increase of 30°C .

- (a) Calculate the following quantities: (1) the compressive force N in the bar; (2) the maximum compressive stress σ_c ; and (3) the displacement δ_C of point C .
 (b) Repeat (a) if the rigid support at A is replaced by an elastic support having spring constant $k = 50 \text{ MN/m}$ (see figure part b; assume that only the bar ACB is subject to the temperature increase).



(a)



(b)

Solution

NUMERICAL DATA

$$d_1 = 50 \text{ mm} \quad d_2 = 75 \text{ mm}$$

$$L_1 = 225 \text{ mm} \quad L_2 = 300 \text{ mm}$$

$$E = 6.0 \text{ GPa} \quad \alpha = 100 \times 10^{-6}/^\circ\text{C}$$

$$\Delta T = 30^\circ\text{C} \quad k = 50 \text{ MN/m}$$

- (a) COMPRESSIVE FORCE N , MAXIMUM COMPRESSIVE STRESS AND DISPLACEMENT OF PT. C

$$A_1 = \frac{\pi}{4} d_1^2 \quad A_2 = \frac{\pi}{4} d_2^2$$

One-degree statically indeterminate—use R_B as redundant

$$\delta_{B1} = \alpha \Delta T (L_1 + L_2)$$

$$\delta_{B2} = R_B \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} \right)$$

Compatibility: $\delta_{B1} = \delta_{B2}$, solve for R_B

$$R_B = \frac{\alpha \Delta T (L_1 + L_2)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2}} \quad N = R_B$$

$$N = 51.8 \text{ kN} \quad \leftarrow$$

Maximum compressive stress in AC since it has the smaller area ($A_1 < A_2$):

$$\sigma_{c\max} = \frac{N}{A_1} \quad \sigma_{c\max} = 26.4 \text{ MPa}$$

Displacement δ_C of point C = superposition of displacements in two released structures at C :

$$\delta_C = \alpha \Delta T (L_1) - R_B \frac{L_1}{EA_1}$$

$$\delta_C = -0.314 \text{ mm} \quad \leftarrow (-) \text{ sign means joint } C \text{ moves left}$$

- (b) COMPRESSIVE FORCE N , MAXIMUM COMPRESSIVE STRESS AND DISPLACEMENT OF PART C FOR ELASTIC SUPPORT CASE

Use R_B as redundant as in part (a):

$$\delta_{B1} = \alpha \Delta T (L_1 + L_2)$$

$$\delta_{B2} = R_B \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k} \right)$$

Now add effect of elastic support; equate δ_{B1} and δ_{B2} then solve for R_B :

$$R_B = \frac{\alpha \Delta T (L_1 + L_2)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k}} \quad N = R_B$$

$$N = 31.2 \text{ kN} \quad \leftarrow$$

$$\sigma_{c\max} = \frac{N}{A_1} \quad \sigma_{c\max} = 15.91 \text{ MPa} \quad \leftarrow$$

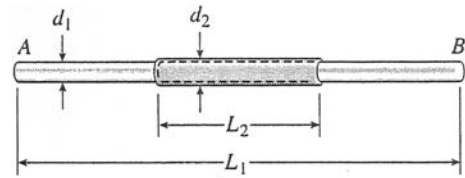
Superposition:

$$\delta_C = \alpha \Delta T (L_1) - R_B \left(\frac{L_1}{EA_1} + \frac{1}{k} \right)$$

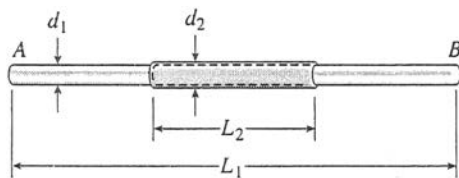
$$\delta_C = -0.546 \text{ mm} \quad \leftarrow (-) \text{ sign means joint } C \text{ moves left}$$

Problem 2.5-7 A circular steel rod AB (diameter $d_1 = 1.0$ in., length $L_1 = 3.0$ ft) has a bronze sleeve (outer diameter $d_2 = 1.25$ in., length $L_2 = 1.0$ ft) shrunk onto it so that the two parts are securely bonded (see figure).

Calculate the total elongation δ of the steel bar due to a temperature rise $\Delta T = 500^\circ\text{F}$. (Material properties are as follows: for steel, $E_s = 30 \times 10^6$ psi and $\alpha_s = 6.5 \times 10^{-6}/^\circ\text{F}$; for bronze, $E_b = 15 \times 10^6$ psi, and $\alpha_b = 11 \times 10^{-6}/^\circ\text{F}$.)



Solution 2.5-7 Steel rod with bronze sleeve



$$L_1 = 36 \text{ in.} \quad L_2 = 12 \text{ in.}$$

ELONGATION OF THE TWO OUTER PARTS OF THE BAR

$$\begin{aligned} \delta_1 &= \alpha_s (\Delta T) (L_1 - L_2) \\ &= (6.5 \times 10^{-6}/^\circ\text{F})(500^\circ\text{F})(36 \text{ in.} - 12 \text{ in.}) \\ &= 0.07800 \text{ in.} \end{aligned}$$

ELONGATION OF THE MIDDLE PART OF THE BAR

The steel rod and bronze sleeve lengthen the same amount, so they are in the same condition as the bolt and sleeve of Example 2-8. Thus, we can calculate the elongation from Eq. (2-23):

$$\delta_2 = \frac{(\alpha_s E_s A_s + \alpha_b E_b A_b)(\Delta T)L_2}{E_s A_s + E_b A_b}$$

SUBSTITUTE NUMERICAL VALUES

$$\alpha_s = 6.5 \times 10^{-6}/^\circ\text{F} \quad \alpha_b = 11 \times 10^{-6}/^\circ\text{F}$$

$$E_s = 30 \times 10^6 \text{ psi} \quad E_b = 15 \times 10^6 \text{ psi}$$

$$d_1 = 1.0 \text{ in.}$$

$$A_s = \frac{\pi}{4} d_1^2 = 0.78540 \text{ in.}^2$$

$$d_2 = 1.25 \text{ in.}$$

$$A_b = \frac{\pi}{4} (d_2^2 - d_1^2) = 0.44179 \text{ in.}^2$$

$$\Delta T = 500^\circ\text{F} \quad L_2 = 12.0 \text{ in.}$$

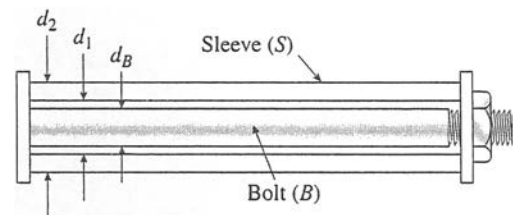
$$\delta_2 = 0.04493 \text{ in.}$$

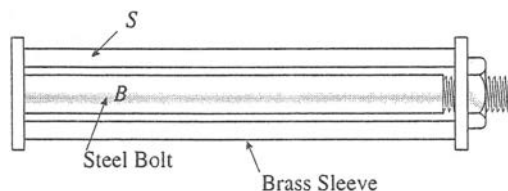
TOTAL ELONGATION

$$\delta = \delta_1 + \delta_2 = 0.123 \text{ in.} \quad \leftarrow$$

Problem 2.5-8 A brass sleeve S is fitted over a steel bolt B (see figure), and the nut is tightened until it is just snug. The bolt has a diameter $d_B = 25$ mm, and the sleeve has inside and outside diameters $d_1 = 26$ mm and $d_2 = 36$ mm, respectively.

Calculate the temperature rise ΔT that is required to produce a compressive stress of 25 MPa in the sleeve. (Use material properties as follows: for the sleeve, $\alpha_s = 21 \times 10^{-6}/^\circ\text{C}$ and $E_s = 100$ GPa; for the bolt, $\alpha_b = 10 \times 10^{-6}/^\circ\text{C}$ and $E_b = 200$ GPa.) (Suggestion: Use the results of Example 2-8.)



Solution 2.5-8 Brass sleeve fitted over a Steel bolt

Subscript S means “sleeve”.

Subscript B means “bolt”.

Use the results of Example 2-8.

σ_S = compressive force in sleeve

EQUATION (2-22a):

$$\sigma_S = \frac{(\alpha_S - \alpha_B)(\Delta T)E_S E_B A_B}{E_S A_S + E_B A_B} \text{ (Compression)}$$

SOLVE FOR ΔT :

$$\Delta T = \frac{\sigma_S(E_S A_S + E_B A_B)}{(\alpha_S - \alpha_B)E_S E_B A_B}$$

or

$$\Delta T = \frac{\sigma_S}{E_S(\alpha_S - \alpha_B)} \left(1 + \frac{E_S A_S}{E_B A_B} \right) \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\sigma_S = 25 \text{ MPa}$$

$$d_2 = 36 \text{ mm} \quad d_1 = 26 \text{ mm} \quad d_B = 25 \text{ mm}$$

$$E_S = 100 \text{ GPa} \quad E_B = 200 \text{ GPa}$$

$$\alpha_S = 21 \times 10^{-6}/^\circ\text{C} \quad \alpha_B = 10 \times 10^{-6}/^\circ\text{C}$$

$$A_S = \frac{\pi}{4} (d_2^2 - d_1^2) = \frac{\pi}{4} (620 \text{ mm}^2)$$

$$A_B = \frac{\pi}{4} (d_B)^2 = \frac{\pi}{4} (625 \text{ mm}^2) \quad 1 + \frac{E_S A_S}{E_B A_B} = 1.496$$

$$\Delta T = \frac{25 \text{ MPa} (1.496)}{(100 \text{ GPa})(11 \times 10^{-6}/^\circ\text{C})}$$

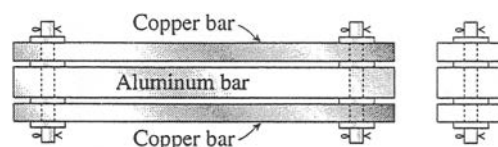
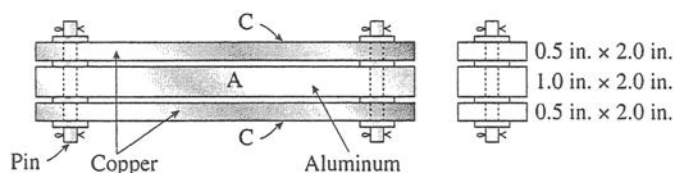
$$\Delta T = 34^\circ\text{C} \quad \leftarrow$$

(Increase in temperature)

Problem 2.5-9 Rectangular bars of copper and aluminum are held by pins at their ends, as shown in the figure. Thin spacers provide a separation between the bars. The copper bars have cross-sectional dimensions 0.5 in. $\times$ 2.0 in., and the aluminum bar has dimensions 1.0 in. $\times$ 2.0 in.

Determine the shear stress in the 7/16 in. diameter pins if the temperature is raised by 100°F. (For copper, $E_c = 18,000$ ksi and $\alpha_c = 9.5 \times 10^{-6}/^\circ\text{F}$; for aluminum, $E_a = 10,000$ ksi, and $\alpha_a = 13 \times 10^{-6}/^\circ\text{F}$.)

Suggestion: Use the results of Example 2-8.

**Solution 2.5-9 Rectangular bars held by pins**

$$\text{Diameter of pin: } d_P = \frac{7}{16} \text{ in.} = 0.4375 \text{ in.}$$

$$\text{Area of pin: } A_P = \frac{\pi}{4} d_P^2 = 0.15033 \text{ in.}^2$$

$$\text{Area of two copper bars: } A_c = 2.0 \text{ in.}^2$$

$$\text{Area of aluminum bar: } A_a = 2.0 \text{ in.}^2$$

$$\Delta T = 100^\circ\text{F}$$

Copper: $E_c = 18,000 \text{ ksi}$ $\alpha_c = 9.5 \times 10^{-6}/^\circ\text{F}$

Aluminum: $E_a = 10,000 \text{ ksi}$

$\alpha_a = 13 \times 10^{-6}/^\circ\text{F}$

Use the results of Example 2-8.

Find the forces P_a and P_c in the aluminum bar and copper bar, respectively, from Eq. (2-21).

Replace the subscript “S” in that equation by “a” (for aluminum) and replace the subscript “B” by “c” (for copper), because α for aluminum is larger than α for copper.

$$P_a = P_c = \frac{(\alpha_a - \alpha_c)(\Delta T)E_a A_a E_c A_c}{E_a A_a + E_c A_c}$$

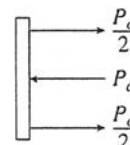
Note that P_a is the compressive force in the aluminum bar and P_c is the combined tensile force in the two copper bars.

$$P_a = P_c = \frac{(\alpha_a - \alpha_c)(\Delta T)E_c A_c}{1 + \frac{E_c A_c}{E_a A_a}}$$

SUBSTITUTE NUMERICAL VALUES:

$$P_a = P_c = \frac{(3.5 \times 10^{-6}/^\circ\text{F})(100^\circ\text{F})(18,000 \text{ ksi})(2 \text{ in.}^2)}{1 + \left(\frac{18}{10}\right)\left(\frac{2.0}{2.0}\right)} = 4,500 \text{ lb}$$

FREE-BODY DIAGRAM OF PIN AT THE LEFT END



V = shear force in pin

$$= P_c/2$$

$$= 2,250 \text{ lb}$$

τ = average shear stress on cross section of pin

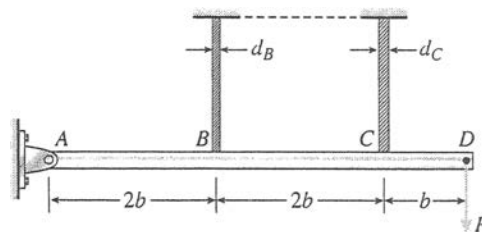
$$\tau = \frac{V}{A_p} = \frac{2,250 \text{ lb}}{0.15033 \text{ in.}^2}$$

$$\tau = 15.0 \text{ ksi} \quad \leftarrow$$

Problem 2.5-10 A rigid bar $ABCD$ is pinned at end A and supported by two cables at points B and C (see figure). The cable at B has nominal diameter $d_B = 12 \text{ mm}$ and the cable at C has nominal diameter $d_C = 20 \text{ mm}$. A load P acts at end D of the bar.

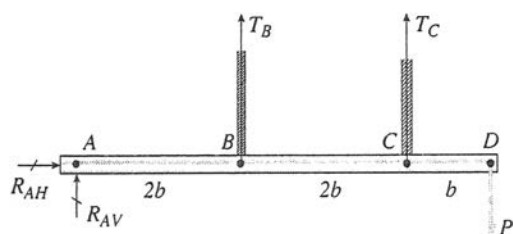
What is the allowable load P if the temperature rises by 60°C and each cable is required to have a factor of safety of at least 5 against its ultimate load?

(Note: The cables have effective modulus of elasticity $E = 140 \text{ GPa}$ and coefficient of thermal expansion $\alpha = 12 \times 10^{-6}/^\circ\text{C}$. Other properties of the cables can be found in Table 2-1, Section 2.2.)



Solution 2.5-10 Rigid bar supported by two cables

FREE-BODY DIAGRAM OF BAR $ABCD$



T_B = force in cable B T_C = force in cable C

$d_B = 12 \text{ mm}$ $d_C = 20 \text{ mm}$

From Table 2-1:

$$A_B = 76.7 \text{ mm}^2 \quad E = 140 \text{ GPa}$$

$$\Delta T = 60^\circ\text{C} \quad A_C = 173 \text{ mm}^2$$

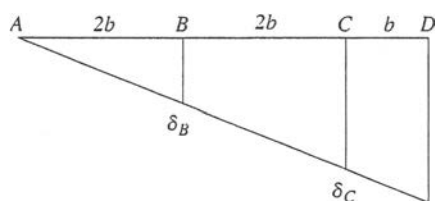
$$\alpha = 12 \times 10^{-6}/^\circ\text{C}$$

EQUATION OF EQUILIBRIUM

$$\Sigma M_A = 0 \quad \curvearrowright \quad T_B(2b) + T_C(4b) - P(5b) = 0$$

$$\text{or } 2T_B + 4T_C = 5P \quad (\text{Eq. 1})$$

DISPLACEMENT DIAGRAM



COMPATIBILITY:

$$\delta_C = 2\delta_B \quad (\text{Eq. 2})$$

FORCE-DISPLACEMENT AND TEMPERATURE-DISPLACEMENT RELATIONS

$$\delta_B = \frac{T_B L}{EA_B} + \alpha(\Delta T)L \quad (\text{Eq. 3})$$

$$\delta_C = \frac{T_C L}{EA_C} + \alpha(\Delta T)L \quad (\text{Eq. 4})$$

SUBSTITUTE EQS. (3) AND (4) INTO EQ. (2):

$$\frac{T_C L}{EA_C} + \alpha(\Delta T)L = \frac{2T_B L}{EA_B} + 2\alpha(\Delta T)L$$

or

$$2T_B A_C - T_C A_B = -E\alpha(\Delta T)A_B A_C \quad (\text{Eq. 5})$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (5):

$$T_B(346) - T_C(76.7) = -1,338,000 \quad (\text{Eq. 6})$$

in which T_B and T_C have units of newtons.

SOLVE SIMULTANEOUSLY EQS. (1) AND (6):

$$T_B = 0.2494 P - 3,480 \quad (\text{Eq. 7})$$

$$T_C = 1.1253 P + 1,740 \quad (\text{Eq. 8})$$

in which P has units of newtons.SOLVE EQS. (7) AND (8) FOR THE LOAD P :

$$P_B = 4.0096 T_B + 13,953 \quad (\text{Eq. 9})$$

$$P_C = 0.8887 T_C - 1,546 \quad (\text{Eq. 10})$$

ALLOWABLE LOADS

From Table 2-1:

$$(T_B)_{\text{ULT}} = 102,000 \text{ N} \quad (T_C)_{\text{ULT}} = 231,000 \text{ N}$$

Factor of safety = 5

$$(T_B)_{\text{allow}} = 20,400 \text{ N} \quad (T_C)_{\text{allow}} = 46,200 \text{ N}$$

$$\text{From Eq. (9): } P_B = (4.0096)(20,400 \text{ N}) + 13,953 \text{ N} = 95,700 \text{ N}$$

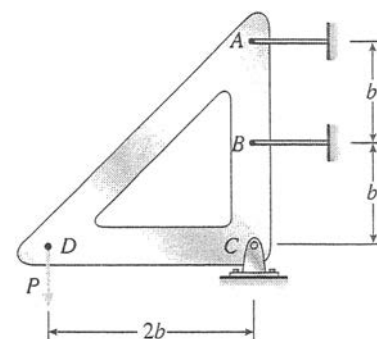
$$\text{From Eq. (10): } P_C = (0.8887)(46,200 \text{ N}) - 1,546 \text{ N} = 39,500 \text{ N}$$

Cable C governs.

$$P_{\text{allow}} = 39.5 \text{ kN} \quad \leftarrow$$

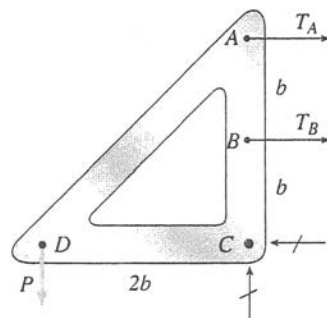
Problem 2.5-11 A rigid triangular frame is pivoted at C and held by two identical horizontal wires at points A and B (see figure). Each wire has axial rigidity $EA = 120 \text{ k}$ and coefficient of thermal expansion $\alpha = 12.5 \times 10^{-6}/^\circ\text{F}$.

- If a vertical load $P = 500 \text{ lb}$ acts at point D , what are the tensile forces T_A and T_B in the wires at A and B , respectively?
- If, while the load P is acting, both wires have their temperatures raised by 180°F , what are the forces T_A and T_B ?
- What further increase in temperature will cause the wire at B to become slack?



Solution 2.5-11 Triangular frame held by two wires

FREE-BODY DIAGRAM OF FRAME

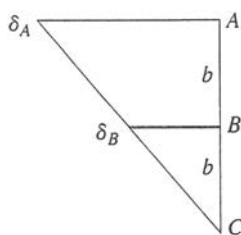


EQUATION OF EQUILIBRIUM

$$\Sigma M_C = 0 \quad \curvearrowright$$

$$P(2b) - T_A(2b) - T_B(b) = 0 \quad \text{or} \quad 2T_A + T_B = 2P \quad (\text{Eq. 1})$$

DISPLACEMENT DIAGRAM



EQUATION OF COMPATIBILITY

$$\delta_A = 2\delta_B \quad (\text{Eq. 2})$$

(a) LOAD P ONLY

Force-displacement relations:

$$\delta_A = \frac{T_AL}{EA} \quad \delta_B = \frac{T_BL}{EA} \quad (\text{Eq. 3, 4})$$

 $(L = \text{length of wires at A and B.})$

Substitute (3) and (4) into Eq. (2):

$$\frac{T_AL}{EA} = \frac{2T_BL}{EA} \quad (\text{Eq. 5})$$

$$\text{or} \quad T_A = 2T_B$$

Solve simultaneously Eqs. (1) and (5):

$$T_A = \frac{4P}{5} \quad T_B = \frac{2P}{5} \quad (\text{Eqs. 6, 7})$$

Numerical values:

$$P = 500 \text{ lb}$$

$$\therefore T_A = 400 \text{ lb} \quad T_B = 200 \text{ lb} \quad \leftarrow$$

(b) LOAD P AND TEMPERATURE INCREASE ΔT

Force-displacement and temperature-displacement relations:

$$\delta_A = \frac{T_AL}{EA} + \alpha(\Delta T)L \quad (\text{Eq. 8})$$

$$\delta_B = \frac{T_BL}{EA} + \alpha(\Delta T)L \quad (\text{Eq. 9})$$

Substitute (8) and (9) into Eq. (2):

$$\frac{T_AL}{EA} + \alpha(\Delta T)L = \frac{2T_BL}{EA} + 2\alpha(\Delta T)L$$

$$\text{or} \quad T_A - 2T_B = EA\alpha(\Delta T) \quad (\text{Eq. 10})$$

Solve simultaneously Eqs. (1) and (10):

$$T_A = \frac{1}{5}[4P + EA\alpha(\Delta T)] \quad (\text{Eq. 11})$$

$$T_B = \frac{2}{5}[P - EA\alpha(\Delta T)] \quad (\text{Eq. 12})$$

Substitute numerical values:

$$P = 500 \text{ lb} \quad EA = 120,000 \text{ lb}$$

$$\Delta T = 180^\circ\text{F}$$

$$\alpha = 12.5 \times 10^{-6}/^\circ\text{F}$$

$$T_A = \frac{1}{5}(2000 \text{ lb} + 270 \text{ lb}) = 454 \text{ lb} \quad \leftarrow$$

$$T_B = \frac{2}{5}(500 \text{ lb} - 270 \text{ lb}) = 92 \text{ lb} \quad \leftarrow$$

(c) WIRE B BECOMES SLACKSet $T_B = 0$ in Eq. (12):

$$P = EA\alpha(\Delta T)$$

or

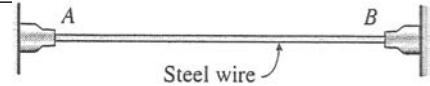
$$\Delta T = \frac{P}{EA\alpha} = \frac{500 \text{ lb}}{(120,000 \text{ lb})(12.5 \times 10^{-6}/^\circ\text{F})} = 333.3^\circ\text{F}$$

Further increase in temperature:

$$\Delta T = 333.3^\circ\text{F} - 180^\circ\text{F} = 153^\circ\text{F} \quad \leftarrow$$

Misfits and Prestrains

Problem 2.5-12 A steel wire AB is stretched between rigid supports (see figure). The initial prestress in the wire is 42 MPa when the temperature is 20°C .



- What is the stress σ in the wire when the temperature drops to 0°C ?
- At what temperature T will the stress in the wire become zero?
(Assume $\alpha = 14 \times 10^{-6}/^\circ\text{C}$ and $E = 200 \text{ GPa}$.)

Solution 2.5-12 Steel wire with initial prestress



Initial prestress: $\sigma_1 = 42 \text{ MPa}$

Initial temperature: $T_1 = 20^\circ\text{C}$

$E = 200 \text{ GPa}$

$\alpha = 14 \times 10^{-6}/^\circ\text{C}$

- STRESS σ WHEN TEMPERATURE DROPS TO 0°C

$$T_2 = 0^\circ\text{C} \quad \Delta T = 20^\circ\text{C}$$

NOTE: Positive ΔT means a *decrease* in temperature and an *increase* in the stress in the wire.

Negative ΔT means an *increase* in temperature and a *decrease* in the stress.

Stress σ equals the initial stress σ_1 plus the additional stress σ_2 due to the temperature drop.

$$\sigma_2 = E\alpha(\Delta T)$$

$$\begin{aligned} \sigma &= \sigma_1 + \sigma_2 = \sigma_1 + E\alpha(\Delta T) \\ &= 42 \text{ MPa} + (200 \text{ GPa})(14 \times 10^{-6}/^\circ\text{C})(20^\circ\text{C}) \\ &= 42 \text{ MPa} + 56 \text{ MPa} = 98 \text{ MPa} \quad \leftarrow \end{aligned}$$

- TEMPERATURE WHEN STRESS EQUALS ZERO

$$\sigma = \sigma_1 + \sigma_2 = 0 \quad \sigma_1 + E\alpha(\Delta T) = 0$$

$$\Delta T = -\frac{\sigma_1}{E\alpha}$$

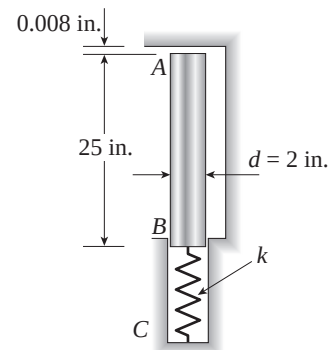
(Negative means increase in temp.)

$$\Delta T = -\frac{42 \text{ MPa}}{(200 \text{ GPa})(14 \times 10^{-6}/^\circ\text{C})} = -15^\circ\text{C}$$

$$T = 20^\circ\text{C} + 15^\circ\text{C} = 35^\circ\text{C} \quad \leftarrow$$

Problem 2.5-13 A copper bar AB of length 25 in. and diameter 2 in. is placed in position at room temperature with a gap of 0.008 in. between end A and a rigid restraint (see figure). The bar is supported at end B by an elastic spring with spring constant $k = 1.2 \times 10^6 \text{ lb/in.}$

- Calculate the axial compressive stress σ_c in the bar if the temperature rises 50°F . (For copper, use $\alpha = 9.6 \times 10^{-6}/^\circ\text{F}$ and $E = 16 \times 10^6 \text{ psi}$.)
- What is the force in the spring? (Neglect gravity effects.)
- Repeat (a) if $k \rightarrow \infty$.



Solution 2.5-13

Numerical data:

$$L = 25 \text{ in.} \quad d = 2 \text{ in.} \quad \delta = 0.008 \text{ in.}$$

$$k = 1.2 \times (10^6) \text{ lb/in.} \quad E = 16 \times (10^6) \text{ psi}$$

$$\alpha = 9.6 \times (10^{-6})/^{\circ}\text{F} \quad \Delta T = 50^{\circ}\text{F}$$

$$A = \frac{\pi}{4}d^2 \quad A = 3.14159 \text{ in.}^2$$

(a) ONE-DEGREE STATICALLY INDETERMINATE IF GAP CLOSSES

$$\Delta = \alpha \Delta T L \quad \Delta = 0.012 \text{ in.} \quad < \text{exceeds gap}$$

Select R_A as redundant and do superposition analysis:

$$\delta_{A1} = \Delta \quad \delta_{A2} = R_A \left(\frac{L}{EA} + \frac{1}{k} \right)$$

$$\text{Compatibility: } \delta_{A1} + \delta_{A2} = \delta \quad \delta_{A2} = \delta - \delta_{A1}$$

$$R_A = \frac{\delta - \Delta}{\frac{L}{EA} + \frac{1}{k}} \quad R_A = -3006 \text{ lb}$$

Compressive stress in bar:

$$\sigma = \frac{R_A}{A} \quad \sigma = -957 \text{ psi}$$

(b) FORCE IN SPRING $F_k = R_C$

STATICS

$$R_A + R_C = 0$$

$$R_C = -R_A$$

$$R_C = 3006 \text{ lb} \quad \leftarrow$$

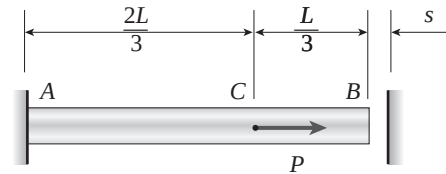
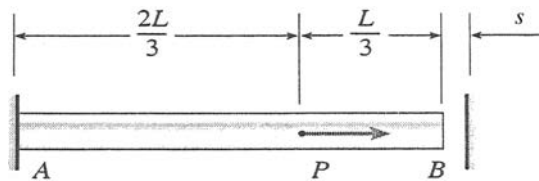
(c) FIND COMPRESSIVE STRESS IN BAR IF k GOES TO INFINITY FROM EXPRESSION FOR R_A ABOVE, $1/k$ GOES TO ZERO

$$R_A = \frac{\delta - \Delta}{\frac{L}{EA}} \quad R_A = -8042 \text{ lb} \quad \sigma = \frac{R_A}{A}$$

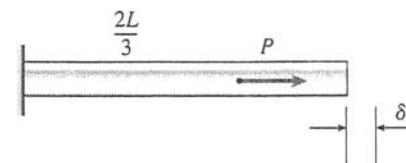
$$\sigma = -2560 \text{ psi} \quad \leftarrow$$

Problem 2.5-14 A bar AB having length L and axial rigidity EA is fixed at end A (see figure). At the other end a small gap of dimension s exists between the end of the bar and a rigid surface. A load P acts on the bar at point C , which is two-thirds of the length from the fixed end.

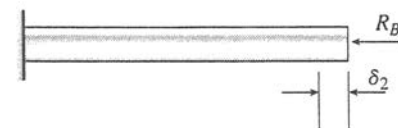
If the support reactions produced by the load P are to be equal in magnitude, what should be the size s of the gap?

**Solution 2.5-14 Bar with a gap (load P)** L = length of bar s = size of gap EA = axial rigidityReactions must be equal; find s .

FORCE-DISPLACEMENT RELATIONS



$$\delta_1 = \frac{P\left(\frac{2L}{3}\right)}{EA}$$



$$\delta_2 = \frac{R_B L}{EA}$$

COMPATIBILITY EQUATION

$$\delta_1 - \delta_2 = s \quad \text{or}$$

$$\frac{2PL}{3EA} - \frac{R_B L}{EA} = s \quad (\text{Eq. 1})$$

EQUILIBRIUM EQUATION

 R_A = reaction at end A (to the left) R_B = reaction at end B (to the left)

$$P = R_A + R_B$$

Reactions must be equal.

$$\therefore R_A = R_B \quad P = 2R_B \quad R_B = \frac{P}{2}$$

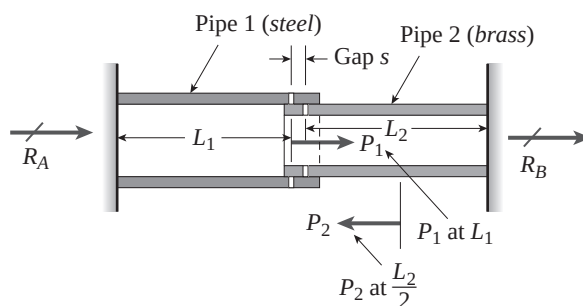
Substitute for R_B in Eq. (1):

$$\frac{2PL}{3EA} - \frac{PL}{2EA} = s \quad \text{or} \quad s = \frac{PL}{6EA} \quad \leftarrow$$

NOTE: The gap closes when the load reaches the value $P/4$. When the load reaches the value P , equal to $6EA s/L$, the reactions are equal ($R_A = R_B = P/2$). When the load is between $P/4$ and P , R_A is greater than R_B . If the load exceeds P , R_B is greater than R_A .

Problem 2.5-15 Pipe 2 has been inserted snugly into Pipe 1, but the holes for a connecting pin do not line up: there is a gap s . The user decides to apply *either* force P_1 to Pipe 1 *or* force P_2 to Pipe 2, whichever is smaller. Determine the following using the numerical properties in the box.

- If only P_1 is applied, find P_1 (kips) required to close gap s ; if a pin is then inserted and P_1 removed, what are reaction forces R_A and R_B for this load case?
- If only P_2 is applied, find P_2 (kips) required to close gap s ; if a pin is inserted and P_2 removed, what are reaction forces R_A and R_B for this load case?
- What is the maximum *shear* stress in the pipes, for the loads in parts (a) and (b)?
- If a temperature increase ΔT is to be applied to the entire structure to close gap s (*instead of applying forces P_1 and P_2*), find the ΔT required to close the gap. If a pin is inserted after the gap has closed, what are reaction forces R_A and R_B for this case?
- Finally, if the structure (with pin inserted) then cools to the *original* ambient temperature, what are reaction forces R_A and R_B ?

**Numerical properties**

$$E_1 = 30,000 \text{ ksi}, E_2 = 14,000 \text{ ksi}$$

$$\alpha_1 = 6.5 \times 10^{-6}/^\circ\text{F}, \alpha_2 = 11 \times 10^{-6}/^\circ\text{F}$$

$$\text{Gap } s = 0.05 \text{ in.}$$

$$L_1 = 56 \text{ in.}, d_1 = 6 \text{ in.}, t_1 = 0.5 \text{ in.}, A_1 = 8.64 \text{ in.}^2$$

$$L_2 = 36 \text{ in.}, d_2 = 5 \text{ in.}, t_2 = 0.25 \text{ in.}, A_2 = 3.73 \text{ in.}^2$$

Solution 2.5-15

- (a) FIND REACTIONS AT A AND B FOR APPLIED FORCE P_1
First compute P_1 , required to close gap:

$$P_1 = \frac{E_1 A_1}{L_1} s \quad P_1 = 231.4 \text{ k} \quad \leftarrow$$

Statically indeterminate analysis with R_B as the redundant:

$$\delta_{B1} = -s \quad \delta_{B2} = R_B \left(\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2} \right)$$

$$\text{Compatibility: } \delta_{B1} + \delta_{B2} = 0$$

$$R_B = \frac{s}{\left(\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2} \right)} \quad R_B = 55.2 \text{ k} \quad \leftarrow$$

$$R_A = -R_B \quad \leftarrow$$

- (b) FIND REACTIONS AT A AND B FOR APPLIED FORCE P_2

$$P_2 = \frac{E_2 A_2}{L_2} s \quad P_2 = 145.1 \text{ k} \quad \leftarrow$$

Analysis after removing P_2 is same as in part (a), so reaction forces are the same

- (c) MAXIMUM SHEAR STRESS IN PIPE 1 OR 2 WHEN EITHER P_1 OR P_2

$$\text{IS APPLIED } \tau_{\max a} = \frac{P_1}{2A_1} \quad \tau_{\max a} = 13.39 \text{ ksi} \quad \leftarrow$$

$$\tau_{\max b} = \frac{P_2}{2A_2} \quad \tau_{\max b} = 19.44 \text{ ksi} \quad \leftarrow$$

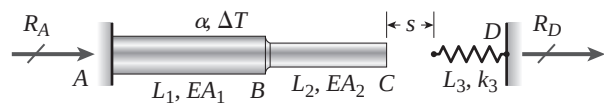
- (d) REQUIRED ΔT AND REACTIONS AT A AND B

$$\Delta T_{\text{reqd}} = \frac{s}{\alpha_1 L_1 + \alpha_2 L_2} \quad \Delta T_{\text{reqd}} = 65.8^\circ\text{F} \quad \leftarrow$$

If pin is inserted but temperature remains at ΔT above ambient temperature, reactions are zero.

- (e) IF TEMPERATURE RETURNS TO ORIGINAL AMBIENT TEMPERATURE, FIND REACTIONS AT A AND B
statically indeterminate analysis with R_B as the redundant Compatibility: $\delta_{B1} + \delta_{B2} = 0$
Analysis is the same as in parts (a) and (b) above since gap s is the same, so reactions are the same.

Problem 2.5-16 A nonprismatic bar ABC made up of segments AB (length L_1 , cross-sectional area A_1) and BC (length L_2 , cross-sectional area A_2) is fixed at end A and free at end C (see figure). The modulus of elasticity of the bar is E . A small gap of dimension s exists between the end of the bar and an elastic spring of length L_3 and spring constant k_3 . If bar ABC only (not the spring) is subjected to temperature increase ΔT determine the following.



- (a) Write an expression for reaction forces R_A and R_D if the elongation of ABC exceeds gap length s .
(b) Find expressions for the displacements of points B and C if the elongation of ABC exceeds gap length s .

Solution 2.5-16

With gap s closed due to ΔT , structure is one-degree statically-indeterminate; select internal force (Q) at juncture of bar and spring as the redundant. Use superposition of two released structures in the solution.

δ_{rel1} = relative displacement between end of bar at C and end of spring due to ΔT

$$\delta_{\text{rel1}} = \alpha \Delta T (L_1 + L_2)$$

δ_{rel1} is greater than gap length s

δ_{rel2} = relative displacement between ends of bar and spring due to pair of forces Q , one on end of bar at C and the other on end of spring

$$\delta_{\text{rel2}} = Q \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} \right) + \frac{Q}{k_3}$$

$$\delta_{\text{rel2}} = Q \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3} \right)$$

$$\text{Compatibility: } \delta_{\text{rel1}} + \delta_{\text{rel2}} = s \quad \delta_{\text{rel2}} = s - \delta_{\text{rel1}}$$

$$\delta_{\text{rel2}} = s - \alpha \Delta T (L_1 + L_2)$$

$$Q = \frac{s - \alpha \Delta T (L_1 + L_2)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}}$$

$$Q = \frac{EA_1 A_2 k_3}{L_1 A_2 k_3 + L_2 A_1 k_3 + EA_1 A_2} [s - \alpha \Delta T (L_1 + L_2)]$$

- (a) REACTIONS AT A AND D

$$\text{Statics: } R_A = -Q \quad R_D = Q$$

$$R_A = \frac{-s + \alpha \Delta T (L_1 + L_2)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}} \quad \leftarrow$$

$$R_D = -R_A \quad \leftarrow$$

(b) DISPLACEMENTS AT B AND C

Use superposition of displacements in the two released structures:

$$\delta_B = \alpha \Delta T (L_1) - R_A \left(\frac{L_1}{EA_1} \right) \quad \leftarrow$$

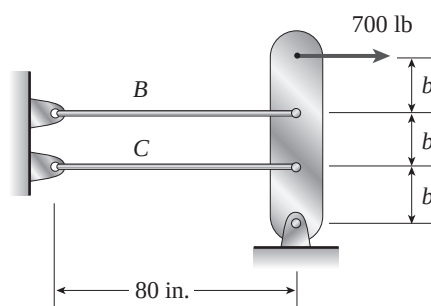
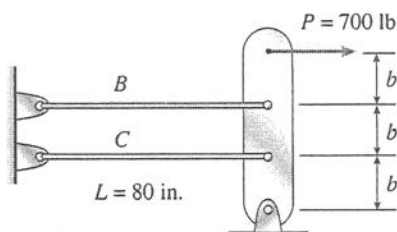
$$\delta_B = \alpha \Delta T (L_1) - \frac{[-s + \alpha \Delta T (L_1 + L_2)] \left(\frac{L_1}{EA_1} \right)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}}$$

$$\delta_C = \alpha \Delta T (L_1 + L_2) - R_A \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} \right) \quad \leftarrow$$

$$\delta_C = \alpha \Delta T (L_1 + L_2) - \frac{[-s + \alpha \Delta T (L_1 + L_2)] \left(\frac{L_1}{EA_1} + \frac{L_2}{EA_2} \right)}{\frac{L_1}{EA_1} + \frac{L_2}{EA_2} + \frac{1}{k_3}}$$

Problem 2.5-17 Wires B and C are attached to a support at the left-hand end and to a pin-supported rigid bar at the right-hand end (see figure). Each wire has cross-sectional area $A = 0.03 \text{ in.}^2$ and modulus of elasticity $E = 30 \times 10^6 \text{ psi}$. When the bar is in a vertical position, the length of each wire is $L = 80 \text{ in.}$ However, before being attached to the bar, the length of wire B was 79.98 in. and of wire C was 79.95 in.

Find the tensile forces T_B and T_C in the wires under the action of a force $P = 700 \text{ lb}$ acting at the upper end of the bar.

**Solution 2.5-17** Wires B and C attached to a bar

$$P = 700 \text{ lb}$$

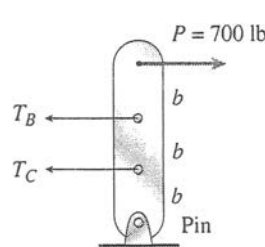
$$A = 0.03 \text{ in.}^2$$

$$E = 30 \times 10^6 \text{ psi}$$

$$L_B = 79.98 \text{ in.}$$

$$L_C = 79.95 \text{ in.}$$

EQUILIBRIUM EQUATION



$$\sum M_{\text{pin}} = 0 \quad \leftarrow$$

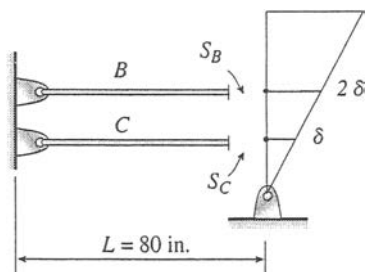
$$T_C(b) + T_B(2b) = P(3b)$$

$$2T_B + T_C = 3P \quad (\text{Eq. 1})$$

DISPLACEMENT DIAGRAM

$$S_B = 80 \text{ in.} - L_B = 0.02 \text{ in.}$$

$$S_C = 80 \text{ in.} - L_C = 0.05 \text{ in.}$$



Elongation of wires:

$$\delta_B = S_B + 2\delta \quad (\text{Eq. 2})$$

$$\delta_C = S_C + \delta \quad (\text{Eq. 3})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_B = \frac{T_B L}{EA} \quad \delta_C = \frac{T_C L}{EA} \quad (\text{Eqs. 4, 5})$$

SOLUTION OF EQUATIONS

Combine Eqs. (2) and (4):

$$\frac{T_B L}{EA} = S_B + 2\delta \quad (\text{Eq. 6})$$

Combine Eqs. (3) and (5):

$$\frac{T_C L}{EA} = S_C + \delta \quad (\text{Eq. 7})$$

Eliminate δ between Eqs. (6) and (7):

$$T_B - 2T_C = \frac{EAS_B}{L} - \frac{2EAS_C}{L} \quad (\text{Eq. 8})$$

Solve simultaneously Eqs. (1) and (8):

$$T_B = \frac{6P}{5} + \frac{EAS_B}{5L} - \frac{2EAS_C}{5L} \quad \leftarrow$$

$$T_C = \frac{3P}{5} - \frac{2EAS_B}{5L} + \frac{4EAS_C}{5L} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{EA}{5L} = 2250 \text{ lb/in.}$$

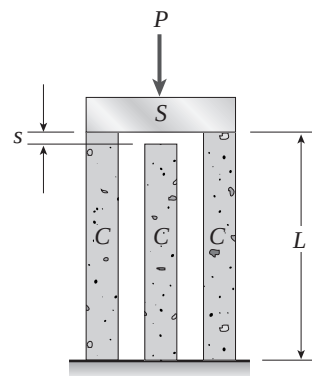
$$T_B = 840 \text{ lb} + 45 \text{ lb} - 225 \text{ lb} = 660 \text{ lb} \quad \leftarrow$$

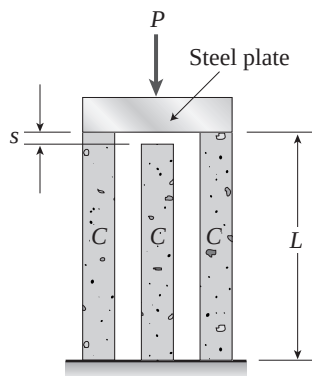
$$T_C = 420 \text{ lb} - 90 \text{ lb} + 450 \text{ lb} = 780 \text{ lb} \quad \leftarrow$$

(Both forces are positive, which means tension, as required for wires.)

Problem 2.5-18 A rigid steel plate is supported by three posts of high-strength concrete each having an effective cross-sectional area $A = 40,000 \text{ mm}^2$ and length $L = 2 \text{ m}$ (see figure). Before the load P is applied, the middle post is shorter than the others by an amount $s = 1.0 \text{ mm}$.

Determine the maximum allowable load P_{allow} if the allowable compressive stress in the concrete is $\sigma_{\text{allow}} = 20 \text{ MPa}$. (Use $E = 30 \text{ GPa}$ for concrete.)



Solution 2.5-18 Plate supported by three posts

s = size of gap = 1.0 mm

L = length of posts = 2.0 m

$A = 40,000 \text{ mm}^2$

$\sigma_{\text{allow}} = 20 \text{ MPa}$

$E = 30 \text{ GPa}$

C = concrete post

DOES THE GAP CLOSE?

Stress in the two outer posts when the gap is just closed:

$$\begin{aligned}\sigma &= E\varepsilon = E\left(\frac{s}{L}\right) = (30 \text{ GPa})\left(\frac{1.0 \text{ mm}}{2.0 \text{ m}}\right) \\ &= 15 \text{ MPa}\end{aligned}$$

Since this stress is less than the allowable stress, the allowable force P will close the gap.

EQUILIBRIUM EQUATION

$$2P_1 + P_2 = P \quad (\text{Eq. 1})$$

COMPATIBILITY EQUATION

δ_1 = shortening of outer posts

δ_2 = shortening of inner post

$$\delta_1 = \delta_2 + s \quad (\text{Eq. 2})$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_1 = \frac{P_1 L}{EA} \quad \delta_2 = \frac{P_2 L}{EA} \quad (\text{Eqs. 3, 4})$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_1 L}{EA} = \frac{P_2 L}{EA} + s \quad \text{or} \quad P_1 - P_2 = \frac{EAs}{L} \quad (\text{Eq. 5})$$

Solve simultaneously Eqs. (1) and (5):

$$P = 3P_1 - \frac{EAs}{L}$$

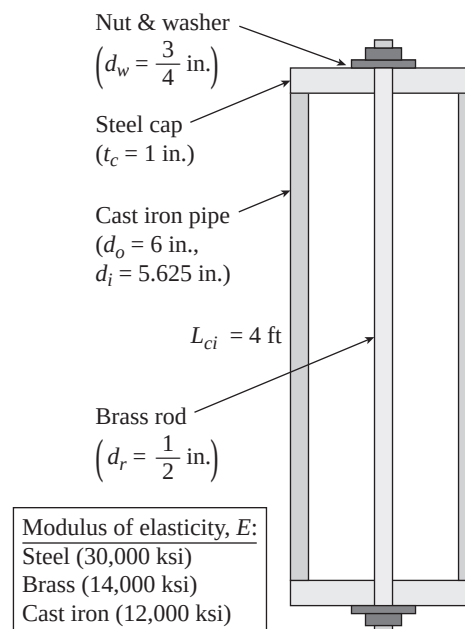
By inspection, we know that P_1 is larger than P_2 .

Therefore, P_1 will control and will be equal to $\sigma_{\text{allow}} A$.

$$\begin{aligned}P_{\text{allow}} &= 3\sigma_{\text{allow}} A - \frac{EAs}{L} \\ &= 2400 \text{ kN} - 600 \text{ kN} = 1800 \text{ kN} \\ &= 1.8 \text{ MN} \quad \leftarrow\end{aligned}$$

Problem 2.5-19 A capped cast-iron pipe is compressed by a brass rod, as shown. The nut is turned until it is just snug, then add an additional quarter turn to pre-compress the CI pipe. The pitch of the threads of the bolt is $p = 52$ mils (a mil is one-thousandth of an inch). Use the numerical properties provided.

- What stresses σ_p and σ_r will be produced in the cast-iron pipe and brass rod, respectively, by the additional quarter turn of the nut?
- Find the bearing stress σ_b beneath the washer and the shear stress τ_c in the steel cap.



Solution 2.5-19

The figure shows a section through the pipe, cap and rod

NUMERICAL PROPERTIES

$$L_{ci} = 48 \text{ in.} \quad E_s = 30000 \text{ ksi} \quad E_b = 14,000 \text{ ksi}$$

$$E_c = 12,000 \text{ ksi} \quad t_c = 1 \text{ in.} \quad p = 52 \times (10^{-3}) \text{ in.} \quad n = \frac{1}{4}$$

$$d_w = \frac{3}{4} \text{ in.} \quad d_r = \frac{1}{2} \text{ in.} \quad d_o = 6 \text{ in.} \quad d_i = 5.625 \text{ in.}$$

(a) FORCES AND STRESSES IN PIPE AND ROD

One degree statically indeterminate—cut rod at cap and use force in rod (Q) as the redundant:

δ_{rel1} = relative displacement between cut ends of rod due to 1/4 turn of nut

$$\delta_{rel1} = -np \quad \text{Ends of rod move apart, not together, so this is (-).}$$

δ_{rel2} = relative displacement between cut ends of rod due pair of forces Q

$$\delta_{rel2} = Q \left(\frac{L + 2t_c}{E_b A_{rod}} + \frac{L_{ci}}{E_c A_{pipe}} \right)$$

$$A_{rod} = \frac{\pi}{4} d_r^2 \quad A_{pipe} = \frac{\pi}{4} (d_o^2 - d_i^2)$$

$$A_{rod} = 0.196 \text{ in.}^2 \quad A_{pipe} = 3.424 \text{ in.}^2$$

$$\text{Compatibility equation: } \delta_{rel1} + \delta_{rel2} = 0$$

$$Q = \frac{np}{\frac{L_{ci} + 2t_c}{E_b A_{rod}} + \frac{L_{ci}}{E_c A_{pipe}}}$$

$$Q = 0.672 \text{ k} \quad F_{rod} = Q$$

$$\text{Statics: } F_{pipe} = -Q$$

$$\text{Stresses: } \sigma_c = \frac{F_{pipe}}{A_{pipe}} \quad \sigma_c = -0.196 \text{ ksi} \quad \leftarrow$$

$$\sigma_b = \frac{F_{rod}}{A_{rod}} \quad \sigma_b = 3.42 \text{ ksi} \quad \leftarrow$$

(b) BEARING AND SHEAR STRESSES IN STEEL CAP

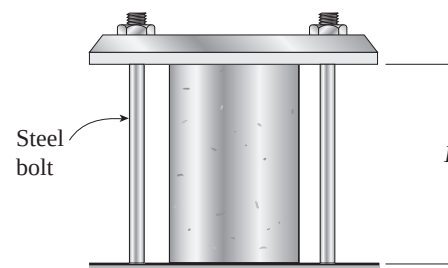
$$\sigma_b = \frac{F_{rod}}{\frac{\pi}{4} (d_w^2 - d_r^2)} \quad \sigma_b = 2.74 \text{ ksi} \quad \leftarrow$$

$$\tau_c = \frac{F_{rod}}{\pi d_w t_c} \quad \tau_c = 0.285 \text{ ksi} \quad \leftarrow$$

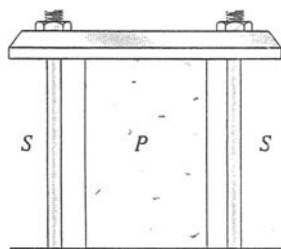
Problem 2.5-20 A plastic cylinder is held snugly between a rigid plate and a foundation by two steel bolts (see figure).

Determine the compressive stress σ_p in the plastic when the nuts on the steel bolts are tightened by one complete turn.

Data for the assembly are as follows: length $L = 200$ mm, pitch of the bolt threads $p = 1.0$ mm, modulus of elasticity for steel $E_s = 200$ GPa, modulus of elasticity for the plastic $E_p = 7.5$ GPa, cross-sectional area of one bolt $A_s = 36.0$ mm², and cross-sectional area of the plastic cylinder $A_p = 960$ mm².



Solution 2.5-20 Plastic cylinder and two steel bolts



$$L = 200 \text{ mm}$$

$$p = 1.0 \text{ mm}$$

$$E_s = 200 \text{ GPa}$$

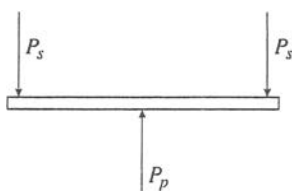
$$A_s = 36.0 \text{ mm}^2 \text{ (for one bolt)}$$

$$E_p = 7.5 \text{ GPa}$$

$$A_p = 960 \text{ mm}^2$$

$$n = 1 \text{ (See Eq. 2-24)}$$

EQUILIBRIUM EQUATION



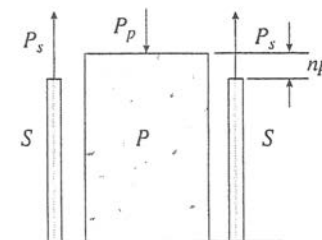
P_s = tensile force in one steel bolt

P_p = compressive force in plastic cylinder

$$P_p = 2P_s$$

(Eq. 1)

COMPATIBILITY EQUATION



δ_s = elongation of steel bolt

δ_p = shortening of plastic cylinder

$$\delta_s + \delta_p = np$$

(Eq. 2)

FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \quad \delta_p = \frac{P_p L}{E_p A_p} \quad (\text{Eq. 3, Eq. 4})$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_s L}{E_s A_s} + \frac{P_p L}{E_p A_p} = np \quad (\text{Eq. 5})$$

Solve simultaneously Eqs. (1) and (5):

$$P_p = \frac{2npE_s A_s E_p A_p}{L(E_p A_p + 2E_s A_s)}$$

STRESS IN THE PLASTIC CYLINDER

$$\sigma_p = \frac{P_p}{A_p} = \frac{2np E_s A_s E_p}{L(E_p A_p + 2E_s A_s)}$$

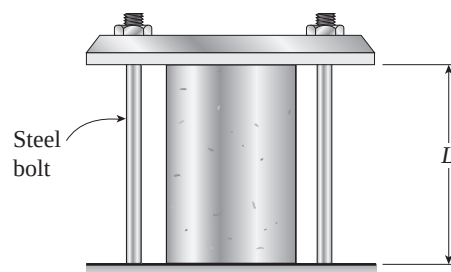
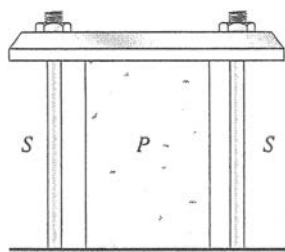
SUBSTITUTE NUMERICAL VALUES:

$$N = E_s A_s E_p = 54.0 \times 10^{15} \text{ N}^2/\text{m}^2$$

$$D = E_p A_p + 2E_s A_s = 21.6 \times 10^6 \text{ N}$$

$$\sigma_p = \frac{2np}{L} \left(\frac{N}{D} \right) = \frac{2(1)(1.0 \text{ mm})}{200 \text{ mm}} \left(\frac{N}{D} \right) = 25.0 \text{ MPa} \quad \leftarrow$$

Problem 2.5-21 Solve the preceding problem if the data for the assembly are as follows: length $L = 10$ in., pitch of the bolt threads $p = 0.058$ in., modulus of elasticity for steel $E_s = 30 \times 10^6$ psi, modulus of elasticity for the plastic $E_p = 500$ ksi, cross-sectional area of one bolt $A_s = 0.06$ in.², and cross-sectional area of the plastic cylinder $A_p = 1.5$ in.²

**Solution 2.5-21 Plastic cylinder and two steel bolts**

$$\begin{aligned} L &= 10 \text{ in.} \\ p &= 0.058 \text{ in.} \\ E_s &= 30 \times 10^6 \text{ psi} \end{aligned}$$

$$A_s = 0.06 \text{ in.}^2 \text{ (for one bolt)}$$

$$E_p = 500 \text{ ksi}$$

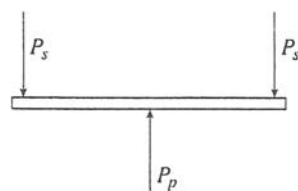
$$A_p = 1.5 \text{ in.}^2$$

$$n = 1 \text{ (see Eq. 2-24)}$$

EQUILIBRIUM EQUATION

 P_s = tensile force in one steel bolt P_p = compressive force in plastic cylinder

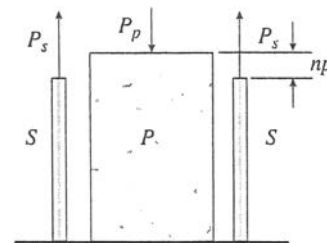
$$P_p = 2P_s \quad (\text{Eq. 1})$$



COMPATIBILITY EQUATION

 δ_s = elongation of steel bolt δ_p = shortening of plastic cylinder

$$\delta_s + \delta_p = np \quad (\text{Eq. 2})$$



FORCE-DISPLACEMENT RELATIONS

$$\delta_s = \frac{P_s L}{E_s A_s} \quad \delta_p = \frac{P_p L}{E_p A_p} \quad (\text{Eq. 3, Eq. 4})$$

SOLUTION OF EQUATIONS

Substitute (3) and (4) into Eq. (2):

$$\frac{P_s L}{E_s A_s} + \frac{P_p L}{E_p A_p} = np \quad (\text{Eq. 5})$$

Solve simultaneously Eqs. (1) and (5):

$$P_p = \frac{2npE_sA_sE_pA_p}{L(E_pA_p + 2E_sA_s)}$$

STRESS IN THE PLASTIC CYLINDER

$$\sigma_p = \frac{P_p}{A_p} = \frac{2npE_sA_sE_p}{L(E_pA_p + 2E_sA_s)} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

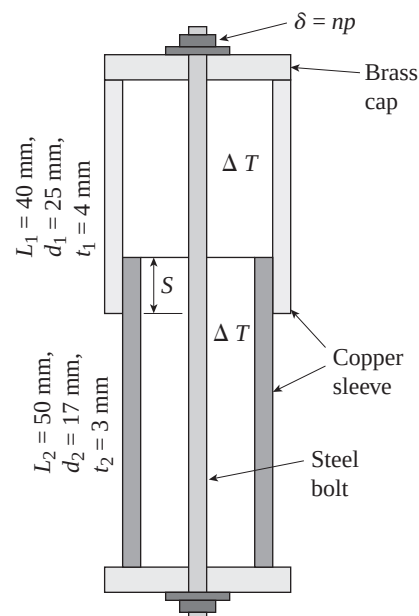
$$N = E_sA_sE_p = 900 \times 10^9 \text{ lb}^2/\text{in.}^2$$

$$D = E_pA_p + 2E_sA_s = 4350 \times 10^3 \text{ lb}$$

$$\sigma_p = \frac{2np}{L} \left(\frac{N}{D} \right) = \frac{2(1)(0.058 \text{ in.})}{10 \text{ in.}} \left(\frac{N}{D} \right) \\ = 2400 \text{ psi} \quad \leftarrow$$

Problem 2.5-22 Consider the sleeve made from two copper tubes joined by tin-lead solder over distance s . The sleeve has brass caps at both ends, which are held in place by a steel bolt and washer with the nut turned just snug at the outset. Then, two “loadings” are applied: $n = 1/2$ turn applied to the nut; at the same time the internal temperature is raised by $\Delta T = 30^\circ\text{C}$.

- Find the forces in the sleeve and bolt, P_s and P_B , due to both the prestress in the bolt and the temperature increase. For copper, use $E_c = 120 \text{ GPa}$ and $\alpha_c = 17 \times 10^{-6}/^\circ\text{C}$; for steel, use $E_s = 200 \text{ GPa}$ and $\alpha_s = 12 \times 10^{-6}/^\circ\text{C}$. The pitch of the bolt threads is $p = 1.0 \text{ mm}$. Assume $s = 26 \text{ mm}$ and bolt diameter $d_b = 5 \text{ mm}$.
- Find the required length of the solder joint, s , if shear stress in the sweated joint cannot exceed the allowable shear stress $\tau_{aj} = 18.5 \text{ MPa}$.
- What is the final elongation of the entire assemblage due to both temperature change ΔT and the initial prestress in the bolt?



Solution 2.5-22

The figure shows a section through the sleeve, cap, and bolt.

NUMERICAL PROPERTIES

$$n = \frac{1}{2} \quad p = 1.0 \text{ mm} \quad \Delta T = 30^\circ\text{C}$$

$$E_c = 120 \text{ GPa} \quad \alpha_c = 17 \times (10^{-6})/^\circ\text{C}$$

$$E_s = 200 \text{ GPa} \quad \alpha_s = 12 \times (10^{-6})/^\circ\text{C}$$

$$\tau_{aj} = 18.5 \text{ MPa} \quad s = 26 \text{ mm} \quad d_b = 5 \text{ mm}$$

$$L_1 = 40 \text{ mm} \quad t_1 = 4 \text{ mm} \quad L_2 = 50 \text{ mm} \quad t_2 = 3 \text{ mm}$$

$$d_1 = 25 \text{ mm} \quad d_1 - 2t_1 = 17 \text{ mm} \quad d_2 = 17 \text{ mm}$$

$$A_b = \frac{\pi}{4} d_b^2 \quad A_1 = \frac{\pi}{4} [d_1^2 - (d_1 - 2t_1)^2]$$

$$A_b = 19.635 \text{ mm}^2 \quad A_1 = 263.894 \text{ mm}^2$$

$$A_2 = \frac{\pi}{4} [d_2^2 - (d_2 - 2t_2)^2] \quad A_2 = 131.947 \text{ mm}^2$$

(a) FORCES IN SLEEVE AND BOLT

One-degree statically indeterminate—cut bolt and use force in bolt (P_B) as redundant (see sketches):

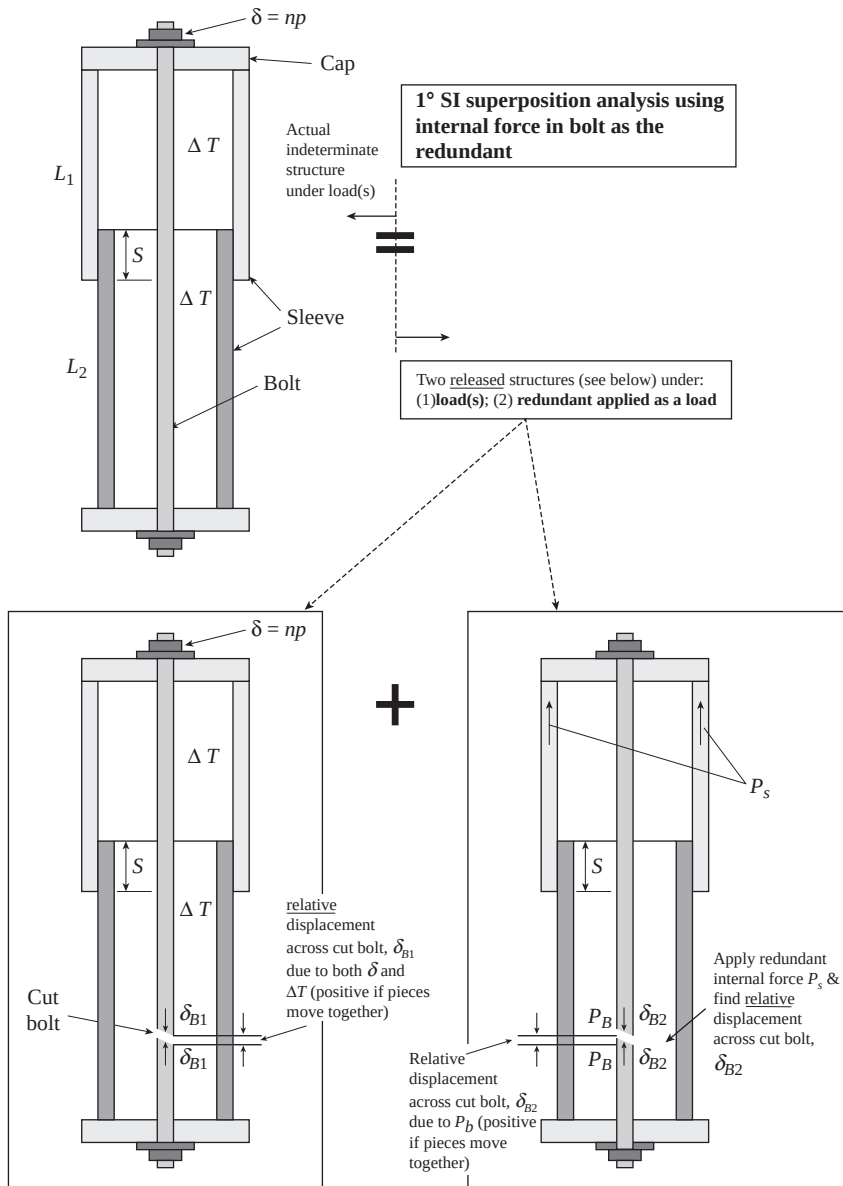
$$\delta_{B1} = -np + \alpha_s \Delta T (L_1 + L_2 - s)$$

$$\delta_{B2} = P_B \left[\frac{L_1 + L_2 - s}{E_s A_b} + \frac{L_1 - s}{E_c A_1} + \frac{L_2 - s}{E_c A_2} + \frac{s}{E_c (A_1 + A_2)} \right]$$

Compatibility: $\delta_{B1} + \delta_{B2} = 0$

$$P_B = \frac{-[-np + \alpha_s \Delta T(L_1 + L_2 - s)]}{\left[\frac{L_1 + L_2 - s}{E_s A_b} + \frac{L_1 - s}{E_c A_1} + \frac{L_2 - s}{E_c A_2} + \frac{s}{E_c (A_1 + A_2)} \right]} \quad P_B = 25.4 \text{ kN} \quad \leftarrow \quad P_s = -P_B \quad \leftarrow$$

Sketches illustrating superposition procedure for statically-indeterminate analysis



(b) REQUIRED LENGTH OF SOLDER JOINT $\approx$

$$\tau = \frac{P}{A_s} \quad A_s = \pi d_2 s$$

$$s_{\text{reqd}} = \frac{P_B}{\pi d_2 \tau_{aj}} \quad s_{\text{reqd}} = 25.7 \text{ mm}$$

(c) FINAL ELONGATION

δ_f = net of elongation of bolt (δ_b) and shortening of sleeve (δ_s)

$$\delta_b = P_B \left(\frac{L_1 + L_2 - s}{E_s A_b} \right) \quad \delta_b = 0.413 \text{ mm}$$

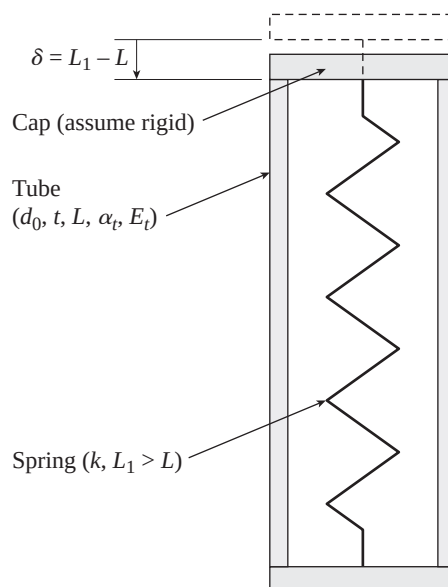
$$\delta_s = P_s \left[\frac{L_1 - s}{E_c A_1} + \frac{L_2 - s}{E_c A_2} + \frac{s}{E_c (A_1 + A_2)} \right]$$

$$\delta_s = -0.064 \text{ mm}$$

$$\delta_f = \delta_b + \delta_s \quad \delta_f = 0.35 \text{ mm} \quad \leftarrow$$

Problem 2.5-23 A polyethylene tube (length L) has a cap which when installed compresses a spring (with undeformed length $L_1 > L$) by amount $\delta = (L_1 - L)$. Ignore deformations of the cap and base. Use the force at the base of the spring as the redundant. Use numerical properties in the boxes given.

- What is the resulting force in the spring, F_k ?
- What is the resulting force in the tube, F_t ?
- What is the final length of the tube, L_f ?
- What temperature change ΔT inside the tube will result in zero force in the spring?



Modulus of elasticity
Polyethylene tube ($E_t = 100$ ksi)
Coefficients of thermal expansion
$\alpha_t = 80 \times 10^{-6}/^\circ\text{F}$, $\alpha_k = 6.5 \times 10^{-6}/^\circ\text{F}$

Properties and dimensions
$d_0 = 6$ in. $t = \frac{1}{8}$ in.
$L_1 = 12.125$ in. $> L = 12$ in. $k = 1.5$ k/in.

Solution 2.5-23

The figure shows a section through the tube, cap, and spring.

Properties and dimensions:

$$d_o = 6 \text{ in.} \quad t = \frac{1}{8} \text{ in.} \quad E_t = 100 \text{ ksi}$$

$$A_t = \frac{\pi}{4} [d_o^2 - (d_o - 2t)^2] \quad A_t = 2.307 \text{ in.}^2$$

$$L_1 = 12.125 \text{ in.} > L = 12 \text{ in.} \quad k = 1.5 \text{ k/in.}$$

Spring is 1/8 in. longer than tube

$$\delta = L_1 - L \quad \delta = 0.125 \text{ in.}$$

$$\alpha_k = 6.5(10^{-6})/^{\circ}\text{F} < \alpha_t = 80 \times (10^{-6})/^{\circ}\text{F}$$

$$\Delta T = 0 \quad < \text{note that } Q \text{ result below is for zero temperature (until part(d))}$$

(a) FORCE IN SPRING $F_k = \text{REDUNDANT } Q$

$$\text{Flexibilities:} \quad f = \frac{1}{k} \quad f_t = \frac{L}{E_t A_t}$$

$\delta_2 = \text{relative displacement across cut spring due to redundant} = Q(f + f_t)$

$\delta_1 = \text{relative displacement across cut spring due to precompression and } \Delta T = \delta + \alpha_k \Delta T L_1 - \alpha_t \Delta T L$

$$\text{Compatibility: } \delta_1 + \delta_2 = 0$$

Solve for redundant Q :

$$Q = \frac{-\delta + \Delta T(-\alpha_k L_1 + \alpha_t L)}{f + f_t} = F_k$$

$$F_k = -0.174 \text{ k} \quad \leftarrow \text{compressive force in spring } (F_k) \text{ and also tensile force in tube}$$

$$(b) F_t = \text{force in tube} = -Q \quad \leftarrow$$

NOTE: If tube is rigid, $F_k = -k\delta = -0.1875 \text{ k}$

(c) FINAL LENGTH OF TUBE

$$L_f = L + \delta_{c1} + \delta_{c2} \quad < \text{i.e., add displacements for the two released structures to initial tube length } L$$

$$L_f = L - Qf_t + \alpha_t(\Delta T)L \quad L_f = 12.01 \text{ in.} \quad \leftarrow$$

(d) SET $Q = 0$ TO FIND ΔT REQUIRED TO REDUCE SPRING FORCE TO ZERO

$$\Delta T_{\text{reqd}} = \frac{\delta}{(-\alpha_k L_1 + \alpha_t L)}$$

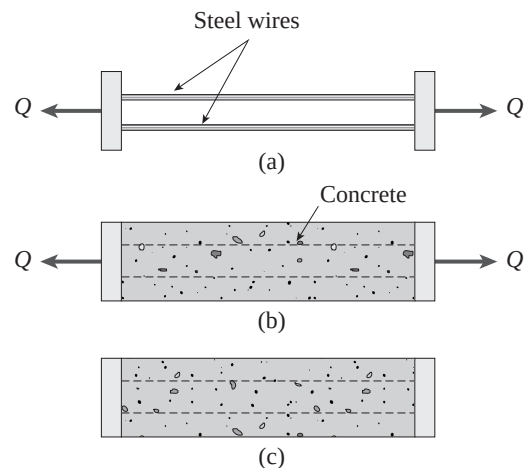
$$\Delta T_{\text{reqd}} = 141.9^{\circ}\text{F}$$

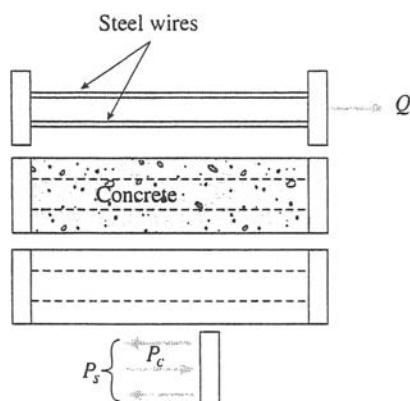
Since $\alpha_t > \alpha_k$, a temp. increase is req'd to expand tube so that spring force goes to zero.

Problem 2.5-24 Prestressed concrete beams are sometimes manufactured in the following manner. High-strength steel wires are stretched by a jacking mechanism that applies a force Q , as represented schematically in part (a) of the figure. Concrete is then poured around the wires to form a beam, as shown in part (b).

After the concrete sets properly, the jacks are released and the force Q is removed [see part (c) of the figure]. Thus, the beam is left in a prestressed condition, with the wires in tension and the concrete in compression.

Let us assume that the prestressing force Q produces in the steel wires an initial stress $\sigma_0 = 620 \text{ MPa}$. If the moduli of elasticity of the steel and concrete are in the ratio 12:1 and the cross-sectional areas are in the ratio 1:50, what are the final stresses σ_s and σ_c in the two materials?



Solution 2.5-24 Prestressed concrete beam

EQUILIBRIUM EQUATION

$$P_s = P_c$$

COMPATIBILITY EQUATION AND
FORCE-DISPLACEMENT RELATIONS δ_1 = initial elongation of steel wires

$$= \frac{QL}{E_s A_s} = \frac{\sigma_0 L}{E_s}$$

 δ_2 = final elongation of steel wires

$$= \frac{P_s L}{E_s A_s}$$

 δ_3 = shortening of concrete

$$= \frac{P_c L}{E_c A_c}$$

 $\delta_1 - \delta_2 = \delta_3$ or

$$\frac{\sigma_0 L}{E_s} - \frac{P_s L}{E_s A_s} = \frac{P_c L}{E_c A_c} \quad (\text{Eq. 2, Eq. 3})$$

Solve simultaneously Eqs. (1) and (3):

$$P_s = P_c = \frac{\sigma_0 A_s}{1 + \frac{E_s A_s}{E_c A_c}}$$

 L = length σ_0 = initial stress in wires

$$= \frac{Q}{A_s} = 620 \text{ MPa}$$

 A_s = total area of steel wires A_c = area of concrete

$$= 50 A_s$$

 $E_s = 12 E_c$ P_s = final tensile force in steel wires P_c = final compressive force in concrete

STRESSES

$$\sigma_s = \frac{P_s}{A_s} = \frac{\sigma_0}{1 + \frac{E_s A_s}{E_c A_c}} \quad \leftarrow$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{\sigma_0}{\frac{A_c}{A_s} + \frac{E_s}{E_c}} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

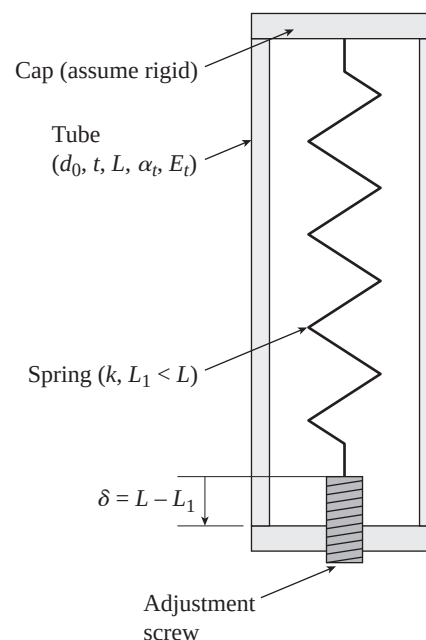
$$\sigma_0 = 620 \text{ MPa} \quad \frac{E_s}{E_c} = 12 \quad \frac{A_s}{A_c} = \frac{1}{50}$$

$$\sigma_s = \frac{620 \text{ MPa}}{1 + \frac{12}{50}} = 500 \text{ MPa (Tension)} \quad \leftarrow$$

$$\sigma_c = \frac{620 \text{ MPa}}{50 + 12} = 10 \text{ MPa (Compression)} \quad \leftarrow$$

Problem 2.5-25 A polyethylene tube (length L) has a cap which is held in place by a spring (with undeformed length $L_1 < L$). After installing the cap, the spring is post-tensioned by turning an adjustment screw by amount δ . Ignore deformations of the cap and base. Use the force at the base of the spring as the redundant. Use numerical properties in the boxes below.

- What is the resulting force in the spring, F_k ?
- What is the resulting force in the tube, F_t ?
- What is the final length of the tube, L_f ?
- What temperature change ΔT inside the tube will result in zero force in the spring?



Modulus of elasticity Polyethylene tube ($E_t = 100$ ksi)
Coefficients of thermal expansion $\alpha_t = 80 \times 10^{-6}/^\circ\text{F}$, $\alpha_k = 6.5 \times 10^{-6}/^\circ\text{F}$

Properties and dimensions
$d_0 = 6$ in. $t = \frac{1}{8}$ in.
$L = 12$ in. $L_1 = 11.875$ in. $k = 1.5$ k/in.

Solution 2.5-25

The figure shows a section through the tube, cap, and spring.

Properties and dimensions:

$$d_o = 6 \text{ in.} \quad t = \frac{1}{8} \text{ in.} \quad E_t = 100 \text{ ksi}$$

$$L = 12 \text{ in.} > L_1 = 11.875 \text{ in.} \quad k = 1.5 \text{ k/in.}$$

$$\alpha_k = 6.5(10^{-6}) < \alpha_t = 80 \times (10^{-6})$$

$$A_t = \frac{\pi}{4} [d_o^2 - (d_o - 2t)^2]$$

$$A_t = 2.307 \text{ in.}^2$$

Pretension and temperature:

Spring is $1/8$ in. shorter than tube.

$$\delta = L - L_1 \quad \delta = 0.125 \text{ in.} \quad \Delta T = 0$$

Note that Q result below is for zero temperature (until part (d)).

$$\text{Flexibilities:} \quad f = \frac{1}{k} \quad f_t = \frac{L}{E_t A_t}$$

(a) FORCE IN SPRING (F_k) = REDUNDANT (Q)

Follow solution procedure outlined in Prob. 2.5-23 solution:

$$Q = \frac{\delta + \Delta T(-\alpha_k L_1 + \alpha_t L)}{f + f_t} = F_k$$

$F_k = 0.174 \text{ k}$ ← also the compressive force in the tube

(b) FORCE IN TUBE $F_t = -Q = -0.174 \text{ k}$ ←

(c) FINAL LENGTH OF TUBE AND SPRING $L_f = L + \delta_{c1} + \delta_{c2}$

$$L_f = L - Qf_t + \alpha_t(\Delta T)L \quad L_f = 11.99 \text{ in.} \quad \leftarrow$$

(d) SET $Q = 0$ TO FIND ΔT REQUIRED TO REDUCE SPRING FORCE TO ZERO

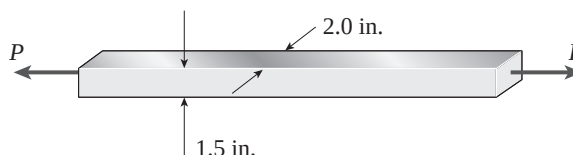
$$\Delta T_{\text{reqd}} = \frac{-\delta}{(-\alpha_k L_1 + \alpha_t L)}$$

$$\Delta T_{\text{reqd}} = -141.6^\circ\text{F}$$

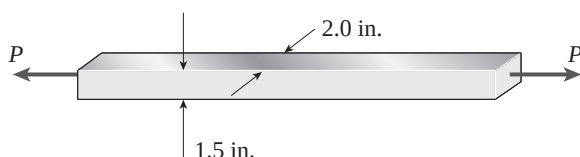
Since $\alpha_t > \alpha_k$, a temperature drop is required to shrink tube so that spring force goes to zero.

Stresses on Inclined Sections

Problem 2.6-1 A steel bar of rectangular cross section (1.5 in. $\times$ 2.0 in.) carries a tensile load P (see figure). The allowable stresses in tension and shear are 14,500 psi and 7,100 psi, respectively. Determine the maximum permissible load $P_{\max}$.



Solution 2.6-1



NUMERICAL DATA

$$A = 3 \text{ in.}^2 \quad \sigma_a = 14500 \text{ psi}$$

$$\tau_a = 7100 \text{ psi}$$

MAXIMUM LOAD—TENSION

$$P_{\max 1} = \sigma_a A \quad P_{\max 1} = 43500 \text{ lbs}$$

MAXIMUM LOAD—SHEAR

$$P_{\max 2} = 2\tau_a A \quad P_{\max 2} = 42,600 \text{ lbs}$$

Because τ_{allow} is less than one-half of σ_{allow} , the shear stress governs.

Problem 2.6-2 A circular steel rod of diameter d is subjected to a tensile force $P = 3.5 \text{ kN}$ (see figure). The allowable stresses in tension and shear are 118 MPa and 48 MPa, respectively. What is the minimum permissible diameter $d_{\min}$ of the rod?



Solution 2.6-2



NUMERICAL DATA $P = 3.5 \text{ kN}$ $\sigma_a = 118 \text{ MPa}$
 $\tau_a = 48 \text{ MPa}$

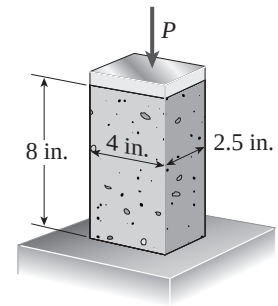
Find $P_{\max}$ then rod diameter.
 since τ_a is less than $1/2$ of σ_a , shear governs.

$$P_{\max} = 2\tau_a \left(\frac{\pi}{4} d_{\min}^2 \right)$$

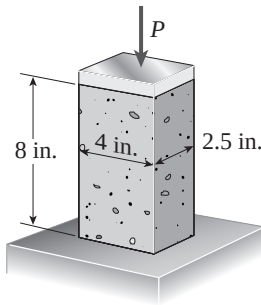
$$d_{\min} = \sqrt{\frac{2}{\pi \tau_a} P}$$

$$d_{\min} = 6.81 \text{ mm} \quad \leftarrow$$

Problem 2.6-3 A standard brick (dimensions 8 in. $\times$ 4 in. $\times$ 2.5 in.) is compressed lengthwise by a force P , as shown in the figure. If the ultimate shear stress for brick is 1200 psi and the ultimate compressive stress is 3600 psi, what force $P_{\max}$ is required to break the brick?



Solution 2.6-3 Standard brick in compression



$$A = 2.5 \text{ in.} \times 4.0 \text{ in.} = 10.0 \text{ in.}^2$$

Maximum normal stress:

$$\sigma_x = \frac{P}{A}$$

Maximum shear stress:

$$\tau_{\max} = \frac{\sigma_x}{2} = \frac{P}{2A}$$

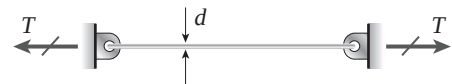
$$\sigma_{\text{ult}} = 3600 \text{ psi} \quad \tau_{\text{ult}} = 1200 \text{ psi}$$

Because τ_{ult} is less than one-half of σ_{ult} , the shear stress governs.

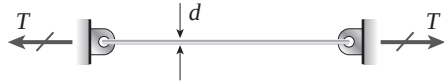
$$\tau_{\max} = \frac{P}{2A} \quad \text{or} \quad P_{\max} = 2A\tau_{\text{ult}}$$

$$P_{\max} = 2(10.0 \text{ in.}^2)(1200 \text{ psi}) = 24,000 \text{ lb} \quad \leftarrow$$

Problem 2.6-4 A brass wire of diameter $d = 2.42 \text{ mm}$ is stretched tightly between rigid supports so that the tensile force is $T = 98 \text{ N}$ (see figure). The coefficient of thermal expansion for the wire is $19.5 \times 10^{-6}/^\circ\text{C}$ and the modulus of elasticity is $E = 110 \text{ GPa}$.



- What is the maximum permissible temperature drop ΔT if the allowable shear stress in the wire is 60 MPa?
- At what temperature changes does the wire go slack?

Solution 2.6-4 Brass wire in tension

NUMERICAL DATA

$$d = 2.42 \text{ mm} \quad T = 98 \text{ N}$$

$$\alpha = 19.5 (10^{-6})/^{\circ}\text{C} \quad E = 110 \text{ GPa}$$

(a) $\Delta T_{\max}$ (DROP IN TEMPERATURE)

$$\sigma = \frac{T}{A} - (E \alpha \Delta T) \quad \tau_{\max} = \frac{\sigma}{2}$$

$$\tau_a = \frac{T}{2A} - \frac{E \alpha \Delta T}{2}$$

$$\tau_a = 60 \text{ MPa} \quad A = \frac{\pi}{4} d^2$$

$$\Delta T_{\max} = \frac{\frac{T}{A} - 2\tau_a}{E \alpha}$$

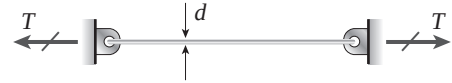
$$\Delta T_{\max} = -46^{\circ}\text{C} \text{ (drop)}$$

(b) ΔT AT WHICH WIRE GOES SLACKIncrease ΔT until $\sigma = 0$:

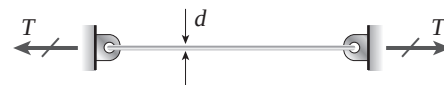
$$\Delta T = \frac{T}{E \alpha A}$$

$$\Delta T = 9.93^{\circ}\text{C} \text{ (increase)}$$

Problem 2.6-5 A brass wire of diameter $d = 1/16$ in. is stretched between rigid supports with an initial tension T of 37 lb (see figure). Assume that the coefficient of thermal expansion is $10.6 \times 10^{-6}/^{\circ}\text{F}$ and the modulus of elasticity is 15×10^6 psi.)



- If the temperature is lowered by 60°F , what is the maximum shear stress $\tau_{\max}$ in the wire?
- If the allowable shear stress is 10,000 psi, what is the maximum permissible temperature drop?
- At what temperature change ΔT does the wire go slack?

Solution 2.6-5

NUMERICAL DATA

$$d = \frac{1}{16} \text{ in.} \quad T = 37 \text{ lb} \quad \alpha = 10.6 \times (10^{-6})/^{\circ}\text{F}$$

$$E = 15 \times (10^6) \text{ psi} \quad \Delta T = -60^{\circ}\text{F}$$

$$A = \frac{\pi}{4} d^2$$

(a) $\tau_{\max}$ (DUE TO DROP IN TEMPERATURE)

$$\tau_{\max} = \frac{\sigma_x}{2} \quad \tau_{\max} = \frac{\frac{T}{A} - (E \alpha \Delta T)}{2}$$

$$\tau_{\max} = 10,800 \text{ psi} \quad \leftarrow$$

(b) $\Delta T_{\max}$ FOR ALLOWABLE SHEAR STRESS

$$\tau_a = 10000 \text{ psi}$$

$$\Delta T_{\max} = \frac{\frac{T}{A} - 2\tau_a}{E \alpha}$$

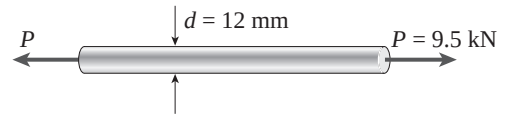
$$\Delta T_{\max} = -49.9^{\circ}\text{F} \quad \leftarrow$$

(c) ΔT AT WHICH WIRE GOES SLACKIncrease ΔT until $\sigma = 0$:

$$\Delta T = \frac{T}{E \alpha A}$$

$$\Delta T = 75.9^{\circ}\text{F} \text{ (increase)} \quad \leftarrow$$

Problem 2.6-6 A steel bar with diameter $d = 12$ mm is subjected to a tensile load $P = 9.5$ kN (see figure).



- What is the maximum normal stress $\sigma_{\max}$ in the bar?
- What is the maximum shear stress $\tau_{\max}$?
- Draw a stress element oriented at 45° to the axis of the bar and show all stresses acting on the faces of this element.
- Repeat part (c) for a stress element oriented at 22.5° to the axis of the bar.

Solution 2.6-6

(a) $d = 12$ mm $P = 9.5$ kN $A = \frac{\pi}{4}d^2 = 1.131 \times 10^{-4} \text{ m}^2$

$$\sigma_x = \frac{P}{A} = 84 \text{ MPa}$$

(b) $\tau_{\max} = \frac{\sigma_x}{2} = 42 \text{ MPa}$ On plane stress element rotated 45°

(c) ROTATED STRESS ELEMENT (45°) HAS NORMAL TENSILE STRESS $\sigma_x/2$ ON ALL FACES, $-T_{\max}$ (CW) ON $+x$ -FACE, AND $+T_{\max}$ (CCW) ON $+y$ -FACE

$$\tau_{xy1y1} = \tau_{\max} \quad \sigma_{x1} = \frac{\sigma_x}{2} \quad \sigma_{y1} = \sigma_{x1}$$

On rotated x -face: $\sigma_{x1} = 42 \text{ MPa}$ $\tau_{x1y1} = 42 \text{ MPa}$

On rotated y -face: $\sigma_{y1} = 42 \text{ MPa}$

(d) $\theta = 22.5^\circ < \text{CCW}$ ROTATION OF ELEMENT

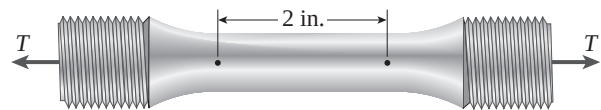
$$\sigma_\theta = \sigma_x \cos^2(\theta) = 71.7 \text{ MPa} < \text{on rotated } x \text{ face} \quad \sigma_y = \sigma_x \cos^2\left(\theta + \frac{\pi}{2}\right) = 12.3 \text{ MPa} < \text{on rotated } y \text{ face}$$

Eq. 2-31b $\tau_\theta = \frac{-\sigma_x}{2} \sin(2\theta) = -29.7 \text{ MPa} < \text{CW on rotated } x\text{-face}$

On rotated x -face: $\sigma_{x1} = 71.7 \text{ MPa}$ $\tau_{x1y1} = -29.7 \text{ MPa}$

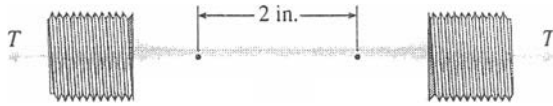
On rotated y -face: $\sigma_{y1} = 12.3 \text{ MPa}$

Problem 2.6-7 During a tension test of a mild-steel specimen (see figure), the extensometer shows an elongation of 0.00120 in. with a gage length of 2 in. Assume that the steel is stressed below the proportional limit and that the modulus of elasticity $E = 30 \times 10^6$ psi.



- What is the maximum normal stress $\sigma_{\max}$ in the specimen?
- What is the maximum shear stress $\tau_{\max}$?
- Draw a stress element oriented at an angle of 45° to the axis of the bar and show all stresses acting on the faces of this element.

Solution 2.6-7 Tension test



Elongation: $\delta = 0.00120$ in.

(2 in. gage length)

$$\text{Strain: } \varepsilon = \frac{\delta}{L} = \frac{0.00120 \text{ in.}}{2 \text{ in.}} = 0.00060$$

$$\text{Hooke's law: } \sigma_x = E\varepsilon = (30 \times 10^6 \text{ psi})(0.00060) = 18,000 \text{ psi}$$

(a) MAXIMUM NORMAL STRESS

σ_x is the maximum normal stress.

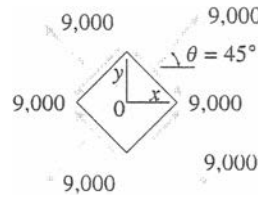
$$\sigma_{\max} = 18,000 \text{ psi} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESS

The maximum shear stress is on a 45° plane and equals $\sigma_x/2$.

$$\tau_{\max} = \frac{\sigma_x}{2} = 9,000 \text{ psi} \quad \leftarrow$$

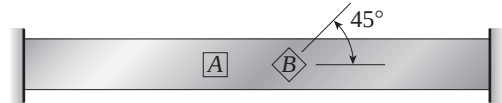
(c) STRESS ELEMENT AT $\theta = 45^\circ$



NOTE: All stresses have units of psi.

Problem 2.6-8 A copper bar with a rectangular cross section is held without stress between rigid supports (see figure). Subsequently, the temperature of the bar is raised 50°C .

- Determine the stresses on all faces of the elements A and B, and show these stresses on sketches of the elements. (Assume $\alpha = 17.5 \times 10^{-6}/^\circ\text{C}$ and $E = 120 \text{ GPa}$.)
- If the shear stress at B is known to be 48 MPa at some inclination θ , find angle θ and show the stresses on a sketch of a properly oriented element.



Solution 2.6-8

(a) $\alpha = 17.5(10^{-6}) \quad \Delta T = 50 \quad E = 120 \text{ GPa}$

$$\sigma_x = -E\alpha\Delta T = -105 \text{ MPa} \quad \tau_{\max} = \frac{\sigma_x}{2} = -52.5 \text{ MPa} \quad < \text{ at } \theta = 45^\circ$$

(compression)

Element A: $\sigma_x = 105 \text{ MPa}$ (compression);

Element B: $\tau_{\max} = 52.5 \text{ MPa}$

(b) $\tau_\theta = 48 \text{ MPa}$

Eq. 2-31b $\tau_\theta = \frac{-\sigma_x}{2} \sin(2\theta)$

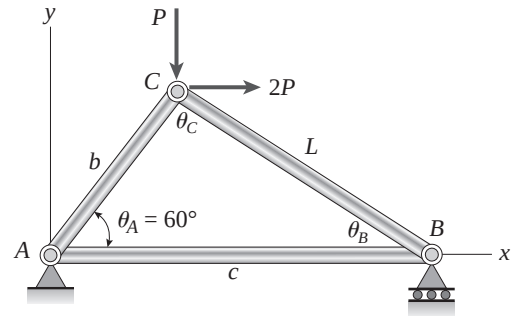
so $\theta = \frac{1}{2} \arcsin\left(\frac{2\tau_\theta}{-\sigma_x}\right) = 33.1^\circ < \text{CCW rotation of element} \quad \boxed{\theta = 33.1^\circ}$

$$\sigma_\theta = \sigma_x \cos^2(\theta) = -73.8 \text{ MPa} \quad < \text{ on rotated x face}$$

$$\sigma_y = \sigma_x \cos^2\left(\theta + \frac{\pi}{2}\right) = -31.2 \text{ MPa} \quad < \text{ on rotated y face}$$

Problem 2.6-9 The plane truss below is assembled from steel C10 × 20 shapes (see Table 3(a) in Appendix F). Assume that $L = 10$ ft and $b = 0.71 L$.

- If load variable $P = 49$ k, what is the maximum shear stress $\tau_{\max}$ in each truss member?
- What is the maximum permissible value of load variable P if the allowable normal stress is 14 ksi and the allowable shear stress is 7.5 ksi?



Solution 2.6-9

NUMERICAL DATA

$$L = 10 \text{ ft} \quad b = 0.71 L \quad P = 49 \text{ k} \quad \sigma_a = 14 \text{ ksi} \quad \tau_a = 7.5 \text{ ksi} \quad A = 5.87 \text{ in.}^2$$

(a) FOR LINEAR ANALYSIS, MEMBER FORCES ARE PROPORTIONAL TO LOADING

$$\text{FROM EXAMPLE 1-1: } F_{AC} = \frac{P}{35} 15.59 = 21.826 \text{ k} \quad F_{AB} = \frac{P}{35} 62.2 = 87.08 \text{ k}$$

(solution for $P = 35$ k)

$$F_{BC} = \frac{P}{35} (-78.9) \quad F_{BC} = -110.46 \text{ k}$$

$$\text{Normal stresses in each member: } \sigma_{AC} = \frac{F_{AC}}{A} = 3.718 \text{ ksi} \quad \sigma_{AB} = \frac{F_{AB}}{A} = 14.835 \text{ ksi}$$

$$\text{From Eq. 2-33: } \sigma_{BC} = \frac{F_{BC}}{A} = -18.818 \text{ ksi}$$

$$\tau_{\max AC} = \frac{\sigma_{AC}}{2} = 1.859 \text{ ksi}$$

$$\tau_{\max AB} = \frac{\sigma_{AB}}{2} = 7.42 \text{ ksi}$$

$$\tau_{\max BC} = \frac{\sigma_{BC}}{2} = -9.41 \text{ ksi}$$

(b) $\sigma_a < 2 \times T_a$ so normal stress will control; lowest value governs here

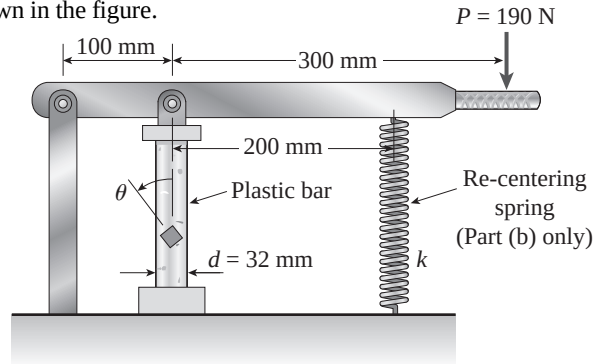
$$\text{MEMBER AC: } P_{\max \sigma} = \frac{P}{F_{AC}} (\sigma_a A) = 184.496 \text{ k} \quad P_{\max \tau} = \frac{P}{F_{AC}} (2 \tau_a A) = 197.675 \text{ k}$$

$$\text{MEMBER AB: } P_{\max \sigma} = \frac{P}{F_{AB}} (\sigma_a A) = 46.243 \text{ k} \quad P_{\max \tau} = \frac{P}{F_{AB}} (2 \tau_a A) = 49.546 \text{ k}$$

$$\text{MEMBER BC: } P_{\max \sigma} = \left| \frac{P}{F_{BC}} \right| (\sigma_a A) = 36.5 \text{ k} \quad P_{\max \tau} = \left| \frac{P}{F_{BC}} \right| (2 \tau_a A) = 39.059 \text{ k}$$

Problem 2.6-10 A plastic bar of diameter $d = 32$ mm is compressed in a testing device by a force $P = 190$ N applied as shown in the figure.

- (a) Determine the normal and shear stresses acting on all faces of stress elements oriented at (1) an angle $\theta = 0^\circ$, (2) an angle $\theta = 22.5^\circ$, and (3) an angle $\theta = 45^\circ$. In each case, show the stresses on a sketch of a properly oriented element. What are $\sigma_{\max}$ and $\tau_{\max}$?
- (b) Find $\sigma_{\max}$ and $\tau_{\max}$ in the plastic bar if a re-centering spring of stiffness k is inserted into the testing device, as shown in the figure. The spring stiffness is $1/6$ of the axial stiffness of the plastic bar.



Solution

NUMERICAL DATA

$$\begin{aligned} d &= 32 \text{ mm} & A &= \frac{\pi}{4} d^2 \\ P &= 190 \text{ N} & A &= 804.25 \text{ mm}^2 \\ a &= 100 \text{ mm} \\ b &= 300 \text{ mm} \end{aligned}$$

- (a) **STATICS**—FIND COMPRESSIVE FORCE F AND STRESSES IN PLASTIC BAR

$$F = \frac{P(a + b)}{a} \quad F = 760 \text{ N}$$

$$\sigma_x = \frac{F}{A} \quad \sigma_x = 0.945 \text{ MPa} \quad \text{or} \quad \sigma_x = 945 \text{ kPa}$$

From (1), (2), and (3) below:

$$\sigma_{\max} = \sigma_x \quad \sigma_{\max} = -945 \text{ kPa}$$

$$\tau_{\max} = 472 \text{ kPa} \quad \frac{\sigma_x}{2} = -472 \text{ kPa}$$

$$(1) \theta = 0^\circ \quad \sigma_x = -945 \text{ kPa} \quad \leftarrow$$

$$(2) \theta = 22.5^\circ$$

On $+x$ -face:

$$\begin{aligned} \sigma_\theta &= \sigma_x \cos^2(\theta) \\ \sigma_\theta &= -807 \text{ kPa} \quad \leftarrow \end{aligned}$$

$$\begin{aligned} \tau_\theta &= -\sigma_x \sin(\theta) \cos(\theta) \\ \tau_\theta &= 334 \text{ kPa} \quad \leftarrow \end{aligned}$$

$$\text{On } +y\text{-face:} \quad \theta = \theta + \frac{\pi}{2}$$

$$\begin{aligned} \sigma_\theta &= \sigma_x \cos^2(\theta) \\ \sigma_\theta &= -138.39 \text{ kPa} \end{aligned}$$

$$\begin{aligned} \tau_\theta &= -\sigma_x \sin(\theta) \cos(\theta) \\ \tau_\theta &= -334.1 \text{ kPa} \end{aligned}$$

$$(3) \theta = 45^\circ$$

On $+x$ -face:

$$\begin{aligned} \sigma_\theta &= \sigma_x \cos^2(\theta) \\ \sigma_\theta &= -472 \text{ kPa} \quad \leftarrow \end{aligned}$$

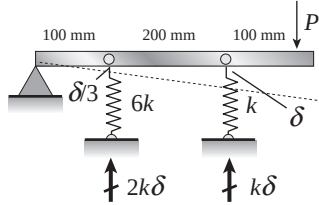
$$\begin{aligned} \tau_\theta &= -\sigma_x \sin(\theta) \cos(\theta) \\ \tau_\theta &= 472 \text{ kPa} \quad \leftarrow \end{aligned}$$

$$\text{On } +y\text{-face:} \quad \theta = \theta + \frac{\pi}{2}$$

$$\sigma_\theta = \sigma_x \cos^2(\theta) \quad \sigma_\theta = -472.49 \text{ kPa}$$

$$\tau_\theta = -\sigma_x \sin(\theta) \cos(\theta) \quad \tau_\theta = -472.49 \text{ kPa}$$

- (b) ADD SPRING—FIND MAXIMUM NORMAL AND SHEAR STRESSES IN PLASTIC BAR



$$\sum M_{\text{pin}} = 0$$

$$P(400) = [2k\delta(100) + k\delta(300)]$$

$$\delta = \frac{4}{5} \frac{P}{k}$$

$$\text{Force in plastic bar: } F = (2k) \left(\frac{4}{5} \frac{P}{k} \right)$$

$$F = \frac{8}{5} P \quad F = 304 \text{ N}$$

Normal and shear stresses in plastic bar:

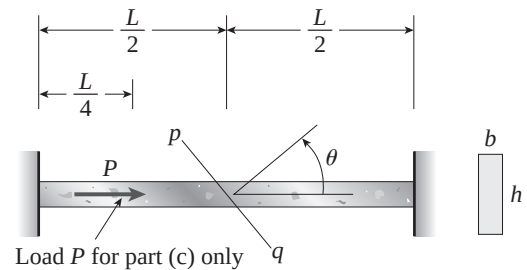
$$\sigma_x = \frac{F}{A} \quad \sigma_x = 0.38$$

$$\sigma_{\text{max}} = -378 \text{ kPa} \quad \leftarrow$$

$$\tau_{\text{max}} = \frac{\sigma_x}{2} \quad \tau_{\text{max}} = -189 \text{ kPa} \quad \leftarrow$$

Problem 2.6-11 A plastic bar of rectangular cross section ($b = 1.5$ in. and $h = 3$ in.) fits snugly between rigid supports at room temperature (68°F) but with no initial stress (see figure). When the temperature of the bar is raised to 160°F , the compressive stress on an inclined plane pq at midspan becomes 1700 psi.

- What is the shear stress on plane pq ? (Assume $\alpha = 60 \times 10^{-6}/^\circ\text{F}$ and $E = 450 \times 10^3$ psi.)
- Draw a stress element oriented to plane pq and show the stresses acting on all faces of this element.
- If the allowable normal stress is 3400 psi and the allowable shear stress is 1650 psi, what is the maximum load P (in $+x$ direction) which can be added at the quarter point (in addition to thermal effects above) without exceeding allowable stress values in the bar?



Solution 2.6-11

NUMERICAL DATA

$$b = 1.5 \text{ in.} \quad h = 3 \text{ in.} \quad A = bh \quad \Delta T = (160 - 68)^\circ\text{F}$$

$$\Delta T = 92^\circ\text{F}$$

$$A = 4.5 \text{ in.}^2 \quad \sigma_{pq} = -1700 \text{ psi}$$

$$\alpha = 60 \times (10^{-6})/^\circ\text{F}$$

$$E = 450 \times (10^3) \text{ psi}$$

- (a) SHEAR STRESS ON PLANE pq

Statically indeterminate analysis gives, for reaction at right support:

$$R = -EA\alpha\Delta T \quad R = -11178 \text{ lb}$$

$$\sigma_x = \frac{R}{A} \quad \sigma_x = -2484 \text{ psi}$$

$$\text{Using } \sigma_\theta = \sigma_x \cos(\theta)^2: \quad \cos(\theta)^2 = \frac{\sigma_{pq}}{\sigma_x}$$

$$\theta = \arccos\left(\sqrt{\frac{\sigma_{pq}}{\sigma_x}}\right) \quad \theta = 34.2^\circ$$

Now with θ , can find shear stress on plane pq :

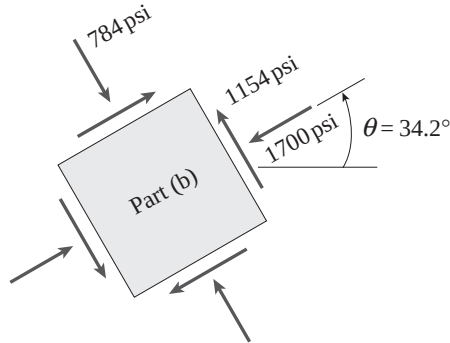
$$\tau_{pq} = -\sigma_x \sin(\theta) \cos(\theta) \quad \tau_{pq} = 1154 \text{ psi} \quad \leftarrow$$

$$\sigma_{pq} = \sigma_x \cos(\theta)^2 \quad \sigma_{pq} = -1700 \text{ psi}$$

Stresses at $\theta + \pi/2$ (y -face):

$$\sigma_y = \sigma_x \cos\left(\theta + \frac{\pi}{2}\right)^2 \quad \sigma_y = -784 \text{ psi}$$

(b) STRESS ELEMENT FOR PLANE PQ



(c) MAXIMUM LOAD AT QUARTER POINT $\sigma_a = 3400$ psi

$$\tau_a = 1650 \text{ psi} \quad 2\tau_a = 3300 < \text{less than } \sigma_a, \text{ so shear controls}$$

Statically indeterminate analysis for P at $L/4$ gives for reactions:

$$R_{R2} = \frac{-P}{4} \quad R_{L2} = \frac{-3}{4} P$$

(tension for 0 to $L/4$ and compression for rest of bar)

From part (a) (for temperature increase ΔT):

$$R_{R1} = -EA\alpha\Delta T \quad R_{L1} = -EA\alpha\Delta T$$

Stresses in bar (0 to $L/4$):

$$\sigma_x = -E\alpha\Delta T + \frac{3P}{4A} \quad \tau_{\max} = \frac{\sigma_x}{2}$$

Set $\tau_{\max} = \tau_a$ and solve for $P_{\max1}$:

$$\tau_a = \frac{-E\alpha\Delta T}{2} + \frac{3P}{8A}$$

$$P_{\max1} = \frac{4A}{3}(2\tau_a + E\alpha\Delta T)$$

$$P_{\max1} = 34,704 \text{ lb}$$

$$\tau_{\max} = \frac{-E\alpha\Delta T}{2} + \frac{3P_{\max1}}{8A}$$

$$\tau_{\max} = 1650 \text{ psi} < \text{check}$$

$$\sigma_x = -E\alpha\Delta T + \frac{3P_{\max1}}{4A}$$

$$\sigma_x = 3300 \text{ psi} < \text{less than } \sigma_a$$

Stresses in bar ($L/4$ to L):

$$\sigma_x = -E\alpha\Delta T - \frac{P}{4A} \quad \tau_{\max} = \frac{\sigma_x}{2}$$

Set $\tau_{\max} = \tau_a$ and solve for $P_{\max2}$:

$$P_{\max2} = -4A(-2\tau_a + E\alpha\Delta T)$$

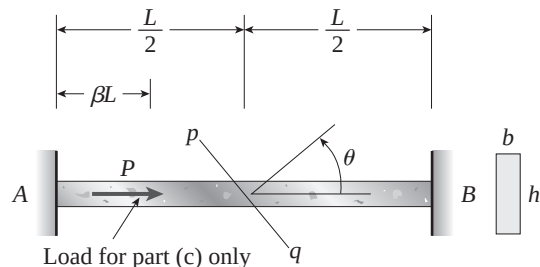
$$P_{\max2} = 14,688 \text{ lb} \leftarrow \text{shear in segment } (L/4 \text{ to } L) \text{ controls}$$

$$\tau_{\max} = \frac{-E\alpha\Delta T}{2} - \frac{P_{\max2}}{8A} \quad \tau_{\max} = -1650 \text{ psi}$$

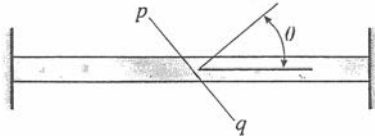
$$\sigma_x = -E\alpha\Delta T - \frac{P_{\max2}}{4A} \quad \sigma_x = -3300 \text{ psi}$$

Problem 2.6-12 A copper bar of rectangular cross section ($b = 18$ mm and $h = 40$ mm) is held snugly (but without any initial stress) between rigid supports (see figure). The allowable stresses on the inclined plane pq at midspan, for which $\theta = 55^\circ$, are specified as 60 MPa in compression and 30 MPa in shear.

- What is the maximum permissible temperature rise ΔT if the allowable stresses on plane pq are not to be exceeded? (Assume $\alpha = 17 \times 10^{-6}/^\circ\text{C}$ and $E = 120$ GPa.)
- If the temperature increases by the maximum permissible amount, what are the stresses on plane pq ?
- If the temperature rise $\Delta T = 28^\circ\text{C}$, how far to the right of end A (distance βL , expressed as a fraction of length L) can load $P = 15$ kN be applied without exceeding allowable stress values in the bar? Assume that $\sigma_a = 75$ MPa and $\tau_a = 35$ MPa.



Solution 2.6-12



NUMERICAL DATA

$$\theta = 55 \left(\frac{\pi}{180} \right) \text{ rad}$$

$$b = 18 \text{ mm} \quad h = 40 \text{ mm}$$

$$A = bh \quad A = 720 \text{ mm}^2$$

$$\sigma_{pqa} = 60 \text{ MPa} \quad \tau_{pqa} = 30 \text{ MPa}$$

$$\alpha = 17 \times (10^{-6})/^{\circ}\text{C} \quad E = 120 \text{ GPa}$$

$$\Delta T = 20^{\circ}\text{C} \quad P = 15 \text{ kN}$$

- (a) FIND $\Delta T_{\max}$ BASED ON ALLOWABLE NORMAL AND SHEAR STRESS VALUES ON PLANE pq

$$\sigma_x = -E\alpha\Delta T_{\max} \quad \Delta T_{\max} = \frac{-\sigma_x}{E\alpha}$$

$$\sigma_{pq} = \sigma_x \cos(\theta)^2 \quad \tau_{pq} = -\sigma_x \sin(\theta) \cos(\theta)$$

Set each equal to corresponding allowable and solve for σ_x :

$$\sigma_{x1} = \frac{\sigma_{pqa}}{\cos(\theta)^2} \quad \sigma_{x1} = 182.38 \text{ MPa}$$

$$\sigma_{x2} = \frac{\tau_{pqa}}{-\sin(\theta) \cos(\theta)} \quad \sigma_{x2} = -63.85 \text{ MPa}$$

Lesser value controls, so allowable shear stress governs.

$$\Delta T_{\max} = \frac{-\sigma_{x2}}{E\alpha} \quad \Delta T_{\max} = 31.3^{\circ}\text{C} \quad \leftarrow$$

- (b) STRESSES ON PLANE pq FOR MAXIMUM TEMPERATURE

$$\sigma_x = -E\alpha\Delta T_{\max} \quad \sigma_x = -63.85 \text{ MPa}$$

$$\sigma_{pq} = \sigma_x \cos(\theta)^2 \quad \sigma_{pq} = -21.0 \text{ MPa} \quad \leftarrow$$

$$\tau_{pq} = -\sigma_x \sin(\theta) \cos(\theta) \quad \tau_{pq} = 30 \text{ MPa} \quad \leftarrow$$

- (c) ADD LOAD P IN $+x$ -DIRECTION TO TEMPERATURE CHANGE AND FIND LOCATION OF LOAD

$$\Delta T = 28^{\circ}\text{C}$$

$P = 15 \text{ kN}$ from one-degree statically indeterminate analysis, reactions R_A and R_B due to load P :

$$R_A = -(1 - \beta)P \quad R_B = \beta P$$

Now add normal stresses due to P to thermal stresses due to ΔT (tension in segment 0 to βL , compression in segment βL to L).

Stresses in bar (0 to βL):

$$\sigma_x = -E\alpha\Delta T + \frac{R_A}{A} \quad \tau_{\max} = \frac{\sigma_x}{2}$$

Shear controls so set $\tau_{\max} = \tau_a$ and solve for β :

$$2\tau_a = -E\alpha\Delta T + \frac{(1 - \beta)P}{A}$$

$$\beta = 1 - \frac{A}{P} [2\tau_a + E\alpha\Delta T]$$

$$\beta = -5.1$$

Impossible so evaluate segment (βL to L):

Stresses in bar (βL to L):

$$\sigma_x = -E\alpha\Delta T - \frac{R_B}{A} \quad \tau_{\max} = \frac{\sigma_x}{2}$$

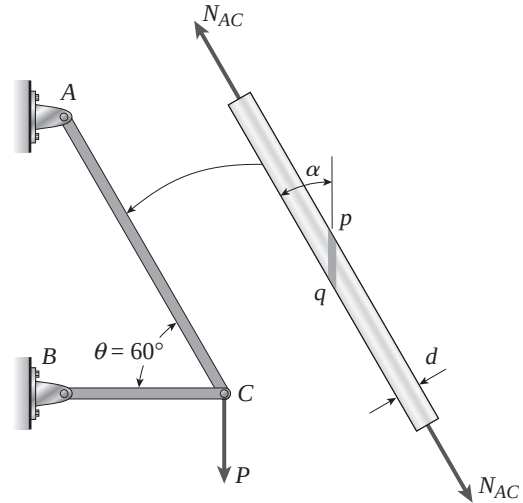
set $\tau_{\max} = \tau_a$ and solve for $P_{\max 2}$

$$2\tau_a = -E\alpha\Delta T - \frac{\beta P}{A}$$

$$\beta = \frac{-A}{P} [-2\tau_a + E\alpha\Delta T]$$

$$\beta = 0.62 \quad \leftarrow$$

Problem 2.6-13 A circular brass bar of diameter d is member AC in truss ABC which has load $P = 5000$ lb applied at joint C . Bar AC is composed of two segments brazed together on a plane pq making an angle $\alpha = 36^\circ$ with the axis of the bar (see figure). The allowable stresses in the brass are 13,500 psi in tension and 6500 psi in shear. On the brazed joint, the allowable stresses are 6000 psi in tension and 3000 psi in shear. What is the tensile force N_{AC} in bar AC ? What is the minimum required diameter $d_{\min}$ of bar AC ?

**Solution 2.6-13**

NUMERICAL DATA

$$P = 5 \text{ k} \quad \alpha = 36^\circ \quad \sigma_a = 13.5 \text{ ksi}$$

$$\tau_a = 6.5 \text{ ksi}$$

$$\theta = \frac{\pi}{2} - \alpha \quad \theta = 54^\circ$$

$$\sigma_{ja} = 6.0 \text{ ksi}$$

$$\tau_{ja} = 3.0 \text{ ksi}$$

Tensile force N_{AC} using Method of Joints at C :

$$N_{AC} = \frac{P}{\sin(60^\circ)} \quad (\text{tension})$$

$$N_{AC} = 5.77 \text{ k} \quad \leftarrow$$

Minimum required diameter of bar AC :

(1) Check tension and shear in bars; $\tau_a < \sigma_a/2$ so shear

$$\text{controls } \tau_{\max} = \frac{\sigma_x}{2}:$$

$$2\tau_a = \frac{N_{AC}}{A} \quad \sigma_x = 2\tau_a = 13 \text{ ksi}$$

$$A_{\text{reqd}} = \frac{N_{AC}}{2\tau_a} \quad A_{\text{reqd}} = 0.44 \text{ in.}^2$$

$$d_{\min} = \sqrt{\frac{4}{\pi} A_{\text{reqd}}} \quad d_{\min} = 0.75 \text{ in.}$$

(2) Check tension and shear on brazed joint:

$$\sigma_x = \frac{N_{AC}}{A} \quad \sigma_x = \frac{N_{AC}}{\frac{\pi}{4} d^2} \quad d_{\text{reqd}} = \sqrt{\frac{4}{\pi} \frac{N_{AC}}{\sigma_x}}$$

Tension on brazed joint:

$$\sigma_\theta = \sigma_x \cos(\theta)^2$$

Set equal to σ_{ja} and solve for σ_x , then d_{reqd} :

$$\sigma_x = \frac{\sigma_{ja}}{\cos(\theta)^2} \quad \sigma_x = 17.37 \text{ ksi}$$

$$d_{\text{reqd}} = \sqrt{\frac{4}{\pi} \frac{N_{AC}}{\sigma_x}} \quad d_{\text{reqd}} = 0.65 \text{ in.}$$

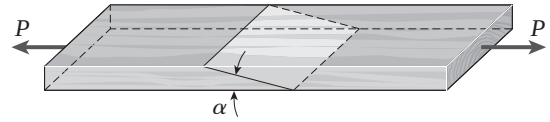
Shear on brazed joint:

$$\tau_\theta = -\sigma_x \sin(\theta) \cos(\theta)$$

$$\sigma_x = \left| \frac{\tau_{ja}}{-(\sin(\theta) \cos(\theta))} \right| \quad \sigma_x = -6.31 \text{ ksi}$$

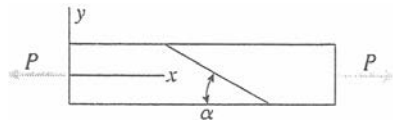
$$d_{\text{reqd}} = \sqrt{\frac{4}{\pi} \frac{N_{AC}}{\sigma_x}} \quad d_{\text{reqd}} = 1.08 \text{ in.} \quad \leftarrow$$

Problem 2.6-14 Two boards are joined by gluing along a scarf joint, as shown in the figure. For purposes of cutting and gluing, the angle α between the plane of the joint and the faces of the boards must be between 10° and 40° . Under a tensile load P , the normal stress in the boards is 4.9 MPa.



- What are the normal and shear stresses acting on the glued joint if $\alpha = 20^\circ$?
- If the allowable shear stress on the joint is 2.25 MPa, what is the largest permissible value of the angle α ?
- For what angle α will the shear stress on the glued joint be numerically equal to twice the normal stress on the joint?

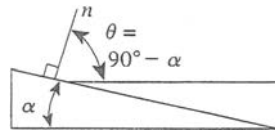
Solution 2.6-14 Two boards joined by a scarf joint



$$10^\circ \leq \alpha \leq 40^\circ$$

Due to load P : $\sigma_x = 4.9$ MPa

- (a) STRESSES ON JOINT WHEN $\alpha = 20^\circ$



$$\theta = 90^\circ - \alpha = 70^\circ$$

$$\begin{aligned}\sigma_\theta &= \sigma_x \cos^2 \theta = (4.9 \text{ MPa})(\cos 70^\circ)^2 \\ &= 0.57 \text{ MPa} \quad \leftarrow\end{aligned}$$

$$\begin{aligned}\tau_\theta &= -\sigma_x \sin \theta \cos \theta \\ &= (-4.9 \text{ MPa})(\sin 70^\circ)(\cos 70^\circ) \\ &= -1.58 \text{ MPa} \quad \leftarrow\end{aligned}$$

- (b) LARGEST ANGLE α IF $\tau_{\text{allow}} = 2.25$ MPa

$$\tau_{\text{allow}} = -\sigma_x \sin \theta \cos \theta$$

The shear stress on the joint has a negative sign. Its numerical value cannot exceed $\tau_{\text{allow}} = 2.25$ MPa. Therefore,

$$-2.25 \text{ MPa} = -(4.9 \text{ MPa})(\sin \theta)(\cos \theta) \text{ or } \sin \theta \cos \theta = 0.4592$$

$$\text{From trigonometry: } \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$$

$$\text{Therefore: } \sin 2\theta = 2(0.4592) = 0.9184$$

$$\text{Solving: } 2\theta = 66.69^\circ \quad \text{or} \quad 113.31^\circ$$

$$\theta = 33.34^\circ \quad \text{or} \quad 56.66^\circ$$

$$\alpha = 90^\circ - \theta \quad \therefore \alpha = 56.66^\circ \quad \text{or} \quad 33.34^\circ$$

Since α must be between 10° and 40° , we select

$$\alpha = 33.3^\circ \quad \leftarrow$$

NOTE: If α is between 10° and 33.3° ,

$$|\tau_\theta| < 2.25 \text{ MPa.}$$

If α is between 33.3° and 40° ,

$$|\tau_\theta| > 2.25 \text{ MPa.}$$

- (c) WHAT IS α IF $\tau_\theta = 2\sigma_\theta$?

Numerical values only:

$$|\tau_\theta| = \sigma_x \sin \theta \cos \theta \quad |\sigma_\theta| = \sigma_x \cos^2 \theta$$

$$\left| \frac{\tau_\theta}{\sigma_\theta} \right| = 2$$

$$\sigma_x \sin \theta \cos \theta = 2\sigma_x \cos^2 \theta$$

$$\sin \theta = 2 \cos \theta \quad \text{or} \quad \tan \theta = 2$$

$$\theta = 63.43^\circ \quad \alpha = 90^\circ - \theta$$

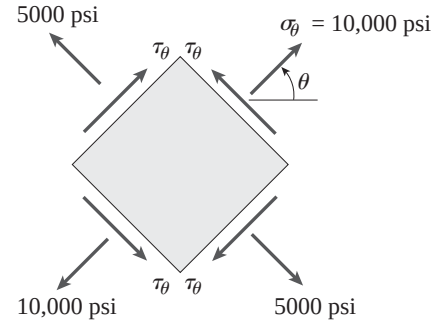
$$\alpha = 26.6^\circ \quad \leftarrow$$

NOTE: For $\alpha = 26.6^\circ$ and $\theta = 63.4^\circ$, we find $\sigma_\theta = 0.98$ MPa and $\tau_\theta = -1.96$ MPa.

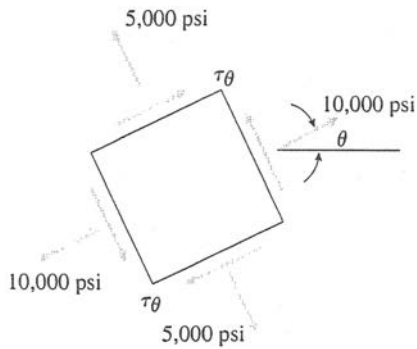
$$\text{Thus, } \left| \frac{\tau_\theta}{\sigma_\theta} \right| = 2 \text{ as required.}$$

Problem 2.6-15 Acting on the sides of a stress element cut from a bar in uniaxial stress are tensile stresses of 10,000 psi and 5,000 psi, as shown in the figure.

- Determine the angle θ and the shear stress τ_θ and show all stresses on a sketch of the element.
- Determine the maximum normal stress $\sigma_{\max}$ and the maximum shear stress $\tau_{\max}$ in the material.



Solution 2.6-15 Bar in uniaxial stress



- (a) ANGLE θ AND SHEAR STRESS τ_θ

$$\sigma_\theta = \sigma_x \cos^2 \theta$$

$$\sigma_\theta = 10,000 \text{ psi}$$

$$\sigma_x = \frac{\sigma_\theta}{\cos^2 \theta} = \frac{10,000 \text{ psi}}{\cos^2 \theta}$$

PLANE AT ANGLE $\theta + 90^\circ$

$$\sigma_{\theta+90^\circ} = \sigma_x [\cos(\theta + 90^\circ)]^2 = \sigma_x [-\sin \theta]^2 = \sigma_x \sin^2 \theta$$

$$\sigma_{\theta+90^\circ} = 5,000 \text{ psi}$$

$$\sigma_x = \frac{\sigma_{\theta+90^\circ}}{\sin^2 \theta} = \frac{5,000 \text{ psi}}{\sin^2 \theta}$$

Equate (1) and (2):

$$\frac{10,000 \text{ psi}}{\cos^2 \theta} = \frac{5,000 \text{ psi}}{\sin^2 \theta}$$

$$\tan^2 \theta = \frac{1}{2} \quad \tan \theta = \frac{1}{\sqrt{2}} \quad \theta = 35.26^\circ \quad \leftarrow$$

From Eq. (1) or (2):

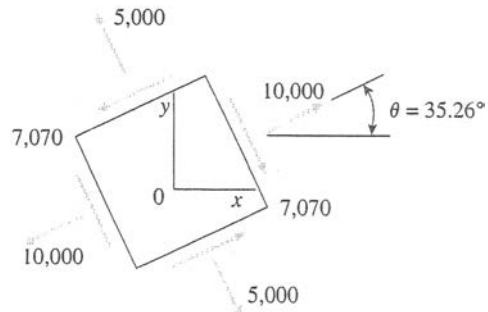
$$\sigma_x = 15,000 \text{ psi}$$

$$\tau_\theta = -\sigma_x \sin \theta \cos \theta$$

$$= (-15,000 \text{ psi})(\sin 35.26^\circ)(\cos 35.26^\circ)$$

$$= -7,070 \text{ psi} \quad \leftarrow$$

Minus sign means that τ_θ acts clockwise on the plane for which $\theta = 35.26^\circ$.



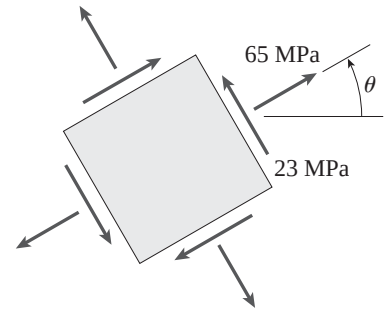
NOTE: All stresses have units of psi.

- (b) MAXIMUM NORMAL AND SHEAR STRESSES

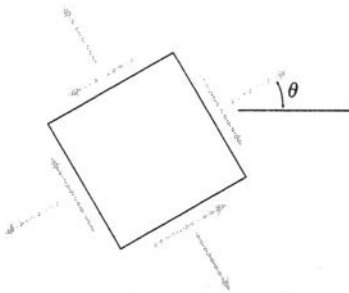
$$\sigma_{\max} = \sigma_x = 15,000 \text{ psi} \quad \leftarrow$$

$$\tau_{\max} = \frac{\sigma_x}{2} = 7,500 \text{ psi} \quad \leftarrow$$

Problem 2.6-16 A prismatic bar is subjected to an axial force that produces a tensile stress $\sigma_\theta = 65$ MPa and a shear stress $\tau_\theta = 23$ MPa on a certain inclined plane (see figure). Determine the stresses acting on all faces of a stress element oriented at $\theta = 30^\circ$ and show the stresses on a sketch of the element.



Solution 2.6-16



Find θ and σ_x for stress state shown in figure.

$$\sigma_\theta = \sigma_x \cos^2(\theta) \quad \cos(\theta) = \sqrt{\frac{\sigma_\theta}{\sigma_x}}$$

$$\text{so} \quad \sin(\theta) = \sqrt{1 - \frac{\sigma_\theta}{\sigma_x}}$$

$$\tau_\theta = -\sigma_x \sin(\theta) \cos(\theta)$$

$$\frac{\tau_\theta}{\sigma_x} = -\sqrt{1 - \frac{\sigma_\theta}{\sigma_x}} \sqrt{\frac{\sigma_\theta}{\sigma_x}}$$

$$\left(\frac{\tau_\theta}{\sigma_x}\right)^2 = \frac{\sigma_\theta}{\sigma_x} - \left(\frac{\sigma_\theta}{\sigma_x}\right)$$

$$\left(\frac{23}{\sigma_x}\right)^2 = \frac{65}{\sigma_x} - \left(\frac{65}{\sigma_x}\right)$$

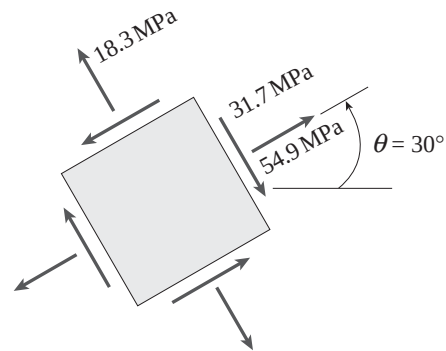
$$\left(\frac{65}{\sigma_x}\right)^2 - \left(\frac{65}{\sigma_x}\right) + \left(\frac{23}{\sigma_x}\right)^2 = 0$$

$$\frac{-(-4754 + 65\sigma_x)}{\sigma_x^2} = 0$$

$$\sigma_x = \frac{4754}{65}$$

$$\sigma_x = 73.1 \text{ MPa} \quad \sigma_\theta = 65 \text{ MPa}$$

$$\theta = \arccos\left(\sqrt{\frac{\sigma_\theta}{\sigma_x}}\right) \quad \theta = 19.5^\circ$$



Now find σ_θ and τ_θ for $\theta = 30^\circ$:

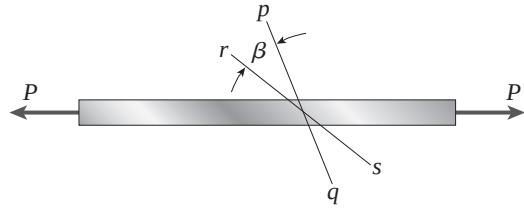
$$\sigma_{\theta 1} = \sigma_x \cos^2(\theta) \quad \sigma_{\theta 1} = 54.9 \text{ MPa} \quad \leftarrow$$

$$\tau_\theta = -\sigma_x \sin(\theta) \cos(\theta) \quad \tau_\theta = -31.7 \text{ MPa} \quad \leftarrow$$

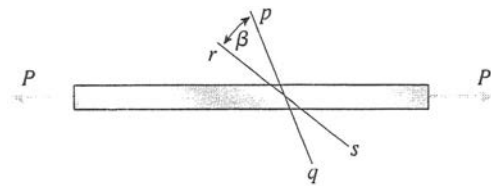
$$\sigma_{\theta 2} = \sigma_x \cos^2\left(\theta + \frac{\pi}{2}\right) \quad \sigma_{\theta 2} = 18.3 \text{ MPa} \quad \leftarrow$$

Problem 2.6-17 The normal stress on plane pq of a prismatic bar in tension (see figure) is found to be 7500 psi. On plane rs , which makes an angle $\beta = 30^\circ$ with plane pq , the stress is found to be 2500 psi.

Determine the maximum normal stress $\sigma_{\max}$ and maximum shear stress $\tau_{\max}$ in the bar.



Solution 2.6-17 Bar in tension



Eq. (2-31a):

$$\sigma_\theta = \sigma_x \cos^2 \theta$$

$$\beta = 30^\circ$$

$$\text{PLANE } pq: \sigma_1 = \sigma_x \cos^2 \theta_1 \quad \sigma_1 = 7500 \text{ psi}$$

$$\text{PLANE } rs: \sigma_2 = \sigma_x \cos^2(\theta_1 + \beta) \quad \sigma_2 = 2500 \text{ psi}$$

Equate σ_x from σ_1 and σ_2 :

$$\sigma_x = \frac{\sigma_1}{\cos^2 \theta_1} = \frac{\sigma_2}{\cos^2(\theta_1 + \beta)} \quad (\text{Eq. 1})$$

or

$$\frac{\cos^2 \theta_1}{\cos^2(\theta_1 + \beta)} = \frac{\sigma_1}{\sigma_2} \frac{\cos \theta_1}{\cos(\theta_1 + \beta)} = \sqrt{\frac{\sigma_1}{\sigma_2}} \quad (\text{Eq. 2})$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (2):

$$\frac{\cos \theta_1}{\cos(\theta_1 + 30^\circ)} = \sqrt{\frac{7500 \text{ psi}}{2500 \text{ psi}}} = \sqrt{3} = 1.7321$$

Solve by iteration or a computer program:

$$\theta_1 = 30^\circ$$

MAXIMUM NORMAL STRESS (FROM EQ. 1)

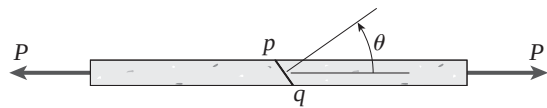
$$\begin{aligned} \sigma_{\max} = \sigma_x &= \frac{\sigma_1}{\cos^2 \theta_1} = \frac{7500 \text{ psi}}{\cos^2 30^\circ} \\ &= 10,000 \text{ psi} \quad \leftarrow \end{aligned}$$

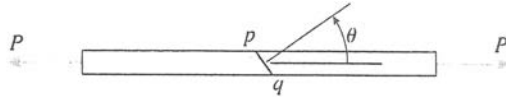
MAXIMUM SHEAR STRESS

$$\tau_{\max} = \frac{\sigma_x}{2} = 5,000 \text{ psi} \quad \leftarrow$$

Problem 2.6-18 A tension member is to be constructed of two pieces of plastic glued along plane pq (see figure). For purposes of cutting and gluing, the angle θ must be between 25° and 45° . The allowable stresses on the glued joint in tension and shear are 5.0 MPa and 3.0 MPa, respectively.

- Determine the angle θ so that the bar will carry the largest load P . (Assume that the strength of the glued joint controls the design.)
- Determine the maximum allowable load $P_{\max}$ if the cross-sectional area of the bar is 225 mm^2 .



Solution 2.6-18 Bar in tension with glued joint

$$25^\circ < \theta < 45^\circ$$

$$A = 225 \text{ mm}^2$$

$$\text{On glued joint: } \sigma_{\text{allow}} = 5.0 \text{ MPa}$$

$$\tau_{\text{allow}} = 3.0 \text{ MPa}$$

ALLOWABLE STRESS σ_x IN TENSION

$$\sigma_\theta = \sigma_x \cos^2 \theta \quad \sigma_x = \frac{\sigma_\theta}{\cos^2 \theta} = \frac{5.0 \text{ MPa}}{\cos^2 \theta} \quad (1)$$

$$\tau_\theta = -\sigma_x \sin \theta \cos \theta$$

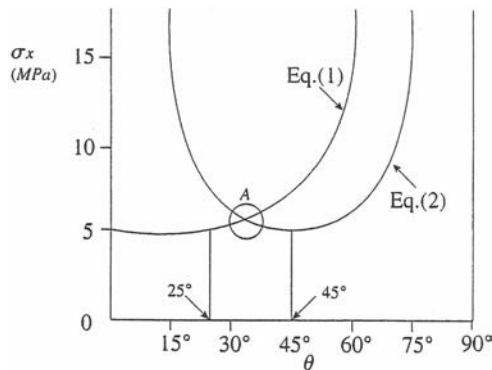
Since the direction of τ_θ is immaterial, we can write:

$$|\tau_\theta| = \sigma_x \sin \theta \cos \theta$$

or

$$\sigma_x = \frac{|\tau_\theta|}{\sin \theta \cos \theta} = \frac{3.0 \text{ MPa}}{\sin \theta \cos \theta} \quad (2)$$

GRAPH OF EQS. (1) AND (2)

**(a) DETERMINE ANGLE θ FOR LARGEST LOAD**

Point A gives the largest value of σ_x and hence the largest load. To determine the angle θ corresponding to point A, we equate Eqs. (1) and (2).

$$\frac{5.0 \text{ MPa}}{\cos^2 \theta} = \frac{3.0 \text{ MPa}}{\sin \theta \cos \theta}$$

$$\tan \theta = \frac{3.0}{5.0} \quad \theta = 30.96^\circ \quad \leftarrow$$

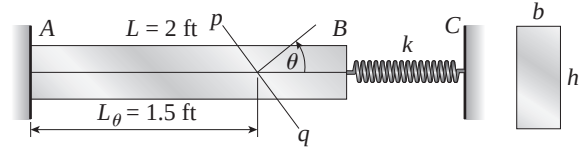
(b) DETERMINE THE MAXIMUM LOAD

From Eq. (1) or Eq. (2):

$$\sigma_x = \frac{5.0 \text{ MPa}}{\cos^2 \theta} = \frac{3.0 \text{ MPa}}{\sin \theta \cos \theta} = 6.80 \text{ MPa}$$

$$P_{\text{max}} = \sigma_x A = (6.80 \text{ MPa})(225 \text{ mm}^2) \\ = 1.53 \text{ kN} \quad \leftarrow$$

Problem 2.6-19 Plastic bar AB of rectangular cross section ($b = 0.75$ in. and $h = 1.5$ in.) and length $L = 2$ ft. is fixed at A and has a spring support ($k = 18$ k/in.) at C (see figure). Initially, the bar and spring have no stress. When the temperature of the bar is raised by 100°F , the compressive stress on an inclined plane pq at $L_\theta = 1.5$ ft becomes 950 psi. Assume the spring is massless and is unaffected by the temperature change. Let $\alpha = 55 \times 10^{-6}/^\circ\text{F}$ and $E = 400$ ksi.



- What is the shear stress τ_θ on plane pq ? What is angle θ ?
- Draw a stress element oriented to plane pq , and show the stresses acting on all faces of this element.
- If the allowable normal stress is ± 1000 psi and the allowable shear stress is ± 560 psi, what is the maximum permissible value of spring constant k if allowable stress values in the bar are not to be exceeded?
- What is the maximum permissible length L of the bar if allowable stress values in the bar are not to be exceeded? (Assume $k = 18$ k/in.)
- What is the maximum permissible temperature increase (ΔT) in the bar if allowable stress values in the bar are not to be exceeded? (Assume $L = 2$ ft and $k = 18$ k/in.)

Solution 2.6-19

NUMERICAL DATA

$$\alpha = 55(10^{-6}) \quad E = 400 \text{ ksi} \quad L = 2 \text{ ft} \quad \Delta T = 100 \quad k = 18 \text{ k/in.} \quad b = 0.75 \text{ in.} \quad h = 1.5 \text{ in.}$$

$$\sigma_\theta = -950 \text{ psi} \quad \sigma_a = -1000 \text{ psi} \quad \tau_a = -560 \text{ psi} \quad L_\theta = 1.5 \text{ ft} \quad A = bh \quad f = \frac{1}{k} = 5.556 \times 10^{-5} \text{ in./lb}$$

- (a) FIND θ AND T_θ

$$R_2 = \text{redundant} \quad R_2 = \frac{-\alpha \Delta T L}{\left(\frac{L}{EA}\right) + f} = -1.212 \times 10^3 \text{ lb} \quad \sigma_x = \frac{R_2}{A} = -1077.551 \text{ psi} \quad \sqrt{\frac{\sigma_\theta}{\sigma_x}} = 0.939$$

$$\theta = \arccos\left(\sqrt{\frac{\sigma_\theta}{\sigma_x}}\right) = 0.351 \quad \cos(2\theta) = 0.763 \quad \theta = 20.124^\circ$$

$$\sigma_x \cos(\theta)^2 = -950 \text{ psi} \quad \text{or} \quad \frac{\sigma_x}{2}(1 + \cos(2\theta)) = -950 \text{ psi} \quad \sigma_y = \sigma_x \cos\left(\theta + \frac{\pi}{2}\right)^2 = -127.551 \text{ psi}$$

$$\theta = 0.351 \quad \theta = 20.124^\circ \quad \sigma_x = -1077.551 \text{ psi} \quad 2\theta = 0.702$$

$$\tau_\theta = -\sigma_x \sin(\theta) \cos(\theta) = 348.1 \text{ psi} \quad \text{or} \quad \tau_\theta = \frac{-\sigma_x}{2} \sin(2\theta) = 348.1 \text{ psi}$$

$$\boxed{\tau_\theta = 348 \text{ psi}} \quad \boxed{\theta = 20.1^\circ}$$

- (b) FIND σ_{x1} AND σ_{y1}

$$\sigma_{x1} = \sigma_x \cos(\theta)^2 \quad \sigma_{y1} = \sigma_x \cos\left(\theta + \frac{\pi}{2}\right)^2$$

$$\boxed{\sigma_{x1} = -950 \text{ psi}} \quad \boxed{\sigma_{y1} = -127.6 \text{ psi}}$$

(c) GIVEN $L = 2$ ft, FIND $k_{\max}$

$$k_{\max 1} = \frac{\sigma_a A}{-\alpha \Delta T L - \sigma_a A \left(\frac{L}{EA} \right)} = 15625 \text{ lb/in.} < \text{controls (based on } \sigma_{\text{allow}})$$

$$\text{or } k_{\max 2} = \frac{2\tau_a A}{-\alpha \Delta T L - 2\tau_a A \left(\frac{L}{EA} \right)} = 19444.444 \text{ lb/in.} < \text{based on allowable shear stress}$$

$$\boxed{k_{\max} = 15625 \text{ lb/in.}}$$

(d) GIVEN ALLOWABLE NORMAL AND SHEAR STRESSES, FIND $L_{\max}$

$$k = 18000 \text{ lb/in.}$$

$$\sigma_x = \frac{R_2}{A} \quad \sigma_a A = \frac{-\alpha \Delta T L}{\left(\frac{L}{EA} \right) + f} \quad L_{\max 1} = \frac{\sigma_a A(f)}{-\left(\alpha \Delta T + \frac{\sigma_a}{E} \right)} = 1.736 \text{ ft} < \text{controls (based on } \sigma_{\text{allow}})$$

$$\text{or } L_{\max 2} = \frac{2\tau_a A(f)}{-\left(\alpha \Delta T + \frac{2\tau_a}{E} \right)} = 2.16 \text{ ft} < \text{based on } T_{\text{allow}}$$

$$\boxed{L_{\max} = 1.736 \text{ ft}}$$

(e) FIND $\Delta T_{\max}$ GIVEN L , k , AND ALLOWABLE STRESSES $k = 18000 \text{ lb/in.}$ $L = 2$ ft $\sigma_a = -1000$ psi
 $\tau_a = -560$ psi

$$\Delta T_{\max 1} = \frac{\left(\frac{L}{EA} + f \right) \sigma_a A}{-\alpha L} = 92.803^\circ\text{F} < \text{based on } \sigma_{\text{allow}} \quad \Delta T = 100$$

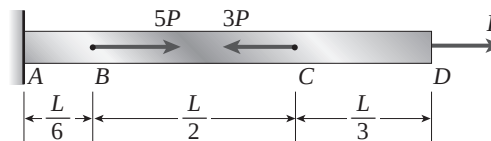
$$\Delta T_{\max 2} = \frac{\left(\frac{L}{EA} + f \right) 2\tau_a A}{-\alpha L} = 103.939^\circ\text{F} < \text{based on } T_{\text{allow}}$$

$$\boxed{\Delta T_{\max} = 92.8^\circ\text{F}}$$

Strain Energy

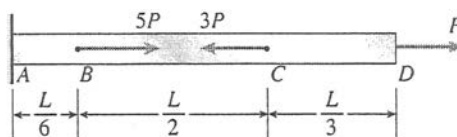
When solving the problems for Section 2.7, assume that the material behaves linearly elastically.

Problem 2.7-1 A prismatic bar AD of length L , cross-sectional area A , and modulus of elasticity E is subjected to loads $5P$, $3P$, and P acting at points B , C , and D , respectively (see figure). Segments AB , BC , and CD have lengths $L/6$, $L/2$, and $L/3$, respectively.



- Obtain a formula for the strain energy U of the bar.
- Calculate the strain energy if $P = 6$ k, $L = 52$ in., $A = 2.76$ in.², and the material is aluminum with $E = 10.4 \times 10^6$ psi.

Solution 2.7-1 Bar with three loads



$$P = 6 \text{ k}$$

$$L = 52 \text{ in.}$$

$$E = 10.4 \times 10^6 \text{ psi}$$

$$A = 2.76 \text{ in.}^2$$

INTERNAL AXIAL FORCES

$$N_{AB} = 3P \quad N_{BC} = -2P \quad N_{CD} = P$$

LENGTHS

$$L_{AB} = \frac{L}{6} \quad L_{BC} = \frac{L}{2} \quad L_{CD} = \frac{L}{3}$$

(a) STRAIN ENERGY OF THE BAR (EQ. 2-40)

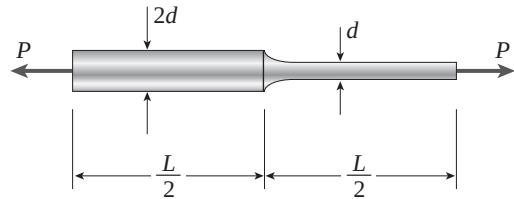
$$\begin{aligned} U &= \sum \frac{N_i^2 L_i}{2E_i A_i} \\ &= \frac{1}{2EA} \left[(3P)^2 \left(\frac{L}{6} \right) + (-2P)^2 \left(\frac{L}{2} \right) + (P)^2 \left(\frac{L}{3} \right) \right] \\ &= \frac{P^2 L}{2EA} \left(\frac{23}{6} \right) = \frac{23P^2 L}{12EA} \quad \leftarrow \end{aligned}$$

(b) SUBSTITUTE NUMERICAL VALUES:

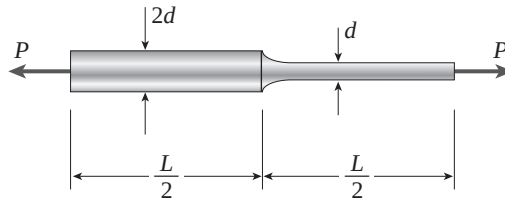
$$\begin{aligned} U &= \frac{23(6 \text{ k})^2(52 \text{ in.})}{12(10.4 \times 10^6 \text{ psi})(2.76 \text{ in.}^2)} \\ &= 125 \text{ in.-lb} \quad \leftarrow \end{aligned}$$

Problem 2.7-2 A bar of circular cross section having two different diameters d and $2d$ is shown in the figure. The length of each segment of the bar is $L/2$ and the modulus of elasticity of the material is E .

- Obtain a formula for the strain energy U of the bar due to the load P .
- Calculate the strain energy if the load $P = 27$ kN, the length $L = 600$ mm, the diameter $d = 40$ mm, and the material is brass with $E = 105$ GPa.



Solution 2.7-2 Bar with two segments



(a) STRAIN ENERGY OF THE BAR

Add the strain energies of the two segments of the bar (see Eq. 2-42).

$$U = \sum_{i=1}^2 \frac{N_i^2 L_i}{2 E_i A_i} = \frac{P^2 (L/2)}{2E} \left[\frac{1}{\frac{\pi}{4}(2d)^2} + \frac{1}{\frac{\pi}{4}(d^2)} \right]$$

$$= \frac{P^2 L}{\pi E} \left(\frac{1}{4d^2} + \frac{1}{d^2} \right) = \frac{5P^2 L}{4\pi E d^2} \quad \leftarrow$$

(b) SUBSTITUTE NUMERICAL VALUES:

$$P = 27 \text{ kN} \quad L = 600 \text{ mm}$$

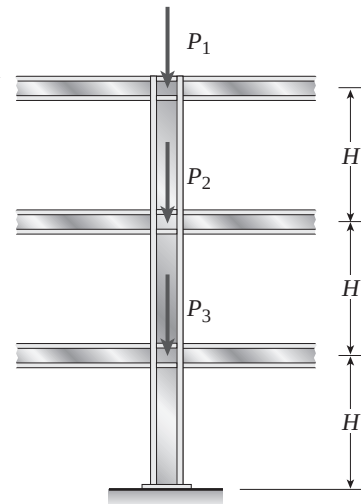
$$d = 40 \text{ mm} \quad E = 105 \text{ GPa}$$

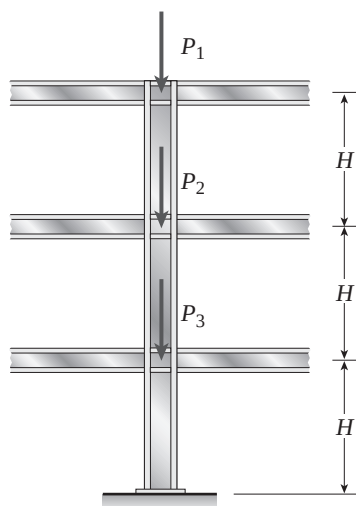
$$U = \frac{5(27 \text{ kN}^2)(600 \text{ mm})}{4\pi(105 \text{ GPa})(40 \text{ mm})^2}$$

$$= 1.036 \text{ N} \cdot \text{m} = 1.036 \text{ J} \quad \leftarrow$$

Problem 2.7-3 A three-story steel column in a building supports roof and floor loads as shown in the figure. The story height H is 10.5 ft, the cross-sectional area A of the column is 15.5 in.^2 , and the modulus of elasticity E of the steel is 30×10^6 psi.

Calculate the strain energy U of the column assuming $P_1 = 40$ k and $P_2 = P_3 = 60$ k.



Solution 2.7-3 Three-story column

$$H = 10.5 \text{ ft} \quad E = 30 \times 10^6 \text{ psi}$$

$$A = 15.5 \text{ in.}^2 \quad P_1 = 40 \text{ k}$$

$$P_2 = P_3 = 60 \text{ k}$$

To find the strain energy of the column, add the strain energies of the three segments (see Eq. 2-42).

$$\text{Upper segment: } N_1 = -P_1$$

$$\text{Middle segment: } N_2 = -(P_1 + P_2)$$

$$\text{Lower segment: } N_3 = -(P_1 + P_2 + P_3)$$

STRAIN ENERGY

$$\begin{aligned} U &= \sum \frac{N_i^2 L_i}{2E_i A_i} \\ &= \frac{H}{2EA} [P_1^2 + (P_1 + P_2)^2 + (P_1 + P_2 + P_3)^2] \\ &= \frac{H}{2EA} [Q] \end{aligned}$$

$$[Q] = (40 \text{ k})^2 + (100 \text{ k})^2 + (160 \text{ k})^2 = 37,200 \text{ k}^2$$

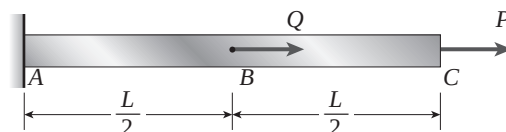
$$2EA = 2(30 \times 10^6 \text{ psi})(15.5 \text{ in.}^2) = 930 \times 10^6 \text{ lb}$$

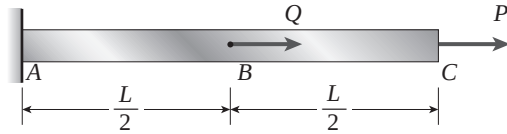
$$U = \frac{(10.5 \text{ ft})(12 \text{ in./ft})}{930 \times 10^6 \text{ lb}} [37,200 \text{ k}^2]$$

$$= 5040 \text{ in.-lb} \quad \leftarrow$$

Problem 2.7-4 The bar ABC shown in the figure is loaded by a force P acting at end C and by a force Q acting at the midpoint B . The bar has constant axial rigidity EA .

- Determine the strain energy U_1 of the bar when the force P acts alone ($Q = 0$).
- Determine the strain energy U_2 when the force Q acts alone ($P = 0$).
- Determine the strain energy U_3 when the forces P and Q act simultaneously upon the bar.



Solution 2.7-4 Bar with two loads(a) FORCE P ACTS ALONE ($Q = 0$)

$$U_1 = \frac{P^2 L}{2EA} \quad \leftarrow$$

(b) FORCE Q ACTS ALONE ($P = 0$)

$$U_2 = \frac{Q^2(L/2)}{2EA} = \frac{Q^2 L}{4EA} \quad \leftarrow$$

(c) FORCES P AND Q ACT SIMULTANEOUSLY

$$\text{Segment } BC: U_{BC} = \frac{P^2(L/2)}{2EA} = \frac{P^2 L}{4EA}$$

$$\begin{aligned} \text{Segment } AB: U_{AB} &= \frac{(P+Q)^2(L/2)}{2EA} \\ &= \frac{P^2 L}{4EA} + \frac{PQL}{2EA} + \frac{Q^2 L}{4EA} \end{aligned}$$

$$U_3 = U_{BC} + U_{AB} = \frac{P^2 L}{2EA} + \frac{PQL}{2EA} + \frac{Q^2 L}{4EA} \quad \leftarrow$$

(Note that U_3 is *not* equal to $U_1 + U_2$. In this case, $U_3 > U_1 + U_2$. However, if Q is reversed in direction, $U_3 < U_1 + U_2$. Thus, U_3 may be larger or smaller than $U_1 + U_2$.)

Problem 2.7-5 Determine the strain energy per unit volume (units of psi) and the strain energy per unit weight (units of in.) that can be stored in each of the materials listed in the accompanying table, assuming that the material is stressed to the proportional limit.

DATA FOR PROBLEM 2.7-5

Material	Weight density (lb/in. ³)	Modulus of elasticity (ksi)	Proportional limit (psi)
Mild steel	0.284	30,000	36,000
Tool steel	0.284	30,000	75,000
Aluminum	0.0984	10,500	60,000
Rubber (soft)	0.0405	0.300	300

Solution 2.7-5 Strain-energy density**DATA:**

Material	Weight density (lb/in. ³)	Modulus of elasticity (ksi)	Proportional limit (psi)
Mild steel	0.284	30,000	36,000
Tool steel	0.284	30,000	75,000
Aluminum	0.0984	10,500	60,000
Rubber (soft)	0.0405	0.300	300

STRAIN ENERGY PER UNIT VOLUME

$$U = \frac{P^2 L}{2EA} \quad \text{Volume } V = AL$$

$$\text{Stress } \sigma = \frac{P}{A}$$

$$u = \frac{U}{V} = \frac{\sigma_{PL}^2}{2E}$$

At the proportional limit:

$u = u_R = \text{modulus of resistance}$

$$u_R = \frac{\sigma_{PL}^2}{2E} \quad (\text{Eq. 1})$$

STRAIN ENERGY PER UNIT WEIGHT

$$U = \frac{P^2 L}{2EA} \quad \text{Weight } W = \gamma AL$$

$\gamma = \text{weight density}$

$$u_W = \frac{U}{W} = \frac{\sigma^2}{2\gamma E}$$

At the proportional limit:

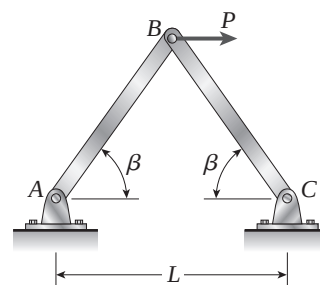
$$u_W = \frac{\sigma_{PL}^2}{2\gamma E} \quad (\text{Eq. 2})$$

RESULTS

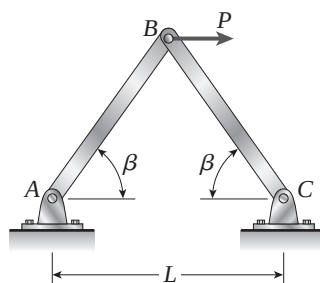
	u_R (psi)	u_W (in.)
Mild steel	22	76
Tool steel	94	330
Aluminum	171	1740
Rubber (soft)	150	3700

Problem 2.7-6 The truss ABC shown in the figure is subjected to a horizontal load P at joint B . The two bars are identical with cross-sectional area A and modulus of elasticity E .

- Determine the strain energy U of the truss if the angle $\beta = 60^\circ$.
- Determine the horizontal displacement δ_B of joint B by equating the strain energy of the truss to the work done by the load.



Solution 2.7-6 Truss subjected to a load P



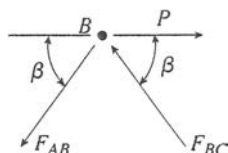
$$\beta = 60^\circ$$

$$L_{AB} = L_{BC} = L$$

$$\sin \beta = \sqrt{3}/2$$

$$\cos \beta = 1/2$$

FREE-BODY DIAGRAM OF JOINT B



$$\Sigma F_{\text{vert}} = 0 \quad \uparrow + \quad \downarrow -$$

$$-F_{AB} \sin \beta + F_{BC} \sin \beta = 0$$

$$F_{AB} = F_{BC} \quad (\text{Eq. 1})$$

$$\Sigma F_{\text{horiz}} = 0 \rightarrow + \quad \leftarrow -$$

$$-F_{AB} \cos \beta - F_{BC} \cos \beta + P = 0$$

$$F_{AB} = F_{BC} = \frac{P}{2 \cos \beta} = \frac{P}{2(1/2)} = P \quad (\text{Eq. 2})$$

Axial forces: $N_{AB} = P$ (tension)

$N_{BC} = -P$ (compression)

(a) STRAIN ENERGY OF TRUSS (EQ. 2-42)

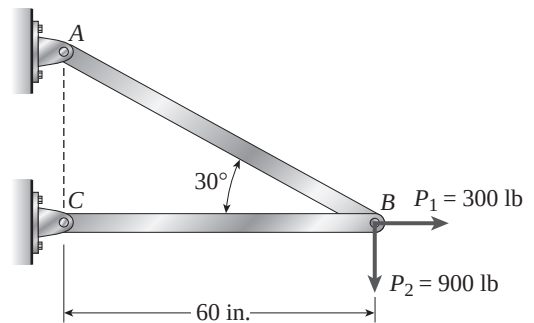
$$U = \sum \frac{N_i^2 L_i}{2E_i A_i} = \frac{(N_{AB})^2 L}{2EA} + \frac{(N_{BC})^2 L}{2EA} = \frac{P^2 L}{EA} \leftarrow$$

(b) HORIZONTAL DISPLACEMENT OF JOINT B (EQ. 2-44)

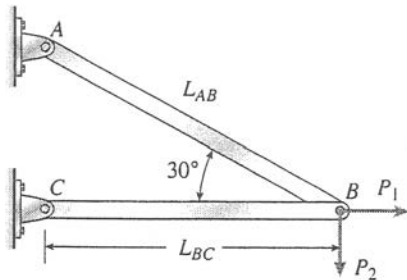
$$\delta_B = \frac{2U}{P} = \frac{2}{P} \left(\frac{P^2 L}{EA} \right) = \frac{2PL}{EA} \leftarrow$$

Problem 2.7-7 The truss ABC shown in the figure supports a horizontal load $P_1 = 300$ lb and a vertical load $P_2 = 900$ lb. Both bars have cross-sectional area $A = 2.4$ in.² and are made of steel with $E = 30 \times 10^6$ psi.

- Determine the strain energy U_1 of the truss when the load P_1 acts alone ($P_2 = 0$).
- Determine the strain energy U_2 when the load P_2 acts alone ($P_1 = 0$).
- Determine the strain energy U_3 when both loads act simultaneously.



Solution 2.7-7 Truss with two loads



$$P_1 = 300 \text{ lb}$$

$$P_2 = 900 \text{ lb}$$

$$A = 2.4 \text{ in.}^2$$

$$E = 30 \times 10^6 \text{ psi}$$

$$L_{BC} = 60 \text{ in.}$$

$$\beta = 30^\circ$$

$$\sin \beta = \sin 30^\circ = \frac{1}{2}$$

$$\cos \beta = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$L_{AB} = \frac{L_{BC}}{\cos 30^\circ} = \frac{120}{\sqrt{3}} \text{ in.} = 69.282 \text{ in.}$$

$$2EA = 2(30 \times 10^6 \text{ psi})(2.4 \text{ in.}^2) = 144 \times 10^6 \text{ lb}$$

FORCES F_{AB} AND F_{BC} IN THE BARS

From equilibrium of joint B:

$$F_{AB} = 2P_2 = 1800 \text{ lb}$$

$$F_{BC} = P_1 - P_2\sqrt{3} = 300 \text{ lb} - 1558.8 \text{ lb}$$

Force	P_1 alone	P_2 alone	P_1 and P_2
F_{AB}	0	1800 lb	1800 lb
F_{BC}	300 lb	-1558.8 lb	-1258.8 lb

(a) LOAD P_1 ACTS ALONE

$$U_1 = \frac{(F_{BC})^2 L_{BC}}{2EA} = \frac{(300 \text{ lb})^2 (60 \text{ in.})}{144 \times 10^6 \text{ lb}}$$

$$= 0.0375 \text{ in.-lb} \leftarrow$$

(b) LOAD P_2 ACTS ALONE

$$\begin{aligned}
 U_2 &= \frac{1}{2EA} \left[(F_{AB})^2 L_{AB} + (F_{BC})^2 L_{BC} \right] \\
 &= \frac{1}{2EA} \left[(1800 \text{ lb})^2 (69.282 \text{ in.}) \right. \\
 &\quad \left. + (-1558.8 \text{ lb})^2 (60 \text{ in.}) \right] \\
 &= \frac{370.265 \times 10^6 \text{ lb}^2\text{-in.}}{144 \times 10^6 \text{ lb}} = 2.57 \text{ in.-lb} \leftarrow
 \end{aligned}$$

(c) LOADS P_1 AND P_2 ACT SIMULTANEOUSLY

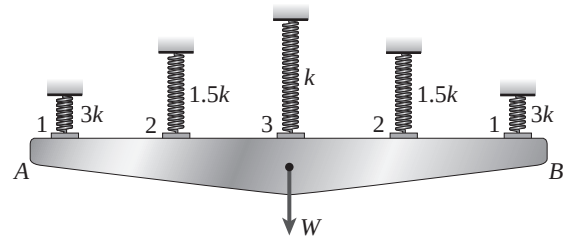
$$U_3 = \frac{1}{2EA} \left[(F_{AB})^2 L_{AB} + (F_{BC})^2 L_{BC} \right]$$

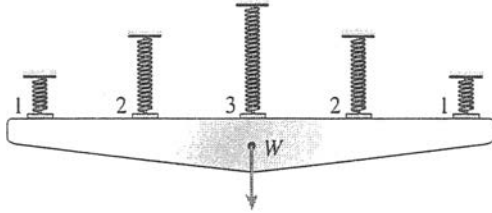
$$\begin{aligned}
 &= \frac{1}{2EA} \left[(1800 \text{ lb})^2 (69.282 \text{ in.}) \right. \\
 &\quad \left. + (-1258.8 \text{ lb})^2 (60 \text{ in.}) \right] \\
 &= \frac{319.548 \times 10^6 \text{ lb}^2\text{-in.}}{144 \times 10^6 \text{ lb}} \\
 &= 2.22 \text{ in.-lb} \leftarrow
 \end{aligned}$$

NOTE: The strain energy U_3 is *not* equal to $U_1 + U_2$.

Problem 2.7-8 The statically indeterminate structure shown in the figure consists of a horizontal rigid bar AB supported by five equally spaced springs. Springs 1, 2, and 3 have stiffnesses $3k$, $1.5k$, and k , respectively. When unstressed, the lower ends of all five springs lie along a horizontal line. Bar AB , which has weight W , causes the springs to elongate by an amount δ .

- Obtain a formula for the total strain energy U of the springs in terms of the downward displacement δ of the bar.
- Obtain a formula for the displacement δ by equating the strain energy of the springs to the work done by the weight W .
- Determine the forces F_1 , F_2 , and F_3 in the springs.
- Evaluate the strain energy U , the displacement δ , and the forces in the springs if $W = 600 \text{ N}$ and $k = 7.5 \text{ N/mm}$.



Solution 2.7-8 Rigid bar supported by springs

$$k_1 = 3k$$

$$k_2 = 1.5k$$

$$k_3 = k$$

δ = downward displacement of rigid bar

For a spring: $U = \frac{k\delta^2}{2}$ Eq. (2-40b)

(a) STRAIN ENERGY U OF ALL SPRINGS

$$U = 2\left(\frac{3k\delta^2}{2}\right) + 2\left(\frac{1.5k\delta^2}{2}\right) + \frac{k\delta^2}{2} = 5k\delta^2 \quad \leftarrow$$

(b) DISPLACEMENT δ

Work done by the weight W equals $\frac{W\delta}{2}$

Strain energy of the springs equals $5k\delta^2$

$$\therefore \frac{W\delta}{2} = 5k\delta^2 \quad \text{and} \quad \delta = \frac{W}{10k} \quad \leftarrow$$

(c) FORCES IN THE SPRINGS

$$F_1 = 3k\delta = \frac{3W}{10} \quad F_2 = 1.5k\delta = \frac{3W}{20} \quad \leftarrow$$

$$F_3 = k\delta = \frac{W}{10} \quad \leftarrow$$

(d) NUMERICAL VALUES

$$W = 600 \text{ N} \quad k = 7.5 \text{ N/mm} = 7500 \text{ N/mm}$$

$$U = 5k\delta^2 = 5k\left(\frac{W}{10k}\right)^2 = \frac{W^2}{20k}$$

$$= 2.4 \text{ N}\cdot\text{m} = 2.4 \text{ J} \quad \leftarrow$$

$$\delta = \frac{W}{10k} = 8.0 \text{ mm} \quad \leftarrow$$

$$F_1 = \frac{3W}{10} = 180 \text{ N} \quad \leftarrow$$

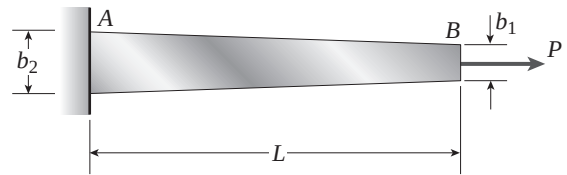
$$F_2 = \frac{3W}{20} = 90 \text{ N} \quad \leftarrow$$

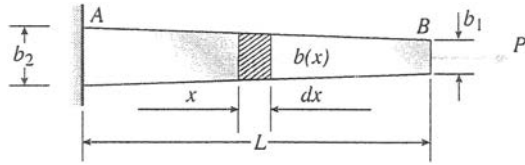
$$F_3 = \frac{W}{10} = 60 \text{ N} \quad \leftarrow$$

NOTE: $W = 2F_1 + 2F_2 + F_3 = 600 \text{ N}$ (Check)

Problem 2.7-9 A slightly tapered bar AB of rectangular cross section and length L is acted upon by a force P (see figure). The width of the bar varies uniformly from b_2 at end A to b_1 at end B . The thickness t is constant.

- Determine the strain energy U of the bar.
- Determine the elongation δ of the bar by equating the strain energy to the work done by the force P .



Solution 2.7-9 Tapered bar of rectangular cross section

$$b(x) = b_2 - \frac{(b_2 - b_1)x}{L}$$

$$A(x) = tb(x)$$

$$= t \left[b_2 - \frac{(b_2 - b_1)x}{L} \right]$$

(a) STRAIN ENERGY OF THE BAR

$$U = \int_0^L \frac{[N(x)]^2 dx}{2EA(x)} \quad (\text{Eq. 2-43})$$

$$= \int_0^L \frac{P^2 dx}{2Et b(x)} = \frac{P^2}{2Et} \int_0^L \frac{dx}{b_2 - (b_2 - b_1)\frac{x}{L}} \quad (1)$$

From Appendix C: $\int \frac{dx}{a + bx} = \frac{1}{b} \ln(a + bx)$

Apply this integration formula to Eq. (1):

$$U = \frac{P^2}{2Et} \left[\frac{1}{-(b_2 - b_1)(\frac{1}{L})} \ln \left[b_2 - \frac{(b_2 - b_1)x}{L} \right] \right]_0^L$$

$$= \frac{P^2}{2Et} \left[\frac{-L}{(b_2 - b_1)} \ln b_1 - \frac{-L}{(b_2 - b_1)} \ln b_2 \right]$$

$$U = \frac{P^2 L}{2Et(b_2 - b_1)} \ln \frac{b_2}{b_1} \quad \leftarrow$$

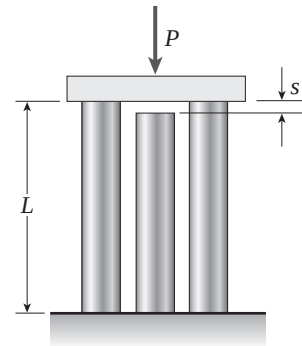
(b) ELONGATION OF THE BAR (Eq. 2-44)

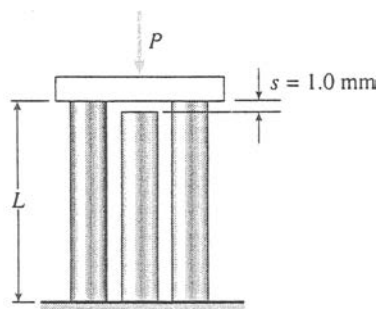
$$\delta = \frac{2U}{P} = \frac{PL}{Et(b_2 - b_1)} \ln \frac{b_2}{b_1} \quad \leftarrow$$

NOTE: This result agrees with the formula derived in Prob. 2.3-13.

Problem 2.7-10 A compressive load P is transmitted through a rigid plate to three magnesium-alloy bars that are identical except that initially the middle bar is slightly shorter than the other bars (see figure). The dimensions and properties of the assembly are as follows: length $L = 1.0$ m, cross-sectional area of each bar $A = 3000 \text{ mm}^2$, modulus of elasticity $E = 45 \text{ GPa}$, and the gap $s = 1.0$ mm.

- Calculate the load P_1 required to close the gap.
- Calculate the downward displacement δ of the rigid plate when $P = 400 \text{ kN}$.
- Calculate the total strain energy U of the three bars when $P = 400 \text{ kN}$.
- Explain why the strain energy U is *not* equal to $P\delta/2$.
(Hint: Draw a load-displacement diagram.)



Solution 2.7-10 Three bars in compression

$$s = 1.0 \text{ mm}$$

$$L = 1.0 \text{ m}$$

For each bar:

$$A = 3000 \text{ mm}^2$$

$$E = 45 \text{ GPa}$$

$$\frac{EA}{L} = 135 \times 10^6 \text{ N/m}$$

(a) LOAD P_1 REQUIRED TO CLOSE THE GAP

$$\text{In general, } \delta = \frac{PL}{EA} \text{ and } P = \frac{EA\delta}{L}$$

For two bars, we obtain:

$$P_1 = 2\left(\frac{EAs}{L}\right) = 2(135 \times 10^6 \text{ N/m})(1.0 \text{ mm})$$

$$P_1 = 270 \text{ kN} \quad \leftarrow$$

(b) DISPLACEMENT δ FOR $P = 400 \text{ kN}$

Since $P > P_1$, all three bars are compressed.

The force P equals P_1 plus the additional force required to compress all three bars by the amount $\delta - s$.

$$P = P_1 + 3\left(\frac{EA}{L}\right)(\delta - s)$$

$$\text{or } 400 \text{ kN} = 270 \text{ kN} + 3(135 \times 10^6 \text{ N/m})(\delta - 0.001 \text{ m})$$

$$\text{Solving, we get } \delta = 1.321 \text{ mm} \quad \leftarrow$$

(c) STRAIN ENERGY U FOR $P = 400 \text{ kN}$

$$U = \sum \frac{EA\delta^2}{2L}$$

$$\text{Outer bars: } \delta = 1.321 \text{ mm}$$

$$\begin{aligned} \text{Middle bar: } \delta &= 1.321 \text{ mm} - s \\ &= 0.321 \text{ mm} \end{aligned}$$

$$U = \frac{EA}{2L}[2(1.321 \text{ mm})^2 + (0.321 \text{ mm})^2]$$

$$= \frac{1}{2}(135 \times 10^6 \text{ N/m})(3.593 \text{ mm}^2)$$

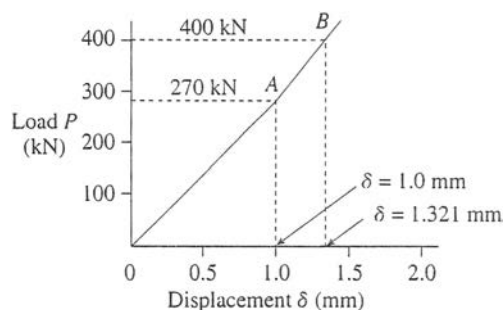
$$= 243 \text{ N}\cdot\text{m} = 243 \text{ J} \quad \leftarrow$$

(d) LOAD-DISPLACEMENT DIAGRAM

$$U = 243 \text{ J} = 243 \text{ N}\cdot\text{m}$$

$$\frac{P\delta}{2} = \frac{1}{2}(400 \text{ kN})(1.321 \text{ mm}) = 264 \text{ N}\cdot\text{m}$$

The strain energy U is *not* equal to $\frac{P\delta}{2}$ = because the load-displacement relation is not linear.

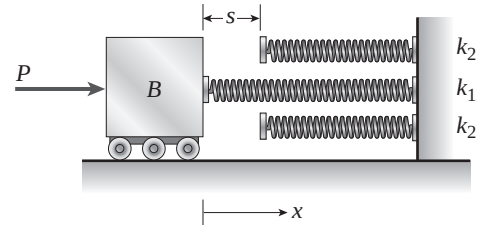


U = area under line OAB .

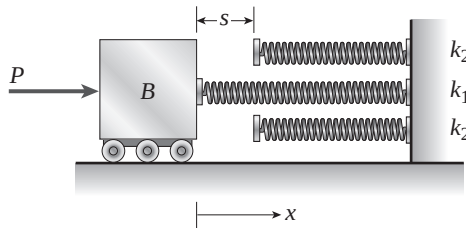
$\frac{P\delta}{2}$ = area under a straight line from O to B , which is larger than U .

Problem 2.7-11 A block B is pushed against three springs by a force P (see figure). The middle spring has stiffness k_1 and the outer springs each have stiffness k_2 . Initially, the springs are unstressed and the middle spring is longer than the outer springs (the difference in length is denoted s).

- Draw a force-displacement diagram with the force P as ordinate and the displacement x of the block as abscissa.
- From the diagram, determine the strain energy U_1 of the springs when $x = 2s$.
- Explain why the strain energy U_1 is not equal to $P\delta/2$, where $\delta = 2s$.



Solution 2.7-11 Block pushed against three springs



Force P_0 required to close the gap:

$$P_0 = k_1 s \quad (1)$$

FORCE-DISPLACEMENT RELATION BEFORE GAP IS CLOSED

$$P = k_1 x \quad (0 \leq x \leq s) \quad (0 \leq P \leq P_0) \quad (2)$$

FORCE-DISPLACEMENT RELATION AFTER GAP IS CLOSED

All three springs are compressed. Total stiffness equals $k_1 + 2k_2$. Additional displacement equals $x - s$. Force P equals P_0 plus the force required to compress all three springs by the amount $x - s$.

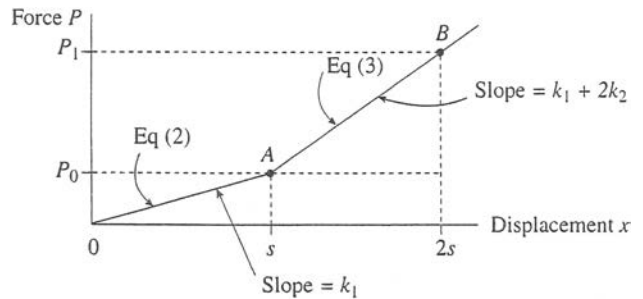
$$\begin{aligned} P &= P_0 + (k_1 + 2k_2)(x - s) \\ &= k_1 s + (k_1 + 2k_2)x - k_1 s - 2k_2 s \\ P &= (k_1 + 2k_2)x - 2k_2 s \quad (x \geq s); \quad (P \geq P_0) \end{aligned} \quad (3)$$

$$P_1 = \text{force } P \text{ when } x = 2s$$

Substitute $x = 2s$ into Eq. (3):

$$P_1 = 2(k_1 + k_2)s \quad (4)$$

(a) FORCE-DISPLACEMENT DIAGRAM



(b) STRAIN ENERGY U_1 WHEN $x = 2s$

$U_1 = \text{Area below force-displacement curve}$

$$= \text{triangle} + \text{rectangle} + \text{triangle}$$

$$= \frac{1}{2}P_0 s + P_0 s + \frac{1}{2}(P_1 - P_0)s = P_0 s + \frac{1}{2}P_1 s$$

$$= k_1 s^2 + (k_1 + k_2)s^2$$

$$U_1 = (2k_1 + k_2)s^2 \quad \leftarrow \quad (5)$$

(c) STRAIN ENERGY U_1 IS NOT EQUAL TO $\frac{P\delta}{2}$

$$\text{For } \delta = 2s: \frac{P\delta}{2} = \frac{1}{2} P_1(2s) = P_1s = 2(k_1 + k_2)s^2$$

(This quantity is greater than U_1 .)

U_1 = area under line OAB .

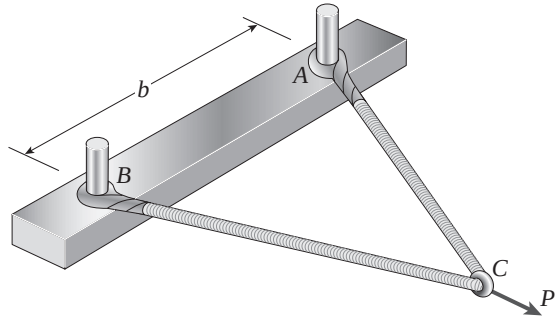
$\frac{P\delta}{2}$ = area under a straight line from O to B , which is larger than U_1 .

Thus, $\frac{P\delta}{2}$ is *not* equal to the strain energy because the force-displacement relation is not linear.

Problem 2.7-12 A bungee cord that behaves linearly elastically has an unstressed length $L_0 = 760$ mm and a stiffness $k = 140$ N/m. The cord is attached to two pegs, distance $b = 380$ mm apart, and pulled at its midpoint by a force $P = 80$ N (see figure).

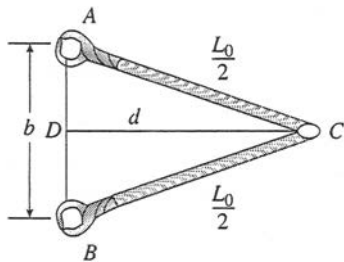
- How much strain energy U is stored in the cord?
- What is the displacement δ_C of the point where the load is applied?
- Compare the strain energy U with the quantity $P\delta_C/2$.

(Note: The elongation of the cord is *not* small compared to its original length.)



Solution 2.7-12 Bungee cord subjected to a load P .

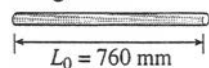
DIMENSIONS BEFORE THE LOAD P IS APPLIED



$$L_0 = 760 \text{ mm} \quad \frac{L_0}{2} = 380 \text{ mm}$$

$$b = 380 \text{ mm}$$

Bungee cord:

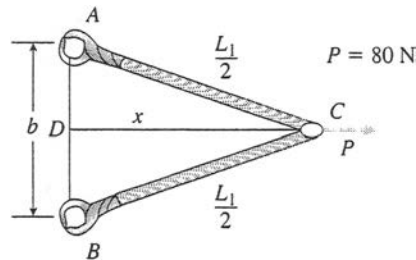


$$k = 140 \text{ N/m}$$

From triangle ACD :

$$d = \frac{1}{2} \sqrt{L_0^2 - b^2} = 329.09 \text{ mm} \quad (1)$$

DIMENSIONS AFTER THE LOAD P IS APPLIED



Let x = distance CD

Let L_1 = stretched length of bungee cord

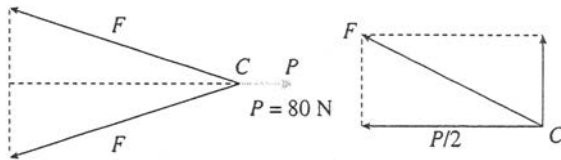
From triangle ACD:

$$\frac{L_1}{2} = \sqrt{\left(\frac{b}{2}\right)^2 + x^2} \quad (2)$$

$$L_1 = \sqrt{b^2 + 4x^2} \quad (3)$$

EQUILIBRIUM AT POINT C

Let F = tensile force in bungee cord



$$\begin{aligned} \frac{F}{P/2} &= \frac{L_1/2}{x} \quad F = \left(\frac{P}{2}\right)\left(\frac{L_1}{2}\right)\left(\frac{1}{x}\right) \\ &= \frac{P}{2}\sqrt{1 + \left(\frac{b}{2x}\right)^2} \end{aligned} \quad (4)$$

ELONGATION OF BUNGEE CORD

Let δ = elongation of the entire bungee cord

$$\delta = \frac{F}{k} = \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} \quad (5)$$

Final length of bungee cord = original length + δ

$$L_1 = L_0 + \delta = L_0 + \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} \quad (6)$$

SOLUTION OF EQUATIONS

Combine Eqs. (6) and (3):

$$L_1 = L_0 + \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} = \sqrt{b^2 + 4x^2}$$

$$\text{or } L_1 = L_0 + \frac{P}{4kx}\sqrt{b^2 + 4x^2} = \sqrt{b^2 + 4x^2}$$

$$L_0 = \left(1 - \frac{P}{4kx}\right)\sqrt{b^2 + 4x^2} \quad (7)$$

This equation can be solved for x .

SUBSTITUTE NUMERICAL VALUES INTO EQ. (7):

$$\begin{aligned} 760 \text{ mm} &= \left[1 - \frac{(80 \text{ N})(1000 \text{ mm/m})}{4(140 \text{ N/m})x}\right] \\ &\quad \times \sqrt{(380 \text{ mm})^2 + 4x^2} \end{aligned} \quad (8)$$

$$760 = \left(1 - \frac{142.857}{x}\right)\sqrt{144,400 + 4x^2} \quad (9)$$

Units: x is in millimeters

Solve for x (Use trial-and-error or a computer program):

$$x = 497.88 \text{ mm}$$

(a) STRAIN ENERGY U OF THE BUNGEE CORD

$$U = \frac{k\delta^2}{2} \quad k = 140 \text{ N/m} \quad P = 80 \text{ N}$$

From Eq. (5):

$$\delta = \frac{P}{2k}\sqrt{1 + \frac{b^2}{4x^2}} = 305.81 \text{ mm}$$

$$U = \frac{1}{2}(140 \text{ N/m})(305.81 \text{ mm})^2 = 6.55 \text{ N}\cdot\text{m}$$

$$U = 6.55 \text{ J} \quad \leftarrow$$

(b) DISPLACEMENT δ_C OF POINT C

$$\begin{aligned} \delta_C &= x - d = 497.88 \text{ mm} - 329.09 \text{ mm} \\ &= 168.8 \text{ mm} \quad \leftarrow \end{aligned}$$

(c) COMPARISON OF STRAIN ENERGY U WITH THE QUANTITY $P\delta_C/2$

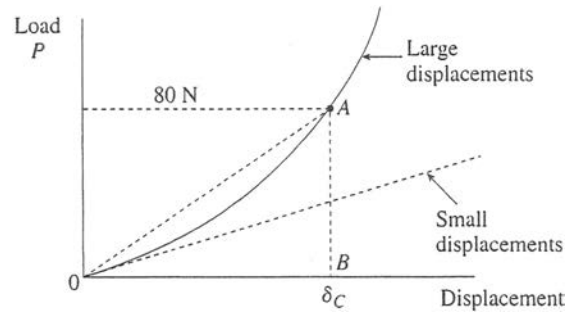
$$U = 6.55 \text{ J}$$

$$\frac{P\delta_C}{2} = \frac{1}{2}(80 \text{ N})(168.8 \text{ mm}) = 6.75 \text{ J}$$

The two quantities are not the same. The work done by the load P is *not* equal to $P\delta_C/2$ because the load-displacement relation (see below) is non-linear when the displacements are large. (The work done by the load P is equal to the strain energy because the bungee cord behaves elastically and there are no energy losses.)

$$U = \text{area } OAB \text{ under the curve } OA.$$

$$\frac{P\delta_C}{2} = \text{area of triangle } OAB, \text{ which is greater than } U.$$

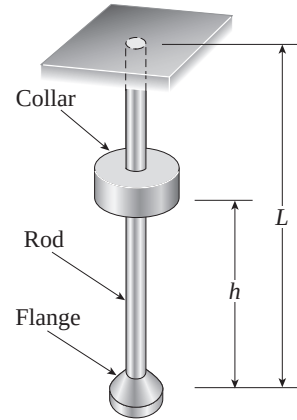


Impact Loading

The problems for Section 2.8 are to be solved on the basis of the assumptions and idealizations described in the text. In particular, assume that the material behaves linearly elastically and no energy is lost during the impact.

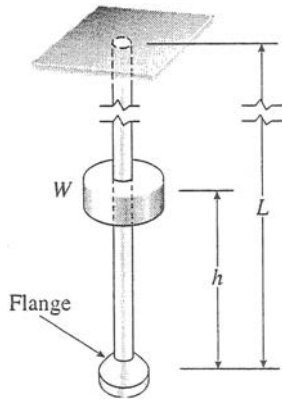
Problem 2.8-1 A sliding collar of weight $W = 150$ lb falls from a height $h = 2.0$ in. onto a flange at the bottom of a slender vertical rod (see figure). The rod has length $L = 4.0$ ft, cross-sectional area $A = 0.75$ in.², and modulus of elasticity $E = 30 \times 10^6$ psi.

Calculate the following quantities: (a) the maximum downward displacement of the flange, (b) the maximum tensile stress in the rod, and (c) the impact factor.



Probs. 2.8-1, 2.8-2, 2.8-3

Solution 2.8-1 Collar falling onto a flange



$$W = 150 \text{ lb}$$

$$h = 2.0 \text{ in.} \quad L = 4.0 \text{ ft} = 48 \text{ in.}$$

$$E = 30 \times 10^6 \text{ psi} \quad A = 0.75 \text{ in.}^2$$

(a) DOWNWARD DISPLACEMENT OF FLANGE

$$\delta_{st} = \frac{WL}{EA} = 0.00032 \text{ in.}$$

Eq. (2-55):

$$\begin{aligned} \delta_{\max} &= \delta_{st} \left[1 + \left(1 + \frac{2h}{\delta_{st}} \right)^{1/2} \right] \\ &= 0.0361 \text{ in.} \quad \leftarrow \end{aligned}$$

(b) MAXIMUM TENSILE STRESS (EQ. 2-57)

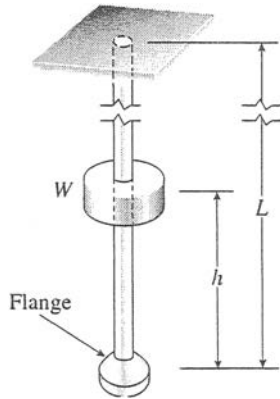
$$\sigma_{\max} = \frac{E\delta_{\max}}{L} = 22,600 \text{ psi} \quad \leftarrow$$

(c) IMPACT FACTOR (EQ. 2-63)

$$\begin{aligned} \text{Impact factor} &= \frac{\delta_{\max}}{\delta_{st}} = \frac{0.0361 \text{ in.}}{0.00032 \text{ in.}} \\ &= 113 \quad \leftarrow \end{aligned}$$

Problem 2.8-2 Solve the preceding problem if the collar has mass $M = 80$ kg, the height $h = 0.5$ m, the length $L = 3.0$ m, the cross-sectional area $A = 350 \text{ mm}^2$, and the modulus of elasticity $E = 170 \text{ GPa}$.

Solution 2.8-2 Collar falling onto a flange



$$\begin{aligned} M &= 80 \text{ kg} \\ W &= Mg = (80 \text{ kg})(9.81 \text{ m/s}^2) \\ &= 784.8 \text{ N} \\ h &= 0.5 \text{ m} \quad L = 3.0 \text{ m} \\ E &= 170 \text{ GPa} \quad A = 350 \text{ mm}^2 \end{aligned}$$

(a) DOWNWARD DISPLACEMENT OF FLANGE

$$\delta_{st} = \frac{WL}{EA} = 0.03957 \text{ mm}$$

$$\begin{aligned} \text{Eq. (2-53): } \delta_{\max} &= \delta_{st} \left[1 + \left(1 + \frac{2h}{\delta_{st}} \right)^{1/2} \right] \\ &= 6.33 \text{ mm} \quad \leftarrow \end{aligned}$$

(b) MAXIMUM TENSILE STRESS (Eq. 2-57)

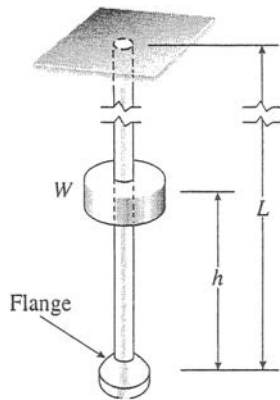
$$\sigma_{\max} = \frac{E\delta_{\max}}{L} = 359 \text{ MPa} \quad \leftarrow$$

(c) IMPACT FACTOR (Eq. 2-63)

$$\begin{aligned} \text{Impact factor} &= \frac{\delta_{\max}}{\delta_{st}} = \frac{6.33 \text{ mm}}{0.03957 \text{ mm}} \\ &= 160 \quad \leftarrow \end{aligned}$$

Problem 2.8-3 Solve Problem 2.8-1 if the collar has weight $W = 50$ lb, the height $h = 2.0$ in., the length $L = 3.0$ ft, the cross-sectional area $A = 0.25 \text{ in.}^2$, and the modulus of elasticity $E = 30,000 \text{ ksi}$.

Solution 2.8-3 Collar falling onto a flange



$$W = 50 \text{ lb} \quad h = 2.0 \text{ in.}$$

$$L = 3.0 \text{ ft} = 36 \text{ in.}$$

$$E = 30,000 \text{ psi} \quad A = 0.25 \text{ in.}^2$$

(a) DOWNWARD DISPLACEMENT OF FLANGE

$$\delta_{st} = \frac{WL}{EA} = 0.00024 \text{ in.}$$

$$\begin{aligned} \text{Eq. (2-55): } \delta_{\max} &= \delta_{st} \left[1 + \left(1 + \frac{2h}{\delta_{st}} \right)^{1/2} \right] \\ &= 0.0312 \text{ in.} \quad \leftarrow \end{aligned}$$

(b) MAXIMUM TENSILE STRESS (EQ. 2-57)

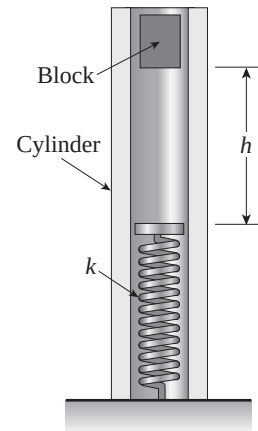
$$\sigma_{\max} = \frac{E\delta_{\max}}{L} = 26,000 \text{ psi} \quad \leftarrow$$

(c) IMPACT FACTOR (EQ. 2-63)

$$\text{Impact factor} = \frac{\delta_{\max}}{\delta_{st}} = \frac{0.0312 \text{ in.}}{0.00024 \text{ in.}} = 130 \quad \leftarrow$$

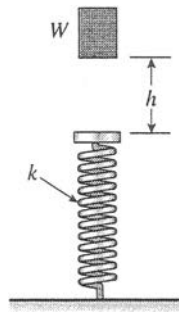
Problem 2.8-4 A block weighing $W = 5.0 \text{ N}$ drops inside a cylinder from a height $h = 200 \text{ mm}$ onto a spring having stiffness $k = 90 \text{ N/m}$ (see figure).

- (a) Determine the maximum shortening of the spring due to the impact and (b) determine the impact factor.



Prob. 2.8-4 and 2.8-5

Solution 2.8-4 Block dropping onto a spring



$$W = 5.0 \text{ N} \quad h = 200 \text{ mm} \quad k = 90 \text{ N/m}$$

(a) MAXIMUM SHORTENING OF THE SPRING

$$\delta_{st} = \frac{W}{k} = \frac{5.0 \text{ N}}{90 \text{ N/m}} = 55.56 \text{ mm}$$

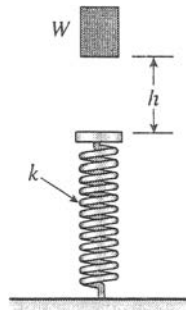
$$\begin{aligned} \text{Eq. (2-55): } \delta_{\max} &= \delta_{st} \left[1 + \left(1 + \frac{2h}{\delta_{st}} \right)^{1/2} \right] \\ &= 215 \text{ mm} \quad \leftarrow \end{aligned}$$

(b) IMPACT FACTOR (EQ. 2-63)

$$\begin{aligned} \text{Impact factor} &= \frac{\delta_{\max}}{\delta_{st}} = \frac{215 \text{ mm}}{55.56 \text{ mm}} \\ &= 3.9 \quad \leftarrow \end{aligned}$$

Problem 2.8-5 Solve the preceding problem if the block weighs $W = 1.0$ lb, $h = 12$ in., and $k = 0.5$ lb/in.

Solution 2.8-5 Block dropping onto a spring



$$W = 1.0 \text{ lb} \quad h = 12 \text{ in.} \quad k = 0.5 \text{ lb/in.}$$

(a) MAXIMUM SHORTENING OF THE SPRING

$$\delta_{st} = \frac{W}{k} = \frac{1.0 \text{ lb}}{0.5 \text{ lb/in.}} = 2.0 \text{ in.}$$

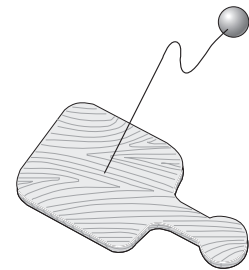
$$\begin{aligned} \text{Eq. (2-55): } \delta_{\max} &= \delta_{st} \left[1 + \left(1 + \frac{2h}{\delta_{st}} \right)^{1/2} \right] \\ &= 9.21 \text{ in.} \quad \leftarrow \end{aligned}$$

(b) IMPACT FACTOR (EQ. 2-63)

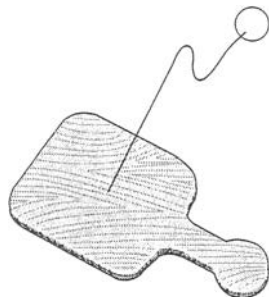
$$\begin{aligned} \text{Impact factor} &= \frac{\delta_{\max}}{\delta_{st}} = \frac{9.21 \text{ in.}}{2.0 \text{ in.}} \\ &= 4.6 \quad \leftarrow \end{aligned}$$

Problem 2.8-6 A small rubber ball (weight $W = 450$ mN) is attached by a rubber cord to a wood paddle (see figure). The natural length of the cord is $L_0 = 200$ mm, its cross-sectional area is $A = 1.6 \text{ mm}^2$, and its modulus of elasticity is $E = 2.0$ MPa. After being struck by the paddle, the ball stretches the cord to a total length $L_1 = 900$ mm.

What was the velocity v of the ball when it left the paddle? (Assume linearly elastic behavior of the rubber cord, and disregard the potential energy due to any change in elevation of the ball.)



Solution 2.8-6 Rubber ball attached to a paddle



$$\begin{aligned} g &= 9.81 \text{ m/s}^2 & E &= 2.0 \text{ MPa} \\ A &= 1.6 \text{ mm}^2 & L_0 &= 200 \text{ mm} \\ L_1 &= 900 \text{ mm} & W &= 450 \text{ mN} \end{aligned}$$

WHEN THE BALL LEAVES THE PADDLE

$$KE = \frac{Wv^2}{2g}$$

WHEN THE RUBBER CORD IS FULLY STRETCHED:

$$U = \frac{EA\delta^2}{2L_0} = \frac{EA}{2L_0}(L_1 - L_0)^2$$

CONSERVATION OF ENERGY

$$KE = U \quad \frac{Wv^2}{2g} = \frac{EA}{2L_0}(L_1 - L_0)^2$$

$$v^2 = \frac{gEA}{WL_0}(L_1 - L_0)^2$$

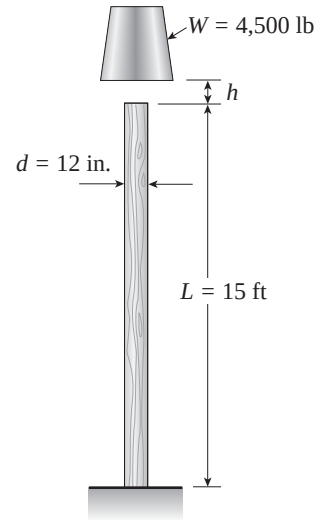
$$v = (L_1 - L_0) \sqrt{\frac{gEA}{WL_0}} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

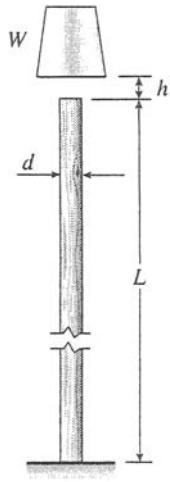
$$\begin{aligned} v &= (700 \text{ mm}) \sqrt{\frac{(9.81 \text{ m/s}^2)(2.0 \text{ MPa})(1.6 \text{ mm}^2)}{(450 \text{ mN})(200 \text{ mm})}} \\ &= 13.1 \text{ m/s} \quad \leftarrow \end{aligned}$$

Problem 2.8-7 A weight $W = 4500$ lb falls from a height h onto a vertical wood pole having length $L = 15$ ft, diameter $d = 12$ in., and modulus of elasticity $E = 1.6 \times 10^6$ psi (see figure).

If the allowable stress in the wood under an impact load is 2500 psi, what is the maximum permissible height h ?



Solution 2.8-7 Weight falling on a wood pole



$$W = 4500 \text{ lb} \quad d = 12 \text{ in.}$$

$$L = 15 \text{ ft} = 180 \text{ in.}$$

$$A = \frac{\pi d^2}{4} = 113.10 \text{ in.}^2$$

$$E = 1.6 \times 10^6 \text{ psi}$$

$$\sigma_{\text{allow}} = 2500 \text{ psi} (= \sigma_{\text{max}})$$

Find h_{max}

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{4500 \text{ lb}}{113.10 \text{ in.}^2} = 39.79 \text{ psi}$$

MAXIMUM HEIGHT h_{max}

$$\text{Eq. (2-61): } \sigma_{\text{max}} = \sigma_{st} \left[1 + \left(1 + \frac{2hE}{L\sigma_{st}} \right)^{1/2} \right]$$

or

$$\frac{\sigma_{\text{max}}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}} \right)^{1/2}$$

Square both sides and solve for h :

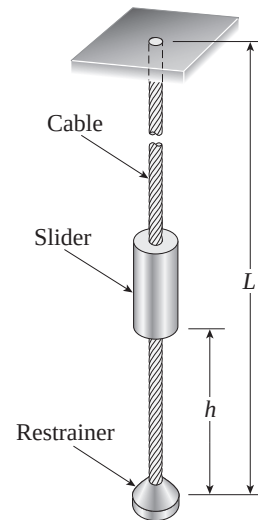
$$h = h_{\text{max}} = \frac{L\sigma_{\text{max}}}{2E} \left(\frac{\sigma_{\text{max}}}{\sigma_{st}} - 2 \right) \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\begin{aligned} h_{\text{max}} &= \frac{(180 \text{ in.})(2500 \text{ psi})}{2(1.6 \times 10^6 \text{ psi})} \left(\frac{2500 \text{ psi}}{39.79 \text{ psi}} - 2 \right) \\ &= 8.55 \text{ in.} \quad \leftarrow \end{aligned}$$

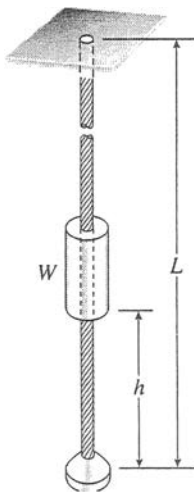
Problem 2.8-8 A cable with a restrainer at the bottom hangs vertically from its upper end (see figure). The cable has an effective cross-sectional area $A = 40 \text{ mm}^2$ and an effective modulus of elasticity $E = 130 \text{ GPa}$. A slider of mass $M = 35 \text{ kg}$ drops from a height $h = 1.0 \text{ m}$ onto the restrainer.

If the allowable stress in the cable under an impact load is 500 MPa , what is the minimum permissible length L of the cable?



Probs. 2.8-8, 2.8-2, 2.8-9

Solution 2.8-8 Slider on a cable



$$W = Mg = (35 \text{ kg})(9.81 \text{ m/s}^2) = 343.4 \text{ N}$$

$$A = 40 \text{ mm}^2 \quad E = 130 \text{ GPa}$$

$$h = 1.0 \text{ m} \quad \sigma_{\text{allow}} = \sigma_{\text{max}} = 500 \text{ MPa}$$

Find minimum length L_{min} .

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{343.4 \text{ N}}{40 \text{ mm}^2} = 8.585 \text{ MPa}$$

MINIMUM LENGTH L_{min}

$$\text{Eq. (2-61): } \sigma_{\text{max}} = \sigma_{st} \left[1 + \left(1 + \frac{2hE}{L\sigma_{st}} \right)^{1/2} \right]$$

or

$$\frac{\sigma_{\text{max}}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}} \right)^{1/2}$$

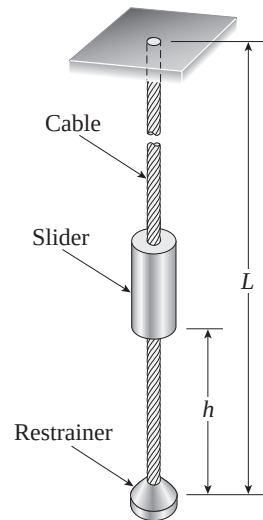
Square both sides and solve for L :

$$L = L_{\text{min}} = \frac{2Eh\sigma_{st}}{\sigma_{\text{max}}(\sigma_{\text{max}} - 2\sigma_{st})} \leftarrow$$

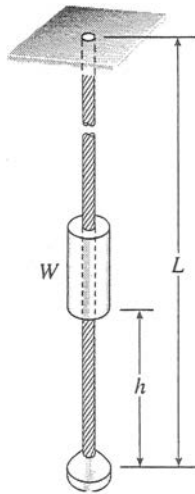
SUBSTITUTE NUMERICAL VALUES:

$$\begin{aligned} L_{\text{min}} &= \frac{2(130 \text{ GPa})(1.0 \text{ m})(8.585 \text{ MPa})}{(500 \text{ MPa})[500 \text{ MPa} - 2(8.585 \text{ MPa})]} \\ &= 9.25 \text{ mm} \leftarrow \end{aligned}$$

Problem 2.8-9 Solve the preceding problem if the slider has weight $W = 100$ lb, $h = 45$ in., $A = 0.080$ in.², $E = 21 \times 10^6$ psi, and the allowable stress is 70 ksi.



Solution 2.8-9 Slider on a cable



$$W = 100 \text{ lb}$$

$$A = 0.080 \text{ in.}^2 \quad E = 21 \times 10^6 \text{ psi}$$

$$h = 45 \text{ in.} \quad \sigma_{\text{allow}} = \sigma_{\text{max}} = 70 \text{ ksi}$$

Find minimum length L_{min} .

STATIC STRESS

$$\sigma_{st} = \frac{W}{A} = \frac{100 \text{ lb}}{0.080 \text{ in.}^2} = 1250 \text{ psi}$$

MINIMUM LENGTH L_{min}

$$\text{Eq. (2-61): } \sigma_{\text{max}} = \sigma_{st} \left[1 + \left(1 + \frac{2hE}{L\sigma_{st}} \right)^{1/2} \right]$$

or

$$\frac{\sigma_{\text{max}}}{\sigma_{st}} - 1 = \left(1 + \frac{2hE}{L\sigma_{st}} \right)^{1/2}$$

Square both sides and solve for L :

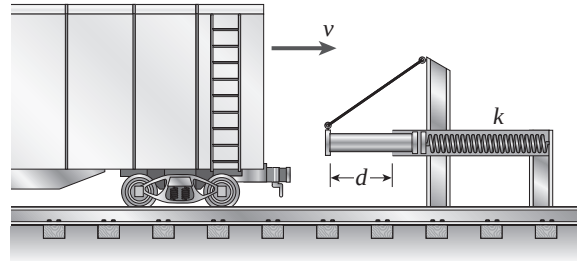
$$L = L_{\text{min}} = \frac{2Eh\sigma_{st}}{\sigma_{\text{max}}(\sigma_{\text{max}} - 2\sigma_{st})} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

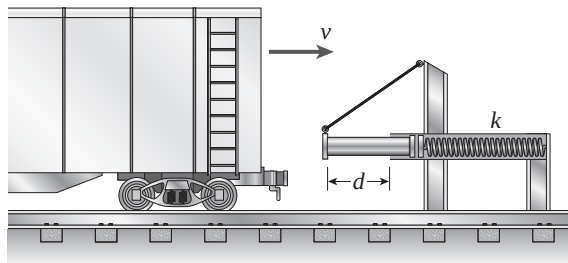
$$\begin{aligned} L_{\text{min}} &= \frac{2(21 \times 10^6 \text{ psi})(45 \text{ in.})(1250 \text{ psi})}{(70,000 \text{ psi})[70,000 \text{ psi} - 2(1250 \text{ psi})]} \\ &= 500 \text{ in.} \quad \leftarrow \end{aligned}$$

Problem 2.8-10 A bumping post at the end of a track in a railway yard has a spring constant $k = 8.0 \text{ MN/m}$ (see figure). The maximum possible displacement d of the end of the striking plate is 450 mm.

What is the maximum velocity $v_{\max}$ that a railway car of weight $W = 545 \text{ kN}$ can have without damaging the bumping post when it strikes it?



Solution 2.8-10 Bumping post for a railway car



$$k = 8.0 \text{ MN/m} \quad W = 545 \text{ kN}$$

$d = \text{maximum displacement of spring}$

$$d = \delta_{\max} = 450 \text{ mm}$$

Find $v_{\max}$.

KINETIC ENERGY BEFORE IMPACT

$$KE = \frac{Mv^2}{2} = \frac{Wv^2}{2g}$$

STRAIN ENERGY WHEN SPRING IS COMPRESSED TO THE MAXIMUM ALLOWABLE AMOUNT

$$U = \frac{k\delta_{\max}^2}{2} = \frac{kd^2}{2}$$

CONSERVATION OF ENERGY

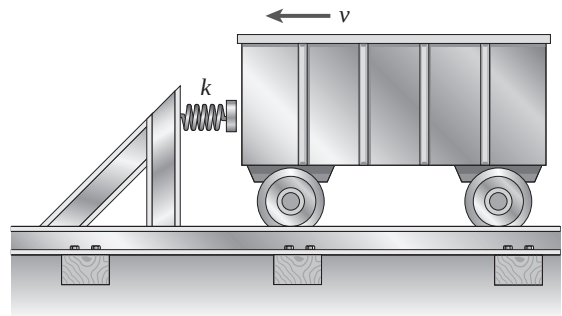
$$KE = U \quad \frac{Wv^2}{2g} = \frac{kd^2}{2} \quad v^2 = \frac{kd^2}{W/g}$$

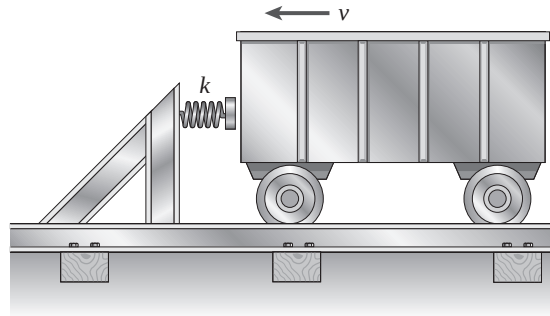
$$v = v_{\max} = d\sqrt{\frac{k}{W/g}} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\begin{aligned} v_{\max} &= (450 \text{ mm})\sqrt{\frac{8.0 \text{ MN/m}}{(545 \text{ kN})/(9.81 \text{ m/s}^2)}} \\ &= 5400 \text{ mm/s} = 5.4 \text{ m/s} \quad \leftarrow \end{aligned}$$

Problem 2.8-11 A bumper for a mine car is constructed with a spring of stiffness $k = 1120 \text{ lb/in.}$ (see figure). If a car weighing 3450 lb is traveling at velocity $v = 7 \text{ mph}$ when it strikes the spring, what is the maximum shortening of the spring?



Solution 2.8-11 Bumper for a mine car

$$k = 1120 \text{ lb/in.} \quad W = 3450 \text{ lb}$$

$$v = 7 \text{ mph} = 123.2 \text{ in./sec}$$

$$g = 32.2 \text{ ft/sec}^2 = 386.4 \text{ in./sec}^2$$

Find the shortening $\delta_{\max}$ of the spring.

KINETIC ENERGY JUST BEFORE IMPACT

$$KE = \frac{Mv^2}{2} = \frac{Wv^2}{2g}$$

STRAIN ENERGY WHEN SPRING IS FULLY COMPRESSED

$$U = \frac{k\delta_{\max}^2}{2}$$

Conservation of energy

$$KE = U \quad \frac{Wv^2}{2g} = \frac{k\delta_{\max}^2}{2}$$

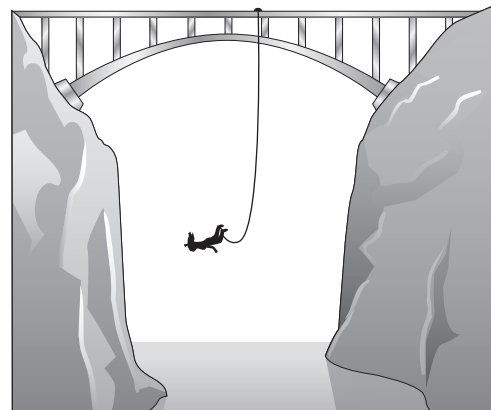
$$\text{Solve for } \delta_{\max}: \delta_{\max} = \sqrt{\frac{Wv^2}{gk}} \quad \leftarrow$$

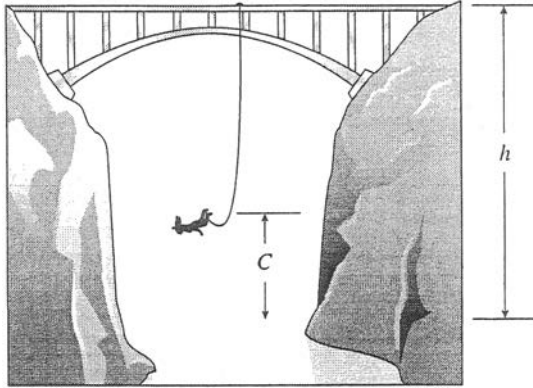
SUBSTITUTE NUMERICAL VALUES:

$$\begin{aligned} \delta_{\max} &= \sqrt{\frac{(3450 \text{ lb})(123.2 \text{ in./sec})^2}{(386.4 \text{ in./sec}^2)(1120 \text{ lb/in.})}} \\ &= 11.0 \text{ in.} \quad \leftarrow \end{aligned}$$

Problem 2.8-12 A bungee jumper having a mass of 55 kg leaps from a bridge, braking her fall with a long elastic shock cord having axial rigidity $EA = 2.3 \text{ kN}$ (see figure).

If the jumpoff point is 60 m above the water, and if it is desired to maintain a clearance of 10 m between the jumper and the water, what length L of cord should be used?



Solution 2.8-12 Bungee jumper

$$W = Mg = (55 \text{ kg})(9.81 \text{ m/s}^2) \\ = 539.55 \text{ N}$$

$$EA = 2.3 \text{ kN}$$

$$\text{Height: } h = 60 \text{ m}$$

$$\text{Clearance: } C = 10 \text{ m}$$

Find length L of the bungee cord.

$P.E.$ = Potential energy of the jumper at the top of bridge (with respect to lowest position)

$$= W(L + \delta_{\max})$$

U = strain energy of cord at lowest position

$$= \frac{EA\delta_{\max}^2}{2L}$$

CONSERVATION OF ENERGY

$$P.E. = U \quad W(L + \delta_{\max}) = \frac{EA\delta_{\max}^2}{2L}$$

$$\text{or} \quad \delta_{\max}^2 - \frac{2WL}{EA}\delta_{\max} - \frac{2WL^2}{EA} = 0$$

SOLVE QUADRATIC EQUATION FOR $\delta_{\max}$:

$$\delta_{\max} = \frac{WL}{EA} + \left[\left(\frac{WL}{EA} \right)^2 + 2L \left(\frac{WL}{EA} \right) \right]^{1/2} \\ = \frac{WL}{EA} \left[1 + \left(1 + \frac{2EA}{W} \right)^{1/2} \right]$$

VERTICAL HEIGHT

$$h = C + L + \delta_{\max}$$

$$h - C = L + \frac{WL}{EA} \left[1 + \left(1 + \frac{2EA}{W} \right)^{1/2} \right]$$

SOLVE FOR L :

$$L = \frac{h - C}{1 + \frac{W}{EA} \left[1 + \left(1 + \frac{2EA}{W} \right)^{1/2} \right]} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{W}{EA} = \frac{539.55 \text{ N}}{2.3 \text{ kN}} = 0.234587$$

$$\text{Numerator} = h - C = 60 \text{ m} - 10 \text{ m} = 50 \text{ m}$$

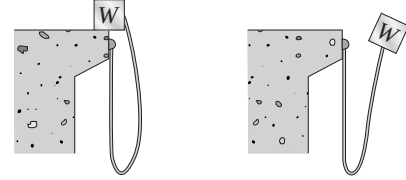
$$\text{Denominator} = 1 + (0.234587)$$

$$\times \left[1 + \left(1 + \frac{2}{0.234587} \right)^{1/2} \right]$$

$$= 1.9586$$

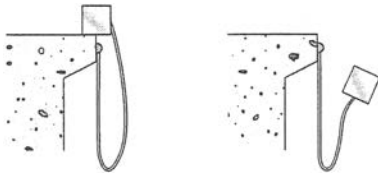
$$L = \frac{50 \text{ m}}{1.9586} = 25.5 \text{ m} \quad \leftarrow$$

Problem 2.8-13 A weight W rests on top of a wall and is attached to one end of a very flexible cord having cross-sectional area A and modulus of elasticity E (see figure). The other end of the cord is attached securely to the wall. The weight is then pushed off the wall and falls freely the full length of the cord.



- Derive a formula for the impact factor.
- Evaluate the impact factor if the weight, when hanging statically, elongates the band by 2.5% of its original length.

Solution 2.8-13 Weight falling off a wall



W = Weight

Properties of elastic cord:

E = modulus of elasticity

A = cross-sectional area

L = original length

$\delta_{\max}$ = elongation of elastic cord

$P.E.$ = potential energy of weight before fall (with respect to lowest position)

$P.E. = W(L + \delta_{\max})$

Let U = strain energy of cord at lowest position.

$$U = \frac{EA\delta_{\max}^2}{2L}$$

CONSERVATION OF ENERGY

$$P.E. = U \quad W(L + \delta_{\max}) = \frac{EA\delta_{\max}^2}{2L}$$

$$\text{or} \quad \delta_{\max}^2 - \frac{2WL}{EA} \delta_{\max} - \frac{2WL^2}{EA} = 0$$

SOLVE QUADRATIC EQUATION FOR $\delta_{\max}$:

$$\delta_{\max} = \frac{WL}{EA} + \left[\left(\frac{WL}{EA} \right)^2 + 2L \left(\frac{WL}{EA} \right) \right]^{1/2}$$

STATIC ELONGATION

$$\delta_{st} = \frac{WL}{EA}$$

IMPACT FACTOR

$$\frac{\delta_{\max}}{\delta_{st}} = 1 + \left[1 + \frac{2EA}{W} \right]^{1/2} \leftarrow$$

NUMERICAL VALUES

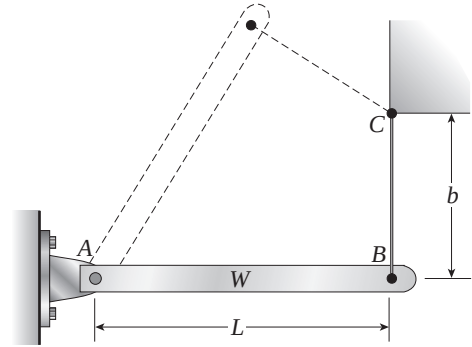
$$\delta_{st} = (2.5\%)(L) = 0.025L$$

$$\delta_{st} = \frac{WL}{EA} \quad \frac{W}{EA} = 0.025 \quad \frac{EA}{W} = 40$$

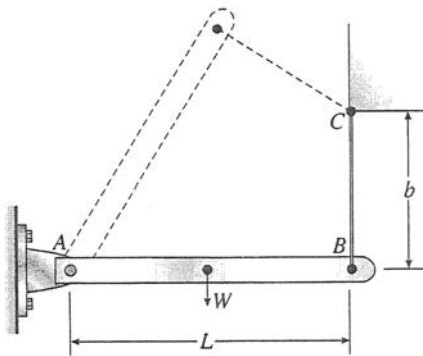
$$\text{Impact factor} = 1 + [1 + 2(40)]^{1/2} = 10 \leftarrow$$

Problem 2.8-14 A rigid bar AB having mass $M = 1.0$ kg and length $L = 0.5$ m is hinged at end A and supported at end B by a nylon cord BC (see figure). The cord has cross-sectional area $A = 30 \text{ mm}^2$, length $b = 0.25$ m, and modulus of elasticity $E = 2.1 \text{ GPa}$.

If the bar is raised to its maximum height and then released, what is the maximum stress in the cord?



Solution 2.8-14 Falling bar AB



RIGID BAR:

$$W = Mg = (1.0 \text{ kg})(9.81 \text{ m/s}^2)$$

$$= 9.81 \text{ N}$$

$$L = 0.5 \text{ m}$$

NYLON CORD:

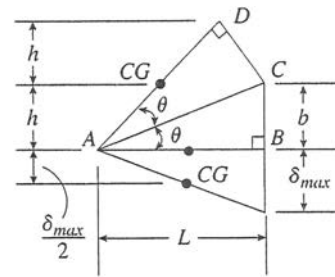
$$A = 30 \text{ mm}^2$$

$$b = 0.25 \text{ m}$$

$$E = 2.1 \text{ GPa}$$

Find maximum stress $\sigma_{\max}$ in cord BC .

GEOMETRY OF BAR AB AND CORD BC



$$\overline{CD} = \overline{CB} = b$$

$$\overline{AD} = \overline{AB} = L$$

h = height of center of gravity of raised bar AD

$\delta_{\max}$ = elongation of cord

$$\text{From triangle } ABC: \sin \theta = \frac{b}{\sqrt{b^2 + L^2}}$$

$$\cos \theta = \frac{L}{\sqrt{b^2 + L^2}}$$

$$\text{From line } AD: \sin 2\theta = \frac{2h}{AD} = \frac{2h}{L}$$

$$\text{From Appendix D: } \sin 2\theta = 2 \sin \theta \cos \theta$$

$$\therefore \frac{2h}{L} = 2 \left(\frac{b}{\sqrt{b^2 + L^2}} \right) \left(\frac{L}{\sqrt{b^2 + L^2}} \right) = \frac{2bL}{b^2 + L^2}$$

$$\text{and } h = \frac{bL^2}{b^2 + L^2} \quad (\text{Eq. 1})$$

CONSERVATION OF ENERGY

P.E. = potential energy of raised bar AD

$$= W \left(h + \frac{\delta_{\max}}{2} \right)$$

$$U = \text{strain energy of stretched cord} = \frac{EA\delta_{\max}^2}{2b}$$

$$P.E. = U \quad W \left(h + \frac{\delta_{\max}}{2} \right) = \frac{EA\delta_{\max}^2}{2b} \quad (\text{Eq. 2})$$

$$\text{For the cord: } \delta_{\max} = \frac{\sigma_{\max}b}{E}$$

Substitute into Eq. (2) and rearrange:

$$\sigma_{\max}^2 - \frac{W}{A} \sigma_{\max} - \frac{2WhE}{bA} = 0 \quad (\text{Eq. 3})$$

Substitute from Eq. (1) into Eq. (3):

$$\sigma_{\max}^2 - \frac{W}{A} \sigma_{\max} - \frac{2WL^2E}{A(b^2 + L^2)} = 0 \quad (\text{Eq. 4})$$

SOLVE FOR $\sigma_{\max}$:

$$\sigma_{\max} = \frac{W}{2A} \left[1 + \sqrt{1 + \frac{8L^2EA}{W(b^2 + L^2)}} \right] \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

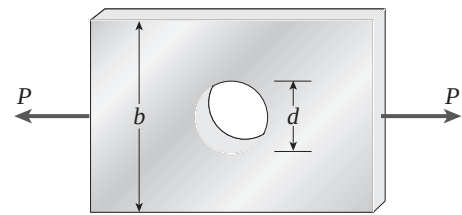
$$\sigma_{\max} = 33.3 \text{ MPa} \quad \leftarrow$$

Stress Concentrations

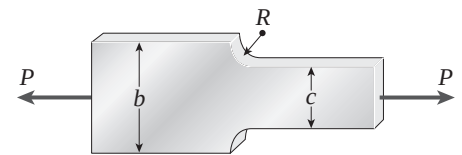
The problems for Section 2.10 are to be solved by considering the stress-concentration factors and assuming linearly elastic behavior.

Problem 2.10-1 The flat bars shown in parts (a) and (b) of the figure are subjected to tensile forces $P = 3.0 \text{ k}$. Each bar has thickness $t = 0.25 \text{ in}$.

- For the bar with a circular hole, determine the maximum stresses for hole diameters $d = 1 \text{ in.}$ and $d = 2 \text{ in.}$ if the width $b = 6.0 \text{ in.}$
- For the stepped bar with shoulder fillets, determine the maximum stresses for fillet radii $R = 0.25 \text{ in.}$ and $R = 0.5 \text{ in.}$ if the bar widths are $b = 4.0 \text{ in.}$ and $c = 2.5 \text{ in.}$

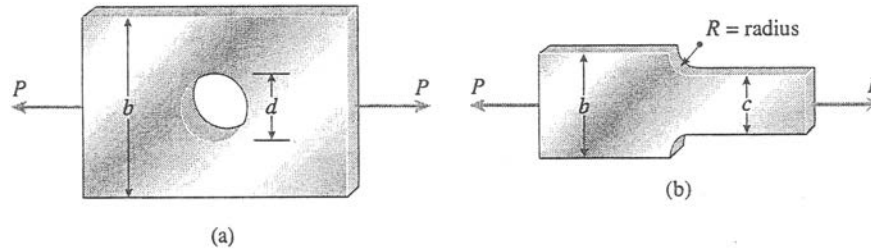


(a)



(b)

Probs. 2.10-1 and 2.10-2

Solution 2.10-1 Flat bars in tension

$$P = 3.0 \text{ k} \quad t = 0.25 \text{ in.}$$

(a) BAR WITH CIRCULAR HOLE ($b = 6 \text{ in.}$)

Obtain K from Fig. 2-63

FOR $d = 1 \text{ in.}$: $c = b - d = 5 \text{ in.}$

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{3.0 \text{ k}}{(5 \text{ in.})(0.25 \text{ in.})} = 2.40 \text{ ksi}$$

$$d/b = \frac{1}{6} \quad K \approx 2.60$$

$$\sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 6.2 \text{ ksi} \quad \leftarrow$$

FOR $d = 2 \text{ in.}$: $c = b - d = 4 \text{ in.}$

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{3.0 \text{ k}}{(4 \text{ in.})(0.25 \text{ in.})} = 3.00 \text{ ksi}$$

$$d/b = \frac{2}{6} \quad K \approx 2.31$$

$$\sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 6.9 \text{ ksi} \quad \leftarrow$$

(b) STEPPED BAR WITH SHOULDER FILLETS

$b = 4.0 \text{ in.}$ $c = 2.5 \text{ in.}$; Obtain k from Fig. 2-65

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{3.0 \text{ k}}{(2.5 \text{ in.})(0.25 \text{ in.})} = 4.80 \text{ ksi}$$

FOR $R = 0.25 \text{ in.}$: $R/c = 0.1$ $b/c = 1.60$

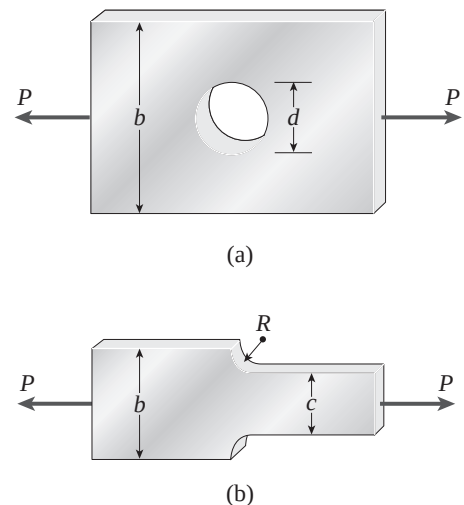
$$k \approx 2.30 \quad \sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 11.0 \text{ ksi} \quad \leftarrow$$

FOR $R = 0.5 \text{ in.}$: $R/c = 0.2$ $b/c = 1.60$

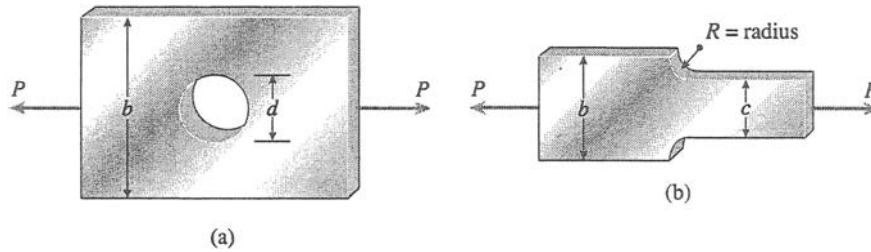
$$K \approx 1.87 \quad \sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 9.0 \text{ ksi} \quad \leftarrow$$

Problem 2.10-2 The flat bars shown in parts (a) and (b) of the figure are subjected to tensile forces $P = 2.5 \text{ kN}$. Each bar has thickness $t = 5.0 \text{ mm}$.

- For the bar with a circular hole, determine the maximum stresses for hole diameters $d = 12 \text{ mm}$ and $d = 20 \text{ mm}$ if the width $b = 60 \text{ mm}$.
- For the stepped bar with shoulder fillets, determine the maximum stresses for fillet radii $R = 6 \text{ mm}$ and $R = 10 \text{ mm}$ if the bar widths are $b = 60 \text{ mm}$ and $c = 40 \text{ mm}$.



Solution 2.10-2 Flat bars in tension



$$P = 2.5 \text{ kN} \quad t = 5.0 \text{ mm}$$

(a) BAR WITH CIRCULAR HOLE ($b = 60 \text{ mm}$)

Obtain K from Fig. 2-63

FOR $d = 12 \text{ mm}$: $c = b - d = 48 \text{ mm}$

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{2.5 \text{ kN}}{(48 \text{ mm})(5 \text{ mm})} = 10.42 \text{ MPa}$$

$$d/b = \frac{1}{5} \quad K \approx 2.51$$

$$\sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 26 \text{ MPa} \quad \leftarrow$$

FOR $d = 20 \text{ mm}$: $c = b - d = 40 \text{ mm}$

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{2.5 \text{ kN}}{(40 \text{ mm})(5 \text{ mm})} = 12.50 \text{ MPa}$$

$$d/b = \frac{1}{3} \quad K \approx 2.31$$

$$\sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 29 \text{ MPa} \quad \leftarrow$$

(b) STEPPED BAR WITH SHOULDER FILLETS

$$b = 60 \text{ mm} \quad c = 40 \text{ mm};$$

Obtain K from Fig. 2-65

$$\sigma_{\text{nom}} = \frac{P}{ct} = \frac{2.5 \text{ kN}}{(40 \text{ mm})(5 \text{ mm})} = 12.50 \text{ MPa}$$

FOR $R = 6 \text{ mm}$: $R/c = 0.15 \quad b/c = 1.5$

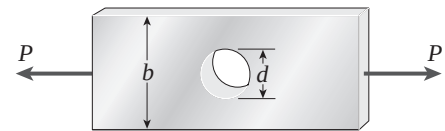
$$K \approx 2.00 \quad \sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 25 \text{ MPa} \quad \leftarrow$$

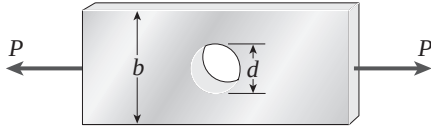
FOR $R = 10 \text{ mm}$: $R/c = 0.25 \quad b/c = 1.5$

$$K \approx 1.75 \quad \sigma_{\text{max}} = K\sigma_{\text{nom}} \approx 22 \text{ MPa} \quad \leftarrow$$

Problem 2.10-3 A flat bar of width b and thickness t has a hole of diameter d drilled through it (see figure). The hole may have any diameter that will fit within the bar.

What is the maximum permissible tensile load P_{max} if the allowable tensile stress in the material is σ_t ?



Solution 2.10-3 Flat bar in tension

t = thickness

σ_t = allowable tensile stress

Find $P_{\max}$

Find K from Fig. 2-63

$$P_{\max} = \sigma_{\text{nom}} ct = \frac{\sigma_{\max}}{K} ct = \frac{\sigma_t}{K} (b - d)t$$

$$= \frac{\sigma_t}{K} bt \left(1 - \frac{d}{b} \right)$$

Because σ_t , b , and t are constants, we write:

$$P^* = \frac{P_{\max}}{\sigma_t bt} = \frac{1}{K} \left(1 - \frac{d}{b} \right)$$

$\frac{d}{b}$	K	P^*
0	3.00	0.333
0.1	2.73	0.330
0.2	2.50	0.320
0.3	2.35	0.298
0.4	2.24	0.268

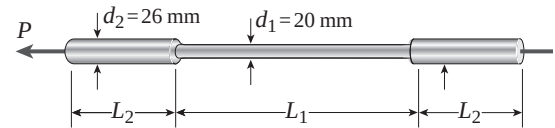
We observe that $P_{\max}$ decreases as d/b increases. Therefore, the maximum load occurs when the hole becomes very small.

$$\left(\frac{d}{b} \rightarrow 0 \quad \text{and} \quad K \rightarrow 3 \right)$$

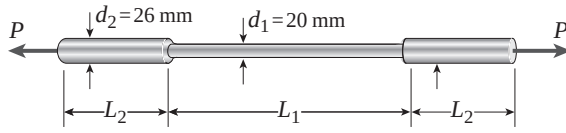
$$P_{\max} = \frac{\sigma_t bt}{3} \leftarrow$$

Problem 2.10-4 A round brass bar of diameter $d_1 = 20$ mm has upset ends of diameter $d_2 = 26$ mm (see figure). The lengths of the segments of the bar are $L_1 = 0.3$ m and $L_2 = 0.1$ m. Quarter-circular fillets are used at the shoulders of the bar, and the modulus of elasticity of the brass is $E = 100$ GPa.

If the bar lengthens by 0.12 mm under a tensile load P , what is the maximum stress $\sigma_{\max}$ in the bar?



Probs. 2.10-4 and 2.10-5

Solution 2.10-4 Round brass bar with upset ends

$$E = 100 \text{ GPa}$$

$$\delta = 0.12 \text{ mm}$$

$$L_2 = 0.1 \text{ m}$$

$$L_1 = 0.3 \text{ m}$$

$$R = \text{radius of fillets} = \frac{26 \text{ mm} - 20 \text{ mm}}{2} = 3 \text{ mm}$$

$$\delta = 2\left(\frac{PL_2}{EA_2}\right) + \frac{PL_1}{EA_1}$$

$$\text{Solve for } P: \quad P = \frac{\delta EA_1 A_2}{2L_2 A_1 + L_1 A_2}$$

Use Fig. 2-66 for the stress-concentration factor:

$$\begin{aligned} \sigma_{\text{nom}} &= \frac{P}{A_1} = \frac{\delta EA_2}{2L_2 A_1 + L_1 A_2} = \frac{\delta E}{2L_2 \left(\frac{A_1}{A_2}\right) + L_1} \\ &= \frac{\delta E}{2L_2 \left(\frac{d_1}{d_2}\right)^2 + L_1} \end{aligned}$$

SUBSTITUTE NUMERICAL VALUES:

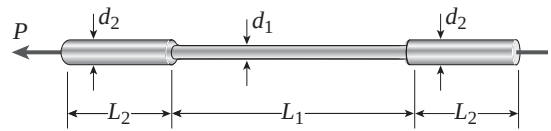
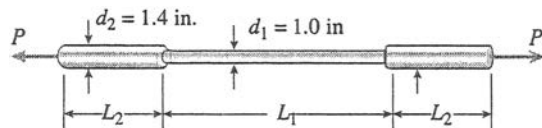
$$\sigma_{\text{nom}} = \frac{(0.12 \text{ mm})(100 \text{ GPa})}{2(0.1 \text{ m})\left(\frac{20}{26}\right)^2 + 0.3 \text{ m}} = 28.68 \text{ MPa}$$

$$\frac{R}{D_1} = \frac{3 \text{ mm}}{20 \text{ mm}} = 0.15$$

Use the dashed curve in Fig. 2-66. $K \approx 1.6$

$$\begin{aligned} \sigma_{\text{max}} &= K\sigma_{\text{nom}} \approx (1.6)(28.68 \text{ MPa}) \\ &\approx 46 \text{ MPa} \quad \leftarrow \end{aligned}$$

Problem 2.10-5 Solve the preceding problem for a bar of monel metal having the following properties: $d_1 = 1.0 \text{ in.}$, $d_2 = 1.4 \text{ in.}$, $L_1 = 20.0 \text{ in.}$, $L_2 = 5.0 \text{ in.}$, and $E = 25 \times 10^6 \text{ psi}$. Also, the bar lengthens by 0.0040 in. when the tensile load is applied.

**Solution 2.10-5 Round bar with upset ends**

$$E = 25 \times 10^6 \text{ psi}$$

$$\delta = 0.0040 \text{ in.}$$

$$L_1 = 20 \text{ in.}$$

$$L_2 = 5 \text{ in.}$$

$$R = \text{radius of fillets} \quad R = \frac{1.4 \text{ in.} - 1.0 \text{ in.}}{2}$$

$$= 0.2 \text{ in.}$$

$$\delta = 2\left(\frac{PL_2}{EA_2}\right) + \frac{PL_1}{EA_1}$$

$$\text{Solve for } P: \quad P = \frac{\delta EA_1 A_2}{2L_2 A_1 + L_1 A_2}$$

Use Fig. 2-66 for the stress-concentration factor.

$$\begin{aligned} \sigma_{\text{nom}} &= \frac{P}{A_1} = \frac{\delta EA_2}{2L_2 A_1 + L_1 A_2} = \frac{\delta E}{2L_2 \left(\frac{A_1}{A_2}\right) + L_1} \\ &= \frac{\delta E}{2L_2 \left(\frac{d_1}{d_2}\right)^2 + L_1} \end{aligned}$$

SUBSTITUTE NUMERICAL VALUES:

$$\sigma_{\text{nom}} = \frac{(0.0040 \text{ in.})(25 \times 10^6 \text{ psi})}{2(5 \text{ in.})\left(\frac{1.0}{1.4}\right)^2 + 20 \text{ in.}} = 3,984 \text{ psi}$$

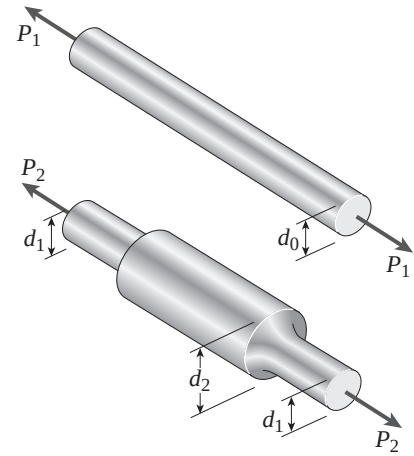
$$\frac{R}{D_1} = \frac{0.2 \text{ in.}}{1.0 \text{ in.}} = 0.2$$

Use the dashed curve in Fig. 2-66. $K \approx 1.53$

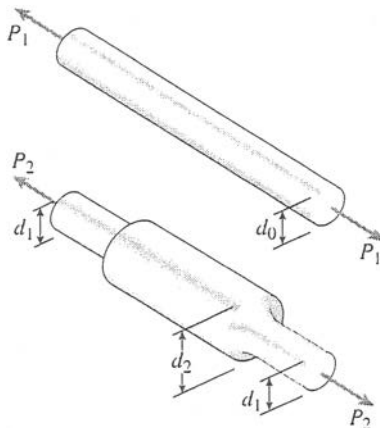
$$\sigma_{\text{max}} = K\sigma_{\text{nom}} \approx (1.53)(3984 \text{ psi}) \approx 6100 \text{ psi} \quad \leftarrow$$

Problem 2.10-6 A prismatic bar of diameter $d_0 = 20 \text{ mm}$ is being compared with a stepped bar of the same diameter ($d_1 = 20 \text{ mm}$) that is enlarged in the middle region to a diameter $d_2 = 25 \text{ mm}$ (see figure). The radius of the fillets in the stepped bar is 2.0 mm .

- Does enlarging the bar in the middle region make it stronger than the prismatic bar? Demonstrate your answer by determining the maximum permissible load P_1 for the prismatic bar and the maximum permissible load P_2 for the enlarged bar, assuming that the allowable stress for the material is 80 MPa .
- What should be the diameter d_0 of the prismatic bar if it is to have the same maximum permissible load as does the stepped bar?



Solution 2.10-6 Prismatic bar and stepped bar



$$d_0 = 20 \text{ mm}$$

$$d_1 = 20 \text{ mm}$$

$$d_2 = 25 \text{ mm}$$

Fillet radius: $R = 2 \text{ mm}$

Allowable stress: $\sigma_t = 80 \text{ MPa}$

(a) COMPARISON OF BARS

$$\begin{aligned} \text{Prismatic bar: } P_1 &= \sigma_t A_0 = \sigma_t \left(\frac{\pi d_0^2}{4} \right) \\ &= (80 \text{ MPa}) \left(\frac{\pi}{4} \right) (20 \text{ mm})^2 = 25.1 \text{ kN} \quad \leftarrow \end{aligned}$$

Stepped bar: See Fig. 2-66 for the stress-concentration factor.

$$\begin{array}{lll} R = 2.0 \text{ mm} & D_1 = 20 \text{ mm} & D_2 = 25 \text{ mm} \\ R/D_1 = 0.10 & D_2/D_1 = 1.25 & K \approx 1.75 \end{array}$$

$$\sigma_{\text{nom}} = \frac{P_2}{\frac{\pi d_1^2}{4}} = \frac{P_2}{A_1} \quad \sigma_{\text{nom}} = \frac{\sigma_{\text{max}}}{K}$$

$$\begin{aligned}
 P_2 &= \sigma_{\text{nom}} A_1 = \frac{\sigma_{\text{max}}}{K} A_1 = \frac{\sigma_t}{K} A_1 \\
 &= \left(\frac{80 \text{ MPa}}{1.75} \right) \left(\frac{\pi}{4} \right) (20 \text{ mm})^2 \\
 &\approx 14.4 \text{ kN} \quad \leftarrow
 \end{aligned}$$

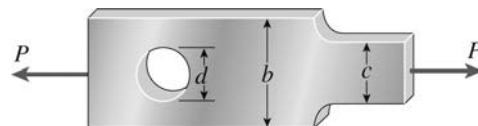
Enlarging the bar makes it *weaker*, not stronger. The ratio of loads is $P_1/P_2 = K = 1.75$

(b) DIAMETER OF PRISMATIC BAR FOR THE SAME ALLOWABLE LOAD

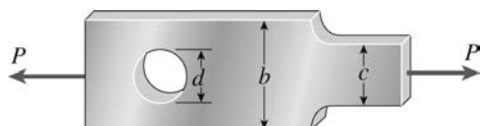
$$\begin{aligned}
 P_1 &= P_2 \quad \sigma_t \left(\frac{\pi d_0^2}{4} \right) = \frac{\sigma_t}{K} \left(\frac{\pi d_1^2}{4} \right) \quad d_0^2 = \frac{d_1^2}{K} \\
 d_0 &= \frac{d_1}{\sqrt{K}} \approx \frac{20 \text{ mm}}{\sqrt{1.75}} \approx 15.1 \text{ mm} \quad \leftarrow
 \end{aligned}$$

Problem 2.10-7 A stepped bar with a hole (see figure) has widths $b = 2.4$ in. and $c = 1.6$ in. The fillets have radii equal to 0.2 in.

What is the diameter d_{max} of the largest hole that can be drilled through the bar without reducing the load-carrying capacity?



Solution 2.10-7 Stepped bar with a hole



$$b = 2.4 \text{ in.}$$

$$c = 1.6 \text{ in.}$$

$$\text{Fillet radius: } R = 0.2 \text{ in.}$$

Find d_{max}

BASED UPON FILLETS (Use Fig. 2-65)

$$b = 2.4 \text{ in.} \quad c = 1.6 \text{ in.} \quad R = 0.2 \text{ in.}$$

$$R/c = 0.125 \quad b/c = 1.5 \quad K \approx 2.10$$

$$\begin{aligned}
 P_{\text{max}} &= \sigma_{\text{nom}} c t = \frac{\sigma_{\text{max}}}{K} c t = \frac{\sigma_{\text{max}}}{K} \left(\frac{c}{b} \right) (b t) \\
 &\approx 0.317 b t \sigma_{\text{max}}
 \end{aligned}$$

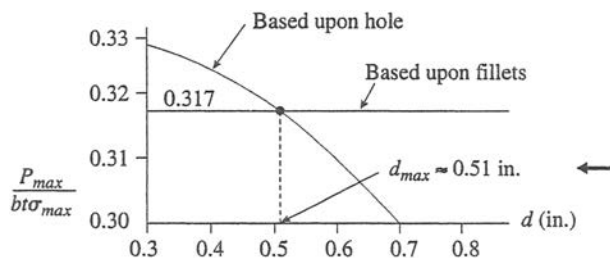
BASED UPON HOLE (Use Fig. 2-63)

$$b = 2.4 \text{ in.} \quad d = \text{diameter of the hole (in.)}$$

$$c_1 = b - d$$

$$\begin{aligned}
 P_{\text{max}} &= \sigma_{\text{nom}} c_1 t = \frac{\sigma_{\text{max}}}{K} (b - d) t \\
 &= \frac{1}{K} \left(1 - \frac{d}{b} \right) b t \sigma_{\text{max}}
 \end{aligned}$$

$d(\text{in.})$	d/b	K	$P_{\text{max}}/b t \sigma_{\text{max}}$
0.3	0.125	2.66	0.329
0.4	0.167	2.57	0.324
0.5	0.208	2.49	0.318
0.6	0.250	2.41	0.311
0.7	0.292	2.37	0.299



Nonlinear Behavior (Changes in Lengths of Bars)

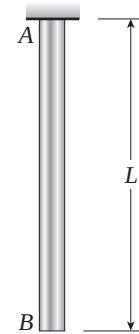
Problem 2.11-1 A bar AB of length L and weight density γ hangs vertically under its own weight (see figure). The stress-strain relation for the material is given by the Ramberg-Osgood equation (Eq. 2-73):

$$\epsilon = \frac{\sigma}{E} + \frac{\sigma_0 \alpha}{E} \left(\frac{\sigma}{\sigma_0} \right)^m$$

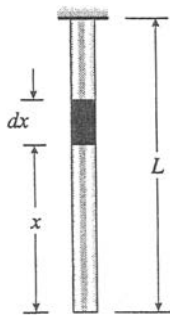
Derive the following formula

$$\delta = \frac{\gamma L^2}{2E} + \frac{\sigma_0 \alpha L}{(m+1)E} \left(\frac{\gamma L}{\sigma_0} \right)^m$$

for the elongation of the bar.



Solution 2.11-1 Bar hanging under its own weight



Let A = cross-sectional area

Let N = axial force at distance x

$$N = \gamma A x$$

$$\sigma = \frac{N}{A} = \gamma x$$

STRAIN AT DISTANCE x

$$\epsilon = \frac{\sigma}{E} + \frac{\sigma_0 \alpha}{E} \left(\frac{\sigma}{\sigma_0} \right)^m = \frac{\gamma x}{E} + \frac{\sigma_0 \alpha}{E} \left(\frac{\gamma x}{\sigma_0} \right)^m$$

ELONGATION OF BAR

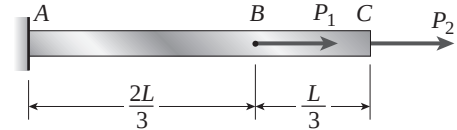
$$\begin{aligned} \delta &= \int_0^L \epsilon dx = \int_0^L \frac{\gamma x}{E} dx + \frac{\sigma_0 \alpha}{E} \int_0^L \left(\frac{\gamma x}{\sigma_0} \right)^m dx \\ &= \frac{\gamma L^2}{2E} + \frac{\sigma_0 \alpha L}{(m+1)E} \left(\frac{\gamma L}{\sigma_0} \right)^m \quad \text{Q.E.D.} \quad \leftarrow \end{aligned}$$

Problem 2.11-2 A prismatic bar of length $L = 1.8$ m and cross-sectional area $A = 480 \text{ mm}^2$ is loaded by forces $P_1 = 30$ kN and $P_2 = 60$ kN (see figure). The bar is constructed of magnesium alloy having a stress-strain curve described by the following Ramberg-Osgood equation:

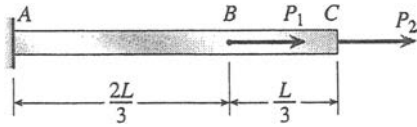
$$\epsilon = \frac{\sigma}{45,000} + \frac{1}{618} \left(\frac{\sigma}{170} \right)^{10} \quad (\sigma = \text{MPa})$$

in which σ has units of megapascals.

- Calculate the displacement δ_C of the end of the bar when the load P_1 acts alone.
- Calculate the displacement when the load P_2 acts alone.
- Calculate the displacement when both loads act simultaneously.



Solution 2.11-2 Axially loaded bar



$$L = 1.8 \text{ m} \quad A = 480 \text{ mm}^2$$

$$P_1 = 30 \text{ kN} \quad P_2 = 60 \text{ kN}$$

Ramberg–Osgood equation:

$$\epsilon = \frac{\sigma}{45,000} + \frac{1}{618} \left(\frac{\sigma}{170} \right)^{10} \quad (\sigma = \text{MPa})$$

Find displacement at end of bar.

(a) P_1 ACTS ALONE

$$AB: \sigma = \frac{P_1}{A} = \frac{30 \text{ kN}}{480 \text{ mm}^2} = 62.5 \text{ MPa}$$

$$\epsilon = 0.001389$$

$$\delta_c = \epsilon \left(\frac{2L}{3} \right) = 1.67 \text{ mm} \quad \leftarrow$$

(b) P_2 ACTS ALONE

$$ABC: \sigma = \frac{P_2}{A} = \frac{60 \text{ kN}}{480 \text{ mm}^2} = 125 \text{ MPa}$$

$$\epsilon = 0.002853$$

$$\delta_c = \epsilon L = 5.13 \text{ mm} \quad \leftarrow$$

(c) BOTH P_1 AND P_2 ARE ACTING

$$AB: \sigma = \frac{P_1 + P_2}{A} = \frac{90 \text{ kN}}{480 \text{ mm}^2} = 187.5 \text{ MPa}$$

$$\epsilon = 0.008477$$

$$\delta_{AB} = \epsilon \left(\frac{2L}{3} \right) = 10.17 \text{ mm}$$

$$BC: \sigma = \frac{P_2}{A} = \frac{60 \text{ kN}}{480 \text{ mm}^2} = 125 \text{ MPa}$$

$$\epsilon = 0.002853$$

$$\delta_{BC} = \epsilon \left(\frac{L}{3} \right) = 1.71 \text{ mm}$$

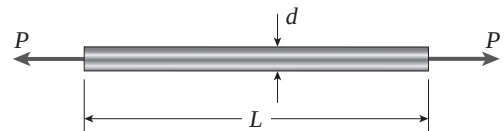
$$\delta_c = \delta_{AB} + \delta_{BC} = 11.88 \text{ mm} \quad \leftarrow$$

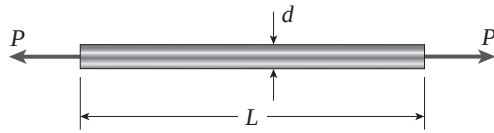
(Note that the displacement when both loads act simultaneously is *not* equal to the sum of the displacements when the loads act separately.)

Problem 2.11-3 A circular bar of length $L = 32$ in. and diameter $d = 0.75$ in. is subjected to tension by forces P (see figure). The wire is made of a copper alloy having the following *hyperbolic stress-strain relationship*:

$$\sigma = \frac{18,000\epsilon}{1 + 300\epsilon} \quad 0 \leq \epsilon \leq 0.03 \quad (\sigma = \text{ksi})$$

- Draw a stress-strain diagram for the material.
- If the elongation of the wire is limited to 0.25 in. and the maximum stress is limited to 40 ksi, what is the allowable load P ?



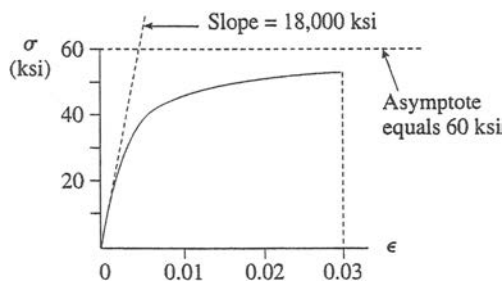
Solution 2.11-3 Copper bar in tension

$$L = 32 \text{ in.} \quad d = 0.75 \text{ in.}$$

$$A = \frac{\pi d^2}{4} = 0.4418 \text{ in.}^2$$

(a) STRESS-STRAIN DIAGRAM

$$\sigma = \frac{18,000\varepsilon}{1 + 300\varepsilon} \quad 0 \leq \varepsilon \leq 0.03 \quad (\sigma = \text{ksi})$$

(b) ALLOWABLE LOAD P Maximum elongation $\delta_{\max} = 0.25 \text{ in.}$ Maximum stress $\sigma_{\max} = 40 \text{ ksi}$

Based upon elongation:

$$\varepsilon_{\max} = \frac{\delta_{\max}}{L} = \frac{0.25 \text{ in.}}{32 \text{ in.}} = 0.007813$$

$$\sigma_{\max} = \frac{18,000\varepsilon_{\max}}{1 + 300\varepsilon_{\max}} = 42.06 \text{ ksi}$$

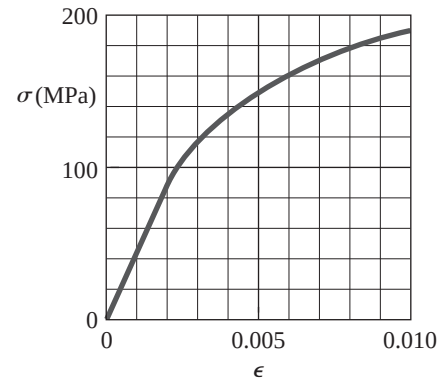
BASED UPON STRESS:

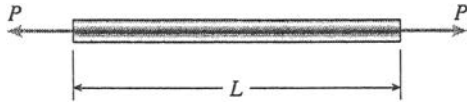
 $\sigma_{\max} = 40 \text{ ksi}$

Stress governs. $P = \sigma_{\max} A = (40 \text{ ksi})(0.4418 \text{ in.}^2)$
 $= 17.7 \text{ k} \quad \leftarrow$

Problem 2.11-4 A prismatic bar in tension has length $L = 2.0 \text{ m}$ and cross-sectional area $A = 249 \text{ mm}^2$. The material of the bar has the stress-strain curve shown in the figure.

Determine the elongation δ of the bar for each of the following axial loads: $P = 10 \text{ kN}$, 20 kN , 30 kN , 40 kN , and 45 kN . From these results, plot a diagram of load P versus elongation δ (load-displacement diagram).



Solution 2.11-4 Bar in tension

$$L = 2.0 \text{ m}$$

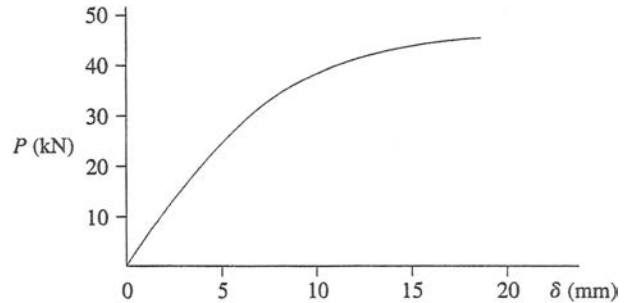
$$A = 249 \text{ mm}^2$$

STRESS-STRAIN DIAGRAM

(See the problem statement for the diagram)

LOAD-DISPLACEMENT DIAGRAM

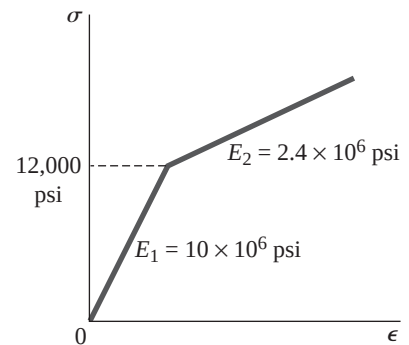
P (kN)	$\sigma = P/A$ (MPa)	ϵ (from diagram)	$\delta = \epsilon L$ (mm)
10	40	0.0009	1.8
20	80	0.0018	3.6
30	120	0.0031	6.2
40	161	0.0060	12.0
45	181	0.0081	16.2

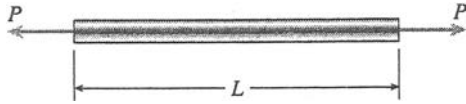


NOTE: The load-displacement curve has the same shape as the stress-strain curve.

Problem 2.11-5 An aluminum bar subjected to tensile forces P has length $L = 150 \text{ in.}$ and cross-sectional area $A = 2.0 \text{ in.}^2$. The stress-strain behavior of the aluminum may be represented approximately by the bilinear stress-strain diagram shown in the figure.

Calculate the elongation δ of the bar for each of the following axial loads: $P = 8 \text{ k}, 16 \text{ k}, 24 \text{ k}, 32 \text{ k},$ and 40 k . From these results, plot a diagram of load P versus elongation δ (load-displacement diagram).

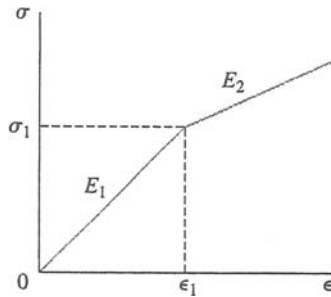


Solution 2.11-5 Aluminum bar in tension

$$L = 150 \text{ in.}$$

$$A = 2.0 \text{ in.}^2$$

STRESS-STRAIN DIAGRAM



$$E_1 = 10 \times 10^6 \text{ psi}$$

$$E_2 = 2.4 \times 10^6 \text{ psi}$$

$$\sigma_1 = 12,000 \text{ psi}$$

$$\begin{aligned} \epsilon_1 &= \frac{\sigma_1}{E_1} = \frac{12,000 \text{ psi}}{10 \times 10^6 \text{ psi}} \\ &= 0.0012 \end{aligned}$$

For $0 \leq \sigma \leq \sigma_1$:

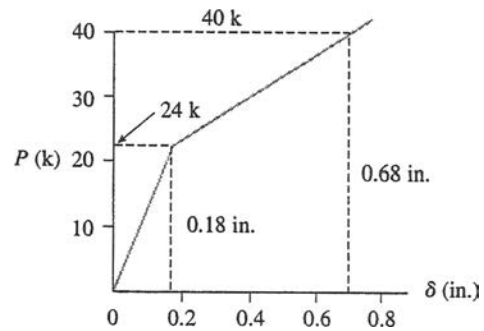
$$\epsilon = \frac{\sigma}{E_1} = \frac{\sigma}{10 \times 10^6 \text{ psi}} \quad (\sigma = \text{psi}) \quad \text{Eq. (1)}$$

For $\sigma \geq \sigma_1$:

$$\begin{aligned} \epsilon &= \epsilon_1 + \frac{\sigma - \sigma_1}{E_2} = 0.0012 + \frac{\sigma - 12,000}{2.4 \times 10^6} \\ &= \frac{\sigma}{2.4 \times 10^6} - 0.0038 \quad (\sigma = \text{psi}) \quad \text{Eq. (2)} \end{aligned}$$

LOAD-DISPLACEMENT DIAGRAM

P (k)	$\sigma = P/A$ (psi)	ϵ (from Eq. 1 or Eq. 2)	$\delta = \epsilon L$ (in.)
8	4,000	0.00040	0.060
16	8,000	0.00080	0.120
24	12,000	0.00120	0.180
32	16,000	0.00287	0.430
40	20,000	0.00453	0.680

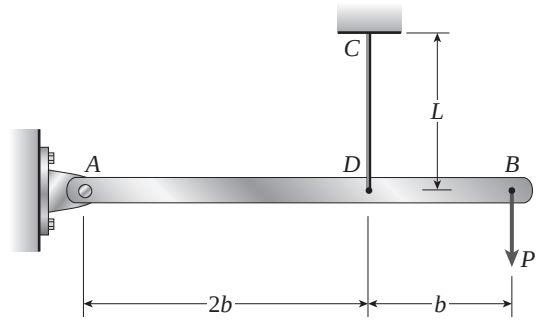


Problem 2.11-6 A rigid bar AB , pinned at end A , is supported by a wire CD and loaded by a force P at end B (see figure). The wire is made of high-strength steel having modulus of elasticity $E = 210$ GPa and yield stress $\sigma_Y = 820$ MPa. The length of the wire is $L = 1.0$ m and its diameter is $d = 3$ mm. The stress-strain diagram for the steel is defined by the *modified power law*, as follows:

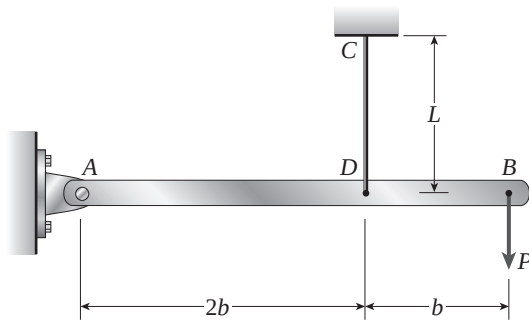
$$\sigma = E\epsilon \quad 0 \leq \sigma \leq \sigma_Y$$

$$\sigma = \sigma_Y \left(\frac{E\epsilon}{\sigma_Y} \right)^n \quad \sigma \geq \sigma_Y$$

- (a) Assuming $n = 0.2$, calculate the displacement δ_B at the end of the bar due to the load P . Take values of P from 2.4 kN to 5.6 kN in increments of 0.8 kN.
- (b) Plot a load-displacement diagram showing P versus δ_B .



Solution 2.11-6 Rigid bar supported by a wire



Wire: $E = 210$ GPa

$\sigma_Y = 820$ MPa

$L = 1.0$ m

$d = 3$ mm

$$A = \frac{\pi d^2}{4} = 7.0686 \text{ mm}^2$$

STRESS-STRAIN DIAGRAM

$$\sigma = E\epsilon \quad (0 \leq \sigma \leq \sigma_Y) \quad (1)$$

$$\sigma = \sigma_Y \left(\frac{E\epsilon}{\sigma_Y} \right)^n \quad (\sigma \geq \sigma_Y) \quad (n = 0.2) \quad (2)$$

(a) DISPLACEMENT δ_B AT END OF BAR

$$\delta = \text{elongation of wire} \quad \delta_B = \frac{3}{2}\delta = \frac{3}{2}\epsilon L \quad (3)$$

Obtain ϵ from stress-strain equations:

$$\text{From Eq. (1): } \epsilon = \frac{\sigma E}{\sigma_Y} \quad (0 \leq \sigma \leq \sigma_Y) \quad (4)$$

$$\text{From Eq. (2): } \epsilon = \frac{\sigma_Y}{E} \left(\frac{\sigma}{\sigma_Y} \right)^{1/n} \quad (5)$$

$$\text{Axial force in wire: } F = \frac{3P}{2}$$

$$\text{Stress in wire: } \sigma = \frac{F}{A} = \frac{3P}{2A} \quad (6)$$

PROCEDURE: Assume a value of P

Calculate σ from Eq. (6)

Calculate ϵ from Eq. (4) or (5)

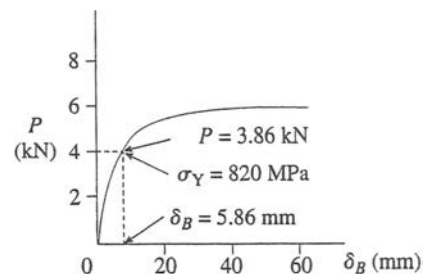
Calculate δ_B from Eq. (3)

P (kN)	σ (MPa) Eq. (6)	ϵ Eq. (4) or (5)	δ_B (mm) Eq. (3)
2.4	509.3	0.002425	3.64
3.2	679.1	0.003234	4.85
4.0	848.8	0.004640	6.96
4.8	1018.6	0.01155	17.3
5.6	1188.4	0.02497	37.5

For $\sigma = \sigma_Y = 820$ MPa:

$$\epsilon = 0.0039048 \quad P = 3.864 \text{ kN} \quad \delta_B = 5.86 \text{ mm}$$

(b) LOAD-DISPLACEMENT DIAGRAM

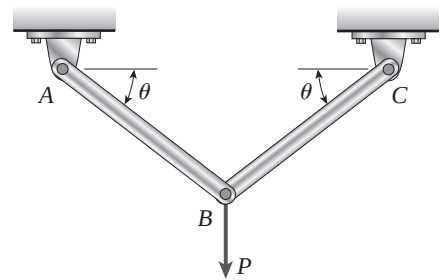


Elastoplastic Analysis

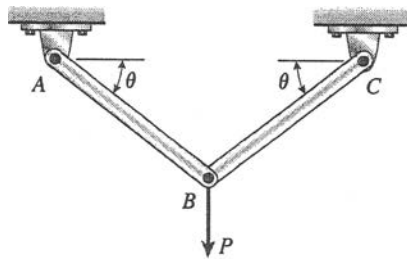
The problems for Section 2.12 are to be solved assuming that the material is elastoplastic with yield stress σ_Y , yield strain ϵ_Y , and modulus of elasticity E in the linearly elastic region (see Fig. 2-72).

Problem 2.12-1 Two identical bars AB and BC support a vertical load P (see figure). The bars are made of steel having a stress-strain curve that may be idealized as elastoplastic with yield stress σ_Y . Each bar has cross-sectional area A .

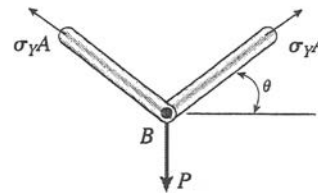
Determine the yield load P_Y and the plastic load P_P .



Solution 2.12-1 Two bars supporting a load P



Structure is statically determinate. The yield load P_Y and the plastic load P_P occur at the same time, namely, when both bars reach the yield stress.



JOINT B

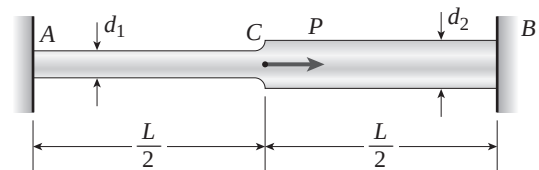
$$\Sigma F_{\text{vert}} = 0$$

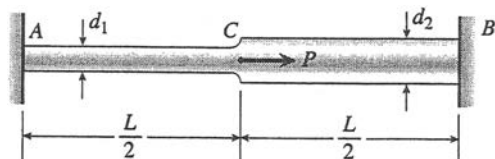
$$(2\sigma_Y A) \sin \theta = P$$

$$P_Y = P_P = 2\sigma_Y A \sin \theta \quad \leftarrow$$

Problem 2.12-2 A stepped bar ACB with circular cross sections is held between rigid supports and loaded by an axial force P at midlength (see figure). The diameters for the two parts of the bar are $d_1 = 20$ mm and $d_2 = 25$ mm, and the material is elastoplastic with yield stress $\sigma_Y = 250$ MPa.

Determine the plastic load P_P .



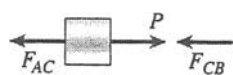
Solution 2.12-2 Bar between rigid supports

$$d_1 = 20 \text{ mm} \quad d_2 = 25 \text{ mm} \quad \sigma_Y = 250 \text{ MPa}$$

DETERMINE THE PLASTIC LOAD P_P :

At the plastic load, all parts of the bar are stressed to the yield stress.

Point C:



$$F_{AC} = \sigma_Y A_1 \quad F_{CB} = \sigma_Y A_2$$

$$P = F_{AC} + F_{CB}$$

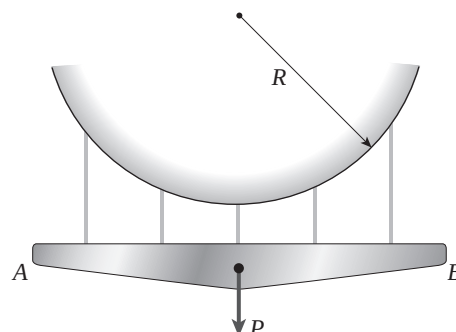
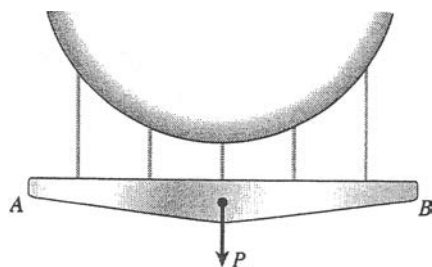
$$P_P = \sigma_Y A_1 + \sigma_Y A_2 = \sigma_Y (A_1 + A_2) \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

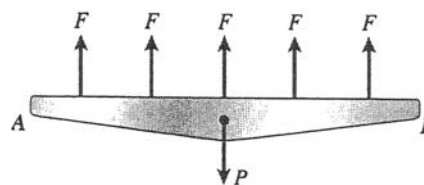
$$\begin{aligned} P_P &= (250 \text{ MPa}) \left(\frac{\pi}{4} \right) (d_1^2 + d_2^2) \\ &= (250 \text{ MPa}) \left(\frac{\pi}{4} \right) [(20 \text{ mm})^2 + (25 \text{ mm})^2] \\ &= 201 \text{ kN} \quad \leftarrow \end{aligned}$$

Problem 2.12-3 A horizontal rigid bar AB supporting a load P is hung from five symmetrically placed wires, each of cross-sectional area A (see figure). The wires are fastened to a curved surface of radius R .

- Determine the plastic load P_P if the material of the wires is elastoplastic with yield stress σ_Y .
- How is P_P changed if bar AB is flexible instead of rigid?
- How is P_P changed if the radius R is increased?

**Solution 2.12-3 Rigid bar supported by five wires**

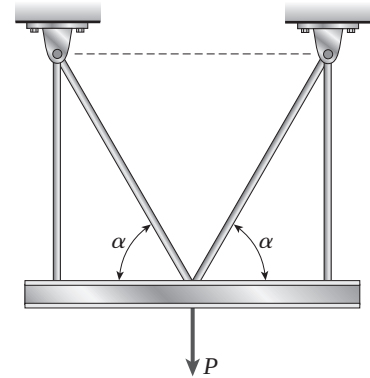
- PLASTIC LOAD P_P**
At the plastic load, each wire is stressed to the yield stress. $\therefore P_P = 5\sigma_Y A \quad \leftarrow$
 $F = \sigma_Y A$



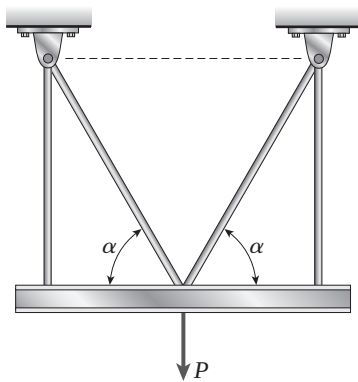
- BAR AB IS FLEXIBLE**
At the plastic load, each wire is stressed to the yield stress, so the plastic load is not changed. $\leftarrow$
- RADIUS R IS INCREASED**
Again, the forces in the wires are not changed, so the plastic load is not changed. $\leftarrow$

Problem 2.12-4 A load P acts on a horizontal beam that is supported by four rods arranged in the symmetrical pattern shown in the figure. Each rod has cross-sectional area A and the material is elastoplastic with yield stress σ_Y .

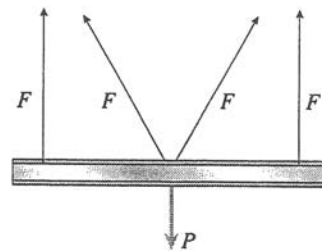
Determine the plastic load P_P .



Solution 2.12-4 Beam supported by four rods



At the plastic load, all four rods are stressed to the yield stress.



$$F = \sigma_Y A$$

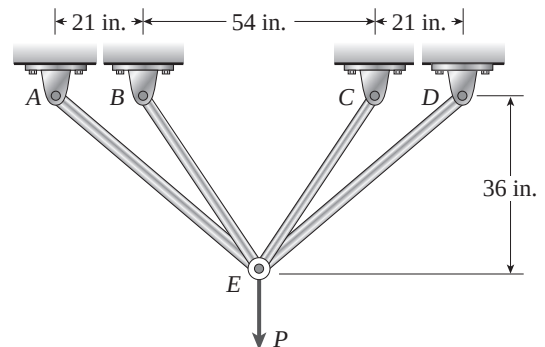
Sum forces in the vertical direction and solve for the load:

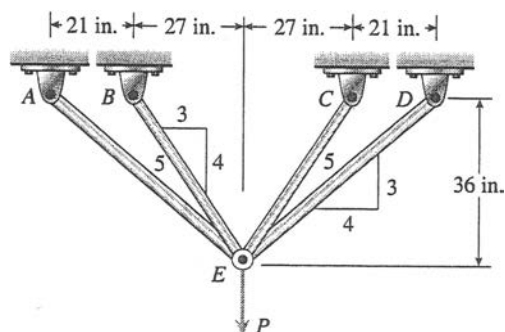
$$P_P = 2F + 2F \sin \alpha$$

$$P_P = 2\sigma_Y A (1 + \sin \alpha) \quad \leftarrow$$

Problem 2.12-5 The symmetric truss $ABCDE$ shown in the figure is constructed of four bars and supports a load P at joint E . Each of the two outer bars has a cross-sectional area of 0.307 in.^2 , and each of the two inner bars has an area of 0.601 in.^2 . The material is elastoplastic with yield stress $\sigma_Y = 36 \text{ ksi}$.

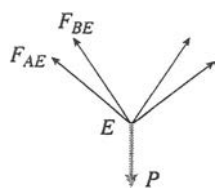
Determine the plastic load P_P .



Solution 2.12-5 Truss with four bars

$$L_{AE} = 60 \text{ in.} \quad L_{BE} = 45 \text{ in.}$$

JOINT E



Equilibrium:

$$2F_{AE}\left(\frac{3}{5}\right) + 2F_{BE}\left(\frac{4}{5}\right) = P$$

or

$$P = \frac{6}{5}F_{AE} + \frac{8}{5}F_{BE}$$

PLASTIC LOAD P_P

At the plastic load, all bars are stressed to the yield stress.

$$F_{AE} = \sigma_Y A_{AE} \quad F_{BE} = \sigma_Y A_{BE}$$

$$P_P = \frac{6}{5}\sigma_Y A_{AE} + \frac{8}{5}\sigma_Y A_{BE} \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

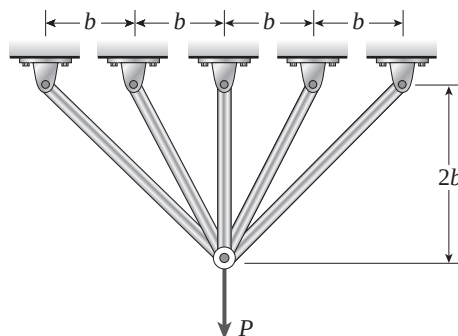
$$A_{AE} = 0.307 \text{ in.}^2 \quad A_{BE} = 0.601 \text{ in.}^2$$

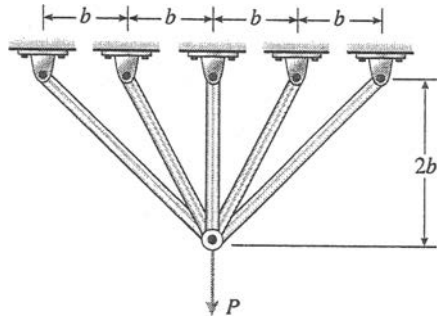
$$\sigma_Y = 36 \text{ ksi}$$

$$P_P = \frac{6}{5}(36 \text{ ksi})(0.307 \text{ in.}^2) + \frac{8}{5}(36 \text{ ksi})(0.601 \text{ in.}^2)$$

$$= 13.26 \text{ k} + 34.62 \text{ k} = 47.9 \text{ k} \quad \leftarrow$$

Problem 2.12-6 Five bars, each having a diameter of 10 mm, support a load P as shown in the figure. Determine the plastic load P_P if the material is elastoplastic with yield stress $\sigma_Y = 250 \text{ MPa}$.

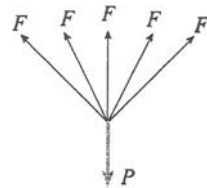


Solution 2.12-6 Truss consisting of five bars

$$d = 10 \text{ mm}$$

$$A = \frac{\pi d^2}{4} = 78.54 \text{ mm}^2$$

$$\sigma_Y = 250 \text{ MPa}$$



At the plastic load, all five bars are stressed to the yield stress

$$F = \sigma_Y A$$

Sum forces in the vertical direction and solve for the load:

$$P_P = 2F\left(\frac{1}{\sqrt{2}}\right) + 2F\left(\frac{2}{\sqrt{5}}\right) + F$$

$$= \frac{\sigma_Y A}{5}(5\sqrt{2} + 4\sqrt{5} + 5)$$

$$= 4.2031\sigma_Y A \quad \leftarrow$$

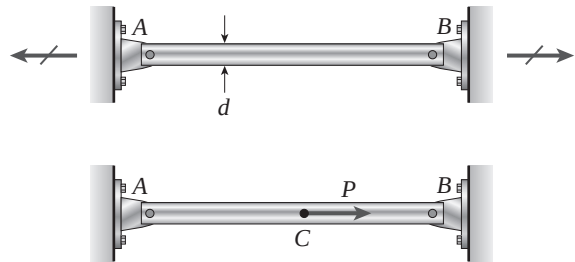
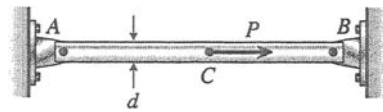
Substitute numerical values:

$$P_P = (4.2031)(250 \text{ MPa})(78.54 \text{ mm}^2)$$

$$= 82.5 \text{ kN} \quad \leftarrow$$

Problem 2.12-7 A circular steel rod AB of diameter $d = 0.60$ in. is stretched tightly between two supports so that initially the tensile stress in the rod is 10 ksi (see figure). An axial force P is then applied to the rod at an intermediate location C.

- Determine the plastic load P_P if the material is elastoplastic with yield stress $\sigma_Y = 36$ ksi.
- How is P_P changed if the initial tensile stress is doubled to 20 ksi?

**Solution 2.12-7 Bar held between rigid supports**

$$d = 0.6 \text{ in.}$$

$$\sigma_Y = 36 \text{ ksi}$$

$$\text{Initial tensile stress} = 10 \text{ ksi}$$

(a) PLASTIC LOAD P_P

The presence of the initial tensile stress does not affect the plastic load. Both parts of the bar must yield in order to reach the plastic load.

POINT C:

$$\leftarrow \sigma_Y A \quad C \quad \rightarrow \sigma_Y A$$

$$P_P = 2\sigma_Y A = (2)(36 \text{ ksi})\left(\frac{\pi}{4}\right)(0.60 \text{ in.})^2$$

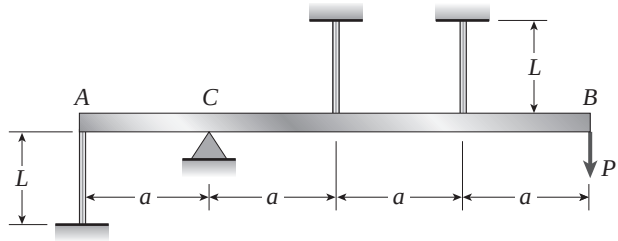
$$= 20.4 \text{ k} \quad \leftarrow$$

(b) INITIAL TENSILE STRESS IS DOUBLED

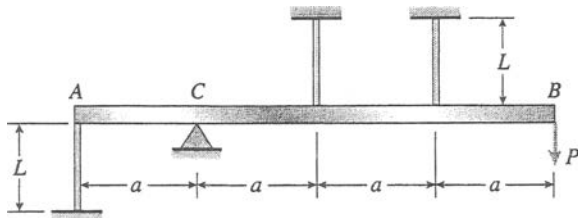
P_P is not changed. $\leftarrow$

Problem 2.12-8 A rigid bar ACB is supported on a fulcrum at C and loaded by a force P at end B (see figure). Three identical wires made of an elastoplastic material (yield stress σ_Y and modulus of elasticity E) resist the load P . Each wire has cross-sectional area A and length L .

- Determine the yield load P_Y and the corresponding yield displacement δ_Y at point B .
- Determine the plastic load P_P and the corresponding displacement δ_P at point B when the load just reaches the value P_P .
- Draw a load-displacement diagram with the load P as ordinate and the displacement δ_B of point B as abscissa.

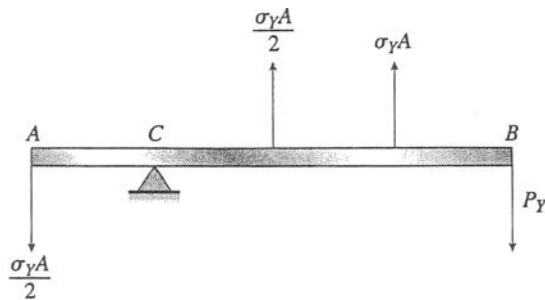


Solution 2.12-8 Rigid bar supported by wires



(a) YIELD LOAD P_Y

Yielding occurs when the most highly stressed wire reaches the yield stress σ_Y



$$\Sigma M_C = 0$$

$$P_Y = \sigma_Y A \quad \leftarrow$$

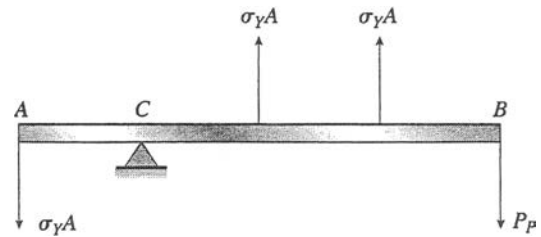
At point A:

$$\delta_A = \left(\frac{\sigma_Y A}{2} \right) \left(\frac{L}{EA} \right) = \frac{\sigma_Y L}{2E}$$

At point B:

$$\delta_B = 3\delta_A = \delta_Y = \frac{3\sigma_Y L}{2E} \quad \leftarrow$$

(b) PLASTIC LOAD P_P



At the plastic load, all wires reach the yield stress.

$$\Sigma M_C = 0$$

$$P_P = \frac{4\sigma_Y A}{3} \quad \leftarrow$$

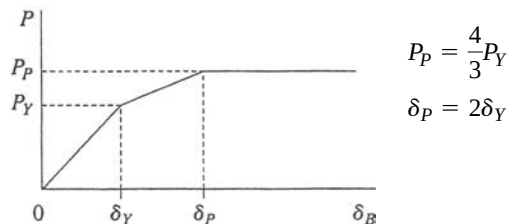
At point A:

$$\delta_A = (\sigma_Y A) \left(\frac{L}{EA} \right) = \frac{\sigma_Y L}{E}$$

At point B:

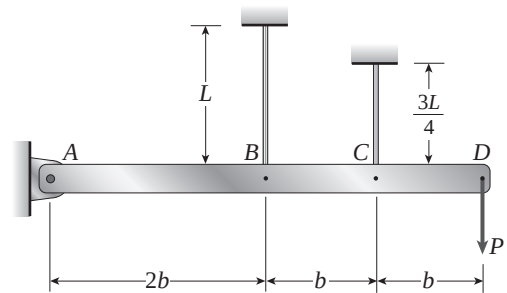
$$\delta_B = 3\delta_A = \delta_P = \frac{3\sigma_Y L}{E} \quad \leftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM

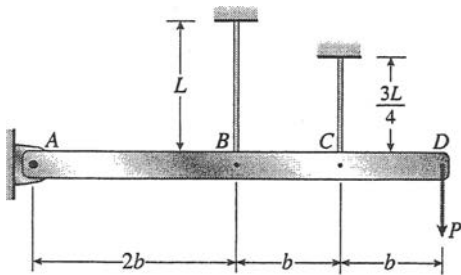


Problem 2.12-9 The structure shown in the figure consists of a horizontal rigid bar $ABCD$ supported by two steel wires, one of length L and the other of length $3L/4$. Both wires have cross-sectional area A and are made of elastoplastic material with yield stress σ_Y and modulus of elasticity E . A vertical load P acts at end D of the bar.

- Determine the yield load P_Y and the corresponding yield displacement δ_Y at point D .
- Determine the plastic load P_P and the corresponding displacement δ_P at point D when the load just reaches the value P_P .
- Draw a load-displacement diagram with the load P as ordinate and the displacement δ_D of point D as abscissa.



Solution 2.12-9 Rigid bar supported by two wires

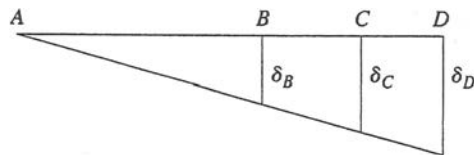


A = cross-sectional area

σ_Y = yield stress

E = modulus of elasticity

DISPLACEMENT DIAGRAM

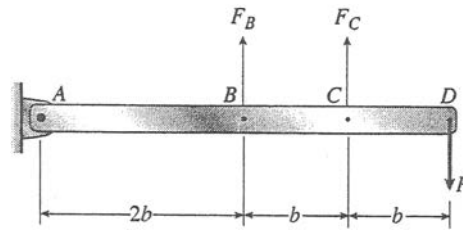


COMPATIBILITY:

$$\delta_C = \frac{3}{2}\delta_B \quad (1)$$

$$\delta_D = 2\delta_B \quad (2)$$

FREE-BODY DIAGRAM



EQUILIBRIUM:

$$\sum M_A = 0 \quad F_B(2b) + F_C(3b) = P(4b) \quad (3)$$

$$2F_B + 3F_C = 4P$$

FORCE-DISPLACEMENT RELATIONS

$$\delta_B = \frac{F_B L}{EA} \quad \delta_C = \frac{F_C \left(\frac{3L}{4}\right)}{EA} \quad (4, 5)$$

Substitute into Eq. (1):

$$\frac{3F_C L}{4EA} = \frac{3F_B L}{2EA} \quad (6)$$

$$F_C = 2F_B$$

STRESSES

$$\sigma_B = \frac{F_B}{A} \quad \sigma_C = \frac{F_C}{A} \quad \sigma_C = 2\sigma_B \quad (7)$$

Wire C has the larger stress. Therefore, it will yield first.

(a) YIELD LOAD

$$\sigma_C = \sigma_Y \quad \sigma_B = \frac{\sigma_C}{2} = \frac{\sigma_Y}{2} \quad (\text{From Eq. 7})$$

$$F_C = \sigma_Y A \quad F_B = \frac{1}{2} \sigma_Y A$$

From Eq. (3):

$$2\left(\frac{1}{2}\sigma_Y A\right) + 3(\sigma_Y A) = 4P$$

$$P = P_Y = \sigma_Y A \quad \leftarrow$$

From Eq. (4):

$$\delta_B = \frac{F_B L}{EA} = \frac{\sigma_Y L}{2E}$$

From Eq. (2):

$$\delta_D = \delta_Y = 2\delta_B = \frac{\sigma_Y L}{E} \quad \leftarrow$$

(b) PLASTIC LOAD

At the plastic load, both wires yield.

$$\sigma_B = \sigma_Y = \sigma_C \quad F_B = F_C = \sigma_Y A$$

From Eq. (3):

$$2(\sigma_Y A) + 3(\sigma_Y A) = 4P$$

$$P = P_P = \frac{5}{4}\sigma_Y A \quad \leftarrow$$

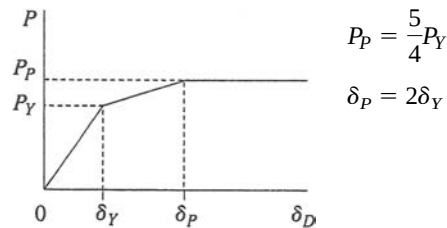
From Eq. (4):

$$\delta_B = \frac{F_B L}{EA} = \frac{\sigma_Y L}{E}$$

From Eq. (2):

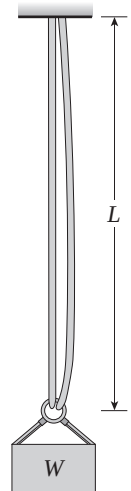
$$\delta_D = \delta_P = 2\delta_B = \frac{2\sigma_Y L}{E} \quad \leftarrow$$

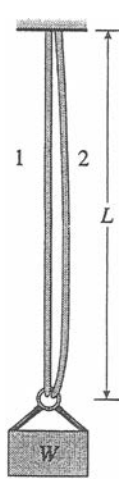
(c) LOAD-DISPLACEMENT DIAGRAM



Problem 2.12-10 Two cables, each having a length L of approximately 40 m, support a loaded container of weight W (see figure). The cables, which have effective cross-sectional area $A = 48.0 \text{ mm}^2$ and effective modulus of elasticity $E = 160 \text{ GPa}$, are identical except that one cable is longer than the other when they are hanging separately and unloaded. The difference in lengths is $d = 100 \text{ mm}$. The cables are made of steel having an elastoplastic stress-strain diagram with $\sigma_Y = 500 \text{ MPa}$. Assume that the weight W is initially zero and is slowly increased by the addition of material to the container.

- Determine the weight W_Y that first produces yielding of the shorter cable. Also, determine the corresponding elongation δ_Y of the shorter cable.
- Determine the weight W_P that produces yielding of both cables. Also, determine the elongation δ_P of the shorter cable when the weight W just reaches the value W_P .
- Construct a load-displacement diagram showing the weight W as ordinate and the elongation δ of the shorter cable as abscissa. (*Hint:* The load displacement diagram is not a single straight line in the region $0 \leq W \leq W_Y$.)



Solution 2.12-10 Two cables supporting a load

$$L = 40 \text{ m} \quad A = 48.0 \text{ mm}^2$$

$$E = 160 \text{ GPa}$$

$$d = \text{difference in length} = 100 \text{ mm}$$

$$\sigma_Y = 500 \text{ MPa}$$

INITIAL STRETCHING OF CABLE 1

Initially, cable 1 supports all of the load.

Let W_1 = load required to stretch cable 1 to the same length as cable 2

$$W_1 = \frac{EA}{L}d = 19.2 \text{ kN}$$

$$\delta_1 = 100 \text{ mm (elongation of cable 1)}$$

$$\sigma_1 = \frac{W_1}{A} = \frac{Ed}{L} = 400 \text{ MPa} \quad (\sigma_1 < \sigma_Y \therefore \text{OK})$$

(a) YIELD LOAD W_Y

Cable 1 yields first. $F_1 = \sigma_Y A = 24 \text{ kN}$

δ_{1Y} = total elongation of cable 1

δ_{1Y} = total elongation of cable 1

$$\delta_{1Y} = \frac{F_1 L}{EA} = \frac{\sigma_Y L}{E} = 0.125 \text{ m} = 125 \text{ mm}$$

$$\delta_Y = \delta_{1Y} = 125 \text{ mm} \quad \leftarrow$$

δ_{2Y} = elongation of cable 2

$$= \delta_{1Y} - d = 25 \text{ mm}$$

$$F_2 = \frac{EA}{L} \delta_{2Y} = 4.8 \text{ kN}$$

$$W_Y = F_1 + F_2 = 24 \text{ kN} + 4.8 \text{ kN}$$

$$= 28.8 \text{ kN} \quad \leftarrow$$

(b) PLASTIC LOAD W_P

$$F_1 = \sigma_Y A \quad F_2 = \sigma_Y A$$

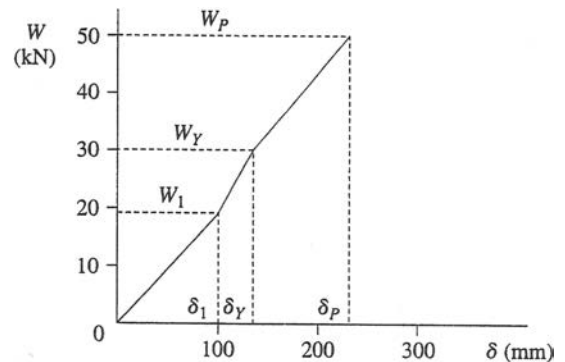
$$W_P = 2\sigma_Y A = 48 \text{ kN} \quad \leftarrow$$

δ_{2P} = elongation of cable 2

$$= F_2 \left(\frac{L}{EA} \right) = \frac{\sigma_Y L}{E} = 0.125 \text{ m} = 125 \text{ mm}$$

$$\delta_{1P} = \delta_{2P} + d = 225 \text{ mm}$$

$$\delta_P = \delta_{1P} = 225 \text{ mm} \quad \leftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM

$$\frac{W_Y}{W_1} = 1.5 \quad \frac{\delta_Y}{\delta_1} = 1.25$$

$$\frac{W_P}{W_Y} = 1.667 \quad \frac{\delta_P}{\delta_Y} = 1.8$$

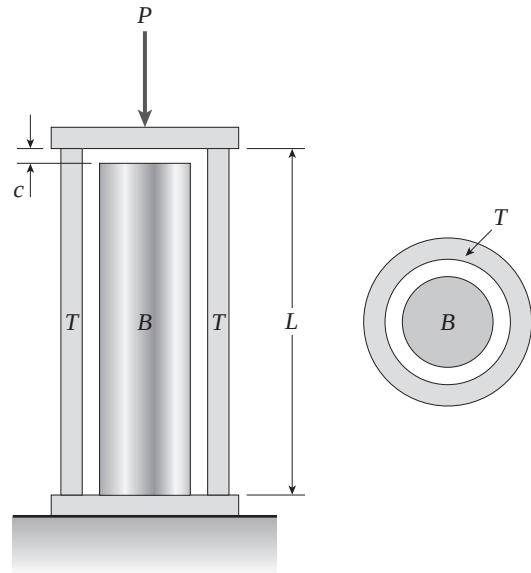
$0 < W < W_1$: slope = 192,000 N/m

$W_1 < W < W_Y$: slope = 384,000 N/m

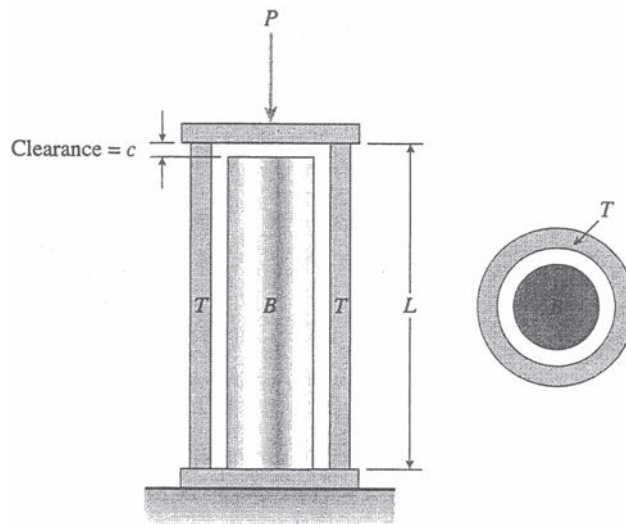
$W_Y < W < W_P$: slope = 192,000 N/m

Problem 2.12-11 A hollow circular tube T of length $L = 15$ in. is uniformly compressed by a force P acting through a rigid plate (see figure). The outside and inside diameters of the tube are 3.0 and 2.75 in., respectively. A concentric solid circular bar B of 1.5 in. diameter is mounted inside the tube. When no load is present, there is a clearance $c = 0.010$ in. between the bar B and the rigid plate. Both bar and tube are made of steel having an elastoplastic stress-strain diagram with $E = 29 \times 10^3$ ksi and $\sigma_Y = 36$ ksi.

- Determine the yield load P_Y and the corresponding shortening δ_Y of the tube.
- Determine the plastic load P_P and the corresponding shortening δ_P of the tube.
- Construct a load-displacement diagram showing the load P as ordinate and the shortening δ of the tube as abscissa. (Hint: The load-displacement diagram is not a single straight line in the region $0 \leq P \leq P_Y$.)



Solution 2.12-11 Tube and bar supporting a load



$$\begin{aligned}
 L &= 15 \text{ in.} \\
 c &= 0.010 \text{ in.} \\
 E &= 29 \times 10^3 \text{ ksi} \\
 \sigma_Y &= 36 \text{ ksi}
 \end{aligned}$$

TUBE:

$$\begin{aligned}
 d_2 &= 3.0 \text{ in.} \\
 d_1 &= 2.75 \text{ in.} \\
 A_T &= \frac{\pi}{4} (d_2^2 - d_1^2) = 1.1290 \text{ in.}^2
 \end{aligned}$$

BAR

$$d = 1.5 \text{ in.}$$

$$A_B = \frac{\pi d^2}{4} = 1.7671 \text{ in.}^2$$

INITIAL SHORTENING OF TUBE T

Initially, the tube supports all of the load.

Let P_1 = load required to close the clearance

$$P_1 = \frac{EA_T}{L}c = 21,827 \text{ lb}$$

Let δ_1 = shortening of tube $\delta_1 = c = 0.010 \text{ in.}$

$$\sigma_1 = \frac{P_1}{A_T} = 19,330 \text{ psi} \quad (\sigma_1 < \sigma_Y \therefore \text{OK})$$

(a) YIELD LOAD P_Y

Because the tube and bar are made of the same material, and because the strain in the tube is larger than the strain in the bar, the tube will yield first.

$$F_T = \sigma_Y A_T = 40,644 \text{ lb}$$

δ_{TY} = shortening of tube at the yield stress

$$\sigma_{TY} = \frac{F_T L}{EA_T} = \frac{\sigma_Y L}{E} = 0.018621 \text{ in.}$$

$$\delta_Y = \delta_{TY} = 0.018621 \text{ in.} \quad \leftarrow$$

δ_{BY} = shortening of bar

$$= \delta_{TY} - c = 0.008621 \text{ in.}$$

$$F_B = \frac{EA_B}{L} \delta_{BY} = 29,453 \text{ lb}$$

$$P_Y = F_T + F_B = 40,644 \text{ lb} + 29,453 \text{ lb}$$

$$= 70,097 \text{ lb}$$

$$P_Y = 70,100 \text{ lb} \quad \leftarrow$$

(b) PLASTIC LOAD P_P

$$F_T = \sigma_Y A_T \quad F_B = \sigma_Y A_B$$

$$P_P = F_T + F_B = \sigma_Y (A_T + A_B)$$

$$= 104,300 \text{ lb} \quad \leftarrow$$

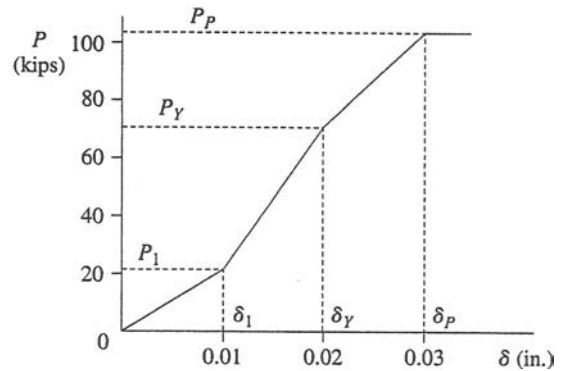
δ_{BP} = shortening of bar

$$= F_B \left(\frac{L}{EA_B} \right) = \frac{\sigma_Y L}{E} = 0.018621 \text{ in.}$$

$$\delta_{TP} = \delta_{BP} + c = 0.028621 \text{ in.}$$

$$\delta_P = \delta_{TP} = 0.02862 \text{ in.} \quad \leftarrow$$

(c) LOAD-DISPLACEMENT DIAGRAM



$$\frac{P_Y}{P_1} = 3.21 \quad \frac{\delta_Y}{\delta_1} = 1.86$$

$$\frac{P_P}{P_Y} = 1.49 \quad \frac{\delta_P}{\delta_Y} = 1.54$$

$$0 < P < P_1: \text{ slope} = 2180 \text{ k/in.}$$

$$P_1 < P < P_Y: \text{ slope} = 5600 \text{ k/in.}$$

$$P_Y < P < P_P: \text{ slope} = 3420 \text{ k/in.}$$

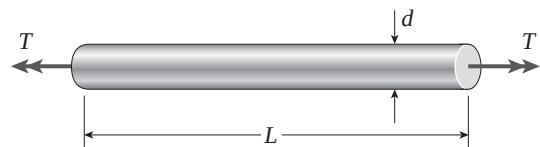
3

Torsion

Torsional Deformations

Problem 3.2-1 A copper rod of length $L = 18.0$ in. is to be twisted by torques T (see figure) until the angle of rotation between the ends of the rod is 3.0° .

- If the allowable shear strain in the copper is 0.0006 rad, what is the maximum permissible diameter of the rod?
- If the rod diameter is 0.5 in., what is the minimum permissible length of the rod?



Probs. 3.2-1 and 3.2-2

Solution 3.2-1

(a) $L = 18$ in. $\phi = 3^\circ$ $\gamma_a = 0.0006$ rad

From Eq. (3-5):

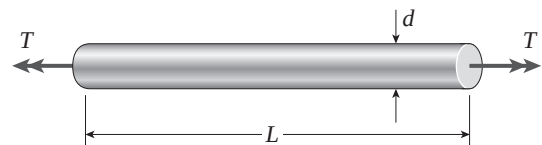
$$\gamma_{\max} = \frac{r\phi}{L} = \frac{d\phi}{2L} \quad \text{Solve for } d: \quad d_{\max} = \frac{2L\gamma_a}{\phi} = 0.413 \text{ in.} \quad \boxed{d_{\max} = 0.413 \text{ in.}}$$

(b) $d = 0.5$ in.

$$\text{Now solve for } L: \quad L_{\min} = \frac{d\phi}{2\gamma_a} = 21.817 \text{ in.} \quad \boxed{L_{\min} = 21.8 \text{ in.}}$$

Problem 3.2-2 A plastic bar of diameter $d = 56$ mm is to be twisted by torques T (see figure) until the angle of rotation between the ends of the bar is 4.0° .

- If the allowable shear strain in the plastic is 0.012 rad, what is the minimum permissible length of the bar?
- If the length of the bar is 200 mm, what is the maximum permissible diameter of the bar?



Solution 3.2-2

(a) $d = 56 \text{ mm}$ $\gamma_a = 0.012 \text{ rad}$ $\phi = 4^\circ$

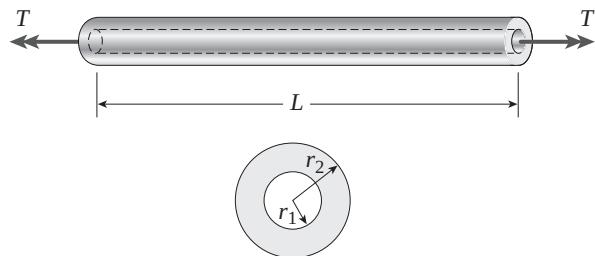
Solution based on Eq. (3-5): $L_{\min} = \frac{d\phi}{2\gamma_a}$ $L_{\min} = \frac{d\phi}{2\gamma_a} = 162.897 \text{ mm}$ $L_{\min} = 162.9 \text{ mm}$

(b) $L = 200 \text{ mm}$

Now solve for d : $d_{\max} = \frac{2L\gamma_a}{\phi} = 68.755 \text{ mm}$ $d_{\max} = 68.8 \text{ mm}$

Problem 3.2-3 A circular aluminum tube subjected to pure torsion by torques T (see figure) has an outer radius r_2 equal to 1.5 times the inner radius r_1 .

- (a) If the maximum shear strain in the tube is measured as $400 \times 10^{-6} \text{ rad}$, what is the shear strain γ_1 at the inner surface?
- (b) If the maximum allowable rate of twist is 0.125 degrees per foot and the maximum shear strain is to be kept at $400 \times 10^{-6} \text{ rad}$ by adjusting the torque T , what is the minimum required outer radius $(r_2)_{\min}$?



Probs. 3.2-3, 3.2-4, and 3.2-5

Solution 3.2-3

NUMERICAL DATA

$$r_2 = 1.5r_1 \quad \gamma_{\max} = 400 \times (10^{-6}) \text{ rad}$$

$$\theta = 0.125 \left(\frac{\pi}{180} \right) \left(\frac{1}{12} \right)$$

$$\theta = 1.818 \times 10^{-4} \text{ rad/m}$$

(a) SHEAR STRAIN AT INNER SURFACE AT RADIUS r_1

$$\gamma_1 = \frac{r_1}{r_2} \gamma_{\max} \quad \gamma_1 = \frac{1}{1.5} \gamma_{\max}$$

$$\gamma_1 = 267 \times 10^{-6} \text{ rad} \quad \leftarrow$$

(b) MINIMUM REQUIRED OUTER RADIUS

$$r_{2\min} = \frac{\gamma_{\max}}{\theta} \quad r_{2\min} = \frac{\gamma_{\max}}{\theta}$$

$$r_{2\min} = 2.2 \text{ in.} \quad \leftarrow$$

Problem 3.2-4 A circular steel tube of length $L = 1.0 \text{ m}$ is loaded in torsion by torques T (see figure).

- (a) If the inner radius of the tube is $r_1 = 45 \text{ mm}$ and the measured angle of twist between the ends is 0.5° , what is the shear strain γ_1 (in radians) at the inner surface?
- (b) If the maximum allowable shear strain is 0.0004 rad and the angle of twist is to be kept at 0.45° by adjusting the torque T , what is the maximum permissible outer radius $(r_2)_{\max}$?

Solution 3.2-4

NUMERICAL DATA

$$L = 1000 \text{ mm}$$

$$r_1 = 45 \text{ mm}$$

$$\phi = 0.5 \left(\frac{\pi}{180} \right) \text{ rad}$$

(a) SHEAR STRAIN AT INNER SURFACE

$$\gamma_1 = r_1 \frac{\phi}{L} \quad \gamma_1 = 393 \times 10^{-6} \text{ rad} \quad \leftarrow$$

(b) MAXIMUM PERMISSIBLE OUTER RADIUS

$$\phi = 0.45 \left(\frac{\pi}{180} \right) \text{ rad} \quad \gamma_{\max} = r_2 \frac{\phi}{L}$$

$$\gamma_{\max} = 0.0004 \text{ rad} \quad r_{2\max} = \gamma_{\max} \frac{L}{\phi}$$

$$r_{2\max} = 50.9 \text{ mm} \quad \leftarrow$$

Problem 3.2-5 Solve the preceding problem if the length $L = 56 \text{ in.}$, the inner radius $r_1 = 1.25 \text{ in.}$, the angle of twist is 0.5° , and the allowable shear strain is 0.0004 rad .

Solution 3.2-5

NUMERICAL DATA

$$L = 56 \text{ in.} \quad r_1 = 1.25 \text{ in.}$$

$$\phi = 0.5 \left(\frac{\pi}{180} \right) \text{ rad}$$

$$\gamma_a = 0.0004 \text{ rad}$$

(a) SHEAR STRAIN γ_1 (IN RADIANS) AT THE INNER SURFACE

$$\gamma_1 = r_1 \frac{\phi}{L} \quad \gamma_1 = 195 \times 10^{-6} \text{ rad} \quad \leftarrow$$

(b) MAXIMUM PERMISSIBLE OUTER RADIUS $(r_2)_{\max}$

$$\phi = 0.5 \left(\frac{\pi}{180} \right) \text{ rad}$$

$$\gamma_{\max} = r_2 \frac{\phi}{L}$$

$$\gamma_a = 0.0004 \text{ rad}$$

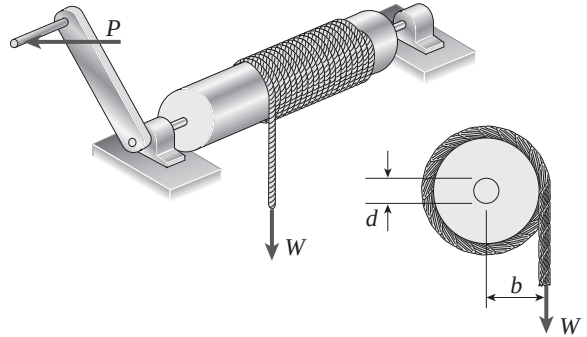
$$r_{2\max} = \gamma_a \frac{L}{\phi}$$

$$r_{2\max} = 2.57 \text{ in.} \quad \leftarrow$$

Circular Bars and Tubes

Problem 3.3-1 A prospector uses a hand-powered winch (see figure) to raise a bucket of ore in his mine shaft. The axle of the winch is a steel rod of diameter $d = 0.625$ in. Also, the distance from the center of the axle to the center of the lifting rope is $b = 4.0$ in.

- If the weight of the loaded bucket is $W = 100$ lb, what is the maximum shear stress in the axle due to torsion?
- If the maximum bucket load is 125 lb, and the allowable shear stress in the axle is 9250 psi, what is the minimum permissible axle diameter?

**Solution 3.3-1**

- (a) MAXIMUM SHEAR STRESS IN THE AXLE

$$W = 100 \text{ lb} \quad b = 4 \text{ in.} \quad d = \frac{5}{8} \text{ in.} = 0.625 \text{ in.} \quad T = Wb = 33.333 \text{ ft-lb}$$

$$\tau_{\max} = \frac{16T}{\pi d^3} = 8344.303 \text{ lb/in.}^2 \quad \boxed{\tau_{\max} = 8344 \text{ psi}}$$

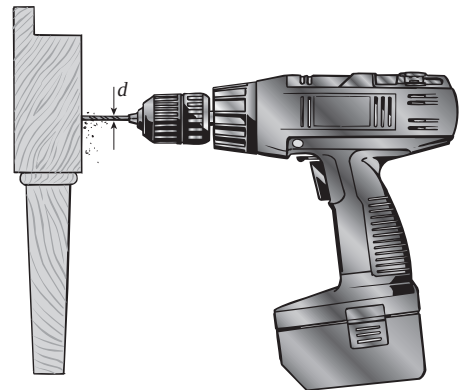
- (b) IF THE MAXIMUM BUCKET LOAD IS 125 LB, AND THE ALLOWABLE SHEAR STRESS IN THE AXLE IS 9250 PSI, WHAT IS THE MINIMUM PERMISSIBLE AXLE DIAMETER?

$$W = 125 \text{ lb} \quad T = Wb \quad \tau_a = 9250 \text{ psi}$$

$$d_{\min} = \left(\frac{16T}{\pi \tau_a} \right)^{\frac{1}{3}} = 0.651 \text{ in.} \quad \boxed{d_{\min} = 0.651 \text{ in.}}$$

Problem 3.3-2 When drilling a hole in a table leg, a furniture maker uses a hand-operated drill (see figure) with a bit of diameter $d = 4.0$ mm.

- If the resisting torque supplied by the table leg is equal to $0.3 \text{ N}\cdot\text{m}$, what is the maximum shear stress in the drill bit?
- If the allowable shear stress in the drill bit is 32 MPa, what is the maximum resisting torque before the drill binds up?
- If the shear modulus of elasticity of the steel is $G = 75 \text{ GPa}$, what is the rate of twist of the drill bit (degrees per meter)?



Solution 3.3-2

(a) MAXIMUM SHEAR STRESS IN DRILL BIT

From Eq. (3-14):

$$\tau_{\max} = \frac{16T}{\pi d^3}$$

$$\tau_{\max} = \frac{16(0.3 \text{ N}\cdot\text{m})}{\pi(4.0 \text{ mm})^3}$$

$$\tau_{\max} = 23.8 \text{ MPa} \quad \leftarrow$$

(b) MAXIMUM RESISTING TORQUE BEFORE THE DRILL BINDS UP

$$\tau_a = 32 \text{ MPa} \quad d = 4 \text{ mm}$$

$$T_{\max} = \frac{\tau_a \pi d^3}{16} = 0.402 \text{ N}\cdot\text{m} \quad \boxed{T_{\max} = 0.402 \text{ N}\cdot\text{m}}$$

(c) RATE OF TWIST

From Eq. (3-16):

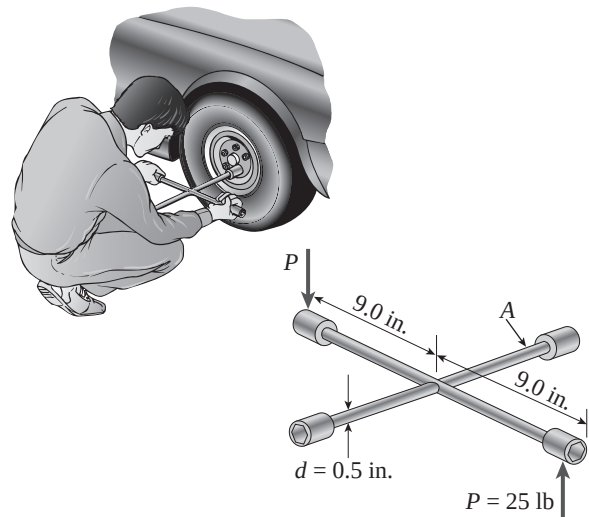
$$\theta = \frac{T}{GI_p}$$

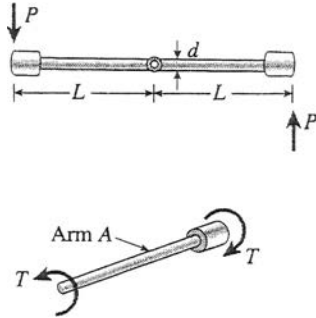
$$\theta = \frac{0.3 \text{ N}\cdot\text{m}}{(75 \text{ GPa}) \left(\frac{\pi}{32} \right) (4.0 \text{ mm})^4}$$

$$\theta = 0.1592 \text{ rad/m} = 9.12^\circ/\text{m} \quad \leftarrow$$

Problem 3.3-3 While removing a wheel to change a tire, a driver applies forces $P = 25 \text{ lb}$ at the ends of two of the arms of a lug wrench (see figure). The wrench is made of steel with shear modulus of elasticity $G = 11.4 \times 10^6 \text{ psi}$. Each arm of the wrench is 9.0 in. long and has a solid circular cross section of diameter $d = 0.5 \text{ in.}$

- Determine the maximum shear stress in the arm that is turning the lug nut (arm A).
- Determine the angle of twist (in degrees) of this same arm.



Solution 3.3-3 Lug wrench

$$\begin{aligned}
 P &= 25 \text{ lb} \\
 L &= 9.0 \text{ in.} \\
 d &= 0.5 \text{ in.} \\
 G &= 11.4 \times 10^6 \text{ psi} \\
 T &= \text{torque acting on arm A} \\
 T &= P(2L) = 2(25 \text{ lb})(9.0 \text{ in.}) \\
 &= 450 \text{ lb-in.}
 \end{aligned}$$

(a) MAXIMUM SHEAR STRESS

From Eq. (3-14):

$$\begin{aligned}
 \tau_{\max} &= \frac{16T}{\pi d^3} = \frac{(16)(450 \text{ lb-in.})}{\pi(0.5 \text{ in.})^3} \\
 \tau_{\max} &= 18,300 \text{ psi} \quad \leftarrow
 \end{aligned}$$

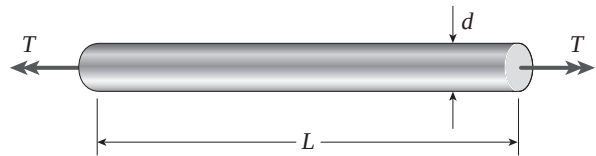
(b) ANGLE OF TWIST

From Eq. (3-17):

$$\begin{aligned}
 \phi &= \frac{TL}{GI_P} = \frac{(450 \text{ lb-in.})(9.0 \text{ in.})}{(11.4 \times 10^6 \text{ psi})\left(\frac{\pi}{32}\right)(0.5 \text{ in.})^4} \\
 \phi &= 0.05790 \text{ rad} = 3.32^\circ \quad \leftarrow
 \end{aligned}$$

Problem 3.3-4 An aluminum bar of solid circular cross section is twisted by torques T acting at the ends (see figure). The dimensions and shear modulus of elasticity are as follows:
 $L = 1.4 \text{ m}$, $d = 32 \text{ mm}$, and $G = 28 \text{ GPa}$.

- Determine the torsional stiffness of the bar.
- If the angle of twist of the bar is 5° , what is the maximum shear stress? What is the maximum shear strain (in radians)?
- If a hole of diameter $d/2$ is drilled longitudinally through the bar, what is the ratio of the torsional stiffnesses of the hollow and solid bars? What is the ratio of their maximum shear stresses if both are acted on by the same torque?
- If the hole diameter remains at $d/2$, what new outside diameter d_2 will result in equal stiffnesses of hollow and solid bars?

**Solution 3.3-4****(a) TORSIONAL STIFFNESS OF SOLID BAR**

$$d = 32 \text{ mm} \quad d_2 = d \quad d_1 = \frac{d}{2} \quad G = 28 \text{ GPa} \quad L = 1.4 \text{ m}$$

$$I_{pS} = \frac{\pi}{32} d^4 = 1.029 \times 10^{-7} \text{ m}^4 \quad k_{T\text{solid}} = \frac{GI_{pS}}{L} \quad \boxed{k_{T\text{solid}} = 2059 \text{ N}\cdot\text{m}}$$

(b) MAXIMUM SHEAR STRESS AND STRAIN OF SOLID BAR

$$\begin{aligned}
 \phi &= 5^\circ \quad T_{\text{solid}} = k_{T\text{solid}} \phi = 179.671 \text{ N}\cdot\text{m} \\
 \tau_{\max S} &= \frac{k_{T\text{solid}} \phi \frac{d}{2}}{I_{pS}} = 27.9 \text{ MPa}
 \end{aligned}$$

$$\gamma_{\max} = \frac{\tau_{\max}}{G} \quad \gamma_{\max} = 997 \times 10^{-6} \text{ rad} \quad \leftarrow$$

(c) STIFFNESS AND SHEAR STRESS RATIOS FOR SOLID AND HOLLOW BARS

$$I_{pH} = \frac{\pi}{32}(d_2^4 - d_1^4) = 9.651 \times 10^{-8} \text{ m}^4 \quad k_{Thollow} = \frac{GI_{pH}}{L} = 1930.195 \text{ N}\cdot\text{m} \quad \boxed{\frac{k_{Thollow}}{k_{Tsolid}} = 0.938}$$

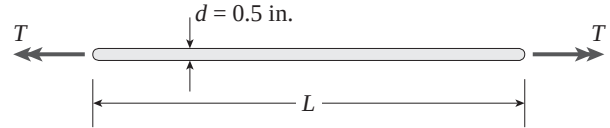
$$\tau_{\max H} = \frac{(T_{\text{solid}}) \frac{d_2}{2}}{I_{pH}} = 29.787 \text{ MPa} \quad \frac{\tau_{\max S}}{\tau_{\max H}} = 0.938 \quad \boxed{\frac{\tau_{\max H}}{\tau_{\max S}} = 1.067}$$

- (d) IF THE HOLE DIAMETER REMAINS AT $d/2$, WHAT NEW OUTSIDE DIAMETER d_2 WILL RESULT IN EQUAL STIFFNESSES OF HOLLOW AND SOLID BARS?

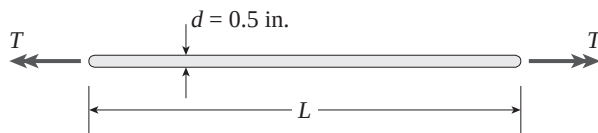
Equate torsional stiffnesses of solid and hollow bars; solve for new value of d_2 for hollow bar.

Must have $d_2^4 - d_1^4 = d^4$, so solving for d_2 gives $\boxed{d_2 = 32.5 \text{ mm}}$

Problem 3.3-5 A high-strength steel drill rod used for boring a hole in the earth has a diameter of 0.5 in. (see figure). The allowable shear stress in the steel is 40 ksi and the shear modulus of elasticity is 11,600 ksi.



- (a) What is the minimum required length of the rod so that one end of the rod can be twisted 30° with respect to the other end without exceeding the allowable stress?
- (b) If the shear strain in part (a) is limited to 3.2×10^{-3} , what is the minimum required length of the drill rod?

Solution 3.3-5 Steel drill rod

(a) MINIMUM REQUIRED LENGTH

$$G = 11,600 \text{ psi}$$

$$d = 0.5 \text{ in.}$$

$$\phi = 30^\circ = (30^\circ) \left(\frac{\pi}{180} \right) \text{ rad} = 0.52360 \text{ rad}$$

$$\tau_{\text{allow}} = 40 \text{ ksi}$$

MINIMUM LENGTH

$$\text{From Eq. (3-14): } \tau_{\max} = \frac{16T}{\pi d^3} \quad (1)$$

$$\text{From Eq. (3-17): } \phi = \frac{TL}{GI_P} = \frac{32TL}{G\pi d^4}$$

$$T = \frac{G\pi d^4 \phi}{32L}, \text{ substitute } T \text{ into Eq. (1):}$$

$$\tau_{\max} = \left(\frac{16}{\pi d^3} \right) \left(\frac{G\pi d^4 \phi}{32L} \right) = \frac{Gd\phi}{2L}$$

$$L_{\min} = \frac{Gd\phi}{2\tau_{\text{allow}}} = \frac{(11,600 \text{ ksi})(0.5 \text{ in.})(0.52360 \text{ rad})}{2(40 \text{ ksi})}$$

$$L_{\min} = 38.0 \text{ in.} \quad \leftarrow$$

- (b) IF THE SHEAR STRAIN IN PART (a) IS LIMITED TO 3.2×10^{-3} , WHAT IS THE MINIMUM REQUIRED LENGTH OF THE DRILL ROD?

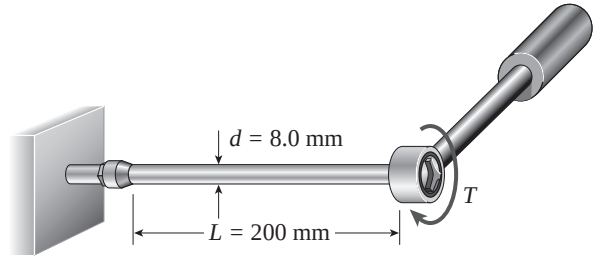
$$G = 11600 \text{ ksi} \quad \phi = 30^\circ \quad \tau_a = 40 \text{ ksi} \quad L = 38 \text{ in.} \quad \gamma_a = 3.2(10^{-3}) \quad d = 0.5 \text{ in.}$$

$$\gamma_{\text{act}} = \frac{d\phi}{2L} = 3.445 \times 10^{-3} \quad \boxed{L_{\min} = \frac{d\phi}{2\gamma_a} = 40.9 \text{ in.}}$$

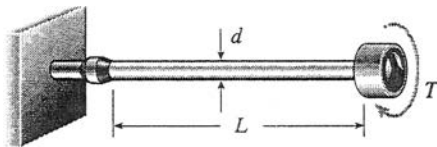
Problem 3.3-6 The steel shaft of a socket wrench has a diameter of 8.0 mm. and a length of 200 mm (see figure).

If the allowable stress in shear is 60 MPa, what is the maximum permissible torque $T_{\max}$ that may be exerted with the wrench?

Through what angle ϕ (in degrees) will the shaft twist under the action of the maximum torque? (Assume $G = 78$ GPa and disregard any bending of the shaft.)



Solution 3.3-6 Socket wrench



$$d = 8.0 \text{ mm} \quad L = 200 \text{ mm}$$

$$\tau_{\text{allow}} = 60 \text{ MPa} \quad G = 78 \text{ GPa}$$

MAXIMUM PERMISSIBLE TORQUE

$$\text{From Eq. (3-14): } \tau_{\max} = \frac{16T}{\pi d^3}$$

$$T_{\max} = \frac{\pi d^3 \tau_{\max}}{16}$$

$$T_{\max} = \frac{\pi (8.0 \text{ mm})^3 (60 \text{ MPa})}{16}$$

$$T_{\max} = 6.03 \text{ N}\cdot\text{m} \quad \leftarrow$$

ANGLE OF TWIST

$$\text{From Eq. (3-17): } \phi = \frac{T_{\max} L}{GI_P}$$

$$\text{From Eq. (3-14): } T_{\max} = \frac{\pi d^3 \tau_{\max}}{16}$$

$$\phi = \left(\frac{\pi d^3 \tau_{\max}}{16} \right) \left(\frac{L}{GI_P} \right) \quad I_P = \frac{\pi d^4}{32}$$

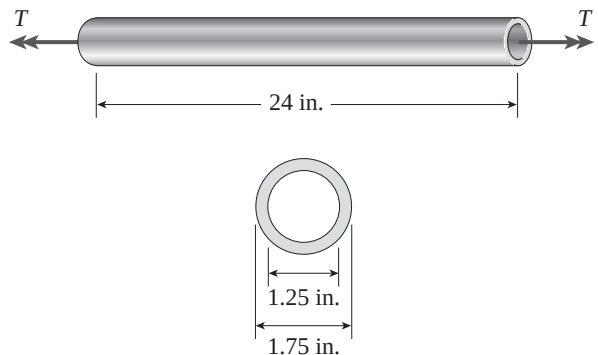
$$\phi = \frac{\pi d^3 \tau_{\max} L (32)}{16G(\pi d^4)} = \frac{2\tau_{\max} L}{Gd}$$

$$\phi = \frac{2(60 \text{ MPa})(200 \text{ mm})}{(78 \text{ GPa})(8.0 \text{ mm})} = 0.03846 \text{ rad}$$

$$\phi = (0.03846 \text{ rad}) \left(\frac{180}{\pi} \text{ }^\circ/\text{rad} \right) = 2.20^\circ \quad \leftarrow$$

Problem 3.3-7 A circular tube of aluminum is subjected to torsion by torques T applied at the ends (see figure). The bar is 24 in. long, and the inside and outside diameters are 1.25 in. and 1.75 in., respectively. It is determined by measurement that the angle of twist is 4° when the torque is 6200 lb-in.

- Calculate the maximum shear stress $\tau_{\max}$ in the tube, the shear modulus of elasticity G , and the maximum shear strain $\gamma_{\max}$ (in radians).
- If the maximum shear strain in the tube is limited to 2.5×10^{-3} and the inside diameter is increased to 1.375 in., what is the maximum permissible torque?



Solution 3.3-7

- (a) MAXIMUM SHEAR STRESS, SHEARING MODULUS, AND MAXIMUM SHEAR STRAIN

NUMERICAL DATA

$$L = 24 \text{ in.} \quad r_2 = \frac{1.75}{2} \text{ in.} \quad r_1 = \frac{1.25}{2} \text{ in.}$$

$$\phi = 4 \left(\frac{\pi}{180} \right) \text{ rad} \quad T = 6200 \text{ lb-in.}$$

$$\text{MAXIMUM SHEAR STRESS} \quad \tau_{\max} = \frac{Tr_2}{I_p}$$

$$I_p = \frac{\pi}{2} (r_2^4 - r_1^4) \quad I_p = 0.681 \text{ in.}^4$$

$$\tau_{\max} = \frac{Tr_2}{I_p} \quad \tau_{\max} = 7965 \text{ psi} \quad \leftarrow$$

$$\text{MAXIMUM SHEAR STRAIN} \quad \gamma_{\max} = \frac{r_2}{L} \phi$$

$$\gamma_{\max} = 0.00255 \text{ rad} \quad \leftarrow$$

SHEAR MODULUS OF ELASTICITY G

$$G = \frac{\tau_{\max}}{\gamma_{\max}}$$

$$G = 3.129 \times 10^6 \text{ psi}$$

$$\text{or } G = \frac{TL}{\phi I_p} \quad G = 3.13 \times 10^6 \text{ psi} \quad \leftarrow$$

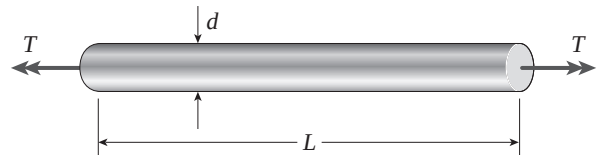
- (b) IF THE MAXIMUM SHEAR STRAIN IN THE TUBE IS LIMITED TO
- 2.5×10^{-3}
- AND THE INSIDE DIAMETER IS INCREASED TO 1.375 in., WHAT IS THE MAXIMUM PERMISSIBLE TORQUE?

$$G = 3.13(10^6) \text{ psi} \quad d_2 = 1.75 \text{ in.} \quad d_1 = 1.375 \text{ in.} \quad I_p = \frac{\pi}{32} (d_2^4 - d_1^4) = 0.56985 \text{ in.}^4$$

$$\gamma_a = 2.5(10^{-3}) \quad \boxed{T_{\max} = \frac{2GI_p}{d_2} \gamma_a = 5096 \text{ lb-in.}}$$

Problem 3.3-8 A propeller shaft for a small yacht is made of a solid steel bar 104 mm in diameter. The allowable stress in shear is 48 MPa, and the allowable rate of twist is 2.0° in 3.5 meters.

- (a) Assuming that the shear modulus of elasticity is $G = 80 \text{ GPa}$, determine the maximum torque $T_{\max}$ that can be applied to the shaft.
- (b) Repeat (a) if the shaft is now hollow with inner diameter of $5d/8$. Compare $T_{\max}$ values to corresponding values from part (a).



Solution 3.3-8

(a) MAXIMUM TORQUE

NUMERICAL DATA

$$d = 104 \text{ mm} \quad \tau_a = 48 \text{ MPa} \quad \theta = \frac{\phi}{L}$$

$$\theta = \frac{2\left(\frac{\pi}{180}\right)}{3.5} \text{ rad/m} \quad G = 80 \text{ GPa}$$

$$I_p = \frac{\pi}{32} d^4 \quad I_p = 1.149 \times 10^7 \text{ mm}^4$$

FIND MAXIMUM TORQUE BASED ON ALLOWABLE RATE OF TWIST

$$T_{\max} = \frac{GI_p \phi}{L} \quad T_{\max} = GI_p \theta$$

$$T_{\max} = 9164 \text{ N}\cdot\text{m} \quad \leftarrow \text{governs}$$

FIND MAXIMUM TORQUE BASED ON ALLOWABLE SHEAR STRESS

$$T_{\max} = \frac{\tau_a I_p}{\frac{d}{2}} \quad T_{\max} = 10,602 \text{ N}\cdot\text{m}$$

(b) REPEAT PART (a) IF THE SHAFT IS NOW HOLLOW WITH INNER DIAMETER OF $5d/8$. COMPARE $T_{\max}$ VALUES TO CORRESPONDING VALUES FROM PART (a).

$$d_2 = 104 \text{ mm} \quad d_1 = \frac{5}{8} d_2 = 65 \text{ mm} \quad G = 80 \text{ GPa} \quad \tau_a = 48 \text{ MPa}$$

$$\theta = \frac{2^\circ}{3.5 \text{ m}} = 9.973 \times 10^{-3} \text{ rad/m} \quad I_p = \frac{\pi}{32} (d_2^4 - d_1^4) = 9.733 \times 10^6 \text{ mm}^4 \quad \frac{9.733}{11.49} = 0.847$$

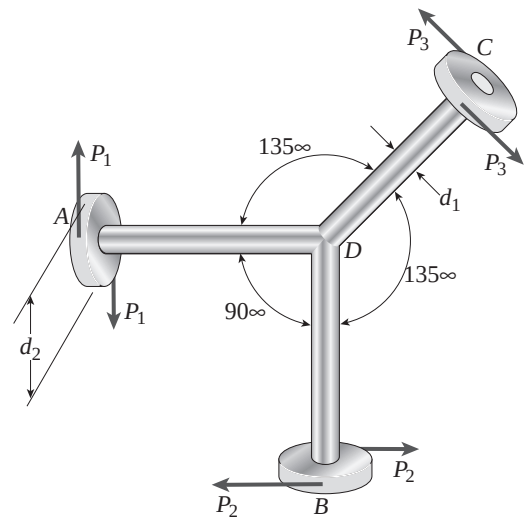
$$T_{\max} \text{ BASED ON ALLOWABLE RATE OF TWIST: } T_{1b} = GI_p \theta = 7765.282 \text{ N}\cdot\text{m} < \text{controls} \quad \frac{T_{1b}}{9164 \text{ N}\cdot\text{m}} = 0.847$$

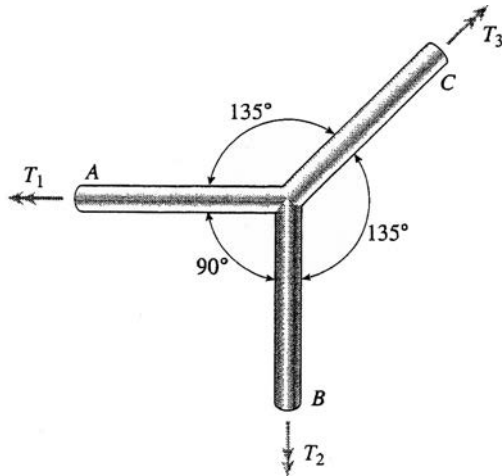
$$T_{\max} \text{ BASED ON ALLOWABLE SHEAR STRESS: } T_{2b} = \frac{\tau_a I_p}{\frac{d_2}{2}} = 8983.919 \text{ N}\cdot\text{m} \quad \frac{T_{2b}}{10602 \text{ N}\cdot\text{m}} = 0.847$$

For hollow shaft: $T_{\max} = T_{1b} = 7765 \text{ N}\cdot\text{m}$

Problem 3.3-9 Three identical circular disks A, B, and C are welded to the ends of three identical solid circular bars (see figure). The bars lie in a common plane and the disks lie in planes perpendicular to the axes of the bars. The bars are welded at their intersection D to form a rigid connection. Each bar has diameter $d_1 = 0.5$ in. and each disk has diameter $d_2 = 3.0$ in.

Forces P_1 , P_2 , and P_3 act on disks A, B, and C, respectively, thus subjecting the bars to torsion. If $P_1 = 28$ lb, what is the maximum shear stress $\tau_{\max}$ in any of the three bars?



Solution 3.3-9 Three circular bars

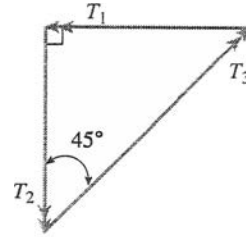
d_1 = diameter of bars
= 0.5 in.

d_2 = diameter of disks
= 3.0 in.

$P_1 = 28$ lb

$T_1 = P_1 d_2 \quad T_2 = P_2 d_2 \quad T_3 = P_3 d_2$

THE THREE TORQUES MUST BE IN EQUILIBRIUM



T_3 is the largest torque:

$$T_3 = T_1 \sqrt{2} = P_1 d_2 \sqrt{2}$$

MAXIMUM SHEAR STRESS (Eq. 3-14)

$$\tau_{\max} = \frac{16T}{\pi d^3} = \frac{16T_3}{\pi d_1^3} = \frac{16P_1 d_2 \sqrt{2}}{\pi d_1^3}$$

$$\tau_{\max} = \frac{16(28 \text{ lb})(3.0 \text{ in.})\sqrt{2}}{\pi(0.5 \text{ in.})^3} = 4840 \text{ psi} \quad \leftarrow$$

Problem 3.3-10 The steel axle of a large winch on an ocean liner is subjected to a torque of 1.65 kN·m (see figure).

- What is the minimum required diameter $d_{\min}$ if the allowable shear stress is 48 MPa and the allowable rate of twist is $0.75^\circ/\text{m}$? (Assume that the shear modulus of elasticity is 80 GPa.)
- Repeat part (a) if the shaft is now hollow with an inner diameter of $5d/8$. Compare $d_{\min}$ values to corresponding values from (a).

**Solution 3.3-10**

(a) MINIMUM REQUIRED DIAMETER

NUMERICAL DATA

$$T = 1.65 \text{ kN}\cdot\text{m} \quad \tau_a = 48 \text{ MPa} \quad G = 80 \text{ GPa}$$

$$\theta_a = 0.75 \left(\frac{\pi}{180} \right) \text{ rad/m}$$

MINIMUM REQUIRED DIAMETER OR SHAFT BASED ON ALLOWABLE RATE OF TWIST

$$\theta = \frac{T}{GI_p} \quad I_p = \frac{T}{G\theta} \quad \frac{\pi}{32} d^4 = \frac{T}{G\theta}$$

$$d^4 = \frac{32T}{\pi G\theta_a} \quad d_{\min} = \left(\frac{32T}{\pi G\theta_a} \right)^{\frac{1}{4}}$$

$$d_{\min} = 0.063 \text{ m} \quad d_{\min} = 63.3 \text{ mm} \leftarrow \text{governs}$$

MINIMUM REQUIRED DIAMETER OR SHAFT BASED ON ALLOWABLE SHEAR STRESS

$$\tau = \frac{Td}{2I_p} \quad \tau = \frac{Td}{2 \left(\frac{\pi}{32} d^4 \right)}$$

$$d_{\min} = \left[\frac{16T}{\pi\tau_a} \right]^{\frac{1}{3}} \quad d_{\min} = 0.056 \text{ m}$$

$$d_{\min} = 55.9 \text{ mm}$$

- (b) REPEAT PART (a) IF THE SHAFT IS NOW HOLLOW WITH INNER DIAMETER OF $5d/8$. COMPARE $d_{\min}$ VALUES TO CORRESPONDING VALUES FROM PART (a).

$$\theta = \frac{0.75^\circ}{1 \text{ m}} = 0.013 \text{ rad/m} \quad I_p = \frac{\pi}{32} \left[d^4 - \left(\frac{5}{8}d \right)^4 \right] = \frac{3471 \pi d^4}{131,072} = \frac{T}{G\theta}$$

$$d_{\min} \text{ BASED ON ALLOWABLE RATE OF TWIST: } d_{\min 1} = \left(\frac{131,072}{3471} \frac{T}{\pi G\theta} \right)^{\frac{1}{4}} = 66 \text{ mm} < \text{controls} \quad \frac{d_{\min 1}}{63.3 \text{ mm}} = 1.042$$

$$d_{\min} \text{ BASED ON ALLOWABLE SHEAR STRESS: } d_{\min 2} = \left(\frac{T}{2\tau_a \frac{3471 \pi}{131,072}} \right)^{\frac{1}{3}} = 59.116 \text{ mm} \quad \frac{d_{\min 2}}{55.9 \text{ mm}} = 1.058$$

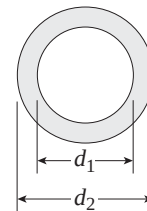
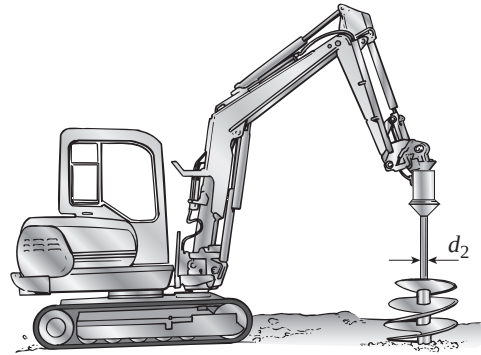
$$d_{\min} = 66 \text{ mm}$$

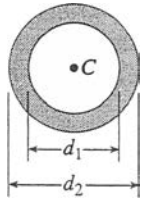
Problem 3.3-11 A hollow steel shaft used in a construction auger has outer diameter $d_2 = 6.0$ in. and inner diameter $d_1 = 4.5$ in. (see figure). The steel has shear modulus of elasticity $G = 11.0 \times 10^6$ psi.

For an applied torque of 150 k-in., determine the following quantities:

- Shear stress τ_2 at the outer surface of the shaft.
- Shear stress τ_1 at the inner surface.
- Rate of twist θ (degrees per unit of length).

Also, draw a diagram showing how the shear stresses vary in magnitude along a radial line in the cross section.



Solution 3.3-11 Construction auger

$$d_2 = 6.0 \text{ in.} \quad r_2 = 3.0 \text{ in.}$$

$$d_1 = 4.5 \text{ in.} \quad r_1 = 2.25 \text{ in.}$$

$$G = 11 \times 10^6 \text{ psi}$$

$$T = 150 \text{ k-in.}$$

$$I_P = \frac{\pi}{32} (d_2^4 - d_1^4) = 86.98 \text{ in.}^4$$

(a) SHEAR STRESS AT OUTER SURFACE

$$\tau_2 = \frac{Tr_2}{I_P} = \frac{(150 \text{ k-in.})(3.0 \text{ in.})}{86.98 \text{ in.}^4} = 5170 \text{ psi} \quad \leftarrow$$

(b) SHEAR STRESS AT INNER SURFACE

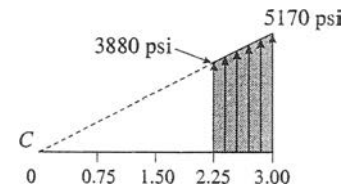
$$\tau_1 = \frac{Tr_1}{I_P} = \frac{r_1}{r_2} \tau_2 = 3880 \text{ psi} \quad \leftarrow$$

(c) RATE OF TWIST

$$\theta = \frac{T}{GI_P} = \frac{(150 \text{ k-in.})}{(11 \times 10^6 \text{ psi})(86.98 \text{ in.}^4)}$$

$$\theta = 157 \times 10^{-6} \text{ rad/in.} = 0.00898^\circ/\text{in.} \quad \leftarrow$$

(d) SHEAR STRESS DIAGRAM



Problem 3.3-12 Solve the preceding problem if the shaft has outer diameter $d_2 = 150 \text{ mm}$ and inner diameter $d_1 = 100 \text{ mm}$. Also, the steel has shear modulus of elasticity $G = 75 \text{ GPa}$ and the applied torque is $16 \text{ kN}\cdot\text{m}$.

Solution 3.3-12 Construction auger

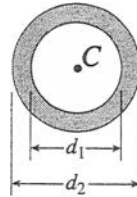
$$d_2 = 150 \text{ mm} \quad r_2 = 75 \text{ mm}$$

$$d_1 = 100 \text{ mm} \quad r_1 = 50 \text{ mm}$$

$$G = 75 \text{ GPa}$$

$$T = 16 \text{ kN}\cdot\text{m}$$

$$I_P = \frac{\pi}{32} (d_2^4 - d_1^4) = 39.88 \times 10^6 \text{ mm}^4$$



(a) SHEAR STRESS AT OUTER SURFACE

$$\tau_2 = \frac{Tr_2}{I_P} = \frac{(16 \text{ kN}\cdot\text{m})(75 \text{ mm})}{39.88 \times 10^6 \text{ mm}^4} = 30.1 \text{ MPa} \quad \leftarrow$$

(b) SHEAR STRESS AT INNER SURFACE

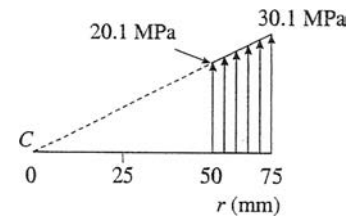
$$\tau_1 = \frac{Tr_1}{I_P} = \frac{r_1}{r_2} \tau_2 = 20.1 \text{ MPa} \quad \leftarrow$$

(c) RATE OF TWIST

$$\theta = \frac{T}{GI_P} = \frac{16 \text{ kN}\cdot\text{m}}{(75 \text{ GPa})(39.88 \times 10^6 \text{ mm}^4)}$$

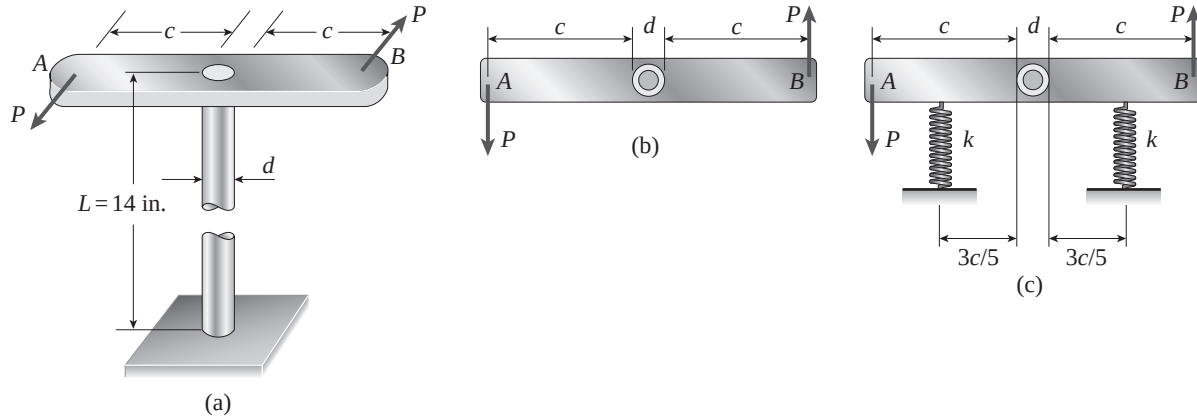
$$\theta = 0.005349 \text{ rad/m} = 0.306^\circ/\text{m} \quad \leftarrow$$

(d) SHEAR STRESS DIAGRAM



Problem 3.3-13 A vertical pole of solid circular cross section is twisted by horizontal forces $P = 1100$ lb acting at the ends of a rigid horizontal arm AB (see figure part a). The distance from the outside of the pole to the line of action of each force is $c = 5.0$ in. (see figure part b) and the pole height $L = 14$ in.

- (a) If the allowable shear stress in the pole is 4500 psi, what is the minimum required diameter $d_{\min}$ of the pole?
 (b) Find the torsional stiffness of the pole (kip-in./rad). Assume that $G = 10,800$ ksi.
 (c) If two translational springs, each with stiffness $k = 33$ kips/in., are added at $2c/5$ from A and B (see figure part c), repeat part (a) to find $d_{\min}$. [Hint: Consider the pole and pair of springs as “springs in parallel.”]



Solution 3.3-13 Vertical pole

NUMERICAL DATA

$$P = 1100 \text{ lb} \quad c = 5 \text{ in.} \quad L = 14 \text{ in.} \quad \tau_a = 4500 \text{ psi} \quad G = 10800 \text{ ksi}$$

$$(a) \quad \tau = \frac{T \frac{d}{2}}{I_p} \quad \tau = \frac{16T}{\pi d^3} \quad T = P(2c + d) \quad P(2c + d) \left(\frac{16}{\pi d^3} \right) = \tau_a \quad \text{Solving gives } \boxed{d = 2.5 \text{ in.}}$$

$$(b) \quad k_T = \frac{GI_p}{L} \quad I_p = \frac{\pi}{32} d^4 = 3.813 \text{ in.}^4 \quad \boxed{k_T = \frac{GI_p}{L} = 2941 \text{ k-in./rad}}$$

$$(c) \quad k = 33 \text{ k/in.}$$

Total torsional stiffness = torsional stiffness of pole plus effect of parallel springs when a unit rotation is applied:

$$k_{T\text{total}} = \frac{GI_p}{L} + 2k \left(\frac{3}{5}c + \frac{d}{2} \right)^2 = 4132.428 \text{ k-in./rad} \quad \phi = \frac{[P(2c + d)]}{k_{T\text{total}}} = 3.326 \times 10^{-3}$$

For pole:

$$\tau_{\text{pole}} = \frac{G\phi d}{2L} = 3202.942 \text{ psi} \quad \phi_a = \frac{T_{\max} L}{GI_p} \quad \text{and} \quad T_{\max} = \frac{\tau_a I_p}{\frac{d}{2}} = \tau_a \left(\frac{\pi d^3}{16} \right)$$

$$\phi_a = \frac{2\tau_a L}{Gd} = 4.673 \times 10^{-3}$$

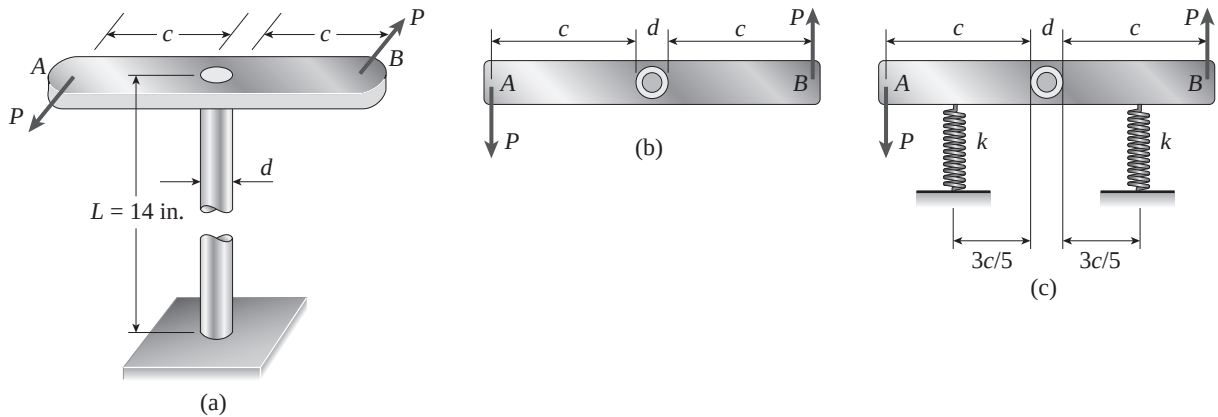
Equate expressions for ϕ and ϕ_a , then solve for $d_{\min}$:

$$\frac{[P(2c + d)]}{G \left(\frac{\pi d^4}{32} \right) \frac{L}{L} + 2k \left(\frac{3}{5}c + \frac{d}{2} \right)^2} = \frac{2\tau_a L}{Gd} \quad \text{OR} \quad \frac{P(2c + d)}{2k \left(\frac{3c}{5} + \frac{d}{2} \right)^2 + \frac{\pi G d^4}{32L}} - \frac{2L\tau_a}{Gd} = 0$$

Solving: $d_{\min} = 1.996 \text{ in.}$

Problem 3.3-14 A vertical pole of solid circular cross section is twisted by horizontal forces $P = 5 \text{ kN}$ acting at the ends of a rigid horizontal arm AB (see figure part a). The distance from the outside of the pole to the line of action of each force is $c = 125 \text{ mm}$ (see figure part b) and the pole height $L = 350 \text{ mm}$.

- If the allowable shear stress in the pole is 30 MPa , what is the minimum required diameter $d_{\min}$ of the pole?
- What is the torsional stiffness of the pole ($\text{kN}\cdot\text{m}/\text{rad}$)?
- If two translational springs each with stiffness $k = 2550 \text{ kN/m}$ are added at $2c/5$ from A and B (see figure part c), repeat part (a) to find $d_{\min}$. (*Hint: Consider the pole and pair of springs as “springs in parallel.”*)



Solution 3.3-14 Vertical pole

NUMERICAL DATA

$$P = 5 \text{ kN} \quad c = 125 \text{ mm} \quad L = 350 \text{ mm} \quad \tau_a = 30 \text{ MPa} \quad G = 28 \text{ GPa}$$

$$(a) \quad \tau = \frac{T \frac{d}{2}}{I_p} \quad \tau = \frac{16T}{\pi d^3} \quad T = P(2c + d) \quad P(2c + d) \left(\frac{16}{\pi d^3} \right) = \tau_a \quad \text{Solving gives: } d = 64.4 \text{ mm}$$

$$(b) \quad k_T = \frac{GI_p}{L} \quad I_p = \frac{\pi}{32} d^4 = 4.052 \text{ in.}^4 \quad k_T = \frac{GI_p}{L} = 134.9 \text{ kN}\cdot\text{m}/\text{rad} \quad \boxed{k_T = 134.9 \text{ kN}\cdot\text{m}/\text{rad}}$$

(c) TOTAL TORSIONAL STIFFNESS

$$k = 2550 \text{ kN/m} \quad k_{T\text{total}} = \frac{GI_p}{L} + 2k \left(\frac{3}{5}c + \frac{d}{2} \right)^2 = 193.533 \text{ kN}\cdot\text{m}/\text{rad} \quad \phi = \frac{[P(2c + d)]}{k_{T\text{total}}} = 8.122 \times 10^{-3}$$

FOR POLE

$$\tau_{\text{pole}} = \frac{G \phi d}{2L} = 20.917 \text{ MPa} \quad \phi_a = \frac{T_{\text{max}} L}{G I_p} \quad \text{and} \quad T_{\text{max}} = \frac{\tau_a I_p}{\frac{d}{2}} = \tau_a \left(\frac{\pi d^3}{16} \right)$$

$$\phi_a = \frac{2 \tau_a L}{G d} = 0.012$$

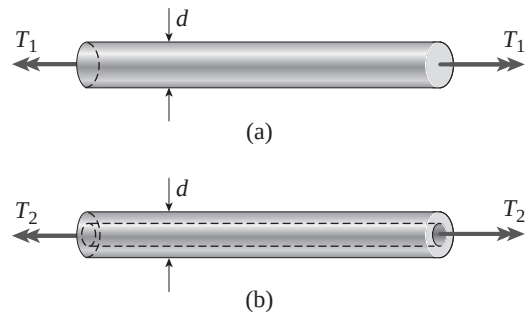
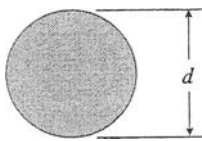
Equate expressions for ϕ and ϕ_a , then solve for d_{min} :

$$\frac{[P(2c + d)]}{\frac{G \left(\frac{\pi d^4}{32} \right)}{L} + 2k \left(\frac{3}{5}c + \frac{d}{2} \right)^2} = \frac{2 \tau_a L}{G d} \quad \text{or} \quad \frac{P(2c + d)}{2k \left(\frac{3}{5}c + \frac{d}{2} \right)^2 + \frac{\pi G d^4}{32L}} - \frac{2L \tau_a}{G d} = 0$$

$$\text{Solving: } \boxed{d_{\text{min}} = 50 \text{ mm}}$$

Problem 3.3-15 A solid brass bar of diameter $d = 1.25$ in. is subjected to torques T_1 , as shown in part (a) of the figure. The allowable shear stress in the brass is 12 ksi.

- What is the maximum permissible value of the torques T_1 ?
- If a hole of diameter 0.625 in. is drilled longitudinally through the bar, as shown in part (b) of the figure, what is the maximum permissible value of the torques T_2 ?
- What is the percent decrease in torque and the percent decrease in weight due to the hole?

**Solution 3.3-15**

- (a) MAXIMUM PERMISSIBLE VALUE OF TORQUE T_1 —SOLID BAR

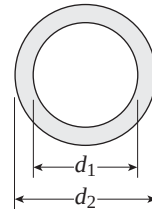
$$T_{1\text{max}} = \frac{\tau_a I_p}{\frac{d}{2}} \quad T_{1\text{max}} = \frac{\tau_a \frac{\pi}{32} d^4}{\frac{d}{2}}$$

$$T_{1\text{max}} = \frac{1}{16} \tau_a \pi d^3$$

$$T_{1\text{max}} = \frac{1}{16} (12) \pi (1.25)^3$$

$$T_{1\text{max}} = 4.60 \text{ in.-k} \quad \leftarrow$$

- (b) MAXIMUM PERMISSIBLE VALUE OF TORQUE T_2 —HOLLOW BAR



$$d_2 = 1.25 \text{ in.} \quad d_1 = 0.625 \text{ in.} \quad \tau_a = 12 \text{ ksi}$$

$$T_{2\text{max}} = \frac{\tau_a \frac{\pi}{32} (d_2^4 - d_1^4)}{\frac{d_2}{2}}$$

$$T_{2\text{max}} = \frac{1}{16} \tau_a \pi \frac{d_2^4 - d_1^4}{d_2}$$

$$T_{2\text{max}} = 4.31 \text{ in.-k} \quad \leftarrow$$

- (c) PERCENT DECREASE IN TORQUE AND PERCENT DECREASE IN WEIGHT DUE TO HOLE IN PART (b)

Percent decrease in torque:

$$\frac{T_{1\max} - T_{2\max}}{T_{1\max}}(100) = 6.25\% \quad \leftarrow$$

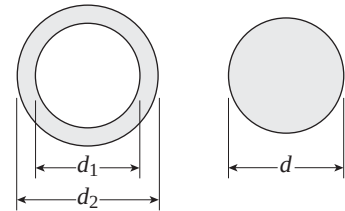
Percent decrease in weight (weight is proportional to x-sec area):

$$A_1 = \frac{\pi}{4}d_2^2 \quad A_2 = \frac{\pi}{4}(d_2^2 - d_1^2)$$

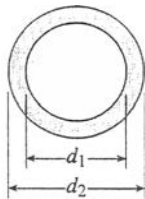
$$\frac{A_1 - A_2}{A_1}(100) = 25\% \quad \leftarrow$$

Problem 3.3-16 A hollow aluminum tube used in a roof structure has an outside diameter $d_2 = 104$ mm and an inside diameter $d_1 = 82$ mm (see figure). The tube is 2.75 m long, and the aluminum has shear modulus $G = 28$ GPa.

- (a) If the tube is twisted in pure torsion by torques acting at the ends, what is the angle of twist (in degrees) when the maximum shear stress is 48 MPa?
 (b) What diameter d is required for a solid shaft (see figure) to resist the same torque with the same maximum stress?
 (c) What is the ratio of the weight of the hollow tube to the weight of the solid shaft?



Solution 3.3-16



NUMERICAL DATA

$$d_2 = 104 \text{ mm}$$

$$d_1 = 82 \text{ mm}$$

$$L = 2.75 \times 10^3 \text{ mm}$$

$$G = 28 \text{ GPa}$$

$$I_p = (\pi/32)(d_2^4 - d_1^4)$$

$$I_p = 7.046 \times 10^6 \text{ mm}^4$$

Set $\tau_{\max}$ expression equal to

$$\frac{Td_2^2}{\frac{\pi}{32}(d_2^4 - d_1^4)} = \frac{32Td_2}{\pi(d_2^4 - d_1^4)}$$

Then solve for d :

$$d^3 = \frac{d_2^4 - d_1^4}{d_2}$$

$$d_{\text{reqd}} = \left(\frac{d_2^4 - d_1^4}{d_2} \right)^{1/3} \quad d_{\text{reqd}} = 88.4 \text{ mm} \quad \leftarrow$$

- (a) FIND ANGLE OF TWIST $\tau_{\max} = 48$ MPa

$$\phi = \frac{TL}{GI_p} \quad \phi = \left(\frac{Td_2}{2I_p} \right) \frac{2L}{Gd_2}$$

$$\phi = (\tau_{\max}) \frac{2L}{Gd_2}$$

$$\phi = 0.091 \text{ rad}$$

$$\phi = 5.19^\circ \quad \leftarrow$$

- (b) REPLACE HOLLOW SHAFT WITH SOLID SHAFT—FIND DIAMETER



$$\tau_{\max} = \frac{T \frac{d}{2}}{\frac{\pi}{32}d^4} \quad \tau_{\max} = \frac{16T}{d^3\pi}$$

- (c) RATIO OF WEIGHTS OF HOLLOW AND SOLID SHAFTS
WEIGHT IS PROPORTIONAL TO CROSS-SECTIONAL AREA

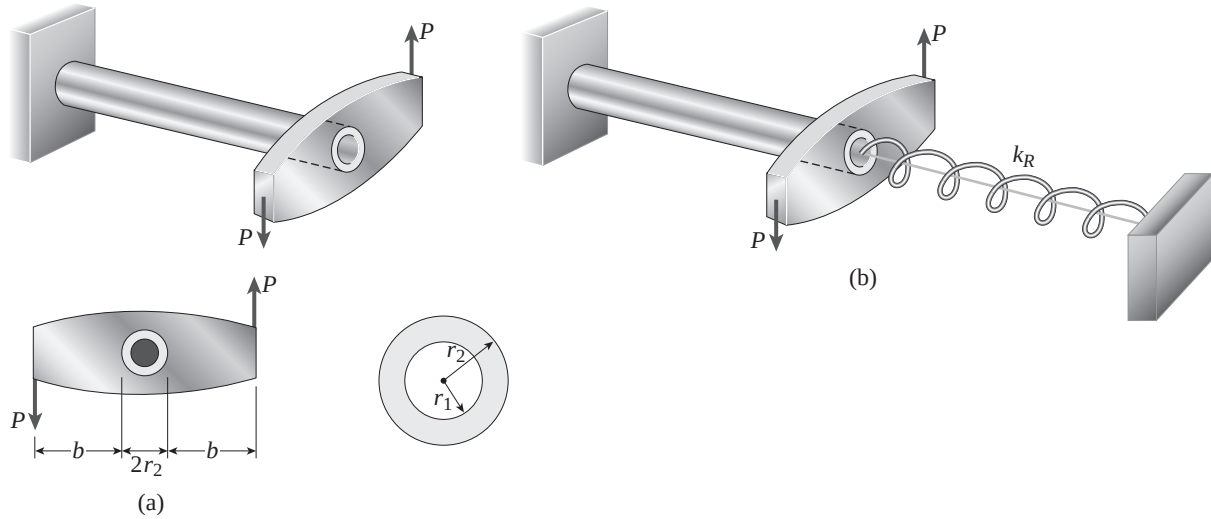
$$A_h = \frac{\pi}{4}(d_2^2 - d_1^2)$$

$$A_s = \frac{\pi}{4}d_{\text{reqd}}^2 \quad \frac{A_h}{A_s} = 0.524 \quad \leftarrow$$

So the weight of the tube is 52% of the solid shaft, but they resist the same torque.

Problem 3.3-17 A circular tube of inner radius r_1 and outer radius r_2 is subjected to a torque produced by forces $P = 900$ lb (see figure part a). The forces have their lines of action at a distance $b = 5.5$ in. from the outside of the tube.

- (a) If the allowable shear stress in the tube is 6300 psi and the inner radius $r_1 = 1.2$ in., what is the minimum permissible outer radius r_2 ?
- (b) If a torsional spring of stiffness $k_R = 450$ kip-in./rad is added at the end of the tube (see figure part b), what is the maximum value of forces P if the allowable shear stress is not to be exceeded? Assume that the tube has length $L = 18$ in., outer radius $r_2 = 1.45$ in., and shear modulus $G = 10,800$ ksi. (Hint: Consider the tube and torsional spring as “springs in parallel.”)



Solution 3.3-17 Circular tube in torsion

NUMERICAL DATA

$$P = 900 \text{ lb} \quad b = 5.5 \text{ in.} \quad \tau_a = 6300 \text{ psi} \quad r_1 = 1.2 \text{ in.} \quad L = 18 \text{ in.}$$

- (a) USING THE TORSION FORMULA; SET SHEAR STRESS EQUAL TO ALLOWABLE VALUE THEN SOLVE FOR RADIUS r_2

$$\tau = \frac{Tr_2}{I_p} \quad \text{or} \quad \frac{P(2b + 2r_2)r_2}{\frac{\pi}{2}(r_2^4 - r_1^4)} = \tau_a \quad \text{Solving: } \boxed{r_2 = 1.399 \text{ in.}}$$

- (b) LET $r_2 = 1.45$ in. $I_p = \frac{\pi}{2}(r_2^4 - r_1^4) = 3.687 \text{ in.}^4$ $G = 10800 \text{ ksi}$ $k_R = 450 \text{ kip-in./rad}$

Allowable twist of shaft ϕ_a based on allowable shear stress T_a :

$$\phi_a = \frac{T_{\max} L}{GI_p} \quad \text{where} \quad T_{\max} = \frac{\tau_a I_p}{r_2} \quad \text{So} \quad \phi_a = \frac{\tau_a L}{Gr_2} = 7.24138 \times 10^{-3}$$

Twist of shaft and torsional spring due to torque $P(2b + 2r_2)$:

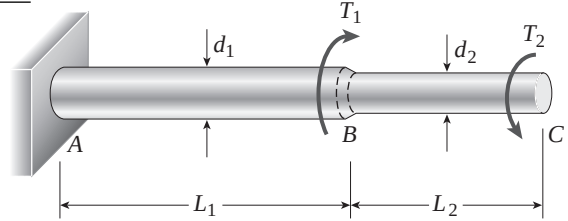
$$\phi = \frac{P(2b + 2r_2)}{\left(\frac{GI_p}{L} + k_R\right)} = 4.7 \times 10^{-3}$$

Equate ϕ and ϕ_a and solve for $P_{\max}$:

$$\frac{P_{\max}(2b + 2r_2)}{\left(\frac{GI_p}{L} + k_R\right)} = \frac{\tau_a L}{Gr_2} \quad \text{Solving for } P_{\max}: \quad \boxed{P_{\max} = 1387 \text{ lb}}$$

Nonuniform Torsion

Problem 3.4-1 A stepped shaft ABC consisting of two solid circular segments is subjected to torques T_1 and T_2 acting in opposite directions, as shown in the figure. The larger segment of the shaft has a diameter of $d_1 = 2.25$ in. and length of $L_1 = 30$ in.; the smaller segment has a diameter of $d_2 = 1.75$ in. and length $L_2 = 20$ in. The material is steel with shear modulus $G = 11 \times 10^6$ psi, and the torques are $T_1 = 20,000$ lb-in. and $T_2 = 8,000$ lb-in.



- Calculate the maximum shear stress $\tau_{\max}$ in the shaft, and the angle of twist ϕ_C (in degrees) at end C .
- If the maximum shear stress in BC must be the same as that in AB , what is the required diameter of segment BC ? What is the resulting twist at end C ?

Solution 3.4-1 Stepped shaft

$$(a) \quad d_2 = 1.75 \text{ in.} \quad d_1 = 2.25 \text{ in.} \quad L_1 = 30 \text{ in.} \quad L_2 = 20 \text{ in.} \quad G = 11(10^6) \text{ psi}$$

$$T_1 = 20,000 \text{ in.-lb} \quad T_2 = 8000 \text{ in.-lb}$$

$$I_{pAB} = \frac{\pi}{32} d_1^4 = 2.516 \text{ in.}^4 \quad I_{pBC} = \frac{\pi}{32} d_2^4 = 0.921 \text{ in.}^4$$

$$\tau_{AB} = \frac{(T_1 - T_2) \left(\frac{d_1}{2} \right)}{I_{pAB}} = 5365 \text{ psi} \quad \tau_{BC} = \frac{T_2 \left(\frac{d_2}{2} \right)}{I_{pBC}} = 7602 \text{ psi}$$

$$\phi_C = \phi_{AB} + \phi_{BC} \quad \phi_{AB} = \frac{(T_2 - T_1)L_1}{GI_{pAB}} = -0.745^\circ \quad \phi_{BC} = \frac{T_2 L_2}{GI_{pBC}} = 0.905^\circ$$

$$\phi_C = \phi_{AB} + \phi_{BC} = 0.16^\circ$$

$$(b) \quad \tau_{BC} = T_2 \left(\frac{16}{\pi d_2^3} \right) \quad \text{So} \quad T_2 \left(\frac{16}{\pi d_2^3} \right) = \tau_{AB} \quad \text{Solving for } d_2: \quad d_{2 \text{ reqd}} = \left(T_2 \frac{16}{\pi \tau_{AB}} \right)^{\frac{1}{3}} = 1.966 \text{ in.}$$

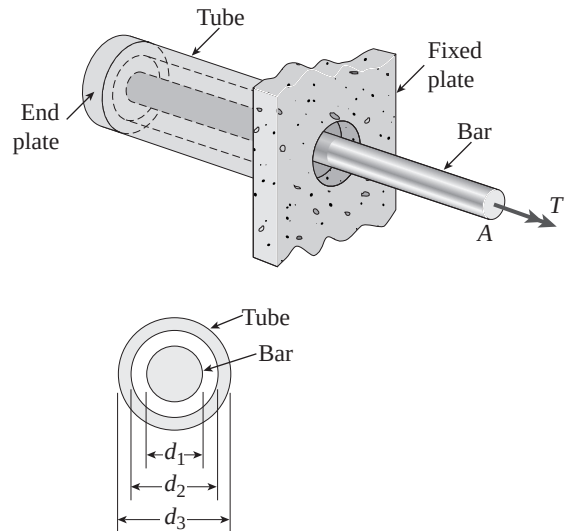
$$\text{Check:} \quad \tau_{BC} = T_2 \left(\frac{16}{\pi d_{2 \text{ reqd}}^3} \right) = 5365 \text{ psi} < \text{same as } \tau_{AB} \text{ above} \quad I_{pBC} = \frac{\pi}{32} d_{2 \text{ reqd}}^4 = 1.465 \text{ in.}^4$$

$$\phi_{BC} = \frac{T_2 L_2}{GI_{pBC}} = 0.569^\circ \quad \phi_C = \phi_{AB} + \phi_{BC} = -0.177^\circ$$

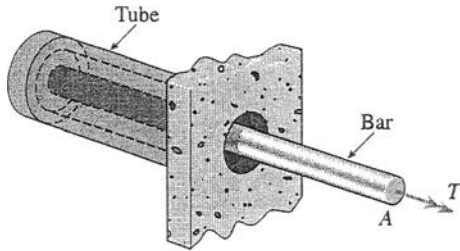
Problem 3.4-2 A circular tube of outer diameter $d_3 = 70$ mm and inner diameter $d_2 = 60$ mm is welded at the right-hand end to a fixed plate and at the left-hand end to a rigid end plate (see figure). A solid circular bar of diameter $d_1 = 40$ mm is inside of, and concentric with, the tube. The bar passes through a hole in the fixed plate and is welded to the rigid end plate.

The bar is 1.0 m long and the tube is half as long as the bar. A torque $T = 1000 \text{ N}\cdot\text{m}$ acts at end A of the bar. Also, both the bar and tube are made of an aluminum alloy with shear modulus of elasticity $G = 27 \text{ GPa}$.

- Determine the maximum shear stresses in both the bar and tube.
- Determine the angle of twist (in degrees) at end A of the bar.



Solution 3.4-2 Bar and tube



TUBE

$$d_3 = 70 \text{ mm} \quad d_2 = 60 \text{ mm}$$

$$L_{\text{tube}} = 0.5 \text{ m} \quad G = 27 \text{ GPa}$$

$$(I_p)_{\text{tube}} = \frac{\pi}{32} (d_3^4 - d_2^4) \\ = 1.0848 \times 10^6 \text{ mm}^4$$

BAR

$$d_1 = 40 \text{ mm} \quad L_{\text{bar}} = 1.0 \text{ m} \quad G = 27 \text{ GPa}$$

$$(I_p)_{\text{bar}} = \frac{\pi d_1^4}{32} = 251.3 \times 10^3 \text{ mm}^4$$

TORQUE

$$T = 1000 \text{ N}\cdot\text{m}$$

(a) MAXIMUM SHEAR STRESSES

$$\text{Bar: } \tau_{\text{bar}} = \frac{16T}{\pi d_1^3} = 79.6 \text{ MPa} \quad \leftarrow$$

$$\text{Tube: } \tau_{\text{tube}} = \frac{T(d_3/2)}{(I_p)_{\text{tube}}} = 32.3 \text{ MPa} \quad \leftarrow$$

(b) ANGLE OF TWIST AT END A

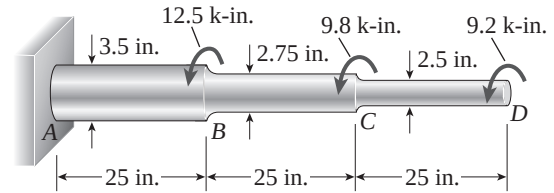
$$\text{Bar: } \phi_{\text{bar}} = \frac{TL_{\text{bar}}}{G(I_p)_{\text{bar}}} = 0.1474 \text{ rad}$$

$$\text{Tube: } \phi_{\text{tube}} = \frac{TL_{\text{tube}}}{G(I_p)_{\text{tube}}} = 0.0171 \text{ rad}$$

$$\phi_A = \phi_{\text{bar}} + \phi_{\text{tube}} = 0.1474 + 0.0171 = 0.1645 \text{ rad}$$

$$\phi_A = 9.43^\circ \quad \leftarrow$$

Problem 3.4-3 A stepped shaft $ABCD$ consisting of solid circular segments is subjected to three torques, as shown in the figure. The torques have magnitudes 12.5 k-in., 9.8 k-in., and 9.2 k-in. The length of each segment is 25 in. and the diameters of the segments are 3.5 in., 2.75 in., and 2.5 in. The material is steel with shear modulus of elasticity $G = 11.6 \times 10^3$ ksi.



- (a) Calculate the maximum shear stress $\tau_{\max}$ in the shaft and the angle of twist ϕ_D (in degrees) at end D .
 (b) If each segment must have the same shear stress, find the required diameter of each segment in part (a) so that all three segments have shear stress $\tau_{\max}$ from part (a). What is the resulting angle of twist at D ?

Solution 3.4-3

(a)

$$T_B = 12.5 \text{ k-in.} \quad T_C = 9.8 \text{ k-in.} \quad T_D = 9.2 \text{ k-in.} \quad L = 25 \text{ in.}$$

$$G = 11.6(10^3) \text{ ksi} \quad d_{AB} = 3.5 \text{ in.} \quad d_{BC} = 2.75 \text{ in.} \quad d_{CD} = 2.5 \text{ in.}$$

$$I_{pAB} = \frac{\pi}{32} d_{AB}^4 \quad I_{pBC} = \frac{\pi}{32} d_{BC}^4 \quad I_{pCD} = \frac{\pi}{32} d_{CD}^4$$

$$\tau_{AB} = \frac{(T_B + T_C + T_D) \left(\frac{d_{AB}}{2} \right)}{I_{pAB}} = 3742 \text{ psi} \quad \tau_{BC} = \frac{(T_C + T_D) \left(\frac{d_{BC}}{2} \right)}{I_{pBC}} = 4653 \text{ psi} < \text{max. shear stress}$$

$$\tau_{CD} = \frac{T_D \left(\frac{d_{CD}}{2} \right)}{I_{pCD}} = 2999 \text{ psi} \quad \text{or} \quad \tau_{CD} = T_D \left(\frac{16}{\pi d_{CD}^3} \right) = 2999 \text{ psi}$$

$$\phi_D = \phi_{AB} + \phi_{BC} + \phi_{CD} \quad \phi_D = \frac{L}{G} \left[\frac{(T_B + T_C + T_D)}{I_{pAB}} + \frac{(T_C + T_D)}{I_{pBC}} + \frac{T_D}{I_{pCD}} \right] = 0.978^\circ \quad \phi_D = 0.017 \text{ rad}$$

(b)

$$\tau_{\max} = \tau_{BC} = 4653 \text{ psi} \quad \text{so} \quad d_{AB\text{reqd}} = \left[(T_B + T_C + T_D) \left(\frac{16}{\pi \tau_{\max}} \right) \right]^{\frac{1}{3}} = 3.25 \text{ in.}$$

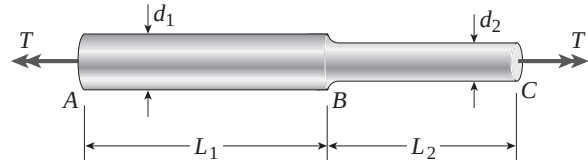
$$d_{BC} = 2.75 \text{ in.} \quad d_{CD\text{reqd}} = \left[T_D \left(\frac{16}{\pi \tau_{\max}} \right) \right]^{\frac{1}{3}} = 2.16 \text{ in.}$$

$$I_{pAB} = \frac{\pi}{32} d_{AB\text{reqd}}^4 \quad I_{pCD} = \frac{\pi}{32} d_{CD\text{reqd}}^4$$

$$\phi_D = \frac{L}{G} \left[\frac{(T_B + T_C + T_D)}{I_{pAB}} + \frac{(T_C + T_D)}{I_{pBC}} + \frac{T_D}{I_{pCD}} \right] = 1.303^\circ \quad \phi_D = 0.023 \text{ rad}$$

Problem 3.4-4 A solid circular bar ABC consists of two segments, as shown in the figure. One segment has diameter $d_1 = 56$ mm and length $L_1 = 1.45$ m; the other segment has diameter $d_2 = 48$ mm and length $L_2 = 1.2$ m.

What is the allowable torque T_{allow} if the shear stress is not to exceed 30 MPa and the angle of twist between the ends of the bar is not to exceed 1.25° ? (Assume $G = 80$ GPa.)



Solution 3.4-4

NUMERICAL DATA

$$d_1 = 56 \text{ mm} \quad d_2 = 48 \text{ mm}$$

$$L_1 = 1450 \text{ mm} \quad L_2 = 1200 \text{ mm} \quad G = 80 \text{ GPa}$$

$$\tau_a = 30 \text{ MPa} \quad \phi_a = 1.25 \left(\frac{\pi}{180} \right) \text{ rad}$$

Allowable torque:

T_{allow} based on shear stress

$$\tau_{\text{max}} = \frac{16T}{d_2^3 \pi} \quad T_{\text{allow}} = \frac{\tau_a \pi d_2^3}{16}$$

$$T_{\text{allow}} = 651.441 \text{ N}\cdot\text{m}$$

T_{allow} based on angle of twist:

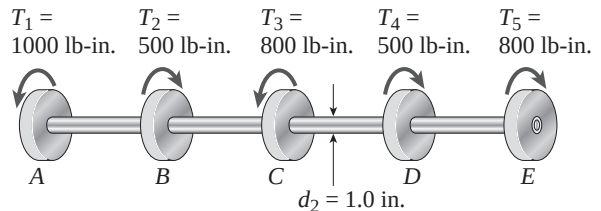
$$\phi_{\text{max}} = \frac{T}{G} \left[\frac{L_1}{\left(\frac{\pi}{32} d_1^4 \right)} + \frac{L_2}{\left(\frac{\pi}{32} d_2^4 \right)} \right]$$

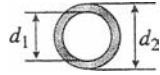
$$T_{\text{allow}} = \frac{G \phi_a}{\frac{L_1}{\left(\frac{\pi}{32} d_1^4 \right)} + \frac{L_2}{\left(\frac{\pi}{32} d_2^4 \right)}}$$

$$T_{\text{allow}} = 459 \text{ N}\cdot\text{m} \quad \leftarrow \text{ governs}$$

Problem 3.4-5 A hollow tube $ABCDE$ constructed of monel metal is subjected to five torques acting in the directions shown in the figure. The magnitudes of the torques are $T_1 = 1000$ lb-in., $T_2 = T_4 = 500$ lb-in., and $T_3 = T_5 = 800$ lb-in. The tube has an outside diameter $d_2 = 1.0$ in. The allowable shear stress is 12,000 psi and the allowable rate of twist is $2.0^\circ/\text{ft}$.

Determine the maximum permissible inside diameter d_1 of the tube.

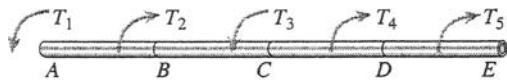


Solution 3.4-5 Hollow tube of monel metal

$$d_2 = 1.0 \text{ in.} \quad \tau_{\text{allow}} = 12,000 \text{ psi}$$

$$\begin{aligned} \theta_{\text{allow}} &= 2^\circ/\text{ft} = 0.16667^\circ/\text{in.} \\ &= 0.002909 \text{ rad/in.} \end{aligned}$$

From Table I-2, Appendix I: $G = 9500 \text{ ksi}$

TORQUES

$$\begin{aligned} T_1 &= 1000 \text{ lb-in.} & T_2 &= 500 \text{ lb-in.} & T_3 &= 800 \text{ lb-in.} \\ T_4 &= 500 \text{ lb-in.} & T_5 &= 800 \text{ lb-in.} \end{aligned}$$

INTERNAL TORQUES

$$T_{AB} = -T_1 = -1000 \text{ lb-in.}$$

$$T_{BC} = -T_1 + T_2 = -500 \text{ lb-in.}$$

$$T_{CD} = -T_1 + T_2 - T_3 = -1300 \text{ lb-in.}$$

$$T_{DE} = -T_1 + T_2 - T_3 + T_4 = -800 \text{ lb-in.}$$

Largest torque (absolute value only):

$$T_{\text{max}} = 1300 \text{ lb-in.}$$

REQUIRED POLAR MOMENT OF INERTIA BASED UPON ALLOWABLE SHEAR STRESS

$$\tau_{\text{max}} = \frac{T_{\text{max}} r}{I_p} \quad I_p = \frac{T_{\text{max}}(d_2/2)}{\tau_{\text{allow}}} = 0.05417 \text{ in.}^4$$

REQUIRED POLAR MOMENT OF INERTIA BASED UPON ALLOWABLE ANGLE OF TWIST

$$\theta = \frac{T_{\text{max}}}{GI_p} \quad I_p = \frac{T_{\text{max}}}{G\theta_{\text{allow}}} = 0.04704 \text{ in.}^4$$

SHEAR STRESS GOVERNS

$$\text{Required } I_p = 0.05417 \text{ in.}^4$$

$$I_p = \frac{\pi}{32} (d_2^4 - d_1^4)$$

$$\begin{aligned} d_1^4 &= d_2^4 - \frac{32I_p}{\pi} = (1.0 \text{ in.})^4 - \frac{32(0.05417 \text{ in.}^4)}{\pi} \\ &= 0.4482 \text{ in.}^4 \end{aligned}$$

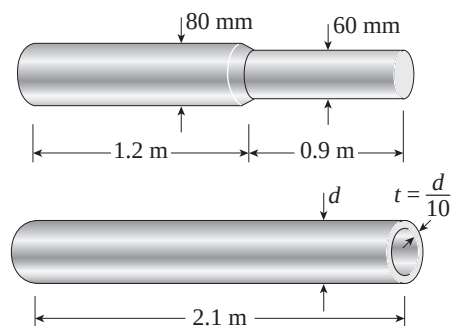
$$d_1 = 0.818 \text{ in.} \quad \leftarrow$$

(Maximum permissible inside diameter)

Problem 3.4-6 A shaft of solid circular cross section consisting of two segments is shown in the first part of the figure. The left-hand segment has a diameter of 80 mm and length of 1.2 m; the right-hand segment has a diameter of 60 mm and length of 0.9 m.

Shown in the second part of the figure is a hollow shaft made of the same material and having the same length. The thickness t of the hollow shaft is $d/10$, where d is the outer diameter. Both shafts are subjected to the same torque.

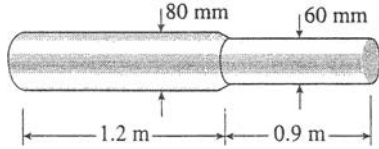
- If the hollow shaft is to have the same torsional stiffness as the solid shaft, what should be its outer diameter d ?
- If torque T is applied at either end of both shafts, and the hollow shaft is to have the same maximum shear stress as the solid shaft, what should be its outer diameter d ?



Solution 3.4-6 Solid and hollow shafts

(a) $L_1 = 1.2 \text{ m}$ $L_2 = 0.9 \text{ m}$ $d_1 = 80 \text{ mm}$ $d_2 = 60 \text{ mm}$ $L = 2.1 \text{ m}$

SOLID SHAFT CONSISTING OF TWO SEGMENTS



$$\begin{aligned}\phi_1 &= \sum \frac{TL_i}{GI_{Pi}} = \frac{T(1.2 \text{ m})}{G\left(\frac{\pi}{32}\right)(80 \text{ mm})^4} + \frac{T(0.9 \text{ m})}{G\left(\frac{\pi}{32}\right)(60 \text{ mm})^4} \\ &= \frac{32T}{\pi G}(29,297 \text{ m}^{-3} + 69,444 \text{ m}^{-3}) \\ &= \frac{32T}{\pi G}(98,741 \text{ m}^{-3})\end{aligned}$$

TORSIONAL STIFFNESS

$$k_T = \frac{T}{\phi} \quad \text{Torque } T \text{ is the same for both shafts.}$$

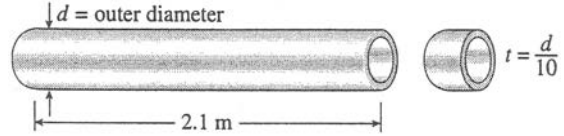
$\therefore$ For equal stiffnesses, $\phi_1 = \phi_2$.

$$98,741 \text{ m}^{-3} = \frac{3.5569 \text{ m}}{d^4}$$

$$d^4 = \frac{3.5569}{98,741} = 36.023 \times 10^{-6} \text{ m}^4$$

$$d = 0.0775 \text{ m} = 77.5 \text{ mm} \quad \leftarrow$$

HOLLOW SHAFT



$$d_0 = \text{inner diameter} = 0.8d$$

$$\begin{aligned}\phi_2 &= \frac{TL}{GI_p} = \frac{T(2.1 \text{ m})}{G\left(\frac{\pi}{32}\right)[d^4 - (0.8d)^4]} \\ &= \frac{32T}{\pi G}\left(\frac{2.1 \text{ m}}{0.5904 d^4}\right) = \frac{32T}{\pi G}\left(\frac{3.5569 \text{ m}}{d^4}\right)\end{aligned}$$

UNITS: d = meters

(b) FIRST, FIND EXPRESSIONS FOR MAXIMUM SHEAR STRESS IN SEGMENTS 1 AND 2 OF THE STEPPED SHAFT

$$\tau_{\max 1} = T\left(\frac{16}{\pi d_1^3}\right) = \frac{T}{32000 \pi \text{ mm}^3} \quad \tau_{\max 2} = T\left(\frac{16}{\pi d_2^3}\right) = \frac{T}{13500 \pi \text{ mm}^3} \quad \begin{array}{l} < \text{larger shear stress} \\ & \text{occurs on surface of} \\ & \text{segment 2 of stepped} \\ & \text{solid shaft} \end{array}$$

Next, find an expression for max. shear stress in the hollow shaft:

$$\tau_{\max \text{Hollow}} = \frac{T\left(\frac{d}{2}\right)}{\frac{\pi}{32}\left[d^4 - \left(\frac{8d}{10}\right)^4\right]}$$

Equate $\tau_{\max}$ for each segment of stepped shaft to $\tau_{\max}$ for the hollow shaft and then solve for required outer diameter d of the hollow shaft. The two solutions are as follows.

(1) If $\tau_{\max 1} = \tau_{\max \text{Hollow}}$, $d = 95.4 \text{ mm}$

(2) If $\tau_{\max 2} = \tau_{\max \text{Hollow}}$, $\boxed{d = 71.5 \text{ mm}} < \text{Use } d = 71.5 \text{ mm (and } t = d/10 = 7.15 \text{ mm)}$

Check:

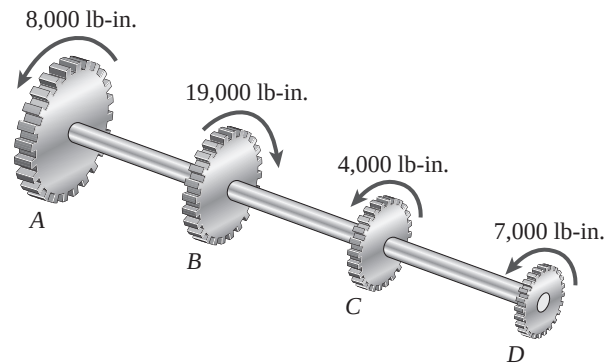
Let $T = 1.0 \text{ kN}\cdot\text{m}$ $d = 71.5 \text{ mm}$

$$\tau_{\max \text{Hollow}} = \frac{T \left(\frac{d}{2} \right)}{\frac{\pi}{32} \left[d^4 - \left(\frac{8d}{10} \right)^4 \right]} = 23.6 \text{ MPa}$$

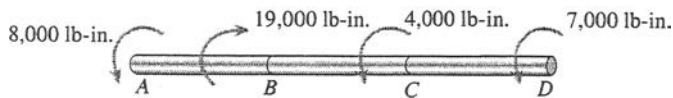
$$\tau_{\max 1} = T \left(\frac{16}{\pi d_1^3} \right) = 9.95 \text{ MPa} \quad \tau_{\max 2} = T \left(\frac{16}{\pi d_2^3} \right) = 23.6 \text{ MPa}$$

Problem 3.4-7 Four gears are attached to a circular shaft and transmit the torques shown in the figure. The allowable shear stress in the shaft is 10,000 psi.

- What is the required diameter d of the shaft if it has a solid cross section?
- What is the required outside diameter d if the shaft is hollow with an inside diameter of 1.0 in.?



Solution 3.4-7 Shaft with four gears



$$\tau_{\text{allow}} = 10,000 \text{ psi} \quad T_{BC} = +11,000 \text{ lb-in.}$$

$$T_{AB} = -8,000 \text{ lb-in.} \quad T_{CD} = +7,000 \text{ lb-in.}$$

(a) SOLID SHAFT

$$\tau_{\max} = \frac{16T}{\pi d^3}$$

$$d^3 = \frac{16T_{\max}}{\pi \tau_{\text{allow}}} = \frac{16(11,000 \text{ lb-in.})}{\pi(10,000 \text{ psi})} = 5.602 \text{ in.}^3$$

$$\text{Required } d = 1.78 \text{ in.} \quad \leftarrow$$

(b) HOLLOW SHAFT

Inside diameter $d_o = 1.0 \text{ in.}$

$$\tau_{\max} = \frac{Tr}{I_p} \quad \tau_{\text{allow}} = \frac{T_{\max} \left(\frac{d}{2} \right)}{I_p}$$

$$10,000 \text{ psi} = \frac{(11,000 \text{ lb-in.}) \left(\frac{d}{2} \right)}{\left(\frac{\pi}{32} \right) [d^4 - (1.0 \text{ in.})^4]}$$

UNITS: $d = \text{inches}$

$$10,000 = \frac{56,023 d}{d^4 - 1}$$

or

$$d^4 - 5.6023 d - 1 = 0$$

Solving: $d = 1.832$

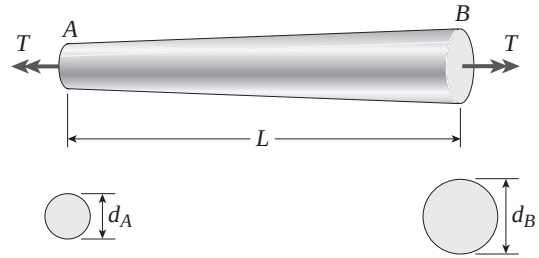
$$\text{Required: } d = 1.83 \text{ in.} \quad \leftarrow$$

Problem 3.4-8 A tapered bar AB of solid circular cross section is twisted by torques T (see figure). The diameter of the bar varies linearly from d_A at the left-hand end to d_B at the right-hand end.

- (a) Confirm that the angle of twist of the tapered bar is

$$\phi = \frac{32TL}{3\pi G(d_B - d_A)} \left(\frac{1}{d_A^3} - \frac{1}{d_B^3} \right)$$

- (b) For what ratio d_B/d_A will the angle of twist of the tapered bar be one-half the angle of twist of a prismatic bar of diameter d_A ? (The prismatic bar is made of the same material, has the same length, and is subjected to the same torque as the tapered bar.)



Problems 3.4-8, 3.4-9 and 3.4-10

Solution 3.4-8 Tapered bar AB

(a) LINEAR VARIATION $d(x) = d_A \left(1 - \frac{x}{L} \right) + d_B \frac{x}{L}$ $d(0) = d_A$ $d(L) = d_B$

$$\phi = \int_0^L \frac{T}{G \frac{\pi}{32} \left[d_A \left(1 - \frac{x}{L} \right) + d_B \frac{x}{L} \right]^4} dx = \frac{32TL}{3\pi G(d_B - d_A)} \left(\frac{1}{d_A^3} - \frac{1}{d_B^3} \right)$$

(b) $\phi_{\text{tapered}} = \frac{32TL}{3\pi G(d_B - d_A)} \left(\frac{1}{d_A^3} - \frac{1}{d_B^3} \right)$ or $\phi_{\text{tapered}} = \frac{TL}{3GI_{PA}} \left(\frac{\beta^2 + \beta + 1}{\beta^3} \right)$

where $\beta = \frac{d_B}{d_A}$ $I_{PA} = \frac{\pi}{32} d_A^4$ and $\phi_{\text{prismatic}} = \frac{TL}{GI_{PA}}$

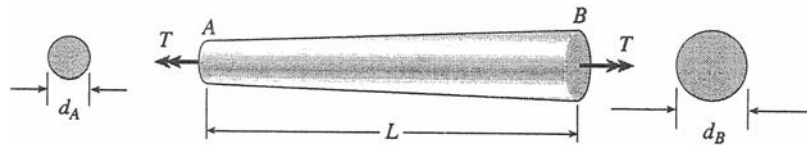
Now set $\phi_{\text{tapered}} = \frac{\phi_{\text{prismatic}}}{2}$ or $\frac{LT(\beta^2 + \beta + 1)}{3GI_{PA}\beta^3} = \frac{LT}{2GI_{PA}}$

or $\frac{(\beta^2 + \beta + 1)}{3\beta^3} = \frac{1}{2}$ Solve numerically: $\boxed{\beta = 1.446}$

Problem 3.4-9 A tapered bar AB of solid circular cross section is twisted by torques $T = 36,000$ lb-in. (see figure). The diameter of the bar varies linearly from d_A at the left-hand end to d_B at the right-hand end. The bar has length $L = 4.0$ ft and is made of an aluminum alloy having shear modulus of elasticity $G = 3.9 \times 10^6$ psi. The allowable shear stress in the bar is 15,000 psi and the allowable angle of twist is 3.0° .

If the diameter at end B is 1.5 times the diameter at end A , what is the minimum required diameter d_A at end A ? [Hint: Use the angle of twist expression from Prob. 3.4-8(a)].

Solution 3.4-9 Tapered bar



$$d_B = 1.5 d_A$$

$$T = 36,000 \text{ lb-in.}$$

$$L = 4.0 \text{ ft} = 48 \text{ in.}$$

$$G = 3.9 \times 10^6 \text{ psi}$$

$$\tau_{\text{allow}} = 15,000 \text{ psi}$$

$$\phi_{\text{allow}} = 3.0^\circ$$

$$= 0.0523599 \text{ rad}$$

MINIMUM DIAMETER BASED UPON ALLOWABLE SHEAR STRESS

$$\begin{aligned} \tau_{\text{max}} &= \frac{16T}{\pi d_A^3} & d_A^3 &= \frac{16T}{\pi \tau_{\text{allow}}} = \frac{16(36,000 \text{ lb-in.})}{\pi(15,000 \text{ psi})} \\ & & &= 12.2231 \text{ in.}^3 \\ d_A &= 2.30 \text{ in.} \end{aligned}$$

MINIMUM DIAMETER BASED UPON ALLOWABLE ANGLE OF TWIST—MODIFYING THE ANGLE OF TWIST EXPRESSION FROM PROB. 3.4-8a:

$$\beta = d_B/d_A = 1.5$$

$$\begin{aligned} \phi &= \frac{TL}{G(I_P)_A} \left(\frac{\beta^2 + \beta + 1}{3\beta^3} \right) = \frac{TL}{G(I_P)_A} (0.469136) \\ &= \frac{(36,000 \text{ lb-in.})(48 \text{ in.})}{(3.9 \times 10^6 \text{ psi}) \left(\frac{\pi}{32} \right) d_A^4} (0.469136) \end{aligned}$$

$$= \frac{2.11728 \text{ in.}^4}{d_A^4}$$

$$d_A^4 = \frac{2.11728 \text{ in.}^4}{\phi_{\text{allow}}} = \frac{2.11728 \text{ in.}^4}{0.0523599 \text{ rad}}$$

$$= 40.4370 \text{ in.}^4$$

$$d_A = 2.52 \text{ in.}$$

ANGLE OF TWIST GOVERNS

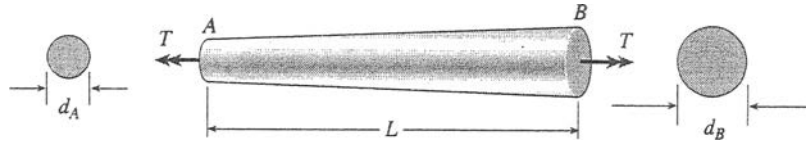
Minimum: $d_A = 2.52 \text{ in.}$ ←

Problem 3.4-10 The bar shown in the figure is tapered linearly from end A to end B and has a solid circular cross section. The diameter at the smaller end of the bar is $d_A = 25$ mm and the length is $L = 300$ mm.

The bar is made of steel with shear modulus of elasticity $G = 82$ GPa.

If the torque $T = 180$ N·m and the allowable angle of twist is 0.3° , what is the minimum allowable diameter d_B at the larger end of the bar? [Hint: Use the angle of twist expression from Prob. 3.4-8(a)].

Solution 3.4-10 Tapered bar



$$d_A = 25 \text{ mm}$$

$$L = 300 \text{ mm}$$

$$G = 82 \text{ GPa}$$

$$T = 180 \text{ N}\cdot\text{m}$$

$$\phi_{\text{allow}} = 0.3^\circ$$

Find d_B .

DIAMETER BASED UPON ALLOWABLE ANGLE OF TWIST—MODIFYING THE ANGLE OF TWIST EXPRESSION FROM PROB. 3.4-8a:

$$\beta = \frac{d_B}{d_A}$$

$$\phi = \frac{TL}{G(I_P)_A} \left(\frac{\beta^2 + \beta + 1}{3\beta^3} \right) \quad (I_P)_A = \frac{\pi}{32} d_A^4$$

$$\begin{aligned} & (0.3^\circ) \left(\frac{\pi}{180} \text{ rad}/^\circ \right) \\ &= \frac{(180 \text{ N}\cdot\text{m})(0.3 \text{ m})}{(82 \text{ GPa}) \left(\frac{\pi}{32} \right) (25 \text{ mm})^4} \left(\frac{\beta^2 + \beta + 1}{3\beta^3} \right) \end{aligned}$$

$$0.304915 = \frac{\beta^2 + \beta + 1}{3\beta^3}$$

$$0.914745\beta^3 - \beta^2 - 1 = 0$$

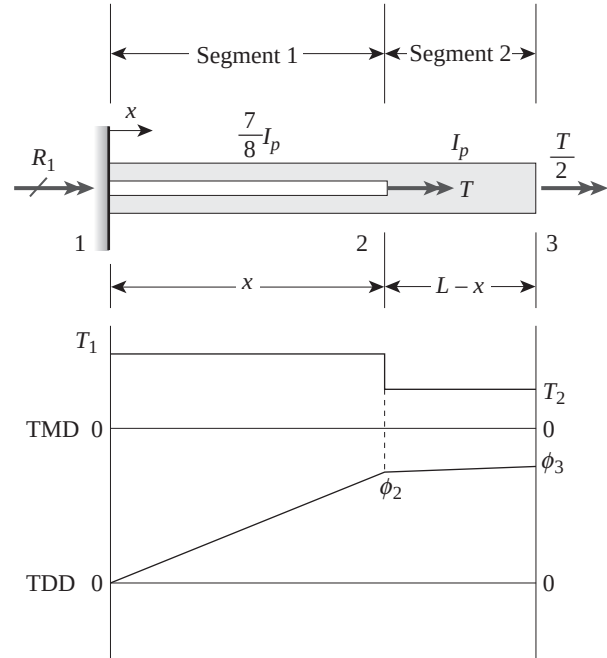
SOLVE NUMERICALLY:

$$\beta = 1.94452$$

$$\text{Minimum: } d_B = \beta d_A = 48.6 \text{ mm} \quad \leftarrow$$

Problem 3.4-11 The nonprismatic cantilever circular bar shown has an internal cylindrical hole from 0 to x , so the net polar moment of inertia of the cross section for segment 1 is $(7/8)I_p$. Torque T is applied at x and torque $T/2$ is applied at $x = L$. Assume that G is constant.

- Find reaction moment R_1 .
- Find internal torsional moments T_i in segments 1 and 2.
- Find x required to obtain twist at joint 3 of $\phi_3 = TL/GI_p$.
- What is the rotation at joint 2, ϕ_2 ?
- Draw the torsional moment (TMD: $T(x)$, $0 \leq x \leq L$) and displacement (TDD: $\phi(x)$, $0 \leq x \leq L$) diagrams.



Solution 3.4-11

- (a) REACTION TORQUE R_1

$$\sum M_x = 0 \quad R_1 = -\left(T + \frac{T}{2}\right) \quad R_1 = -\frac{3}{2}T \quad \leftarrow$$

- (b) INTERNAL MOMENTS IN SEGMENTS 1 AND 2

$$T_1 = -R_1 \quad T_1 = 1.5T \quad T_2 = \frac{T}{2}$$

- (c) FIND x REQUIRED TO OBTAIN TWIST AT JOINT 3

$$\phi_3 = \sum \frac{T_i L_i}{GI_{pi}}$$

$$\frac{TL}{GI_p} = \frac{T_1 x}{G\left(\frac{7}{8}I_p\right)} + \frac{T_2(L-x)}{GI_p}$$

$$\frac{TL}{GI_p} = \frac{\left(\frac{3}{2}T\right)x}{G\left(\frac{7}{8}I_p\right)} + \frac{\left(\frac{T}{2}\right)(L-x)}{GI_p}$$

$$L = \frac{\left(\frac{3}{2}\right)x}{\left(\frac{7}{8}\right)} + \frac{1}{2}(L-x)$$

$$L = \frac{17}{14}x + \frac{1}{2}L$$

$$x = \frac{14}{17}\left(\frac{L}{2}\right) \quad x = \frac{7}{17}L \quad \leftarrow$$

- (d) ROTATION AT JOINT 2 FOR x VALUE IN PART (c)

$$\phi_2 = \frac{T_1 x}{G\left(\frac{7}{8}I_p\right)} \quad \phi_2 = \frac{\left(\frac{3}{2}T\right)\left(\frac{7}{17}L\right)}{G\left(\frac{7}{8}I_p\right)}$$

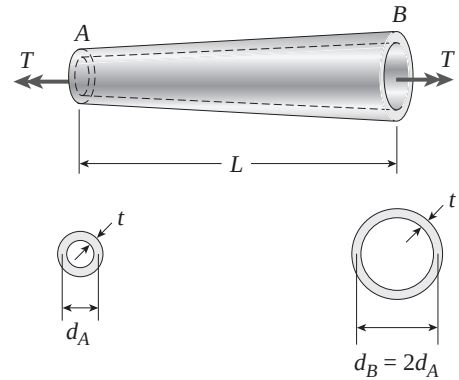
$$\phi_2 = \frac{12TL}{17GI_p} \quad \leftarrow$$

- (e) TMD AND TDD—SEE PLOTS ABOVE

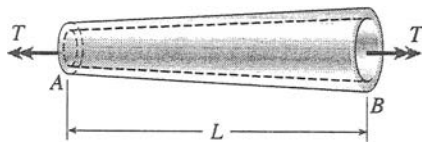
TMD is constant— T_1 for 0 to x and T_2 for x to L ; hence TDD is linear—zero at joint 1, ϕ_2 at joint 2 and ϕ_3 at joint 3

Problem 3.4-12 A uniformly tapered tube AB of hollow circular cross section is shown in the figure. The tube has constant wall thickness t and length L . The average diameters at the ends are d_A and $d_B = 2d_A$. The polar moment of inertia may be represented by the approximate formula $I_p \approx \pi d^3 t / 4$ (see Eq. 3-21).

Derive a formula for the angle of twist ϕ of the tube when it is subjected to torques T acting at the ends.



Solution 3.4-12 Tapered tube

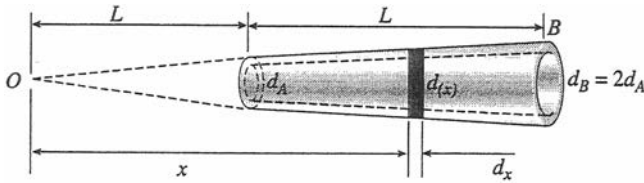


t = thickness (constant)

d_A, d_B = average diameters at the ends

$$d_B = 2d_A \quad I_p = \frac{\pi d^3 t}{4} \text{ (approximate formula)}$$

ANGLE OF TWIST



Take the origin of coordinates at point O .

$$d(x) = \frac{x}{2L} (d_B) = \frac{x}{L} d_A$$

$$I_p(x) = \frac{\pi [d(x)]^3 t}{4} = \frac{\pi t d_A^3}{4L^3} x^3$$

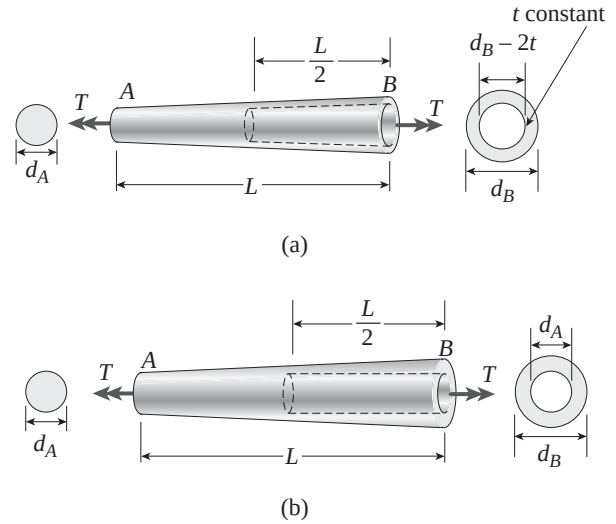
For element of length dx :

$$d\phi = \frac{T dx}{G I_p(x)} = \frac{T dx}{G \left(\frac{\pi t d_A^3}{4L^3} \right) x^3} = \frac{4TL^3 dx}{\pi G t d_A^3 x^3}$$

$$\phi = \int_L^{2L} d\phi = \frac{4TL^3}{\pi G t d_A^3} \int_L^{2L} \frac{dx}{x^3} = \frac{3TL}{2\pi G t d_A^3} \quad \leftarrow$$

Problem 3.4-13 A uniformly tapered aluminum-alloy tube AB of circular cross section and length L is shown in the figure. The outside diameters at the ends are d_A and $d_B = 2d_A$. A hollow section of length $L/2$ and constant thickness $t = d_A/10$ is cast into the tube and extends from B halfway toward A .

- (a) Find the angle of twist ϕ of the tube when it is subjected to torques T acting at the ends. Use numerical values as follows: $d_A = 2.5$ in., $L = 48$ in., $G = 3.9 \times 10^6$ psi, and $T = 40,000$ in.-lb.
- (b) Repeat part (a) if the hollow section has constant diameter d_A . (See figure part b.)



Solution 3.4-13

PART (a)—CONSTANT THICKNESS

Use x as integration variable measured from B toward A .

FROM B TO CENTERLINE

Outer and inner diameters as function of x .

$$0 \leq x \leq \frac{L}{2} \quad d_0(x) = d_B - \left(\frac{d_B - d_A}{L} \right) x$$

$$d_0(x) = 2d_A - \frac{xd_A}{L}$$

$$d_i(x) = (d_B - 2t) - \frac{[(2d_A - 2t) - (d_A - 2t)]}{L} x$$

$$d_i(x) = \frac{-1}{5} d_A \frac{-9L + 5x}{L}$$

SOLID FROM CENTERLINE TO A

$$\frac{L}{2} \leq x \leq L \quad d_0(x) = 2d_A - \frac{xd_A}{L}$$

$$\phi = \frac{T}{G} \left(\frac{32}{\pi} \right) \left(\int_0^{L/2} \frac{1}{d_0^4 - d_i^4} dx + \int_{L/2}^L \frac{1}{d_0^4} dx \right)$$

$$\phi = \frac{T}{G} \left(\frac{32}{\pi} \right) \left[\int_0^{\frac{L}{2}} \frac{1}{\left(2d_A - \frac{xd_A}{L} \right)^4 - \left(\frac{-1}{5} d_A - \frac{9L + 5x}{L} \right)^4} dx + \int_{\frac{L}{2}}^L \frac{1}{\left(2d_A - \frac{xd_A}{L} \right)^4} dx \right]$$

$$\phi = 32 \frac{T}{G\pi} \left(\frac{-125}{2} L \frac{3\ln(2) + 2\ln(7) - \ln(197)}{d_A^4} - \frac{125}{2} L \frac{-2\ln(19) + \ln(181)}{d_A^4} + \frac{19}{81d_A^4} L \right)$$

$$\text{Simplifying: } \phi = \frac{16TL}{81G\pi d_A^4} \left(38 + 10125 \ln \left(\frac{71117}{70952} \right) \right) \quad \text{or} \quad \phi_a = 3.868 \frac{TL}{Gd_A^4}$$

$$\text{Use numerical properties: } L = 48 \text{ in.} \quad G = 3.9 \times 10^6 \text{ psi} \quad d_A = 2.5 \text{ in.} \quad t = \frac{d_A}{10} \quad T = 40000 \text{ in.-lb}$$

$$\phi_a = 0.049 \text{ rad} \quad \phi_a = 2.79^\circ \quad \leftarrow$$

PART (b)—CONSTANT HOLE DIAMETER

$$0 \leq x \leq \frac{L}{2} \quad d_0(x) = d_B - \left(\frac{d_B - d_A}{L} \right) x \quad d_0(x) = 2d_A - \frac{xd_A}{L} \quad d_i(x) = d_A$$

$$\frac{L}{2} \leq x \leq L \quad d_0(x) = 2d_A - \frac{xd_A}{L}$$

$$\phi = \frac{T}{G} \left(\frac{32}{\pi} \right) \left(\int_0^{\frac{L}{2}} \frac{1}{d_0^4 - d_i^4} dx + \int_{\frac{L}{2}}^L \frac{1}{d_0^4} dx \right)$$

$$\phi = \frac{T}{G} \left(\frac{32}{\pi} \right) \left[\int_0^{\frac{L}{2}} \frac{1}{\left(2d_A - \frac{xd_A}{L} \right)^4 - d_A^4} dx + \int_{\frac{L}{2}}^L \frac{1}{\left(2d_A - \frac{xd_A}{L} \right)^4} dx \right]$$

$$\phi_b = 32 \frac{T}{G\pi} \left(\frac{1}{4} L \frac{\ln(5) + 2 \arctan \left(\frac{3}{2} \right)}{d_A^4} - \frac{1}{4} L \frac{\ln(3) + 2 \arctan(2)}{d_A^4} + \frac{19}{81d_A^4} L \right)$$

$$\text{Simplifying: } \phi_b = 3.057 \frac{TL}{Gd_A^4}$$

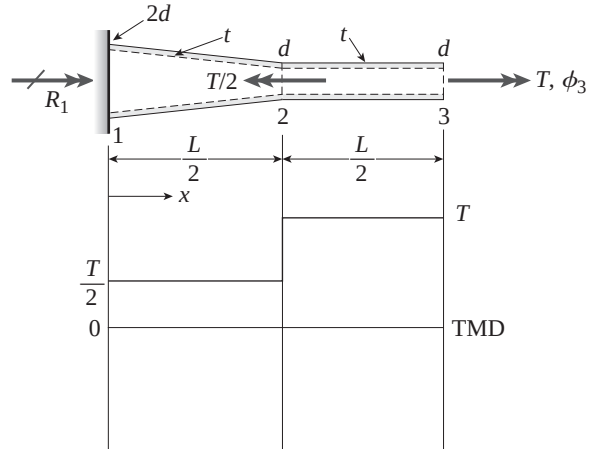
Use numerical properties given above:

$$\phi_b = 0.039 \text{ rad} \quad \phi_b = 2.21^\circ \quad \leftarrow$$

$$\frac{\phi_a}{\phi_b} = 1.265 \quad \text{So tube (a) is more flexible than tube (b).}$$

Problem 3.4-14 For the *thin* nonprismatic steel pipe of constant thickness t and variable diameter d shown with applied torques at joints 2 and 3, determine the following.

- Find reaction moment R_1 .
- Find an expression for twist rotation ϕ_3 at joint 3. Assume that G is constant.
- Draw the torsional moment diagram (TMD: $T(x)$, $0 \leq x \leq L$).



Solution 3.4-14

- (a) REACTION TORQUE R_1

Statics: $\sum T = 0$

$$R_1 - \frac{T}{2} + T = 0 \quad R_1 = \frac{-T}{2} \quad \leftarrow$$

- (b) ROTATION AT JOINT 3

$$d_{12}(x) = 2d \left(1 - \frac{x}{L} \right) \quad 0 \leq x \leq \frac{L}{2}$$

$$d_{23}(x) = d \quad \frac{L}{2} \leq x \leq L$$

$$\begin{aligned} \phi_3 &= \int_0^{\frac{L}{2}} \frac{\frac{T}{2}}{G \left(\frac{\pi}{4} d_{12}(x)^3 t \right)} dx \\ &\quad + \int_{\frac{L}{2}}^L \frac{T}{G \left(\frac{\pi}{4} d_{23}(x)^3 t \right)} dx \end{aligned}$$

Use I_p expression for thin walled tubes.

$$\begin{aligned} \phi_3 &= \frac{2T}{G\pi t} \int_0^{\frac{L}{2}} \frac{1}{\left[2d \left(1 - \frac{x}{L} \right) \right]^3} dx \\ &\quad + \frac{4T}{G\pi d^3 t} \int_{\frac{L}{2}}^L dx \end{aligned}$$

$$\begin{aligned} \phi_3 &= \frac{2T}{G\pi t} \int_0^{\frac{L}{2}} \frac{1}{\left[2d \left(1 - \frac{x}{L} \right) \right]^3} dx \\ &\quad + \frac{2TL}{G\pi d^3 t} \end{aligned}$$

$$\phi_3 = \frac{3TL}{8G\pi d^3 t} + \frac{2TL}{G\pi d^3 t}$$

$$\phi_3 = \frac{19TL}{8G\pi d^3 t} \quad \leftarrow$$

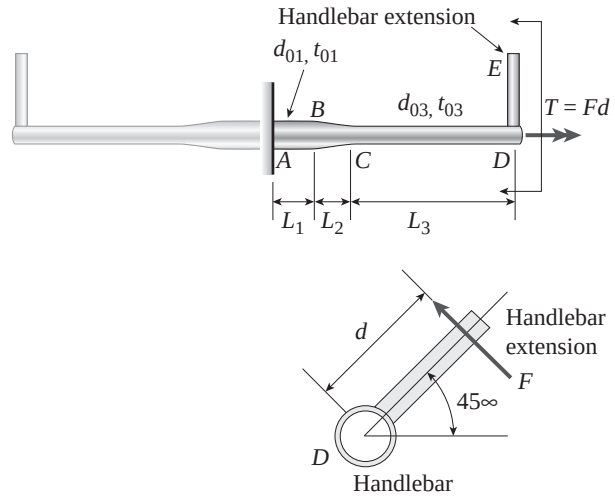
- (c) TMD

TMD is piecewise constant: $T(x) = +T/2$ for segments 1 and 2 and $T(x) = +T$ for segments 2 and 3 (see plot above).

Problem 3.4-15 A mountain-bike rider going uphill applies torque $T = Fd$ ($F = 15$ lb, $d = 4$ in.) to the end of the handlebars $ABCD$ (by pulling on the handlebar extenders DE). Consider the right half of the handlebar assembly only (assume the bars are fixed at the fork at A). Segments AB and CD are prismatic with lengths $L_1 = 2$ in. and $L_3 = 8.5$ in., and with outer diameters and thicknesses $d_{01} = 1.25$ in., $t_{01} = 0.125$ in., and $d_{03} = 0.87$ in., $t_{03} = 0.115$ in., respectively as shown. Segment BC of length $L_2 = 1.2$ in., however, is tapered, and outer diameter and thickness vary linearly between dimensions at B and C .

Consider torsion effects only. Assume $G = 4000$ ksi is constant.

Derive an integral expression for the angle of twist ϕ_D of half of the handlebar tube when it is subjected to torque $T = Fd$ acting at the end. Evaluate ϕ_D for the given numerical values.



Solution 3.4-15

ASSUME THIN WALLED TUBES

Segments AB and CD :

$$I_{P1} = \frac{\pi}{4} d_{01}^3 t_{01} \quad I_{P3} = \frac{\pi}{4} d_{03}^3 t_{03}$$

Segment BC : $0 \leq x \leq L_2$

$$d_{02}(x) = d_{01} \left(1 - \frac{x}{L_2} \right) + d_{03} \left(\frac{x}{L_2} \right)$$

$$d_{02}(x) = \frac{d_{01}L_2 - d_{01}x + d_{03}x}{L_2}$$

$$t_{02}(x) = t_{01} \left(1 - \frac{x}{L_2} \right) + t_{03} \left(\frac{x}{L_2} \right)$$

$$t_{02}(x) = \frac{t_{01}L_2 - t_{01}x + t_{03}x}{L_2}$$

$$\phi_D = \frac{Fd}{G} \left(\frac{L_1}{I_{P1}} + \int_0^{L_2} \frac{1}{\frac{\pi}{4} d_{02}(x)^3 t_{02}(x)} dx + \frac{L_3}{I_{P3}} \right)$$

$$\phi_D = \frac{4Fd}{G\pi} \left[\frac{L_1}{d_{01}^3 t_{01}} + \int_0^{L_2} \frac{L_2^4}{(d_{01}L_2 - d_{01}x + d_{03}x)^3 \times (t_{01}L_2 - t_{01}x + t_{03}x)} dx + \frac{L_3}{d_{03}^3 t_{03}} \right] \leftarrow$$

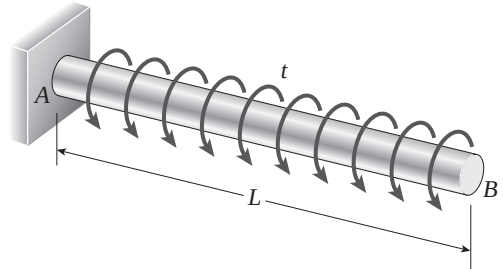
NUMERICAL DATA

$$\begin{array}{lll} L_1 = 2 \text{ in.} & L_2 = 1.2 \text{ in.} & L_3 = 8.5 \text{ in.} \\ t_{01} = 0.125 \text{ in.} & t_{03} = 0.115 \text{ in.} & d_{01} = 1.25 \text{ in.} \\ d_{03} = 0.87 \text{ in.} & F = 15 \text{ lb} & d = 4 \text{ in.} \\ G = 4 \times (10^6) \text{ psi} & & \end{array}$$

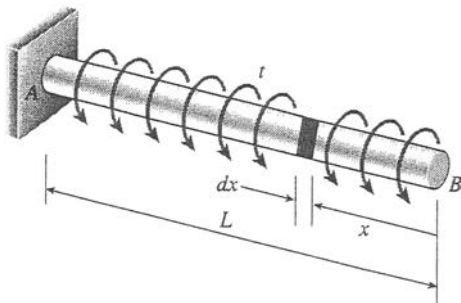
$$\phi_D = 0.142^\circ \leftarrow$$

Problem 3.4-16 A prismatic bar AB of length L and solid circular cross section (diameter d) is loaded by a distributed torque of constant intensity t per unit distance (see figure).

- Determine the maximum shear stress $\tau_{\max}$ in the bar.
- Determine the angle of twist ϕ between the ends of the bar.



Solution 3.4-16 Bar with distributed torque



t = intensity of distributed torque

d = diameter

G = shear modulus of elasticity

- MAXIMUM SHEAR STRESS

$$T_{\max} = tL \quad \tau_{\max} = \frac{16T_{\max}}{\pi d^3} = \frac{16tL}{\pi d^3} \quad \leftarrow$$

- ANGLE OF TWIST

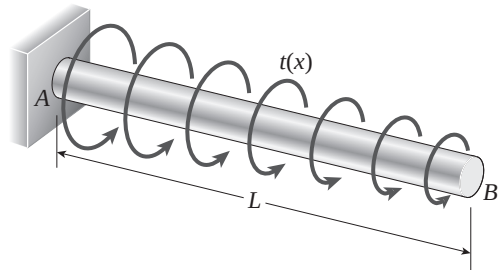
$$T(x) = tx \quad I_P = \frac{\pi d^4}{32}$$

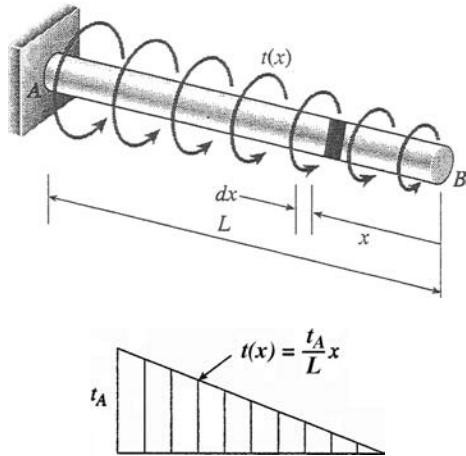
$$d\phi = \frac{T(x)dx}{GI_P} = \frac{32txdx}{\pi Gd^4}$$

$$\phi = \int_0^L d\phi = \frac{32t}{\pi Gd^4} \int_0^L x dx = \frac{16tL^2}{\pi Gd^4} \quad \leftarrow$$

Problem 3.4-17 A prismatic bar AB of solid circular cross section (diameter d) is loaded by a distributed torque (see figure). The intensity of the torque, that is, the torque per unit distance, is denoted $t(x)$ and varies linearly from a maximum value t_A at end A to zero at end B . Also, the length of the bar is L and the shear modulus of elasticity of the material is G .

- Determine the maximum shear stress $\tau_{\max}$ in the bar.
- Determine the angle of twist ϕ between the ends of the bar.



Solution 3.4-17 Bar with linearly varying torque

$t(x)$ = intensity of distributed torque
 t_A = maximum intensity of torque
 d = diameter
 G = shear modulus
 T_A = maximum torque
 $= \frac{1}{2} t_A L$

(a) Maximum shear stress

$$\tau_{\max} = \frac{16T_{\max}}{\pi d^3} = \frac{16T_A}{\pi d^3} = \frac{8t_A L}{\pi d^3} \leftarrow$$

(b) ANGLE OF TWIST

 $T(x)$ = torque at distance x from end B

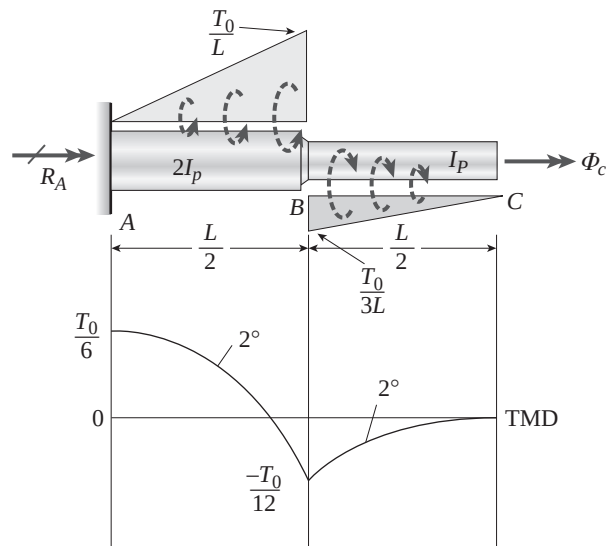
$$T(x) = \frac{t(x)x}{2} = \frac{t_A x^2}{2L} \quad I_P = \frac{\pi d^4}{32}$$

$$d\phi = \frac{T(x) dx}{GI_P} = \frac{16t_A x^2 dx}{\pi GLd^4}$$

$$\phi = \int_0^L d\phi = \frac{16t_A}{\pi GLd^4} \int_0^L x^2 dx = \frac{16t_A L^2}{3\pi Gd^4} \leftarrow$$

Problem 3.4-18 A nonprismatic bar ABC of solid circular cross section is loaded by distributed torques (see figure). The intensity of the torques, that is, the torque per unit distance, is denoted $t(x)$ and varies linearly from zero at A to a maximum value T_0/L at B . Segment BC has linearly distributed torque of intensity $t(x) = T_0/3L$ of opposite sign to that applied along AB . Also, the polar moment of inertia of AB is twice that of BC , and the shear modulus of elasticity of the material is G .

- Find reaction torque R_A .
- Find internal torsional moments $T(x)$ in segments AB and BC .
- Find rotation ϕ_C .
- Find the maximum shear stress $\tau_{\max}$ and its location along the bar.
- Draw the torsional moment diagram (TMD: $T(x)$, $0 \leq x \leq L$).



Solution 3.4-18(a) TORQUE REACTION R_A

$$\text{STATICS:} \quad \Sigma T = 0$$

$$R_A + \frac{1}{2} \left(\frac{T_0}{L} \right) \left(\frac{L}{2} \right) - \frac{1}{2} \left(\frac{T_0}{3L} \right) \left(\frac{L}{2} \right) = 0$$

$$R_A + \frac{1}{6} T_0 = 0 \quad R_A = \frac{-T_0}{6} \quad \leftarrow$$

(b) INTERNAL TORSIONAL MOMENTS IN AB AND BC

$$T_{AB}(x) = \frac{T_0}{6} - \left(\frac{x}{L} \right) \left(\frac{T_0}{L} \right) \frac{x}{2}$$

$$T_{AB}(x) = \left(\frac{T_0}{6} - \frac{x^2}{L^2} T_0 \right) \quad 0 \leq x \leq \frac{L}{2} \quad \leftarrow$$

$$T_{BC}(x) = \frac{-(L-x)}{\frac{L}{2}} \left(\frac{T_0}{3L} \right) \frac{(L-x)}{2}$$

$$T_{BC}(x) = - \left[\left(\frac{x-L}{L} \right)^2 \frac{T_0}{3} \right]$$

$$\frac{L}{2} \leq x \leq L \quad \leftarrow$$

(c) ROTATION AT C

$$\phi_C = \int_0^{\frac{L}{2}} \frac{T_{AB}(x)}{G(2I_P)} dx + \int_{\frac{L}{2}}^L \frac{T_{BC}(x)}{GI_P} dx$$

$$\phi_C = \int_0^{\frac{L}{2}} \frac{\frac{T_0}{6} - \frac{x^2}{3L^2} T_0}{G(2I_P)} dx$$

$$+ \int_{\frac{L}{2}}^L \frac{- \left[\left(\frac{x-L}{L} \right)^2 \frac{T_0}{3} \right]}{GI_P} dx$$

$$\phi_C = \frac{T_0 L}{48GI_P} - \frac{T_0 L}{72GI_P}$$

$$\phi_C = \frac{T_0 L}{144GI_P} \quad \leftarrow$$

(d) MAXIMUM SHEAR STRESS ALONG BAR

$$\text{For } AB: 2I_P = \frac{\pi}{32} d_{AB}^4$$

$$\text{For } BC: I_P = \frac{\pi}{32} d_{BC}^4$$

$$d_{BC} = \left(\frac{1}{2} \right)^{\frac{1}{4}} d_{AB}$$

$$\text{At } A, T = T_0/6 \quad \tau_{\max} = \frac{\frac{T_0}{6} \frac{d_{AB}}{2}}{\frac{\pi}{32} d_{AB}^4}$$

$$\tau_{\max} = \frac{8T_0}{3\pi d_{AB}^3} \quad \leftarrow \text{controls}$$

Just to right of B , $T = -T_0/12$

$$\tau_{\max} = \frac{\frac{T_0}{12} \left(\frac{d_{BC}}{2} \right)}{\frac{\pi}{32} d_{BC}^4}$$

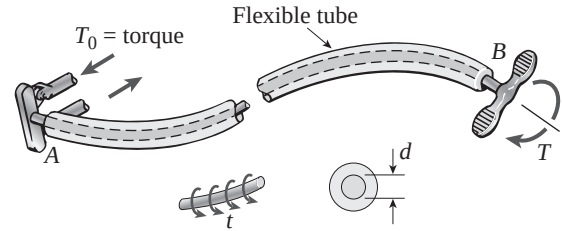
$$\tau_{\max} = \frac{\frac{T_0}{12} \left(\frac{0.841 d_{AB}}{2} \right)}{\frac{\pi}{32} (0.841 d_{AB})^4}$$

$$\tau_{\max} = \frac{2.243 T_0}{\pi d_{AB}^3}$$

(e) TMD = two second-degree curves: from $T_0/6$ at A , to $-T_0/12$ at B , to zero at C (with zero slopes at A and C , since slope on TMD is proportional to ordinate on torsional loading)—see plot of $T(x)$ above.

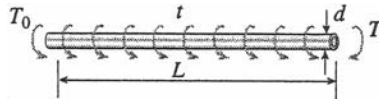
Problem 3.4-19 A magnesium-alloy wire of diameter $d = 4$ mm and length L rotates inside a flexible tube in order to open or close a switch from a remote location (see figure). A torque T is applied manually (either clockwise or counterclockwise) at end B , thus twisting the wire inside the tube. At the other end A , the rotation of the wire operates a handle that opens or closes the switch.

A torque $T_0 = 0.2$ N·m is required to operate the switch. The torsional stiffness of the tube, combined with friction between the tube and the wire, induces a distributed torque of constant intensity $t = 0.04$ N·m/m (torque per unit distance) acting along the entire length of the wire.



- If the allowable shear stress in the wire is $\tau_{\text{allow}} = 30$ MPa, what is the longest permissible length L_{max} of the wire?
- If the wire has length $L = 4.0$ m and the shear modulus of elasticity for the wire is $G = 15$ GPa, what is the angle of twist ϕ (in degrees) between the ends of the wire?

Solution 3.4-19 Wire inside a flexible tube



$$\begin{aligned} d &= 4 \text{ mm} \\ T_0 &= 0.2 \text{ N}\cdot\text{m} \\ t &= 0.04 \text{ N}\cdot\text{m/m} \end{aligned}$$

(a) MAXIMUM LENGTH L_{max}

$$\tau_{\text{allow}} = 30 \text{ MPa}$$

$$\text{Equilibrium: } T = tL + T_0$$

$$\text{From Eq. (3-14): } \tau_{\text{max}} = \frac{16T}{\pi d^3} \quad T = \frac{\pi d^3 \tau_{\text{max}}}{16}$$

$$tL + T_0 = \frac{\pi d^3 \tau_{\text{max}}}{16}$$

$$L = \frac{1}{16t} (\pi d^3 \tau_{\text{max}} - 16T_0)$$

$$L_{\text{max}} = \frac{1}{16t} (\pi d^3 \tau_{\text{allow}} - 16T_0) \quad \leftarrow$$

$$\text{Substitute numerical values: } L_{\text{max}} = 4.42 \text{ m} \quad \leftarrow$$

(b) ANGLE OF TWIST ϕ

$$L = 4 \text{ m} \quad G = 15 \text{ GPa}$$

$$\phi_1 = \text{angle of twist due to distributed torque } t$$

$$= \frac{16tL^2}{\pi G d^4} \quad (\text{from Problem 3.4-16})$$

$$\phi_2 = \text{angle of twist due to torque } T_0$$

$$= \frac{T_0 L}{GI_P} = \frac{32 T_0 L}{\pi G d^4} \quad (\text{from Eq. 3-17})$$

$$\phi = \text{total angle of twist}$$

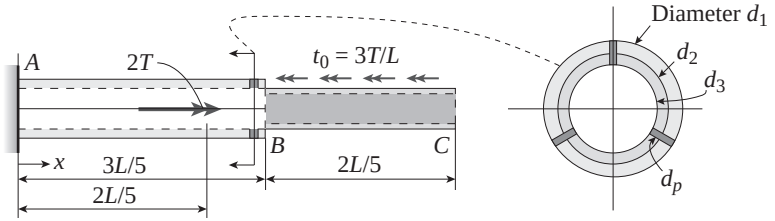
$$= \phi_1 + \phi_2$$

$$\phi = \frac{16L}{\pi G d^4} (tL + 2T_0) \quad \leftarrow$$

Substitute numerical values:

$$\phi = 2.971 \text{ rad} = 170^\circ \quad \leftarrow$$

Problem 3.4-20 Two tubes (AB, BC) of the same material are connected by three pins (pin diameter = d_p) just left of B as shown in the figure. Properties and dimensions for each tube are given in the figure. Torque $2T$ is applied at $x = 2L/5$ and uniformly distributed torque intensity $t_0 = 3T/L$ is applied on tube BC. [Hint: See Example 3-5 for torsional moment and displacement diagrams]



- (a) Find the maximum value of load variable $T(\text{N}\cdot\text{m})$ based on allowable shear (τ_a) and bearing (σ_{ba}) stresses in the three pins which connect the two tubes at B. Use the following numerical properties: $L = 1.5 \text{ m}$, $E = 74 \text{ GPa}$, $\nu = 0.33$, $d_p = 18 \text{ mm}$, $\tau_a = 45 \text{ MPa}$, $\sigma_{ba} = 90 \text{ MPa}$, $d_1 = 85 \text{ mm}$, $d_2 = 73 \text{ mm}$, and $d_3 = 60 \text{ mm}$.
- (b) What is the maximum shear stress in the tubes for the applied torque in part(a)?

Solution 3.4-20

See Example 3-5 for determination of internal torques and TMD

- (a) FIND THE MAXIMUM VALUE OF LOAD VARIABLE $T(\text{N}\cdot\text{m})$

Based on allowable shear (τ_a) and bearing (σ_{ba}) stresses in the three pins which connect the two tubes at B; use the following numerical properties: $L = 1.5 \text{ m}$, $E = 74 \text{ GPa}$, $\nu = 0.33$, $d_p = 18 \text{ mm}$, $\tau_a = 45 \text{ MPa}$, $\sigma_{ba} = 90 \text{ MPa}$, $d_1 = 85 \text{ mm}$, $d_2 = 73 \text{ mm}$, $d_3 = 60 \text{ mm}$

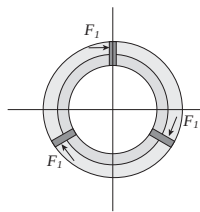
$$\text{From Example 3-5, } T_{\max} = 6T/5 \quad E = 74 \text{ GPa} \quad \nu = 0.33 \quad G = \frac{E}{2(1 + \nu)} = 27.82 \text{ GPa}$$

$$d_p = 18 \text{ mm} \quad d_1 = 85 \text{ mm} \quad d_2 = 73 \text{ mm} \quad d_3 = 60 \text{ mm} \quad \tau_a = 45 \text{ MPa} \quad \sigma_{ba} = 90 \text{ MPa}$$

$$t_1 = \frac{d_1 - d_2}{2} = 6 \text{ mm} \quad t_2 = \frac{d_2 - d_3}{2} = 6.5 \text{ mm}$$

BEARING STRESSES ON PIN

Convert $T_{\max}$ above to force-couple at distribution ($d_1 - t_1$) or ($d_2 - t_2$) [see figure]; set σ_b to σ_{ba} and solve for $T_{\max}$.



SOLUTION: Convert maximum torque into bearing forces on pin: either F_1 or F_2 ; do same for shear on interface of two tubes.

$$F_1 = \frac{\left(\frac{6T}{5}\right)}{3\left(\frac{d_1}{2} - \frac{t_1}{2}\right)} \quad \sigma_{b1} = \frac{F_1}{d_p t_1} \quad F_2 = \frac{\frac{6T}{5}}{3\left(\frac{d_2}{2} - \frac{t_2}{2}\right)} \quad \sigma_{b2} = \frac{F_2}{d_p t_2}$$

$$T_{1\max} = (\sigma_{ba} d_p t_1) \frac{3}{2} (d_1 - t_1) \frac{5}{6} = 960 \text{ N}\cdot\text{m} \quad T_{2\max} = (\sigma_{ba} d_p t_2) \frac{3}{2} (d_2 - t_2) \frac{5}{6} = \boxed{875 \text{ N}\cdot\text{m}} \quad \begin{array}{l} < \text{lowest controls;} \\ \text{here for bearing} \end{array}$$

SHEAR STRESSES ON PIN

Convert $T_{\max}$ to force-couple at distribution d_2 ; set τ to τ_a and solve for $T_{\max}$:

$$F_{\tau} = \frac{\frac{6T}{5}}{3\left(\frac{d_2}{2}\right)} \quad \tau = \frac{F_{\tau}}{\frac{\pi}{4} d_p^2} \quad T_{3\max} = \left[\tau_a \left(\frac{\pi}{4} d_p^2 \right) \right] \frac{3}{2} d_2 \frac{5}{6} = 1045 \text{ N}\cdot\text{m}$$

(b) WHAT IS THE MAXIMUM SHEAR STRESS IN THE TUBES FOR THE APPLIED TORQUE IN PART (a)?

$$T_{\max} = \frac{6}{5} T_{2\max} = 1050.4 \text{ N}\cdot\text{m}$$

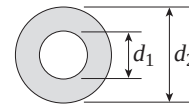
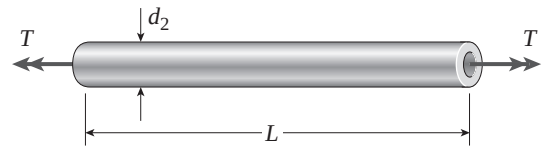
$$I_{p1} = \frac{\pi}{32} (d_1^4 - d_2^4) = 2.337 \times 10^{-6} \text{ m}^4 \quad I_{p2} = \frac{\pi}{32} (d_2^4 - d_3^4) = 1.516 \times 10^{-6} \text{ m}^4 \quad \frac{I_{p1}}{I_{p2}} = 1.542$$

$$\tau_{\max AB} = \frac{T_{\max} \left(\frac{d_1}{2} \right)}{I_{p1}} = 19.1 \text{ MPa} \quad \tau_{\max BC} = \frac{T_{\max} \left(\frac{d_2}{2} \right)}{I_{p2}} = 25.3 \text{ MPa} < \text{both are less than } 45 \text{ MPa}$$

Pure Shear

Problem 3.5-1 A hollow aluminum shaft (see figure) has outside diameter $d_2 = 4.0$ in. and inside diameter $d_1 = 2.0$ in. When twisted by torques T , the shaft has an angle of twist per unit distance equal to $0.54^\circ/\text{ft}$. The shear modulus of elasticity of the aluminum is $G = 4.0 \times 10^6$ psi.

- Determine the maximum tensile stress $\sigma_{\max}$ in the shaft.
- Determine the magnitude of the applied torques T .



Probs. 3.5-1, 3.5-2, and 3.5-3

Solution 3.5-1 Hollow aluminum shaft



$$d_2 = 4.0 \text{ in.} \quad d_1 = 2.0 \text{ in.} \quad \theta = 0.54^\circ/\text{ft}$$

$$G = 4.0 \times 10^6 \text{ psi}$$

MAXIMUM SHEAR STRESS

$$\tau_{\max} = Gr\theta \text{ (from Eq. 3-9a)}$$

$$r = d_2/2 = 2.0 \text{ in.}$$

$$\theta = (0.54^\circ/\text{ft}) \left(\frac{1 \text{ ft}}{12 \text{ in.}} \right) \left(\frac{\pi \text{ rad}}{180^\circ} \right)$$

$$= 785.40 \times 10^{-6} \text{ rad/in.}$$

$$\tau_{\max} = (4.0 \times 10^6 \text{ psi})(2.0 \text{ in.})(785.40 \times 10^{-6} \text{ rad/in.})$$

$$= 6283.2 \text{ psi}$$

- MAXIMUM TENSILE STRESS

$\sigma_{\max}$ occurs on a 45° plane and is equal to $\tau_{\max}$.

$$\sigma_{\max} = \tau_{\max} = 6280 \text{ psi} \quad \leftarrow$$

- APPLIED TORQUE

Use the torsion formula $\tau_{\max} = \frac{Tr}{I_P}$

$$T = \frac{\tau_{\max} I_P}{r} \quad I_P = \frac{\pi}{32} [(4.0 \text{ in.})^4 - (2.0 \text{ in.})^4]$$

$$= 23.562 \text{ in.}^4$$

$$T = \frac{(6283.2 \text{ psi})(23.562 \text{ in.}^4)}{2.0 \text{ in.}}$$

$$= 74,000 \text{ lb-in.} \quad \leftarrow$$

Problem 3.5-2 A hollow steel bar ($G = 80 \text{ GPa}$) is twisted by torques T (see figure). The twisting of the bar produces a maximum shear strain $\gamma_{\max} = 640 \times 10^{-6} \text{ rad}$. The bar has outside and inside diameters of 150 mm and 120 mm, respectively.

- Determine the maximum tensile strain in the bar.
- Determine the maximum tensile stress in the bar.
- What is the magnitude of the applied torques T ?

Solution 3.5-2 Hollow steel bar



$$G = 80 \text{ GPa} \quad \gamma_{\max} = 640 \times 10^{-6} \text{ rad}$$

$$d_2 = 150 \text{ mm} \quad d_1 = 120 \text{ mm}$$

$$I_P = \frac{\pi}{32} (d_2^4 - d_1^4)$$

$$= \frac{\pi}{32} [(150 \text{ mm})^4 - (120 \text{ mm})^4]$$

$$= 29.343 \times 10^6 \text{ mm}^4$$

- (a) MAXIMUM TENSILE STRAIN

$$\epsilon_{\max} = \frac{\gamma_{\max}}{2} = 320 \times 10^{-6} \quad \leftarrow$$

- (b) MAXIMUM TENSILE STRESS

$$\tau_{\max} = G\gamma_{\max} = (80 \text{ GPa})(640 \times 10^{-6})$$

$$= 51.2 \text{ MPa}$$

$$\sigma_{\max} = \tau_{\max} = 51.2 \text{ MPa} \quad \leftarrow$$

- (c) APPLIED TORQUES

$$\text{Torsion formula: } \tau_{\max} = \frac{Tr}{I_P} = \frac{Td_2}{2I_P}$$

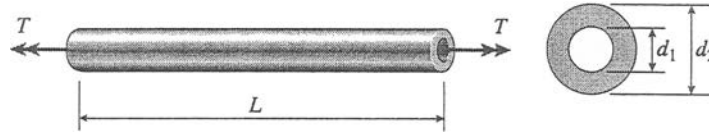
$$T = \frac{2I_P\tau_{\max}}{d_2} = \frac{2(29.343 \times 10^6 \text{ mm}^4)(51.2 \text{ MPa})}{150 \text{ mm}}$$

$$= 20,030 \text{ N}\cdot\text{m}$$

$$= 20.0 \text{ kN}\cdot\text{m} \quad \leftarrow$$

Problem 3.5-3 A tubular bar with outside diameter $d_2 = 4.0 \text{ in.}$ is twisted by torques $T = 70.0 \text{ k-in.}$ (see figure). Under the action of these torques, the maximum tensile stress in the bar is found to be 6400 psi.

- Determine the inside diameter d_1 of the bar.
- If the bar has length $L = 48.0 \text{ in.}$ and is made of aluminum with shear modulus $G = 4.0 \times 10^6 \text{ psi}$, what is the angle of twist ϕ (in degrees) between the ends of the bar?
- Determine the maximum shear strain $\gamma_{\max}$ (in radians)?

Solution 3.5-3 Tubular bar

$$d_2 = 4.0 \text{ in.} \quad T = 70.0 \text{ k-in.} = 70,000 \text{ lb-in.}$$

$$\sigma_{\max} = 6400 \text{ psi} \quad \tau_{\max} = \sigma_{\max} = 6400 \text{ psi}$$

(a) INSIDE DIAMETER d_1

$$\text{Torsion formula: } \tau_{\max} = \frac{Tr}{I_p} = \frac{Td_2}{2I_p}$$

$$I_p = \frac{Td_2}{2\tau_{\max}} = \frac{(70.0 \text{ k-in.})(4.0 \text{ in.})}{2(6400 \text{ psi})}$$

$$= 21.875 \text{ in.}^4$$

$$\text{Also, } I_p = \frac{\pi}{32} (d_2^4 - d_1^4) = \frac{\pi}{32} [(4.0 \text{ in.})^4 - d_1^4]$$

Equate formulas:

$$\frac{\pi}{32} [256 \text{ in.}^4 - d_1^4] = 21.875 \text{ in.}^4$$

$$\text{Solve for } d_1: d_1 = 2.40 \text{ in.} \quad \leftarrow$$

(b) ANGLE OF TWIST ϕ

$$L = 48 \text{ in.} \quad G = 4.0 \times 10^6 \text{ psi}$$

$$\phi = \frac{TL}{GI_p}$$

$$\text{From torsion formula, } T = \frac{2I_p\tau_{\max}}{d_2}$$

$$\therefore \phi = \frac{2I_p\tau_{\max}}{d_2} \left(\frac{L}{GI_p} \right) = \frac{2L\tau_{\max}}{Gd_2}$$

$$= \frac{2(48 \text{ in.})(6400 \text{ psi})}{(4.0 \times 10^6 \text{ psi})(4.0 \text{ in.})} = 0.03840 \text{ rad}$$

$$\phi = 2.20^\circ \quad \leftarrow$$

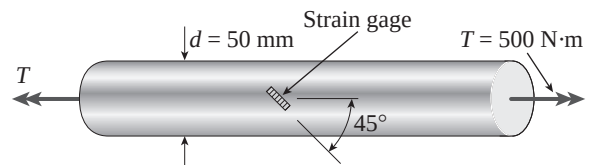
(c) MAXIMUM SHEAR STRAIN

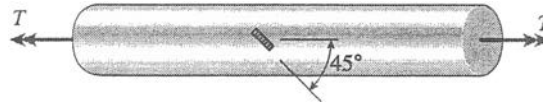
$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{6400 \text{ psi}}{4.0 \times 10^6 \text{ psi}}$$

$$= 1600 \times 10^{-6} \text{ rad} \quad \leftarrow$$

Problem 3.5-4 A solid circular bar of diameter $d = 50 \text{ mm}$ (see figure) is twisted in a testing machine until the applied torque reaches the value $T = 500 \text{ N}\cdot\text{m}$. At this value of torque, a strain gage oriented at 45° to the axis of the bar gives a reading $\epsilon = 339 \times 10^{-6}$.

What is the shear modulus G of the material?



Solution 3.5-4 Bar in a testing machine

Strain gage at 45°:

$$\epsilon_{\max} = 339 \times 10^{-6}$$

$$d = 50 \text{ mm}$$

$$T = 500 \text{ N}\cdot\text{m}$$

SHEAR STRAIN (FROM Eq. 3-32)

$$\gamma_{\max} = 2\epsilon_{\max} = 678 \times 10^{-6}$$

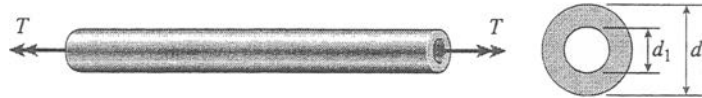
SHEAR STRESS (FROM Eq. 3-14)

$$\tau_{\max} = \frac{16T}{\pi d^3} = \frac{16(500 \text{ N}\cdot\text{m})}{\pi(0.050 \text{ m})^3} = 20.372 \text{ MPa}$$

SHEAR MODULUS

$$G = \frac{\tau_{\max}}{\gamma_{\max}} = \frac{20.372 \text{ MPa}}{678 \times 10^{-6}} = 30.0 \text{ GPa} \quad \leftarrow$$

Problem 3.5-5 A steel tube ($G = 11.5 \times 10^6$ psi) has an outer diameter $d_2 = 2.0$ in. and an inner diameter $d_1 = 1.5$ in. When twisted by a torque T , the tube develops a maximum normal strain of 170×10^{-6} . What is the magnitude of the applied torque T ?

Solution 3.5-5 Steel tube

$$G = 11.5 \times 10^6 \text{ psi} \quad d_2 = 2.0 \text{ in.} \quad d_1 = 1.5 \text{ in.}$$

$$\epsilon_{\max} = 170 \times 10^{-6}$$

$$I_P = \frac{\pi}{32} (d_2^4 - d_1^4) = \frac{\pi}{32} [(2.0 \text{ in.})^4 - (1.5 \text{ in.})^4] \\ = 1.07379 \text{ in.}^4$$

SHEAR STRAIN (FROM Eq. 3-32)

$$\gamma_{\max} = 2\epsilon_{\max} = 340 \times 10^{-6}$$

SHEAR STRESS (FROM TORSION FORMULA)

$$\tau_{\max} = \frac{Tr}{I_P} = \frac{Td_2}{2I_P}$$

$$\text{Also, } \tau_{\max} = G\gamma_{\max}$$

Equate expressions:

$$\frac{Td_2}{2I_P} = G\gamma_{\max}$$

SOLVE FOR TORQUE

$$T = \frac{2GI_P\gamma_{\max}}{d_2} \\ = \frac{2(11.5 \times 10^6 \text{ psi})(1.07379 \text{ in.}^4)(340 \times 10^{-6})}{2.0 \text{ in.}} \\ = 4200 \text{ lb}\cdot\text{in.} \quad \leftarrow$$

Problem 3.5-6 A solid circular bar of steel ($G = 78 \text{ GPa}$) transmits a torque $T = 360 \text{ N}\cdot\text{m}$. The allowable stresses in tension, compression, and shear are 90 MPa , 70 MPa , and 40 MPa , respectively. Also, the allowable tensile strain is 220×10^{-6} .

- (a) Determine the minimum required diameter d of the bar.
 (b) If the bar diameter $d = 40 \text{ mm}$, what is $T_{\max}$?

Solution 3.5-6 Solid circular bar of steel

$$(a) \quad G = 78 \text{ GPa} \quad \sigma_{aT} = 90 \text{ MPa} \quad \sigma_{aC} = 70 \text{ MPa} \quad \tau_{a1} = 40 \text{ MPa} \quad \varepsilon_a = 220(10^{-6}) \quad T = 360 \text{ N}\cdot\text{m}$$

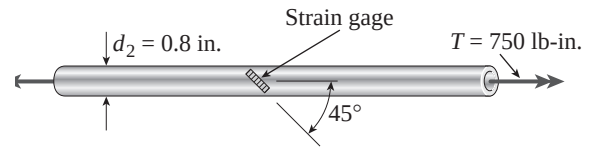
$$\gamma_a = 2\varepsilon_a \quad \tau_{a2} = G\gamma_a = 34.32 \text{ MPa} \quad < \text{controls}$$

$$\tau_{\max} = T \left(\frac{16}{\pi d^3} \right) \quad T \left(\frac{16}{\pi d^3} \right) = \tau_{a2} \quad \text{Solving for } d: \quad \boxed{d = 37.7 \text{ mm}}$$

- (b) At 45° , $\sigma_{\max} = \text{plus or minus } \tau$ for pure shear (Fig. 3-29b), so lowest allowable shear stress governs

$$d = 40 \text{ mm} \quad T_{\max} \left(\frac{16}{\pi d^3} \right) = \tau_{a2} \quad \boxed{T_{\max} = \tau_{a2} \frac{\pi d^3}{16} = 431 \text{ N}\cdot\text{m}}$$

Problem 3.5-7 The normal strain in the 45° direction on the surface of a circular tube (see figure) is 880×10^{-6} when the torque $T = 750 \text{ lb}\cdot\text{in.}$ The tube is made of copper alloy with $G = 6.2 \times 10^6 \text{ psi}$ and $\nu = 0.35$.



- (a) If the outside diameter d_2 of the tube is 0.8 in. , what is the inside diameter d_1 ?
 (b) If the allowable normal stress in the tube is 14 ksi , what is the maximum permissible inside diameter d_1 ?

Solution 3.5-7 Circular tube with strain gage

$$(a) \quad \varepsilon = 880(10^{-6}) \quad T = 750 \text{ lb}\cdot\text{in.} \quad G = 6.2(10^6) \text{ psi} \quad d_2 = 0.8 \text{ in.} \quad \nu = 0.35$$

$$\gamma = 2\varepsilon = 1.76 \times 10^{-3} \quad \tau = G\gamma = 10.912 \text{ ksi}$$

$$\frac{T \left(\frac{d_2}{2} \right)}{\frac{\pi}{32} (d_2^4 - d_1^4)} = \tau \quad \text{solving for } d_1 \quad \boxed{d_1 = 0.6 \text{ in.}}$$

- (b) $\sigma_a = 14 \text{ ksi}$ pure shear, so $\tau_a = \sigma_a$: $\tau_a = 14 \text{ ksi}$

$$\frac{T \left(\frac{d_2}{2} \right)}{\frac{\pi}{32} (d_2^4 - d_{1\max}^4)} = \tau_a \quad \text{Solving:} \quad d_{1\max} = \left[d_2^4 - \frac{\frac{32}{\pi} T \left(\frac{d_2}{2} \right)}{\tau_a} \right]^{\frac{1}{4}} = 0.661 \text{ in.}$$

$$\boxed{d_{1\max} = 0.661 \text{ in.}}$$

Problem 3.5-8 An aluminum tube has inside diameter $d_1 = 50$ mm, shear modulus of elasticity $G = 27$ GPa, $\nu = 0.33$, and torque $T = 4.0$ kN·m. The allowable shear stress in the aluminum is 50 MPa and the allowable normal strain is 900×10^{-6} .

- Determine the required outside diameter d_2 .
- Re-compute the required outside diameter d_2 if allowable normal stress is 62 MPa and allowable shear strain is 1.7×10^{-3} .

Solution 3.5-8 Aluminum tube

(a) $d_1 = 50$ mm $G = 27$ GPa $\nu = 0.33$ $T = 4$ kN·m $\tau_a = 50$ MPa $\varepsilon_a = 900 (10^{-6})$

$\gamma_a = 2\varepsilon_a = 1.8 \times 10^{-3}$ $\tau_{a2} = G\gamma_a = 48.6$ MPa $<$ controls

$$\frac{T \left(\frac{d_2}{2} \right)}{\frac{\pi}{32} (d_2^4 - d_1^4)} = \tau_{a2} \quad \text{Solving for } d_2: \quad \boxed{d_2 = 79.3 \text{ mm}}$$

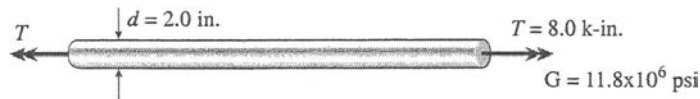
(b) $\sigma_a = 62$ MPa Pure shear, so $\tau_a = \sigma_a = 62$ MPa However, $\tau_a = G\gamma_a = 45.9$ MPa

$$\frac{T \left(\frac{d_2}{2} \right)}{\frac{\pi}{32} (d_2^4 - d_1^4)} = \tau_a \quad \text{Solving for } d_2: \quad \boxed{d_2 = 80.5 \text{ mm}}$$

Problem 3.5-9 A solid steel bar ($G = 11.8 \times 10^6$ psi) of diameter $d = 2.0$ in. is subjected to torques $T = 8.0$ k-in. acting in the directions shown in the figure.

- Determine the maximum shear, tensile, and compressive stresses in the bar and show these stresses on sketches of properly oriented stress elements.
- Determine the corresponding maximum strains (shear, tensile, and compressive) in the bar and show these strains on sketches of the deformed elements.



Solution 3.5-9 Solid steel bar

$$T = 8.0 \text{ k-in.}$$

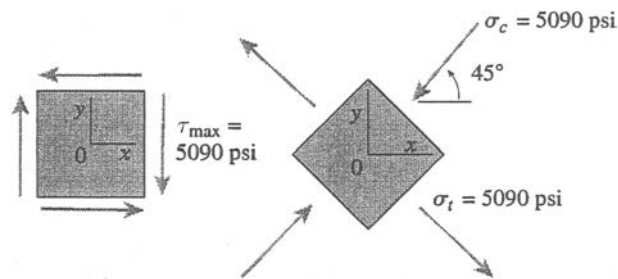
$$G = 11.8 \times 10^6 \text{ psi}$$

(a) MAXIMUM STRESSES

$$\tau_{\max} = \frac{16T}{\pi d^3} = \frac{16(8000 \text{ lb-in.})}{\pi(2.0 \text{ in.})^3}$$

$$= 5093 \text{ psi} \quad \leftarrow$$

$$\sigma_t = 5090 \text{ psi} \quad \sigma_c = -5090 \text{ psi} \quad \leftarrow$$

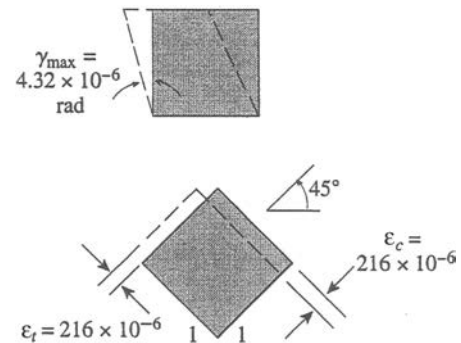
**(b) MAXIMUM STRAINS**

$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{5093 \text{ psi}}{11.8 \times 10^6 \text{ psi}}$$

$$= 432 \times 10^{-6} \text{ rad} \quad \leftarrow$$

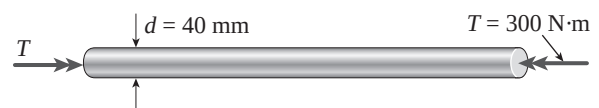
$$\varepsilon_{\max} = \frac{\gamma_{\max}}{2} = 216 \times 10^{-6}$$

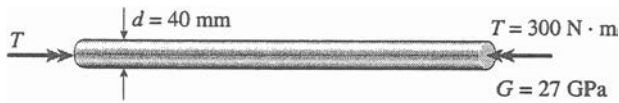
$$\varepsilon_t = 216 \times 10^{-6} \quad \varepsilon_c = -216 \times 10^{-6} \quad \leftarrow$$



Problem 3.5-10 A solid aluminum bar ($G = 27$ GPa) of diameter $d = 40$ mm is subjected to torques $T = 300$ N·m acting in the directions shown in the figure.

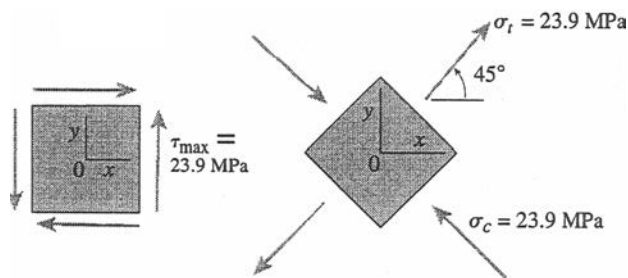
- Determine the maximum shear, tensile, and compressive stresses in the bar and show these stresses on sketches of properly oriented stress elements.
- Determine the corresponding maximum strains (shear, tensile, and compressive) in the bar and show these strains on sketches of the deformed elements.



Solution 3.5-10 Solid aluminum bar

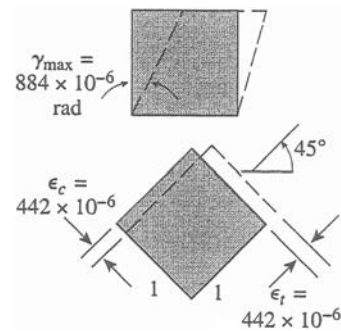
(a) MAXIMUM STRESSES

$$\begin{aligned}\tau_{\max} &= \frac{16T}{\pi d^3} = \frac{16(300 \text{ N}\cdot\text{m})}{\pi(0.040 \text{ m})^3} \\ &= 23.87 \text{ MPa} \quad \leftarrow \\ \sigma_t &= 23.9 \text{ MPa} \quad \sigma_c = -23.9 \text{ MPa} \quad \leftarrow\end{aligned}$$



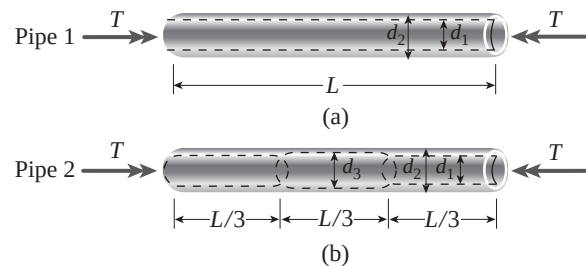
(b) MAXIMUM STRAINS

$$\begin{aligned}\gamma_{\max} &= \frac{\tau_{\max}}{G} = \frac{23.87 \text{ MPa}}{27 \text{ GPa}} \\ &= 884 \times 10^{-6} \text{ rad} \quad \leftarrow \\ \epsilon_{\max} &= \frac{\gamma_{\max}}{2} = 442 \times 10^{-6} \\ \epsilon_t &= 442 \times 10^{-6} \quad \epsilon_c = -442 \times 10^{-6} \quad \leftarrow\end{aligned}$$



Problem 3.5-11 Two circular aluminum pipes of equal length $L = 24$ in. are loaded by torsional moments T (see figure). Pipe 1 has outside and inside diameters $d_2 = 3$ in. and $d_1 = 2.5$ in., respectively. Pipe 2 has a constant outer diameter of d_2 along its entire length L and an inner diameter of d_1 but has an increased inner diameter of $d_3 = 2.65$ in. over the middle third.

Assume that $E = 10,400$ ksi, $\nu = 0.33$, and allowable shear stress $\tau_a = 6500$ psi.



- Find the maximum acceptable torques that can be applied to Pipe 1; repeat for Pipe 2.
- If the maximum twist ϕ of Pipe 2 cannot exceed $5/4$ of that of Pipe 1, what is the maximum acceptable length of the middle segment? Assume both pipes have total length L and the same applied torque T .
- Find the new value of inner diameter d_3 of Pipe 2 if the maximum torque carried by Pipe 2 is to be $7/8$ of that for Pipe 1.
- If the maximum normal strain in each pipe is known to be $\epsilon_{\max} = 811 \times 10^{-6}$, what is the applied torque on each pipe? Also, what is the maximum twist of each pipe? Use original properties and dimensions.

Solution 3.5-11 Numerical data

$$L = 24 \text{ in.} \quad d_2 = 3 \text{ in.} \quad d_1 = 2.5 \text{ in.} \quad d_3 = 2.65 \text{ in.} \quad E = 10400 \text{ ksi} \quad \nu = 0.33$$

$$\tau_a = 6500 \text{ psi} \qquad G = \frac{E}{2(1 + \nu)} = 3910 \text{ ksi}$$

(a) FIND $T_{\max 1}$ AND $T_{\max 2}$

$$I_{p1} = \frac{\pi}{32} (d_2^4 - d_1^4) = 4.1172 \text{ in.}^4 \quad I_{p2} = \frac{\pi}{32} (d_2^4 - d_3^4) = 3.111 \text{ in.}^4 \quad \frac{I_{p1}}{I_{p2}} = 1.324$$

Based on torsion formula:

$$T_{\max 1} = I_{p1} \tau_a \left(\frac{2}{d_2} \right) = 17.841 \text{ k-in.} \quad T_{\max 2} = I_{p2} \tau_a \left(\frac{2}{d_2} \right) = 13.479 \text{ k-in.} \quad \frac{T_{\max 1}}{T_{\max 2}} = 1.324$$

$$T_{1\text{allow}} = 17.84 \text{ k-in.} \quad T_{2\text{allow}} = 13.48 \text{ k-in.}$$

(b) MAX. LENGTH OF MIDDLE SEGMENT OF PIPE 2 BASED ON ALLOWABLE TWIST

Using the torque-displacement relation:

$$\frac{L - L_{\text{mid}}}{I_{p1}} + \frac{L_{\text{mid}}}{I_{p2}} = \frac{5}{4} \frac{L}{I_{p1}} \quad \text{Solving for } L_{\text{mid}}: \quad L_{\text{mid}} = 18.54 \text{ in.} \quad \frac{L_{\text{mid}}}{L} = 0.772$$

(c) REQUIRED NEW DIAMETER d_3 SO THAT $T_{\max 2} = 7/8 T_{\max 1}$

$$I_{p2} = \frac{7}{8} I_{p1} \quad \text{so} \quad d_2^4 - d_3^4 = \frac{7}{8} (d_2^4 - d_1^4)$$

$$\text{Solving for } d_3: \quad d_{3\text{new}} = \left(\frac{d_2^4}{8} + \frac{7}{8} d_1^4 \right)^{\frac{1}{4}} = 2.57996 \text{ in.} \quad d_{3\text{new}} = 2.58 \text{ in.}$$

(d) GIVEN MAXIMUM STRAIN IN EACH PIPE, FIND APPLIED TORQUE IN EACH; ALSO MAXIMUM TWIST

$$\varepsilon_{\max} = 811 (10^{-6}) \quad \gamma_{\max} = 2 \varepsilon_{\max} = 1.622 \times 10^{-3} < \text{for pure shear only} \quad G \gamma_{\max} = 6.342 \text{ ksi}$$

Using the expressions above for $T_{\max 1}$ and $T_{\max 2}$:

$$T_{\max 1} = G \gamma_{\max} I_{p1} \left(\frac{2}{d_2} \right) = 17.407 \text{ k-in.} \quad T_{\max 2} = G \gamma_{\max} I_{p2} \left(\frac{2}{d_2} \right) = 13.153 \text{ kip in.} \quad \frac{T_{\max 1}}{T_{\max 2}} = 1.323$$

$$T_{\max 1} = 17.41 \text{ k-in.} \quad T_{\max 2} = 13.15 \text{ kip in.}$$

$$\phi_{\max 1} = \frac{T_{\max 1} L}{G I_{p1}} = 0.026 \text{ rad} \quad \text{or} \quad \phi_{\max 1} = \gamma_{\max} \left(\frac{2}{d_2} \right) L = 0.026$$

$$\phi_{\max 2} = \frac{T_{\max 2}}{G} \left(\frac{2L}{3} \frac{1}{I_{p1}} + \frac{L}{3} \frac{1}{I_{p2}} \right) = 0.022$$

$$\frac{\phi_{\max 1}}{\phi_{\max 2}} = 1.195 < \text{only middle third of Pipe 2 is at max. strain so it twists less; outer third of Pipe 2 on either end has lower strain (see below)}$$

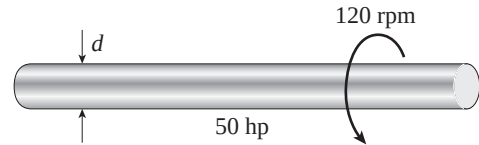
$$\phi_{\max 1} = 1.487^\circ \quad \phi_{\max 2} = 1.245^\circ$$

$$\gamma_{\text{out}2} = \frac{T_{\max 2} \left(\frac{d_2}{2} \right)}{G I_{p1}} = 1.226 \times 10^{-3} \quad \frac{\gamma_{\text{out}2}}{\gamma_{\max}} = 0.756$$

Transmission of Power

Problem 3.7-1 A generator shaft in a small hydroelectric plant turns at 120 rpm and delivers 50 hp (see figure).

- If the diameter of the shaft is $d = 3.0$ in., what is the maximum shear stress $\tau_{\max}$ in the shaft?
- If the shear stress is limited to 4000 psi, what is the minimum permissible diameter $d_{\min}$ of the shaft?



Solution 3.7-1 Generator shaft

$$n = 120 \text{ rpm} \quad H = 50 \text{ hp} \quad d = \text{diameter}$$

TORQUE

$$H = \frac{2\pi n T}{33,000} \quad H = \text{hp} \quad n = \text{rpm} \quad T = \text{lb-ft}$$

$$T = \frac{33,000 H}{2\pi n} = \frac{(33,000)(50 \text{ hp})}{2\pi(120 \text{ rpm})}$$

$$= 2188 \text{ lb-ft} = 26,260 \text{ lb-in.}$$

- (a) MAXIMUM SHEAR STRESS $\tau_{\max}$

$$d = 3.0 \text{ in.}$$

$$\tau_{\max} = \frac{16T}{\pi d^3} = \frac{16(26,260 \text{ lb-in.})}{\pi (3.0 \text{ in.})^3}$$

$$\tau_{\max} = 4950 \text{ psi} \quad \leftarrow$$

- (b) MINIMUM DIAMETER $d_{\min}$

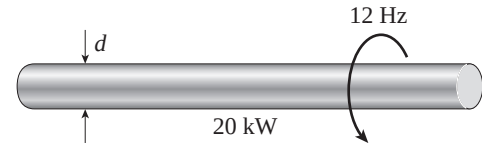
$$\tau_{\text{allow}} = 4000 \text{ psi}$$

$$d^3 = \frac{16T}{\pi \tau_{\text{allow}}} = \frac{16(26,260 \text{ lb-in.})}{\pi (4000 \text{ psi})} = 33.44 \text{ in.}^3$$

$$d_{\min} = 3.22 \text{ in.} \quad \leftarrow$$

Problem 3.7-2 A motor drives a shaft at 12 Hz and delivers 20 kW of power (see figure).

- If the shaft has a diameter of 30 mm, what is the maximum shear stress $\tau_{\max}$ in the shaft?
- If the maximum allowable shear stress is 40 MPa, what is the minimum permissible diameter $d_{\min}$ of the shaft?



Solution 3.7-2 Motor-driven shaft

$$f = 12 \text{ Hz} \quad P = 20 \text{ kW} = 20,000 \text{ N}\cdot\text{m/s}$$

TORQUE

$$P = 2\pi f T \quad P = W \quad f = \text{Hz} = \text{s}^{-1}$$

$T =$ Newton meters

$$T = \frac{P}{2\pi f} = \frac{20,000 \text{ W}}{2\pi(12 \text{ Hz})} = 265.3 \text{ N}\cdot\text{m}$$

- (a) MAXIMUM SHEAR STRESS $\tau_{\max}$

$$d = 30 \text{ mm}$$

$$\begin{aligned} \tau_{\max} &= \frac{16T}{\pi d^3} = \frac{16(265.3 \text{ N}\cdot\text{m})}{\pi(0.030 \text{ m})^3} \\ &= 50.0 \text{ MPa} \quad \leftarrow \end{aligned}$$

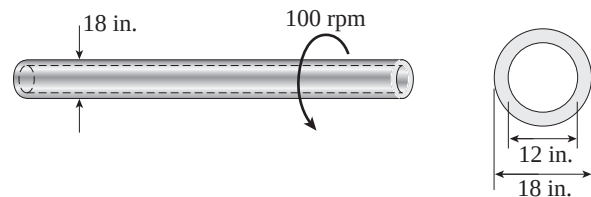
- (b) MINIMUM DIAMETER $d_{\min}$

$$\tau_{\text{allow}} = 40 \text{ MPa}$$

$$\begin{aligned} d^3 &= \frac{16T}{\pi \tau_{\text{allow}}} = \frac{16(265.3 \text{ N}\cdot\text{m})}{\pi(40 \text{ MPa})} \\ &= 33.78 \times 10^{-6} \text{ m}^3 \\ d_{\min} &= 0.0323 \text{ m} = 32.3 \text{ mm} \quad \leftarrow \end{aligned}$$

Problem 3.7-3 The propeller shaft of a large ship has outside diameter 18 in. and inside diameter 12 in., as shown in the figure. The shaft is rated for a maximum shear stress of 4500 psi.

- If the shaft is turning at 100 rpm, what is the maximum horsepower that can be transmitted without exceeding the allowable stress?
- If the rotational speed of the shaft is doubled but the power requirements remain unchanged, what happens to the shear stress in the shaft?



Solution 3.7-3 Hollow propeller shaft

$$d_2 = 18 \text{ in.} \quad d_1 = 12 \text{ in.} \quad \tau_{\text{allow}} = 4500 \text{ psi}$$

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 8270.2 \text{ in.}^4$$

TORQUE

$$\tau_{\max} = \frac{T(d_2/2)}{I_P} \quad T = \frac{2\tau_{\text{allow}}I_P}{d_2}$$

$$T = \frac{2(4500 \text{ psi})(8270.2 \text{ in.}^4)}{18 \text{ in.}}$$

$$= 4.1351 \times 10^6 \text{ lb}\cdot\text{in.}$$

$$= 344,590 \text{ lb}\cdot\text{ft.}$$

- (a) HORSEPOWER

$$n = 100 \text{ rpm} \quad H = \frac{2\pi n T}{33,000}$$

$$n = \text{rpm} \quad T = \text{lb}\cdot\text{ft} \quad H = \text{hp}$$

$$H = \frac{2\pi(100 \text{ rpm})(344,590 \text{ lb}\cdot\text{ft})}{33,000}$$

$$= 6560 \text{ hp} \quad \leftarrow$$

(b) ROTATIONAL SPEED IS DOUBLED

$$H = \frac{2\pi nT}{33,000}$$

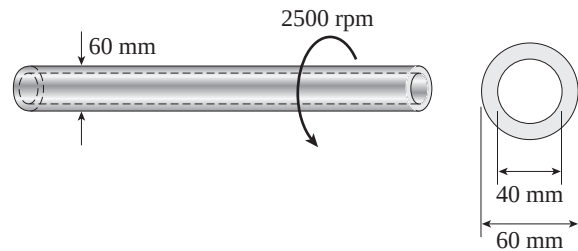
If n is doubled but H remains the same, then T is halved.

If T is halved, so is the maximum shear stress.

$\therefore$ Shear stress is halved $\leftarrow$

Problem 3.7-4 The drive shaft for a truck (outer diameter 60 mm and inner diameter 40 mm) is running at 2500 rpm (see figure).

- If the shaft transmits 150 kW, what is the maximum shear stress in the shaft?
- If the allowable shear stress is 30 MPa, what is the maximum power that can be transmitted?



Solution 3.7-4 Drive shaft for a truck

$$d_2 = 60 \text{ mm} \quad d_1 = 40 \text{ mm} \quad n = 2500 \text{ rpm}$$

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 1.0210 \times 10^{-6} \text{ m}^4$$

(a) MAXIMUM SHEAR STRESS $\tau_{\max}$

$$P = \text{power (watts)} \quad P = 150 \text{ kW} = 150,000 \text{ W}$$

$$T = \text{torque (newton meters)} \quad n = \text{rpm}$$

$$P = \frac{2\pi nT}{60} \quad T = \frac{60P}{2\pi n}$$

$$T = \frac{60(150,000 \text{ W})}{2\pi(2500 \text{ rpm})} = 572.96 \text{ N}\cdot\text{m}$$

$$\begin{aligned} \tau_{\max} &= \frac{Td_2}{2I_P} = \frac{(572.96 \text{ N}\cdot\text{m})(0.060 \text{ m})}{2(1.0210 \times 10^{-6} \text{ m}^4)} \\ &= 16.835 \text{ MPa} \end{aligned}$$

$$\tau_{\max} = 16.8 \text{ MPa} \quad \leftarrow$$

(b) MAXIMUM POWER $P_{\max}$

$$\tau_{\text{allow}} = 30 \text{ MPa}$$

$$\begin{aligned} P_{\max} &= P \frac{\tau_{\text{allow}}}{\tau_{\max}} = (150 \text{ kW}) \left(\frac{30 \text{ MPa}}{16.835 \text{ MPa}} \right) \\ &= 267 \text{ kW} \quad \leftarrow \end{aligned}$$

Problem 3.7-5 A hollow circular shaft for use in a pumping station is being designed with an inside diameter equal to 0.75 times the outside diameter. The shaft must transmit 400 hp at 400 rpm without exceeding the allowable shear stress of 6000 psi.

Determine the minimum required outside diameter d .

Solution 3.7-5 Hollow shaft d = outside diameter d_o = inside diameter

$$= 0.75 d$$

$$H = 400 \text{ hp} \quad n = 400 \text{ rpm}$$

$$\tau_{\text{allow}} = 6000 \text{ psi}$$

$$I_p = \frac{\pi}{32} [d^4 - (0.75 d)^4] = 0.067112 d^4$$

TORQUE

$$H = \frac{2\pi n T}{33,000}$$

$$H = \text{hp} \quad n = \text{rpm} \quad T = \text{lb-ft}$$

$$T = \frac{33,000 H}{2\pi n} = \frac{(33,000)(400 \text{ hp})}{2\pi(400 \text{ rpm})}$$

$$= 5252.1 \text{ lb-ft} = 63,025 \text{ lb-in.}$$

MINIMUM OUTSIDE DIAMETER

$$\tau_{\text{max}} = \frac{Td}{2I_p} \quad I_p = \frac{Td}{2\tau_{\text{max}}} = \frac{Td}{2\tau_{\text{allow}}}$$

$$0.067112 d^4 = \frac{(63,025 \text{ lb-in.})(d)}{2(6000 \text{ psi})}$$

$$d^3 = 78.259 \text{ in.}^3 \quad d_{\text{min}} = 4.28 \text{ in.} \quad \leftarrow$$

Problem 3.7-6 A tubular shaft being designed for use on a construction site must transmit 120 kW at 1.75 Hz. The inside diameter of the shaft is to be one-half of the outside diameter.

If the allowable shear stress in the shaft is 45 MPa, what is the minimum required outside diameter d ?

Solution 3.7-6 Tubular shaft d = outside diameter d_o = inside diameter

$$= 0.5 d$$

$$P = 120 \text{ kW} = 120,000 \text{ W} \quad f = 1.75 \text{ Hz}$$

$$\tau_{\text{allow}} = 45 \text{ MPa}$$

$$I_p = \frac{\pi}{32} [d^4 - (0.5 d)^4] = 0.092039 d^4$$

TORQUE

$$P = 2\pi f T \quad P = \text{watts} \quad f = \text{Hz}$$

 T in newton meters

$$T = \frac{P}{2\pi f} = \frac{120,000 \text{ W}}{2\pi(1.75 \text{ Hz})} = 10,913.5 \text{ N}\cdot\text{m}$$

MINIMUM OUTSIDE DIAMETER

$$\tau_{\text{max}} = \frac{Td}{2I_p} \quad I_p = \frac{Td}{2\tau_{\text{max}}} = \frac{Td}{2\tau_{\text{allow}}}$$

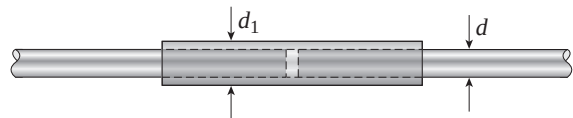
$$0.092039 d^4 = \frac{(10,913.5 \text{ N}\cdot\text{m})(d)}{2(45 \text{ MPa})}$$

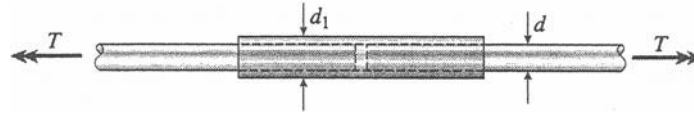
$$d^3 = 0.0013175 \text{ m}^3 \quad d = 0.1096 \text{ m}$$

$$d_{\text{min}} = 110 \text{ mm} \quad \leftarrow$$

Problem 3.7-7 A propeller shaft of solid circular cross section and diameter d is spliced by a collar of the same material (see figure). The collar is securely bonded to both parts of the shaft.

What should be the minimum outer diameter d_1 of the collar in order that the splice can transmit the same power as the solid shaft?



Solution 3.7-7 Splice in a propeller shaft

SOLID SHAFT

$$\tau_{\max} = \frac{16 T_1}{\pi d^3} \quad T_1 = \frac{\pi d^3 \tau_{\max}}{16}$$

HOLLOW COLLAR

$$I_P = \frac{\pi}{32}(d_1^4 - d^4) \quad \tau_{\max} = \frac{T_2 r}{I_P} = \frac{T_2(d_1/2)}{I_P}$$

$$T_2 = \frac{2\tau_{\max} I_P}{d_1} = \frac{2\tau_{\max}}{d_1} \left(\frac{\pi}{32} \right) (d_1^4 - d^4)$$

$$= \frac{\pi \tau_{\max}}{16 d_1} (d_1^4 - d^4)$$

EQUATE TORQUES

For the same power, the torques must be the same.
For the same material, both parts can be stressed to the same maximum stress.

$$\therefore T_1 = T_2 \quad \frac{\pi d^3 \tau_{\max}}{16} = \frac{\pi \tau_{\max}}{16 d_1} (d_1^4 - d^4)$$

$$\text{or } \left(\frac{d_1}{d} \right)^4 - \frac{d_1}{d} - 1 = 0 \quad (\text{Eq. 1})$$

MINIMUM OUTER DIAMETER

Solve Eq. (1) numerically:

$$\text{Min. } d_1 = 1.221 d \quad \leftarrow$$

Problem 3.7-8 What is the maximum power that can be delivered by a hollow propeller shaft (outside diameter 50 mm, inside diameter 40 mm, and shear modulus of elasticity 80 GPa) turning at 600 rpm if the allowable shear stress is 100 MPa and the allowable rate of twist is $3.0^\circ/\text{m}$?

Solution 3.7-8 Hollow propeller shaft

$$d_2 = 50 \text{ mm} \quad d_1 = 40 \text{ mm}$$

$$G = 80 \text{ GPa} \quad n = 600 \text{ rpm}$$

$$\tau_{\text{allow}} = 100 \text{ MPa} \quad \theta_{\text{allow}} = 3.0^\circ/\text{m}$$

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 362.3 \times 10^{-9} \text{ m}^4$$

BASED UPON ALLOWABLE SHEAR STRESS

$$\tau_{\max} = \frac{T_1(d_2/2)}{I_P} \quad T_1 = \frac{2\tau_{\text{allow}} I_P}{d_2}$$

$$T_1 = \frac{2(100 \text{ MPa})(362.3 \times 10^{-9} \text{ m}^4)}{0.050 \text{ m}}$$

$$= 1449 \text{ N}\cdot\text{m}$$

BASED UPON ALLOWABLE RATE OF TWIST

$$\theta = \frac{T_2}{G I_P} \quad T_2 = G I_P \theta_{\text{allow}}$$

$$T_2 = (80 \text{ GPa})(362.3 \times 10^{-9} \text{ m}^4)(3.0^\circ/\text{m})$$

$$\times \left(\frac{\pi}{180} \text{ rad/degree} \right)$$

$$T_2 = 1517 \text{ N}\cdot\text{m}$$

SHEAR STRESS GOVERNS

$$T_{\text{allow}} = T_1 = 1449 \text{ N}\cdot\text{m}$$

MAXIMUM POWER

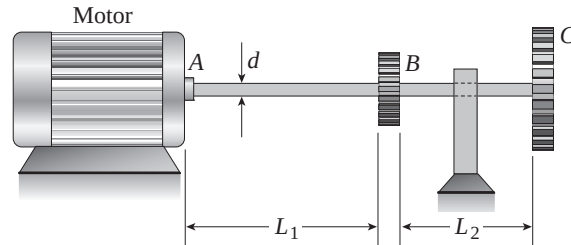
$$P = \frac{2\pi n T}{60} = \frac{2\pi(600 \text{ rpm})(1449 \text{ N}\cdot\text{m})}{60}$$

$$P = 91,047 \text{ W}$$

$$P_{\max} = 91.0 \text{ kW} \quad \leftarrow$$

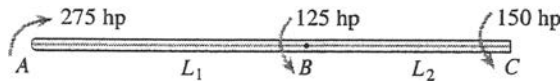
Problem 3.7-9 A motor delivers 275 hp at 1000 rpm to the end of a shaft (see figure). The gears at B and C take out 125 and 150 hp, respectively.

Determine the required diameter d of the shaft if the allowable shear stress is 7500 psi and the angle of twist between the motor and gear C is limited to 1.5° . (Assume $G = 11.5 \times 10^6$ psi, $L_1 = 6$ ft, and $L_2 = 4$ ft.)



PROBS. 3.7-9 and 3.7-10

Solution 3.7-9 Motor-driven shaft



$$L_1 = 6 \text{ ft}$$

$$L_2 = 4 \text{ ft}$$

$$d = \text{diameter}$$

$$n = 1000 \text{ rpm}$$

$$\tau_{\text{allow}} = 7500 \text{ psi}$$

$$(\phi_{AC})_{\text{allow}} = 1.5^\circ = 0.02618 \text{ rad}$$

$$G = 11.5 \times 10^6 \text{ psi}$$

TORQUES ACTING ON THE SHAFT

$$H = \frac{2\pi nT}{33,000} \quad H = \text{hp} \quad n = \text{rpm} \quad T = \text{lb-ft}$$

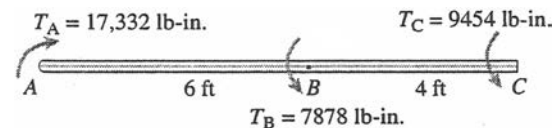
$$T = \frac{33,000 H}{2\pi n}$$

$$\begin{aligned} \text{At point A: } T_A &= \frac{33,000(275 \text{ hp})}{2\pi(1000 \text{ rpm})} \\ &= 1444 \text{ lb-ft} \\ &= 17,332 \text{ lb-in.} \end{aligned}$$

$$\text{At point B: } T_B = \frac{125}{275} T_A = 7878 \text{ lb-in.}$$

$$\text{At point C: } T_C = \frac{150}{275} T_A = 9454 \text{ lb-in.}$$

FREE-BODY DIAGRAM



$$T_A = 17,332 \text{ lb-in.}$$

$$T_C = 9454 \text{ lb-in.}$$

$$d = \text{diameter}$$

$$T_B = 7878 \text{ lb-in.}$$

INTERNAL TORQUES

$$T_{AB} = 17,332 \text{ lb-in.}$$

$$T_{BC} = 9454 \text{ lb-in.}$$

DIAMETER BASED UPON ALLOWABLE SHEAR STRESS

The larger torque occurs in segment AB

$$\begin{aligned} \tau_{\text{max}} &= \frac{16T_{AB}}{\pi d^3} \quad d^3 = \frac{16T_{AB}}{\pi \tau_{\text{allow}}} \\ &= \frac{16(17,332 \text{ lb-in.})}{\pi(7500 \text{ psi})} = 11.77 \text{ in.}^3 \end{aligned}$$

$$d = 2.27 \text{ in.}$$

DIAMETER BASED UPON ALLOWABLE ANGLE OF TWIST

$$I_P = \frac{\pi d^4}{32} \quad \phi = \frac{TL}{GI_P} = \frac{32TL}{\pi G d^4}$$

Segment AB:

$$\begin{aligned}\phi_{AB} &= \frac{32T_{AB}L_{AB}}{\pi Gd^4} \\ &= \frac{32(17,330 \text{ lb} \cdot \text{in.})(6 \text{ ft})(12 \text{ in./ft})}{\pi(11.5 \times 10^6 \text{ psi})d^4} \\ \phi_{AB} &= \frac{1.1052}{d^4}\end{aligned}$$

Segment BC:

$$\begin{aligned}\phi_{BC} &= \frac{32T_{BC}L_{BC}}{\pi Gd^4} \\ &= \frac{32(9450 \text{ lb} \cdot \text{in.})(4 \text{ ft})(12 \text{ in./ft})}{\pi(11.5 \times 10^6 \text{ psi})d^4}\end{aligned}$$

$$\phi_{BC} = \frac{0.4018}{d^4}$$

$$\text{From A to C: } \phi_{AC} = \phi_{AB} + \phi_{BC} = \frac{1.5070}{d^4}$$

$$(\phi_{AC})_{\text{allow}} = 0.02618 \text{ rad}$$

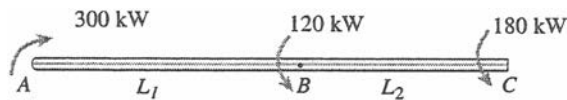
$$\therefore 0.02618 = \frac{1.5070}{d^4} \quad \text{and} \quad d = 2.75 \text{ in.}$$

Angle of twist governs

$$d = 2.75 \text{ in.} \quad \leftarrow$$

Problem 3.7-10 The shaft ABC shown in the figure is driven by a motor that delivers 300 kW at a rotational speed of 32 Hz. The gears at B and C take out 120 and 180 kW, respectively. The lengths of the two parts of the shaft are $L_1 = 1.5 \text{ m}$ and $L_2 = 0.9 \text{ m}$.

Determine the required diameter d of the shaft if the allowable shear stress is 50 MPa, the allowable angle of twist between points A and C is 4.0° , and $G = 75 \text{ GPa}$.

Solution 3.7-10 Motor-driven shaft

$$L_1 = 1.5 \text{ m}$$

$$L_2 = 0.9 \text{ m}$$

 d = diameter

$$f = 32 \text{ Hz}$$

$$\tau_{\text{allow}} = 50 \text{ MPa}$$

$$G = 75 \text{ GPa}$$

$$(\phi_{AC})_{\text{allow}} = 4^\circ = 0.06981 \text{ rad}$$

TORQUES ACTING ON THE SHAFT

$$P = 2\pi fT \quad P = \text{watts} \quad f = \text{Hz}$$

 T in newton meters

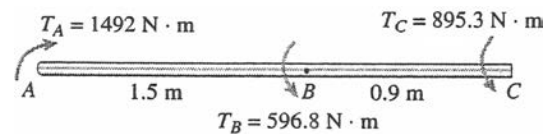
$$T = \frac{P}{2\pi f}$$

$$\text{At point A: } T_A = \frac{300,000 \text{ W}}{2\pi(32 \text{ Hz})} = 1492 \text{ N} \cdot \text{m}$$

$$\text{At point B: } T_B = \frac{120}{300} T_A = 596.8 \text{ N} \cdot \text{m}$$

$$\text{At point C: } T_C = \frac{180}{300} T_A = 895.3 \text{ N} \cdot \text{m}$$

FREE-BODY DIAGRAM



$$T_A = 1492 \text{ N} \cdot \text{m}$$

$$T_B = 596.8 \text{ N} \cdot \text{m}$$

$$T_C = 895.3 \text{ N} \cdot \text{m}$$

 d = diameter

INTERNAL TORQUES

$$T_{AB} = 1492 \text{ N} \cdot \text{m}$$

$$T_{BC} = 895.3 \text{ N} \cdot \text{m}$$

DIAMETER BASED UPON ALLOWABLE SHEAR STRESS

The larger torque occurs in segment AB

$$\tau_{\max} = \frac{16 T_{AB}}{\pi d^3} \quad d^3 = \frac{16 T_{AB}}{\pi \tau_{\text{allow}}} = \frac{16(1492 \text{ N}\cdot\text{m})}{\pi(50 \text{ MPa})}$$

$$d^3 = 0.0001520 \text{ m}^3 \quad d = 0.0534 \text{ m} = 53.4 \text{ mm}$$

DIAMETER BASED UPON ALLOWABLE ANGLE OF TWIST

$$I_P = \frac{\pi d^4}{32} \quad \phi = \frac{TL}{GI_P} = \frac{32TL}{\pi Gd^4}$$

Segment AB:

$$\phi_{AB} = \frac{32 T_{AB} L_{AB}}{\pi G d^4} = \frac{32(1492 \text{ N}\cdot\text{m})(1.5 \text{ m})}{\pi(75 \text{ GPa})d^4}$$

$$\phi_{AB} = \frac{0.3039 \times 10^{-6}}{d^4}$$

Segment BC:

$$\phi_{BC} = \frac{32 T_{BC} L_{BC}}{\pi G d^4} = \frac{32(895.3 \text{ N}\cdot\text{m})(0.9 \text{ m})}{\pi(75 \text{ GPa})d^4}$$

$$\phi_{BC} = \frac{0.1094 \times 10^{-6}}{d^4}$$

$$\text{From A to C: } \phi_{AC} = \phi_{AB} + \phi_{BC} = \frac{0.4133 \times 10^{-6}}{d^4}$$

$$(\phi_{AC})_{\text{allow}} = 0.06981 \text{ rad}$$

$$\therefore 0.06981 = \frac{0.1094 \times 10^{-6}}{d^4}$$

$$\text{and } d = 0.04933 \text{ m}$$

$$= 49.3 \text{ mm}$$

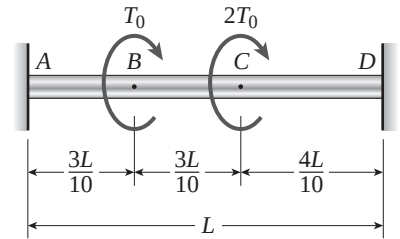
SHEAR STRESS GOVERNS

$$d = 53.4 \text{ mm} \quad \leftarrow$$

Statically Indeterminate Torsional Members

Problem 3.8-1 A solid circular bar ABCD with fixed supports is acted upon by torques T_0 and $2T_0$ at the locations shown in the figure.

- Obtain a formula for the maximum angle of twist $\phi_{\max}$ of the bar. [Hint: Use Eqs. (3-50a and b) of Example 3-9 to obtain the reactive torques.]
- What is $\phi_{\max}$ if the applied torque T_0 at B is reversed in direction?



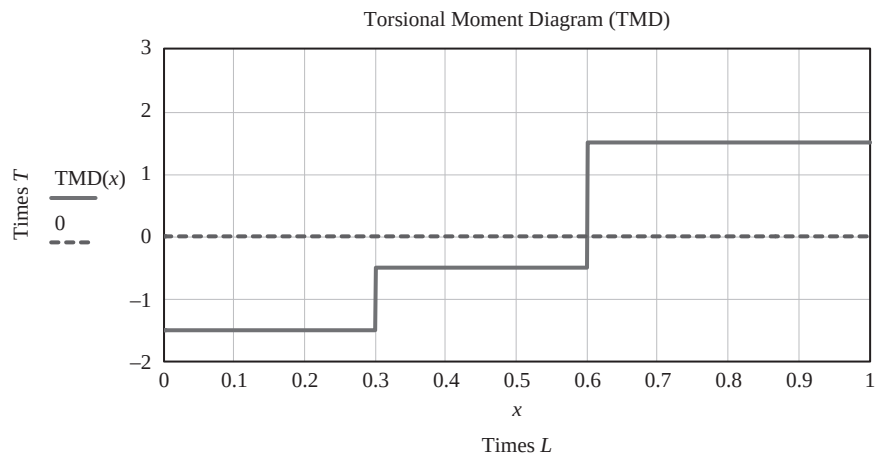
Solution 3.8-1 Circular bar with fixed ends

- SELECT T_D AS THE REDUNDANT; USE SUPERPOSITION OF RELEASED STRUCTURES TO FIND TWIST ANGLES DUE TO APPLIED TORQUES AT B AND C (i.e., ϕ_1) AND ALSO FOR APPLIED REDUNDANT T_D APPLIED AT D (i.e., ϕ_2)

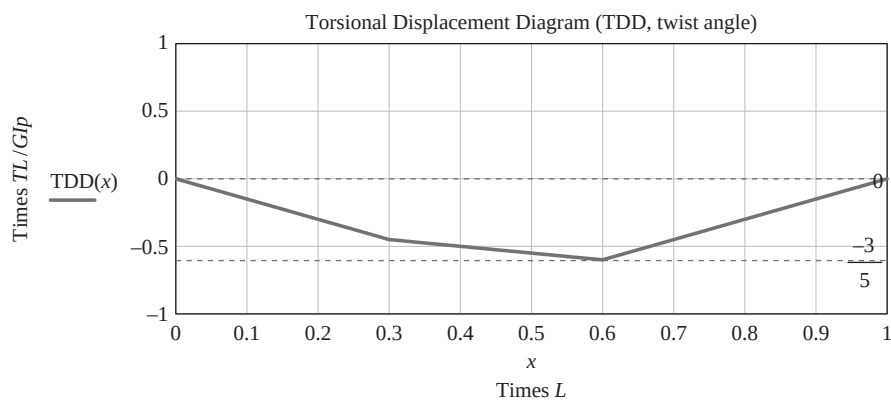
$$\phi_1 = \frac{-3T_0\left(\frac{3L}{10}\right)}{GI_P} + \frac{(-2T_0)\left(\frac{3L}{10}\right)}{GI_P} = -\frac{3LT_0}{2GI_P} \quad \phi_2 = T_D\left(\frac{L}{GI_P}\right)$$

$$\text{Compatibility} \quad \phi_1 + \phi_2 = 0 \quad \text{so} \quad T_D = \frac{GI_P}{L} \left(\frac{3LT_0}{2GI_P} \right) = \frac{3T_0}{2} \quad \text{and} \quad T_A = 3T_0 - T_D = \frac{3T_0}{2}$$

$$\text{TORSIONAL MOMENT DIAGRAM (TMD)} \quad T_A = \frac{3T_0}{2} \quad T_D = T_A$$



TORSIONAL DISPLACEMENT DIAGRAM (TDD)



So maximum twist is at $x = 3L/5$ and is equal to $\phi_{\max} = \frac{T_D}{GI_p} \frac{4L}{10} = \frac{3LT_0}{5GI_p} \leftarrow$
 $T_{A1} = -T_0 + 2T_0 = T_0$

(B) REPEAT SUPERPOSITION PROCEDURE USING REDUNDANT T_D BUT REVERSE SIGN OF TORQUE APPLIED AT B

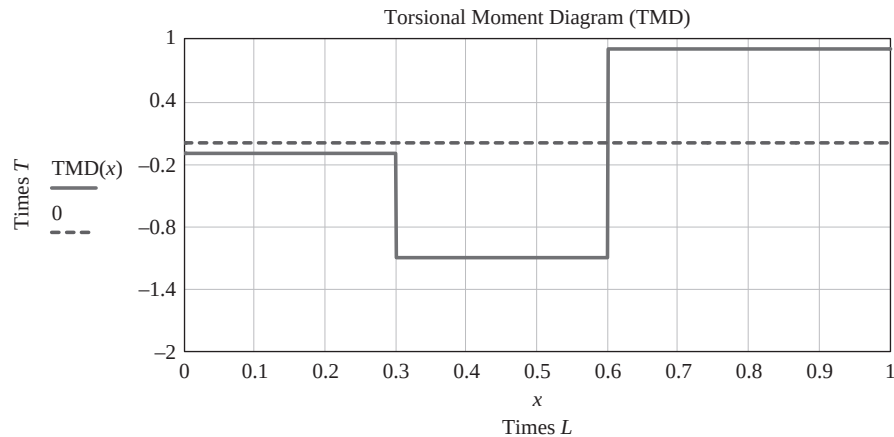
$$\phi_1 = \frac{-T_0 \left(\frac{3L}{10} \right)}{GI_p} + \frac{(-2T_0) \left(\frac{3L}{10} \right)}{GI_p} = -\frac{9LT_0}{10GI_p} \quad \phi_2 = T_D \left(\frac{L}{GI_p} \right)$$

compatibility $\phi_1 + \phi_2 = 0 \quad T_D = \frac{GI_p}{L} \left(\frac{9LT_0}{10GI_p} \right) = \frac{9T_0}{10}$

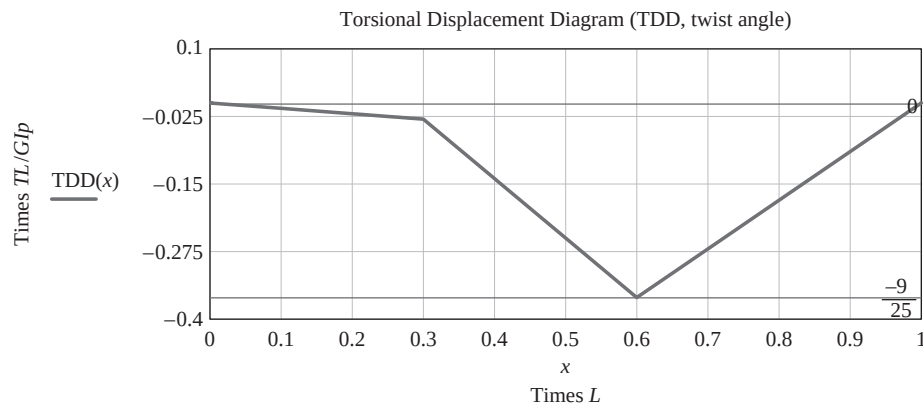
$$T_A = T_0 - T_D = \frac{T_0}{10}$$

TORSIONAL MOMENT DIAGRAM (TMD)

$$T_A = \frac{T_0}{10} \quad T_D = \frac{9}{10} T_0$$



TORSIONAL DISPLACEMENT DIAGRAM (TDD)

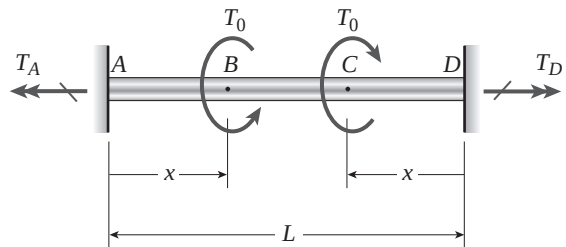


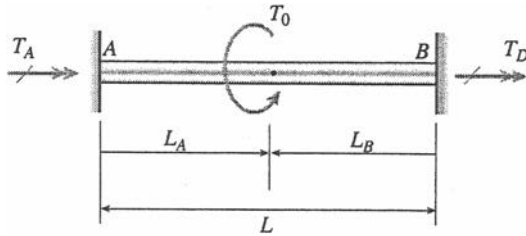
So maximum twist is at $x = 3L/5$ and is equal to $\phi_{\max} = \frac{T_D}{GI_p} \frac{4L}{10} = \frac{9L T_0}{25}$ ←

$$\frac{9}{25} = 0.36$$

Problem 3.8-2 A solid circular bar $ABCD$ with fixed supports at ends A and D is acted upon by two equal and oppositely directed torques T_0 , as shown in the figure. The torques are applied at points B and C , each of which is located at distance x from one end of the bar. (The distance x may vary from zero to $L/2$.)

- For what distance x will the angle of twist at points B and C be a maximum?
- What is the corresponding angle of twist $\phi_{\max}$?
[Hint: Use Eqs. (3-50a and b) of Example 3-9 to obtain the reactive torques.]



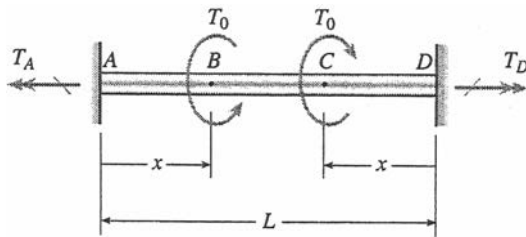
Solution 3.8-2 Circular bar with fixed ends

From Eqs. (3-50a and b):

$$T_A = \frac{T_0 L_B}{L}$$

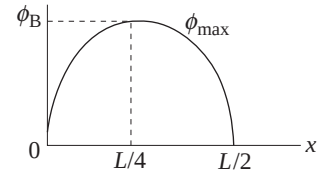
$$T_B = \frac{T_0 L_A}{L}$$

APPLY THE ABOVE FORMULAS TO THE GIVEN BAR:



$$T_A = \frac{T_0(L-x)}{L} - \frac{T_0 x}{L} = \frac{T_0}{L}(L-2x) \quad T_D = T_A$$

(a) ANGLE OF TWIST AT SECTIONS B AND C



$$\phi_B = \phi_{AB} = \frac{T_A x}{G I_P} = \frac{T_0}{G I_P L} (L-2x)(x)$$

$$\frac{d\phi_B}{dx} = \frac{T_0}{G I_P L} (L-4x)$$

$$\frac{d\phi_B}{dx} = 0; L-4x = 0$$

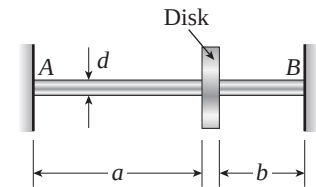
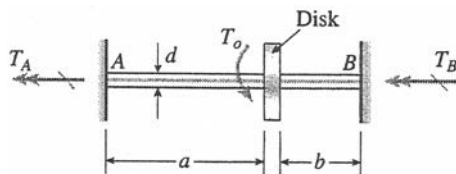
$$\text{or } x = \frac{L}{4} \leftarrow$$

(b) MAXIMUM ANGLE OF TWIST

$$\phi_{\max} = (\phi_B)_{\max} = (\phi_B)_{x=L/4} = \frac{T_0 L}{8 G I_P} \leftarrow$$

Problem 3.8-3 A solid circular shaft AB of diameter d is fixed against rotation at both ends (see figure). A circular disk is attached to the shaft at the location shown.

What is the largest permissible angle of rotation $\phi_{\max}$ of the disk if the allowable shear stress in the shaft is τ_{allow} ? (Assume that $a > b$. Also, use Eqs. 3-50a and b of Example 3-9 to obtain the reactive torques.)

**Solution 3.8-3 Shaft fixed at both ends**

$$L = a + b$$

$$a > b$$

Assume that a torque T_0 acts at the disk.

The reactive torques can be obtained from Eqs. (3-50a and b):

$$T_A = \frac{T_0 b}{L} \quad T_B = \frac{T_0 a}{L}$$

Since $a > b$, the larger torque (and hence the larger stress) is in the right hand segment.

$$\tau_{\max} = \frac{T_B(d/2)}{I_P} = \frac{T_0 ad}{2LI_P}$$

$$T_0 = \frac{2LI_P\tau_{\max}}{ad} \quad (T_0)_{\max} = \frac{2L I_P\tau_{\text{allow}}}{ad}$$

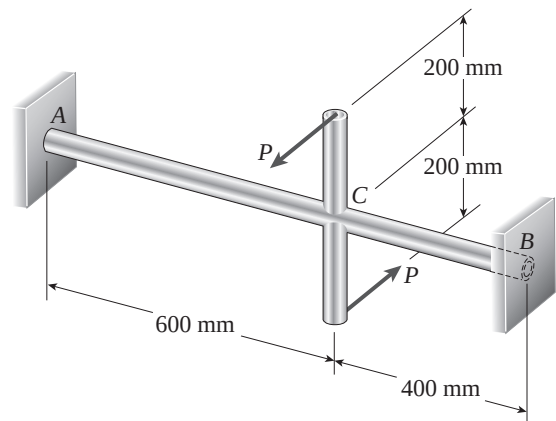
ANGLE OF ROTATION OF THE DISK (FROM Eq. 3-53)

$$\phi = \frac{T_0 ab}{GLI_P}$$

$$\phi_{\max} = \frac{(T_0)_{\max} ab}{GLI_P} = \frac{2b\tau_{\text{allow}}}{Gd} \quad \leftarrow$$

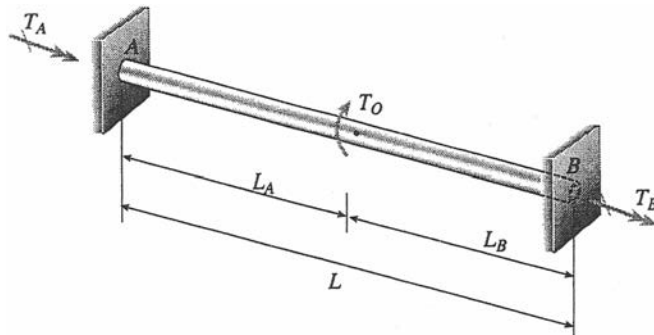
Problem 3.8-4 A hollow steel shaft *ACB* of outside diameter 50 mm and inside diameter 40 mm is held against rotation at ends *A* and *B* (see figure). Horizontal forces *P* are applied at the ends of a vertical arm that is welded to the shaft at point *C*.

Determine the allowable value of the forces *P* if the maximum permissible shear stress in the shaft is 45 MPa. (Hint: Use Eqs. 3-50a and b of Example 3-9 to obtain the reactive torques.)

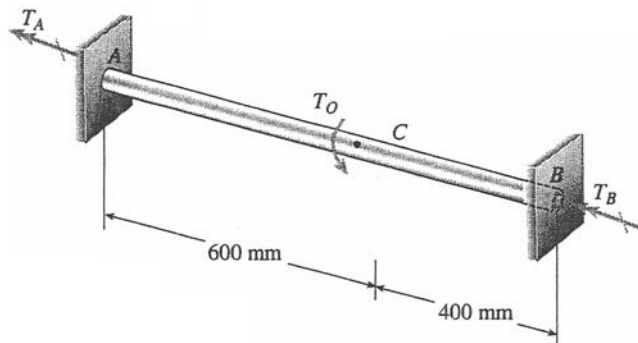


Solution 3.8-4 Hollow shaft with fixed ends

GENERAL FORMULAS:



APPLY THE ABOVE FORMULAS TO THE GIVEN SHAFT



$$T_0 = P(400 \text{ mm})$$

$$L_B = 400 \text{ mm}$$

$$L_A = 600 \text{ mm}$$

$$L = L_A + L_B = 1000 \text{ mm}$$

$$d_2 = 50 \text{ mm} \quad d_1 = 40 \text{ mm}$$

$$\tau_{\text{allow}} = 45 \text{ MPa}$$

$$T_A = \frac{T_0 L_B}{L} = \frac{P(0.4 \text{ m})(400 \text{ mm})}{1000 \text{ mm}} = 0.16 P$$

$$T_B = \frac{T_0 L_A}{L} = \frac{P(0.4 \text{ m})(600 \text{ mm})}{1000 \text{ mm}} = 0.24 P$$

UNITS: *P* in Newtons *T* in Newton meters

From Eqs. (3-50a and b):

$$T_A = \frac{T_0 L_B}{L}$$

$$T_B = \frac{T_0 L_A}{L}$$

The larger torque, and hence the larger shear stress, occurs in part *CB* of the shaft.

$$\therefore T_{\max} = T_B = 0.24 P$$

SHEAR STRESS IN PART CB

$$\tau_{\max} = \frac{T_{\max}(d/2)}{I_P} \quad T_{\max} = \frac{2\tau_{\max}I_P}{d} \quad (\text{Eq. 1})$$

UNITS: Newtons and meters

$$\tau_{\max} = 45 \times 10^6 \text{ N/m}^2$$

$$I_P = \frac{\pi}{32} (d_2^4 - d_1^4) = 362.26 \times 10^{-9} \text{ m}^4$$

$$d = d_2 = 0.05 \text{ m}$$

Substitute numerical values into (Eq. 1):

$$0.24P = \frac{2(45 \times 10^6 \text{ N/m}^2)(362.26 \times 10^{-9} \text{ m}^4)}{0.05 \text{ m}}$$

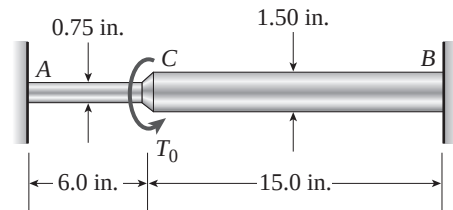
$$= 652.07 \text{ N}\cdot\text{m}$$

$$P = \frac{652.07 \text{ N}\cdot\text{m}}{0.24 \text{ m}} = 2717 \text{ N}$$

$$P_{\text{allow}} = 2720 \text{ N} \quad \leftarrow$$

Problem 3.8-5 A stepped shaft ACB having solid circular cross sections with two different diameters is held against rotation at the ends (see figure).

- (a) If the allowable shear stress in the shaft is 6000 psi, what is the maximum torque $(T_0)_{\max}$ that may be applied at section C? [Hint: Use Eqs. (3-49a and b) of Example 3-9 to obtain the reactive torques.]
- (b) Find $(T_0)_{\max}$ if the maximum angle of twist is limited to 0.55° . Let $G = 10,600 \text{ ksi}$.



Solution 3.8-5

(a) NUMERICAL DATA

$$G = 10600 \text{ ksi} \quad d_A = \frac{3}{4} \text{ in.} \quad d_B = \frac{3}{2} \text{ in.} \quad L_A = 6 \text{ in.} \quad L_B = 15 \text{ in.} \quad \tau_a = 6 \text{ ksi}$$

$$I_{PA} = \frac{\pi}{32} (d_A^4) \quad I_{PB} = \frac{\pi}{32} d_B^4$$

Using Eqs 3-49a, b

$$T_A = \left(\frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}} \right) T_0 \rightarrow \frac{5 T_0}{37} \quad T_B = \left(\frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}} \right) T_0 \rightarrow \frac{32 T_0}{37}$$

Now use torsion formula to find shear stresses in each segment; set each shear equal to the allowable value then solve for the max. permissible value of torque T_0 .

$$\tau_A = T_A \left(\frac{16}{\pi d_A^3} \right) \quad \tau_B = T_B \left(\frac{16}{\pi d_B^3} \right)$$

From segment 1:

$$T_{0\max 1} = \frac{\tau_a \left(\frac{\pi d_A^3}{16} \right)}{\frac{5}{37}} = 3678 \text{ lb-in.}$$

From segment 2:

$$T_{0\max 2} = \frac{\tau_a \left(\frac{\pi d_B^3}{16} \right)}{\frac{32}{37}} = 4597 \text{ lb-in.}$$

$$< \text{controls}$$

- (b) THE MAX. TWIST ANGLE IS AT C; BOTH SEGMENTS TWIST THE SAME AMOUNT. USE THE TORQUE-DISPL. RELATION, SUBSTITUTE FOR T_A AND T_B IN TERMS OF T_0

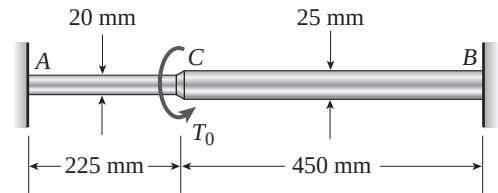
$$\phi_{\text{allow}} = 0.55^\circ$$

$$\phi_{AB} = \frac{T_A L_A}{G I_{pA}} \quad \phi_{BC} = \frac{T_B L_B}{G I_{pB}}$$

$$\boxed{T_{0\text{max}} = \frac{G I_{pA}}{L_A \left(\frac{5}{37} \right)} \phi_{\text{allow}} = 3898 \text{ lb-in.}} \quad \text{or} \quad T_{0\text{max}} = \frac{G I_{pB}}{L_B \left(\frac{32}{37} \right)} \phi_{\text{allow}} = 3898 \text{ lb-in.}$$

Problem 3.8-6 A stepped shaft ACB having solid circular cross sections with two different diameters is held against rotation at the ends (see figure).

- (a) If the allowable shear stress in the shaft is 43 MPa, what is the maximum torque $(T_0)_{\text{max}}$ that may be applied at section C? [Hint: Use Eqs. (3-49a and b) of Example 3-9 to obtain the reactive torques.]
- (b) Find $(T_0)_{\text{max}}$ if the maximum angle of twist is limited to 1.85° . Let $G = 28 \text{ GPa}$.



Solution 3.8-6

NUMERICAL DATA

$$G = 28 \text{ GPa} \quad d_A = 20 \text{ mm} \quad d_B = 25 \text{ mm}$$

$$L_A = 225 \text{ mm} \quad L_B = 450 \text{ mm} \quad \tau_a = 43 \text{ MPa}$$

$$I_{pA} = \frac{\pi}{32} (d_A^4) \quad I_{pB} = \frac{\pi}{32} d_B^4$$

- (a) Using Eqs 3-49a, b

$$T_A = \left(\frac{L_B I_{pA}}{L_B I_{pA} + L_A I_{pB}} \right) T_0 = \frac{512 T_0}{1137} \quad T_B = \left(\frac{L_A I_{pB}}{L_B I_{pA} + L_A I_{pB}} \right) T_0 = \frac{625 T_0}{1137}$$

Now use torsion formula to find shear stresses in each segment; set each shear equal to the allowable value then solve for the maximum permissible value of torque T_0 .

$$\tau_A = T_A \left(\frac{16}{\pi d_A^3} \right) \quad \tau_B = T_B \left(\frac{16}{\pi d_B^3} \right)$$

From segment 1:

$$\boxed{T_{0\text{max}1} = \frac{\tau_a \left(\frac{\pi d_A^3}{16} \right)}{\frac{512}{1137}} = 150 \text{ N}\cdot\text{m}}$$

From segment 2:

$$< \text{controls} \quad T_{0\text{max}2} = \frac{\tau_a \left(\frac{\pi d_B^3}{16} \right)}{\frac{625}{1137}} = 240 \text{ N}\cdot\text{m}$$

- (b) The maximum twist angle is at C; both segments twist the same amount. Use the torque-displacement relation, substitute for T_A and T_B in terms of T_0 .

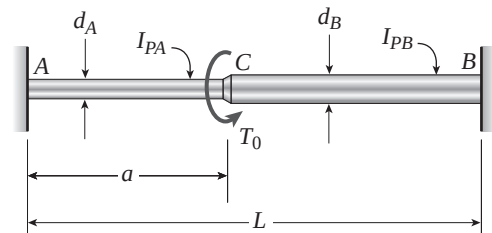
$$\phi_{\text{allow}} = 1.85^\circ$$

$$\phi_{AB} = \frac{T_A L_A}{G I_{PA}} \quad \phi_{BC} = \frac{T_B L_B}{G I_{PB}}$$

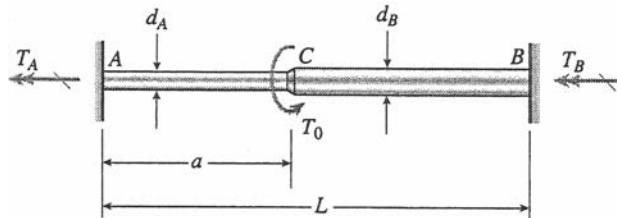
$$T_{0\text{max}} = \frac{G I_{PA}}{L_A \left(\frac{512}{1137} \right)} \phi_{\text{allow}} = 140 \text{ N}\cdot\text{m} \quad \text{or} \quad T_{0\text{max}} = \frac{G I_{PB}}{L_B \left(\frac{625}{1137} \right)} \phi_{\text{allow}} = 140 \text{ N}\cdot\text{m}$$

Problem 3.8-7 A stepped shaft ACB is held against rotation at ends A and B and subjected to a torque T_0 acting at section C (see figure). The two segments of the shaft (AC and CB) have diameters d_A and d_B , respectively, and polar moments of inertia I_{PA} and I_{PB} , respectively. The shaft has length L and segment AC has length a .

- (a) For what ratio a/L will the maximum shear stresses be the same in both segments of the shaft?
 (b) For what ratio a/L will the internal torques be the same in both segments of the shaft? (*Hint:* Use Eqs. 3-49a and b of Example 3-9 to obtain the reactive torques.)



Solution 3.8-7 Stepped shaft



SEGMENT AC: $d_A, I_{PA} \quad L_A = a$

SEGMENT CB: $d_B, I_{PB} \quad L_B = L - a$

REACTIVE TORQUES (from Eqs. 3-49a and b)

$$T_A = T_0 \left(\frac{L_B I_{PA}}{L_B I_{PA} + L_A I_{PB}} \right); \quad T_B = T_0 \left(\frac{L_A I_{PB}}{L_B I_{PA} + L_A I_{PB}} \right)$$

- (a) EQUAL SHEAR STRESSES

$$\tau_{AC} = \frac{T_A (d_A/2)}{I_{PA}} \quad \tau_{CB} = \frac{T_B (d_B/2)}{I_{PB}}$$

$$\tau_{AC} = \tau_{CB} \quad \text{or} \quad \frac{T_A d_A}{I_{PA}} = \frac{T_B d_B}{I_{PB}} \quad (\text{Eq. 1})$$

Substitute T_A and T_B into Eq. (1):

$$\frac{L_B I_{PA} d_A}{I_{PA}} = \frac{L_A I_{PB} d_B}{I_{PB}} \quad \text{or} \quad L_B d_A = L_A d_B$$

$$\text{or} \quad (L - a) d_A = a d_B$$

$$\text{Solve for } a/L: \quad \frac{a}{L} = \frac{d_A}{d_A + d_B} \quad \leftarrow$$

- (b) EQUAL TORQUES

$$T_A = T_B \quad \text{or} \quad L_B I_{PA} = L_A I_{PB}$$

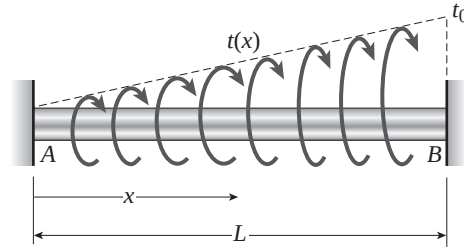
$$\text{or} \quad (L - a) I_{PA} = a I_{PB}$$

$$\text{Solve for } a/L: \quad \frac{a}{L} = \frac{I_{PA}}{I_{PA} + I_{PB}}$$

$$\text{or} \quad \frac{a}{L} = \frac{d_A^4}{d_A^4 + d_B^4} \quad \leftarrow$$

Problem 3.8-8 A circular bar AB of length L is fixed against rotation at the ends and loaded by a distributed torque $t(x)$ that varies linearly in intensity from zero at end A to t_0 at end B (see figure).

- (a) Obtain formulas for the fixed-end torques T_A and T_B .
 (b) Find an expression for the angle of twist $\phi(x)$. What is $\phi_{\max}$ and where does it occur along the bar?



Solution 3.8-8

- (a) USE SUPERPOSITION WITH T_B AS THE REDUNDANT. TWIST AT B DUE TO LOADING $t(x)$ IS ϕ_{B1} , AND TWIST AT B DUE TO REDUNDANT T_B IS ϕ_{B2}

$$\phi_{B1} = \int_0^L \frac{\left[\frac{-t_0 L}{2} + \frac{1}{2} \left(t_0 \frac{x}{L} \right) x \right]}{G I_p} dx = -\frac{L^2 t_0}{3 G I_p} \quad \phi_{B2} = \frac{T_B L}{G I_p}$$

$$\text{Compatibility} \quad \phi_{B1} + \phi_{B2} = 0 \quad \boxed{T_B = \frac{G I_p}{L} \left(\frac{L^2 t_0}{3 G I_p} \right) = \frac{L t_0}{3}}$$

$$\text{Statics} \quad \boxed{T_A = \frac{t_0 L}{2} \quad T_B = \frac{L t_0}{6}}$$

- (b) USE INTEGRAL EXPRESSION FOR TORQUE-DISPLACEMENT RELATION TO FIND $\phi(x)$

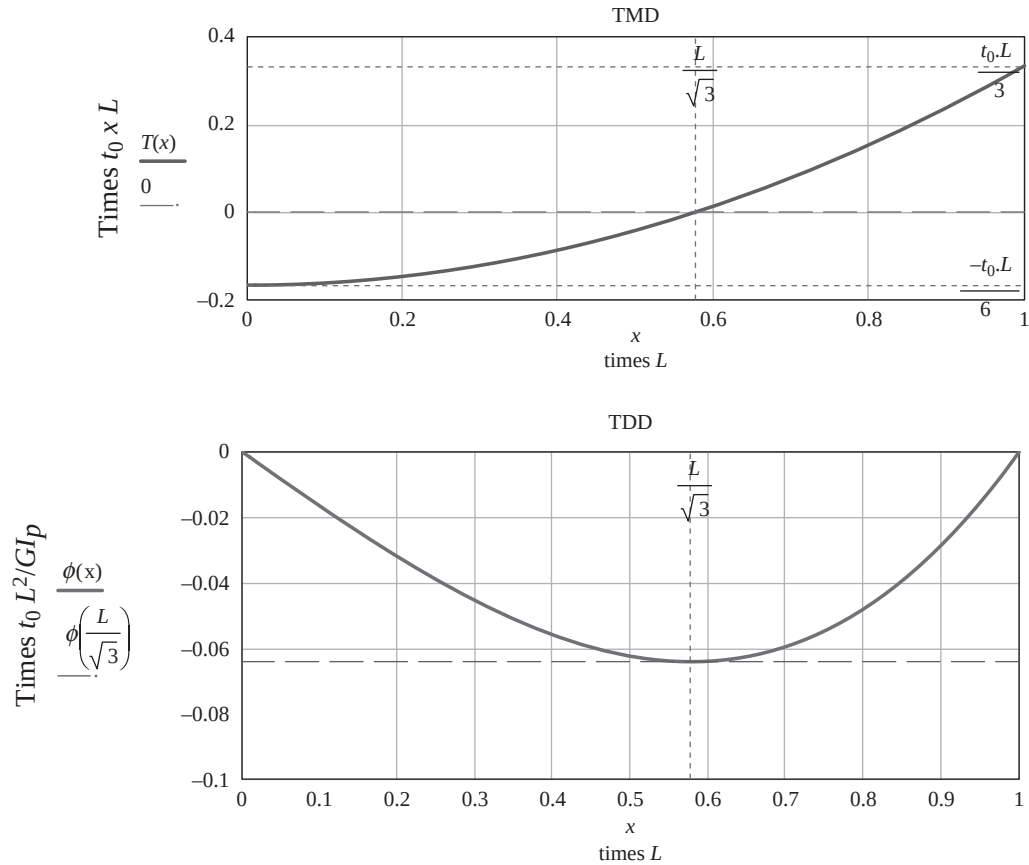
$$\phi(x) = \frac{1}{G I_p} \left[\int_0^x \left[\frac{-t_0 L}{6} + \frac{1}{2} t_0 \left(\frac{\xi}{L} \right) \xi \right] d\xi \right] = -\frac{t_0 x (L^2 - x^2)}{6 G I_p L} \quad \leftarrow$$

Differentiate expression for $\phi(x)$ and set equal to zero to find location x_m at which $\phi_{\max}$ occurs.

$$\frac{d}{dx} \phi(x) = \frac{t_0 x^2}{3 G I_p L} - \frac{t_0 (L^2 - x^2)}{6 G I_p L} = \frac{t_0 x^2}{3 G I_p L} - \frac{t_0 (L^2 - x^2)}{6 G I_p L} = 0 \quad \text{Solving: } x_m = \frac{L}{\sqrt{3}}$$

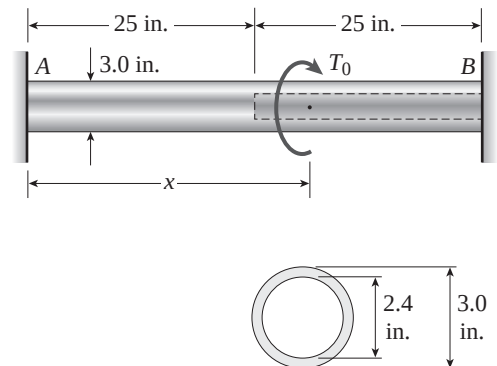
$$\boxed{\phi_{\max} = \phi\left(\frac{L}{\sqrt{3}}\right) = -\frac{\sqrt{3} L^2 t_0}{27 G I_p}} \quad \frac{\sqrt{3}}{27} = 0.064 \quad \frac{1}{\sqrt{3}} = 0.577$$

Plot torsional moment (TMD) and torsional displacement (TDD) diagrams showing variation of $T(x)$ and $\phi(x)$, respectively, over the length of the bar from A to B



Problem 3.8-9 A circular bar AB with ends fixed against rotation has a hole extending for half of its length (see figure). The outer diameter of the bar is $d_2 = 3.0$ in. and the diameter of the hole is $d_1 = 2.4$ in. The total length of the bar is $L = 50$ in.

- At what distance x from the left-hand end of the bar should a torque T_0 be applied so that the reactive torques at the supports will be equal?
- Based on the solution for x in part (a), what is $\phi_{\max}$, and where does it occur? Assume that $T_0 = 87.4$ kip-in. and $G = 10,600$ ksi.



Solution 3.8-9(a) SUPERPOSITION WITH T_B AS THE REDUNDANT

$$\phi_{B1} = -T_0 \left[\frac{25 \text{ in.}}{G \left[\frac{\pi (3 \text{ in.})^4}{32} \right]} + \frac{x - 25 \text{ in.}}{G \left[\frac{\pi}{32} [(3 \text{ in.})^4 - (2.4 \text{ in.})^4] \right]} \right] = -T_0 \left[\frac{800}{81 \pi G \text{ in.}^3} + \frac{32(x - 25 \text{ in.})}{\pi G (81 \text{ in.}^4 - 33.1776 \text{ in.}^4)} \right]$$

$$T_B = \frac{T_0}{2}$$

$$\phi_{B2} = T_B \left[\frac{25 \text{ in.}}{G \left[\frac{\pi (3 \text{ in.})^4}{32} \right]} + \frac{25 \text{ in.}}{G \left[\frac{\pi}{32} [(3 \text{ in.})^4 - (2.4 \text{ in.})^4] \right]} \right] = \frac{T_0 \left[\frac{800}{81 \pi G \text{ in.}^3} + \frac{800 \text{ in.}}{\pi G (81 \text{ in.}^4 - 33.1776 \text{ in.}^4)} \right]}{2}$$

$$\phi_{B1} + \phi_{B2} = 0 \quad \text{Solve: } x = 30.12 \text{ in.}$$

(b) $x = 30.12 \text{ in.}$ $G = 10600 \text{ ksi}$ $T_0 = 87.4 \text{ k-in.}$ $T_A = \frac{T_0}{2}$ $T_B = T_B$ $L = 50 \text{ in.}$ $\phi_{\max}$ occurs at x , point of application of torque T_0 (TMD has zero ordinate so TDD has horiz. tangent)

$$<\phi_{\max} = -T_A \left[\frac{25 \text{ in.}}{G \left[\frac{\pi (3 \text{ in.})^4}{32} \right]} + \frac{x - 25 \text{ in.}}{G \left[\frac{\pi}{32} [(3 \text{ in.})^4 - (2.4 \text{ in.})^4] \right]} \right] = -1^\circ < \text{CW when viewed from } B \text{ toward } A$$

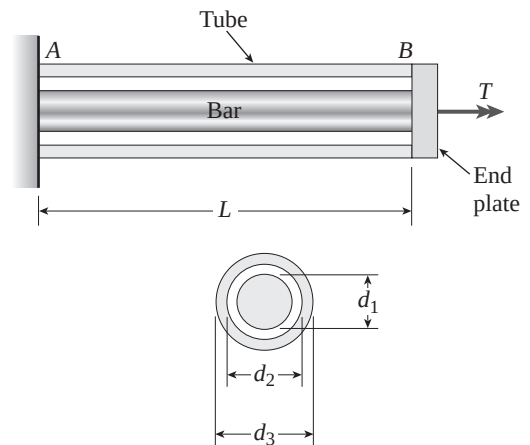
or

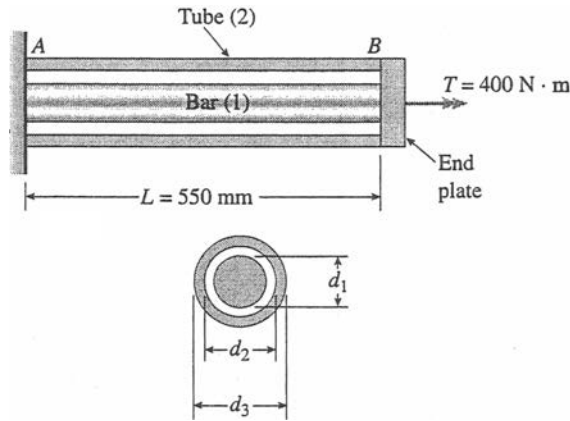
$$\phi_{\max} = -T_B \left[\frac{L - x}{G \left[\frac{\pi}{32} [(3 \text{ in.})^4 - (2.4 \text{ in.})^4] \right]} \right] = -1^\circ$$

$$\phi_{\max} = -1^\circ \quad \text{at } x = 30.12 \text{ in.}$$

Problem 3.8-10 A solid steel bar of diameter $d_1 = 25.0 \text{ mm}$ is enclosed by a steel tube of outer diameter $d_3 = 37.5 \text{ mm}$ and inner diameter $d_2 = 30.0 \text{ mm}$ (see figure). Both bar and tube are held rigidly by a support at end A and joined securely to a rigid plate at end B. The composite bar, which has a length $L = 550 \text{ mm}$, is twisted by a torque $T = 400 \text{ N} \cdot \text{m}$ acting on the end plate.

- Determine the maximum shear stresses τ_1 and τ_2 in the bar and tube, respectively.
- Determine the angle of rotation ϕ (in degrees) of the end plate, assuming that the shear modulus of the steel is $G = 80 \text{ GPa}$.
- Determine the torsional stiffness k_T of the composite bar.
(Hint: Use Eqs. 3-48a and b to find the torques in the bar and tube.)



Solution 3.8-10 Bar enclosed in a tube

$$d_1 = 25.0 \text{ mm} \quad d_2 = 30.0 \text{ mm} \quad d_3 = 37.5 \text{ mm}$$

$$G = 80 \text{ GPa}$$

POLAR MOMENTS OF INERTIA

$$\text{Bar: } I_{P1} = \frac{\pi}{32} d_1^4 = 38.3495 \times 10^{-9} \text{ m}^4$$

$$\text{Tube: } I_{P2} = \frac{\pi}{32} (d_3^4 - d_2^4) = 114.6229 \times 10^{-9} \text{ m}^4$$

TORQUES IN THE BAR (1) AND TUBE (2)
FROM EQS. (3-44A AND B)

$$\text{Bar: } T_1 = T \left(\frac{I_{P1}}{I_{P1} + I_{P2}} \right) = 100.2783 \text{ N}\cdot\text{m}$$

$$\text{Tube: } T_2 = T \left(\frac{I_{P2}}{I_{P1} + I_{P2}} \right) = 299.7217 \text{ N}\cdot\text{m}$$

(a) MAXIMUM SHEAR STRESSES

$$\text{Bar: } \tau_1 = \frac{T_1(d_1/2)}{I_{P1}} = 32.7 \text{ MPa} \quad \leftarrow$$

$$\text{Tube: } \tau_2 = \frac{T_2(d_3/2)}{I_{P2}} = 49.0 \text{ MPa} \quad \leftarrow$$

(b) ANGLE OF ROTATION OF END PLATE

$$\phi = \frac{T_1 L}{G I_{P1}} = \frac{T_2 L}{G I_{P2}} = 0.017977 \text{ rad}$$

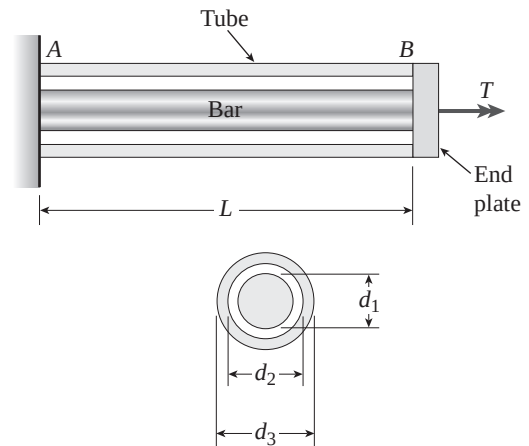
$$\phi = 1.03^\circ \quad \leftarrow$$

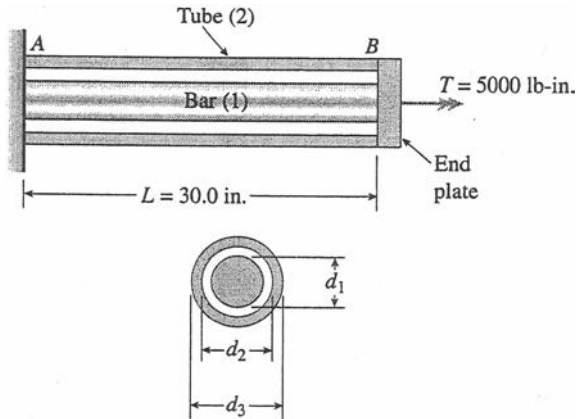
(c) TORSIONAL STIFFNESS

$$k_T = \frac{T}{\phi} = 22.3 \text{ kN}\cdot\text{m} \quad \leftarrow$$

Problem 3.8-11 A solid steel bar of diameter $d_1 = 1.50$ in. is enclosed by a steel tube of outer diameter $d_3 = 2.25$ in. and inner diameter $d_2 = 1.75$ in. (see figure). Both bar and tube are held rigidly by a support at end A and joined securely to a rigid plate at end B. The composite bar, which has length $L = 30.0$ in., is twisted by a torque $T = 5000$ lb-in. acting on the end plate.

- Determine the maximum shear stresses τ_1 and τ_2 in the bar and tube, respectively.
- Determine the angle of rotation ϕ (in degrees) of the end plate, assuming that the shear modulus of the steel is $G = 11.6 \times 10^6$ psi.
- Determine the torsional stiffness k_T of the composite bar. (Hint: Use Eqs. 3-48a and b to find the torques in the bar and tube.)



Solution 3.8-11 Bar enclosed in a tube

$$d_1 = 1.50 \text{ in.} \quad d_2 = 1.75 \text{ in.} \quad d_3 = 2.25 \text{ in.}$$

$$G = 11.6 \times 10^6 \text{ psi}$$

POLAR MOMENTS OF INERTIA

$$\text{Bar: } I_{P1} = \frac{\pi}{32} d_1^4 = 0.497010 \text{ in.}^4$$

$$\text{Tube: } I_{P2} = \frac{\pi}{32} (d_3^4 - d_2^4) = 1.595340 \text{ in.}^4$$

TORQUES IN THE BAR (1) AND TUBE (2)
FROM EQS. (3-48a AND b)

$$\text{Bar: } T_1 = T \left(\frac{I_{P1}}{I_{P1} + I_{P2}} \right) = 1187.68 \text{ lb-in.}$$

$$\text{Tube: } T_2 = T \left(\frac{I_{P2}}{I_{P1} + I_{P2}} \right) = 3812.32 \text{ lb-in.}$$

(a) MAXIMUM SHEAR STRESSES

$$\text{Bar: } \tau_1 = \frac{T_1(d_1/2)}{I_{P1}} = 1790 \text{ psi} \quad \leftarrow$$

$$\text{Tube: } \tau_2 = \frac{T_2(d_3/2)}{I_{P2}} = 2690 \text{ psi} \quad \leftarrow$$

(b) ANGLE OF ROTATION OF END PLATE

$$\phi = \frac{T_1 L}{G I_{P1}} = \frac{T_2 L}{G I_{P2}} = 0.006180015 \text{ rad}$$

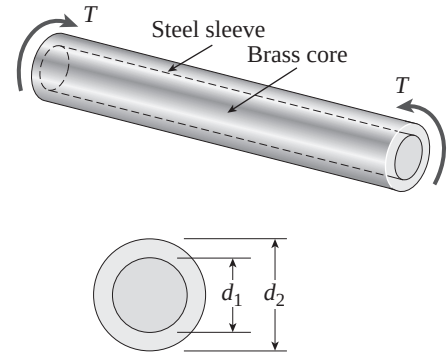
$$\phi = 0.354^\circ \quad \leftarrow$$

(c) TORSIONAL STIFFNESS

$$k_T = \frac{T}{\phi} = 809 \text{ k-in.} \quad \leftarrow$$

Problem 3.8-12 The composite shaft shown in the figure is manufactured by shrink-fitting a steel sleeve over a brass core so that the two parts act as a single solid bar in torsion. The outer diameters of the two parts are $d_1 = 40 \text{ mm}$ for the brass core and $d_2 = 50 \text{ mm}$ for the steel sleeve. The shear moduli of elasticity are $G_b = 36 \text{ GPa}$ for the brass and $G_s = 80 \text{ GPa}$ for the steel.

- Assuming that the allowable shear stresses in the brass and steel are $\tau_b = 48 \text{ MPa}$ and $\tau_s = 80 \text{ MPa}$, respectively, determine the maximum permissible torque $T_{\max}$ that may be applied to the shaft. [Hint: Use Eqs. (3-48a and b) to find the torques.]
- If the applied torque $T = 2500 \text{ kN}\cdot\text{m}$, find the required diameter d_2 so that allowable shear stress τ_s is reached in the steel.

**Probs. 3.8-12 and 3.8-13****Solution 3.8-12**

NUMERICAL DATA

$$(a) \quad d_1 = 40 \text{ mm} \quad d_2 = 50 \text{ mm} \quad G_b = 36 \text{ GPa} \quad G_s = 80 \text{ GPa} \quad \tau_b = 48 \text{ MPa} \quad \tau_s = 80 \text{ MPa}$$

$$I_{PB} = \frac{\pi}{32} d_1^4 \quad I_{PS} = \frac{\pi}{32} (d_2^4 - d_1^4)$$

FROM EQ. (3-48a, b)

$$T_B = T \left(\frac{G_B I_{pB}}{G_B I_{pB} + G_S I_{pS}} \right) \quad \text{and} \quad T_S = T \left(\frac{G_S I_{pS}}{G_B I_{pB} + G_S I_{pS}} \right)$$

And

$$T_{B\max} = \frac{2\tau_b I_{pB}}{d_1} \quad T_{S\max} = \frac{2\tau_s I_{pS}}{d_2}$$

Solving for $T_{\max}$ for either brass or steel

$$T_{\max B} = \frac{2\tau_b I_{pB}}{d_1} \left(\frac{G_B I_{pB} + G_S I_{pS}}{G_B I_{pB}} \right) = 2535.265 \text{ N}\cdot\text{m}$$

$$T_{\max S} = \frac{2\tau_s I_{pS}}{d_2} \left(\frac{G_B I_{pB} + G_S I_{pS}}{G_S I_{pS}} \right) = 1521.159 \text{ N}\cdot\text{m} < \text{controls so } \boxed{T_{\max} = 1521 \text{ N}\cdot\text{m}}$$

(b) Let $T = 2500 \text{ N}\cdot\text{m}$ so $T_{\max S} = T$

Expand I_{pS} and I_{pB} in above expression for $T_{\max S}$ then for required diameter d_2

$$\frac{2\tau_s \left[\frac{\pi}{32} (d_2^4 - d_1^4) \right]}{d_2} \left[\frac{G_B I_{pB} + G_S \left[\frac{\pi}{32} (d_2^4 - d_1^4) \right]}{G_S \left[\frac{\pi}{32} (d_2^4 - d_1^4) \right]} \right] = T \quad \text{Solving: } \begin{matrix} d_2 = 56.868 \text{ mm} \\ \boxed{d_2 = 56.9 \text{ mm}} \end{matrix}$$

Check

$$\text{Recompute: } I_{pS} = \frac{\pi}{32} (d_2^4 - d_1^4) = 7.778 \times 10^{-7} \text{ m}^4$$

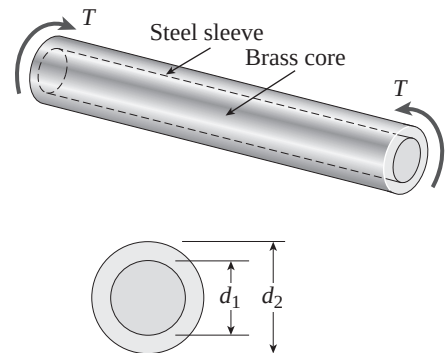
$$T_S = T \left(\frac{G_S I_{pS}}{G_B I_{pB} + G_S I_{pS}} \right) \quad \tau_s = \frac{T_S \left(\frac{d_2}{2} \right)}{I_{pS}} = 80 \text{ MPa} < \text{OK, at specified allowable value}$$

Find associated max. shear stress in brass core:

$$T_B = (T) \left(\frac{G_B I_{pB}}{G_B I_{pB} + G_S I_{pS}} \right) \quad \tau_b = \frac{T_B \left(\frac{d_1}{2} \right)}{I_{pB}} = 25.257 \text{ MPa} < \text{OK, less than allowable value of 48 MPa}$$

Problem 3.8-13 The composite shaft shown in the figure is manufactured by shrink-fitting a steel sleeve over a brass core so that the two parts act as a single solid bar in torsion. The outer diameters of the two parts are $d_1 = 1.6$ in. for the brass core and $d_2 = 2.0$ in. for the steel sleeve. The shear moduli of elasticity are $G_b = 5400$ ksi for the brass and $G_s = 12,000$ ksi for the steel.

- Assuming that the allowable shear stresses in the brass and steel are $\tau_b = 4500$ psi and $\tau_s = 7500$ psi, respectively, determine the maximum permissible torque $T_{\max}$ that may be applied to the shaft. [Hint: Use Eqs. (3-48a and b) to find the torques.]
- If the applied torque $T = 15$ kip-in., find the required diameter d_2 so that allowable shear stress τ_s is reached in the steel.



Solution 3.8-13

NUMERICAL DATA

(a) $d_1 = 1.6$ in. $d_2 = 2$ in. $G_B = 5400$ ksi $G_S = 12000$ ksi $\tau_b = 4500$ psi $\tau_s = 7500$ psi

$$I_{pB} = \frac{\pi}{32} d_1^4 \quad I_{pS} = \frac{\pi}{32} (d_2^4 - d_1^4)$$

FROM Eqs. (3-48a, and b)

$$T_B = T \left(\frac{G_B I_{pB}}{G_B I_{pB} + G_S I_{pS}} \right) \quad \text{and} \quad T_S = T \left(\frac{G_S I_{pS}}{G_B I_{pB} + G_S I_{pS}} \right)$$

And

$$T_{B\max} = \frac{2\tau_b I_{pB}}{d_1} \quad T_{S\max} = \frac{2\tau_s I_{pS}}{d_2}$$

Solving for $T_{\max}$ for either brass or steel:

$$T_{\max B} = \frac{2\tau_b I_{pB}}{d_1} \left(\frac{G_B I_{pB} + G_S I_{pS}}{G_B I_{pB}} \right) = 15.212 \text{ k-in.}$$

$$T_{\max S} = \frac{2\tau_s I_{pS}}{d_2} \left(\frac{G_B I_{pB} + G_S I_{pS}}{G_S I_{pS}} \right) = 9.127 \text{ k-in.} < \text{controls, so } \boxed{T_{\max} = 9.13 \text{ k-in.}}$$

(b) Let $T = 15$ k-in. so $T_{\max S} = T$ Expand I_{pS} and I_{pB} in above expression for $T_{\max S}$ then for required diameter d_2 :

$$\frac{2\tau_s \left[\frac{\pi}{32} (d_2^4 - d_1^4) \right]}{d_2} \left[\frac{G_B I_{pB} + G_S \left[\frac{\pi}{32} (d_2^4 - d_1^4) \right]}{G_S \left[\frac{\pi}{32} (d_2^4 - d_1^4) \right]} \right] = T \quad \text{Solving: } d_2 = 2.275 \text{ in.}$$

$$\boxed{d_2 = 2.27 \text{ in.}}$$

Check

$$\text{Recomputed: } I_{pS} = \frac{\pi}{32} (d_2^4 - d_1^4) = 9.574 \times 10^{-5} \text{ ft}^4$$

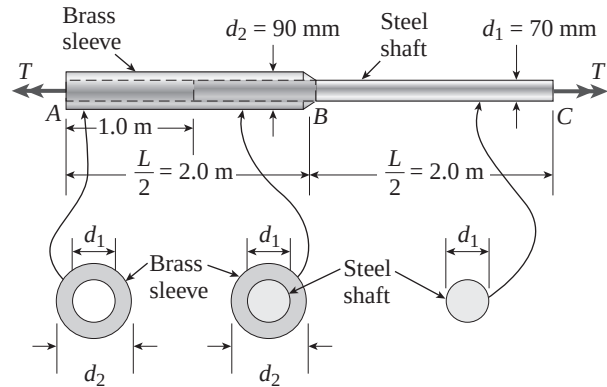
$$T_S = T \left(\frac{G_S I_{pS}}{G_B I_{pB} + G_S I_{pS}} \right) \quad \tau_s = \frac{T_S \left(\frac{d_2}{2} \right)}{I_{pS}} = 7500 \text{ psi} < \text{OK, at specified allowable value}$$

Find associated max. shear stress in brass core:

$$T_B = (T) \left(\frac{G_B I_{pB}}{G_B I_{pB} + G_S I_{pS}} \right) \quad \frac{T_B \left(\frac{d_1}{2} \right)}{I_{pB}} = 2374 \text{ psi} < \text{OK, less than allowable value of 4500 psi}$$

Problem 3.8-14 A steel shaft ($G_s = 80$ GPa) of total length $L = 3.0$ m is encased for one-third of its length by a brass sleeve ($G_b = 40$ GPa) that is securely bonded to the steel (see figure). The outer diameters of the shaft and sleeve are $d_1 = 70$ mm and $d_2 = 90$ mm, respectively.

- (a) Determine the allowable torque T_1 that may be applied to the ends of the shaft if the angle of twist between the ends is limited to 8.0° .
- (b) Determine the allowable torque T_2 if the shear stress in the brass is limited to $\tau_b = 70$ MPa.
- (c) Determine the allowable torque T_3 if the shear stress in the steel is limited to $\tau_s = 110$ MPa.
- (d) What is the maximum allowable torque $T_{\max}$ if all three of the preceding conditions must be satisfied?

**Solution 3.8-14**

- (a) ALLOWABLE TORQUE T_1 BASED ON TWIST AT ENDS OF 8 DEGREES
first find torques in steel (T_s) & brass (T_b) in segment in which they are joined – 1 degree statically indeterminate; use T_s as the internal redundant; see Eq. 3-48a and b

$$T_s = T_1 \left(\frac{G_s I_{Ps}}{G_s I_{Ps} + G_b I_{Pb}} \right) \quad \text{statics} > \quad T_b = T_1 - T_s \quad T_b = T_1 \left(\frac{G_b I_{Pb}}{G_s I_{Ps} + G_b I_{Pb}} \right)$$

Now find twist of three segments:

$$\phi = \frac{T_1 \frac{L}{4}}{G_b I_{Pb}} + \frac{T_s \frac{L}{4}}{G_s I_{Ps}} + \frac{T_1 \frac{L}{2}}{G_s I_{Ps}} \quad \text{for middle term, brass sleeve and steel shaft twist the same so could use } T_b(L/4)/(G_b I_{Pb}) \text{ instead}$$

$$\phi_a = \frac{T_1 \frac{L}{4}}{G_b I_{Pb}} + \frac{T_1 \left(\frac{G_s I_{Ps}}{G_s I_{Ps} + G_b I_{Pb}} \right) \frac{L}{4}}{G_s I_{Ps}} + \frac{T_1 \frac{L}{2}}{G_s I_{Ps}} \quad \phi_a = \frac{T_1 \frac{L}{4}}{G_b I_{Pb}} + \frac{T_1 \frac{L}{4}}{G_s I_{Ps} + G_b I_{Pb}} + \frac{T_1 \frac{L}{2}}{G_s I_{Ps}}$$

$$\phi_a = T_1 \frac{L}{4} \left(\frac{1}{G_b I_{Pb}} + \frac{1}{G_s I_{Ps} + G_b I_{Pb}} + \frac{2}{G_s I_{Ps}} \right)$$

$$\phi_a = T_1 \frac{L}{4} \left[\frac{G_s^2 I_{Ps}^2 + 4 G_b I_{Pb} G_s I_{Ps} + 2 G_b^2 I_{Pb}^2}{G_b I_{Pb} (G_s I_{Ps} + G_b I_{Pb}) G_s I_{Ps}} \right]$$

$$T_{1\text{allow}} = \frac{4 \phi_a}{L} \left[\frac{G_b I_{Pb} (G_s I_{Ps} + G_b I_{Pb}) G_s I_{Ps}}{G_s^2 I_{Ps}^2 + 4 G_b I_{Pb} G_s I_{Ps} + 2 G_b^2 I_{Pb}^2} \right]$$

NUMERICAL VALUES $\phi_a = 8 \left(\frac{\pi}{180} \right) \text{ rad} \quad G_s = 80 \text{ GPa} \quad G_b = 40 \text{ GPa} \quad L = 4.0 \text{ m}$

$$d_1 = 70 \text{ mm} \quad d_2 = 90 \text{ mm}$$

$$I_{Ps} = \frac{\pi}{32} \left(\frac{d_1}{1000} \right)^4 \quad I_{Ps} = 2.357 \times 10^{-6} \text{ m}^4 \quad I_{Pb} = \frac{\pi}{32} \frac{(d_2^4 - d_1^4)}{(10^3)^4} \quad I_{Pb} = 4.084 \times 10^{-6} \text{ m}^4$$

$$T_{1\text{allow}} = \frac{4 \phi_a}{L} \left[\frac{G_b I_{Pb} (G_s I_{Ps} + G_b I_{Pb}) G_s I_{Ps}}{G_s^2 I_{Ps}^2 + 4 G_b I_{Pb} G_s I_{Ps} + 2 G_b^2 I_{Pb}^2} \right] \quad T_{1\text{allow}} (10^9) = 7135.197 \text{ N}\cdot\text{m}$$

$$\boxed{T_{1\text{allow}} = 7.14 \text{ kN}\cdot\text{m}}$$

- (b) ALLOWABLE TORQUE
- T_1
- BASED ON ALLOWABLE SHEAR STRESS IN BRASS,
- τ_b
- $\tau_b = 70$
- MPa

FIRST CHECK HOLLOW SEGMENT 1 (BRASS SLEEVE ONLY)

$$\tau = \frac{T_2 \frac{d_2}{2}}{I_{pb}} \quad T_{2\text{allow}} = \frac{2\tau_b I_{pb}}{d_2} \quad T_{2\text{allow}}(10^9) = 6.353 \times 10^3 \text{ N}\cdot\text{m}$$

$$\boxed{T_{2\text{allow}} = 6.35 \text{ kN}\cdot\text{m}} < \text{controls over } T_2 \text{ below}$$

ALSO CHECK SEGMENT 2 WITH BRASS SLEEVE OVER STEEL SHAFT

$$\tau = \frac{T_b \frac{d_2}{2}}{I_{pb}} \quad \text{where from statically indeterminate analysis above} \quad T_b = T_2 \left(\frac{G_b I_{pb}}{G_s I_{ps} + G_b I_{pb}} \right)$$

$$T_{2\text{allow}} = \frac{2\tau_b (G_s I_{ps} + G_b I_{pb})}{d_2 G_b} \quad T_{2\text{allow}}(10^9) = 1.369 \times 10^4 \text{ N}\cdot\text{m}$$

$$T_{2\text{allow}} = 13.69 \text{ kN}\cdot\text{m}$$

- (c) ALLOWABLE TORQUE
- T_1
- BASED ON ALLOWABLE SHEAR STRESS IN STEEL,
- τ_s
- $\tau_s = 110$
- MPa

FIRST CHECK SEGMENT 2 WITH BRASS SLEEVE OVER STEEL SHAFT

$$\tau = \frac{T_s \frac{d_1}{2}}{I_{ps}} \quad \text{where from statically indeterminate analysis above} \quad T_s = T_3 \left(\frac{G_s I_{ps}}{G_s I_{ps} + G_b I_{pb}} \right)$$

$$T_{3\text{allow}} = \frac{2\tau_s (G_s I_{ps} + G_b I_{pb})}{d_1 G_s} \quad T_{3\text{allow}}(10^9) = 1.383 \times 10^4 \text{ N}\cdot\text{m}$$

$$T_{3\text{allow}} = 13.83 \text{ kN}\cdot\text{m}$$

ALSO CHECK SEGMENT 3 WITH STEEL SHAFT ALONE

$$\tau = \frac{T_3 \frac{d_1}{2}}{I_{ps}} \quad T_{3\text{allow}} = \frac{2\tau_s I_{ps}}{d_1} \quad T_{3\text{allow}}(10^9) = 7.408 \times 10^3 \text{ N}\cdot\text{m}$$

$$\boxed{T_{3\text{allow}} = 7.41 \text{ kN}\cdot\text{m}} < \text{controls over } T_3 \text{ above}$$

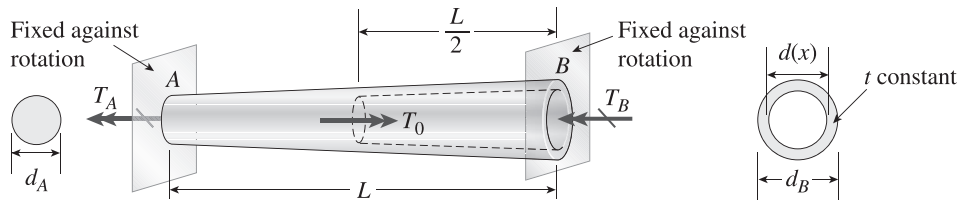
- (d)
- T_{MAX}
- IF ALL PRECEDING CONDITIONS MUST BE CONSIDERED

From part (b):

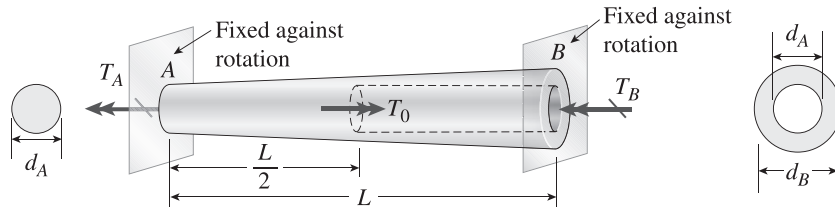
$$\boxed{T_{\text{max}} = 6.35 \text{ kN}\cdot\text{m}} < \text{max. shear stress in hollow brass sleeve in segment 1 controls overall}$$

Problem 3.8-15 A uniformly tapered aluminum-alloy tube AB of circular cross section and length L is fixed against rotation at A and B , as shown in the figure. The outside diameters at the ends are d_A and $d_B = 2d_A$. A hollow section of length $L/2$ and constant thickness $t = d_A/10$ is cast into the tube and extends from B halfway toward A . Torque T_0 is applied at $L/2$.

- (a) Find the reactive torques at the supports, T_A and T_B . Use numerical values as follows: $d_A = 2.5$ in., $L = 48$ in., $G = 3.9 \times 10^6$ psi, $T_0 = 40,000$ in.-lb.
 (b) Repeat (a) if the hollow section has constant diameter d_A .



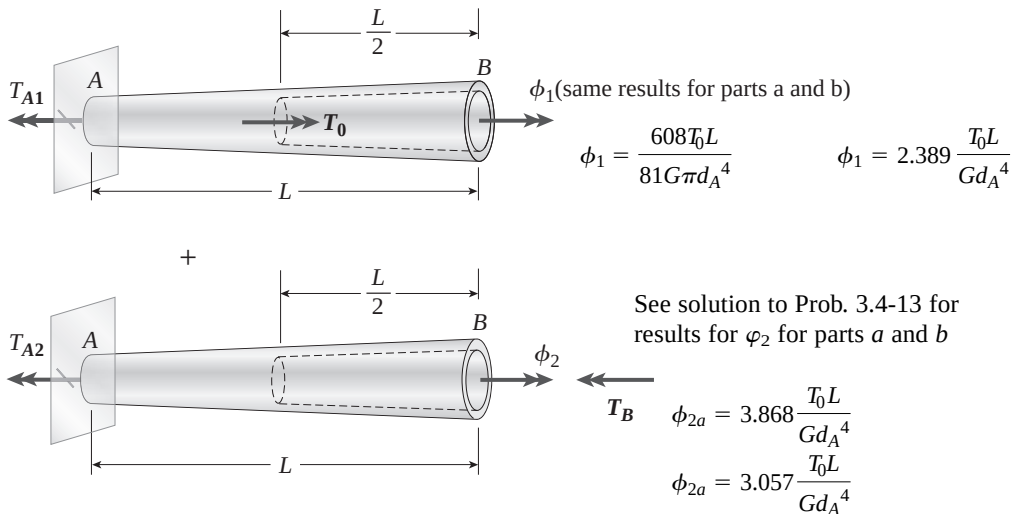
(a)



(b)

Solution 3.8-15

Solution approach-superposition: select T_B as the redundant (1° SI)



- (a) REACTIVE TORQUES, T_A AND T_B , FOR CASE OF CONSTANT THICKNESS OF HOLLOW SECTION OF TUBE

compatibility equation: $\phi_1 - \phi_2 = 0$

$T_B = \text{redundant}$

$T_0 = 40000 \text{ in.-lb}$

$$T_B = \left(\frac{608T_0L}{81G\pi d_A^4} \right) \left(\frac{Gd_A^4}{3.86804L} \right)$$

$$T_B = 1.94056 \frac{T_0}{\pi} \quad T_B = 24708 \text{ in.-lb} \quad \leftarrow$$

$$T_A = T_0 - T_B \quad T_A = 15292 \text{ in.-lb} \quad \leftarrow$$

$$T_A + T_B = 40,000 \text{ in.-lb (check)}$$

- (b) REACTIVE TORQUES, T_A AND T_B , FOR CASE OF CONSTANT DIAMETER OF HOLE

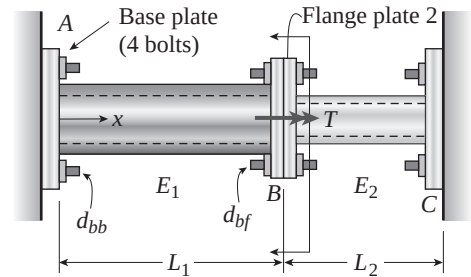
$$T_B = \left(\frac{608T_0L}{81G\pi d_A^4} \right) \left(\frac{Gd_A^4}{3.05676L} \right)$$

$$T_B = 2.45560 \frac{T_0}{\pi} \quad T_B = 31266 \text{ in.-lb} \quad \leftarrow$$

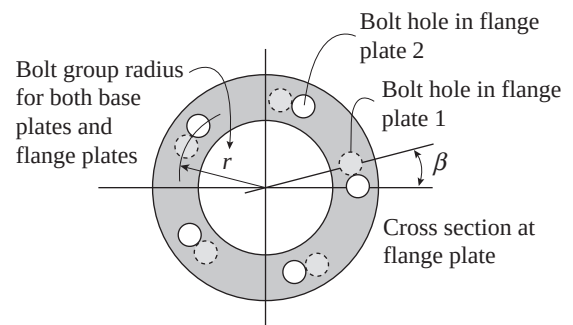
$$T_A = T_0 - T_B \quad T_A = 8734 \text{ in.-lb} \quad \leftarrow$$

$$T_A + T_B = 40,000 \text{ in.-lb (check)}$$

Problem 3.8-16 Two pipes ($L_1 = 2.5 \text{ m}$ and $L_2 = 1.5 \text{ m}$) are joined at B by flange plates (thickness $t_f = 14 \text{ mm}$) with five bolts ($d_{bf} = 13 \text{ mm}$) arranged in a circular pattern (see figure). Also, each pipe segment is attached to a wall (at A and C , see figure) using a base plate ($t_b = 15 \text{ mm}$) and four bolts ($d_{bb} = 16 \text{ mm}$). All bolts are tightened until just snug. Assume $E_1 = 110 \text{ GPa}$, $E_2 = 73 \text{ GPa}$, $\nu_1 = 0.33$, and $\nu_2 = 0.25$. Neglect the self-weight of the pipes, and assume the pipes are in a stress-free state initially. The cross-sectional areas of the pipes are $A_1 = 1500 \text{ mm}^2$ and $A_2 = (3/5)A_1$. The outer diameter of Pipe 1 is 60 mm . The outer diameter of Pipe 2 is equal to the inner diameter of Pipe 1. The bolt radius $r = 64 \text{ mm}$ for both base and flange plates.



- If torque T is applied at $x = L_1$, find an expression for reactive torques R_1 and R_2 in terms of T .
- Find the maximum load variable T (i.e., $T_{\max}$) if allowable torsional stress in the two pipes is $\tau_{\text{allow}} = 65 \text{ MPa}$.
- Draw torsional moment (TMD) and torsional displacement (TDD) diagrams. Label all key ordinates. What is $\phi_{\max}$?
- Find $T_{\max}$ if allowable shear and bearing stresses in the base plate and flange bolts cannot be exceeded. Assume allowable stresses in shear and bearing for all bolts are $\tau_{\text{allow}} = 45 \text{ MPa}$ and $\sigma_{\text{allow}} = 90 \text{ MPa}$.
- Remove torque T at $x = L_1$. Now assume the flange-plate bolt holes are misaligned by some angle β (see figure). Find the expressions for reactive torques R_1 and R_2 if the pipes are twisted to align the flange-plate bolt holes, bolts are then inserted, and the pipes released.
- What is the maximum permissible misalignment angle $\beta_{\max}$ if allowable stresses in shear and bearing for all bolts (from part (d) above) are not to be exceeded?



Solution 3.8-16

NUMERICAL DATA AND CROSS SECTIONAL AND MATERIAL PROPERTIES

$$A_1 = 1500 \text{ mm}^2 \quad A_2 = \frac{3}{5} A_1 = 900 \text{ mm}^2 \quad d_{o1} = 60 \text{ mm}$$

$$d_{i1} = \sqrt{\frac{-4}{\pi} A_1 + d_{o1}^2} = 41.111 \text{ mm} \quad \frac{\pi}{4} (d_{o1}^2 - d_{i1}^2) = 1500 \text{ mm}^2$$

$$d_{o2} = d_{i1} = 41.111 \text{ mm} \quad d_{i2} = \sqrt{d_{o2}^2 - \left(\frac{4}{\pi}\right) A_2} = 23.329 \text{ mm}$$

$$\frac{\pi}{4} (d_{o2}^2 - d_{i2}^2) = 900 \text{ mm}^2$$

$$t_1 = \frac{d_{o1} - d_{i1}}{2} = 9.444 \text{ mm} \quad t_2 = \frac{d_{o2} - d_{i2}}{2} \quad t_2 = 8.891 \text{ mm} \quad L_1 = 2.5 \text{ m} \quad L_2 = 1.5 \text{ m}$$

$$I_{p1} = \frac{\pi}{32} (d_{o1}^4 - d_{i1}^4) = 9.919 \times 10^{-7} \text{ m}^4 \quad r = 64 \text{ mm}$$

$$I_{p2} = \frac{\pi}{32} (d_{o2}^4 - d_{i2}^4) = 2.514 \times 10^{-7} \text{ m}^4 \quad t_f = 14 \text{ mm} \quad d_{bf} = 13 \text{ mm} \quad t_b = 15 \text{ mm} \quad d_{bb} = 16 \text{ mm}$$

$$E_1 = 110 \text{ GPa} \quad E_2 = 73 \text{ GPa} \quad \nu_1 = 0.33 \quad \nu_2 = 0.25$$

$$\frac{I_{p1}}{I_{p2}} = 3.946$$

$$G_1 = \frac{E_1}{2(1 + \nu_1)} = 41.4 \text{ GPa} \quad G_2 = \frac{E_2}{2(1 + \nu_2)} = 29.2 \text{ GPa}$$

- (a) IF TORQUE T IS APPLIED AT $x = L_1$, FIND AN EXPRESSION FOR REACTIVE TORQUES R_1 AND R_2 IN TERMS OF APPLIED T FROM ONE-DEGREE STATICALLY-INDETERMINATE ANALYSIS WITH REACTION AT RIGHT SUPPORT R_2 AS THE REDUNDANT:

$$\phi_a = T f_{T1} \quad f_{T1} = \frac{L_1}{G_1 I_{p1}} \quad \phi_b = R_2 (f_{T1} + f_{T2}) \quad f_{T2} = \frac{L_2}{G_2 I_{p2}} \quad < \text{using } f_T = \text{segment torsional flexibility}$$

$$R_2 = \frac{-T f_{T1}}{f_{T1} + f_{T2}} = -0.23 T \quad R_1 = -T - R_2 = -0.77 T \quad \frac{f_{T1}}{(f_{T1} + f_{T2})} = 0.23 \quad \frac{f_{T2}}{(f_{T1} + f_{T2})} = 0.77$$

$$\boxed{R_1 = -0.77 T} \quad \boxed{R_2 = -0.23 T}$$

- (b) FIND MAX. LOAD VARIABLE T_{MAX} IF ALLOWABLE TORSIONAL STRESS IN THE TWO PIPES IS

$$\tau_{\text{allow}} = 65 \text{ MPa} \quad \tau_{\text{allow}} = 65 \text{ MPa}$$

$$T_{\text{max1}} = \frac{\tau_{\text{allow}} I_{p1}}{\frac{d_{o1}}{2}} = 2.149 \text{ kN}\cdot\text{m} \quad T_{\text{max2}} = \frac{\tau_{\text{allow}} I_{p2}}{\frac{d_{o2}}{2}} = 0.795 \text{ kN}\cdot\text{m}$$

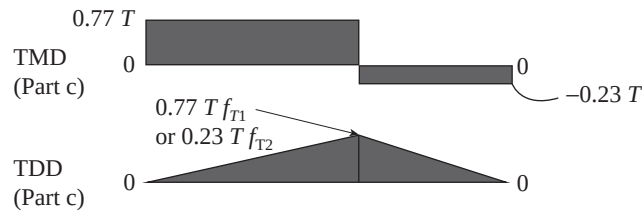
Now, must relate these T_{max} values above back to TMD to find max. APPLIED (load variable) T (i.e., see reaction moment expressions above)

$$\text{ON SEGMENT 1: } T_{\text{max}} = \frac{T_{\text{max1}}}{0.77} = 2.791 \text{ kN}\cdot\text{m} \quad \text{ON SEGMENT 2: } T_{\text{max}} = \frac{T_{\text{max2}}}{0.23} = 3.456 \text{ kN}\cdot\text{m}$$

^ controls

$$\boxed{T_{\text{max}} = 2.79 \text{ kN}\cdot\text{m}}$$

- (c) DRAW TORSIONAL MOMENT (TMD) AND TORSIONAL DISPLACEMENT (TDD) DIAGRAMS; LABEL ALL KEY ORDINATES; WHAT IS $\phi_{\max}$?



$$\phi_{\max} \frac{f_{T1} f_{T2}}{f_{T1} + f_{T2}} T_{\max} = 7.51^\circ$$

or

$$\phi_{\max} = \frac{L_1 L_2 T_{\max}}{G_1 I_{p1} L_2 + G_2 + I_{p2} L_1} = 7.51^\circ$$

- (d) FIND $T_{\max}$ IF ALLOWABLE SHEAR AND BEARING STRESSES IN THE BASE PLATE AND FLANGE PLATE BOLTS CANNOT BE EXCEEDED; ASSUME ALLOWABLE STRESSES IN SHEAR AND BEARING FOR ALL BOLTS ARE $\tau_{\text{allow}} = 45 \text{ MPa}$ and $\sigma_{\text{ballow}} = 90 \text{ MPa}$

$$\tau_{\text{allow}} = 45 \text{ MPa} \quad \sigma_{\text{ballow}} = 90 \text{ MPa} \quad t_f = 14 \text{ mm} \quad t_b = 15 \text{ mm} \quad d_{bf} = 13 \text{ mm} \quad d_{bb} = 16 \text{ mm}$$

$$A_{bb} = \frac{\pi}{4} d_{bb}^2 = 201.062 \text{ mm}^2 \quad A_{bf} = \frac{\pi}{4} d_{bf}^2 = 132.732 \text{ mm}^2 \quad \begin{matrix} \wedge \text{diameter} \\ \text{of flange} \\ \text{plate bolts} \end{matrix} \quad \begin{matrix} \wedge \text{diameter of} \\ \text{base plate} \\ \text{bolts} \end{matrix}$$

Bolt radius in both base and flange plates: $r = 64 \text{ mm}$

Number of base plate bolts: $n_b = 4$ Number of flange plate bolts: $n_f = 5$

Replace torque by n forces F_b in the bolts at distributed r (radius) from center of shaft (shaft axis).

$$\text{TORQUE REACTION } R_1 \text{ CONTROLS (0.77 } T\text{): AT Base plate } R_1 = -0.77 T \quad \frac{f_{T2}}{f_{T1} + f_{T2}} = 0.77$$

FIRST, CHECK SHEAR STRESS IN BASE AND FLANGE PLATE BOLTS; MAXIMUM FORCE IN EACH BOLT = $T_{\text{allow}} \times \text{AREA OF BOLT}$

$$\frac{f_{T2}}{f_{T1} + f_{T2}} (T_{\max}) = n_b r (\tau_{\text{allow}} A_{bb}) \quad T_{\max1} = \frac{n_b r (\tau_{\text{allow}} A_{bb})}{\frac{f_{T2}}{f_{T1} + f_{T2}}} = 3.007 \text{ kN}\cdot\text{m}$$

$$T_{\max2} = \frac{n_f r (\tau_{\text{allow}} A_{bf})}{\frac{f_{T2}}{f_{T1} + f_{T2}}} = 2.481 \text{ kN}\cdot\text{m} < \text{shear in flange plate bolts controls over that in base plate bolts}$$

THEN, CHECK BEARING STRESS IN BASE AND FLANGE PLATE BOLTS; MAXIMUM FORCE IN EACH BOLT = $\sigma_{\text{ballow}} \times A_{\text{brg-bolt}}$

$$A_{\text{brgBasePl}} = d_{bb} t_b = 240 \text{ mm}^2$$

$\wedge$ bearing stress area between base plate and each bolt

$$A_{\text{brgFlangePl}} = d_{bf} t_f = 182 \text{ mm}^2$$

$\wedge$ bearing stress area between flange plate and each bolt

$$T_{\max3} = \frac{n_b r (\sigma_{\text{ballow}} A_{\text{brgBasePl}})}{\frac{f_{T2}}{f_{T1} + f_{T2}}} = 7.179 \text{ kN}\cdot\text{m}$$

$$T_{\max4} = \frac{n_f r (\sigma_{\text{ballow}} A_{\text{brgFlangePl}})}{\frac{f_{T2}}{f_{T1} + f_{T2}}} = 6.805 \text{ kN}\cdot\text{m}$$

Both $T_{\max3}$ and $T_{\max4}$ exceed $T_{\max1}$ and $T_{\max2}$ above (based on bolt shear) so

$$T_{\max} = T_{\max2} = 2.48 \text{ kN}\cdot\text{m}$$

- (e) REMOVE TORQUE T AT $x = L_1$; NOW ASSUME THE FLANGE PLATE BOLT HOLES ARE MISALIGNED BY SOME ANGLE β (SEE FIGURE); FIND EXPRESSIONS FOR REACTIVE TORQUES R_1 AND R_2 IF THE PIPES ARE TWISTED TO ALIGN THE FLANGE PLATE BOLT HOLES, BOLTS ARE THEN INSERTED AND THE PIPES RELEASED

Use superposition analysis with R_2 as the redundant. Twist due to misfit = $\phi_1 = -\beta$ and twist due to redundant R_2 is ϕ_2 where

$$\phi_2 = R_2(f_{T1} + f_{T2}) \quad \text{and the compatibility equation is} \quad \phi_1 + \phi_2 = 0$$

$$\text{Solving for } R_2 \text{ gives } R_2 = \frac{\beta}{f_{T1} + f_{T2}} \quad \text{then statics gives } R_1 = -R_2$$

$$\text{where } f_{T1} = \frac{L_1}{G_1 I_{P1}} \quad \text{and} \quad f_{T2} = \frac{L_2}{G_2 I_{P2}}$$

- (f) WHAT IS THE MAXIMUM PERMISSIBLE MISALIGNMENT ANGLE $\beta_{\max}$ IF ALLOWABLE STRESSES IN SHEAR AND BEARING FOR ALL BOLTS (FROM PART (d) ABOVE) ARE NOT TO BE EXCEEDED?

Maximum reaction torque = $n \times r \times (\text{allow stress} \times \text{bolt or bearing area})$, so $\beta_{\max} = (f_{T1} + f_{T2}) \times [n \times r \times (\text{allow stress} \times \text{bolt or bearing area})]$.

CHECK SHEAR IN BOTH BASE PLATE AND FLANGE PLATE BOLTS

$$\beta_{\max1} = (f_{T1} + f_{T2})[n_b r (\tau_{\text{allow}} A_{bb})] = 0.615$$

$$\beta_{\max2} = (f_{T1} + f_{T2})[n_f r (\tau_{\text{allow}} A_{bf})] = 0.507 \quad < \text{shear in flange plate bolts controls; associated reactive torques at each end:}$$

$$\frac{\beta_{\max2}}{f_{T1} + f_{T2}} = 1.911 \text{ kN}\cdot\text{m} \quad \frac{\beta_{\max2}}{L_1 + L_2} = 7.264^\circ/\text{m}$$

CHECK BEARING STRESS IN BOTH BASE PLATE AND FLANGE PLATE BOLTS

$$\beta_{\max3} = (f_{T1} + f_{T2})[n_b r (\sigma_{\text{ballow}} A_{\text{brgBaseP1}})] = 1.467$$

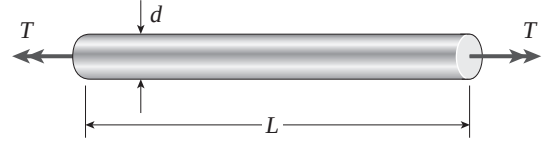
$$\beta_{\max4} = (f_{T1} + f_{T2})[n_f r (\sigma_{\text{ballow}} A_{\text{brgFlangeP1}})] = 1.391$$

$$\beta_{\max} = \beta_{\max2} = 29.1^\circ$$

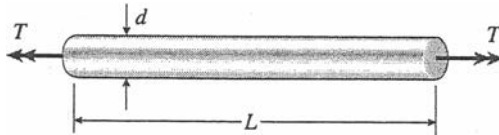
Strain Energy in Torsion

Problem 3.9-1 A solid circular bar of steel ($G = 11.4 \times 10^6$ psi) with length $L = 30$ in. and diameter $d = 1.75$ in. is subjected to pure torsion by torques T acting at the ends (see figure).

- Calculate the amount of strain energy U stored in the bar when the maximum shear stress is 4500 psi.
- From the strain energy, calculate the angle of twist ϕ (in degrees).



Solution 3.9-1 Steel bar



$$G = 11.4 \times 10^6 \text{ psi}$$

$$L = 30 \text{ in.}$$

$$d = 1.75 \text{ in.}$$

$$\tau_{\max} = 4500 \text{ psi}$$

$$\tau_{\max} = \frac{16 T}{\pi d^3} \quad T = \frac{\pi d^3 \tau_{\max}}{16}$$

$$I_P = \frac{\pi d^4}{32} \quad (\text{Eq. 1})$$

(a) STRAIN ENERGY

$$U = \frac{T^2 L}{2 G I_P} = \left(\frac{\pi d^3 \tau_{\max}}{16} \right)^2 \left(\frac{L}{2 G} \right) \left(\frac{32}{\pi d^4} \right)$$

$$= \frac{\pi d^2 L \tau_{\max}^2}{16 G} \quad (\text{Eq. 2})$$

Substitute numerical values:

$$U = 32.0 \text{ in.-lb} \quad \leftarrow$$

(b) ANGLE OF TWIST

$$U = \frac{T \phi}{2} \quad \phi = \frac{2U}{T}$$

Substitute for T and U from Eqs. (1) and (2):

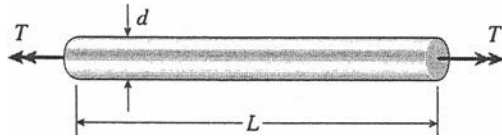
$$\phi = \frac{2L \tau_{\max}}{G d} \quad (\text{Eq. 3})$$

Substitute numerical values:

$$\phi = 0.013534 \text{ rad} = 0.775^\circ \quad \leftarrow$$

Problem 3.9-2 A solid circular bar of copper ($G = 45$ GPa) with length $L = 0.75$ m and diameter $d = 40$ mm is subjected to pure torsion by torques T acting at the ends (see figure).

- Calculate the amount of strain energy U stored in the bar when the maximum shear stress is 32 MPa.
- From the strain energy, calculate the angle of twist ϕ (in degrees)

Solution 3.9-2 Copper bar

$$G = 45 \text{ GPa}$$

$$L = 0.75 \text{ m}$$

$$d = 40 \text{ mm}$$

$$\tau_{\max} = 32 \text{ MPa}$$

$$\tau_{\max} = \frac{16T}{\pi d^3} \quad T = \frac{\pi d^3 \tau_{\max}}{16}$$

$$I_P = \frac{\pi d^4}{32} \quad (\text{Eq. 1})$$

(a) STRAIN ENERGY

$$U = \frac{T^2 L}{2GI_P} = \left(\frac{\pi d^3 \tau_{\max}}{16} \right)^2 \left(\frac{L}{2G} \right) \left(\frac{32}{\pi d^4} \right)$$

$$= \frac{\pi d^2 L \tau_{\max}^2}{16G} \quad (\text{Eq. 2})$$

Substitute numerical values:

$$U = 5.36 \text{ J} \quad \leftarrow$$

(b) ANGLE OF TWIST

$$U = \frac{T\phi}{2} \quad \phi = \frac{2U}{T}$$

Substitute for T and U from Eqs. (1) and (2):

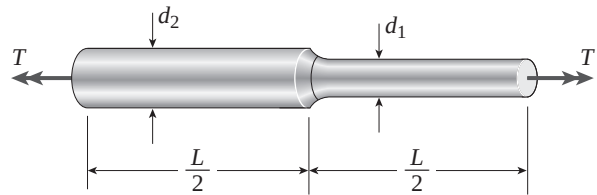
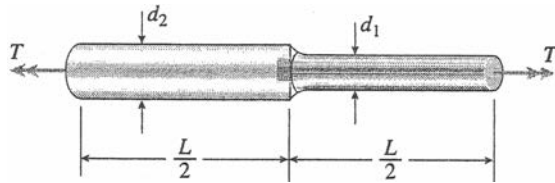
$$\phi = \frac{2L\tau_{\max}}{Gd} \quad (\text{Eq. 3})$$

Substitute numerical values:

$$\phi = 0.026667 \text{ rad} = 1.53^\circ \quad \leftarrow$$

Problem 3.9-3 A stepped shaft of solid circular cross sections (see figure) has length $L = 45 \text{ in.}$, diameter $d_2 = 1.2 \text{ in.}$, and diameter $d_1 = 1.0 \text{ in.}$ The material is brass with $G = 5.6 \times 10^6 \text{ psi}$.

Determine the strain energy U of the shaft if the angle of twist is 3.0° .

**Solution 3.9-3 Stepped shaft**

$$d_1 = 1.0 \text{ in.}$$

$$d_2 = 1.2 \text{ in.}$$

$$L = 45 \text{ in.}$$

$$G = 5.6 \times 10^6 \text{ psi (brass)}$$

$$\phi = 3.0^\circ = 0.0523599 \text{ rad}$$

STRAIN ENERGY

$$U = \sum \frac{T^2 L}{2GI_P} = \frac{16 T^2 (L/2)}{\pi G d_2^4} + \frac{16 T^2 (L/2)}{\pi G d_1^4}$$

$$= \frac{8T^2 L}{\pi G} \left(\frac{1}{d_2^4} + \frac{1}{d_1^4} \right) \quad (\text{Eq. 1})$$

$$\text{Also, } U = \frac{T\phi}{2} \quad (\text{Eq. 2})$$

Equate U from Eqs. (1) and (2) and solve for T :

$$T = \frac{\pi G d_1^4 d_2^4 \phi}{16L(d_1^4 + d_2^4)}$$

$$U = \frac{T\phi}{2} = \frac{\pi G \phi^2}{32L} \left(\frac{d_1^4 d_2^4}{d_1^4 + d_2^4} \right) \quad \phi \text{ in radians}$$

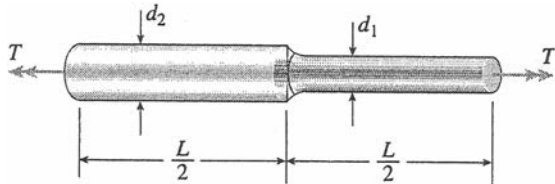
SUBSTITUTE NUMERICAL VALUES:

$$U = 22.6 \text{ in.-lb} \quad \leftarrow$$

Problem 3.9-4 A stepped shaft of solid circular cross sections (see figure) has length $L = 0.80$ m, diameter $d_2 = 40$ mm, and diameter $d_1 = 30$ mm. The material is steel with $G = 80$ GPa.

Determine the strain energy U of the shaft if the angle of twist is 1.0° .

Solution 3.9-4 Stepped shaft



$$\begin{aligned} d_1 &= 30 \text{ mm} & d_2 &= 40 \text{ mm} \\ L &= 0.80 \text{ m} & G &= 80 \text{ GPa (steel)} \\ \phi &= 1.0^\circ = 0.0174533 \text{ rad} \end{aligned}$$

STRAIN ENERGY

$$\begin{aligned} U &= \sum \frac{T^2 L}{2GI_P} = \frac{16T^2(L/2)}{\pi G d_2^4} + \frac{16T^2(L/2)}{\pi G d_1^4} \\ &= \frac{8T^2 L}{\pi G} \left(\frac{1}{d_2^4} + \frac{1}{d_1^4} \right) \end{aligned} \quad (\text{Eq. 1})$$

$$\text{Also, } U = \frac{T\phi}{2} \quad (\text{Eq. 2})$$

Equate U from Eqs. (1) and (2) and solve for T :

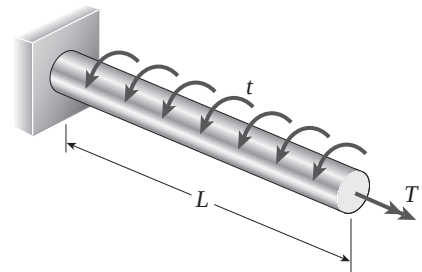
$$\begin{aligned} T &= \frac{\pi G d_1^4 d_2^4 \phi}{16L(d_1^4 + d_2^4)} \\ U &= \frac{T\phi}{2} = \frac{\pi G \phi^2}{32L} \left(\frac{d_1^4 d_2^4}{d_1^4 + d_2^4} \right) \quad \phi \text{ in radians} \end{aligned}$$

SUBSTITUTE NUMERICAL VALUES:

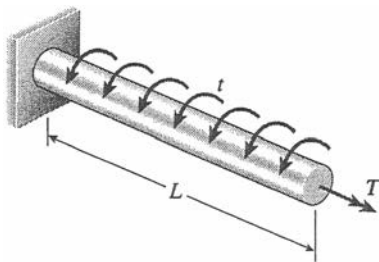
$$U = 1.84 \text{ J} \quad \leftarrow$$

Problem 3.9-5 A cantilever bar of circular cross section and length L is fixed at one end and free at the other (see figure). The bar is loaded by a torque T at the free end and by a distributed torque of constant intensity t per unit distance along the length of the bar.

- What is the strain energy U_1 of the bar when the load T acts alone?
- What is the strain energy U_2 when the load t acts alone?
- What is the strain energy U_3 when both loads act simultaneously?



Solution 3.9-5 Cantilever bar with distributed torque



G = shear modulus

I_P = polar moment of inertia

T = torque acting at free end

t = torque per unit distance

(a) LOAD T ACTS ALONE (Eq. 3-55a)

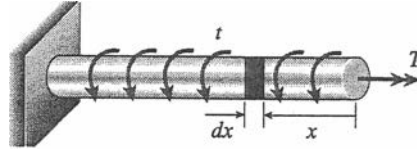
$$U_1 = \frac{T^2 L}{2GI_P} \quad \leftarrow$$

(b) LOAD t ACTS ALONE

From Eq. (3-64) of Example 3-11:

$$U_2 = \frac{t^2 L^3}{6GI_P} \quad \leftarrow$$

(c) BOTH LOADS ACT SIMULTANEOUSLY

At distance x from the free end:

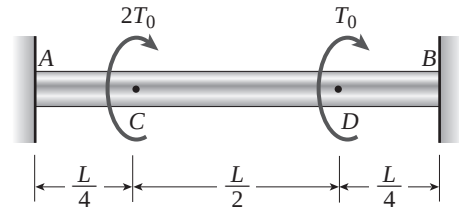
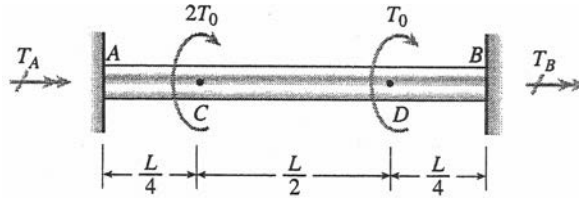
$$T(x) = T + tx$$

$$\begin{aligned} U_3 &= \int_0^L \frac{[T(x)]^2}{2GI_P} dx = \frac{1}{2GI_P} \int_0^L (T + tx)^2 dx \\ &= \frac{T^2 L}{2GI_P} + \frac{TtL^2}{2GI_P} + \frac{t^2 L^3}{6GI_P} \quad \leftarrow \end{aligned}$$

NOTE: U_3 is not the sum of U_1 and U_2 .

Problem 3.9-6 Obtain a formula for the strain energy U of the statically indeterminate circular bar shown in the figure. The bar has fixed supports at ends A and B and is loaded by torques $2T_0$ and T_0 at points C and D , respectively.

Hint: Use Eqs. 3-50a and b of Example 3-9, Section 3.8, to obtain the reactive torques.

**Solution 3.9-6** Statically indeterminate bar**REACTIVE TORQUES**

From Eq. (3-50a):

$$T_A = \frac{(2T_0)\left(\frac{3L}{4}\right)}{L} + \frac{T_0\left(\frac{L}{4}\right)}{L} = \frac{7T_0}{4}$$

$$T_B = 3T_0 - T_A = \frac{5T_0}{4}$$

INTERNAL TORQUES

$$T_{AC} = -\frac{7T_0}{4} \quad T_{CD} = \frac{T_0}{4} \quad T_{DB} = \frac{5T_0}{4}$$

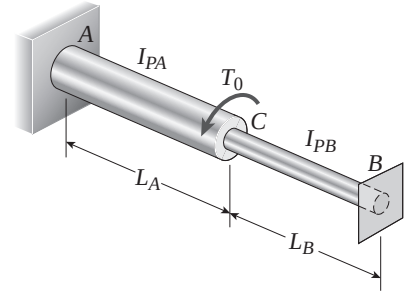
STRAIN ENERGY (from Eq. 3-57)

$$\begin{aligned} U &= \sum_{i=1}^n \frac{T_i^2 L_i}{2GI_{Pi}} \\ &= \frac{1}{2GI_P} \left[T_{AC}^2 \left(\frac{L}{4}\right) + T_{CD}^2 \left(\frac{L}{2}\right) + T_{DB}^2 \left(\frac{L}{4}\right) \right] \\ &= \frac{1}{2GI_P} \left[\left(-\frac{7T_0}{4}\right)^2 \left(\frac{L}{4}\right) + \left(\frac{T_0}{4}\right)^2 \left(\frac{L}{2}\right) + \left(\frac{5T_0}{4}\right)^2 \left(\frac{L}{4}\right) \right] \\ U &= \frac{19T_0^2 L}{32GI_P} \quad \leftarrow \end{aligned}$$

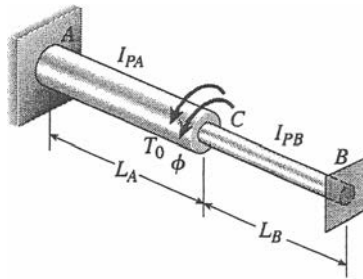
Problem 3.9-7 A statically indeterminate stepped shaft ACB is fixed at ends A and B and loaded by a torque T_0 at point C (see figure). The two segments of the bar are made of the same material, have lengths L_A and L_B , and have polar moments of inertia I_{PA} and I_{PB} .

Determine the angle of rotation ϕ of the cross section at C by using strain energy.

[Hint: Use Eq. 3-55b to determine the strain energy U in terms of the angle ϕ . Then equate the strain energy to the work done by the torque T_0 . Compare your result with Eq. (3-52) of Example 3-9, Section 3.8.]



Solution 3.9-7 Statically indeterminate bar



STRAIN ENERGY (FROM Eq. 3-55b)

$$U = \sum_{i=1}^n \frac{GI_{Pi}\phi_i^2}{2L_i} = \frac{GI_{PA}\phi^2}{2L_A} + \frac{GI_{PB}\phi^2}{2L_B}$$

$$= \frac{G\phi^2}{2} \left(\frac{I_{PA}}{L_A} + \frac{I_{PB}}{L_B} \right)$$

WORK DONE BY THE TORQUE T_0

$$W = \frac{T_0\phi}{2}$$

EQUATE U AND W AND SOLVE FOR ϕ

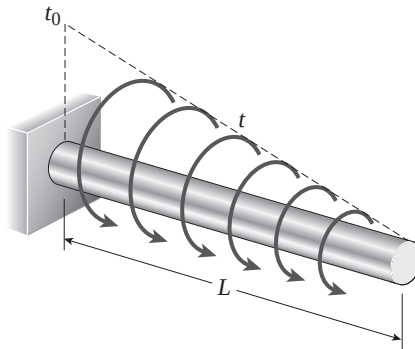
$$\frac{G\phi^2}{2} \left(\frac{I_{PA}}{L_A} + \frac{I_{PB}}{L_B} \right) = \frac{T_0\phi}{2}$$

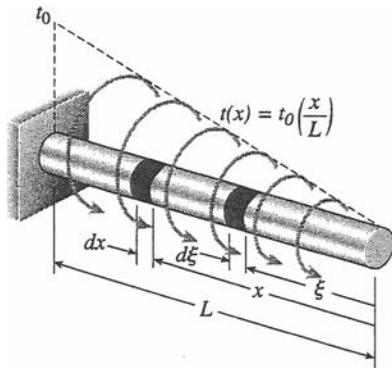
$$\phi = \frac{T_0 L_A L_B}{G(L_B I_{PA} + L_A I_{PB})} \quad \leftarrow$$

(This result agrees with Eq. (3-52) of Example 3-9, Section 3.8.)

Problem 3.9-8 Derive a formula for the strain energy U of the cantilever bar shown in the figure.

The bar has circular cross sections and length L . It is subjected to a distributed torque of intensity t per unit distance. The intensity varies linearly from $t = 0$ at the free end to a maximum value $t = t_0$ at the support.

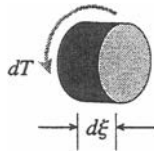


Solution 3.9-8 Cantilever bar with distributed torque

x = distance from right-hand end of the bar

ELEMENT $d\xi$

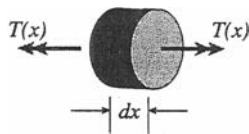
Consider a differential element $d\xi$ at distance ξ from the right-hand end.



dT = external torque acting on this element

$$dT = t(\xi)d\xi$$

$$= t_0\left(\frac{\xi}{L}\right)d\xi$$

ELEMENT dx AT DISTANCE x 

$T(x)$ = internal torque acting on this element

$T(x)$ = total torque from $x = 0$ to $x = x$

$$\begin{aligned} T(x) &= \int_0^x dT = \int_0^x t_0\left(\frac{\xi}{L}\right)d\xi \\ &= \frac{t_0 x^2}{2L} \end{aligned}$$

STRAIN ENERGY OF ELEMENT dx

$$\begin{aligned} dU &= \frac{[T(x)]^2 dx}{2GI_p} = \frac{1}{2GI_p} \left(\frac{t_0}{2L}\right)^2 x^4 dx \\ &= \frac{t_0^2}{8L^2 GI_p} x^4 dx \end{aligned}$$

STRAIN ENERGY OF ENTIRE BAR

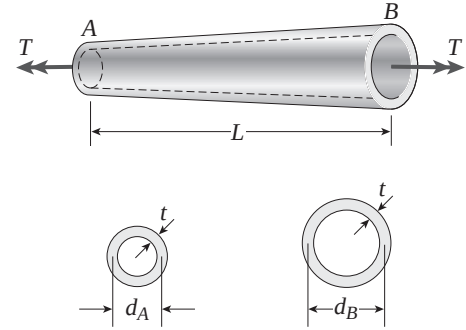
$$\begin{aligned} U &= \int_0^L dU = \frac{t_0^2}{8L^2 GI_p} \int_0^L x^4 dx \\ &= \frac{t_0^2}{8L^2 GI_p} \left(\frac{L^5}{5}\right) \end{aligned}$$

$$U = \frac{t_0^2 L^3}{40GI_p} \quad \leftarrow$$

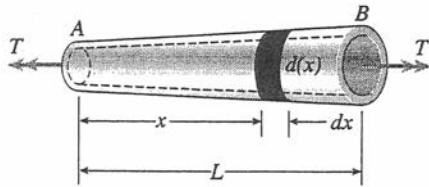
Problem 3.9-9 A thin-walled hollow tube AB of conical shape has constant thickness t and average diameters d_A and d_B at the ends (see figure).

- Determine the strain energy U of the tube when it is subjected to pure torsion by torques T .
- Determine the angle of twist ϕ of the tube.

Note: Use the approximate formula $I_P \approx \pi d^3 t / 4$ for a thin circular ring; see Case 22 of Appendix E.



Solution 3.9-9 Thin-walled, hollow tube



t = thickness

d_A = average diameter at end A

d_B = average diameter at end B

$d(x)$ = average diameter at distance x from end A

$$d(x) = d_A + \left(\frac{d_B - d_A}{L} \right) x$$

POLAR MOMENT OF INERTIA

$$I_P = \frac{\pi d^3 t}{4}$$

$$I_P(x) = \frac{\pi [d(x)]^3 t}{4} = \frac{\pi t}{4} \left[d_A + \left(\frac{d_B - d_A}{L} \right) x \right]^3$$

- (a) STRAIN ENERGY (FROM Eq. 3-58)

$$\begin{aligned} U &= \int_0^L \frac{T^2 dx}{2GI_P(x)} \\ &= \frac{T^2}{\pi Gt} \int_0^L \frac{dx}{\left[d_A + \left(\frac{d_B - d_A}{L} \right) x \right]^3} \quad (\text{Eq. 1}) \end{aligned}$$

From Appendix D:

$$\int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}$$

Therefore,

$$\begin{aligned} &\int_0^L \frac{dx}{\left[d_A + \left(\frac{d_B - d_A}{L} \right) x \right]^3} \\ &= -\frac{1}{\frac{2(d_B - d_A)}{L} \left[d_A + \left(\frac{d_B - d_A}{L} \right) x \right]^2} \bigg|_0^L \\ &= -\frac{L}{2(d_B - d_A)(d_B)^2} + \frac{L}{2(d_B - d_A)(d_A)^2} \\ &= \frac{L(d_A + d_B)}{2d_A^2 d_B^2} \end{aligned}$$

Substitute this expression for the integral into the equation for U (Eq. 1):

$$U = \frac{2T^2 L(d_A + d_B)}{\pi Gt \cdot 2d_A^2 d_B^2} = \frac{T^2 L}{\pi Gt} \left(\frac{d_A + d_B}{d_A^2 d_B^2} \right) \quad \leftarrow$$

- (b) ANGLE OF TWIST

$$\text{Work of the torque } T: W = \frac{T\phi}{2}$$

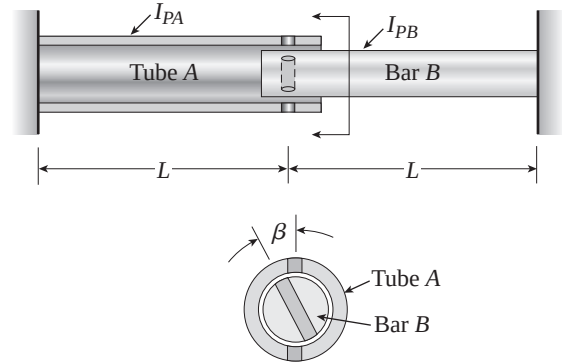
$$W = U \quad \frac{T\phi}{2} = \frac{T^2 L(d_A + d_B)}{\pi Gt d_A^2 d_B^2}$$

Solve for ϕ :

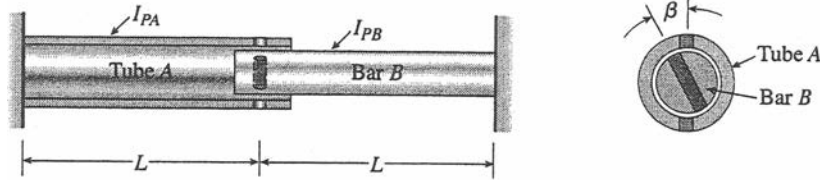
$$\phi = \frac{2TL(d_A + d_B)}{\pi Gt d_A^2 d_B^2} \quad \leftarrow$$

Problem 3.9-10 A hollow circular tube A fits over the end of a solid circular bar B, as shown in the figure. The far ends of both bars are fixed. Initially, a hole through bar B makes an angle β with a line through two holes in tube A. Then bar B is twisted until the holes are aligned, and a pin is placed through the holes.

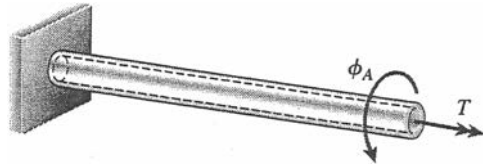
When bar B is released and the system returns to equilibrium, what is the total strain energy U of the two bars? (Let I_{PA} and I_{PB} represent the polar moments of inertia of bars A and B, respectively. The length L and shear modulus of elasticity G are the same for both bars.)



Solution 3.9-10 Circular tube and bar



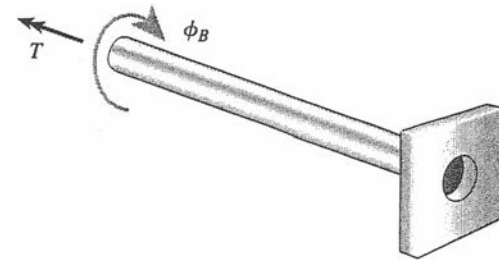
TUBE A



T = torque acting on the tube

ϕ_A = angle of twist

BAR B



T = torque acting on the bar

ϕ_B = angle of twist

COMPATIBILITY

$$\phi_A + \phi_B = \beta$$

FORCE-DISPLACEMENT RELATIONS

$$\phi_A = \frac{TL}{GI_{PA}} \quad \phi_B = \frac{TL}{GI_{PB}}$$

Substitute into the equation of compatibility and solve for T :

$$T = \frac{\beta G}{L} \left(\frac{I_{PA} I_{PB}}{I_{PA} + I_{PB}} \right)$$

STRAIN ENERGY

$$\begin{aligned} U &= \sum \frac{T^2 L}{2GI_P} = \frac{T^2 L}{2GI_{PA}} + \frac{T^2 L}{2GI_{PB}} \\ &= \frac{T^2 L}{2G} \left(\frac{1}{I_{PA}} + \frac{1}{I_{PB}} \right) \end{aligned}$$

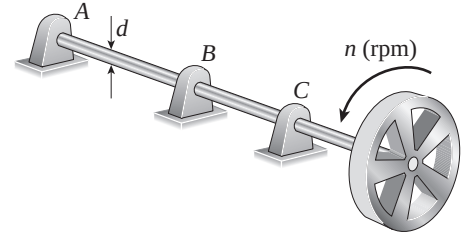
Substitute for T and simplify:

$$U = \frac{\beta^2 G}{2L} \left(\frac{I_{PA} I_{PB}}{I_{PA} + I_{PB}} \right) \quad \leftarrow$$

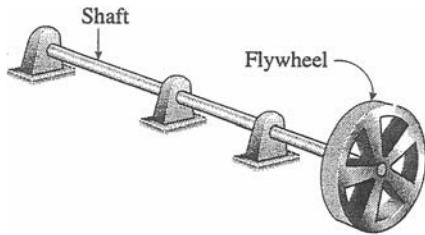
Problem 3.9-11 A heavy flywheel rotating at n revolutions per minute is rigidly attached to the end of a shaft of diameter d (see figure). If the bearing at A suddenly freezes, what will be the maximum angle of twist ϕ of the shaft? What is the corresponding maximum shear stress in the shaft?

(Let L = length of the shaft, G = shear modulus of elasticity, and I_m = mass moment of inertia of the flywheel about the axis of the shaft. Also, disregard friction in the bearings at B and C and disregard the mass of the shaft.)

Hint: Equate the kinetic energy of the rotating flywheel to the strain energy of the shaft.



Solution 3.9-11 Rotating flywheel



d = diameter

n = rpm

KINETIC ENERGY OF FLYWHEEL

$$\text{K.E.} = \frac{1}{2} I_m v^2$$

$$v = \frac{2\pi n}{60}$$

n = rpm

$$\begin{aligned} \text{K.E.} &= \frac{1}{2} I_m \left(\frac{2\pi n}{60} \right)^2 \\ &= \frac{\pi^2 n^2 I_m}{1800} \end{aligned}$$

UNITS:

$$I_m = (\text{force})(\text{length})(\text{second})^2$$

ω = radians per second

$$\text{K.E.} = (\text{length})(\text{force})$$

STRAIN ENERGY OF SHAFT (FROM Eq. 3-55b)

$$U = \frac{GI_p \phi^2}{2L}$$

$$I_p = \frac{\pi}{32} d^4$$

d = diameter of shaft

$$U = \frac{\pi G d^4 \phi^2}{64L}$$

UNITS:

$$G = (\text{force})/(\text{length})^2$$

$$I_p = (\text{length})^4$$

ϕ = radians

L = length

$$U = (\text{length})(\text{force})$$

EQUATE KINETIC ENERGY AND STRAIN ENERGY

$$\text{K.E.} = U \quad \frac{\pi^2 n^2 I_m}{1800} = \frac{\pi G d^4 \phi^2}{64L}$$

Solve for ϕ :

$$\phi = \frac{2n}{15d^2} \sqrt{\frac{2\pi I_m L}{G}} \quad \leftarrow$$

MAXIMUM SHEAR STRESS

$$\tau = \frac{T(d/2)}{I_p} \quad \phi = \frac{TL}{GI_p}$$

Eliminate T :

$$\tau = \frac{Gd\phi}{2L}$$

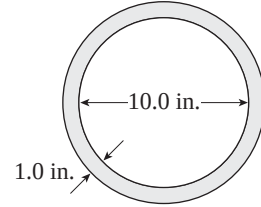
$$\tau_{\max} = \frac{Gd2n}{2L15d^2} \sqrt{\frac{2\pi I_m L}{G}}$$

$$\tau_{\max} = \frac{n}{15d} \sqrt{\frac{2\pi G I_m}{L}} \quad \leftarrow$$

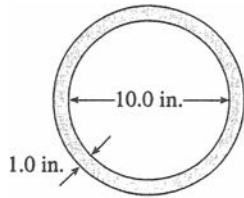
Thin-Walled Tubes

Problem 3.11-1 A hollow circular tube having an inside diameter of 10.0 in. and a wall thickness of 1.0 in. (see figure) is subjected to a torque $T = 1200$ k-in.

Determine the maximum shear stress in the tube using (a) the approximate theory of thin-walled tubes, and (b) the exact torsion theory. Does the approximate theory give conservative or nonconservative results?



Solution 3.11-1 Hollow circular tube



$$T = 1200 \text{ k-in.}$$

$$t = 1.0 \text{ in.}$$

$$r = \text{radius to median line}$$

$$r = 5.5 \text{ in.}$$

$$d_2 = \text{outside diameter} = 12.0 \text{ in.}$$

$$d_1 = \text{inside diameter} = 10.0 \text{ in.}$$

APPROXIMATE THEORY (Eq. 3-83)

$$\tau_1 = \frac{T}{2\pi r^2 t} = \frac{1200 \text{ k-in.}}{2\pi (5.5 \text{ in.})^2 (1.0 \text{ in.})} = 6314 \text{ psi}$$

$$\tau_{\text{approx}} = 6310 \text{ psi} \quad \leftarrow$$

EXACT THEORY (Eq. 3-13)

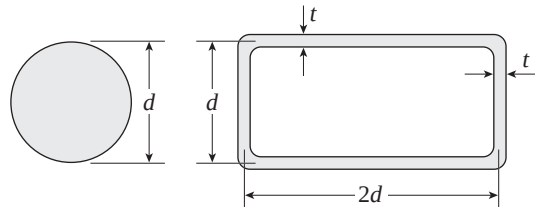
$$\begin{aligned} \tau_2 &= \frac{T(d_2/2)}{I_P} = \frac{Td_2}{2\left(\frac{\pi}{32}\right)(d_2^4 - d_1^4)} \\ &= \frac{16(1200 \text{ k-in.})(12.0 \text{ in.})}{\pi[(12.0 \text{ in.})^4 - (10.0 \text{ in.})^4]} \\ &= 6831 \text{ psi} \end{aligned}$$

$$\tau_{\text{exact}} = 6830 \text{ psi} \quad \leftarrow$$

Because the approximate theory gives stresses that are too low, it is nonconservative. Therefore, the approximate theory should only be used for very thin tubes.

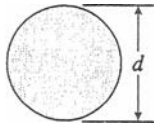
Problem 3.11-2 A solid circular bar having diameter d is to be replaced by a rectangular tube having cross-sectional dimensions $d \times 2d$ to the median line of the cross section (see figure).

Determine the required thickness $t_{\min}$ of the tube so that the maximum shear stress in the tube will not exceed the maximum shear stress in the solid bar.



Solution 3.11-2 Bar and tube

SOLID BAR



$$\tau_{\max} = \frac{16T}{\pi d^3} \quad (\text{Eq. 3-14})$$

$$A_m = (d)(2d) = 2d^2 \quad (\text{Eq. 3-84})$$

$$\tau_{\max} = \frac{T}{2tA_m} = \frac{T}{4td^2} \quad (\text{Eq. 3-81})$$

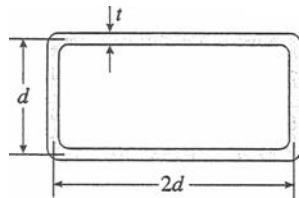
EQUATE THE MAXIMUM SHEAR STRESSES AND SOLVE FOR t

$$\frac{16T}{\pi d^3} = \frac{T}{4td^2}$$

$$t_{\min} = \frac{\pi d}{64} \quad \leftarrow$$

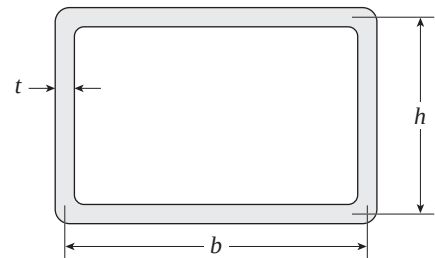
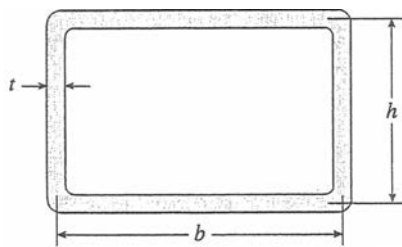
If $t > t_{\min}$, the shear stress in the tube is less than the shear stress in the bar.

RECTANGULAR TUBE



Problem 3.11-3 A thin-walled aluminum tube of rectangular cross section (see figure) has a centerline dimensions $b = 6.0$ in. and $h = 4.0$ in. The wall thickness t is constant and equal to 0.25 in.

- Determine the shear stress in the tube due to a torque $T = 15$ k-in.
- Determine the angle of twist (in degrees) if the length L of the tube is 50 in. and the shear modulus G is 4.0×10^6 psi.

**Probs. 3.11-3 and 3.11-4****Solution 3.11-3 Thin-walled tube**

$$b = 6.0 \text{ in.}$$

$$h = 4.0 \text{ in.}$$

$$t = 0.25 \text{ in.}$$

$$T = 15 \text{ k-in.}$$

$$L = 50 \text{ in.}$$

$$G = 4.0 \times 10^6 \text{ psi}$$

$$\text{Eq. (3-84): } A_m = bh = 24.0 \text{ in.}^2$$

$$\text{Eq. (3-71) with } t_1 = t_2 = t: \quad J = \frac{2b^2h^2t}{b + h}$$

$$J = 28.8 \text{ in.}^4$$

(a) SHEAR STRESS (Eq. 3-81)

$$\tau = \frac{T}{2tA_m} = 1250 \text{ psi} \quad \leftarrow$$

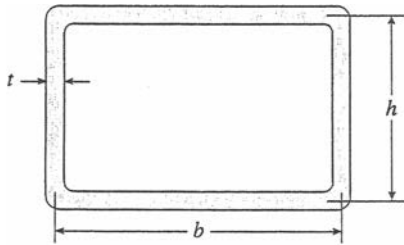
(b) ANGLE OF TWIST (Eq. 3-17)

$$\begin{aligned} \phi &= \frac{TL}{GJ} = 0.0065104 \text{ rad} \\ &= 0.373^\circ \quad \leftarrow \end{aligned}$$

Problem 3.11-4 A thin-walled steel tube of rectangular cross section (see figure) has centerline dimensions $b = 150$ mm and $h = 100$ mm. The wall thickness t is constant and equal to 6.0 mm.

- (a) Determine the shear stress in the tube due to a torque $T = 1650$ N·m.
 (b) Determine the angle of twist (in degrees) if the length L of the tube is 1.2 m and the shear modulus G is 75 GPa.

Solution 3.11-4 Thin-walled tube



$$\begin{aligned} b &= 150 \text{ mm} \\ h &= 100 \text{ mm} \\ t &= 6.0 \text{ mm} \\ T &= 1650 \text{ N}\cdot\text{m} \\ L &= 1.2 \text{ m} \\ G &= 75 \text{ GPa} \end{aligned}$$

(a) SHEAR STRESS (Eq. 3-81)

$$\tau = \frac{T}{2tA_m} = 9.17 \text{ MPa} \quad \leftarrow$$

(b) ANGLE OF TWIST (Eq. 3-17)

$$\begin{aligned} \phi &= \frac{TL}{GJ} = 0.002444 \text{ rad} \\ &= 0.140^\circ \quad \leftarrow \end{aligned}$$

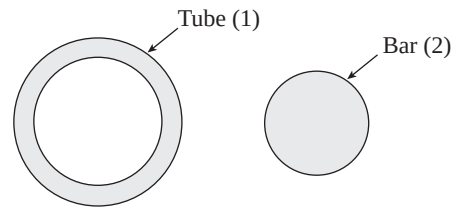
$$\text{Eq. (3-84): } A_m = bh = 0.015 \text{ m}^2$$

$$\text{Eq. (3-94) with } t_1 = t_2 = t: \quad J = \frac{2b^2h^2t}{b+h}$$

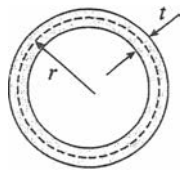
$$J = 10.8 \times 10^{-6} \text{ m}^4$$

Problem 3.11-5 A thin-walled circular tube and a solid circular bar of the same material (see figure) are subjected to torsion. The tube and bar have the same cross-sectional area and the same length.

What is the ratio of the strain energy U_1 in the tube to the strain energy U_2 in the solid bar if the maximum shear stresses are the same in both cases? (For the tube, use the approximate theory for thin-walled bars.)



Solution 3.11-5 THIN-WALLED TUBE (1)



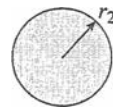
$$\begin{aligned} A_m &= \pi r^2 \quad J = 2\pi r^3 t \quad A = 2\pi r t \\ \tau_{\max} &= \frac{T}{2tA_m} = \frac{T}{2\pi r^2 t} \\ T &= 2\pi r^2 t \tau_{\max} \end{aligned}$$

$$\begin{aligned} U_1 &= \frac{T^2 L}{2GJ} = \frac{(2\pi r^2 t \tau_{\max})^2 L}{2G(2\pi r^3 t)} \\ &= \frac{\pi r t \tau_{\max}^2 L}{G} \end{aligned}$$

$$\text{But } rt = \frac{A}{2\pi}$$

$$\therefore U_1 = \frac{A \tau_{\max}^2 L}{2G}$$

SOLID BAR (2)



$$\begin{aligned} A &= \pi r_2^2 \quad I_p = \frac{\pi}{2} r_2^4 \\ \tau_{\max} &= \frac{Tr_2}{I_p} = \frac{2T}{\pi r_2^3} \quad T = \frac{\pi r_2^3 \tau_{\max}}{2} \end{aligned}$$

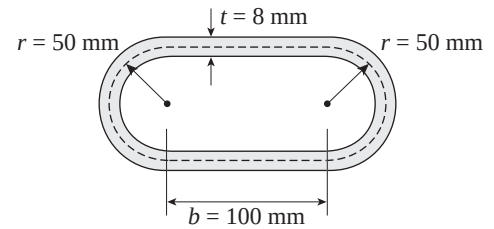
$$U_2 = \frac{T^2 L}{2GI_p} = \frac{(\pi r_2^3 \tau_{\max})^2 L}{8G \left(\frac{\pi}{2} r_2^4 \right)} = \frac{\pi r_2^2 \tau_{\max}^2 L}{4G}$$

$$\text{But } \pi r_2^2 = A \quad \therefore U_2 = \frac{A \tau_{\max}^2 L}{4G}$$

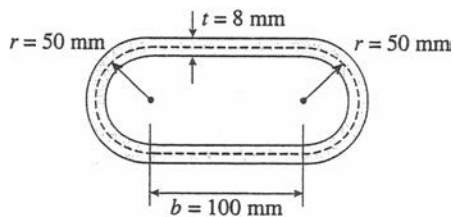
RATIO

$$\frac{U_1}{U_2} = 2 \quad \leftarrow$$

Problem 3.11-6 Calculate the shear stress τ and the angle of twist ϕ (in degrees) for a steel tube ($G = 76 \text{ GPa}$) having the cross section shown in the figure. The tube has length $L = 1.5 \text{ m}$ and is subjected to a torque $T = 10 \text{ kN}\cdot\text{m}$.



Solution 3.11-6 Steel tube



$$\begin{aligned} G &= 76 \text{ GPa} \\ L &= 1.5 \text{ m} \\ T &= 10 \text{ kN}\cdot\text{m} \end{aligned}$$

$$\begin{aligned} A_m &= \pi r^2 + 2br \\ A_m &= \pi (50 \text{ mm})^2 + 2(100 \text{ mm})(50 \text{ mm}) \\ &= 17,850 \text{ mm}^2 \\ L_m &= 2b + 2\pi r \\ &= 2(100 \text{ mm}) + 2\pi(50 \text{ mm}) \\ &= 514.2 \text{ mm} \\ J &= \frac{4tA_m^2}{L_m} = \frac{4(8 \text{ mm})(17,850 \text{ mm}^2)^2}{514.2 \text{ mm}} \\ &= 19.83 \times 10^6 \text{ mm}^4 \end{aligned}$$

SHEAR STRESS

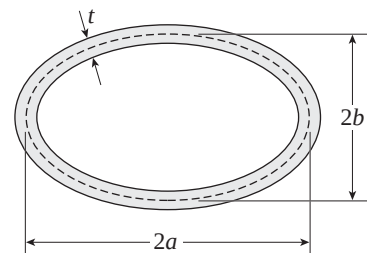
$$\begin{aligned} \tau &= \frac{T}{2tA_m} = \frac{10 \text{ kN}\cdot\text{m}}{2(8 \text{ mm})(17,850 \text{ mm}^2)} \\ &= 35.0 \text{ MPa} \quad \leftarrow \end{aligned}$$

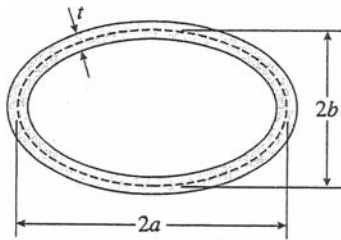
ANGLE OF TWIST

$$\begin{aligned} \phi &= \frac{TL}{GJ} = \frac{(10 \text{ kN}\cdot\text{m})(1.5 \text{ m})}{(76 \text{ GPa})(19.83 \times 10^6 \text{ mm}^4)} \\ &= 0.00995 \text{ rad} \\ &= 0.570^\circ \quad \leftarrow \end{aligned}$$

Problem 3.11-7 A thin-walled steel tube having an elliptical cross section with constant thickness t (see figure) is subjected to a torque $T = 18 \text{ k-in.}$

Determine the shear stress τ and the rate of twist θ (in degrees per inch) if $G = 12 \times 10^6 \text{ psi}$, $t = 0.2 \text{ in.}$, $a = 3 \text{ in.}$, and $b = 2 \text{ in.}$ (Note: See Appendix E, Case 16, for the properties of an ellipse.)



Solution 3.11-7 Elliptical tube

$$T = 18 \text{ k-in.}$$

$$G = 12 \times 10^6 \text{ psi}$$

$$t = \text{constant}$$

$$t = 0.2 \text{ in.} \quad a = 3.0 \text{ in.} \quad b = 2.0 \text{ in.}$$

FROM APPENDIX E, CASE 16:

$$A_m = \pi ab = \pi(3.0 \text{ in.})(2.0 \text{ in.}) = 18.850 \text{ in.}^2$$

$$L_m \approx \pi[1.5(a + b) - \sqrt{ab}]$$

$$= \pi[1.5(5.0 \text{ in.}) - \sqrt{6.0 \text{ in.}^2}] = 15.867 \text{ in.}$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4(0.2 \text{ in.})(18.850 \text{ in.}^2)^2}{15.867 \text{ in.}}$$

$$= 17.92 \text{ in.}^4$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{18 \text{ k-in.}}{2(0.2 \text{ in.})(18.850 \text{ in.}^2)}$$

$$= 2390 \text{ psi} \quad \leftarrow$$

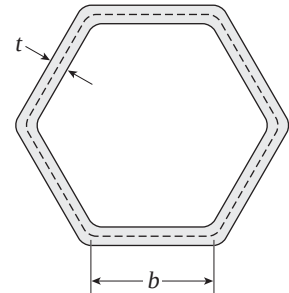
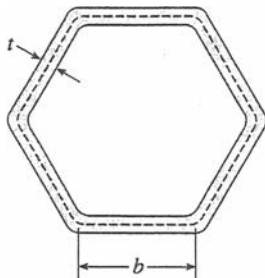
ANGLE OF TWIST PER UNIT LENGTH (RATE OF TWIST)

$$\theta = \frac{\phi}{L} = \frac{T}{GJ} = \frac{18 \text{ k-in.}}{(12 \times 10^6 \text{ psi})(17.92 \text{ in.}^4)}$$

$$\theta = 83.73 \times 10^{-6} \text{ rad/in.} = 0.0048^\circ/\text{in.} \quad \leftarrow$$

Problem 3.11-8 A torque T is applied to a thin-walled tube having a cross section in the shape of a regular hexagon with constant wall thickness t and side length b (see figure).

Obtain formulas for the shear stress τ and the rate of twist θ .

**Solution 3.11-8 Regular hexagon**

b = Length of side

t = Thickness

$$L_m = 6b$$

FROM APPENDIX E, CASE 25:

$$\beta = 60^\circ \quad n = 6$$

$$A_m = \frac{nb^2}{4} \cot \frac{\beta}{2} = \frac{6b^2}{4} \cot 30^\circ$$

$$= \frac{3\sqrt{3}b^2}{2}$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{T\sqrt{3}}{9b^2t} \quad \leftarrow$$

ANGLE OF TWIST PER UNIT LENGTH (RATE OF TWIST)

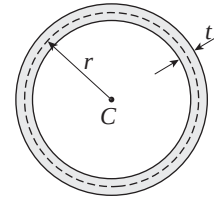
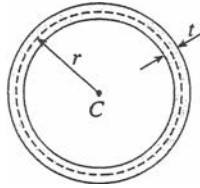
$$J = \frac{4A_m^2 t}{\int_0^{L_m} \frac{ds}{t}} = \frac{4A_m^2 t}{L_m} = \frac{9b^3 t}{2}$$

$$\theta = \frac{T}{GJ} = \frac{2T}{G(9b^3 t)} = \frac{2T}{9Gb^3 t} \quad \leftarrow$$

(radians per unit length)

Problem 3.11-9 Compare the angle of twist ϕ_1 for a thin-walled circular tube (see figure) calculated from the approximate theory for thin-walled bars with the angle of twist ϕ_2 calculated from the exact theory of torsion for circular bars.

- Express the ratio ϕ_1/ϕ_2 in terms of the nondimensional ratio $\beta = r/t$.
- Calculate the ratio of angles of twist for $\beta = 5, 10$, and 20 . What conclusion about the accuracy of the approximate theory do you draw from these results?

**Solution 3.11-9 Thin-walled tube**

APPROXIMATE THEORY

$$\phi_1 = \frac{TL}{GJ} \quad J = 2\pi r^3 t \quad \phi_1 = \frac{TL}{2\pi G r^3 t}$$

EXACT THEORY

$$\phi_2 = \frac{TL}{GI_p} \quad \text{From Eq. (3-17): } I_p = \frac{\pi r t}{2} (4r^2 + t^2)$$

$$\phi_2 = \frac{TL}{GI_p} = \frac{2TL}{\pi G r t (4r^2 + t^2)}$$

(a) RATIO

$$\frac{\phi_1}{\phi_2} = \frac{4r^2 + t^2}{4r^2} = 1 + \frac{t^2}{4r^2}$$

$$\text{Let } \beta = \frac{r}{t} \quad \frac{\phi_1}{\phi_2} = 1 + \frac{1}{4\beta^2} \quad \leftarrow$$

(b)

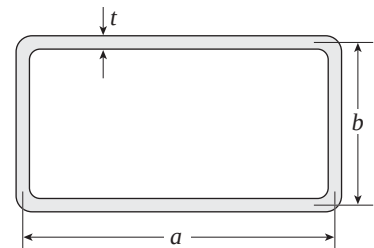
β	ϕ_1/ϕ_2
5	1.0100
10	1.0025
20	1.0006

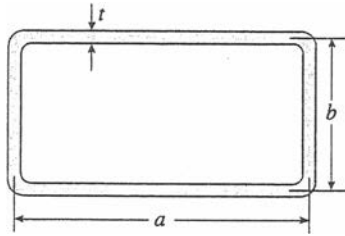
As the tube becomes thinner and β becomes larger, the ratio ϕ_1/ϕ_2 approaches unity. Thus, the thinner the tube, the more accurate the approximate theory becomes.

Problem 3.11-10 A thin-walled rectangular tube has uniform thickness t and dimensions $a \times b$ to the median line of the cross section (see figure).

How does the shear stress in the tube vary with the ratio $\beta = a/b$ if the total length L_m of the median line of the cross section and the torque T remain constant?

From your results, show that the shear stress is smallest when the tube is square ($\beta = 1$).



Solution 3.11-10 Rectangular tube

t = thickness (constant)

a, b = dimensions of the tube

$$\beta = \frac{a}{b}$$

$$L_m = 2(a + b) = \text{constant}$$

$$T = \text{constant}$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} \quad A_m = ab = \beta b^2$$

$$L_m = 2b(1 + \beta) = \text{constant}$$

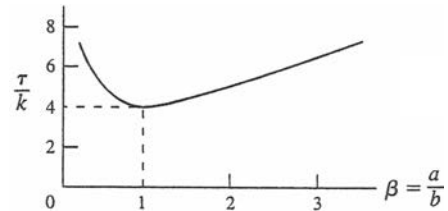
$$b = \frac{L_m}{2(1 + \beta)} \quad A_m = \beta \left[\frac{L_m}{2(1 + \beta)} \right]^2$$

$$A_m = \frac{\beta L_m^2}{4(1 + \beta)^2}$$

$$\tau = \frac{T}{2tA_m} = \frac{T(4)(1 + \beta)^2}{2t\beta L_m^2} = \frac{2T(1 + \beta)^2}{tL_m^2\beta} \quad \leftarrow$$

T, t , and L_m are constants.

$$\text{Let } k = \frac{2T}{tL_m^2} = \text{constant} \quad \tau = k \frac{(1 + \beta)^2}{\beta}$$



$$\left(\frac{\tau}{k} \right)_{\min} = 4 \quad \tau_{\min} = \frac{8T}{tL_m^2}$$

From the graph, we see that τ is minimum when $\beta = 1$ and the tube is square.

ALTERNATE SOLUTION

$$\tau = \frac{2T}{tL_m^2} \left[\frac{(1 + \beta)^2}{\beta} \right]$$

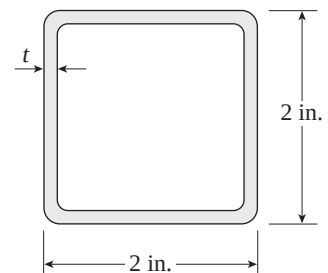
$$\frac{d\tau}{d\beta} = \frac{2T}{tL_m^2} \left[\frac{\beta(2)(1 + \beta) - (1 + \beta)^2(1)}{\beta^2} \right] = 0$$

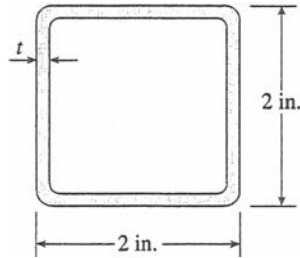
$$\text{or } 2\beta(1 + \beta) - (1 + \beta)^2 = 0 \quad \therefore \beta = 1$$

Thus, the tube is square and τ is either a minimum or a maximum. From the graph, we see that τ is a minimum.

Problem 3.11-11 A tubular aluminum bar ($G = 4 \times 10^6$ psi) of square cross section (see figure) with outer dimensions 2 in. $\times$ 2 in. must resist a torque $T = 3000$ lb-in.

Calculate the minimum required wall thickness $t_{\min}$ if the allowable shear stress is 4500 psi and the allowable rate of twist is 0.01 rad/ft.



Solution 3.11-11 Square aluminum tube

Outer dimensions:

$$2.0 \text{ in.} \times 2.0 \text{ in.}$$

$$G = 4 \times 10^6 \text{ psi}$$

$$T = 3000 \text{ lb-in.}$$

$$\tau_{\text{allow}} = 4500 \text{ psi}$$

$$\theta_{\text{allow}} = 0.01 \text{ rad/ft} = \frac{0.01}{12} \text{ rad/in.}$$

Let b = outer dimension

$$= 2.0 \text{ in.}$$

$$\text{Centerline dimension} = b - t$$

$$A_m = (b - t)^2 \quad L_m = 4(b - t)$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4t(b - t)^4}{4(b - t)} = t(b - t)^3$$

THICKNESS t BASED UPON SHEAR STRESS

$$\tau = \frac{T}{2tA_m} \quad tA_m = \frac{T}{2\tau} \quad t(b - t)^2 = \frac{T}{2\tau}$$

$$\text{UNITS: } t = \text{in.} \quad b = \text{in.} \quad T = \text{lb-in.} \quad \tau = \text{psi}$$

$$t(2.0 \text{ in.} - t)^2 = \frac{3000 \text{ lb-in.}}{2(4500 \text{ psi})} = \frac{1}{3} \text{ in.}^3$$

$$3t(2 - t)^2 - 1 = 0$$

$$\text{Solve for } t: t = 0.0915 \text{ in.}$$

THICKNESS t BASED UPON RATE OF TWIST

$$\theta = \frac{T}{GJ} = \frac{T}{Gt(b - t)^3} \quad t(b - t)^3 = \frac{T}{G\theta}$$

$$\text{UNITS: } t = \text{in.} \quad G = \text{psi} \quad \theta = \text{rad/in.}$$

$$t(2.0 \text{ in.} - t)^3 = \frac{3000 \text{ lb-in.}}{(4 \times 10^6 \text{ psi})(0.01/12 \text{ rad/in.})}$$

$$= \frac{9}{10}$$

$$10t(2 - t)^3 - 9 = 0$$

Solve for t :

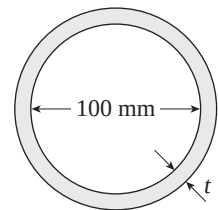
$$t = 0.140 \text{ in.}$$

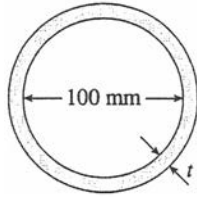
ANGLE OF TWIST GOVERNS

$$t_{\min} = 0.140 \text{ in.} \quad \leftarrow$$

Problem 3.11-12 A thin tubular shaft of circular cross section (see figure) with inside diameter 100 mm is subjected to a torque of 5000 N·m.

If the allowable shear stress is 42 MPa, determine the required wall thickness t by using (a) the approximate theory for a thin-walled tube, and (b) the exact torsion theory for a circular bar.



Solution 3.11-12 Thin tube

$$T = 5,000 \text{ N}\cdot\text{m} \quad d_1 = \text{inner diameter} = 100 \text{ mm}$$

$$\tau_{\text{allow}} = 42 \text{ MPa}$$

t is in millimeters.

r = average radius

$$= 50 \text{ mm} + \frac{t}{2}$$

r_1 = inner radius

$$= 50 \text{ mm}$$

r_2 = Outer radius

$$= 50 \text{ mm} + t \quad A_m = \pi r^2$$

(a) APPROXIMATE THEORY

$$\tau = \frac{T}{2tA_m} = \frac{T}{2t(\pi r^2)} = \frac{T}{2\pi r^2 t}$$

$$42 \text{ MPa} = \frac{5,000 \text{ N}\cdot\text{m}}{2\pi \left(50 + \frac{t}{2}\right)^2 t}$$

or

$$t \left(50 + \frac{t}{2}\right)^2 = \frac{5,000 \text{ N}\cdot\text{m}}{2\pi(42 \text{ MPa})} = \frac{5 \times 10^6}{84\pi} \text{ mm}^3$$

Solve for t :

$$t = 6.66 \text{ mm} \quad \leftarrow$$

(b) EXACT THEORY

$$\tau = \frac{Tr_2}{I_p}$$

$$I_p = \frac{\pi}{2} (r_2^4 - r_1^4) = \frac{\pi}{2} [(50 + t)^4 - (50)^4]$$

$$42 \text{ MPa} = \frac{(5,000 \text{ N}\cdot\text{m})(50 + t)}{\frac{\pi}{2} [(50 + t)^4 - (50)^4]}$$

$$\frac{(50 + t)^4 - (50)^4}{50 + t} = \frac{(5,000 \text{ N}\cdot\text{m})(2)}{(\pi)(42 \text{ MPa})}$$

$$= \frac{5 \times 10^6}{21\pi} \text{ mm}^3$$

Solve for t :

$$t = 7.02 \text{ mm} \quad \leftarrow$$

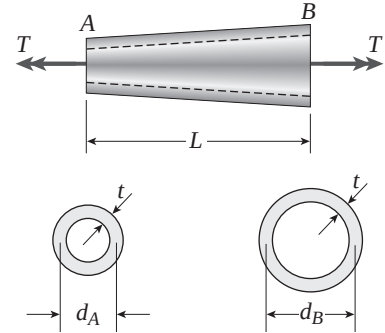
The approximate result is 5% less than the exact result. Thus, the approximate theory is nonconservative and should only be used for thin-walled tubes.

Problem 3.11-13 A long, thin-walled tapered tube AB of circular cross section (see figure) is subjected to a torque T . The tube has length L and constant wall thickness t . The diameter to the median lines of the cross sections at the ends A and B are d_A and d_B , respectively.

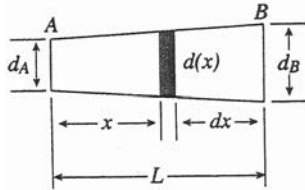
Derive the following formula for the angle of twist of the tube:

$$\phi = \frac{2TL}{\pi Gt} \left(\frac{d_A + d_B}{d_A^2 d_B^2} \right)$$

Hint: If the angle of taper is small, we may obtain approximate results by applying the formulas for a thin-walled prismatic tube to a differential element of the tapered tube and then integrating along the axis of the tube.



Solution 3.11-13 Thin-walled tapered tube



t = thickness

d_A = average diameter at end A

d_B = average diameter at end B

T = torque

$d(x)$ = average diameter at distance x from end A .

$$d(x) = d_A + \left(\frac{d_B - d_A}{L} \right) x$$

$$J = 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

$$J(x) = \frac{\pi t}{4} [d(x)]^3 = \frac{\pi t}{4} \left[d_A + \left(\frac{d_B - d_A}{L} \right) x \right]^3$$

For element of length dx :

$$d\phi = \frac{T dx}{GJ(x)} = \frac{4T dx}{\pi t \left[d_A + \left(\frac{d_B - d_A}{L} \right) x \right]^3}$$

For entire tube:

$$\phi = \frac{4T}{\pi Gt} \int_0^L \frac{dx}{\left[d_A + \left(\frac{d_B - d_A}{L} \right) x \right]^3}$$

From table of integrals (see Appendix D):

$$\int \frac{dx}{(a + bx)^3} = -\frac{1}{2b(a + bx)^2}$$

$$\begin{aligned} \phi &= \frac{4T}{\pi Gt} \left[-\frac{1}{2 \left(\frac{d_B - d_A}{L} \right) \left(d_A + \frac{d_B - d_A}{L} x \right)^2} \right]_0^L \\ &= \frac{4T}{\pi Gt} \left[-\frac{L}{2(d_B - d_A)d_B^2} + \frac{L}{2(d_B - d_A)d_A^2} \right] \end{aligned}$$

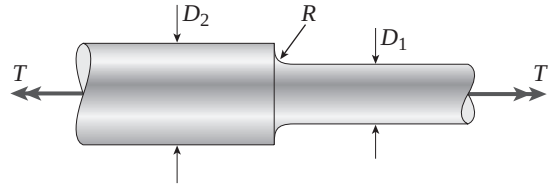
$$\phi = \frac{2TL}{\pi Gt} \left(\frac{d_A + d_B}{d_A^2 d_B^2} \right) \quad \leftarrow$$

Stress Concentrations in Torsion

The problems for Section 3.12 are to be solved by considering the stress-concentration factors.

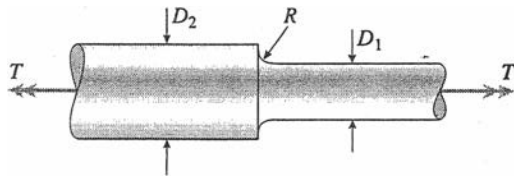
Problem 3.12-1 A stepped shaft consisting of solid circular segments having diameters $D_1 = 2.0$ in. and $D_2 = 2.4$ in. (see figure) is subjected to torques T . The radius of the fillet is $R = 0.1$ in.

If the allowable shear stress at the stress concentration is 6000 psi, what is the maximum permissible torque $T_{\max}$?



Probs. 3.12-1 through 3.12-5

Solution 3.12-1 Stepped shaft in torsion



$$D_1 = 2.0 \text{ in.}$$

$$D_2 = 2.4 \text{ in.}$$

$$R = 0.1 \text{ in.}$$

$$\tau_{\text{allow}} = 6000 \text{ psi}$$

USE FIG. 3-59 FOR THE STRESS-CONCENTRATION FACTOR

$$\frac{R}{D_1} = \frac{0.1 \text{ in.}}{2.0 \text{ in.}} = 0.05 \quad \frac{D_2}{D_1} = \frac{2.4 \text{ in.}}{2.0 \text{ in.}} = 1.2$$

$$K \approx 1.52 \quad \tau_{\max} = K\tau_{\text{nom}} = K \left(\frac{16 T_{\max}}{\pi D_1^3} \right)$$

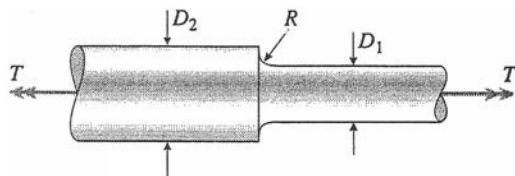
$$T_{\max} = \frac{\pi D_1^3 \tau_{\max}}{16K} = \frac{\pi (2.0 \text{ in.})^3 (6000 \text{ psi})}{16(1.52)} = 6200 \text{ lb-in.}$$

$$\therefore T_{\max} \approx 6200 \text{ lb-in.} \quad \leftarrow$$

Problem 3.12-2 A stepped shaft with diameters $D_1 = 40$ mm and $D_2 = 60$ mm is loaded by torques $T = 1100 \text{ N}\cdot\text{m}$ (see figure).

If the allowable shear stress at the stress concentration is 120 MPa, what is the smallest radius $R_{\min}$ that may be used for the fillet?

Solution 3.12-2 Stepped shaft in torsion



$$D_1 = 40 \text{ mm}$$

$$D_2 = 60 \text{ mm}$$

$$T = 1100 \text{ N}\cdot\text{m}$$

$$\tau_{\text{allow}} = 120 \text{ MPa}$$

USE FIG. 3-59 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{\max} = K\tau_{\text{nom}} = K \left(\frac{16T}{\pi D_1^3} \right)$$

$$K = \frac{\pi D_1^3 \tau_{\max}}{16T} = \frac{\pi (40 \text{ mm})^3 (120 \text{ MPa})}{16(1100 \text{ N}\cdot\text{m})} = 1.37$$

$$\frac{D_2}{D_1} = \frac{60 \text{ mm}}{40 \text{ mm}} = 1.5$$

From Fig. (3-59) with $\frac{D_2}{D_1} = 1.5$ and $K = 1.37$,

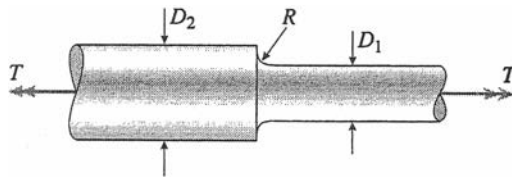
$$\text{we get } \frac{R}{D_1} \approx 0.10$$

$$\therefore R_{\min} \approx 0.10(40 \text{ mm}) = 4.0 \text{ mm} \quad \leftarrow$$

Problem 3.12-3 A full quarter-circular fillet is used at the shoulder of a stepped shaft having diameter $D_2 = 1.0$ in. (see figure). A torque $T = 500$ lb-in. acts on the shaft.

Determine the shear stress $\tau_{\max}$ at the stress concentration for values as follows: $D_1 = 0.7, 0.8$, and 0.9 in. Plot a graph showing $\tau_{\max}$ versus D_1 .

Solution 3.12-3 Stepped shaft in torsion



D_1 (in.)	D_2/D_1	R (in.)	R/D_1	K	$\tau_{\max}$ (psi)
0.7	1.43	0.15	0.214	1.20	8900
0.8	1.25	0.10	0.125	1.29	6400
0.9	1.11	0.05	0.056	1.41	4900

$$D_2 = 1.0 \text{ in.}$$

$$T = 500 \text{ lb-in.}$$

$$D_1 = 0.7, 0.8, \text{ and } 0.9 \text{ in.}$$

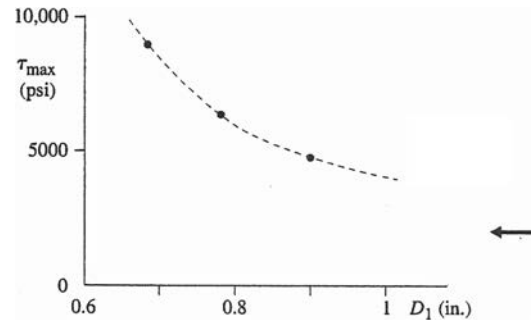
Full quarter-circular fillet ($D_2 = D_1 + 2R$):

$$R = \frac{D_2 - D_1}{2} = 0.5 \text{ in.} - \frac{D_1}{2}$$

USE FIG. 3-59 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{\max} = K\tau_{\text{nom}} = K \left(\frac{16T}{\pi D_1^3} \right)$$

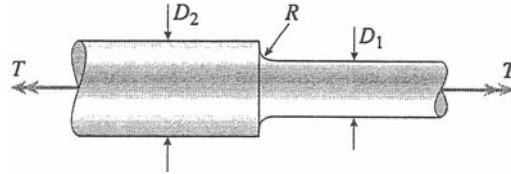
$$= K \frac{16(500 \text{ lb-in.})}{\pi D_1^3} = 2546 \frac{K}{D_1^3}$$



Note: $\tau_{\max}$ gets smaller as D_1 gets larger, even though K is increasing.

Problem 3.12-4 The stepped shaft shown in the figure is required to transmit 600 kW of power at 400 rpm. The shaft has a full quarter-circular fillet, and the smaller diameter $D_1 = 100$ mm.

If the allowable shear stress at the stress concentration is 100 MPa, at what diameter D_2 will this stress be reached? Is this diameter an upper or a lower limit on the value of D_2 ?

Solution 3.12-4 Stepped shaft in torsion

$$P = 600 \text{ kW} \quad D_1 = 100 \text{ mm}$$

$$n = 400 \text{ rpm} \quad \tau_{\text{allow}} = 100 \text{ MPa}$$

Full quarter-circular fillet

$$\text{POWER } P = \frac{2\pi n T}{60} \quad (\text{Eq. 3-42 of Section 3.7})$$

P in watts, n in rpm T in Newton meters.

$$T = \frac{60P}{2\pi n} = \frac{60(600 \times 10^3 \text{ W})}{2\pi(400 \text{ rpm})} = 14,320 \text{ N}\cdot\text{m}$$

USE FIG. 3-59 FOR THE STRESS-CONCENTRATION FACTOR

$$\tau_{\text{max}} = K\tau_{\text{nom}} = K\left(\frac{16T}{\pi D_1^3}\right)$$

$$K = \frac{\tau_{\text{max}}(\pi D_1^3)}{16T}$$

$$= \frac{(100 \text{ MPa})(\pi)(100 \text{ mm})^3}{16(14,320 \text{ N}\cdot\text{m})} = 1.37$$

Use the dashed line for a full quarter-circular fillet.

$$\frac{R}{D_1} \approx 0.075 \quad R \approx 0.075 D_1 = 0.075 (100 \text{ mm})$$

$$= 7.5 \text{ mm}$$

$$D_2 = D_1 + 2R = 100 \text{ mm} + 2(7.5 \text{ mm}) = 115 \text{ mm}$$

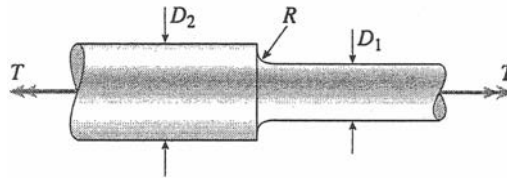
$$\therefore D_2 \approx 115 \text{ mm} \quad \leftarrow$$

This value of D_2 is a *lower limit* $\leftarrow$

(If D_2 is less than 115 mm, R/D_1 is smaller, K is larger, and τ_{max} is larger, which means that the allowable stress is exceeded.)

Problem 3.12-5 A stepped shaft (see figure) has diameter $D_2 = 1.5$ in. and a full quarter-circular fillet. The allowable shear stress is 15,000 psi and the load $T = 4800$ lb-in.

What is the smallest permissible diameter D_1 ?

Solution 3.12-5 Stepped shaft in torsion

$$D_2 = 1.5 \text{ in.}$$

$$\tau_{\text{allow}} = 15,000 \text{ psi}$$

$$T = 4800 \text{ lb-in.}$$

Full quarter-circular fillet $D_2 = D_1 + 2R$

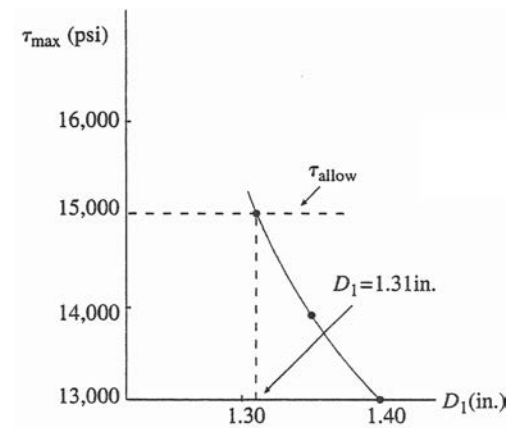
$$R = \frac{D_2 - D_1}{2} = 0.75 \text{ in.} - \frac{D_1}{2}$$

USE FIG. 3-59 FOR THE STRESS-CONCENTRATION FACTOR

$$\begin{aligned} \tau_{\text{max}} &= K\tau_{\text{nom}} = K\left(\frac{16T}{\pi D_1^3}\right) \\ &= \frac{K}{D_1^3} \left[\frac{16(4800 \text{ lb-in.})}{\pi} \right] \\ &= 24,450 \frac{K}{D_1^3} \end{aligned}$$

Use trial-and-error. Select trial values of D_1

D_1 (in.)	R (in.)	R/D_1	K	τ_{max} (psi)
1.30	0.100	0.077	1.38	15,400
1.35	0.075	0.056	1.41	14,000
1.40	0.050	0.036	1.46	13,000



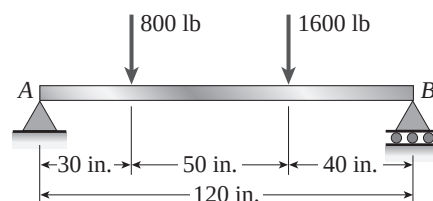
From the graph, minimum $D_1 \approx 1.31 \text{ in.}$ ←

4

Shear Forces and Bending Moments

Shear Forces and Bending Moments

Problem 4.3-1 Calculate the shear force V and bending moment M at a cross section just to the left of the 1600-lb load acting on the simple beam AB shown in the figure.

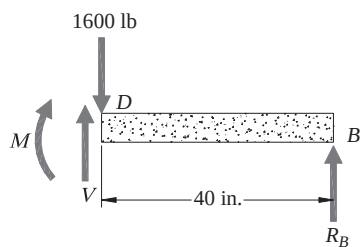


Solution 4.3-1

$$\sum M_A = 0: R_B = \frac{3800}{3} = 1267 \text{ lb}$$

$$\sum M_B = 0: R_A = \frac{3400}{3} = 1133 \text{ lb}$$

FREE-BODY DIAGRAM OF SEGMENT DB

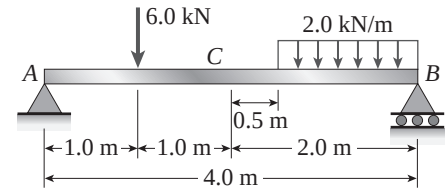


$$\sum F_{\text{VERT}} = 0: V = 1600 \text{ lb} - 1267 \text{ lb} = 333 \text{ lb} \quad \leftarrow$$

$$\sum M_D = 0: M = (1267 \text{ lb})(40 \text{ in.})$$

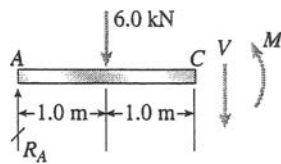
$$= \frac{152000}{3} \text{ lb}\cdot\text{in} = 50667 \text{ lb}\cdot\text{in.} \quad \leftarrow$$

Problem 4.3-2 Determine the shear force V and bending moment M at the midpoint C of the simple beam AB shown in the figure.



Solution 4.3-2

FREE-BODY DIAGRAM OF SEGMENT AC



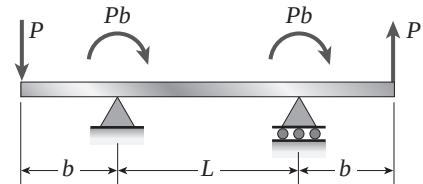
$$\sum M_A = 0: R_B = 3.9375 \text{ kN}$$

$$\sum M_B = 0: R_A = 5.0625 \text{ kN}$$

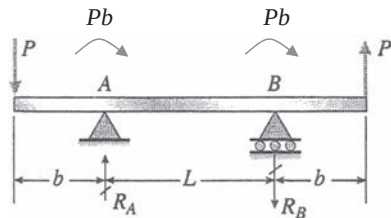
$$\sum F_{\text{VERT}} = 0: V = R_A - 6 = -0.938 \text{ kN} \quad \leftarrow$$

$$\begin{aligned} \sum M_C = 0: M &= R_A \cdot 2 \text{ m} - 6 \text{ kN} \cdot 1 \text{ m} \\ &= 4.12 \text{ kN} \cdot \text{m} \quad \leftarrow \end{aligned}$$

Problem 4.3-3 Determine the shear force V and bending moment M at the midpoint of the beam with overhangs (see figure). Note that one load acts downward and the other upward. Also clockwise moments Pb are applied at each support.

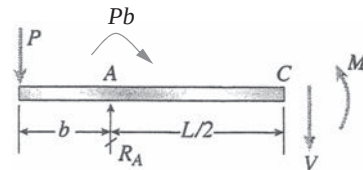


Solution 4.3-3



$$\begin{aligned} \sum M_B = 0: R_A &= \frac{1}{L} (2Pb - (b + L)P - Pb) \\ &= P \text{ (upward)} \end{aligned}$$

$$\sum M_A = 0: R_B = P \text{ (downward)}$$

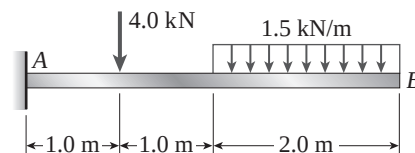


FREE-BODY DIAGRAM (C IS THE MIDPOINT)

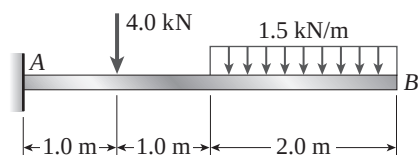
$$\sum F_{\text{VERT}} = 0: V = R_A - P = 0 \quad \leftarrow$$

$$\begin{aligned} \sum M_C = 0: M &= -P \left(b + \frac{L}{2} \right) \\ &\quad + R_A \frac{L}{2} + Pb = 0 \quad \leftarrow \end{aligned}$$

Problem 4.3-4 Calculate the shear force V and bending moment M at a cross section located 0.5 m from the fixed support of the cantilever beam AB shown in the figure.

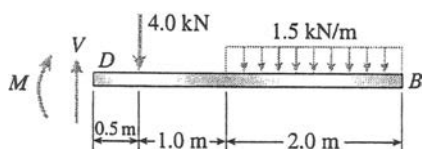


Solution 4.3-4 Cantilever beam



FREE-BODY DIAGRAM OF SEGMENT DB

Point D is 0.5 m from support A .



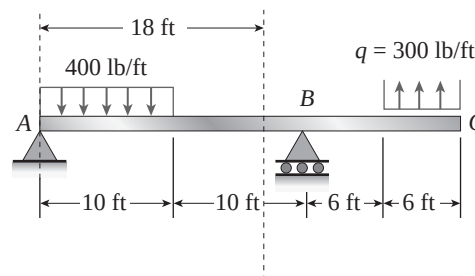
$$\sum F_{\text{VERT}} = 0:$$

$$\begin{aligned} V &= 4.0 \text{ kN} + (1.5 \text{ kN/m})(2.0 \text{ m}) \\ &= 4.0 \text{ kN} + 3.0 \text{ kN} = 7.0 \text{ kN} \quad \leftarrow \end{aligned}$$

$$\begin{aligned} \sum M_D = 0: \quad M &= -(4.0 \text{ kN})(0.5 \text{ m}) \\ &\quad - (1.5 \text{ kN/m})(2.0 \text{ m})(2.5 \text{ m}) \\ &= -2.0 \text{ kN}\cdot\text{m} - 7.5 \text{ kN}\cdot\text{m} \\ &= -9.5 \text{ kN}\cdot\text{m} \quad \leftarrow \end{aligned}$$

Problem 4.3-5 Consider the beam with an overhang shown in the figure.

- Determine the shear force V and bending moment M at a cross section located 18 ft from the left-hand end A .
- Find the required magnitude of load intensity q , acting on the right half of member BC that will result in a zero shear force on the cross section 18 ft from A .



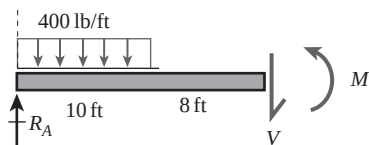
Solution 4.3-5

(a) V AND M AT $x = 18 \text{ FT}$

$$\sum M_B = 0 \quad R_A = \frac{1}{20 \text{ ft}} \left[400 \text{ lb/ft} (10 \text{ ft}) \left(10 \text{ ft} + \frac{10 \text{ ft}}{2} \right) + q (6 \text{ ft}) \left(6 \text{ ft} + \frac{6 \text{ ft}}{2} \right) \right] = 3810 \text{ lb}$$

$$V_{18} = R_A - 400 \text{ lb/ft} (10 \text{ ft}) = -190 \text{ lb}$$

$$M_{18} = R_A (18 \text{ ft}) - 400 \text{ lb/ft} (10 \text{ ft}) (18 \text{ ft} - 5 \text{ ft}) = 16,580 \text{ lb}\cdot\text{ft}$$



(b) REQUIRED q SO THAT SHEAR AT $x = 18$ FT IS ZERO

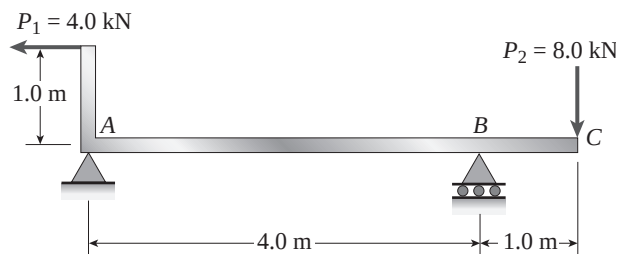
$$R_A = \frac{1}{20 \text{ ft}} \left[400 \text{ lb/ft} (10 \text{ ft}) \left(10 \text{ ft} + \frac{10 \text{ ft}}{2} \right) + q (6 \text{ ft}) \left(6 \text{ ft} + \frac{6 \text{ ft}}{2} \right) \right] = \frac{54 q \text{ ft}^2 + 60,000 \text{ lb ft}}{20 \text{ ft}}$$

$$V_{18} = R_A - 400 \text{ lb/ft} (10 \text{ ft}) = \frac{54 q \text{ ft}^2 + 60,000 \text{ lb ft}}{20 \text{ ft}} - 4000 \text{ lb}$$

$$V_{18} = 0 \text{ solve, } q = \frac{10,000 \text{ lb}}{27 \text{ ft}} = 370.4 \text{ lb/ft} < q \text{ is upward}$$

Problem 4.3-6 The beam ABC shown in the figure is simply supported at A and B and has an overhang from B to C . The loads consist of a horizontal force $P_1 = 4.0$ kN acting at the end of a vertical arm and a vertical force $P_2 = 8.0$ kN acting at the end of the overhang.

- Determine the shear force V and bending moment M at a cross section located 3.0 m from the left-hand support. (Note: Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)
- Find the value of load P_2 that results in $V = 0$ at a cross section located 2.0 m from the left-hand support.
- If $P_2 = 8$ kN, find the value of load P_1 that results in $M = 0$ at a cross section located 2.0 m from the left-hand support.

**Solution 4.3-6**

$$P_1 = 4 \quad P_2 = 8$$

units: kN, m

(a) RHFB

$$R_B = \frac{1}{4}(-P_1 1 + P_2 5) = 9 \quad V_3 = -R_B + P_2 = -1 \text{ kN} \quad M_3 = R_B 1 - P_2 2 = -7 \text{ kN}\cdot\text{m}$$

(b)

$$R_B = \frac{1}{4}(-4 1 + P_2 5) = \frac{5P_2}{4} - 1$$

$$V_2 = -R_B + P_2 = 1 - \frac{P_2}{4} \quad V_2 = 0 \quad \text{gives} \quad P_2 = 4 \text{ kN}$$

(c) LHFB

$$P_2 = 8$$

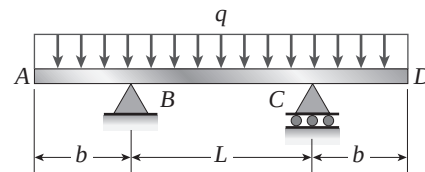
$$R_B = \frac{1}{4}(-P_1 1 + P_2 5) = 10 - \frac{P_1}{4} \quad R_A = P_2 - R_B = \frac{P_1}{4} - 2$$

$$M_2 = R_A 2 - P_1 1 = -\frac{P_1}{2} - 4$$

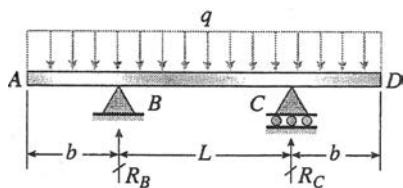
$$\text{Let } M_2 = 0 \text{ so } P_1 = -8 \text{ kN (acting to the right)}$$

Problem 4.3-7 The beam $ABCD$ shown in the figure has overhangs at each end and carries a uniform load of intensity q .

For what ratio b/L will the bending moment at the midpoint of the beam be zero?



Solution 4.3-7 Beam with overhangs

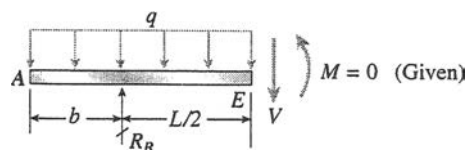


From symmetry and equilibrium of vertical forces:

$$R_B = R_C = q\left(b + \frac{L}{2}\right)$$

FREE-BODY DIAGRAM OF LEFT-HAND HALF OF BEAM:

Point E is at the midpoint of the beam.



$$\sum M_E = 0 \quad \curvearrowright \quad \curvearrowleft$$

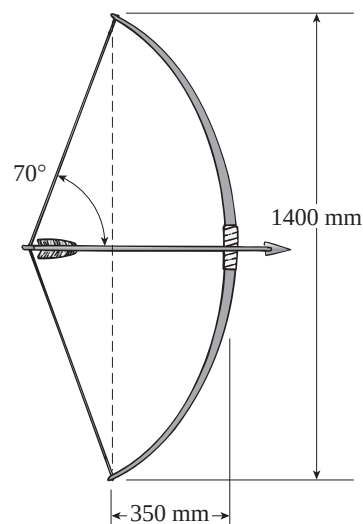
$$-R_B\left(\frac{L}{2}\right) + q\left(\frac{1}{2}\right)\left(b + \frac{L}{2}\right)^2 = 0$$

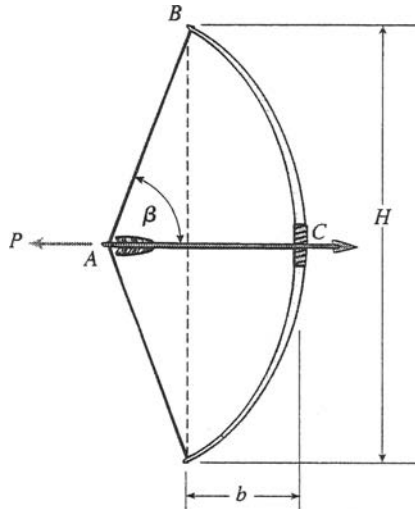
$$-q\left(b + \frac{L}{2}\right)\left(\frac{L}{2}\right) + q\left(\frac{1}{2}\right)\left(b + \frac{L}{2}\right)^2 = 0$$

Solve for b/L :

$$\frac{b}{L} = \frac{1}{2} \quad \leftarrow$$

Problem 4.3-8 At full draw, an archer applies a pull of 130 N to the bowstring of the bow shown in the figure. Determine the bending moment at the midpoint of the bow.



Solution 4.3-8 Archer's bow

$$P = 130 \text{ N}$$

$$\beta = 70^\circ$$

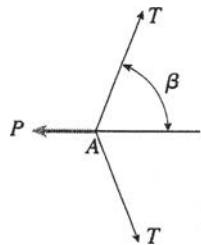
$$H = 1400 \text{ mm}$$

$$= 1.4 \text{ m}$$

$$b = 350 \text{ mm}$$

$$= 0.35 \text{ m}$$

FREE-BODY DIAGRAM OF POINT A

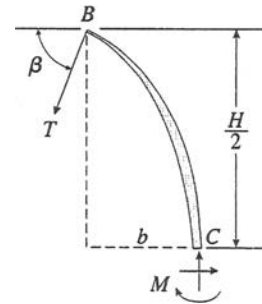


T = tensile force in the bowstring

$$\Sigma F_{\text{HORIZ}} = 0: 2T \cos \beta - P = 0$$

$$T = \frac{P}{2 \cos \beta}$$

FREE-BODY DIAGRAM OF SEGMENT BC



$$\Sigma M_C = 0 \quad \curvearrowright \quad \curvearrowleft$$

$$T(\cos \beta) \left(\frac{H}{2} \right) + T(\sin \beta)(b) - M = 0$$

$$M = T \left(\frac{H}{2} \cos \beta + b \sin \beta \right)$$

$$= \frac{P}{2} \left(\frac{H}{2} + b \tan \beta \right)$$

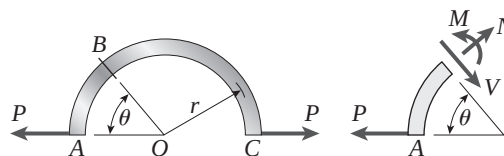
SUBSTITUTE NUMERICAL VALUES:

$$M = \frac{130 \text{ N}}{2} \left[\frac{1.4 \text{ m}}{2} + (0.35 \text{ m})(\tan 70^\circ) \right]$$

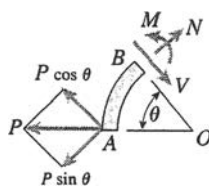
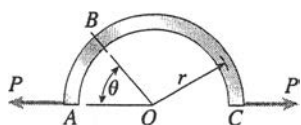
$$M = 108 \text{ N}\cdot\text{m} \quad \leftarrow$$

Problem 4.3-9 A curved bar ABC is subjected to loads in the form of two equal and opposite forces P , as shown in the figure. The axis of the bar forms a semicircle of radius r .

Determine the axial force N , shear force V , and bending moment M acting at a cross section defined by the angle θ .



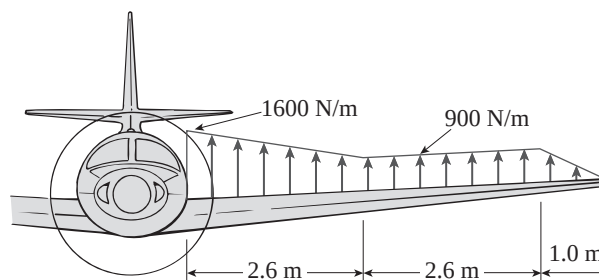
Solution 4.3-9 Curved bar



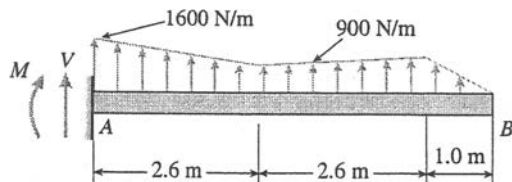
$$\begin{aligned}\sum F_N = 0 \quad \nearrow + \quad \nwarrow - \quad N - P \sin \theta &= 0 \\ N &= P \sin \theta \quad \leftarrow \\ \sum F_V = 0 \quad \searrow + \quad \swarrow - \quad V - P \cos \theta &= 0 \\ V &= P \cos \theta \quad \leftarrow \\ \sum M_O = 0 \quad \curvearrowright + \quad \curvearrowleft - \quad M - Nr &= 0 \\ M &= Nr = Pr \sin \theta \quad \leftarrow\end{aligned}$$

Problem 4.3-10 Under cruising conditions the distributed load acting on the wing of a small airplane has the idealized variation shown in the figure.

Calculate the shear force V and bending moment M at the inboard end of the wing.



Solution 4.3-10 Airplane wing



SHEAR FORCE

$$\sum F_{\text{VERT}} = 0 \quad \uparrow + \quad \downarrow -$$

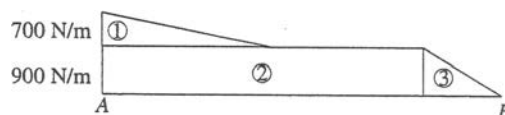
$$V + \frac{1}{2} (700 \text{ N/m})(2.6 \text{ m}) + (900 \text{ N/m})(5.2 \text{ m})$$

$$+ \frac{1}{2} (900 \text{ N/m})(1.0 \text{ m}) = 0$$

$$V = -6040 \text{ N} = -6.04 \text{ kN} \quad \leftarrow$$

(Minus means the shear force acts opposite to the direction shown in the figure.)

LOADING (IN THREE PARTS)



BENDING MOMENT

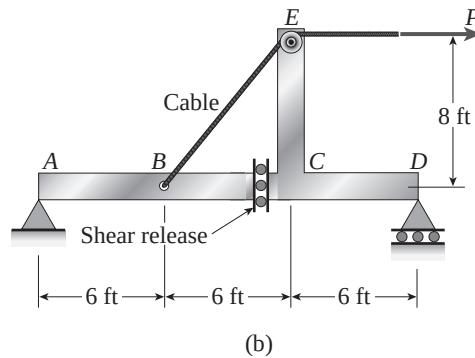
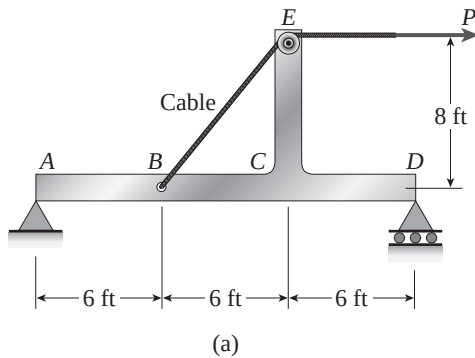
$$\sum M_A = 0 \quad \curvearrowright$$

$$\begin{aligned} -M + \frac{1}{2}(700 \text{ N/m})(2.6 \text{ m})\left(\frac{2.6 \text{ m}}{3}\right) \\ + (900 \text{ N/m})(5.2 \text{ m})(2.6 \text{ m}) \\ + \frac{1}{2}(900 \text{ N/m})(1.0 \text{ m})\left(5.2 \text{ m} + \frac{1.0 \text{ m}}{3}\right) = 0 \end{aligned}$$

$$\begin{aligned} M &= 788.67 \text{ N}\cdot\text{m} + 12,168 \text{ N}\cdot\text{m} + 2490 \text{ N}\cdot\text{m} \\ &= 15,450 \text{ N}\cdot\text{m} \\ &= 15.45 \text{ kN}\cdot\text{m} \quad \leftarrow \end{aligned}$$

Problem 4.3-11 A beam $ABCD$ with a vertical arm CE is supported as a simple beam at A and D (see figure a). A cable passes over a small pulley that is attached to the arm at E . One end of the cable is attached to the beam at point B .

- (a) What is the force P in the cable if the bending moment in the beam just to the left of point C is equal numerically to 640 lb-ft? (Note: Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)
 (b) Repeat (a) if a roller support is added at C and a shear release is inserted just left of C (see figure part b).



Solution 4.3-11

(a) LHFB

$$R_A = \frac{1}{18}(-8P) = -\frac{4P}{9}$$

$$M_C = R_A 12 + \frac{4}{5}P 6 = -\frac{8P}{15}$$

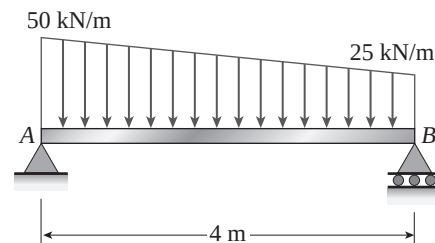
$$\frac{-8P}{15} = -640 \quad \text{Solve: } P = 1200 \text{ lb}$$

(b) LHFB $V_C = R_A + \frac{4}{5}P = 0$ so $R_A = -\frac{4}{5}P$

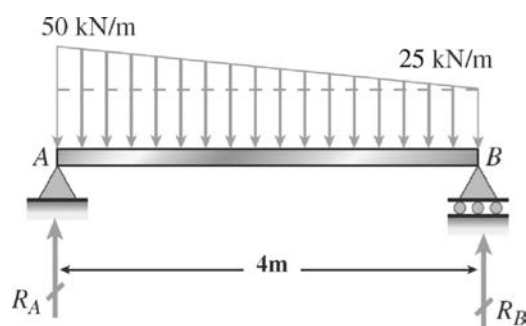
$$M_C = R_A 12 + \frac{4}{5}P 6 = -640 \quad \frac{-4}{5}P 12 + \frac{4}{5}P 6 = -640 \quad \text{Solve: } P = \frac{400}{3} \text{ lb} \quad \frac{400}{3} = 133.333$$

Problem 4.3-12 A simply supported beam AB supports a trapezoidally distributed load (see figure). The intensity of the load varies linearly from 50 kN/m at support A to 25 kN/m at support B .

Calculate the shear force V and bending moment M at the midpoint of the beam.



Solution 4.3-12



$$\sum M_B = 0: -R_A(4\text{ m}) + (25 \text{ kN/m})(4\text{ m})(2\text{ m}) + (25 \text{ kN/m})(4\text{ m})\left(\frac{1}{2}\right)\left(4\text{ m}\frac{2}{3}\right) = 0$$

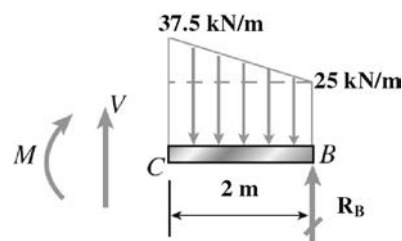
$$R_A = 83.33 \text{ kN}$$

$$\sum F_{\text{VERT}} = 0: R_A + R_B - \frac{1}{2}(50 \text{ kN/m} + 25 \text{ kN/m})(4\text{ m}) = 0$$

$$R_B = 66.67 \text{ kN}$$

FREE-BODY DIAGRAM OF SECTION CB

Point C is at the midpoint of the beam.



$$\sum F_{\text{VERT}} = 0: V - (25 \text{ kN/m})(2\text{ m}) - (12.5 \text{ kN/m})(2\text{ m})\frac{1}{2} + R_B = 0$$

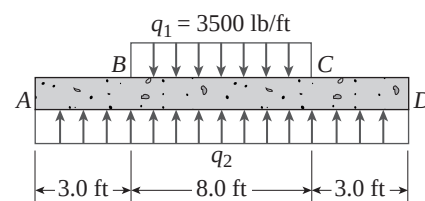
$$V = -4.17 \text{ kN} \quad \leftarrow$$

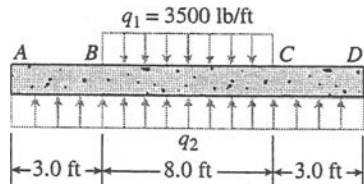
$$\sum M_C = 0: -M - (25 \text{ kN/m})(2\text{ m})(1\text{ m}) - (12.5 \text{ kN/m})(2\text{ m})\frac{1}{2}\left(2\text{ m}\frac{1}{3}\right) + R_B(2\text{ m}) = 0$$

$$M = 75 \text{ kN}\cdot\text{m} \quad \leftarrow$$

Problem 4.3-13 Beam $ABCD$ represents a reinforced-concrete foundation beam that supports a uniform load of intensity $q_1 = 3500 \text{ lb/ft}$ (see figure). Assume that the soil pressure on the underside of the beam is uniformly distributed with intensity q_2 .

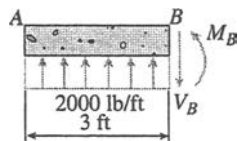
- Find the shear force V_B and bending moment M_B at point B .
- Find the shear force V_m and bending moment M_m at the midpoint of the beam.



Solution 4.3-13 Foundation beam

$$\sum F_{\text{VERT}} = 0: q_2(14 \text{ ft}) = q_1(8 \text{ ft})$$

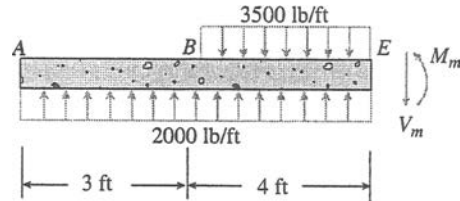
$$\therefore q_2 = \frac{8}{14} q_1 = 2000 \text{ lb/ft}$$

(a) *V* AND *M* AT POINT *B*

$$\sum F_{\text{VERT}} = 0:$$

$$V_B = 6000 \text{ lb} \quad \leftarrow$$

$$\sum M_B = 0: M_B = 9000 \text{ lb-ft} \quad \leftarrow$$

(b) *V* AND *M* AT MIDPOINT *E*

$$\sum F_{\text{VERT}} = 0: V_m = (2000 \text{ lb/ft})(7 \text{ ft}) - (3500 \text{ lb/ft})(4 \text{ ft})$$

$$V_m = 0 \quad \leftarrow$$

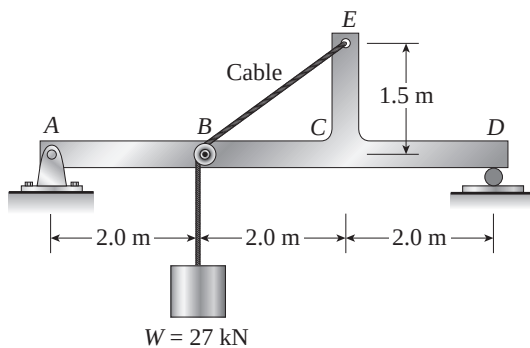
$$\sum M_E = 0:$$

$$M_m = (2000 \text{ lb/ft})(7 \text{ ft})(3.5 \text{ ft}) - (3500 \text{ lb/ft})(4 \text{ ft})(2 \text{ ft})$$

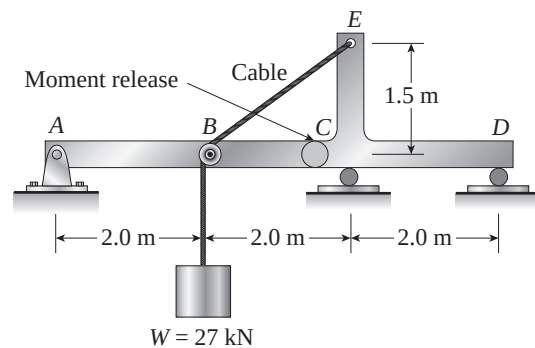
$$M_m = 21,000 \text{ lb-ft} \quad \leftarrow$$

Problem 4.3-14 The simply-supported beam *ABCD* is loaded by a weight $W = 27 \text{ kN}$ through the arrangement shown in the figure part a. The cable passes over a small frictionless pulley at *B* and is attached at *E* to the end of the vertical arm.

- (a) Calculate the axial force *N*, shear force *V*, and bending moment *M* at section *C*, which is just to the left of the vertical arm. (*Note*: Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)
 (b) Repeat (a) if a roller support is added at *C* and a moment release is inserted just left of *C* (see figure part b).



(a)



(b)

Solution 4.3-14

(a) SUM MOMENT ABOUT A TO FIND R_C , THEN USE RHFB TO FIND N , V , AND M AT C

$$R_{Dy} = \frac{1}{6}(2W) = \frac{W}{3} \quad V_C = -R_{Dy} + \frac{3}{5}W \quad \text{Simplify} \rightarrow \frac{4W}{15} \quad N_C = \frac{4}{5}W \quad (\text{compression})$$

$$M_C = R_{Dy}(2) + \frac{4}{5}W(1.5) = 1.8667W$$

$$W = 27 \text{ kN} \quad N_C = \frac{-4}{5}W = -21.6 \text{ kN} \quad V_C = \frac{4}{15}W = 7.2 \text{ kN} \quad M_C = 50.4 \text{ kN}\cdot\text{m}$$

(b)

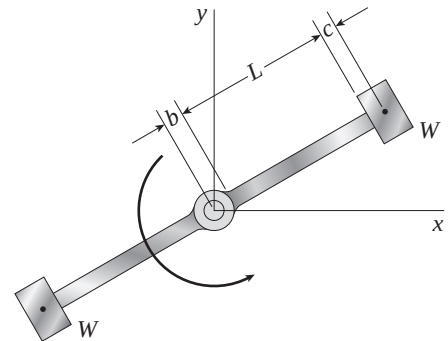
$$\text{RHFB: } \sum M_{\text{Mrel}} = 0 \quad R_{Dy} = \frac{-1}{2}\left(\frac{4}{5}W(1.5)\right) = -16.2 \text{ kN}$$

$$\text{Entire FBD: } \sum M_A = 0 \quad R_{Cy} = \frac{-R_{Dy}(6) + W(2)}{4} = 37.8 \text{ kN}$$

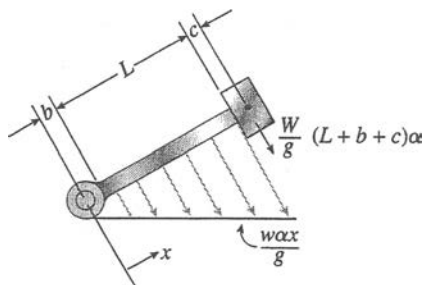
$$\text{RHFB: } N_C = -21.6 \text{ kN} \quad V_C = -(R_{Cy} + R_{Dy}) + \frac{3}{5}W = -5.4 \text{ kN} \quad M_C = 0 \quad (\text{at moment release})$$

Problem 4.3-15 The centrifuge shown in the figure rotates in a horizontal plane (the xy plane) on a smooth surface about the z axis (which is vertical) with an angular acceleration α . Each of the two arms has weight w per unit length and supports a weight $W = 2.0 wL$ at its end.

Derive formulas for the maximum shear force and maximum bending moment in the arms, assuming $b = L/9$ and $c = L/10$.



Solution 4.3-15 Rotating centrifuge

Tangential acceleration = $r\alpha$

$$\text{Inertial force } Mr\alpha = \frac{W}{g} r\alpha$$

Maximum V and M occur at $x = b$.

$$\begin{aligned} V_{\max} &= \frac{W}{g}(L + b + c)\alpha + \int_b^{L+b} \frac{w\alpha}{g} x \, dx \\ &= \frac{W\alpha}{g}(L + b + c) \\ &\quad + \frac{wL\alpha}{2g}(L + 2b) \quad \leftarrow \end{aligned}$$

$$\begin{aligned} M_{\max} &= \frac{W\alpha}{g}(L + b + c)(L + c) \\ &\quad + \int_b^{L+b} \frac{w\alpha}{g} x(x - b) \, dx \\ &= \frac{W\alpha}{g}(L + b + c)(L + c) \\ &\quad + \frac{wL^2\alpha}{6g}(2L + 3b) \quad \leftarrow \end{aligned}$$

SUBSTITUTE NUMERICAL DATA:

$$W = 2.0 \, wL \quad b = \frac{L}{9} \quad c = \frac{L}{10}$$

$$V_{\max} = \frac{91wL^2\alpha}{30g} \quad \leftarrow$$

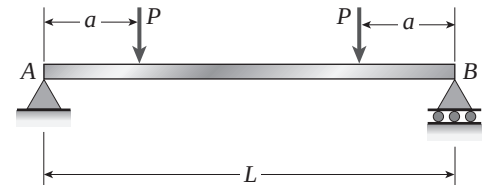
$$M_{\max} = \frac{229wL^3\alpha}{75g} \quad \leftarrow$$

Shear-Force and Bending-Moment Diagrams

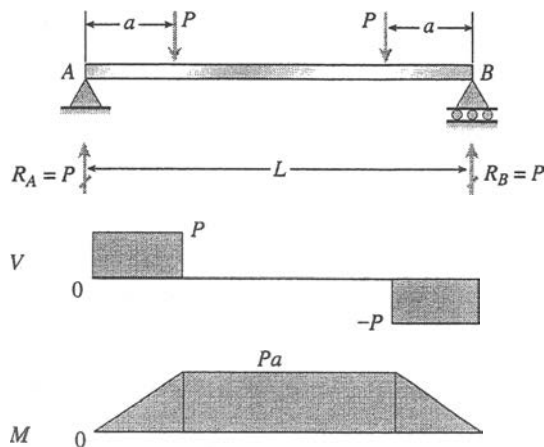
When solving the problems for Section 4.5, draw the shear-force and bending-moment diagrams approximately to scale and label all critical ordinates, including the maximum and minimum values.

Problems 4.5-1 through 4.5-10 are symbolic problems and Problems 4.5-11 through 4.5-24 are mostly numerical problems. The remaining problems (4.5-25 through 4.5-40) involve specialized topics, such as optimization, beams with hinges, and moving loads.

Problem 4.5-1 Draw the shear-force and bending-moment diagrams for a simple beam AB supporting two equal concentrated loads P (see figure).

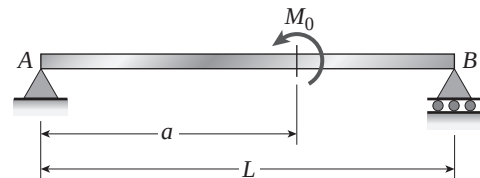


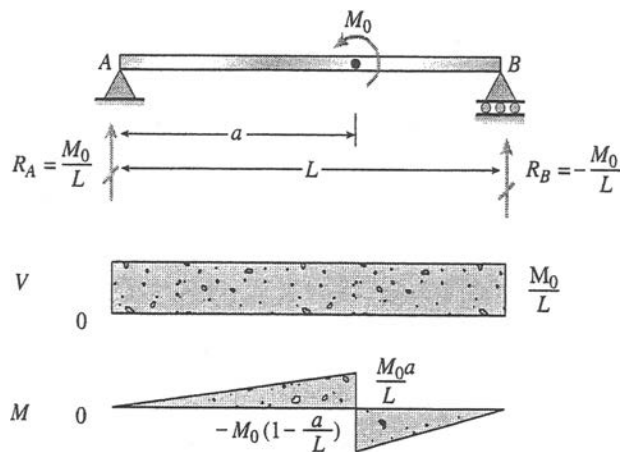
Solution 4.5-1 Simple beam



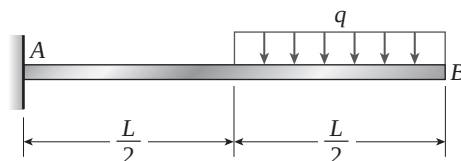
Problem 4.5-2 A simple beam AB is subjected to a counterclockwise couple of moment M_0 acting at distance a from the left-hand support (see figure).

Draw the shear-force and bending-moment diagrams for this beam.

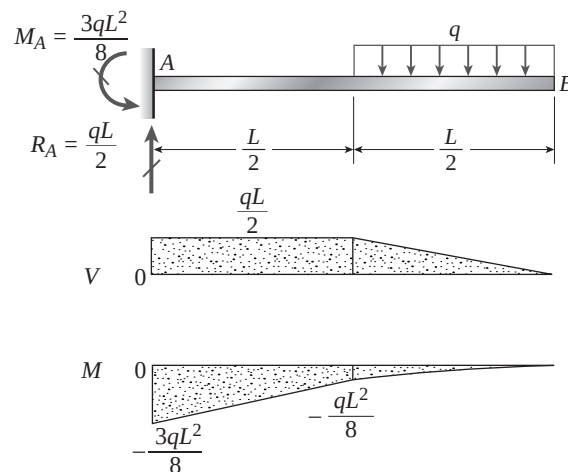


Solution 4.5-2 Simple beam

Problem 4.5-3 Draw the shear-force and bending-moment diagrams for a cantilever beam AB carrying a uniform load of intensity q over one-half of its length (see figure).

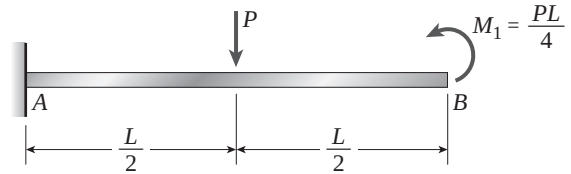
**Solution 4.5-3 Cantilever beam**

Moment varies linearly from A to $L/2$ on moment diagram.

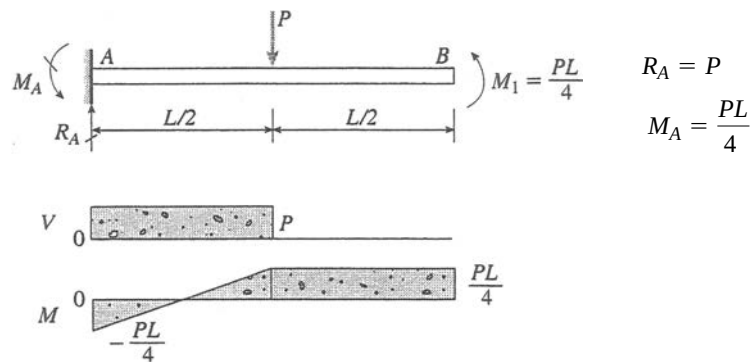


Problem 4.5-4 The cantilever beam AB shown in the figure is subjected to a concentrated load P at the midpoint and a counterclockwise couple of moment $M_1 = PL/4$ at the free end.

Draw the shear-force and bending-moment diagrams for this beam.

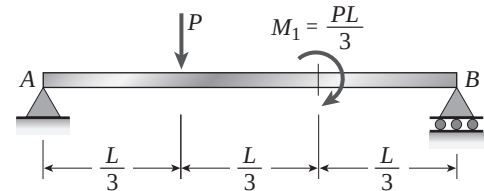


Solution 4.5-4 Cantilever beam

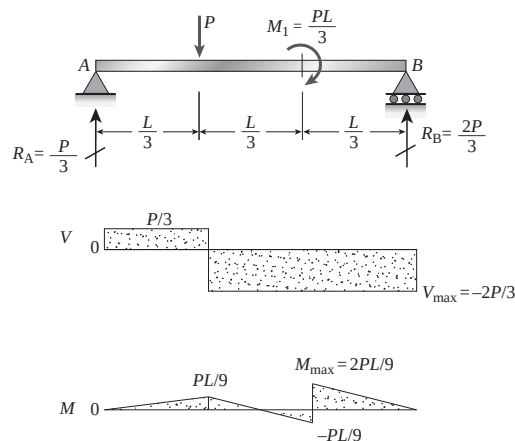


Problem 4.5-5 The simple beam AB shown in the figure is subjected to a concentrated load P and a clockwise couple $M_1 = PL/3$ acting at the third points.

Draw the shear-force and bending-moment diagrams for this beam.

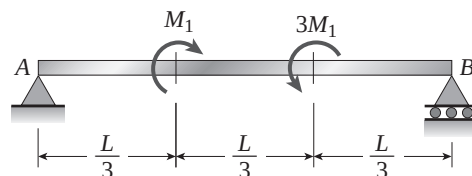


Solution 4.5-5

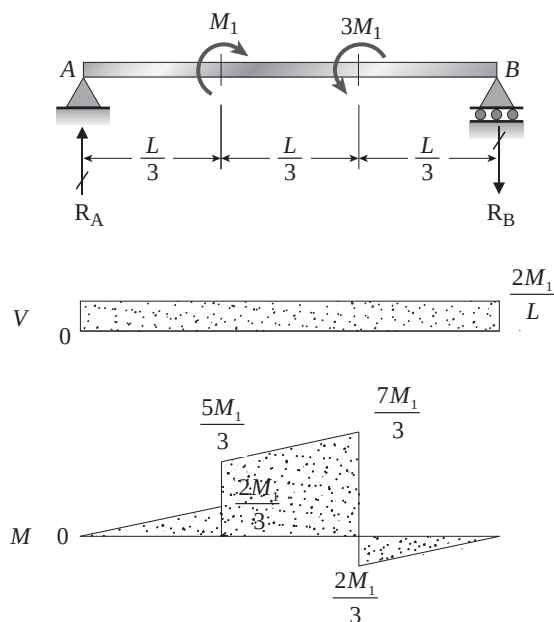


Problem 4.5-6 A simple beam AB subjected to couples M_1 and $3M_1$ acting at the third points is shown in the figure.

Draw the shear-force and bending-moment diagrams for this beam.

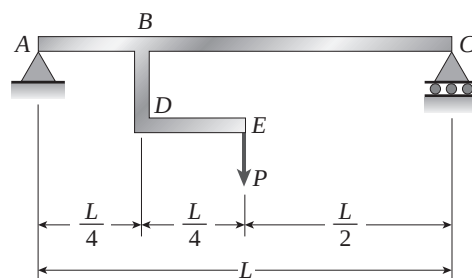


Solution 4.5-6



Problem 4.5-7 A simply supported beam ABC is loaded by a vertical load P acting at the end of a bracket BDE (see figure).

- Draw the shear-force and bending-moment diagrams for beam ABC .
- Now assume that load P at E is directed to the right. Vertical dimension BD is $L/5$. Draw axial-force, shear-force and bending-moment diagrams for ABC .



Solution 4.5-7

- (a) REPLACE DOWNWARD LOAD
- P
- AT
- E
- WITH STATICALLY-EQUIVALENT LOAD
- P
- AND CW MOMENT
- $PL/4$
- AT
- B

$$\sum M_A = 0 \quad R_C = \frac{1}{L} \left(P \frac{L}{4} + \frac{PL}{4} \right) = \frac{P}{2} \quad \sum F_y = 0 \quad R_A = P - R_C = \frac{P}{2}$$

$$V_{\max} = \frac{P}{2} \quad \text{on } AB \quad M_{\max} = R_C \left(\frac{3L}{4} \right) = \frac{3LP}{8} \quad (\text{just right of } B)$$

- (b) REMOVE DOWNWARD LOAD
- P
- ; REPLACE HORIZONTAL LOAD
- P
- AT
- E
- WITH STATICALLY-EQUIVALENT X-LOAD
- P
- AT
- B
- , AND CCW MOMENT
- $PL/5$
- AT
- B

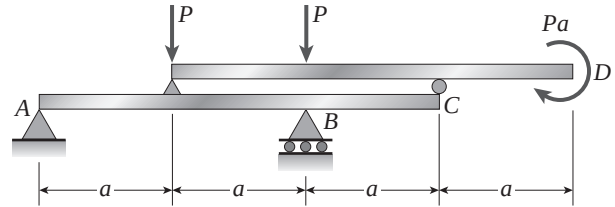
$$\sum M_A = 0 \quad R_{Cy} = \frac{-1}{L} \left(\frac{PL}{5} \right) = -\frac{P}{5} \quad (\text{downward})$$

$$\sum F_y = 0 \quad R_{Ay} = -R_{Cy} = \frac{P}{5} \quad (\text{upward})$$

$$\sum F_x = 0 \quad R_{Ax} = -P \quad (\text{leftward})$$

$$N_{\max} = P \quad (\text{tension on } AB) \quad V_{\max} = \frac{P}{5} \quad M_{\max} = \frac{-P}{5} \left(\frac{3L}{4} \right) = \frac{3LP}{20} \quad (\text{just right of } B)$$

Problem 4.5-8 A beam ABC is simply supported at A and B and has an overhang BC (see figure). The beam is loaded by two forces P and a clockwise couple of moment Pa at D that act through the arrangement shown.



- (a) Draw the shear-force and bending-moment diagrams for beam ABC .
 (b) If moment Pa at D is replaced by moment M , find an expression for M in terms of variables P and a so that the reaction at B goes to zero. Plot the associated shear-force and bending-moment diagrams for beam ABC .

Solution 4.5-8

- (a)

$$\text{UFBD:} \quad \sum M_O = 0 \quad V_C = \frac{1}{2a}(Pa + Pa) = P$$

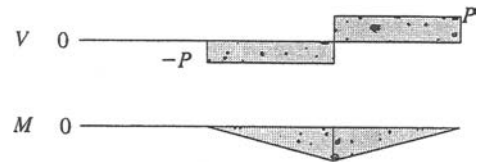
$$\sum F_V = 0 \quad V_O = 2P - P = P$$

$$\text{LFBD:} \quad \text{DOWNWARD LOAD } P \text{ AT } O \text{ AND ALSO AT } C$$

$$\sum M_A = 0 \quad R_B = \frac{1}{2a}[Pa + P(3a)] = 2P$$

$$\sum F_V = 0 \quad R_A = -R_B + 2P = 0$$

$$V_{\max} = P, \quad M_{\max} = -Pa$$



(b)

$$\text{UFBD: } V_C = \frac{1}{2a}(Pa + M) = \frac{M + Pa}{2a} \quad < \text{sum moments about left end of upper beam to find downward load } V_C \text{ applied on lower beam}$$

$$V_O = 2P - V_C \quad < \text{sum vertical forces for upper beam to find downward load } V_O \text{ applied downward on lower beam by upper beam}$$

$$\text{LFBD: } R_B = \frac{1}{2a}[V_O a + V_C(3a)] = \frac{\frac{3M}{2} + \frac{3Pa}{2} + a\left(2P - \frac{M + Pa}{2a}\right)}{2a} \quad < \text{find reaction at B}$$

$$\boxed{R_B = 0 \text{ solve, } M = -3Pa} \quad < \text{CCW} \quad M = -3Pa \quad R_B = 0$$

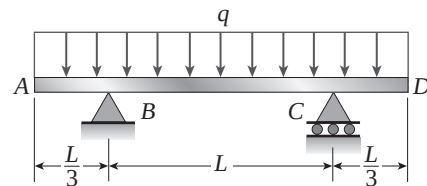
$$R_A = -R_B + V_O + V_C = 2P$$

$$V_C = \frac{1}{2a}(Pa + M) = -P$$

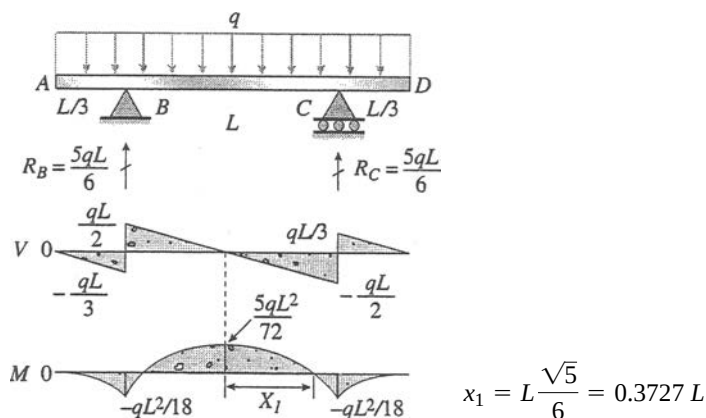
$$V_O = 2P - V_C = 3P$$

Problem 4.5-9 Beam $ABCD$ is simply supported at B and C and has overhangs at each end (see figure). The span length is L and each overhang has length $L/3$. A uniform load of intensity q acts along the entire length of the beam.

Draw the shear-force and bending-moment diagrams for this beam.

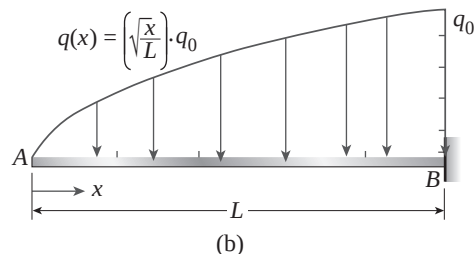
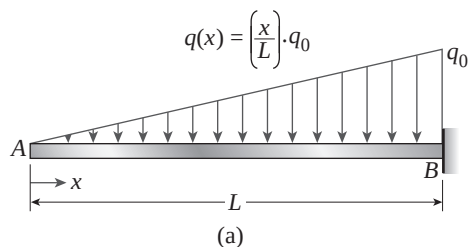


Solution 4.5-9 Beam with overhangs



Problem 4.5-10 Draw the shear-force and bending-moment diagrams for a cantilever beam AB acted upon by two different load cases:

- (a) A distributed load with linear variation and maximum intensity q_0 (see figure part a).
- (b) A distributed load with parabolic variation and maximum intensity q_0 (see figure part b).



Solution 4.5-10

(a) SHEAR DIAGRAM IS QUADRATIC; MOMENT DIAGRAM IS CUBIC

$$V_{\max} = - \int_0^L \frac{x}{L} q_0 dx = - \frac{L q_0}{2} \quad \text{at B} \quad M(x) = - \int_0^x \frac{\zeta}{L} q_0 (x - \zeta) d\zeta = - \frac{q_0 x^3}{6L}$$

$$M_{\max} = M(L) = - \frac{L^2 q_0}{6} \quad M_{\max} = - \frac{L^2 q_0}{6} \quad \text{at B}$$

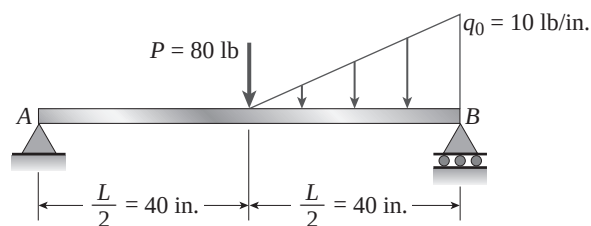
(b) SHEAR DIAGRAM IS CUBIC; MOMENT DIAGRAM IS QUARTIC

$$V_{\max} = - \int_0^L \sqrt{\frac{x}{L}} q_0 dx = - \frac{2L q_0}{3} \quad \text{at B} \quad M(x) = - \int_0^x \sqrt{\frac{\zeta}{L}} q_0 (x - \zeta) d\zeta = - \frac{4 q_0 x^2 \sqrt{x/L}}{15}$$

$$V_{\max} = - \frac{2L q_0}{3} \quad M_{\max} = M(L) = - \frac{4L^2 q_0}{15} \quad M_{\max} = - \frac{4L^2 q_0}{15} \quad \text{at B}$$

Problem 4.5-11 The simple beam AB supports a triangular load of maximum intensity $q_0 = 10 \text{ lb/in.}$ acting over one-half of the span and a concentrated load $P = 80 \text{ lb}$ acting at midspan (see figure).

Draw the shear-force and bending-moment diagrams for this beam.



Solution 4.5-11 Simple beam

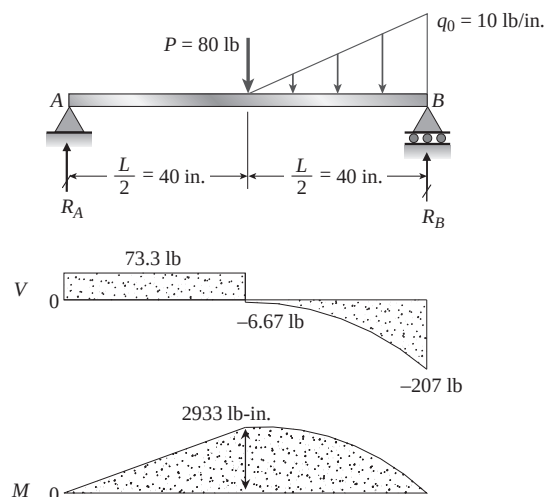
$$\Sigma M_A = 0: R_B(80 \text{ in.}) - (80 \text{ lb})(40 \text{ in.})$$

$$- (10 \text{ lb/in.})(\frac{1}{2})(40 \text{ in.})(40 + 40\frac{2}{3} \text{ in.}) = 0$$

$$R_B = 206.7 \text{ lb}$$

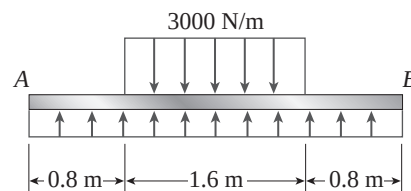
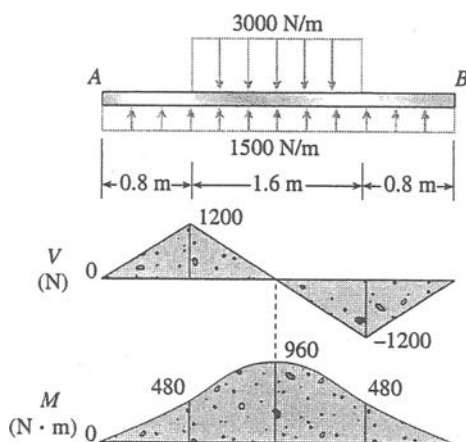
$$\Sigma F_{\text{VERT}} = 0: R_A + R_B - 80 \text{ lb} - \left(10 \text{ lb/in.}\right)\left(\frac{1}{2}\right)(40 \text{ in.}) = 0$$

$$R_A = 73.3 \text{ lb}$$



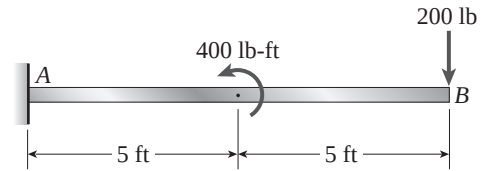
Problem 4.5-12 The beam AB shown in the figure supports a uniform load of intensity 3000 N/m acting over half the length of the beam. The beam rests on a foundation that produces a uniformly distributed load over the entire length.

Draw the shear-force and bending-moment diagrams for this beam.

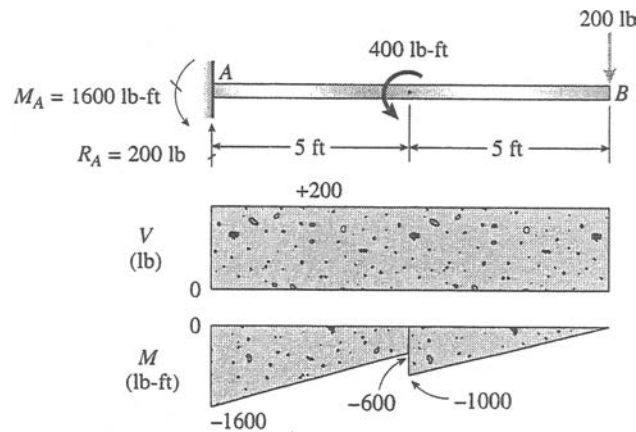
**Solution 4.5-12 Beam with distributed loads**

Problem 4.5-13 A cantilever beam AB supports a couple and a concentrated load, as shown in the figure.

Draw the shear-force and bending-moment diagrams for this beam.

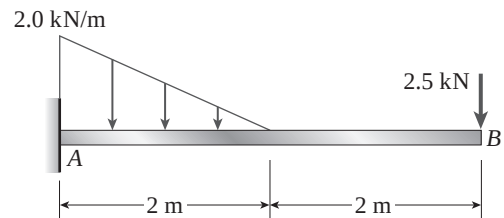


Solution 4.5-13 Cantilever beam

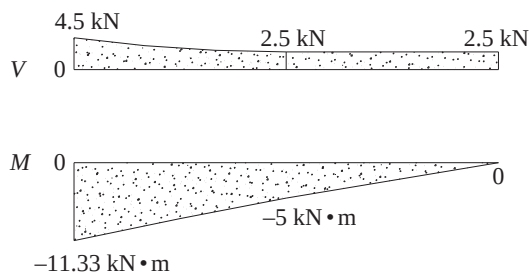


Problem 4.5-14 The cantilever beam AB shown in the figure is subjected to a triangular load acting throughout one-half of its length and a concentrated load acting at the free end.

Draw the shear-force and bending-moment diagrams for this beam.

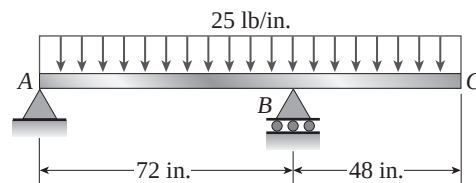


Solution 4.5-14

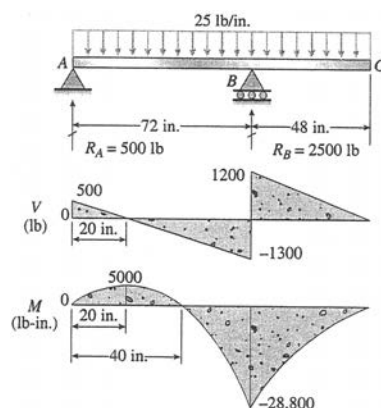


Problem 4.5-15 The uniformly loaded beam ABC has simple supports at A and B and an overhang BC (see figure).

Draw the shear-force and bending-moment diagrams for this beam.

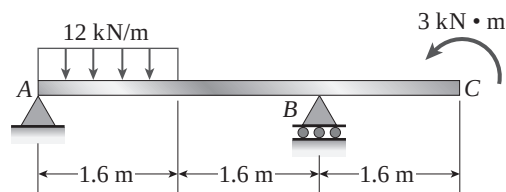


Solution 4.5-15 Beam with an overhang

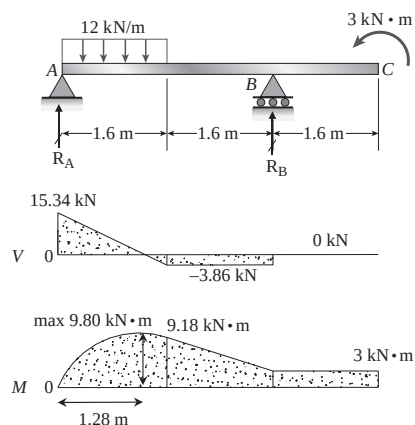


Problem 4.5-16 A beam ABC with an overhang at one end supports a uniform load of intensity 12 kN/m and a concentrated moment of magnitude $3 \text{ kN}\cdot\text{m}$ at C (see figure).

Draw the shear-force and bending-moment diagrams for this beam.

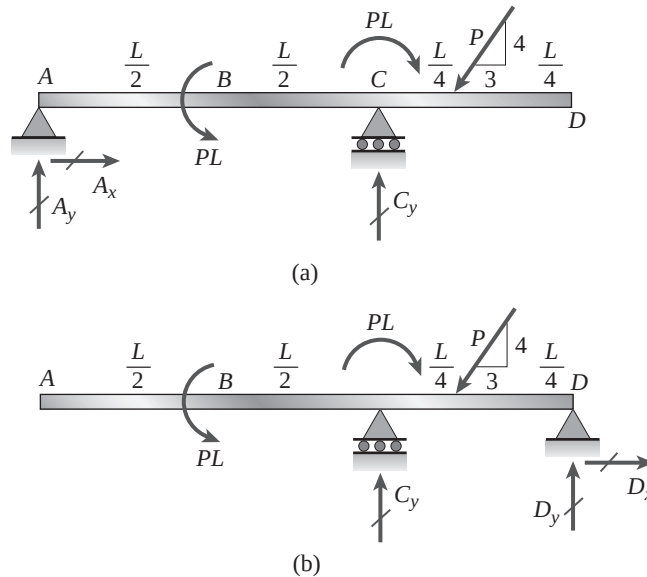


Solution 4.5-16 Beam with an overhang



Problem 4.5-17 Consider the two beams below; they are loaded the same but have different support conditions. Which beam has the larger maximum moment?

First, find support reactions, then plot axial force (N), shear (V) and moment (M) diagrams for both beams. Label all critical N , V , and M values and also the distance to points where N , V , and/or M is zero.



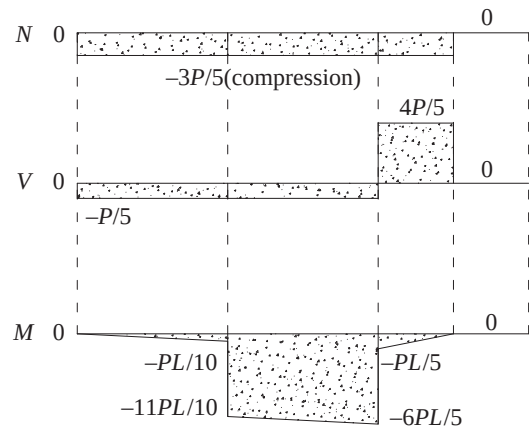
Solution 4.5-17

BEAM (a):

$$\sum M_A = 0: C_y = \frac{1}{L} \left(\frac{4}{5} P \frac{5}{4} L \right) = P \text{ (upward)}$$

$$\sum F_V = 0: A_y = \frac{4}{5} P - C_y = -\frac{P}{5} \text{ (downward)}$$

$$\sum F_H = 0: A_x = \frac{3}{5} P \text{ (right)}$$



BEAM (b):

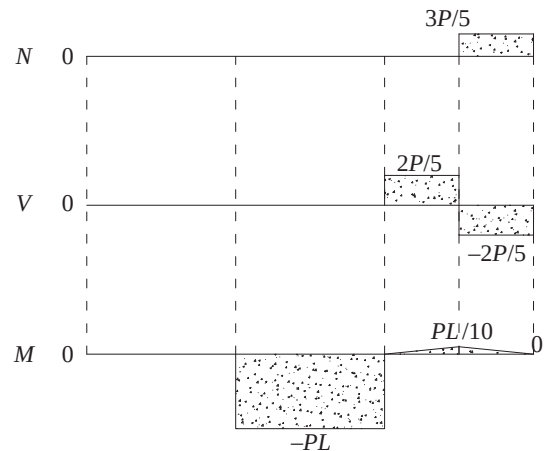
$$\sum M_D = 0: C_y = \frac{2}{L} \left(\frac{4}{5} P \frac{1}{4} L \right) = \frac{2}{5} P \text{ (upward)}$$

$$\sum F_V = 0: D_y = \frac{4}{5} P - C_y = \frac{2}{5} P \text{ (upward)}$$

$$\sum F_H = 0: D_x = \frac{3}{5} P \text{ (right)}$$

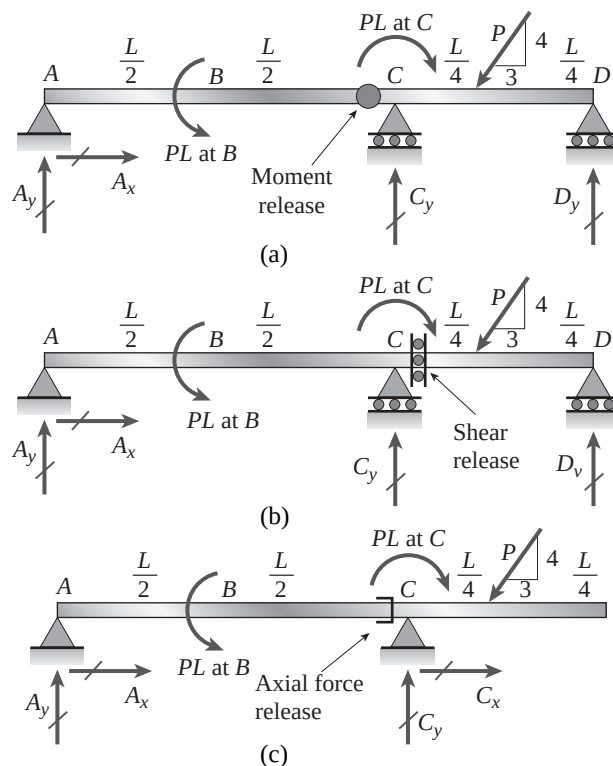
$\therefore$ The first case has the larger maximum moment

$$\left(\frac{6}{5} PL \right) \leftarrow$$



Problem 4.5-18 The three beams below are loaded the same and have the same support conditions. However, one has a *moment release* just to the left of C, the second has a *shear release* just to the right of C, and the third has an *axial release* just to the left of C. Which beam has the largest maximum moment?

First, find support reactions, then plot axial force (N), shear (V) and moment (M) diagrams for all three beams. Label all critical N , V , and M values and also the *distance* to points where N , V , and/or M is zero.



Solution 4.5-18

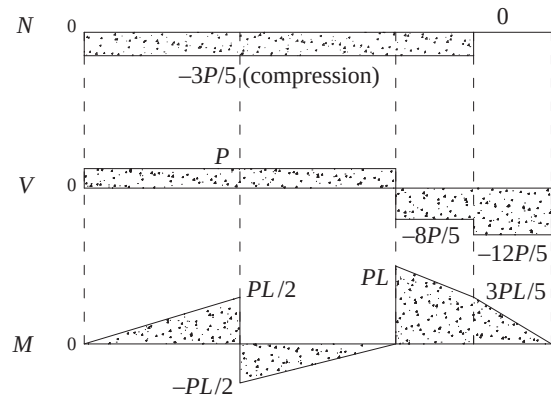
BEAM (a): MOMENT RELEASE

$$A_y = P \text{ (upward)}$$

$$C_y = -\frac{13}{5}P \text{ (downward)}$$

$$D_y = \frac{12}{5}P \text{ (upward)}$$

$$A_x = \frac{3}{5}P \text{ (right)}$$



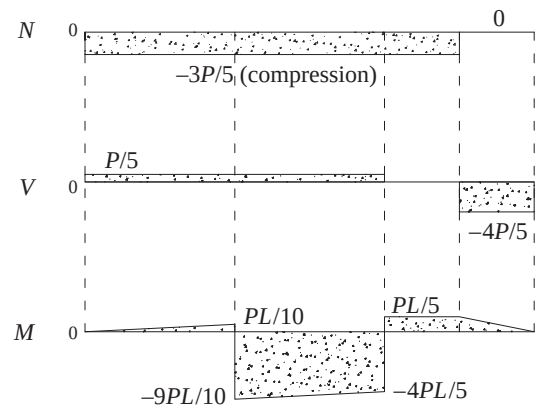
BEAM (b): SHEAR RELEASE

$$A_y = \frac{1}{5}P \text{ (upward)}$$

$$C_y = -\frac{1}{5}P \text{ (downward)}$$

$$D_y = \frac{4}{5}P \text{ (upward)}$$

$$A_x = \frac{3}{5}P \text{ (right)}$$



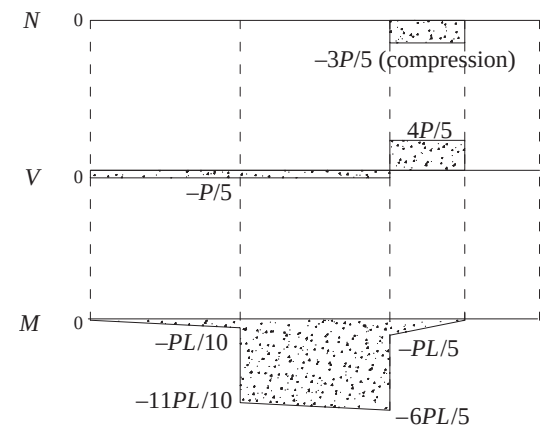
BEAM (c): AXIAL RELEASE

$$A_y = -\frac{1}{5}P \text{ (downward)}$$

$$C_y = P \text{ (upward)}$$

$$A_x = 0$$

$$C_x = \frac{3}{5}P \text{ (right)}$$

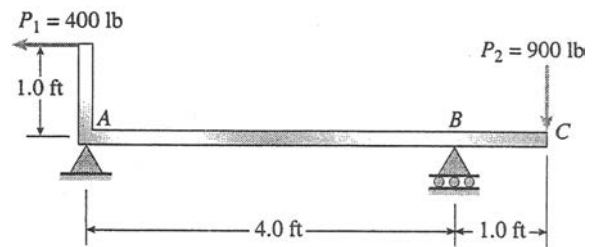


∴ The third case has the largest maximum moment

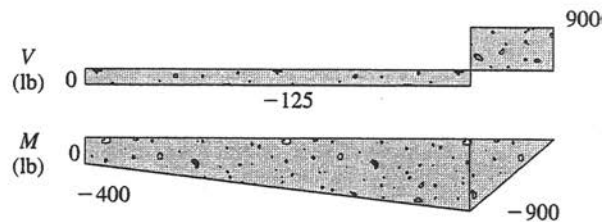
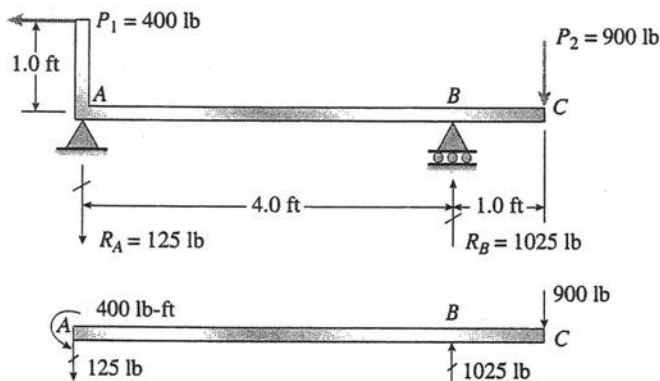
$$\left(\frac{6}{5}PL\right) \leftarrow$$

Problem 4.5-19 A beam $ABCD$ shown in the figure is simply supported at A and B and has an overhang from B to C . The loads consist of a horizontal force $P_1 = 400$ lb acting at the end of the vertical arm and a vertical force $P_2 = 900$ lb acting at the end of the overhang.

Draw the shear-force and bending-moment diagrams for this beam. (Note: Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)

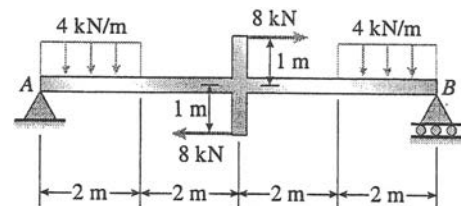


Solution 4.5-19 Beam with vertical arm

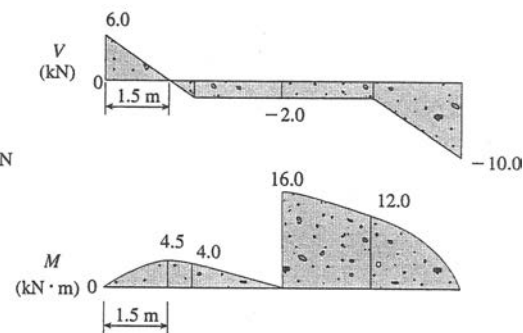
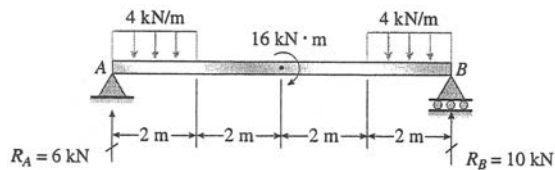


Problem 4.5-20 A simple beam AB is loaded by two segments of uniform load and two horizontal forces acting at the ends of a vertical arm (see figure).

Draw the shear-force and bending-moment diagrams for this beam.

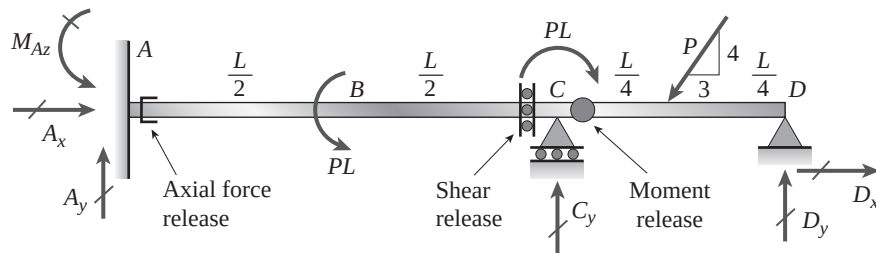


Solution 4.5-20 Simple beam

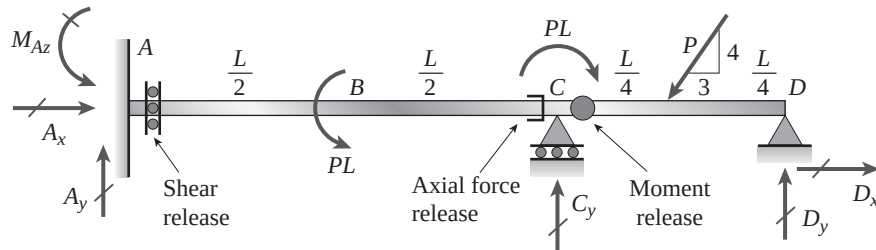


Problem 4.5-21 The two beams below are loaded the same and have the same support conditions. However, the location of internal *axial*, *shear* and *moment* releases is different for each beam (see figures). Which beam has the larger maximum moment?

First, find support reactions, then plot axial force (N), shear (V) and moment (M) diagrams for both beams. Label all critical N , V , and M values and also the *distance* to points where N , V , and/or M is zero.



(a)



(b)

Solution 4.5-21

Support reactions for both beams:

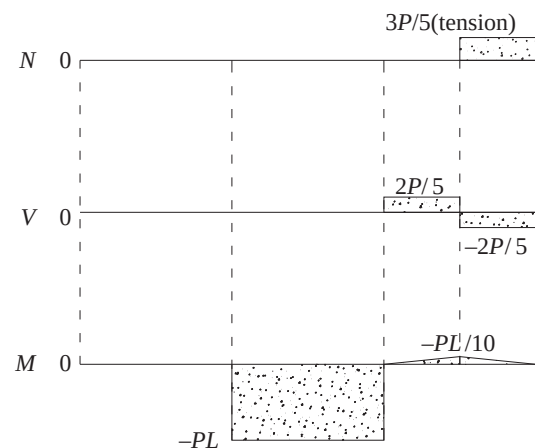
$$M_{Az} = 0, A_x = 0, A_y = 0$$

$$C_y = \frac{2}{5}P \text{ (upward)}, D_y = \frac{2}{5}P \text{ (upward)}$$

$$D_x = \frac{3}{5}P \text{ (rightward)}$$

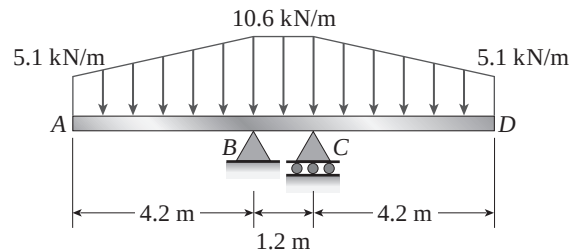
$\therefore$ These two cases have the same maximum moment (PL) $\leftarrow$

(Both beams have the same N , V , and M diagrams)

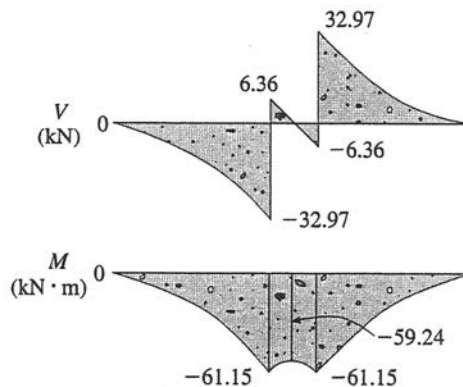
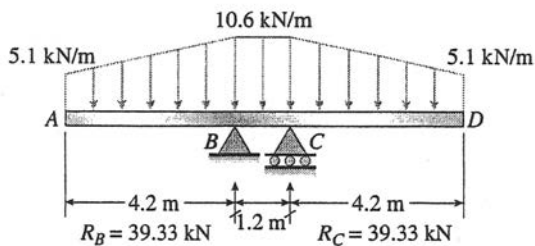


Problem 4.5-22 The beam $ABCD$ shown in the figure has overhangs that extend in both directions for a distance of 4.2 m from the supports at B and C , which are 1.2 m apart.

Draw the shear-force and bending-moment diagrams for this overhanging beam.

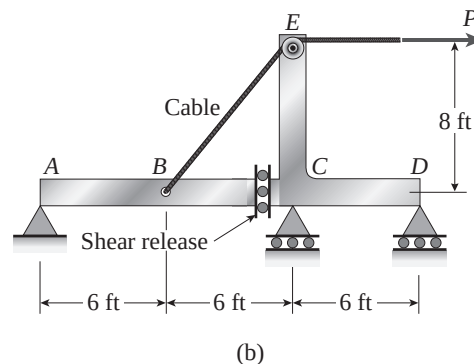
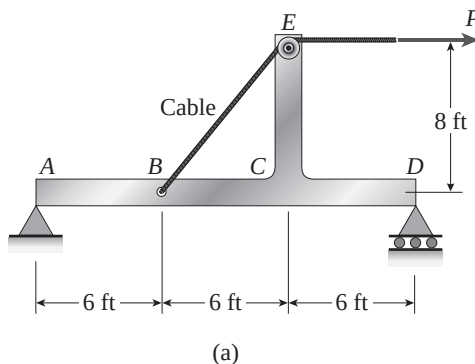


Solution 4.5-22 Beam with overhangs



Problem 4.5-23 A beam $ABCD$ with a vertical arm CE is supported as a simple beam at A and D (see figure). A cable passes over a small pulley that is attached to the arm at E . One end of the cable is attached to the beam at point B . The tensile force in the cable is 1800 lb.

- Draw the shear-force and bending-moment diagrams for beam $ABCD$. (Note: Disregard the widths of the beam and vertical arm and use centerline dimensions when making calculations.)
- Repeat part (a) if a roller support is added at C and a shear release is inserted just left of C (see figure part b).

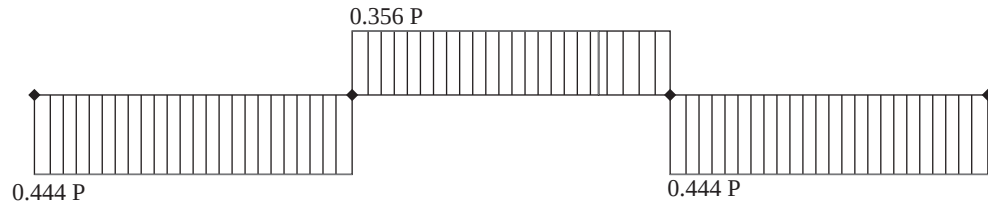


Solution 4.5-23

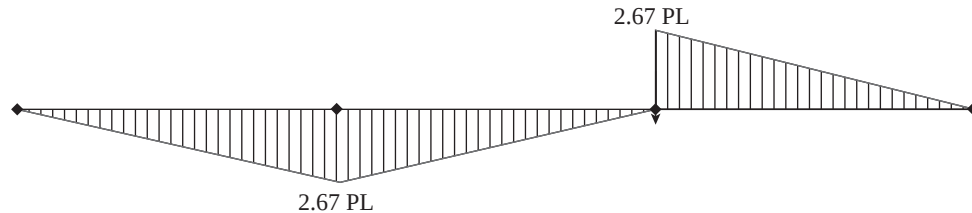
(a) NO SHEAR RELEASE

$$R_A = \frac{1}{18}(-8P) = -\frac{4P}{9} \quad R_D = -R_A = \frac{4P}{9} \quad V_{BC} = R_A + 0.8P = 0.35556P$$

SHEAR DIAGRAM



MOMENT DIAGRAM



(b) ADD SUPPORT AT C AND SHEAR RELEASE JUST LEFT OF C

$$\text{LHFB: } V_C = R_A + \frac{4}{5}P = 0 \quad \text{so} \quad R_A = -\frac{4}{5}P$$

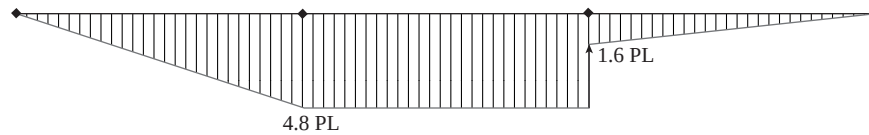
$$\text{Entire FBD: } \sum M_C = 0 \quad R_D = \frac{1}{6}(R_A 12 + P 8) = -\frac{4P}{15}$$

$$\sum F_y = 0 \quad R_C = -(R_A + R_D) = \frac{16P}{15}$$

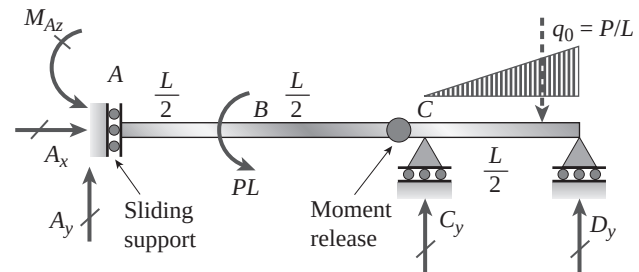
SHEAR DIAGRAM



MOMENT DIAGRAM



Problem 4.5-24 Beams ABC and CD are supported at A , C , and D , and are joined by a hinge (or *moment release*) just to the left of C and a *shear release* just to the right of C . The support at A is a sliding support (hence reaction $A_y = 0$ for the loading shown below). Find all support reactions then plot shear (V) and moment (M) diagrams. Label all critical V and M values and also the *distance* to points where either V and/or M is zero.

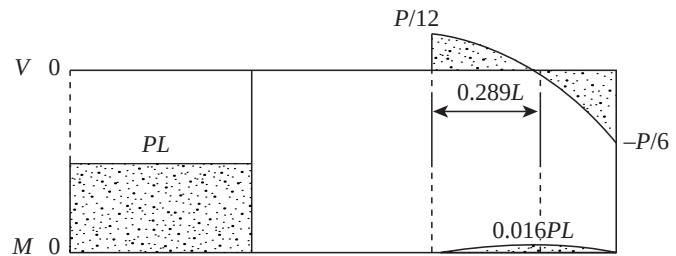


Solution 4.5-24

$$M_{Az} = -PL \text{ (clockwise), } A_x = 0, A_y = 0 \quad \leftarrow$$

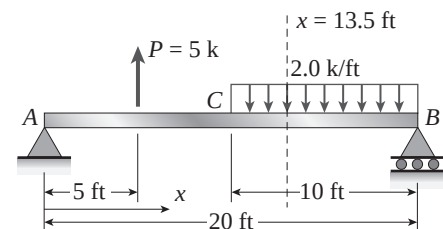
$$C_y = \frac{1}{12}P \text{ (upward), } D_y = \frac{1}{6}P \text{ (upward)} \quad \leftarrow$$

$$V_{\max} = \frac{P}{6} \quad M_{\max} = PL \quad \leftarrow$$



Problem 4.5-25 The simple beam AB shown in the figure supports a concentrated load and a segment of uniform load.

- Draw the shear-force and bending-moment diagrams for this beam.
- Find the value of P that will result in zero shear at $x = 13.5$ ft. Draw shear-force and bending-moment diagrams for this case.



Solution 4.5-25

(a)

$$R_B = \frac{1}{20}(-5 \times 5 + 2 \times 10 \times 15) = 13.75 \text{ k} \quad R_A = -5 - R_B + 2 \times 10 = 1.25 \text{ k}$$

$$V_{13a} = -R_B + 26.5 = -0.75$$

(b) 6.5 FT FROM B, USE RHFB

$$R_B = \frac{1}{20}(-5P + 2 \times 10 \times 15) = 15 - \frac{P}{4} \quad V_{13b} = -R_B + 2 \times 6.5 = \frac{P}{4} - 2.0 \quad 20 - 6.5 = 13.5$$

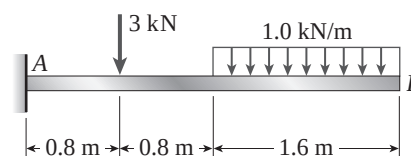
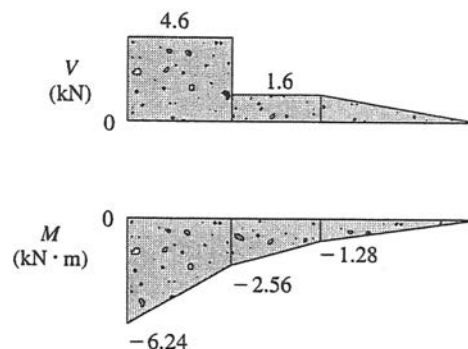
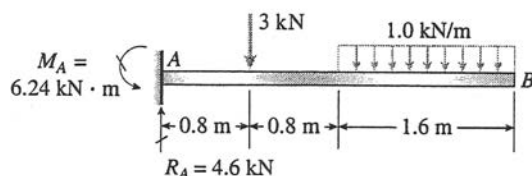
$$P = 8$$

So apply $P = 8 \text{ k}$ upward.

$$R_B = \frac{1}{20}(-5P + 2 \times 10 \times 15) = 13 \quad R_A = -R_B - P + 2(10) = -1$$

Problem 4.5-26 The cantilever beam shown in the figure supports a concentrated load and a segment of uniform load.

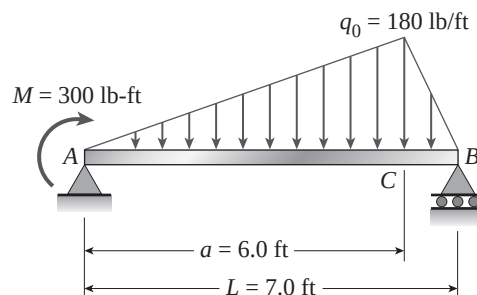
Draw the shear-force and bending-moment diagrams for this cantilever beam.

**Solution 4.5-26 Cantilever beam**

Problem 4.5-27 The simple beam ACB shown in the figure is subjected to a triangular load of maximum intensity $q_0 = 180 \text{ lb/ft}$ at $a = 6.0 \text{ ft}$, and a concentrated moment $M = 300 \text{ lb-ft}$ at A.

- Draw the shear-force and bending-moment diagrams for this beam.
- Find the value of distance a that results in the maximum moment occurring at $L/2$. Draw the shear-force and bending-moment diagrams for this case.
- Find the value of distance a for which $M_{\max}$ is the largest possible value.

$$a = 6 \quad L = 7 \quad M_0 = 300 \quad q = 180 \quad (\text{Units in feet and pounds})$$



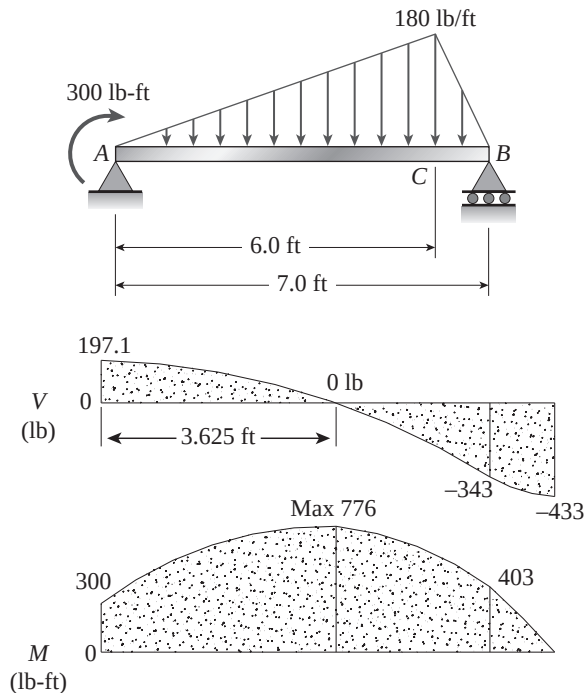
Solution 4.5-27

(a) FIND REACTIONS THEN USE TO FORM EXPRESSIONS FOR $V(x)$ AND $M(x)$; $M = M_{\max}$ AT LOCATION OF ZERO V

$$R_B = \frac{1}{L} \left[M_0 + \frac{1}{2} q a \left(\frac{2}{3} \right) a + \frac{1}{2} q (L - a) \left(a + \frac{L - a}{3} \right) \right] = 432.85 \text{ lb} \quad R_A = \frac{1}{2} q L - R_B = 197.143 \text{ lb}$$

$$V = R_A - \frac{1}{2} q \left(\frac{x}{a} \right) x \quad \text{If } V = 0: \quad x_m = \sqrt{\frac{2 R_A a}{q}} = 3.625 \text{ ft} \quad x = x_m$$

$$M_{\max} = M_0 + R_A x - \frac{1}{2} q \left(\frac{x}{a} \right) x \frac{x}{3} = 776.469 \text{ lb-ft}$$



(b) FIND VALUE OF DISTANCE a FOR WHICH SHEAR V IS ZERO (i.e., LOCATION OF $M_{\max}$) AT $x = L/2 = 3.5$ ft

$$V = 0 \quad \text{when} \quad \frac{1}{2} q a = R_A \text{—substitute expression for } R_A, \text{ then solve numerically for } a.$$

$$\sqrt{\frac{2 \left[\frac{1}{2} q L - \frac{1}{L} \left[M_0 + \frac{1}{2} q a \left(\frac{2}{3} \right) a + \frac{1}{2} q (L - a) \left(a + \frac{L - a}{3} \right) \right] \right] a}{q}} = 3.5$$

$$\text{Solve: } a = \left(\frac{7.9471}{4.624328} \right)$$

Second solution for a is valid: $a = 4.624328$ ft

$$R_B = \frac{1}{L} \left[M_0 + \frac{1}{2} q a \left(\frac{2}{3} \right) a + \frac{1}{2} q (L - a) \left(a + \frac{L - a}{3} \right) \right] = 391.58 \text{ lb} \quad R_A = \frac{1}{2} q L - R_B = 238.413 \text{ lb}$$

$$x_m = \sqrt{\frac{2 R_A a}{q}} = 3.5 \text{ ft} \quad M_{\max} = M_0 + R_A x_m - \frac{1}{2} q \left(\frac{x_m}{a} \right) x_m \frac{x_m}{3} = 856.29 \text{ lb-ft}$$

(c) FIND VALUE OF a (i.e., LOCATION WHERE $Q(x)$ HAS PEAK INTENSITY) FOR WHICH $M_{\max}$ IS LARGEST POSSIBLE

(1) Write expressions for reactions and maximum moment in terms of variable a :

$$R_B = \frac{1}{L} \left[M_0 + \frac{1}{2} q a \left(\frac{2}{3} \right) a + \frac{1}{2} q (L - a) \left(a + \frac{L - a}{3} \right) \right] = 30a + \frac{1770}{7}$$

$$R_A = \frac{1}{2} q L - R_B = \frac{2640}{7} - 30a$$

$$x_m = \sqrt{\frac{2R_A a}{q}} = \frac{\sqrt{21} \sqrt{88a - 7a^2}}{21} \quad x = x_m$$

$$M_{\max} = M_0 + R_A x - \frac{1}{2} q \left(\frac{x}{a} \right) x \frac{x}{3} = \frac{1760 \sqrt{1848a - 147a^2}}{147} - \frac{20a \sqrt{1848a - 147a^2}}{21} + 300$$

(2) Differentiate expressions for $M_{\max}$ with respect to a . Set expression equal to zero and solve for a :

$$\frac{d}{da} M_{\max} \rightarrow \frac{10a(294a - 1848)}{21 \sqrt{1848a - 147a^2}} - \frac{20 \sqrt{1848a - 147a^2}}{21} - \frac{880(294a - 1848)}{147 \sqrt{1848a - 147a^2}}$$

$$\text{Solve numerically: } a = \frac{22}{7} = 3.143$$

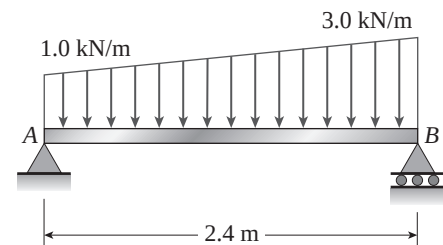
Substitute a into reaction and moment expressions to get target possible $M_{\max}$:

$$R_B = 30a + \frac{1770}{7} = 347.143 \text{ lb} \quad R_A = \frac{1}{2} q L - R_B = 282.857 \text{ lb} \quad x_m = \sqrt{\frac{2R_A a}{q}} = 3.143 \text{ ft}$$

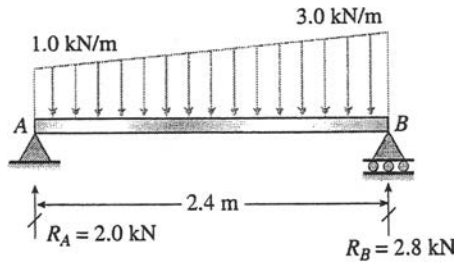
$$M_{\max} = M_0 + R_A x - \frac{1}{2} q \left(\frac{x}{a} \right) x \frac{x}{3} = 892.653 \text{ lb-ft}$$

Problem 4.5-28 A beam with simple supports is subjected to a trapezoidally distributed load (see figure). The intensity of the load varies from 1.0 kN/m at support A to 3.0 kN/m at support B.

Draw the shear-force and bending-moment diagrams for this beam.

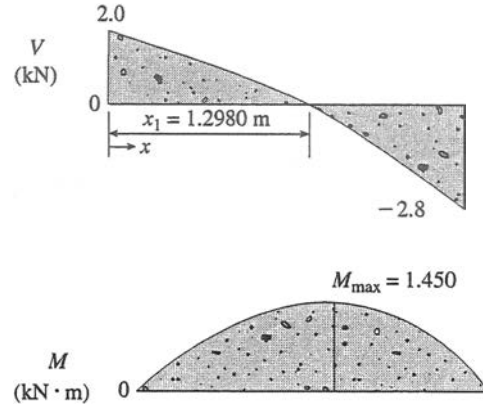


Solution 4.5-28 Simple beam



$$V = 2.0 - x - \frac{x^2}{2.4} \quad (x = \text{meters}; V = \text{kN})$$

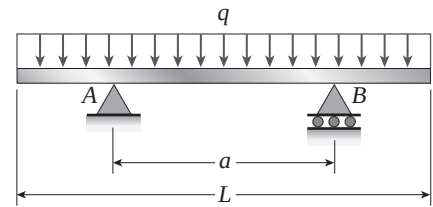
$$\text{Set } V = 0: \quad x_1 = 1.2980 \text{ m}$$



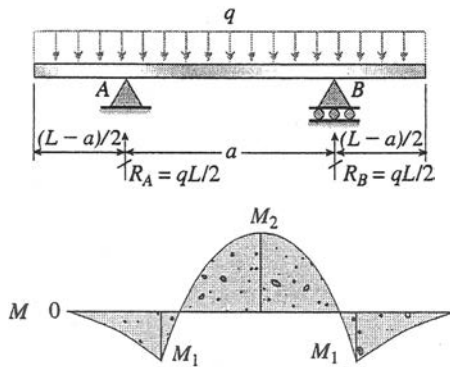
Problem 4.5-29 A beam of length L is being designed to support a uniform load of intensity q (see figure). If the supports of the beam are placed at the ends, creating a simple beam, the maximum bending moment in the beam is $qL^2/8$. However, if the supports of the beam are moved symmetrically toward the middle of the beam (as pictured), the maximum bending moment is reduced.

Determine the distance a between the supports so that the maximum bending moment in the beam has the smallest possible numerical value.

Draw the shear-force and bending-moment diagrams for this condition.



Solution 4.5-29 Beam with overhangs



The maximum bending moment is smallest when $M_1 = M_2$ (numerically).

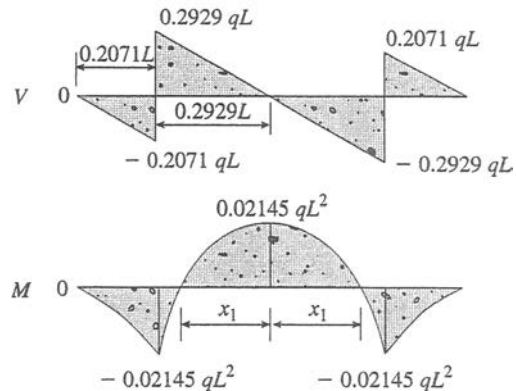
$$M_1 = \frac{q(L-a)^2}{8}$$

$$M_2 = R_A \left(\frac{a}{2} \right) - \frac{qL^2}{8} = \frac{qL}{8} (2a - L)$$

$$M_1 = M_2 \quad (L-a)^2 = L(2a-L)$$

$$\text{Solve for } a: \quad a = (2 - \sqrt{2})L = 0.5858L \quad \leftarrow$$

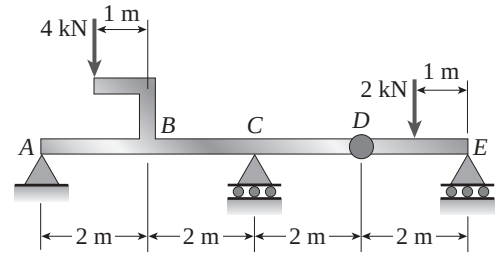
$$\begin{aligned} M_1 &= M_2 = \frac{q}{8} (L-a)^2 \\ &= \frac{qL^2}{8} (3 - 2\sqrt{2}) = 0.02145qL^2 \quad \leftarrow \end{aligned}$$



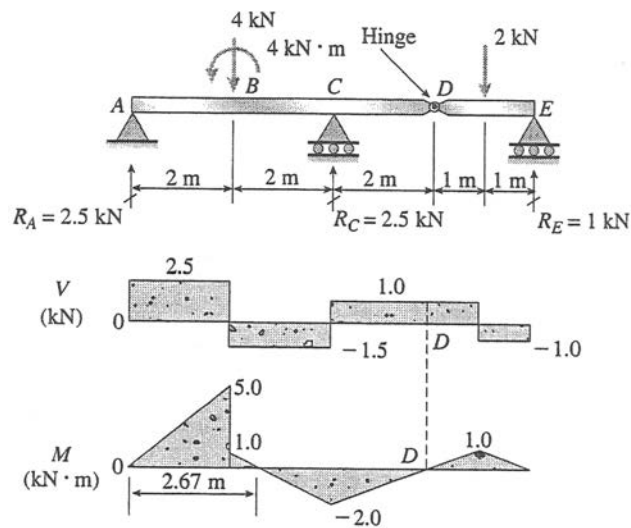
$$\begin{aligned} x_1 &= 0.3536a \\ &= 0.2071L \end{aligned}$$

Problem 4.5-30 The compound beam $ABCDE$ shown in the figure consists of two beams (AD and DE) joined by a hinged connection at D . The hinge can transmit a shear force but not a bending moment. The loads on the beam consist of a 4-kN force at the end of a bracket attached at point B and a 2-kN force at the midpoint of beam DE .

Draw the shear-force and bending-moment diagrams for this compound beam.

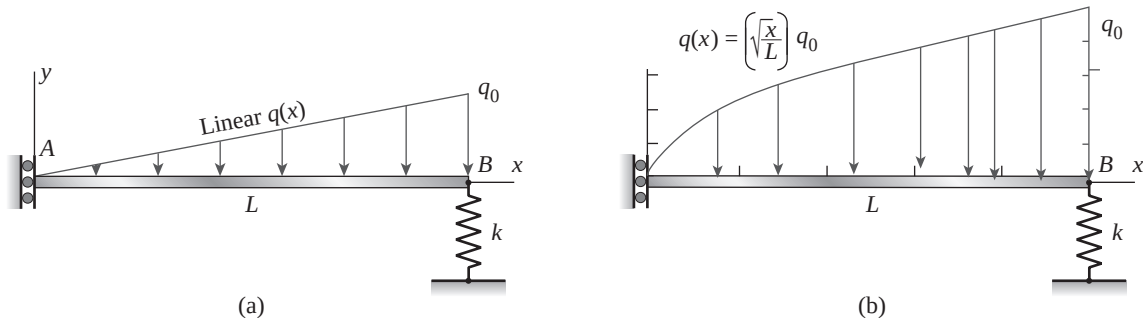


Solution 4.5-30 Compound beam



Problem 4.5-31 Draw the shear-force and bending-moment diagrams for beam AB , with a sliding support at A and an elastic support with spring constant k at B acted upon by two different load cases.

- A distributed load with linear variation and maximum intensity q_0 (see figure part a).
- A distributed load with parabolic variation and maximum intensity q_0 (see figure part b).



Solution 4.5-31

(a) SHEAR DIAGRAM IS QUADRATIC; MOMENT DIAGRAM IS CUBIC

$$R_B = \frac{1}{2}q_0L \quad M_A = -R_B L + \frac{1}{2}q_0L\left(\frac{2}{3}L\right) = -\frac{L^2q_0}{6}$$

$$V_{\max} = -R_B = -\frac{Lq_0}{2} \quad M_{\max} = -M_A = \frac{L^2q_0}{6}$$

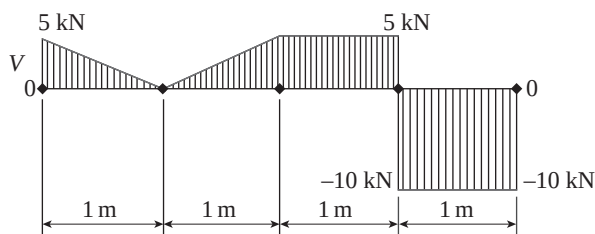
(b) SHEAR DIAGRAM IS CUBIC; MOMENT DIAGRAM IS QUARTIC

$$R_B = \int_0^L \sqrt{\frac{x}{L}} q_0 dx = \frac{2Lq_0}{3} \quad M_A = -R_B L + R_B \frac{3L}{5} = -\frac{4L^2q_0}{15}$$

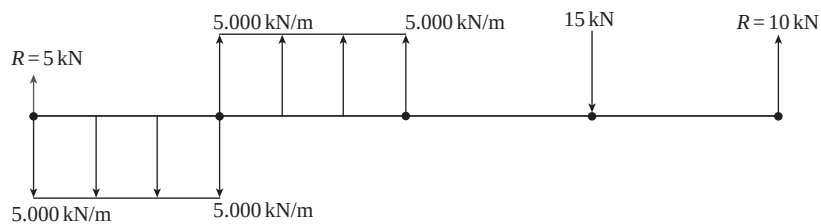
$$V_{\max} = -R_B = -\frac{2Lq_0}{3} \quad M_{\max} = -M_A = \frac{4L^2q_0}{15}$$

Problem 4.5-32 The shear-force diagram for a simple beam is shown in the figure.

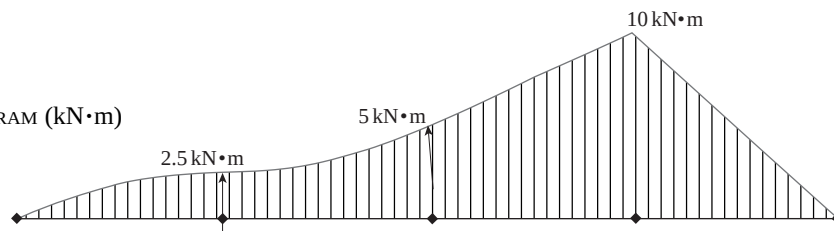
Determine the loading on the beam and draw the bending-moment diagram, assuming that no couples act as loads on the beam.

**Solution 4.5-32**

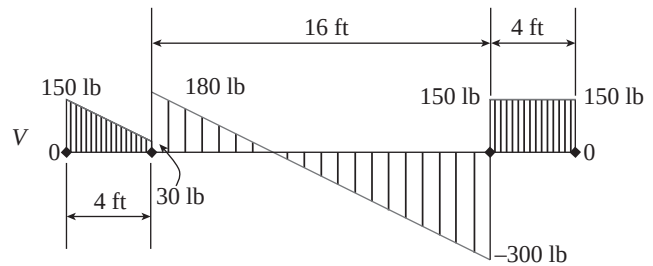
BEAM WITH LOADING AND REACTION FORCES (kN, meters)



MOMENT DIAGRAM (kN·m)

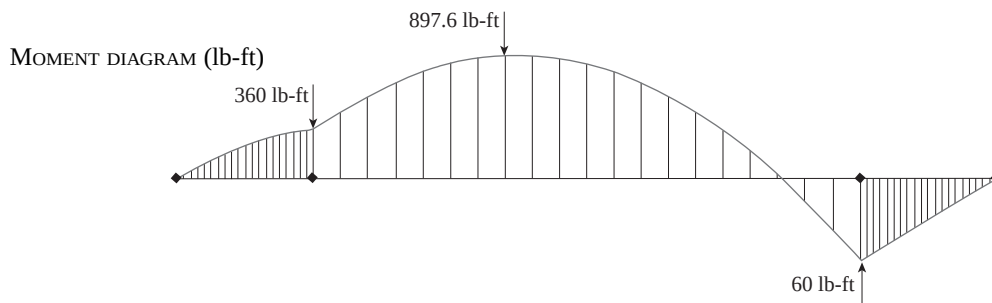
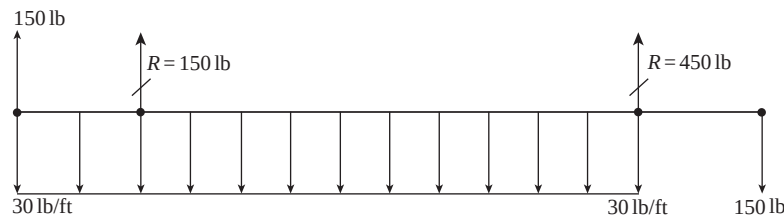


Problem 4.5-33 The shear-force diagram for a beam is shown in the figure. Assuming that no couples act as loads on the beam, determine the forces acting on the beam and draw the bending-moment diagram.



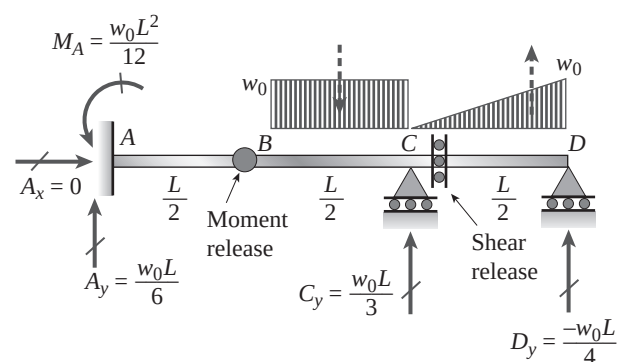
Solution 4.5-33

BEAM WITH LOADING AND REACTION FORCES (lbs, ft)



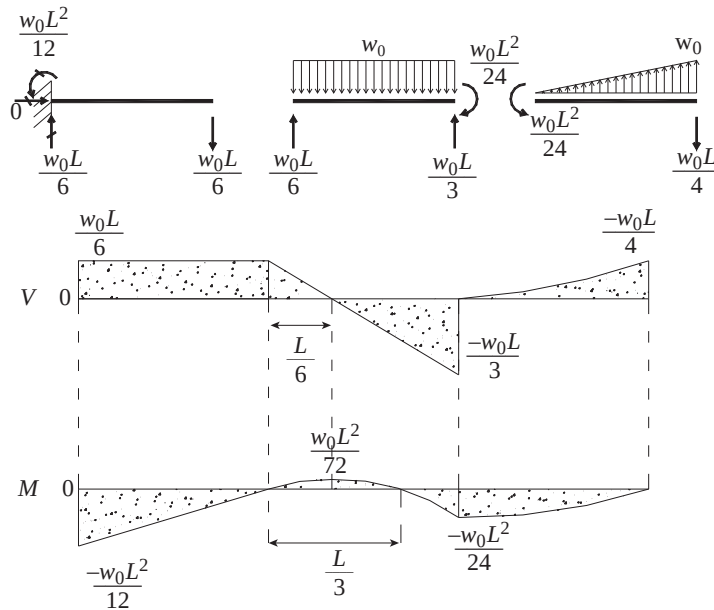
Problem 4.5-34 The compound beam below has an internal *moment release* just to the left of *B* and a *shear release* just to the right of *C*. Reactions have been computed at *A*, *C*, and *D* and are shown in the figure.

First, confirm the reaction expressions using statics, then plot shear (*V*) and moment (*M*) diagrams. Label all critical *V* and *M* values and also the *distance* to points where either *V* and/or *M* is zero.



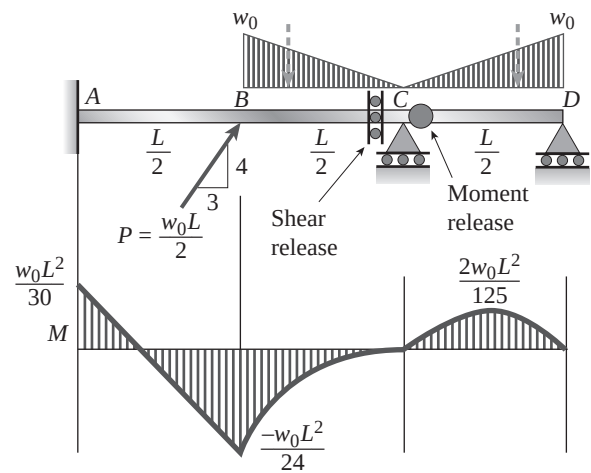
Solution 4.5-34

FREE-BODY DIAGRAM



Problem 4.5-35 The compound beam below has an *shear release* just to the left of C and a *moment release* just to the right of C. A plot of the moment diagram is provided below for applied load P at B and triangular distributed loads $w(x)$ on segments BC and CD.

First, solve for reactions using statics, then plot axial force (N) and shear (V) diagrams. Confirm that the moment diagram is that shown below. Label all critical N , V , and M values and also the distance to points where N , V , and/or M is zero.



Solution 4.5-35

Solve for reactions using statics.

$$M_A = -w_0 L^2/30 \text{ (clockwise), } \leftarrow$$

$$A_x = -\frac{3}{10} w_0 L \text{ (left) } \leftarrow$$

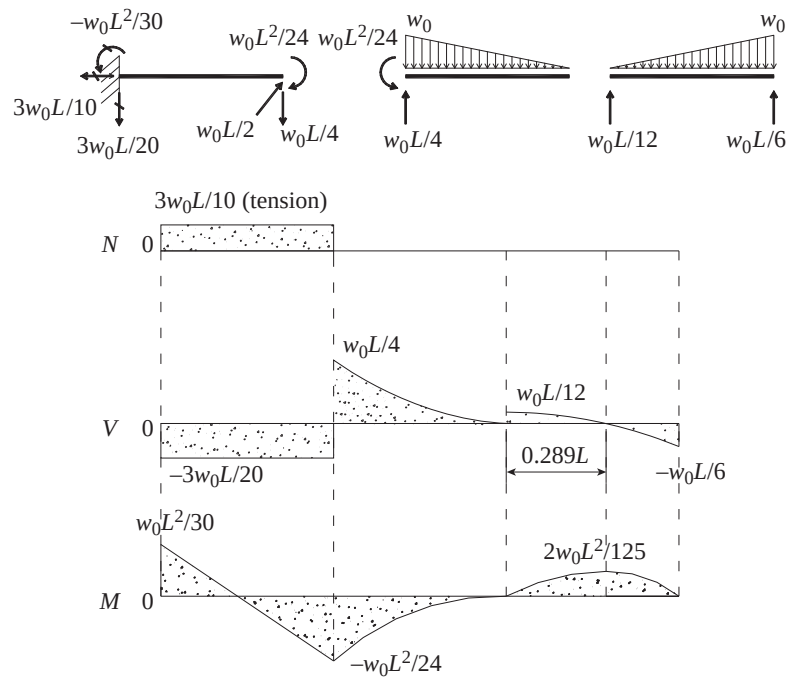
$$A_y = -\frac{3}{20} w_0 L \text{ (downward) } \leftarrow$$

$$C_y = \frac{w_0}{12} L \text{ (upward) } \leftarrow$$

$$D_y = \frac{w_0}{6} L \text{ (upward) } \leftarrow$$

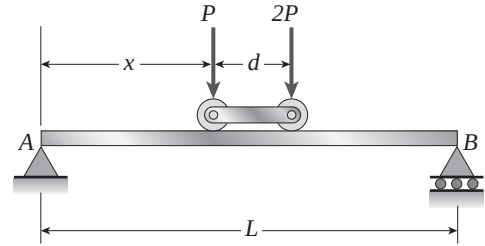
$$V_{\max} = w_0 L/4, M_{\max} = -w_0 L^2/24 \text{ at } B$$

FREE-BODY DIAGRAM

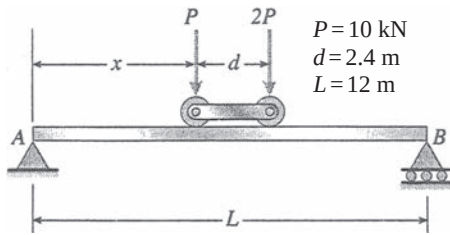


Problem 4.5-36 A simple beam AB supports two connected wheel loads P and $2P$ that are distance d apart (see figure). The wheels may be placed at any distance x from the left-hand support of the beam.

- Determine the distance x that will produce the maximum shear force in the beam, and also determine the maximum shear force $V_{\max}$.
- Determine the distance x that will produce the maximum bending moment in the beam, and also draw the corresponding bending-moment diagram. (Assume $P = 10 \text{ kN}$, $d = 2.4 \text{ m}$, and $L = 12 \text{ m}$.)

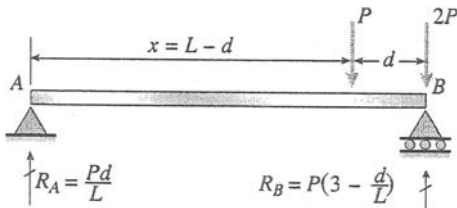


Solution 4.5-36 Moving loads on a beam



(a) MAXIMUM SHEAR FORCE

By inspection, the maximum shear force occurs at support B when the larger load is placed close to, but not directly over, that support.

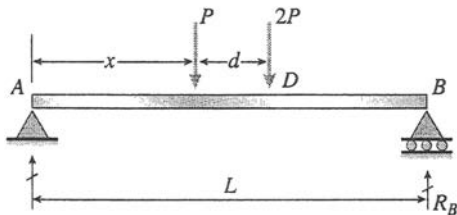


$$x = L - d = 9.6 \text{ m} \quad \leftarrow$$

$$V_{\max} = R_B = P \left(3 - \frac{d}{L} \right) = 28 \text{ kN} \quad \leftarrow$$

(b) MAXIMUM BENDING MOMENT

By inspection, the maximum bending moment occurs at point D , under the larger load $2P$.



Reaction at support B :

$$R_B = \frac{P}{L}x + \frac{2P}{L}(x + d) = \frac{P}{L}(2d + 3x)$$

Bending moment at D :

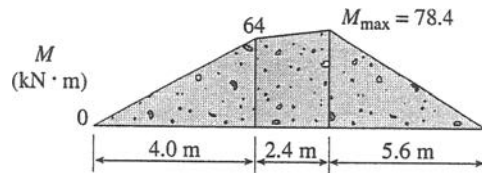
$$\begin{aligned} M_D &= R_B(L - x - d) \\ &= \frac{P}{L}(2d + 3x)(L - x - d) \\ &= \frac{P}{L}[-3x^2 + (3L - 5d)x + 2d(L - d)] \quad \text{Eq.(1)} \end{aligned}$$

$$\frac{dM_D}{dx} = \frac{P}{L}(-6x + 3L - 5d) = 0$$

$$\text{Solve for } x: \quad x = \frac{L}{6} \left(3 - \frac{5d}{L} \right) = 4.0 \text{ m} \quad \leftarrow$$

Substitute x into Eq (1):

$$\begin{aligned} M_{\max} &= \frac{P}{L} \left[-3 \left(\frac{L}{6} \right)^2 \left(3 - \frac{5d}{L} \right)^2 + (3L - 5d) \right. \\ &\quad \left. \times \left(\frac{L}{6} \right) \left(3 - \frac{5d}{L} \right) + 2d(L - d) \right] \\ &= \frac{PL}{12} \left(3 - \frac{d}{L} \right)^2 = 78.4 \text{ kN}\cdot\text{m} \quad \leftarrow \end{aligned}$$

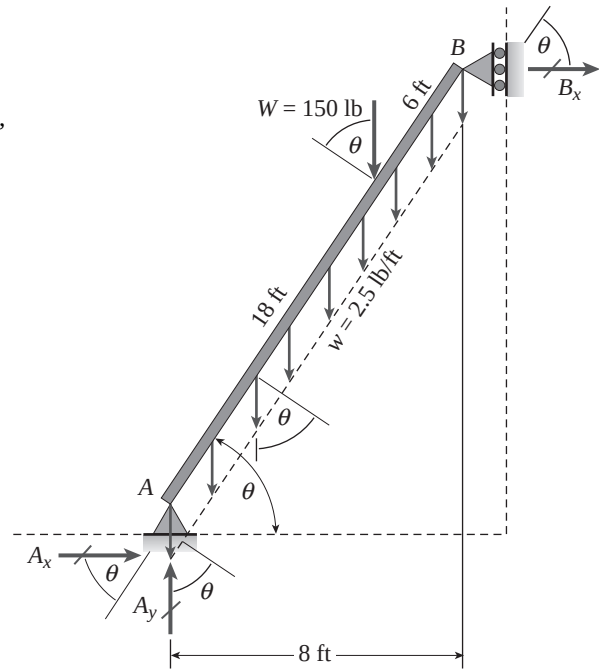


$$\text{Note: } R_A = \frac{P}{2} \left(3 + \frac{d}{L} \right) = 16 \text{ kN}$$

$$R_B = \frac{P}{2} \left(3 - \frac{d}{L} \right) = 14 \text{ kN}$$

Problem 4.5-37 The inclined beam represents a ladder with the following applied loads: the weight (W) of the house painter and the distributed weight (w) of the ladder itself.

- Find support reactions at A and B , then plot axial force (N), shear (V), and moment (M) diagrams. Label all critical N , V , and M values and also the distance to points where any critical ordinates are zero. Plot N , V , and M diagrams normal to the inclined ladder.
- Repeat part (a) for the case of the ladder suspended from a pin at B and traveling on a roller support perpendicular to the floor at A .



Solution 4.5-37

(a) LADDER WITH PIN AT BASE

$$\cos \theta = \frac{8}{18 + 6} = \frac{1}{3}, \quad \sin \theta = \frac{2\sqrt{2}}{3}$$

Solution procedure:

- (1) Use statics to find reaction forces at A and B .

$$\sum F_V = 0: A_y = 150 + 2.5(18 + 6) = 210 \text{ lb}$$

$$A_y = 210 \text{ lb (upward)} \quad \leftarrow$$

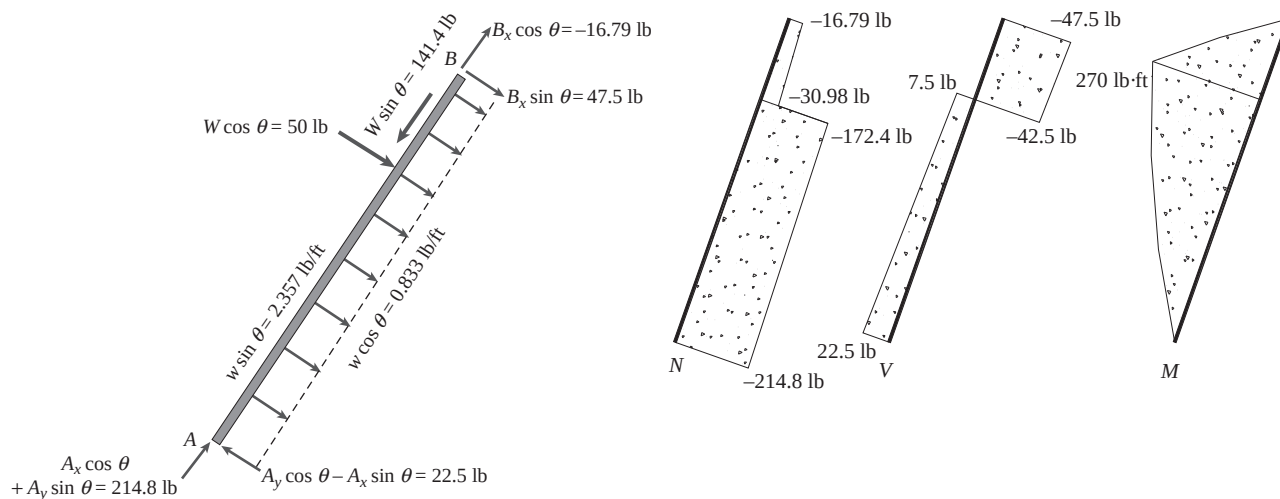
$$\sum M_A = 0: B_x(24 \sin \theta) + 150 \times 6 + 2.5 \times 24 \times 4 = 0$$

$$B_x = -50.38 \text{ lb (left)} \quad \leftarrow$$

$$\sum F_H = 0: A_x = 50.38 \text{ lb (right)} \quad \leftarrow$$

- (2) Use θ to find forces at ends A and B which are along and perpendicular to member AB (see free-body diagram); also resolve forces W and w into components along and perpendicular to member AB .

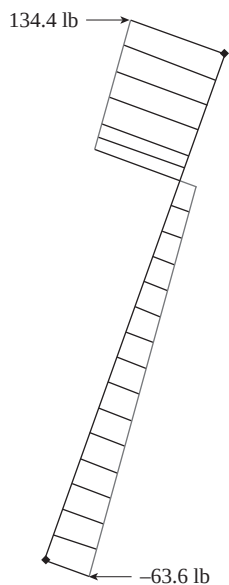
(3) Starting at end A, plot N , V , and M diagrams (see plots).



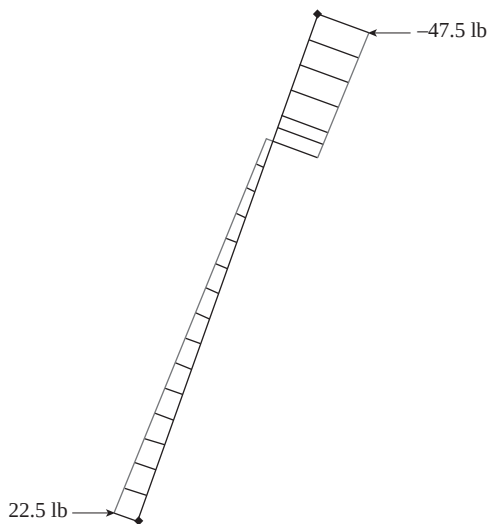
(b) LADDER WITH PIN AT TOP Use statics to find reactions at A and B.

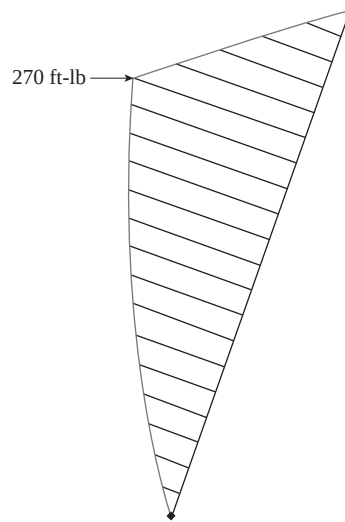
$$A_x = 0, A_y = 67.5 \text{ lb}, B_x = 0, B_y = 142.5 \text{ lb}$$

N DIAGRAM



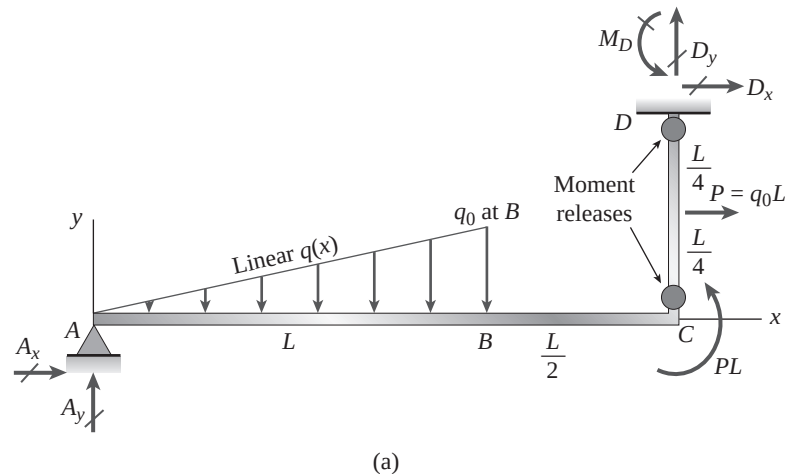
V DIAGRAM



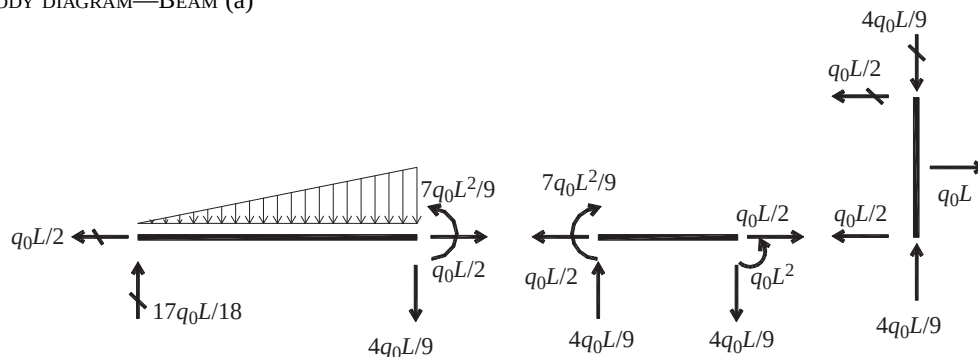
M DIAGRAM

Problem 4.5-38 Beam ABC is supported by a tie rod CD as shown (see Prob. 10.4-15). Two configurations are possible: pin support at A and downward triangular load on AB , or pin at B and upward load on AB . Which has the larger maximum moment?

First, find all support reactions, then plot axial force (N), shear (V) and moment (M) diagrams for ABC only and label all critical N , V , and M values. Label the distance to points where any critical ordinates are zero.

**Solution 4.5-38**

FREE-BODY DIAGRAM—BEAM (a)



Use statics to find reactions at A and D for beam (a).

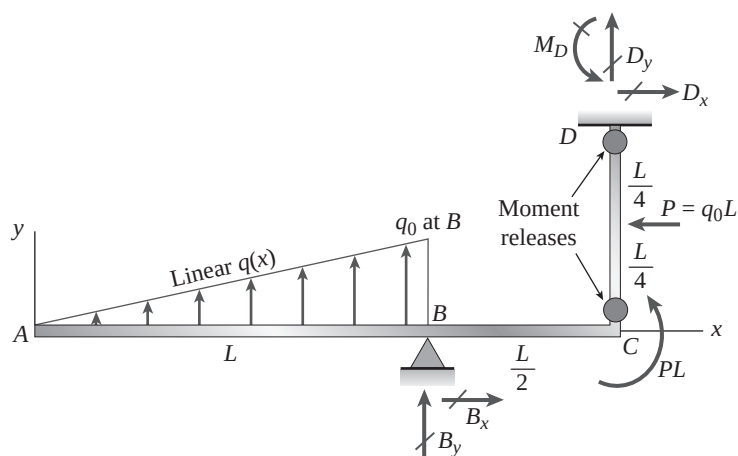
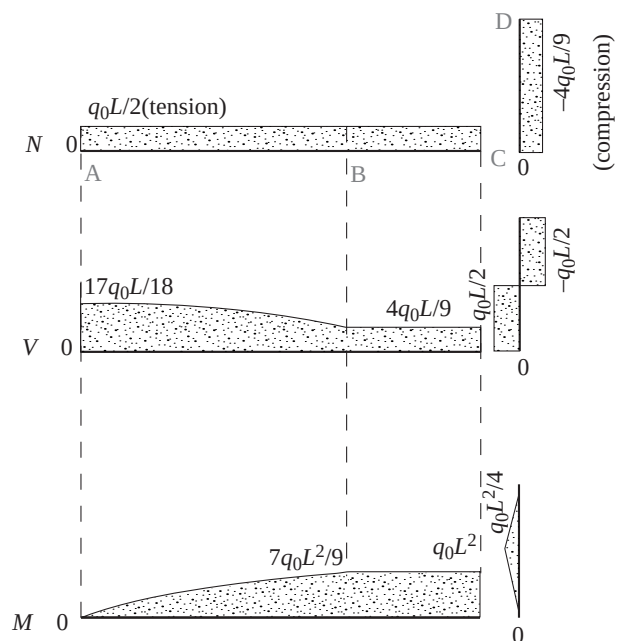
$$A_x = -\frac{1}{2} q_0 L \text{ (left)} \quad \leftarrow$$

$$A_y = \frac{17}{18} q_0 L \text{ (upward)} \quad \leftarrow$$

$$D_x = -\frac{1}{2} q_0 L \text{ (left)} \quad \leftarrow$$

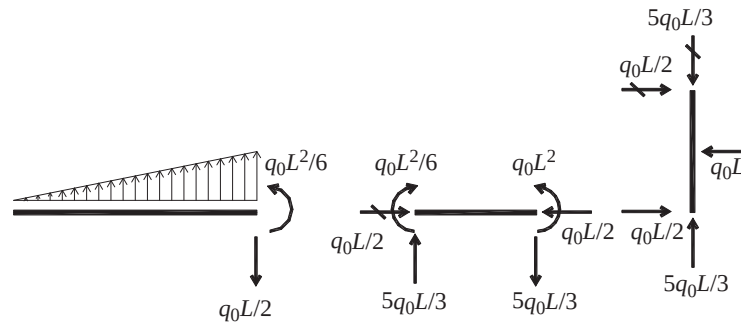
$$D_y = -\frac{4}{9} q_0 L \text{ (downward)} \quad \leftarrow$$

$$M_D = 0 \quad \leftarrow$$



(b)

FREE-BODY DIAGRAM—BEAM (b)



Use statics to find reactions at B and D for beam (b).

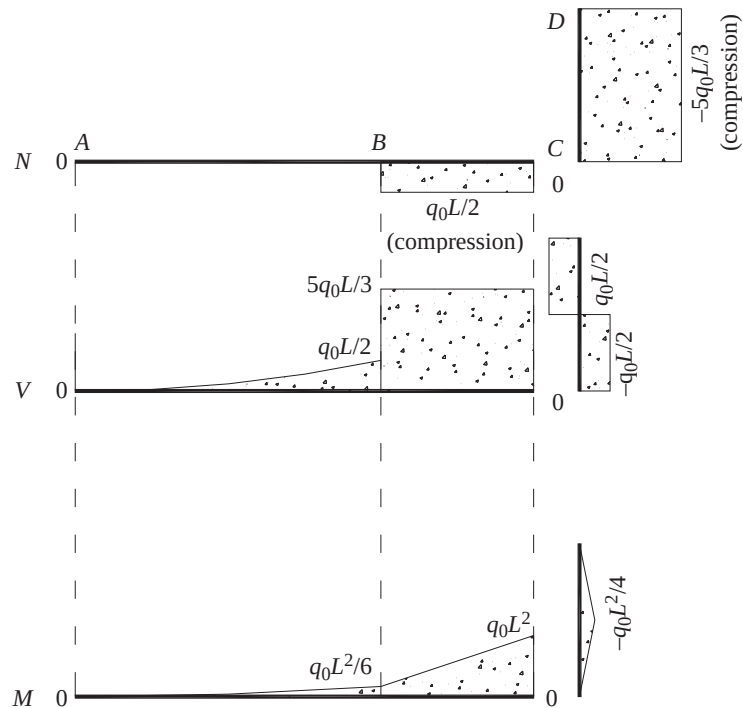
$$B_x = \frac{1}{2} q_0 L \text{ (right)} \quad \leftarrow$$

$$B_y = -\frac{1}{2} q_0 L + \frac{5}{3} q_0 L = \frac{7}{6} q_0 L \text{ (upward)} \quad \leftarrow$$

$$D_x = \frac{1}{2} q_0 L \text{ (right)} \quad \leftarrow$$

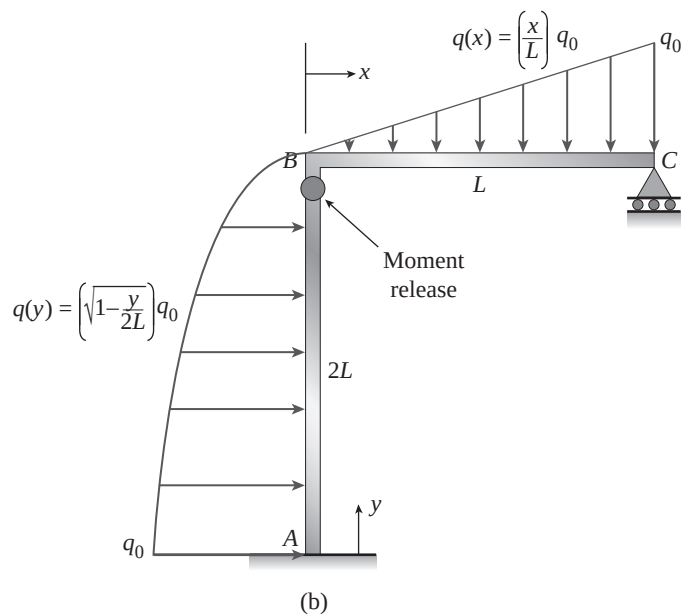
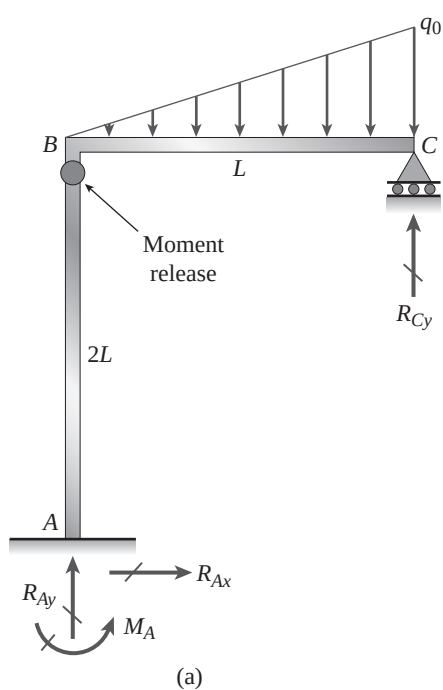
$$D_y = -\frac{5}{3} q_0 L \text{ (downward)} \quad \leftarrow$$

$$M_D = 0 \quad \leftarrow$$



Problem 4.5-39 The plane frame below consists of column AB and beam BC which carries a triangular distributed load (see figure part a). Support A is fixed, and there is a roller support at C . Column AB has a moment release just below joint B .

- Find support reactions at A and C , then plot axial force (N), shear-force (V), and bending-moment (M) diagrams for both members. Label all critical N , V , and M values and also the distance to points where any critical ordinates are zero.
- Repeat part (a) if a parabolic lateral load acting to the right is now added on column AB (figure part b).



Solution 4.5-39

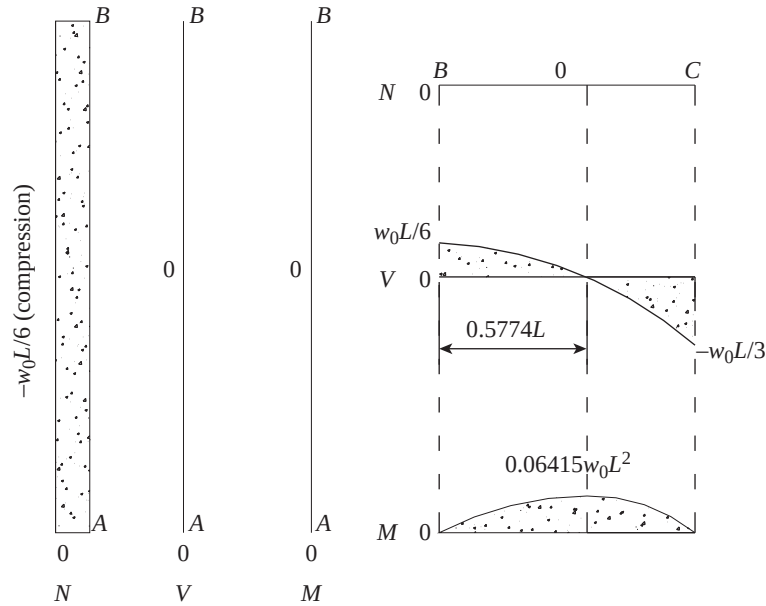
(a) USE STATICS TO FIND REACTIONS AT A AND C

$$M_A = 0 \quad \leftarrow$$

$$R_{Ay} = \frac{q_0}{6} L \text{ (upward)} \quad \leftarrow$$

$$R_{Cy} = \frac{q_0}{3} L \text{ (upward)} \quad \leftarrow$$

$$R_{Ax} = 0 \quad \leftarrow$$



(b) REACTIONS AT A AND C

(1) Sum moments about moment release of FBD of BC to find reaction R_{Cy} , then sum vertical forces to get R_{Ay} .

$$R_{Cy} = \frac{1}{L} \left[\frac{1}{2} q_0 L \left(\frac{2}{3} L \right) \right] = \frac{L q_0}{3} \quad R_{Ay} = \frac{1}{2} q_0 L - R_{Cy} = \frac{L q_0}{6}$$

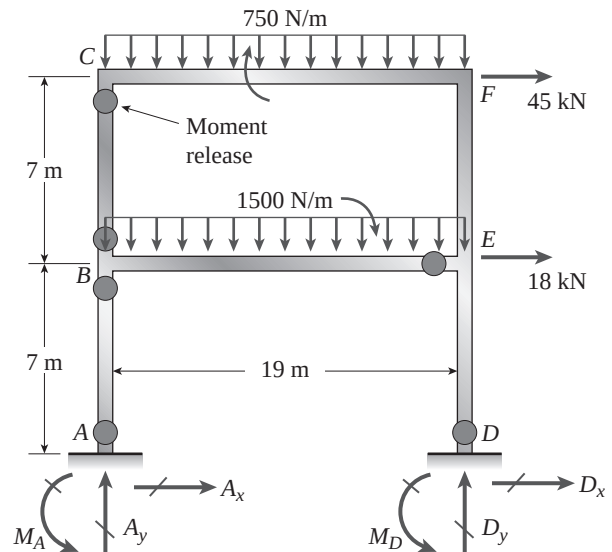
(2) Sum forces in x-direction for entire FBD to get R_{Ax} .

$$R_{Ax} = - \int_0^{2L} \left(\sqrt{1 - \frac{y}{2L}} \right) q_0 dy = - \frac{4L q_0}{3}$$

(3) Sum moments about A for entire FBD to get reaction moment M_A .

$$M_A = \int_0^{2L} \left(\sqrt{1 - \frac{y}{2L}} \right) q_0 y dy + \frac{1}{2} q_0 L \left(\frac{2}{3} L \right) - R_{Cy} L = \frac{16L^2 q_0}{15}$$

Problem 4.5-40 The plane frame shown below is part of an elevated freeway system. Supports at A and D are fixed but there are *moment releases* at the base of both columns (AB and DE), as well near in column BC and at the end of beam BE . Find all support reactions, then plot axial force (N), shear (V) and moment (M) diagrams for all beam and column members. Label all critical N , V , and M values and also the distance to points where any critical ordinates are zero.



Solution 4.5-40

Solution procedure:

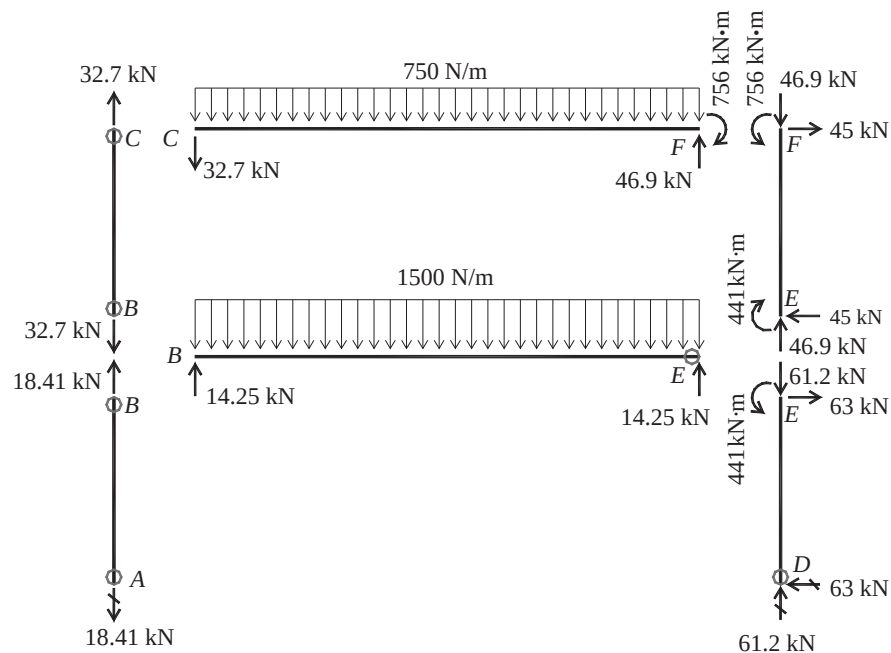
- (1) $M_A = M_D = 0$ due to moment releases
- (2) $\sum M_A = 0$: $D_y = 61,164 \text{ N} = 61.2 \text{ kN}$
- (3) $\sum F_y = 0$: $A_y = -18,414 \text{ N} = -18.41 \text{ kN}$

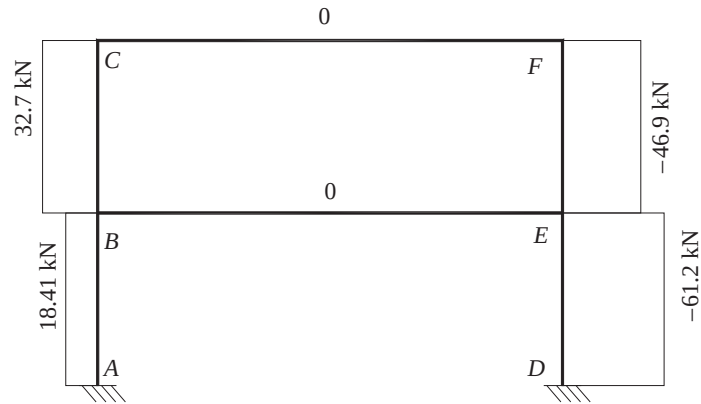
$$(4) \sum M_B = 0 \text{ for } AB: A_x = 0$$

$$(5) \sum F_H = 0: D_x = -63 \text{ kN}$$

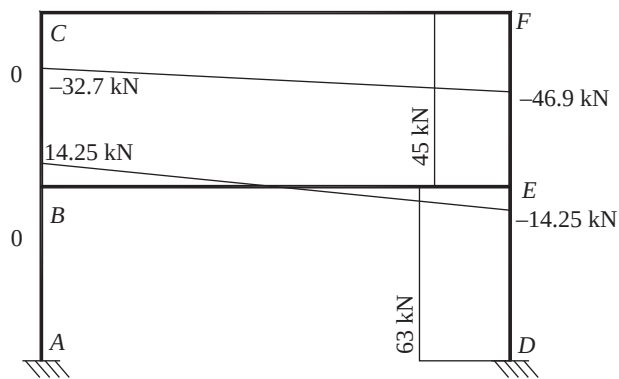
- (6) Draw separate FBD's of each member (see below) to find N , V , and M for each member; plot diagrams (see below).

FREE-BODY DIAGRAM

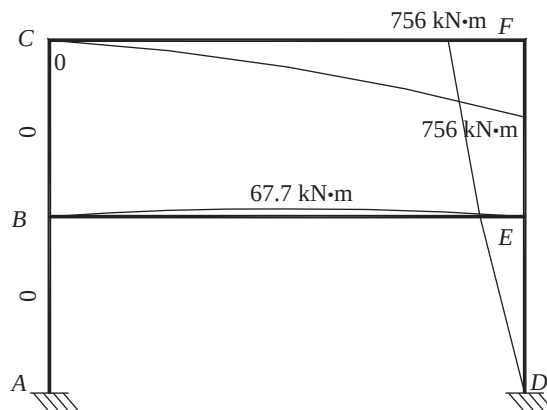




AXIAL FORCE DIAGRAM
(-) COMPRESSION



SHEAR FORCE DIAGRAM



BENDING MOMENT DIAGRAM

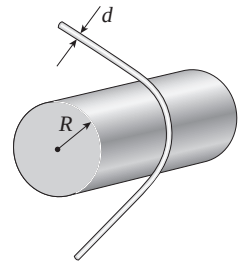
5

Stresses in Beams (Basic Topics)

Longitudinal Strains in Beams

Problem 5.4-1 A steel wire of diameter $d = 1/16$ in. is bent around a cylindrical drum of radius $R = 36$ in. (see figure).

- Determine the maximum normal strain $\epsilon_{\max}$.
- What is the minimum acceptable radius of the drum if the maximum normal strain must remain below yield? Assume $E = 30,000$ ksi and $\sigma_Y = 100$ ksi.
- If $R = 36$ in., what is the maximum acceptable diameter of the wire if the maximum normal strain must remain below yield?



Solution 5.4-1 Steel wire

$$\begin{aligned} \text{(a)} \quad \sigma_Y &= 100 \text{ ksi} & E &= 30,000 \text{ ksi} & \epsilon_Y &= \frac{\sigma_Y}{E} = 3.333 \times 10^{-3} \\ R &= 36 \text{ in.} & d &= \frac{1}{16} \text{ in.} & \rho &= R + \frac{d}{2} = 915.194 \text{ mm} \end{aligned}$$

$$\kappa = \frac{1}{\rho} = 1.093 \times 10^{-3} \frac{1}{\text{mm}} \quad \epsilon_{\max} = \frac{y}{\rho} \quad \boxed{\epsilon_{\max} = \frac{\frac{d}{2}}{\rho} = 8.673 \times 10^{-4}}$$

$$\text{(b)} \quad \frac{\frac{d}{2}}{R_{\min} + \frac{d}{2}} = \epsilon_Y \quad \text{solving for } R_{\min} = 9.34375 \text{ in.} \quad \boxed{R_{\min} = 9.35 \text{ in.}}$$

$$\text{CHECK} \quad \epsilon = \frac{\frac{d}{2}}{R_{\min} + \frac{d}{2}} = 3.331 \times 10^{-3}$$

(c) $R = 36$ in.

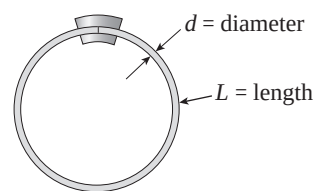
$$\frac{\frac{d_{\max}}{2}}{R + \frac{d_{\max}}{2}} = \varepsilon_Y \quad \text{solving for } d_{\max} \text{ gives } \boxed{d_{\max} = 0.24 \text{ in.}}$$

CHECK

$$\frac{\frac{d_{\max}}{2}}{R + \frac{d_{\max}}{2}} = 3.322 \times 10^{-3}$$

Problem 5.4-2 A copper wire having diameter $d = 4$ mm is bent into a circle and held with the ends just touching (see figure).

- (a) If the maximum permissible strain in the copper is $\varepsilon_{\max} = 0.0024$, what is the shortest length L of wire that can be used?
- (b) If $L = 5.5$ m, what is the maximum acceptable diameter of the wire if the maximum normal strain must remain below yield? Assume $E = 120$ GPa and $\sigma_Y = 300$ MPa.



Solution 5.4-2 Copper wire

$$\varepsilon = \frac{\frac{d}{2}}{\rho} \quad \varepsilon = \frac{\frac{d}{2}}{\frac{L}{2\pi}} = \frac{\pi d}{L}$$

(a) $\varepsilon_{\max} = 0.0024 \quad \rho = \frac{L}{2\pi} \quad d = 4 \text{ mm}$

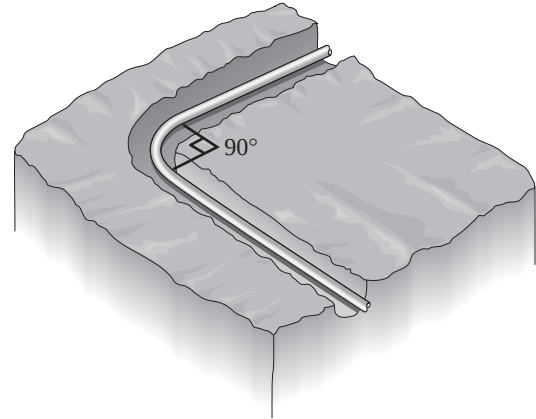
$$\boxed{L_{\min} = \frac{\pi d}{\varepsilon_{\max}} = 5.24 \text{ m}}$$

(b) $E = 120 \text{ GPa} \quad \sigma_Y = 300 \text{ MPa} \quad \varepsilon_Y = \frac{\sigma_Y}{E} = 2.5 \times 10^{-3} \quad L = 5.5 \text{ m}$

$$\boxed{d_{\max} = \frac{\varepsilon_Y L}{\pi} = 4.38 \text{ mm}}$$

Problem 5.4-3 A 4.75-in. outside diameter polyethylene pipe designed to carry chemical wastes is placed in a trench and bent around a quarter-circular 90° bend (see figure). The bent section of the pipe is 52 ft long.

- Determine the maximum compressive strain $\epsilon_{\max}$ in the pipe.
- If the normal strain cannot exceed 6.1×10^{-3} , what is the maximum diameter of the pipe?
- If $d = 4.75$ in., what is the minimum acceptable length of the bent section of the pipe?



Solution 5.4-3 Polyethylene pipe

$$(a) \quad L = 52 \text{ ft} \quad \rho = \frac{L}{\frac{\pi}{2}} = 397.251 \text{ in.} \quad d = 4.75 \text{ in.} \quad \epsilon = \frac{\frac{d}{2}}{\left(\frac{2L}{\pi}\right)} = \frac{\pi d}{4L}$$

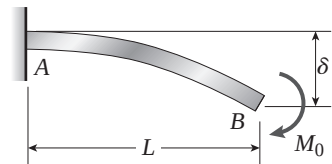
$$\epsilon_{\max} = \frac{\pi d}{4L} = 5.98 \times 10^{-3}$$

$$(b) \quad \epsilon_a = 6.1 (10^{-3}) \quad d_{\max} = \frac{\epsilon_a (4L)}{\pi} = 4.85 \text{ in.}$$

$$(c) \quad d = 4.75 \text{ in.} \quad L_{\min} = \frac{\pi d}{4\epsilon_a} = 51 \text{ ft}$$

Problem 5.4-4 A cantilever beam AB is loaded by a couple M_0 at its free end (see figure). The length of the beam is $L = 2.0$ m, and the longitudinal normal strain at the top surface is $\epsilon = 0.0010$. The distance from the top surface of the beam to the neutral surface is $c = 85$ mm.

- Calculate the radius of curvature ρ , the curvature κ , and the vertical deflection δ at the end of the beam.
- If allowable strain $\epsilon_a = 0.0008$, what is the maximum acceptable depth of the beam? (Assume that the curvature is unchanged from Part (a)).
- If allowable strain $\epsilon_a = 0.0008$, $c = 85$ mm, and $L = 4$ m, what is deflection δ ?



Solution 5.4-4Let $b = 40$ mm.

$$(a) \quad \varepsilon = 0.0010 \quad c = 85 \text{ mm} \quad L = 2 \text{ m} \quad h = 2c = 170 \text{ mm} \quad E = 210 \text{ GPa}$$

$$\boxed{\rho = \frac{c}{\varepsilon} = 85 \text{ m}} \quad \boxed{\kappa = \frac{1}{\rho} = 0.01176 \frac{1}{\text{m}}} \quad \text{so} \quad M_{0a} = E \left(\frac{bh^3}{12} \right) \kappa = 40.46 \text{ kN} \cdot \text{m}$$

$$\theta = \frac{L}{\rho} = 0.02353 \quad \delta = L \tan\left(\frac{\theta}{2}\right) \quad \boxed{\delta = 23.5 \text{ mm}} \quad \frac{\delta}{L} = 0.012 \quad \frac{L}{\delta} = 84.996$$

$$\text{or} \quad \delta = \rho(1 - \cos(\theta)) = 23.5 \text{ mm}$$

$$(b) \quad \varepsilon_b = 0.0008 \quad c_{\max} = \rho \varepsilon_b = 68 \text{ mm} \quad \boxed{h_{\max} = 2c_{\max} = 136 \text{ mm}} \quad < \text{ASSUMES THAT CURVATURE IS SAME AS IN (a)}$$

$$\text{so} \quad M_{0b} = E \left(\frac{bh_{\max}^3}{12} \right) \kappa = 20.716 \text{ kN} \cdot \text{m}$$

ALTERNATE SOLUTION: assume that APPLIED MOMENT M_0 IS SAME AS IN (A) (minus sign cancels in final result)

$$\varepsilon_a = \left(\frac{M_0}{EI_a} \right) c_a \quad \text{so} \quad M_0 = \frac{EI_a}{c_a} \varepsilon_a$$

$$\varepsilon_b = \left(\frac{M_0}{EI_b} \right) c_b \quad c_b = \frac{EI_b}{M_0} \varepsilon_b \quad \text{or} \quad c_b = \frac{I_b \varepsilon_b}{I_a \varepsilon_a} c_a \quad c_b = \frac{\frac{b(2c_b)^3}{12}}{\left[\frac{b(2c_a)^3}{12} \right]} \left(\frac{\varepsilon_b}{\varepsilon_a} \right) c_a$$

$$c_b = \frac{\frac{b(2c_b)^3}{12}}{\left[\frac{b(2c_a)^3}{12} \right]} \left(\frac{\varepsilon_b}{\varepsilon_a} \right) c_a = \frac{c_b^3 \varepsilon_b}{c_a^2 \varepsilon_a} \quad \text{so} \quad c_b^2 = \frac{\varepsilon_a}{\varepsilon_b} c_a^2 \quad \text{or} \quad c_b = c_a \sqrt{\frac{\varepsilon_a}{\varepsilon_b}}$$

$$h_b = 2c_b \quad \boxed{h_b = 2(85 \text{ mm}) \sqrt{\frac{0.0010}{0.0008}} = 190.1 \text{ mm}} \quad \text{where} \quad c_b = (85 \text{ mm}) \sqrt{\frac{0.0010}{0.0008}} = 95.033 \text{ mm}$$

$$\text{or} \quad \frac{h_b}{2} = 95.033 \text{ mm}$$

Now in alternate solution:

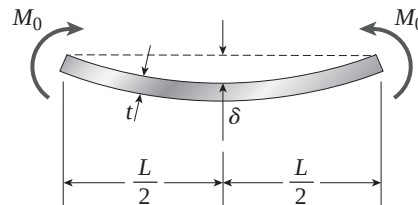
$$M_{0a} = \frac{E \left[\frac{b(170 \text{ mm})^3}{12} \right]}{85 \text{ mm}} (0.0010) = 40.46 \text{ kN} \cdot \text{m} \quad M_{0b} = \frac{E \left[\frac{b(h_b)^3}{12} \right]}{\frac{h_b}{2}} (0.0008) = 40.46 \text{ kN} \cdot \text{m}$$

$$(c) \quad L = 4 \text{ m} \quad \varepsilon_a = 0.0008 \quad \rho = \frac{c}{\varepsilon_a} = 106.25 \text{ m} \quad \theta = \frac{L}{\rho} = 0.038$$

$$\delta = L \tan\left(\frac{\theta}{2}\right) \quad \boxed{\delta = 75.3 \text{ mm}} \quad \text{or} \quad \delta = \rho(1 - \cos(\theta)) = 75.285 \text{ mm}$$

$$\frac{\delta}{L} = 0.019 \quad \frac{L}{\delta} = 53.131$$

Problem 5.4-5 A thin strip of steel with a length of $L = 19$ in. and thickness of $t = 0.275$ in. is bent by couples M_0 (see figure). The deflection at the midpoint of the strip (measured from a line joining its end points) is found to be 0.30 in.



- Determine the longitudinal normal strain ε at the top surface of the strip.
- If allowable strain $\varepsilon_a = 0.0008$, what is the maximum acceptable thickness of the strip?
- If allowable strain $\varepsilon_a = 0.0008$, $t = 0.275$ in., and $L = 32$ in., what is deflection δ ?
- If allowable strain $\varepsilon_a = 0.0008$, $t = 0.275$ in., and the deflection cannot exceed 1.0 in., what is the maximum permissible length of the strip?

Solution 5.4-5

$$(a) \quad L = 19 \text{ in.} \quad t = 0.275 \text{ in.} \quad \delta = 0.30 \text{ in.} \quad \varepsilon = \frac{\frac{t}{2}}{\rho} = \frac{0.1375 \text{ in.}}{\rho}$$

$$\delta = \rho(1 - \cos(\theta)) \quad \sin(\theta) = \frac{L}{2\rho} \quad \text{or approx.} \quad \theta = \frac{L}{2\rho} \quad \text{so} \quad \delta = \rho\left(1 - \cos\left(\frac{L}{2\rho}\right)\right)$$

SOLVE NUMERICALLY FOR ρ :

$$\rho\left(1 - \cos\left(\frac{L}{2\rho}\right)\right) = \delta$$

$$\rho = 150.367 \text{ in.}$$

$$\varepsilon = \frac{\frac{t}{2}}{\rho} = 9.144 \times 10^{-4}$$

compressive

$$(b) \quad \varepsilon_a = 0.0008 \quad t_{\max} = 2\rho\varepsilon_a = 0.241 \text{ in.}$$

$$\frac{\varepsilon}{\varepsilon_a} = 1.143$$

$$(c) \quad \rho = \frac{\frac{t}{2}}{\varepsilon_a} = 171.875 \text{ in.} \quad L = 32 \text{ in.}$$

$$\delta = \rho\left(1 - \cos\left(\frac{L}{2\rho}\right)\right) = 0.744 \text{ in.}$$

$$\frac{\delta}{L} = 0.023 \quad \frac{L}{\delta} = 43$$

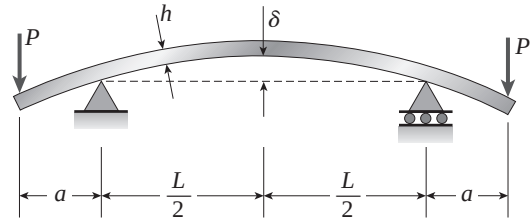
$$(d) \quad \varepsilon_a = 0.0008 \quad t = 0.275 \text{ in.} \quad \delta_{\max} = 1.0 \text{ in.} \quad \rho = \frac{t}{2\varepsilon_a} = 171.875 \text{ in.}$$

SOLVE NUMERICALLY FOR L :

$$\rho\left(1 - \cos\left(\frac{L}{2\rho}\right)\right) = \delta_{\max}$$

$$L_{\max} = 37.1 \text{ in.}$$

Problem 5.4-6 A bar of rectangular cross section is loaded and supported as shown in the figure. The distance between supports is $L = 1.75$ m, and the height of the bar is $h = 140$ mm. The deflection at the midpoint is measured as 2.5 mm.



- What is the maximum normal strain ϵ at the top and bottom of the bar?
- If allowable strain $\epsilon_a = 0.0006$ and the deflection cannot exceed 4.3 mm, what is the maximum permissible length of the bar?

Solution 5.4-6

$$(a) \quad L = 1.75 \text{ m} \quad h = 140 \text{ mm} \quad \delta = 2.5 \text{ mm} \quad \epsilon = \frac{\frac{h}{2}}{\rho} = \frac{h}{2\rho}$$

$$\delta = \rho(1 - \cos(\theta)) \quad \sin(\theta) = \frac{\frac{L}{2}}{\rho} \quad \text{or approx.} \quad \theta = \frac{L}{2\rho} \quad \text{so} \quad \delta = \rho\left(1 - \cos\left(\frac{L}{2\rho}\right)\right)$$

SOLVE NUMERICALLY FOR ρ :

$$\rho\left(1 - \cos\left(\frac{L}{2\rho}\right)\right) = \delta$$

$$\rho = 153.125 \text{ m}$$

$$\epsilon = \frac{\frac{h}{2}}{\rho} = 4.57 \times 10^{-4}$$

$$(b) \quad \epsilon_a = 0.0006 \quad \rho = \frac{h}{2\epsilon_a} = 116.667 \text{ m} \quad \delta_{\max} = 4.3 \text{ mm}$$

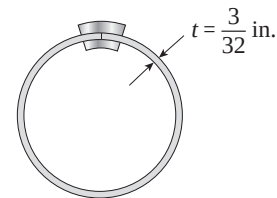
SOLVE NUMERICALLY FOR L :

$$\rho\left(1 - \cos\left(\frac{L}{2\rho}\right)\right) = \delta_{\max}$$

$$L_{\max} = 2 \text{ m}$$

Normal Stresses in Beams

Problem 5.5-1 A thin strip of hard copper ($E = 16,000$ ksi) having length $L = 90$ in. and thickness $t = 3/32$ in. is bent into a circle and held with the ends just touching (see figure).



- Calculate the maximum bending stress $\sigma_{\max}$ in the strip.
- By what percent does the stress increase or decrease if the thickness of the strip is increased by $1/32$ in.?
- Find the new length of the strip so that the stress in part (b) ($t = 1/8$ in. and $L = 90$ in.) is equal to that in part (a) ($t = 3/32$ in. and $L = 90$ in.).

Solution 5.5-1

$$E = 16,000 \text{ ksi} \quad L = 90 \text{ in.} \quad t = \frac{3}{32} \text{ in.}$$

$$\rho = \frac{L}{2\pi} = 14.324 \text{ in.}$$

$$(a) \quad \sigma_{\max} = E \left(\frac{\frac{t}{2}}{\rho} \right) = 52.4 \text{ ksi}$$

$$(b) \quad \sigma_{\max\text{new}} = \frac{E}{\rho} \left(\frac{t + \frac{1}{32} \text{ in.}}{2} \right) = 69.813 \text{ ksi}$$

$$\frac{\sigma_{\max\text{new}} - \sigma_{\max}}{\sigma_{\max}} = 33.3\% \quad \text{same as} \quad \frac{\left(t + \frac{1}{32} \text{ in.} \right) - t}{t} = 33.3\%$$

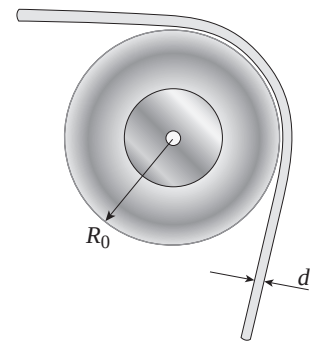
$$(c) \quad E \left(\frac{t + \frac{1}{32} \text{ in.}}{\frac{2}{\frac{L_{\text{new}}}{2\pi}}} \right) = \sigma_{\max} \quad \text{solving}$$

$$\boxed{L_{\text{new}} = 120 \text{ in.}}$$

$$\frac{L_{\text{new}} - L}{L} = 33.3\%$$

Problem 5.5-2 A steel wire ($E = 200 \text{ GPa}$) of diameter $d = 1.25 \text{ mm}$ is bent around a pulley of radius $R_0 = 500 \text{ mm}$ (see figure).

- What is the maximum stress $\sigma_{\max}$ in the wire?
- By what percent does the stress increase or decrease if the radius of the pulley is increased by 25%?
- By what percent does the stress increase or decrease if the diameter of the wire increased by 25% while the pulley radius remains at $R_0 = 500 \text{ mm}$?



Solution 5.5-2

$$E = 200 \text{ GPa} \quad d = 1.25 \text{ mm} \quad R_0 = 500 \text{ mm}$$

$$(a) \quad \sigma_{\max} = E \left(\frac{\frac{d}{2}}{R_0 + \frac{d}{2}} \right) = 250 \text{ MPa}$$

$$(b) \quad \sigma_{\max\text{new}} = E \left(\frac{\frac{d}{2}}{1.25R_0 + \frac{d}{2}} \right) = 199.8 \text{ MPa}$$

$$\frac{\sigma_{\max\text{new}} - \sigma_{\max}}{\sigma_{\max}} = -19.98\%$$

(c)

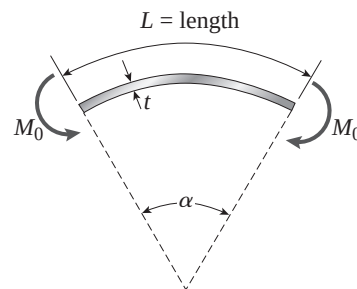
$$1.25d = 1.563 \text{ mm}$$

$$\sigma_{\max\text{new}} = E \left(\frac{\frac{1.25d}{2}}{R_0 + \frac{1.25d}{2}} \right) = 312.012 \text{ MPa}$$

$$\frac{\sigma_{\max\text{new}} - \sigma_{\max}}{\sigma_{\max}} = 25\%$$

Problem 5.5-3 A thin, high-strength steel rule ($E = 30 \times 10^6$ psi) having thickness $t = 0.175$ in. and length $L = 48$ in. is bent by couples M_0 into a circular arc subtending a central angle $\alpha = 40^\circ$ (see figure).

- What is the maximum bending stress $\sigma_{\max}$ in the rule?
- By what percent does the stress increase or decrease if the central angle is increased by 10%?
- What percent increase or decrease in rule thickness will result in the maximum stress reaching the allowable value of 42 ksi?

**Solution 5.5-3**

$$E = 30,000 \text{ ksi}$$

$$t = 0.175 \text{ in.} \quad L = 48 \text{ in.}$$

$$\alpha = 40^\circ$$

$$(a) \quad \rho = \frac{L}{\alpha} = 68.755 \text{ in.}$$

$$\sigma_{\max} = E \left(\frac{\frac{t}{2}}{\rho} \right) = 38.2 \text{ ksi}$$

$$(b) \quad \sigma_{\max\text{new}} = E \left(\frac{\frac{t}{2}}{\frac{L}{1.1\alpha}} \right) = 41.997 \text{ ksi}$$

$$\frac{\sigma_{\max\text{new}} - \sigma_{\max}}{\sigma_{\max}} = 10\%$$

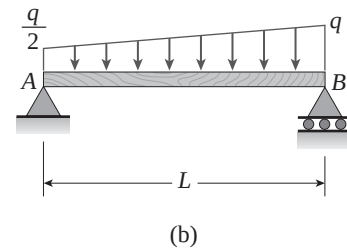
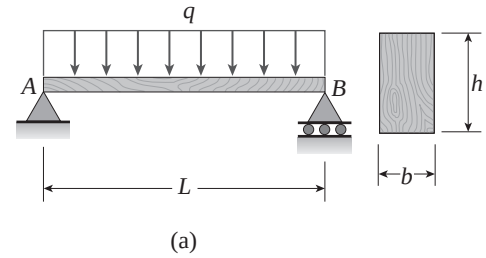
$$(c) \quad \sigma_a = 42 \text{ ksi} \quad t = 0.175 \text{ in.} \quad \alpha = 40^\circ \quad L = 48 \text{ in.}$$

$$E \left(\frac{\frac{t}{2}}{\frac{L}{\alpha}} \right) = \sigma_a \quad t_{\text{new}} = 0.193 \text{ in.}$$

$$\frac{t_{\text{new}} - t}{t} = 10.01\%$$

Problem 5.5-4 A simply supported wood beam AB with span length $L = 4$ m carries a uniform load of intensity $q = 5.8$ kN/m (see figure).

- Calculate the maximum bending stress $\sigma_{\max}$ due to the load q if the beam has a rectangular cross section with width $b = 140$ mm and height $h = 240$ mm.
- Repeat part (a) but use the trapezoidal distributed load shown in the figure part b.



Solution 5.5-4

(a) MAX. BENDING STRESS DUE TO UNIFORM LOAD q

$$M_{\max} = \frac{qL^2}{8} \quad S = \frac{I}{\frac{h}{2}}$$

$$S = \frac{\frac{bh^3}{12}}{\frac{h}{2}} \quad S = \frac{1}{6}bh^2$$

$$\sigma_{\max} = \frac{M_{\max}}{S} \quad \sigma_{\max} = \frac{\frac{qL^2}{8}}{\left(\frac{1}{6}bh^2\right)}$$

$$\sigma_{\max} = \frac{3}{4}q \frac{L^2}{bh^2}$$

$$q = 5.8 \frac{\text{kN}}{\text{m}} \quad L = 4 \text{ m} \quad b = 140 \text{ mm}$$

$$h = 240 \text{ mm}$$

$$M_{\max} = \frac{qL^2}{8}$$

$$M_{\max} = 11.6 \text{ kN} \cdot \text{m}$$

$$\sigma_{\max} = 8.63 \text{ MPa} \quad \leftarrow$$

(b) MAX. BENDING STRESS DUE TO TRAPEZOIDAL LOAD q

$$R_A = \left[\frac{1}{2} \left(\frac{q}{2} \right) L + \frac{1}{3} \left(\frac{q}{2} \right) \left(\frac{q}{2} \right) L \right]$$

uniform load ($q/2$) & triang. load ($q/2$)

$$R_A = \frac{1}{3}qL$$

find x = location of zero shear

$$R_A - \frac{q}{2}x - \frac{1}{2} \left(\frac{x}{L} \frac{q}{2} \right) x = 0$$

$$3x^2 + 6Lx - 4L^2 = 0$$

$$x = \frac{-6L - \sqrt{(84L^2)}}{2(3)}$$

$$\frac{x}{L} = \left(-1 + \frac{1}{6}\sqrt{84} \right)$$

$$x_{\max} = 0.52753 L$$

$$M_{\max} = R_A x_{\max} - \frac{q}{2} \frac{x_{\max}^2}{2} - \frac{1}{2} \left(\frac{x_{\max}}{L} \frac{q}{2} \right) \frac{x_{\max}^2}{3}$$

$$M_{\max} = 9.40376 \times 10^{-2} qL^2$$

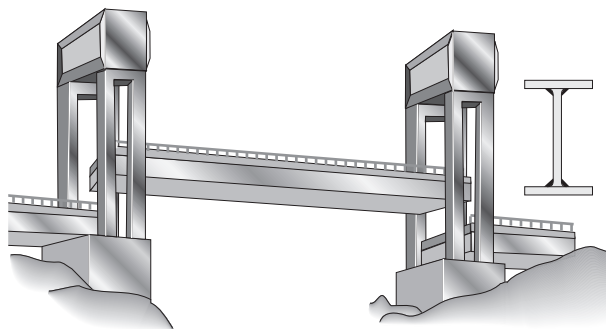
$$M_{\max} = 8.727 \text{ kN} \cdot \text{m}$$

$$\sigma_{\max} = \frac{M_{\max}}{S}$$

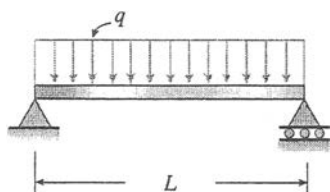
$$\sigma_{\max} = 6.493 \times 10^3 \frac{\text{N}}{\text{m}^2} \quad \sigma_{\max} = 6.49 \text{ MPa} \quad \leftarrow$$

Problem 5.5-5 Each girder of the lift bridge (see figure) is 180 ft long and simply supported at the ends. The design load for each girder is a uniform load of intensity 1.6 k/ft. The girders are fabricated by welding three steel plates so as to form an I-shaped cross section (see figure) having section modulus $S = 3600 \text{ in.}^3$.

What is the maximum bending stress $\sigma_{\max}$ in a girder due to the uniform load?



Solution 5.5-5 Bridge girder



$$L = 180 \text{ ft} \quad q = 1.6 \text{ k/ft}$$

$$S = 3600 \text{ in.}^3$$

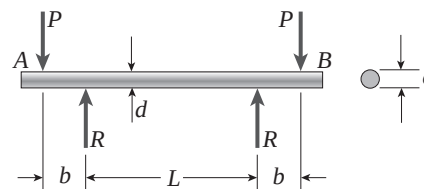
$$M_{\max} = \frac{qL^2}{8}$$

$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{qL^2}{8S}$$

$$\sigma_{\max} = \frac{(1.6 \text{ k/ft})(180 \text{ ft})^2(12 \text{ in./ft})}{8(3600 \text{ in.}^3)} = 21.6 \text{ ksi} \quad \leftarrow$$

Problem 5.5-6 A freight-car axle AB is loaded approximately as shown in the figure, with the forces P representing the car loads (transmitted to the axle through the axle boxes) and the forces R representing the rail loads (transmitted to the axle through the wheels). The diameter of the axle is $d = 82 \text{ mm}$, the distance between centers of the rails is L , and the distance between the forces P and R is $b = 200 \text{ mm}$.

Calculate the maximum bending stress $\sigma_{\max}$ in the axle if $P = 47 \text{ kN}$.



Solution 5.5-6

NUMERICAL DATA

$$d = 82 \text{ mm} \quad b = 220 \text{ mm}$$

$$P = 50 \text{ kN}$$

$$I = \frac{\pi d^4}{64} \quad I = 2.219 \times 10^{-6} \text{ m}^4$$

$$M_{\max} = Pb \quad M_{\max} = 11 \text{ kN} \cdot \text{m}$$

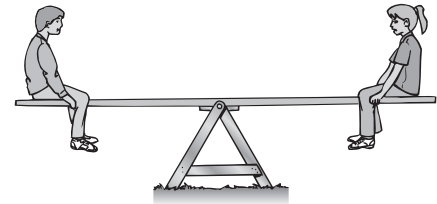
MAX. BENDING STRESS

$$\sigma_{\max} = \frac{Md}{2I}$$

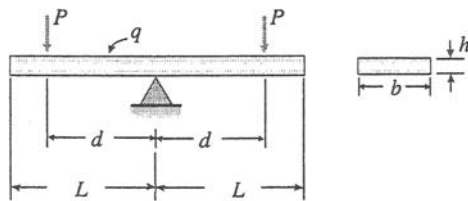
$$\sigma_{\max} = 203 \text{ MPa} \quad \leftarrow$$

Problem 5.5-7 A seesaw weighing 3 lb/ft of length is occupied by two children, each weighing 90 lb (see figure). The center of gravity of each child is 8 ft from the fulcrum. The board is 19 ft long, 8 in. wide, and 1.5 in. thick.

What is the maximum bending stress in the board?



Solution 5.5-7 Seesaw



$$b = 8 \text{ in.} \quad h = 1.5 \text{ in.}$$

$$q = 3 \text{ lb/ft} \quad P = 90 \text{ lb} \quad d = 8.0 \text{ ft} \quad L = 9.5 \text{ ft}$$

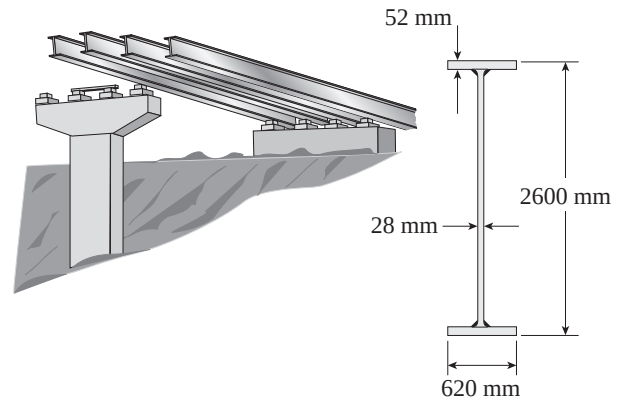
$$M_{\max} = Pd + \frac{qL^2}{2} = 720 \text{ lb-ft} + 135.4 \text{ lb-ft} \\ = 855.4 \text{ lb-ft} = 10,264 \text{ lb-in.}$$

$$S = \frac{bh^2}{6} = 3.0 \text{ in.}^3$$

$$\sigma_{\max} = \frac{M}{S} = \frac{10,264 \text{ lb-in.}}{3.0 \text{ in.}^3} = 3420 \text{ psi} \quad \leftarrow$$

Problem 5.5-8 During construction of a highway bridge, the main girders are cantilevered outward from one pier toward the next (see figure). Each girder has a cantilever length of 48 m and an I-shaped cross section with dimensions shown in the figure. The load on each girder (during construction) is assumed to be 9.5 kN/m, which includes the weight of the girder.

Determine the maximum bending stress in a girder due to this load.



Solution 5.5-8

NUMERICAL DATA

$$t_f = 52 \text{ mm} \quad t_w = 28 \text{ mm}$$

$$h = 2600 \text{ mm} \quad b_f = 620 \text{ mm}$$

$$L = 48 \text{ m} \quad q = 9.5 \frac{\text{kN}}{\text{m}}$$

$$I = \frac{1}{12} (b_f) h^3 - \frac{1}{12} (b_f - t_w) [h - 2(t_f)]^3$$

$$I = 1.41 \times 10^{11} \text{ mm}^4$$

$$M_{\max} = qL \left(\frac{L}{2} \right) \quad M_{\max} = 1.094 \times 10^4 \text{ kN} \cdot \text{m}$$

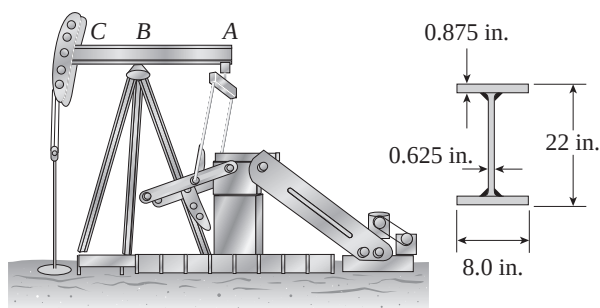
$$\sigma_{\max} = \frac{M_{\max} h}{2I}$$

$$\sigma_{\max} = 101 \text{ MPa} \quad \leftarrow$$

Problem 5.5-9 The horizontal beam ABC of an oil-well pump has the cross section shown in the figure. If the vertical pumping force acting at end C is 9 k and if the distance from the line of action of that force to point B is 16 ft, what is the maximum bending stress in the beam due to the pumping force?



Horizontal beam transfers loads as part of oil well pump



Solution 5.5-9

NUMERICAL DATA

$$F_C = 9 \text{ k} \quad BC = 16 \text{ ft}$$

$$M_{\max} = F_C(BC) \quad M_{\max} = 144 \text{ k-ft}$$

$$I = \frac{1}{12} (8)(22)^3 - \frac{1}{12} (8 - 0.625) \times [22 - 2(0.875)]^3 \quad I = 1.995 \times 10^3 \text{ in.}^4$$

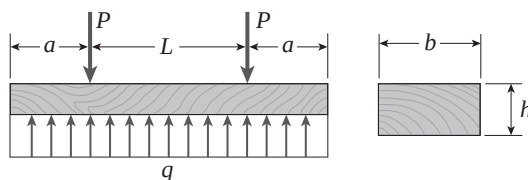
MAX. BENDING STRESS AT B

$$\sigma_{\max} = \frac{M_{\max} (12) \left(\frac{22}{2} \right)}{I}$$

$$\sigma_{\max} = 9.53 \text{ ksi} \quad \leftarrow$$

Problem 5.5-10 A railroad tie (or *sleeper*) is subjected to two rail loads, each of magnitude $P = 175 \text{ kN}$, acting as shown in the figure. The reaction q of the ballast is assumed to be uniformly distributed over the length of the tie, which has cross-sectional dimensions $b = 300 \text{ mm}$ and $h = 250 \text{ mm}$.

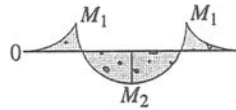
Calculate the maximum bending stress $\sigma_{\max}$ in the tie due to the loads P , assuming the distance $L = 1500 \text{ mm}$ and the overhang length $a = 500 \text{ mm}$.



Solution 5.5-10 Railroad tie (or sleeper)DATA $P = 175 \text{ kN}$ $b = 300 \text{ mm}$ $h = 250 \text{ mm}$ $L = 1500 \text{ mm}$ $a = 500 \text{ mm}$

$$q = \frac{2P}{L + 2a} \quad S = \frac{bh^2}{6} = 3.125 \times 10^{-3} \text{ m}^3$$

BENDING-MOMENT DIAGRAM



$$M_1 = \frac{qa^2}{2} = \frac{Pa^2}{L + 2a}$$

$$\begin{aligned} M_2 &= \frac{q}{2} \left(\frac{L}{2} + a \right)^2 - \frac{PL}{2} \\ &= \frac{P}{L + 2a} \left(\frac{L}{2} + a \right)^2 - \frac{PL}{2} \\ &= \frac{P}{4} (2a - L) \end{aligned}$$

Substitute numerical values:

$$M_1 = 17,500 \text{ N} \cdot \text{m} \quad M_2 = -21,875 \text{ N} \cdot \text{m}$$

$$M_{\max} = 21,875 \text{ N} \cdot \text{m}$$

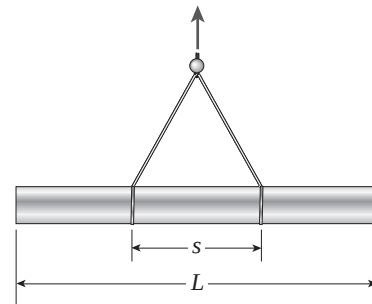
MAXIMUM BENDING STRESS

$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{21,875 \text{ N} \cdot \text{m}}{3.125 \times 10^{-3} \text{ m}^3} = 7.0 \text{ MPa} \quad \leftarrow$$

(Tension on top; compression on bottom)

Problem 5.5-11 A fiberglass pipe is lifted by a sling, as shown in the figure. The outer diameter of the pipe is 6.0 in., its thickness is 0.25 in., and its weight density is 0.053 lb/in.³ The length of the pipe is $L = 36 \text{ ft}$ and the distance between lifting points is $s = 11 \text{ ft}$.

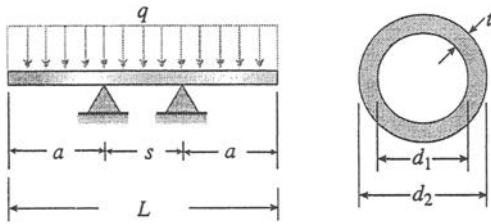
- Determine the maximum bending stress in the pipe due to its own weight.
- Find the spacing s between lift points which will minimize the bending stress. What is the minimum bending stress?
- What spacing s will lead to maximum bending stress? What is that stress?

**Solution 5.5-11**

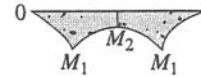
NUMERICAL DATA AND CROSS-SECTIONAL PROPERTIES OF PIPE

$$L = 36 \text{ ft} \quad d_2 = 6 \text{ in.} \quad t = 0.25 \text{ in.} \quad d_1 = d_2 - 2t = 5.5 \text{ in.} \quad A = \frac{\pi}{4} (d_2^2 - d_1^2) = 4.516 \text{ in.}^2$$

$$\gamma = 0.053 \frac{\text{lb}}{\text{in.}^3} \quad q = \gamma A = 0.239 \frac{\text{lb}}{\text{in.}} \quad s = 11 \text{ ft} \quad a = \frac{L - s}{2} = 12.5 \text{ ft} \quad I = \frac{\pi}{64} (d_2^4 - d_1^4) = 18.699 \text{ in.}^4$$



BENDING-MOMENT DIAGRAM



$$M_1 = -\frac{qa^2}{2} = -2,692.7 \text{ lb-in.}$$

$$M_2 = -\frac{qL}{4}\left(\frac{L}{2} - s\right) = -2,171.4 \text{ lb-in.}$$

$$M_{\max} = 2,692.7 \text{ lb-in.}$$

(a) MAXIMUM BENDING STRESS

$$\sigma_{\max} = \frac{M_{\max}c}{I} \quad c = \frac{d_2}{2} = 3.0 \text{ in.}$$

$$\sigma_{\max} = \frac{(2,692.7 \text{ lb-in.})(3.0 \text{ in.})}{18.699 \text{ in.}^4} = 432 \text{ psi} \quad \leftarrow$$

(Tension on top)

(b) IF $M_1 + M_2 = 0$, THE BENDING STRESS IS MINIMIZED

$$M_1(s) = -q\frac{\left(\frac{L-s}{2}\right)^2}{2} \quad M_2(s) = \frac{-qL}{4}\left(\frac{L}{2} - s\right) \quad \text{Let } M_1(s) + M_2(s) = 0 \quad \text{solving } s = 0.58579L$$

$$M_1(0.58579L) = -0.021446L^2q \quad M_2(0.58579L) = 0.021447L^2q \quad \boxed{s = 21.1 \text{ ft}}$$

MINIMUM BENDING STRESS

$$c = \frac{d_2}{2} \quad \sigma_{\min 1} = \frac{-M_1(s) - c}{I} = -153.692 \text{ psi} \quad \sigma_{\min 2} = \frac{-M_2(s)(-c)}{I} = 153.701 \text{ psi} \quad \boxed{\sigma_{\min} = \sigma_{\min 1} = -153.7 \text{ psi}}$$

Compare to solution for stresses in Part (a):

$$\sigma_{\max 1} = \frac{-M_1(11 \text{ ft})c}{I} = 432 \text{ psi} \quad \sigma_{\max 2} = \frac{-M_2(11 \text{ ft})(c)}{I} = 348 \text{ psi}$$

(c) EITHER $M_{1,\max}$ ($s = 0$) OR $M_{2,\max}$ ($s = L$) WILL LEAD TO MAXIMUM BENDING STRESS

$$(1) \text{ Support at } L/2 \text{ so } 1/2 \text{ of beam is a cantilever with max. moment } \frac{qL^2}{8} \quad \text{so } \frac{qL^2}{8} \frac{c}{I} = 895.795 \text{ psi}$$

$$\text{and } \sigma_{\max 1} = \frac{-M_1(0)c}{I} = 895.795 \text{ psi} \quad \sigma_{\max 2} = \frac{-M_2(0)(c)}{I} = 895.795 \text{ psi}$$

$$(2) \text{ OR simply supported beam } (s = L) \text{ under uniform load, so max. moment is once again } \frac{qL^2}{8}$$

$$\text{and } \sigma_{\max 1} = \frac{-M_1(L)(c)}{I} = 0 \text{ psi} \quad \sigma_{\max 2} = \frac{-M_2(L)(c)}{I} = -895.795 \text{ psi}$$

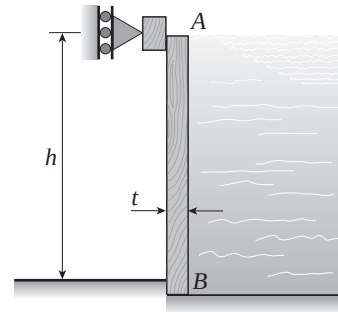
^ moment is zero
at pin support

^ moment is $qL^2/8$ at $L/2$

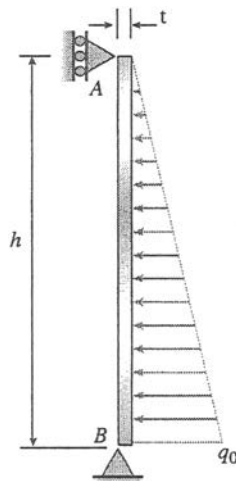
$$\text{so maximum bending stress is } \boxed{\sigma_{\max} = 896 \text{ psi}}$$

Problem 5.5-12 A small dam of height $h = 2.0$ m is constructed of vertical wood beams AB of thickness $t = 120$ mm, as shown in the figure. Consider the beams to be simply supported at the top and bottom.

Determine the maximum bending stress $\sigma_{\max}$ in the beams, assuming that the weight density of water is $\gamma = 9.81$ kN/m³



Solution 5.5-12 Vertical wood beam



$$h = 2.0 \text{ m}$$

$$t = 120 \text{ mm}$$

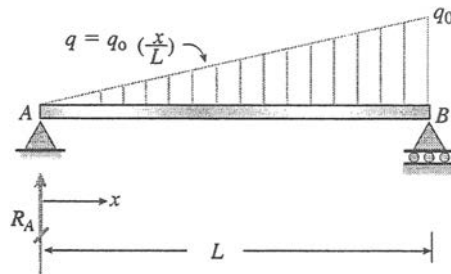
$$\gamma = 9.81 \text{ kN/m}^3 \text{ (water)}$$

Let b = width of beam perpendicular to the plane of the figure

Let q_0 = maximum intensity of distributed load

$$q_0 = \gamma b h \quad S = \frac{b t^2}{6}$$

MAXIMUM BENDING MOMENT



$$R_A = \frac{q_0 L}{6}$$

$$M = R_A x - \frac{q_0 x^3}{6L}$$

$$= \frac{q_0 L x}{6} - \frac{q_0 x^3}{6L}$$

$$\frac{dM}{dx} = \frac{q_0 L}{6} - \frac{q_0 x^2}{2L} = 0 \quad x = \frac{L}{\sqrt{3}}$$

Substitute $x = L/\sqrt{3}$ into the equation for M :

$$M_{\max} = \frac{q_0 L}{6} \left(\frac{L}{\sqrt{3}} \right) - \frac{q_0}{6L} \left(\frac{L^3}{3\sqrt{3}} \right) = \frac{q_0 L^2}{9\sqrt{3}}$$

$$\text{For the vertical wood beam: } L = h; M_{\max} = \frac{q_0 h^2}{9\sqrt{3}}$$

MAXIMUM BENDING STRESS

$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{2q_0 h^2}{3\sqrt{3} b t^2} = \frac{2\gamma h^3}{3\sqrt{3} t^2}$$

Substitute numerical values:

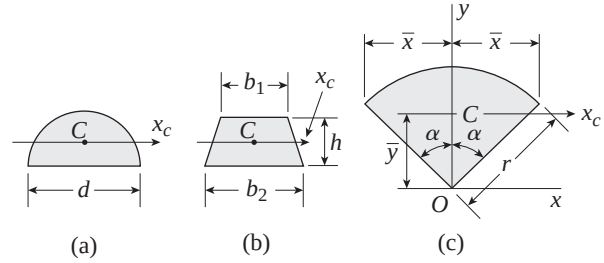
$$\sigma_{\max} = 2.10 \text{ MPa} \quad \leftarrow$$

NOTE: For $b = 1.0$ m, we obtain $q_0 = 19,620$ N/m, $S = 0.0024$ m³,

$M_{\max} = 5,034.5$ N·m, and $\sigma_{\max} = M_{\max}/S = 2.10$ MPa

Problem 5.5-13 Determine the maximum tensile stress σ_t (due to pure bending about a horizontal axis through C by positive bending moments M) for beams having cross sections as follows (see figure).

- (a) A semicircle of diameter d
- (b) An isosceles trapezoid with bases $b_1 = b$ and $b_2 = 4b/3$, and altitude h
- (c) A circular sector with $\alpha = \pi/3$ and $r = d/2$



Solution 5.5-13

MAX. TENSILE STRESS DUE TO POSITIVE BENDING MOMENT IS ON BOTTOM OF BEAM CROSS-SECTION

- (a) SEMICIRCLE

From Appendix E, Case 10:

$$I_c = \frac{(9\pi^2 - 64)r^4}{72\pi} = \frac{(9\pi^2 - 64)d^4}{1152\pi}$$

$$c = \frac{4r}{3\pi} = \frac{2d}{3\pi}$$

$$\sigma_t = \frac{Mc}{I_c} = \frac{768M}{(9\pi^2 - 64)d^3} = 30.93 \frac{M}{d^3} \quad \leftarrow$$

- (b) ISOSCELES TRAPEZOID

From Appendix E, Case 8:

$$I_C = \frac{h^3(b_1^2 + 4b_1b_2 + b_2^2)}{36(b_1 + b_2)}$$

$$= \frac{73bh^3}{756}$$

$$c = \frac{h(2b_1 + b_2)}{3(b_1 + b_2)} = \frac{10h}{21}$$

$$\sigma_t = \frac{Mc}{I_c} = \frac{360M}{73bh^2} \quad \leftarrow$$

- (c) CIRCULAR SECTOR WITH $\alpha = \pi/3$, $r = d/2$

From Appendix E, Case 13:

$$A = r^2(\alpha)$$

$$I_x = \frac{r^4}{4}(\alpha + \sin(\alpha)\cos(\alpha))$$

$$y_{\text{bar}} = \frac{2r}{3} \left(\frac{\sin(\alpha)}{\alpha} \right) \quad c = y_{\text{bar}}$$

$$d = 1$$

$$\text{For } \alpha = \pi/3, r = d/2: \quad A = \left(\frac{d}{2} \right)^2 \left(\frac{\pi}{3} \right)$$

$$A = d^2 \left(\frac{\pi}{12} \right) \quad A = 0.2618 d^2$$

$$c = \frac{2 \left(\frac{d}{2} \right) \left(\sin \left(\frac{\pi}{3} \right) \right)}{\frac{\pi}{3}} \quad c = 0.276 d$$

$$I_x = \frac{\left(\frac{d}{2} \right)^4}{4} \left(\frac{\pi}{3} + \sin \left(\frac{\pi}{3} \right) \cos \left(\frac{\pi}{3} \right) \right)$$

$$I_x = 0.02313 d^4$$

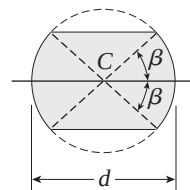
$$I_C = I_x - Ay_{\text{bar}}^2$$

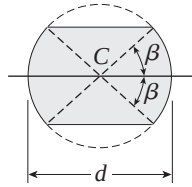
$$I_C = \left[d^4 \frac{(4\pi + 3\sqrt{3})}{768} - d^2 \left(\frac{\pi}{12} \right) \left[\frac{d}{2} \left(\frac{\sqrt{3}}{\pi} \right) \right]^2 \right]$$

$$I_C = 3.234 \times 10^{-3} d^4$$

$$\text{max. tensile stress } \sigma_t = \frac{Mc}{I_C} \quad \sigma_t = 85.24 \frac{M}{d^3} \quad \leftarrow$$

Problem 5.5-14 Determine the maximum bending stress σ_{max} (due to pure bending by a moment M) for a beam having a cross section in the form of a circular core (see figure). The circle has diameter d and the angle $\beta = 60^\circ$. (Hint: Use the formulas given in Appendix E, Cases 9 and 15.)



Solution 5.5-14 Circular core

From Appendix E, Cases 9 and 15:

$$I_y = \frac{\pi r^4}{4} - \frac{r^4}{2} \left(\alpha - \frac{ab}{r^2} + \frac{2ab^3}{r^4} \right)$$

$$r = \frac{d}{2} \quad \alpha = \frac{\pi}{2} - \beta$$

$$\begin{aligned} &= \frac{\pi d^4}{64} - \frac{d^4}{32} \left(\frac{\pi}{2} - \beta + \frac{1}{4} \sin 4\beta \right) \\ &= \frac{d^4}{128} (4\beta - \sin 4\beta) \end{aligned}$$

MAXIMUM BENDING STRESS

$$\sigma_{\max} = \frac{Mc}{I_y} \quad c = r \sin \beta = \frac{d}{2} \sin \beta$$

$$\sigma_{\max} = \frac{64M \sin \beta}{d^3 (4\beta - \sin 4\beta)} \quad \leftarrow$$

For $\beta = 60^\circ = \pi/3$ rad:

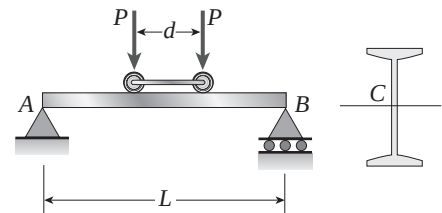
$$\sigma_{\max} = \frac{576M}{(8\pi\sqrt{3} + 9)d^3} = 10.96 \frac{M}{d^3} \quad \leftarrow$$

$$\beta = \text{radians} \quad \alpha = \text{radians} \quad a = r \sin \beta \quad b = r \cos \beta$$

$$\begin{aligned} I_y &= \frac{\pi d^4}{64} - \frac{d^4}{32} \left(\frac{\pi}{2} - \beta - \sin \beta \cos \beta + 2 \sin \beta \cos^3 \beta \right) \\ &= \frac{\pi d^4}{64} - \frac{d^4}{32} \left(\frac{\pi}{2} - \beta - (\sin \beta \cos \beta)(1 - 2 \cos^2 \beta) \right) \\ &= \frac{\pi d^4}{64} - \frac{d^4}{32} \left(\frac{\pi}{2} - \beta - \left(\frac{1}{2} \sin 2\beta \right) (-\cos 2\beta) \right) \end{aligned}$$

Problem 5.5-15 A simple beam AB of span length $L = 24$ ft is subjected to two wheel loads acting at distance $d = 5$ ft apart (see figure). Each wheel transmits a load $P = 3.0$ k, and the carriage may occupy any position on the beam.

- Determine the maximum bending stress $\sigma_{\max}$ due to the wheel loads if the beam is an I-beam having section modulus $S = 16.2$ in.³
- If $d = 5$ ft, find the required span length L to reduce the maximum stress in part (a) to 18 ksi.
- If $L = 24$ ft, find the required wheel spacing s to reduce the maximum stress in part (a) to 18 ksi.



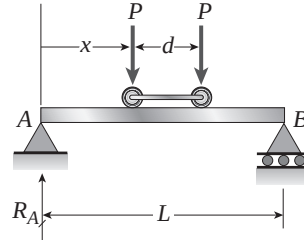
Solution 5.5-15 Wheel loads on a beam

NUMERICAL DATA

$$L = 24 \text{ ft} \quad d = 5 \text{ ft} \quad P = 3 \text{ kip}$$

$$x = \frac{L}{2} - \frac{d}{4} = 10.75 \text{ ft}$$

$$x + \frac{d}{2} = 13.25 \text{ ft} \quad S = 16.2 \text{ in.}^3$$



MAXIMUM BENDING MOMENT

$$R_A = \frac{P}{L}L - x + \frac{P}{L}(L - x - d) = \frac{P}{L}(2L - d - 2x)$$

$$M = R_A x = \frac{P}{L}(2Lx - dx - 2x^2)$$

$$\frac{dM}{dx} = \frac{P}{L}(2L - d - 4x) = 0 \quad x = \frac{L}{2} - \frac{d}{4}$$

(a) SUBSTITUTE x INTO THE EQUATION FOR M :

$$M_{\max} = \frac{P}{2L} \left(L - \frac{d}{2} \right)^2$$

MAXIMUM BENDING STRESS

$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{P}{2LS} \left(L - \frac{d}{2} \right)^2 \quad \leftarrow$$

Substitute numerical values:

$$\begin{aligned} \sigma_{\max} &= \frac{3k}{2(288 \text{ in.})(16.2 \text{ in.}^3)} (288 \text{ in.} - 30 \text{ in.})^2 \\ &= 21.4 \text{ ksi} \quad \leftarrow \end{aligned}$$

(b) MAX. MOMENT AND STRESS EXPRESSIONS FROM ABOVE

$$M_{\max} = \frac{P}{2L} \left(L - \frac{d}{2} \right)^2 \quad \sigma_{\max}(L, d) = \frac{P}{2LS} \left(L - \frac{d}{2} \right)^2$$

Set $\sigma_{\max}$ expression equal to 18 ksi, then solve for L ; only first solution below is valid.

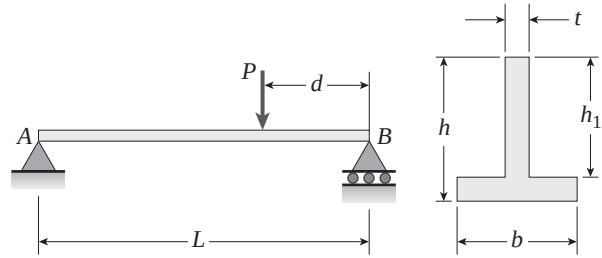
$$\frac{P}{2LS} \left(L - \frac{d}{2} \right)^2 = 18 \text{ ksi} \quad \text{solving for } L = \left[\frac{Pd + 36S \text{ ksi} + 6\sqrt{2} \sqrt{S \text{ ksi} (Pd + 18S \text{ ksi})}}{2P} \right] = \left(\frac{20.901}{0.299} \right) \text{ ft} \quad \boxed{L_{\text{reqd}} = 20.9 \text{ ft}}$$

(c) IF $L = 24 \text{ FT}$, FIND REQUIRED WHEEL SPACING d TO REDUCE THE MAXIMUM STRESS IN (A) TO 18 KSISet $\sigma_{\max}$ expression equal to 18 ksi, then solve for d :

$$\frac{P}{2LS} \left(L - \frac{d}{2} \right)^2 = 18 \text{ ksi} \quad \text{Solving gives } d = 2L - 12\sqrt{\frac{LS}{P}} = 8.564 \text{ ft} \quad \text{so } \boxed{d_{\text{reqd}} = 8.56 \text{ ft}}$$

Problem 5.5-16 Determine the maximum tensile stress σ_t and maximum compressive stress σ_c due to the load P acting on the simple beam AB (see figure).

- (a) Data are as follows: $P = 6.2 \text{ kN}$, $L = 3.2 \text{ m}$, $d = 1.25 \text{ m}$, $b = 80 \text{ mm}$, $t = 25 \text{ mm}$, $h = 120 \text{ mm}$, and $h_1 = 90 \text{ mm}$.
 (b) Find the value of d for which tensile and compressive stresses will be largest. What are these stresses?



Solution 5.5-16

NUMERICAL DATA AND CROSS SECTION PROPERTIES

$$P = 6.2 \text{ kN} \quad L = 3.2 \text{ m} \quad d = 1.25 \text{ m} \quad b = 80 \text{ mm} \quad t = 25 \text{ mm} \quad h = 120 \text{ mm} \quad h_1 = 90 \text{ mm}$$

Sum moments about base to find NA :

$$c_2 = \frac{b \frac{(h - h_1)^2}{2} + h_1 t \left(h - h_1 + \frac{h_1}{2} \right)}{b(h - h_1) + h_1 t} = 44.032 \text{ mm} \quad c_1 = h - c_2 = 75.968 \text{ mm}$$

$$I = b \frac{(h - h_1)^3}{12} + b(h - h_1) \left[c_2 - \left(\frac{h - h_1}{2} \right) \right]^2 + \frac{th_1^3}{12} + th_1 \left(c_1 - \frac{h_1}{2} \right)^2 = 5.879 \times 10^{-6} \text{ m}^4$$

(a) MAX. TENSILE AND COMPRESSIVE STRESSES

$$M_{\max} = \frac{Pd}{L} (L - d) = 4.72 \text{ kN} \cdot \text{m} \quad \sigma_c = \frac{M_{\max} c_1}{I} = 61 \text{ MPa} \quad \sigma_t = \frac{M_{\max} c_2}{I} = 35.4 \text{ MPa}$$

(b) FIND EXPRESSION FOR MOMENT AS A FUNCTION OF DISTANCE d , THEN TAKE FIRST DERIVATION TO FIND VALUE OF DISTANCE d AT WHICH MOMENT IS MAXIMUM

$$M(d) = \frac{Pd}{L} (L - d) \quad \frac{d}{dd} M(d) = 0 \quad \text{solving for } d = 0.5L \quad \text{so } d_{\max} = \frac{L}{2} \quad \text{and } M_{\max} = \frac{Pd_{\max}}{L} (L - d_{\max}) = 4.96 \text{ kN} \cdot \text{m}$$

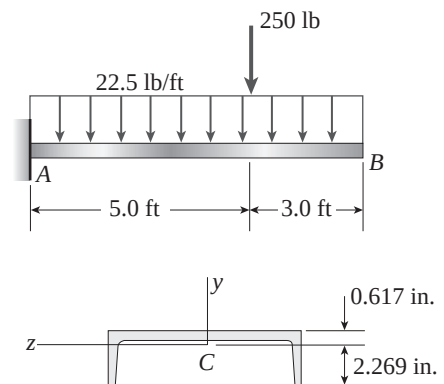
Tensile and compressive stresses are:

$$\sigma_t = \frac{M_{\max} c_2}{I} = 37.1 \text{ MPa}$$

$$\sigma_c = \frac{M_{\max} c_1}{I} = 64.1 \text{ MPa}$$

Problem 5.5-17 A cantilever beam AB , loaded by a uniform load and a concentrated load (see figure), is constructed of a channel section.

- (a) Find the maximum tensile stress σ_t and maximum compressive stress σ_c if the cross section has the dimensions indicated and the moment of inertia about the z axis (the neutral axis) is $I = 3.36 \text{ in.}^4$ (Note: The uniform load represents the weight of the beam.)
 (b) Find the maximum value of the concentrated load if the maximum tensile stress cannot exceed 4 ksi and the maximum compressive stress is limited to 14.5 ksi.
 (c) How far from A can load $P = 250 \text{ lb}$ be positioned if the maximum tensile stress cannot exceed 4 ksi and the maximum compressive stress is limited to 14.5 ksi?



Solution 5.5-17

NUMERICAL DATA

$$c_1 = 0.617 \text{ in.} \quad c_2 = 2.269 \text{ in.} \quad I = 3.36 \text{ in.}^4 \quad a = 5 \text{ ft} \quad b = 3 \text{ ft} \quad q = 22.5 \frac{\text{lb}}{\text{ft}} \quad L = a + b \quad P = 250 \text{ lb}$$

$$(a) \quad M_{\max} = \frac{q(L)^2}{2} + Pa \quad \boxed{\sigma_c = \frac{M_{\max} c_2}{I} = 15.96 \text{ ksi}} \quad \boxed{\sigma_t = \frac{M_{\max} c_1}{I} = 4.34 \text{ ksi}}$$

- (b) FIND AN EXPRESSION FOR THE MAX. TENSILE STRESS IN TERMS OF LOAD P , EQUATE IT TO THE ALLOWABLE VALUE, THEN SOLVE FOR $P_{\max}$

$$\frac{\left[\frac{q(L)^2}{2} + Pa \right] c_1}{I} = 4 \text{ ksi} \quad \text{solving for } P = \frac{4.0I \text{ ksi} - 32.0c_1 \text{ ft}^2 q}{ac_1} = 219 \text{ lb}$$

Repeat for max. compressive stress:

$$\frac{\left[\frac{q(L)^2}{2} + Pa \right] c_2}{I} = 14.5 \text{ ksi} \quad \text{solving for } P = \frac{0.5(29.0I \text{ ksi} - 64.0c_2 \text{ ft}^2 q)}{ac_2} = 214 \text{ lb} < \text{compression on bottom controls} \quad \boxed{P_{\max} = 214 \text{ lb}}$$

- (c) FIND AN EXPRESSION FOR THE MAX. TENSILE STRESS IN TERMS OF DISTANCE a TO LOAD P FROM SUPPORT A , EQUATE IT TO THE ALLOWABLE VALUE, THEN SOLVE FOR $a_{\max}$

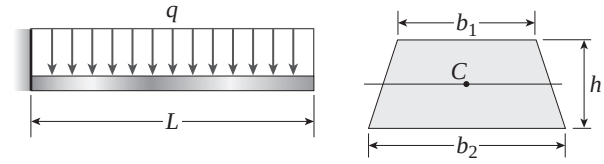
$$\frac{\left[\frac{q(L)^2}{2} + Pa \right] c_1}{I} = 4 \text{ ksi} \quad \text{solving for } a = \frac{4.0I \text{ ksi} - 32.0c_1 \text{ ft}^2 q}{Pc_1} = 4.38 \text{ ft}$$

Repeat for max. compressive stress:

$$\frac{\left[\frac{q(L)^2}{2} + Pa \right] c_2}{I} = 14.5 \text{ ksi} \quad \text{solving for } a = \frac{0.5(29.0I \text{ ksi} - 64.0c_2 \text{ ft}^2 q)}{Pc_2} = 4.28 \text{ ft} < \text{compression on bottom controls} \quad \boxed{a = 4.28 \text{ ft}}$$

Problem 5.5-18 A cantilever beam AB of isosceles trapezoidal cross section has length $L = 0.8$ m, dimensions $b_1 = 80$ mm, $b_2 = 90$ mm, and height $h = 110$ mm (see figure). The beam is made of brass weighing 85 kN/m³.

- Determine the maximum tensile stress σ_t and maximum compressive stress σ_c due to the beam's own weight.
- If the width b_1 is doubled, what happens to the stresses?
- If the height h is doubled, what happens to the stresses?



Solution 5.5-18

NUMERICAL DATA

$$L = 0.8 \text{ m} \quad \gamma = 85 \frac{\text{kN}}{\text{m}^3}$$

$$b_1 = 80 \text{ mm} \quad b_2 = 90 \text{ mm}$$

$$h = 110 \text{ mm}$$

- (a) MAX. STRESSES DUE TO BEAM'S OWN WEIGHT

$$M_{\max} = \frac{qL^2}{2} \quad q = \gamma A \quad A = \frac{1}{2}(b_1 + b_2)h$$

$$A = 9.35 \times 10^3 \text{ mm}^2$$

$$q = 7.9475 \times 10^2 \frac{\text{N}}{\text{m}}$$

$$M_{\max} = 254.32 \text{ N} \cdot \text{m}$$

$$y_{\text{bar}} = \frac{h(2b_1 + b_2)}{3(b_1 + b_2)} \quad y_{\text{bar}} = 53.922 \text{ mm}$$

$$I = h^3 \frac{(b_1^2 + 4b_1b_2 + b_2^2)}{36(b_1 + b_2)}$$

$$I = 9.417 \times 10^6 \text{ mm}^4$$

MAX. TENSILE STRESS AT SUPPORT (TOP)

$$\sigma_t = \frac{M_{\max}(h - y_{\text{bar}})}{I} \quad \sigma_t = 1.514 \text{ MPa} \quad \leftarrow$$

MAX. COMPRESSIVE STRESS AT SUPPORT (BOTTOM)

$$\sigma_c = \frac{M_{\max}y_{\text{bar}}}{I} \quad \sigma_c = 1.456 \text{ MPa} \quad \leftarrow$$

- (b) DOUBLE b_1 & RECOMPUTE STRESSES

$$b_1 = 160 \text{ mm}$$

$$A = \frac{1}{2}(b_1 + b_2)h \quad A = 1.375 \times 10^4 \text{ mm}^2$$

$$q = \gamma A \quad q = 1.169 \times 10^3 \frac{\text{N}}{\text{m}}$$

$$M_{\max} = \frac{qL^2}{2}$$

$$M_{\max} = 374 \text{ N} \cdot \text{m}$$

$$y_{\text{bar}} = \frac{h(2b_1 + b_2)}{3(b_1 + b_2)} \quad y_{\text{bar}} = 60.133 \text{ mm}$$

$$I = h^3 \frac{(b_1^2 + 4b_1b_2 + b_2^2)}{36(b_1 + b_2)}$$

$$I = 1.35 \times 10^7 \text{ mm}^4$$

MAX. TENSILE STRESS AT SUPPORT (TOP)

$$\sigma_t = \frac{M_{\max}(h - y_{\text{bar}})}{I} \quad \sigma_t = 1.381 \text{ MPa} \quad \leftarrow$$

MAX. COMPRESSIVE STRESS AT SUPPORT (BOTTOM)

$$\sigma_c = \frac{M_{\max}y_{\text{bar}}}{I} \quad \sigma_c = 1.666 \text{ MPa} \quad \leftarrow$$

(c) DOUBLE h & RECOMPUTE STRESSES

$$b_1 = 80 \text{ mm} \quad h = 220 \text{ mm}$$

$$A = \frac{1}{2}(b_1 + b_2)h \quad A = 1.87 \times 10^4 \text{ mm}^2$$

$$q = \gamma A \quad q = 1.589 \times 10^3 \frac{\text{N}}{\text{m}}$$

$$M_{\max} = \frac{qL^2}{2} \quad M_{\max} = 508.64 \text{ N} \cdot \text{m}$$

$$y_{\text{bar}} = \frac{h(2b_1 + b_2)}{3(b_1 + b_2)} \quad y_{\text{bar}} = 107.843 \text{ mm}$$

$$I = h^3 \frac{(b_1^2 + 4b_1b_2 + b_2^2)}{36(b_1 + b_2)}$$

$$I = 7.534 \times 10^7 \text{ mm}^4$$

MAX. TENSILE STRESS AT SUPPORT (TOP)

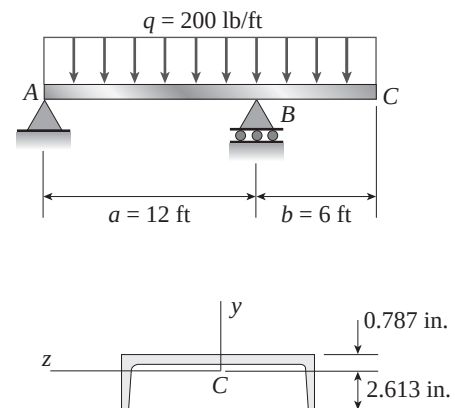
$$\sigma_t = \frac{M_{\max}(h - y_{\text{bar}})}{I} \quad \sigma_t = 0.757 \text{ MPa} \quad \leftarrow$$

MAX. COMPRESSIVE STRESS AT SUPPORT (BOTTOM)

$$\sigma_c = \frac{M_{\max}y_{\text{bar}}}{I} \quad \sigma_c = 0.728 \text{ MPa} \quad \leftarrow$$

Problem 5.5-19 A beam ABC with an overhang from B to C supports a uniform load of 200 lb/ft throughout its length (see figure). The beam is a channel section with dimensions as shown in the figure. The moment of inertia about the z axis (the neutral axis) equals 8.13 in.^4

- Calculate the maximum tensile stress σ_t and maximum compressive stress σ_c due to the uniform load.
- Find required span length a that results in the ratio of larger to smaller compressive stress being equal to the ratio of larger to smaller tensile stress for the beam. Assume that the total length $L = a + b = 18 \text{ ft}$ remains unchanged.



Solution 5.5-19

NUMERICAL DATA

$$q = 200 \frac{\text{lb}}{\text{ft}} \quad I = 8.13 \text{ in.}^4$$

$$a = 12 \text{ ft} \quad b = 6 \text{ ft} \quad L = a + b$$

$$c_1 = 0.787 \text{ in.} \quad c_2 = 2.613 \text{ in.}$$

(a) STATICS: Find reactions and moment at B and max. moment on span AB .

$$R_B = \frac{qL^2}{2} \quad R_A = qL - R_B \quad x_{\max} = \frac{R_A}{q} \quad x_{\max} = \frac{qL - \frac{qL^2}{2a}}{q} = \frac{9 \text{ ft}}{2}$$

$$M_B = \frac{qb^2}{2} \quad M_{\max AB} = R_A x_{\max} - q \frac{x_{\max}^2}{2}$$

$$M_{\max AB} = \left(qL - \frac{qL^2}{2a} \right) \left(L - \frac{L^2}{2a} \right) - q \frac{\left(L - \frac{L^2}{2a} \right)^2}{2} = 2025 \text{ ft-lb} \quad M_{\max AB} = \frac{L^2 q (L - 2a)^2}{8a^2}$$

COMPUTE MAX. TENSILE AND COMPRESSIVE STRESSES on span AB and at B :

$$\sigma_{cAB} = \frac{M_{\max AB} c_1}{I} = 2352 \text{ psi} < \text{comp.} \quad \boxed{\sigma_{tAB} = \frac{M_{\max AB} c_2}{I} = 7810 \text{ psi} < \text{tens.}}$$

$$\sigma_{tB} = \frac{M_B c_1}{I} = 4182 \text{ psi} < \text{tens.} \quad \boxed{\sigma_{cB} = \frac{M_B c_2}{I} = 13,885 \text{ psi} < \text{comp.}}$$

(b) THE STRESS RATIOS FROM PART (A) ARE: $\frac{\sigma_{cB}}{\sigma_{cAB}} = 5.903$ $\frac{\sigma_{tAB}}{\sigma_{tB}} = 1.868$ < in Part (b), **we want these ratios to be the same for tension and compression**

Write general expression for ratio of compressive stresses, then repeat for tensile stresses.

$$\frac{\sigma_{cB}}{\sigma_{cAB}} = \frac{\frac{M_B c_2}{I}}{\frac{M_{\max AB} c_1}{I}} \quad \frac{\sigma_{cB}}{\sigma_{cAB}} = \frac{\frac{\frac{qb^2}{2} c_2}{I}}{\frac{\frac{L^2 q (L - 2a)^2}{8a^2} c_1}{I}} = \frac{4a^2 b^2 c_2}{L^2 c_1 (L - 2a)^2} < \text{compression}$$

$$\frac{\sigma_{tAB}}{\sigma_{tB}} = \frac{\frac{M_{\max AB} c_2}{I}}{\frac{M_B c_1}{I}} \quad \frac{\sigma_{tAB}}{\sigma_{tB}} = \frac{\frac{\frac{L^2 q (L - 2a)^2}{8a^2} c_2}{I}}{\frac{\frac{qb^2}{2} c_1}{I}} = \frac{L^2 c_2 (L - 2a)^2}{4a^2 b^2 c_2} < \text{tension}$$

Equate above expressions and then solve for a : $\frac{4a^2(L-a)^2c_2}{L^2c_1(L-2a)^2} = \frac{L^2c_2(L-2a)^2}{4a^2(L-a)^2c_1}$ solving $a = \frac{L}{\sqrt{2}} = 12.728 \text{ ft}$

$$a = 12.73 \text{ ft}$$

CONFIRM THAT STRESS RATIOS ARE EQUAL IF $a = 12.73 \text{ ft}$

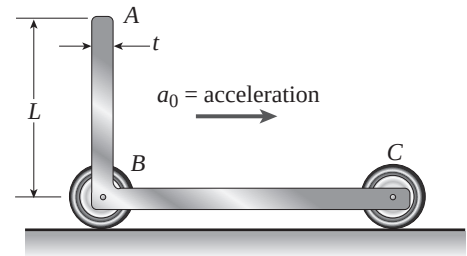
$$a = \frac{L}{\sqrt{2}} = 12.728 \text{ ft} \quad b = L - a = 5.272 \text{ ft} \quad L = a + b = 18 \text{ ft} \quad M_{\max AB} = \frac{L^2 q (L - 2a)^2}{8a^2} \quad M_B = \frac{qb^2}{2}$$

$$\sigma_{cAB} = \frac{M_{\max AB} c_1}{I} = 3229 \text{ psi} < \text{comp.} \quad \sigma_{tB} = \frac{M_B c_1}{I} = 3229 \text{ psi} < \text{tens.}$$

$$\sigma_{tAB} = \frac{M_{\max AB} c_2}{I} = 10,720 \text{ psi} < \text{tens.} \quad \sigma_{cB} = \frac{M_B c_2}{I} = 10,720 \text{ psi} < \text{comp.}$$

$$\text{So ratios are: } \frac{\sigma_{cB}}{\sigma_{cAB}} = 3.32 \quad \frac{\sigma_{tAB}}{\sigma_{tB}} = 3.32$$

Problem 5.5-20 A frame ABC travels horizontally with an acceleration a_0 (see figure). Obtain a formula for the maximum stress $\sigma_{\max}$ in the vertical arm AB, which had length L , thickness t , and mass density ρ .



Solution 5.5-20 Accelerating frame

L = length of vertical arm

t = thickness of vertical arm

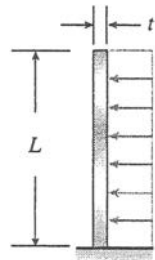
ρ = mass density

a_0 = acceleration

Let b = width of arm perpendicular to the plane of the figure

Let q = inertia force per unit distance along vertical arm

VERTICAL ARM



$$q = \rho b t a_0 \quad M_{\max} = \frac{q L^2}{2} = \frac{\rho b t a_0 L^2}{2}$$

$$S = \frac{b t^2}{6} \quad \sigma_{\max} = \frac{M_{\max}}{S} = \frac{3 \rho L^2 a_0}{t} \quad \leftarrow$$

TYPICAL UNITS FOR USE
IN THE PRECEDING EQUATION

SI units: $\rho = \text{kg/m}^3 = \text{N} \cdot \text{s}^2/\text{m}^4$

L = meters (m)

$a_0 = \text{m/s}^2$

t = meters (m)

$\sigma_{\max} = \text{N/m}^2$ (pascals)

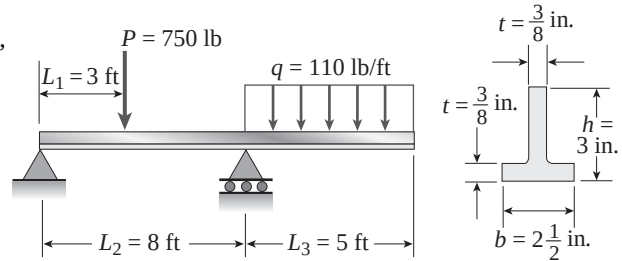
USCS units: $\rho = \text{slug/ft}^3 = \text{lb} \cdot \text{s}^2/\text{ft}^4$

$L = \text{ft} \quad a_0 = \text{ft/s}^2 \quad t = \text{ft}$

$\sigma_{\max} = \text{lb/ft}^2$ (Divide by 144 to obtain psi.)

Problem 5.5-21 A beam of T-section is supported and loaded as shown in the figure. The cross section has width $b = 2\frac{1}{2}$ in., height $h = 3$ in., and thickness $t = \frac{3}{8}$ in.

- Determine the maximum tensile and compressive stresses in the beam.
- If the allowable stresses in tension and compression are 18 ksi and 12 ksi, respectively, what is the required depth h of the beam? Assume that thickness t remains at $\frac{3}{8}$ in. and that flange width $b = 2.5$ in.
- Find the new values of loads P and q so that allowable tension (18 ksi) and compression (12 ksi) stresses are reached simultaneously for the beam. Use the beam cross section in part (a) (see figure) and assume that L_1 , L_2 and L_3 are unchanged.



Solution 5.5-21

NUMERICAL DATA

$$P = 750 \text{ lb} \quad q = 110 \frac{\text{lb}}{\text{ft}} \quad L_1 = 3 \text{ ft} \quad L_2 = 8 \text{ ft} \quad L_3 = 5 \text{ ft} \quad t = \frac{3}{8} \text{ in.} \quad b = 2.5 \text{ in.} \quad h = 3 \text{ in.}$$

LOCATE NEUTRAL AXIS AND COMPUTE MOMENT OF INERTIA I

$$c_2 = \frac{bt \frac{t}{2} + t(h-t) \left(\frac{h-t}{2} + t \right)}{bt + (h-t)t} = 0.956 \text{ in.} \quad c_1 = h - c_2 = 2.044 \text{ in.}$$

$$I = \frac{1}{12} bt^3 + bt \left(c_2 - \frac{t}{2} \right)^2 + \frac{1}{12} t(h-t)^3 + t(h-t) \left[c_1 - \left(\frac{h-t}{2} \right) \right]^2 = 1.657 \text{ in.}^4$$

STATICS

$$R_B = \frac{1}{L_2} \left[PL_1 + qL_3 \left(L_2 + \frac{L_3}{2} \right) \right] = 1.003 \times 10^3 \text{ lb} \quad R_A = P + qL_3 - R_B = 296.875 \text{ lb}$$

$$\text{MOMENT AT LOAD } P \quad M_P = R_A L_1 = 890.625 \text{ ft-lb} \quad \text{MOMENT AT SUPPORT } B \quad M_B = \frac{qL_3^2}{2} = 1375 \text{ ft-lb}$$

$$\text{STRESSES AT } P \text{ AND AT } B \quad \sigma_{cP} = \frac{M_P c_1}{I} = 13,188 \text{ psi} \quad \sigma_{tP} = \frac{M_P c_2}{I} = 6166 \text{ psi}$$

$$\sigma_{cB} = \frac{M_B c_2}{I} = 9520 \text{ psi} \quad \sigma_{tB} = \frac{M_B c_1}{I} = 20,360 \text{ psi}$$

(b) FOR COMPRESSION, USE TOP OF BEAM AT P

$$M_P \left(\frac{c_1}{I} \right) = 12 \text{ ksi} \quad \text{where} \quad c_1 = h - \frac{bt \frac{t}{2} + t(h-t) \left(\frac{h-t}{2} + t \right)}{bt + (h-t)t}$$

and

$$I = \frac{1}{12} bt^3 + bt \left[\frac{bt \frac{t}{2} + t(h-t) \left(\frac{h-t}{2} + t \right)}{bt + (h-t)t} - \frac{t}{2} \right]^2 + \frac{1}{12} t(h-t)^3 + t(h-t) \left[\left[h - \frac{bt \frac{t}{2} + t(h-t) \left(\frac{h-t}{2} + t \right)}{bt + (h-t)t} \right] - \left(\frac{h-t}{2} \right) \right]^2$$

$$M_P \left(\frac{c_1}{I} \right) = 12 \text{ ksi} \quad < \text{solve this expression numerically for depth } h: \quad h = 3.1486 \text{ in.} \quad < \text{compression controls}$$

$$h = 3.15 \text{ in.}$$

Substitute value of h to confirm compressive stress value

$$c_2 = \frac{bt \frac{t}{2} + t(h-t) \left(\frac{h-t}{2} + t \right)}{bt + (h-t)t} = 1.015 \text{ in.} \quad c_1 = h - c_2 = 2.133 \text{ in.}$$

$$I = \frac{1}{12} bt^3 + bt \left[\frac{bt \frac{t}{2} + t(h-t) \left(\frac{h-t}{2} + t \right)}{bt + (h-t)t} - \frac{t}{2} \right]^2 + \frac{1}{12} t(h-t)^3 + t(h-t) \left[\left[h - \frac{bt \frac{t}{2} + t(h-t) \left(\frac{h-t}{2} + t \right)}{bt + (h-t)t} \right] - \left(\frac{h-t}{2} \right) \right]^2$$

$$M_P = 890.625 \text{ ft-lb} \quad \frac{M_P c_1}{I} = 12,000 \text{ psi}$$

FOR TENSION, USE TOP OF BEAM AT B

$$M_B \frac{c_1}{I} = 18 \text{ ksi} \quad \text{solving for } h \quad h = 3.196 \text{ in.} \quad \text{so value here based on tension } (h = 3.20 \text{ in.}) \text{ controls}$$

$$\boxed{h = 3.20 \text{ in.}}$$

- (c) MUST HAVE RATIO $M_B/M_P = 18/12 = 1.5$ IF ALLOWABLE TENSION AND COMPRESSION STRESSES ARE TO BE REACHED SIMULTANEOUSLY

$$\frac{\sigma_{aT}}{\sigma_{aC}} = \frac{\frac{M_B c_1}{I}}{\frac{M_P c_1}{I}} = \frac{1.0 M_B}{M_P} = \frac{18}{12} = \frac{3}{2} \quad \text{Use expression above for } M_B \text{ and } M_P.$$

$$\frac{\frac{qL_3^2}{2}}{\left[P + qL_3 - \frac{1}{L_2} \left[PL_1 + qL_3 \left(L_2 + \frac{L_3}{2} \right) \right] \right] L_1} = \frac{88}{57} \quad \frac{88}{57} = 1.544 < \text{we want ratio to be } 1.50.$$

Divide through by q in above expression to get the following where $\alpha = P/q$, then solve for α :

$$\frac{\frac{(L - L_2)^2}{2}}{\left[\alpha + (L - L_2) - \frac{1}{L_2} \left[\alpha L_1 + (L - L_2) \left(L_2 + \frac{L - L_2}{2} \right) \right] \right] L_1} = \frac{3}{2} \quad \text{solving numerically gives } \alpha = 6.944 \text{ ft}$$

Now find q so that tension at top at B is 18 ksi, then use that q to find P so that compression at top at P is 12 ksi:

$$\frac{qL_3^2}{2} \frac{c_1}{I} = 18 \text{ ksi} \quad \text{solving for } q \text{ gives } q = 97.249 \frac{\text{lb}}{\text{ft}} \quad \text{so } P = \alpha q = 675.297 \text{ lb}$$

$$\boxed{q = 97.2 \frac{\text{lb}}{\text{ft}}}$$

$$\boxed{P = 675 \text{ lb}}$$

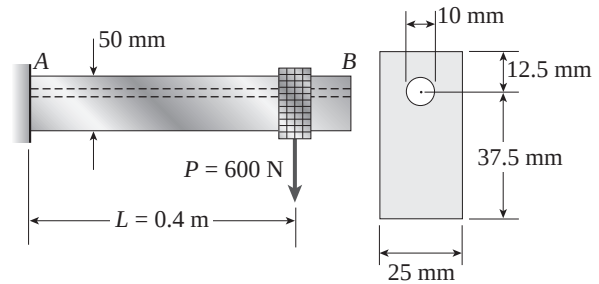
$$\frac{P}{q} = 6.944 \text{ ft}$$

Final check on stresses at B and P :

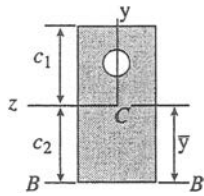
$$\frac{qL_3^2}{2} \frac{c_1}{I} = 18 \text{ ksi} \quad \left[P + qL_3 - \frac{1}{L_2} \left[PL_1 + qL_3 \left(L_3 + \frac{L_3}{2} \right) \right] \right] L_1 \frac{c_1}{I} = 12 \text{ ksi} \quad \text{OK}$$

Problem 5.5-22 A cantilever beam AB with a rectangular cross section has a longitudinal hole drilled throughout its length (see figure). The beam supports a load $P = 600$ N. The cross section is 25 mm wide and 50 mm high, and the hole has a diameter of 10 mm.

Find the bending stresses at the top of the beam, at the top of the hole, and at the bottom of the beam.



Solution 5.5-22 Rectangular beam with a hole



MAXIMUM BENDING MOMENT

$$M = PL = (600 \text{ N})(0.4 \text{ m}) = 240 \text{ N} \cdot \text{m}$$

PROPERTIES OF THE CROSS SECTION

A_1 = area of rectangle

$$= (25 \text{ mm})(50 \text{ mm}) = 1250 \text{ mm}^2$$

A_2 = area of hole

$$= \frac{\pi}{4} (10 \text{ mm})^2 = 78.54 \text{ mm}^2$$

A = area of cross section

$$= A_1 - A_2 = 1171.5 \text{ mm}^2$$

Using line $B-B$ as reference axis:

$$\sum A_i y_i = A_1(25 \text{ mm}) - A_2(37.5 \text{ mm}) = 28,305 \text{ mm}^3$$

$$\bar{y} = \frac{\sum A_i y_i}{A} = \frac{28,305 \text{ mm}^3}{1171.5 \text{ mm}^2} = 24.162 \text{ mm}$$

Distances to the centroid C :

$$c_2 = \bar{y} = 24.162 \text{ mm}$$

$$c_1 = 50 \text{ mm} - c_2 = 25.838 \text{ mm}$$

MOMENT OF INERTIA ABOUT THE NEUTRAL AXIS
(THE z AXIS)

All dimensions in millimeters.

Rectangle:

$$\begin{aligned} I_z &= I_c + Ad^2 \\ &= \frac{1}{12} (25)(50)^3 + (25)(50)(25 - 24.162)^2 \\ &= 260,420 + 878 = 261,300 \text{ mm}^4 \end{aligned}$$

Hole:

$$\begin{aligned} I_z &= I_c + Ad^2 = \frac{\pi}{64} (10)^4 + (78.54)(37.5 - 24.162)^2 \\ &= 490.87 + 13,972 = 14,460 \text{ mm}^4 \end{aligned}$$

Cross-section:

$$I = 261,300 - 14,460 = 246,800 \text{ mm}^4$$

STRESS AT THE TOP OF THE BEAM

$$\begin{aligned} \sigma_1 &= \frac{Mc_1}{I} = \frac{(240 \text{ N} \cdot \text{m})(25.838 \text{ mm})}{246,800 \text{ mm}^4} \\ &= 25.1 \text{ MPa} \quad \leftarrow \\ &\quad \text{(tension)} \end{aligned}$$

STRESS AT THE TOP OF THE HOLE

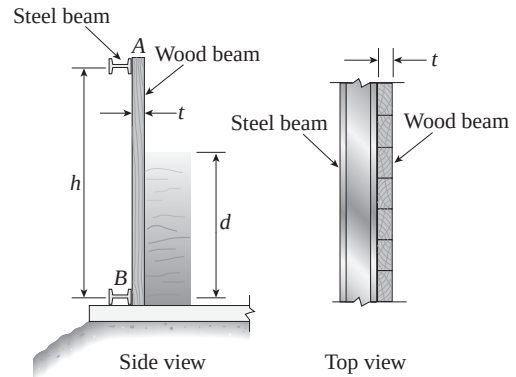
$$\begin{aligned} \sigma_2 &= \frac{My}{I} \quad y = c_1 - 7.5 \text{ mm} = 18.338 \text{ mm} \\ \sigma_2 &= \frac{(240 \text{ N} \cdot \text{m})(18.338 \text{ mm})}{246,800 \text{ mm}^4} = 17.8 \text{ MPa} \quad \leftarrow \\ &\quad \text{(tension)} \end{aligned}$$

STRESS AT THE BOTTOM OF THE BEAM

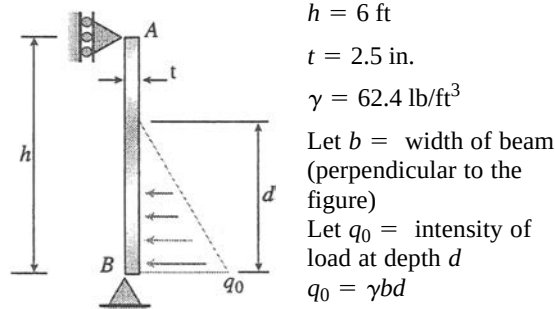
$$\begin{aligned} \sigma_3 &= -\frac{Mc_2}{I} = -\frac{(240 \text{ N} \cdot \text{m})(24.162 \text{ mm})}{246,800 \text{ mm}^4} \\ &= -23.5 \text{ MPa} \quad \leftarrow \\ &\quad \text{(compression)} \end{aligned}$$

Problem 5.5-23 A small dam of height $h = 6$ ft is constructed of vertical wood beams AB , as shown in the figure. The wood beams, which have thickness $t = 2.5$ in., are simply supported by horizontal steel beams at A and B .

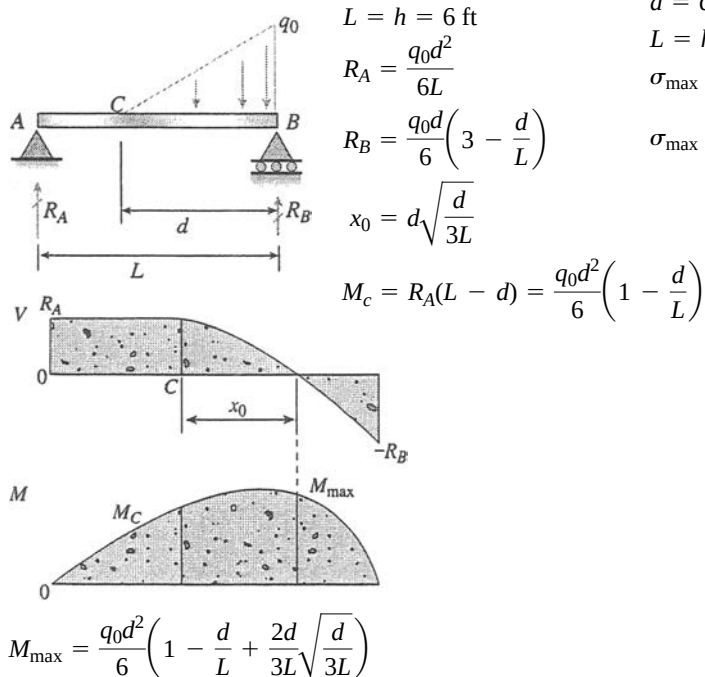
Construct a graph showing the maximum bending stress $\sigma_{\max}$ in the wood beams versus the depth d of the water above the lower support at B . Plot the stress $\sigma_{\max}$ (psi) as the ordinate and the depth d (ft) as the abscissa. (Note: The weight density γ of water equals 62.4 lb/ft³.)



Solution 5.5-23 Vertical wood beam in a dam



ANALYSIS OF BEAM



MAXIMUM BENDING STRESS

Section modulus: $S = \frac{1}{6} b t^2$

$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{6}{b t^2} \left[\frac{q_0 d^2}{6} \left(1 - \frac{d}{L} + \frac{2d}{3L} \sqrt{\frac{d}{3L}} \right) \right]$$

$$q_0 = \gamma b d$$

$$\sigma_{\max} = \frac{\gamma d^3}{t^2} \left(1 - \frac{d}{L} + \frac{2d}{3L} \sqrt{\frac{d}{3L}} \right) \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

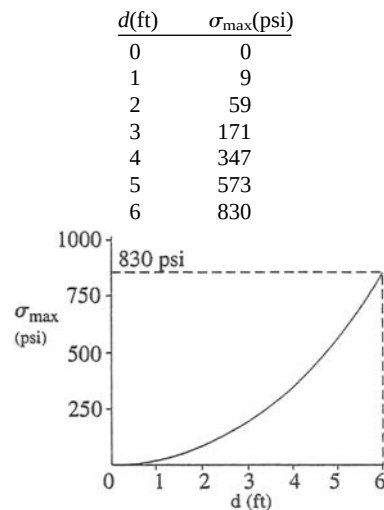
$$d = \text{depth of water (ft)} \quad (\text{Max. } d = h = 6 \text{ ft})$$

$$L = h = 6 \text{ ft} \quad \gamma = 62.4 \text{ lb/ft}^3 \quad t = 2.5 \text{ in.}$$

$$\sigma_{\max} = \text{psi}$$

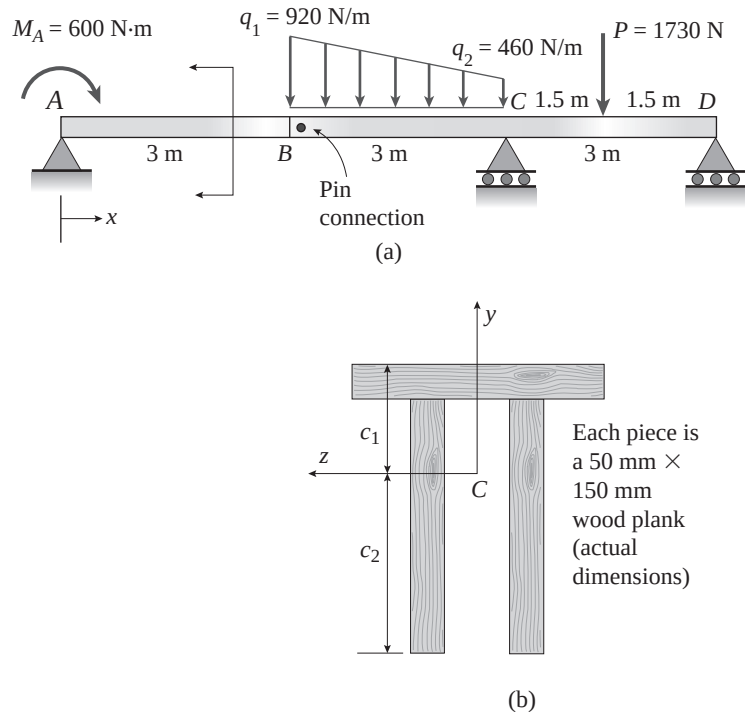
$$\sigma_{\max} = \frac{(62.4)d^3}{(2.5)^2} \left(1 - \frac{d}{6} + \frac{d}{9} \sqrt{\frac{d}{18}} \right)$$

$$= 0.1849d^3(54 - 9d + d\sqrt{2d}) \leftarrow$$



Problem 5.5-24 Consider the compound beam with segments AB and BCD joined by a pin connection (moment release) just right of B (see figure part a). The beam cross section is a double-Tee made up from three $50 \text{ mm} \times 150 \text{ mm}$ wood members (actual dimensions, see figure part b).

- Find the centroid C of the double-T cross section (c_1, c_2), then compute the moment of inertia I_z (mm^4).
- Find the maximum *tensile* normal stress σ_t and maximum *compressive* normal stress σ_c (kPa) for the loading shown. (Ignore the weight of the beam.)



Solution 5.5-24

(a) CROSS-SECTIONAL PROPERTIES

$$b = 50 \text{ mm} \quad h = 150 \text{ mm} \quad A_x = bh = 7.5 \times 10^{-3} \text{ m}^2$$

CENTROID

$$c_2 = \frac{2A_x\left(\frac{h}{2}\right) + A_x\left(h + \frac{b}{2}\right)}{3A_x} = 108.333 \text{ mm}$$

$$c_1 = h + b - c_2 = 91.667 \text{ mm}$$

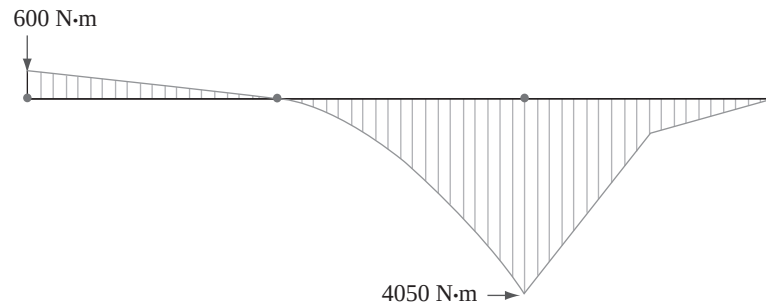
$$\boxed{c_1 = 91.7 \text{ mm}} \quad \boxed{c_2 = 108.3 \text{ mm}}$$

$$I_z = 2\left(\frac{1}{12}\right)b(h)^3 + 2A_x\left(c_2 - \frac{h}{2}\right)^2 + \frac{1}{12}(h)(b)^3 + A_x\left(c_1 - \frac{b}{2}\right)^2 \quad \boxed{I_z = 7.969 \times 10^7 \text{ mm}^4}$$

SECTION MODULUS for top and also for bottom of beam

$$S_{\text{top}} = \frac{I_z}{c_1} = 8.693 \times 10^5 \text{ mm}^3 \quad S_{\text{bot}} = \frac{I_z}{c_2} = 7.356 \times 10^5 \text{ mm}^3$$

STATICS: find reactions, then shear and MOMENT DIAGRAMS for this beam. (See Prob. 1.2-3 for similar problem.)



$M_A = 600 \text{ N} \cdot \text{m}$ $M_C = 4050 \text{ N} \cdot \text{m}$ $<$ max. moments at A (compression on top) and C (compression on bottom)

STRESSES AT A

$$\sigma_{A\text{top}} = \frac{M_A}{S_{\text{top}}} = 690 \text{ kPa} \quad \text{compression}$$

$$\sigma_{A\text{bot}} = \frac{M_A}{S_{\text{bot}}} = 816 \text{ kPa} \quad \text{tension}$$

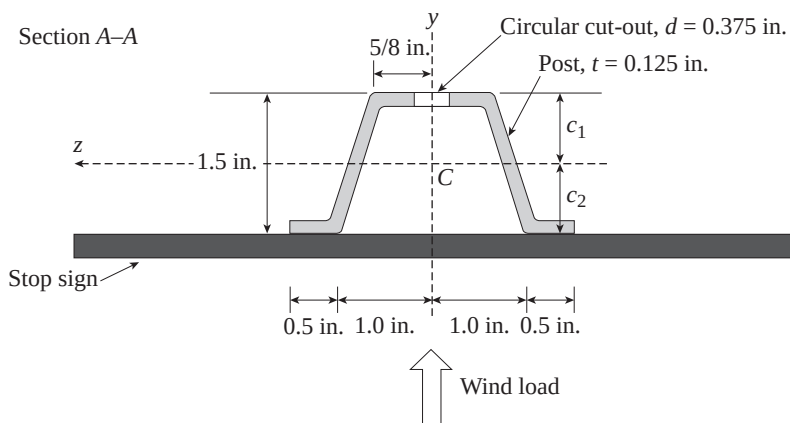
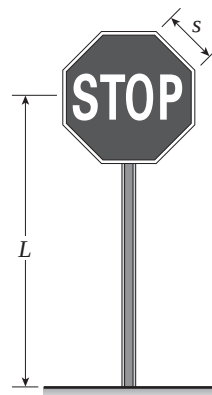
STRESSES AT C: maximum tension and compression occur at C.

$\sigma_{C\text{top}} = \frac{M_C}{S_{\text{top}}} = 4659 \text{ kPa}$	tension

$\sigma_{C\text{bot}} = \frac{M_C}{S_{\text{bot}}} = 5506 \text{ kPa}$	compression

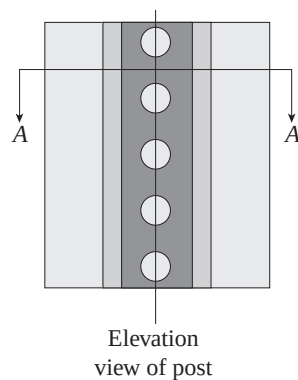
Problem 5.5-25 A steel post ($E = 30 \times 10^6$ psi) having thickness $t = 1/8$ in. and height $L = 72$ in. supports a stop sign (see figure: $s = 12.5$ in.). The height of the post L is measured from the base to the centroid of the sign. The stop sign is subjected to wind pressure $p = 20$ lb/ft² normal to its surface. Assume that the post is fixed at its base.

- What is the resultant load on the sign? [See Appendix E, Case 25, for properties of an octagon, $n = 8$].
- What is the maximum bending stress $\sigma_{\max}$ in the post?
- Repeat part (b) if the circular cutouts are eliminated over the height of the post.



Numerical properties of post

$$A = 0.578 \text{ in.}^2, c_1 = 0.769 \text{ in.}, c_2 = 0.731 \text{ in.}, \\ I_y = 0.44867 \text{ in.}^4, I_z = 0.16101 \text{ in.}^4$$



Solution 5.5-25

(a) RESULTANT LOAD F ON SIGN $p = 20 \text{ psf}$ $s = 12.5 \text{ in.}$ $n = 8$ $\beta = \frac{360}{n} \left(\frac{\pi}{180} \right)$ $\beta = 0.785 \text{ rad}$

$$A = \frac{ns^2}{4} \cot\left(\frac{\beta}{2}\right) \quad A = 754.442 \text{ in.}^2 \quad A = \frac{A}{144} \quad A = 5.239 \text{ ft}^2$$

$$F = pA \quad F = 104.784 \text{ lb} \quad \boxed{F_{\text{res}} = 104.8 \text{ lb}}$$

(b) MAX. BENDING STRESS IN POST $L = 72 \text{ in.}$ $I_z = 0.16101 \text{ in.}^4$ $c_1 = 0.769 \text{ in.}$ $c_2 = 0.731 \text{ in.}$

$$M_{\text{max}} = FL \quad M_{\text{max}} = 628.7 \text{ ft-lb}$$

$$\sigma_c = \frac{M_{\text{max}} c_1}{I_z} \quad \boxed{\sigma_c = 36.0} \text{ ksi} < \text{max. bending stress at base of post}$$

$$\sigma_t = \frac{M_{\text{max}} c_2}{I_z} \quad \sigma_t = 34.2 \text{ ksi}$$

(c) REPEAT (b) IF CUTOUT HOLES ARE ELIMINATED FROM POST

Re-compute cross sectional properties (see below): $c_1 = 0.716 \text{ in.}$ $c_2 = 0.784 \text{ in.}$ $I_z = 0.18269 \text{ in.}^4$

$$p = 20 \frac{\text{lb}}{\text{ft}^2} \quad A = 754.442 \text{ in.}^2 \quad F = pA \quad L = 6 \text{ ft} \quad M_{\text{max}} = FL = 628.702 \text{ ft-lb}$$

$$\sigma_c = \frac{M_{\text{max}} c_1}{I_z} = 29.6 \text{ ksi} \quad \boxed{\sigma_t = \frac{M_{\text{max}} c_2}{I_z} = 32.4 \text{ ksi}} \quad \text{tension at base}$$

Re-compute cross-sectional properties **with holes eliminated**:

$$A = t(2a_1 + 2b + 2a_2 - d) \quad A = 0.625 \text{ in.}^2 \quad b \cos(\beta_0) = 0.375$$

CENTROID

$$c_1 = \frac{\frac{t}{2} t(2a_2 - d) + 2bt\left(\frac{1.5}{2}\right) + 2a_1\left(1.5 - \frac{t}{2}\right)}{A} \quad c_1 = 0.716 \text{ in.}$$

$$c_2 = 1.5 - c_1 \quad c_2 = 0.784 \text{ in.}$$

MOMENTS OF INERTIA WRT Y & Z AXES

$$I_y = \frac{1}{12} t[(2a_2)^3 - d^3] + \left(\frac{1}{12} 2ta_1^3\right) + 2(a_1 t) \left(1.5 - \frac{a_1}{2}\right)^2 + 2 \frac{tb^3}{12} \cos^2(\beta_0) + 2bt \left(a_2 + \frac{1}{2} \frac{3}{8}\right)^2$$

$$I_y = 0.44922 \text{ in.}^4$$

$$I_z = \frac{1}{12} t^3(2a_2 - d) + t(2a_2 - d) \left(c_1 - \frac{t}{2}\right)^2 + 2a_1 \frac{t^3}{12} + t(2a_1) \left(c_2 - \frac{t}{2}\right)^2 + \frac{2tb^3}{12} \sin^2(\beta_0) + 2bt \left(c_1 - \frac{1.5}{2}\right)^2$$

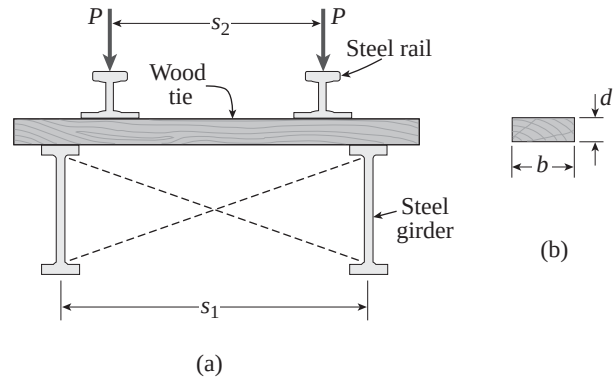
$$I_z = 0.18269 \text{ in.}^4$$

Design of Beams

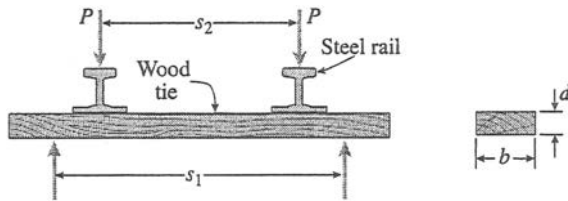
Problem 5.6-1 The cross section of a narrow-gage railway bridge is shown in part a of the figure. The bridge is constructed with longitudinal steel girders that support the wood cross ties. The girders are restrained against lateral buckling by diagonal bracing, as indicated by the dashed lines.

The spacing of the girders is $s_1 = 50$ in. and the spacing of the rails is $s_2 = 30$ in. The load transmitted by each rail to a single tie is $P = 1500$ lb. The cross section of a tie, shown in part b of the figure, has width $b = 5.0$ in. and depth d .

Determine the minimum value of d based upon an allowable bending stress of 1125 psi in the wood tie. (Disregard the weight of the tie itself.)



Solution 5.6-1 Railway cross tie



$$s_1 = 50 \text{ in.} \quad b = 5.0 \text{ in.} \quad s_2 = 30 \text{ in.}$$

$$d = \text{depth of tie} \quad P = 1500 \text{ lb} \quad \sigma_{\text{allow}} = 1125 \text{ psi}$$

$$M_{\text{max}} = \frac{P(s_1 - s_2)}{2} = 15,000 \text{ lb-in.}$$

$$S = \frac{bd^2}{6} = \frac{1}{6} (5.0 \text{ in.})(d^2) = \frac{5d^2}{6} \quad d = \text{inches}$$

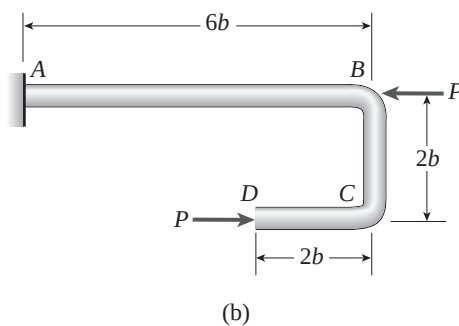
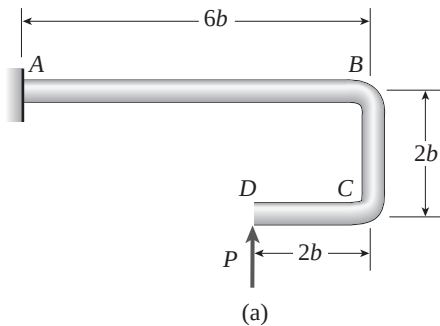
$$M_{\text{max}} = \sigma_{\text{allow}} S \quad 15,000 = (1125) \left(\frac{5d^2}{6} \right)$$

$$\text{Solving, } d^2 = 16.0 \text{ in.} \quad d_{\text{min}} = 4.0 \text{ in.} \quad \leftarrow$$

$$\text{NOTE: Symbolic solution: } d^2 = \frac{3P(s_1 - s_2)}{b\sigma_{\text{allow}}}$$

Problem 5.6-2 A fiberglass bracket $ABCD$ of solid circular cross section has the shape and dimensions shown in the figure. A vertical load $P = 40$ N acts at the free end D .

- Determine the minimum permissible diameter d_{min} of the bracket if the allowable bending stress in the material is 30 MPa and $b = 37$ mm. (Note: Disregard the weight of the bracket itself.)
- If $d = 10$ mm, $b = 37$ mm, and $\sigma_{\text{allow}} = 30$ MPa, what is the maximum value of load P if vertical load P at D is replaced with horizontal loads P at B and D (see figure part b)?



Solution 5.6-2(a) NORMAL STRESS AT A DUE TO FLEXURE (COMPRESSION AT TOP)

$$\sigma_a = \frac{(4Pb) \left(\frac{d_{\min}}{2} \right)}{\left(\frac{\pi d_{\min}^4}{64} \right)} = \frac{128 Pb}{\pi d_{\min}^3}$$

Solve above equation for required diameter d :

$$d_{\min} = \left(\frac{128 Pb}{\pi \sigma_a} \right)^{\frac{1}{3}}$$

SUBSTITUTE NUMERICAL VALUES THEN SOLVE FOR d

$$P = 40 \text{ N} \quad b = 37 \text{ mm} \quad \sigma_a = 30 \text{ MPa}$$

$$d_{\min} = \left[\frac{128(40 \text{ N})(37 \text{ mm})}{\pi(30 \text{ MPa})} \right]^{\frac{1}{3}} \quad \boxed{d_{\min} = 12.62 \text{ mm}}$$

(b) REACTIVE MOMENT AT A IS $P(2b)$; HORIZONTAL AND VERTICAL REACTIONS AT A ARE BOTH ZERO

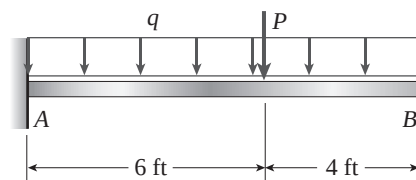
$$M_A = P(2b) \quad \text{so} \quad \sigma_{\text{allow}} = \frac{(2Pb) \left(\frac{d}{2} \right)}{\left(\frac{\pi d^4}{64} \right)} = \frac{94,720 \text{ N} \cdot \text{mm}}{\pi d^3}$$

$$\text{solving for } P \quad P_{\max} = \frac{\sigma_{\text{allow}} \pi d^3}{64b} \quad \text{and} \quad \boxed{P_{\max} = \frac{(30 \text{ MPa}) \pi (10 \text{ mm})^3}{64(37 \text{ mm})} = 39.8 \text{ N}}$$

Problem 5.6-3 A cantilever beam AB is loaded by a uniform load q and a concentrated load P as shown in the figure.

- Select the most economical steel C shape from Table F-3(a) in Appendix F; use $q = 20 \text{ lb/ft}$ and $P = 300 \text{ lb}$ (assume allowable normal stress is $\sigma_{\delta} = 18 \text{ ksi}$).
- Select the most economical steel S shape from Table F-2(a) in Appendix F; use $q = 45 \text{ lb/ft}$ and $P = 2000 \text{ lb}$ (assume allowable normal stress is $\sigma_{\delta} = 20 \text{ ksi}$).
- Select the most economical steel W shape from Table F-1(a) in Appendix F; use $q = 45 \text{ lb/ft}$ and $P = 2000 \text{ lb}$ (assume allowable normal stress is $\sigma_a = 20 \text{ ksi}$). *However*, assume that the design requires that the W shape must be used in weak axis bending, i.e., it must bend about the 2-2 (or y) axis of the cross section.

Note: For parts (a), (b), and (c), revise your initial beam selection as needed to include the distributed weight of the beam in addition to uniform load q .



Solution 5.6-3

(a) SELECT THE MOST ECONOMICAL STEEL C SHAPE FROM TABLE F-3(a) IN APPENDIX F

$$L = 10 \text{ ft}$$

$$q = 20 \frac{\text{lb}}{\text{ft}} \quad P = 300 \text{ lb} \quad M_{\max 1} = (q) \frac{L^2}{2} + P \left(\frac{3}{5} L \right) = 28 \text{ kip-ft} \quad < \text{does not yet include beam self-weight}$$

$$\sigma_{as} = 18 \text{ ksi}$$

$$\text{TRIAL \#1} \quad S_{\text{reqd1}} = \frac{M_{\max 1}}{\sigma_{as}} \quad S_{\text{reqd1}} = 1.867 \text{ in.}^3 \quad < \text{use 2.2 axis in Table F-3(a); try C12} \times \mathbf{30}, S_{22} = 2.05 \text{ in.}^3$$

$$w = 30 \frac{\text{lb}}{\text{ft}} \quad M_{\max 2} = (q + w) \frac{L^2}{2} + P \left(\frac{3}{5} L \right) = 4.3 \text{ kip-ft} \quad S_{\text{reqd2}} = \frac{M_{\max 2}}{\sigma_{as}} = 2.867 \text{ in.}^3$$

< C12 × 30 will not work

$$\text{TRIAL \#2: Try C15} \times \mathbf{33.9} (S_{22} = 3.09 \text{ in.}^3)$$

$$w = 33.9 \frac{\text{lb}}{\text{ft}} \quad M_{\max 3} = (q + w) \frac{L^2}{2} + P \left(\frac{3}{5} L \right) = 4.495 \text{ kip-ft}$$

$$S_{\text{reqd3}} = \frac{M_{\max 3}}{\sigma_{as}} = 2.997 \text{ in.}^3 \quad \boxed{\text{< C15} \times \mathbf{33.9} \text{ will work}}$$

(b) SELECT THE MOST ECONOMICAL STEEL S SHAPE FROM TABLE F-2(a) IN APP. F:

$$q = 45 \frac{\text{lb}}{\text{ft}} \quad P = 2000 \text{ lb} \quad M_{\max 1} = (q) \frac{L^2}{2} + P \left(\frac{3}{5} L \right) = 14.25 \text{ kip-ft} \quad < \text{does not yet include beam self-weight}$$

$$\sigma_{as} = 20 \text{ ksi}$$

$$\text{TRIAL \#1} \quad S_{\text{reqd1}} = \frac{M_{\max 1}}{\sigma_{as}} \quad S_{\text{reqd1}} = 8.55 \text{ in.}^3 \quad < \text{use 1-1 axis in Table F-2(a); try S6} \times \mathbf{17.2},$$

$$S_{11} = 8.74 \text{ in.}^3$$

$$w = 17.3 \frac{\text{lb}}{\text{ft}} \quad M_{\max 2} = (q + w) \frac{L^2}{2} + P \left(\frac{3}{5} L \right) = 15.115 \text{ kip-ft} \quad S_{\text{reqd2}} = \frac{M_{\max 2}}{\sigma_{as}} = 9.069 \text{ in.}^3$$

< C6 × 17.2 will not work

$$\text{TRIAL \#2: TRY S8} \times \mathbf{18.4} (S_{11} = 14.4 \text{ in.}^3)$$

$$w = 18.4 \frac{\text{lb}}{\text{ft}} \quad M_{\max 3} = (q + w) \frac{L^2}{2} + P \left(\frac{3}{5} L \right) = 15.17 \text{ kip-ft} \quad S_{\text{reqd3}} = \frac{M_{\max 3}}{\sigma_{as}} = 9.102 \text{ in.}^3$$

< S8 × 18.4 will work

(c) SELECT THE MOST ECONOMICAL STEEL W SHAPE FROM TABLE F-1(a) IN APP. F:

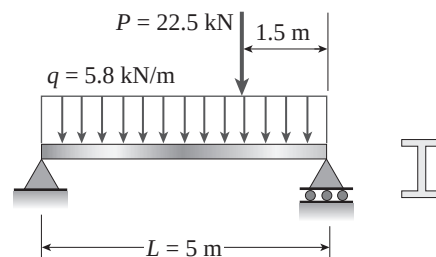
From (b), assume that required S will be approx. 10 in.^3 so from Table F-1(a), select W shape from S_{22} column.

$$\text{TRY W8} \times \mathbf{35}: S_{22} = 10.6 \text{ in.}^3 \quad w = 35 \frac{\text{lb}}{\text{ft}}$$

$$M_{\max} = (q + w) \frac{L^2}{2} + P \left(\frac{3}{5} L \right) = 16 \text{ kip-ft} \quad S_{22\text{reqd}} = \frac{M_{\max}}{\sigma_{as}} = 9.6 \text{ in.}^3 \quad \boxed{\text{< W8} \times \mathbf{35} \text{ will work}}$$

Problem 5.6-4 A simple beam of length $L = 5$ m carries a uniform load of intensity $q = 5.8$ kN/m and a concentrated load 22.5 kN (see figure).

- (a) Assuming $\sigma_{\text{allow}} = 110$ MPa, calculate the required section modulus S . Then select the most economical wide-flange beam (W shape) from Table F-1(b) in Appendix F, and recalculate S , taking into account the weight of the beam. Select a new beam if necessary.
- (b) Repeat part (a), but now assume that the design requires that the W shape must be used in weak axis bending, (i.e., it must bend about the 2-2 (or y) axis of the cross section).



Solution 5.6-4

NUMERICAL DATA

$$L = 5 \text{ m} \quad q = 5.8 \frac{\text{kN}}{\text{m}} \quad P = 22.5 \text{ kN} \quad b = 1.5 \text{ m} \quad a = L - b \quad a = 3.5 \text{ m}$$

$$\sigma_{\text{allow}} = 110 \text{ MPa}$$

$$\begin{aligned} \text{(a) STATICS} \quad R_A &= \frac{qL}{2} + \frac{Pb}{L} \quad R_A = 21.25 \text{ kN} \quad R_B = \frac{qL}{2} + \frac{Pa}{L} \quad R_B = 30.25 \text{ kN} \\ qL + P &= 51.5 \quad R_A + R_B = 51.5 \end{aligned}$$

$$\text{LOCATE POINT OF ZERO SHEAR} \quad x_m = \frac{R_A}{q} \quad x_m = 3.664 \text{ m} < \text{greater than dist. } a \text{ to load } P \text{ so zero shear is at load point}$$

$$M_{\text{max}} = R_A a - \frac{qa^2}{2} \quad M_{\text{max}} = 38.85 \text{ kN} \cdot \text{m}$$

$$\begin{aligned} \text{FIND REQUIRED SECTION MODULUS} \quad S_{\text{reqd}} &= \frac{M_{\text{max}} (10^3)^2}{\sigma_{\text{allow}}} \quad \frac{S_{\text{reqd}}}{10^3} = 353.182 \quad 10^3 \text{ mm}^3 \\ &\text{^select } \boxed{\text{W360} \times \text{39}} \quad (S_{\text{act}} = 578 \times 10^3 \text{ mm}^3) \end{aligned}$$

RECOMPUTE MAX. MOMENT WITH BEAM MASS INCLUDED & THEN CHECK ALLOWABLE STRESS

$$w = (39)9.81 \frac{\text{kg}}{\text{m}} \frac{\text{m}}{\text{s}^2} < = \text{N/m} \quad w = 382.59 \frac{\text{N}}{\text{m}} \quad S_{\text{act}} = 578(10^3) \text{ mm}^3$$

$$\begin{aligned} R_A &= \frac{\left(q + \frac{w}{1000}\right)L}{2} + \frac{Pb}{L} \quad R_A = 22.206 \text{ kN} \quad x_m = \frac{R_A}{q + \frac{w}{1000}} \quad x_m = 3.592 \\ &\text{^ greater than } a \text{ so max. moment at load pt.} \end{aligned}$$

$$M_{\text{max}} = R_A a - \frac{\left(q + \frac{w}{1000}\right)a^2}{2} \quad M_{\text{max}} = 39.854 \text{ kN} \cdot \text{m}$$

$$\sigma_{\text{max}} = \frac{M_{\text{max}}(10^6)}{S_{\text{act}}} \quad \sigma_{\text{max}} = 68.952 < \text{OK, less than 110 MPa}$$

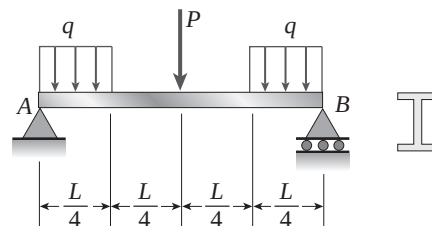
CHAPTER 5 Stresses in Beams (Basic Topics)

- (b) REPEAT (a) BUT ASSUME WEAK AXIS BENDING $S_{\text{reqd}} = 353(10^3) \text{ mm}^3$

$$\text{Select } \boxed{W250 \times 89} \quad S_{\text{act}} = 377(10^3) \text{ mm}^3 \quad M_{\text{max}} = 41.142 \text{ kN} \cdot \text{m} \quad \frac{M_{\text{max}}(10^6)}{S_{\text{act}}} = 109.13 \text{ MPa}$$

Problem 5.6-5 A simple beam AB is loaded as shown in the figure.

- Calculate the required section modulus S if $\sigma_{\text{allow}} = 18,000$ psi, $L = 32$ ft, $P = 2900$ lb, and $q = 450$ lb/ft.
Then select a suitable I-beam (**S** shape) from Table F-2(a), Appendix F, and recalculate S taking into account the weight of the beam. Select a new beam size if necessary.
- What is the maximum load P that can be applied to your final beam selection in part (a)?



Solution 5.6-5

$$(a) \quad R_A = \frac{qL}{4} + \frac{P}{2} \quad M_{\max} = R_A \frac{L}{2} - q \frac{L}{4} \left(\frac{L}{4} + \frac{L}{8} \right) = \frac{L \left(\frac{P}{2} + \frac{Lq}{4} \right)}{2} - \frac{3L^2 q}{32}$$

$$\sigma_a = 18 \text{ ksi} \quad L = 32 \text{ ft} \quad P = 2900 \text{ lb} \quad q = 450 \frac{\text{lb}}{\text{ft}}$$

$$S_{\text{reqd}} = \frac{M_{\text{max}}}{\sigma_a} \quad S_{\text{reqd}} = \frac{\frac{L\left(\frac{P}{2} + \frac{Lq}{4}\right)}{2} - \frac{3L^2q}{32}}{\sigma_a} = 25.067 \text{ in.}^3 \quad < \text{try } \mathbf{S10} \times \mathbf{35} (S_{\text{act}} = 29.4 \text{ in.}^3)$$

Recompute $M_{\max}$ including beam weight $w = 35 \frac{\text{lb}}{\text{ft}}$

$$R_A = \frac{qL}{4} + \frac{P}{2} + \frac{wL}{2} \quad \text{so} \quad M_{\max} = R_A \frac{L}{2} - q \frac{L}{4} \left(\frac{L}{4} + \frac{L}{8} \right) - w \frac{L}{2} \frac{L}{4} = 42,080 \text{ ft-lb}$$

$$S_{\text{reqd2}} = \frac{M_{\text{max}}}{\sigma_d} = 28.053 \text{ in.}^3 < \text{less than } S_{\text{act}} \text{ for } S10 \times 35 \text{ so } \mathbf{S10 \times 35 \text{ will work}}$$

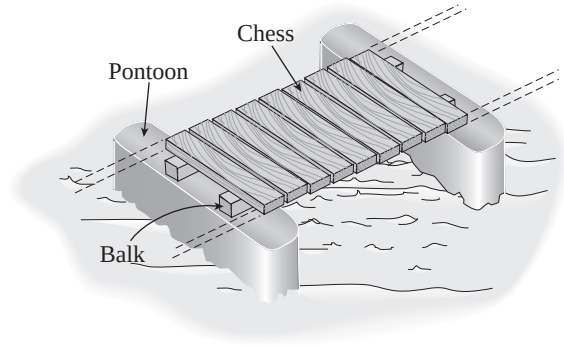
$$(b) \quad S_{\text{act}} = 29.4 \text{ in.}^3 \quad M_{\text{max}} = \sigma_a S_{\text{act}} = 44,100 \text{ ft-lb} \quad R_A = \frac{qL}{4} + \frac{P}{2} + \frac{wL}{2} \quad \text{so}$$

$$M_{\max} = R_A \frac{L}{2} - q \frac{L}{4} \left(\frac{L}{4} + \frac{L}{8} \right) - w \frac{L}{2} \frac{L}{4} = \frac{L \left(\frac{P}{2} \frac{Lq}{4} + \frac{Lw}{2} \right)}{2} - \frac{L^2 w}{8} - \frac{3L^2 q}{32}$$

$$\frac{L \left(\frac{P}{2} + \frac{Lq}{4} + \frac{Lw}{2} \right)}{2} - \frac{L^2 w}{8} - \frac{3L^2 q}{32} = \sigma_a S_{\text{act}} \quad \text{solving for } P = \frac{L^2 q + 4L^2 w - 32 S_{\text{act}} \sigma_a}{8L}$$

$$P_{\max} = -\frac{L^2 q + 4L^2 w - 32S_{\text{act}}\sigma_a}{8L} = 3152 \text{ lb} \quad \boxed{P_{\max} = 3152 \text{ lb}}$$

Problem 5.6-6 A pontoon bridge (see figure) is constructed of two longitudinal wood beams, known as *balks*, that span between adjacent pontoons and support the transverse floor beams, which are called *chesses*. For purposes of design, assume that a uniform floor load of 7.5 kPa acts over the chesses. (This load includes an allowance for the weights of the chesses and balks.) Also, assume that the chesses are 2.5 m long and that the balks are simply supported with a span of 3.0 m. The allowable bending stress in the wood is 15 MPa.



- If the balks have a square cross section, what is their minimum required width $b_{\min}$?
- Repeat part (a) if the balk width is $1.5b$ and the balk depth is b ; compare the cross-sectional areas of the two designs.

Solution 5.6-6

(a) $L_c = 2.5 \text{ m}$ $L_b = 3 \text{ m}$ $w = 7.5 \text{ kPa}$ $\sigma_a = 15 \text{ MPa}$

$$q = \frac{wL_c}{2} = 9.375 \frac{\text{kN}}{\text{m}} \quad M_{\max} = \frac{qL_b^2}{8} \quad \text{and} \quad M_{\max} = \sigma_a S \quad M_{\max} = \sigma_a \left(\frac{b^3}{6} \right)$$

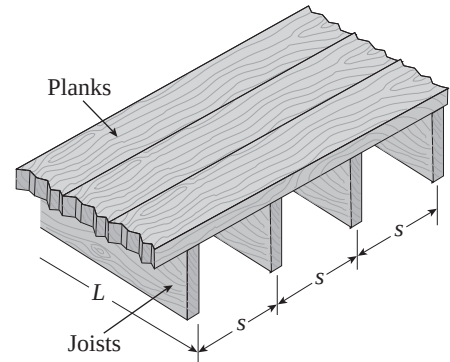
$$b_{\min} = \left(\frac{6}{8} \frac{qL_b^2}{\sigma_a} \right)^{\frac{1}{3}} = 161.6 \text{ mm} \quad A_a = b_{\min}^2 = 2.611 \times 10^4 \text{ mm}^2$$

(b) $S = \frac{1.5b(b^2)}{6} \rightarrow 0.25b^3 \quad M_{\max} = \sigma_a \left(\frac{b^3}{4} \right)$

$$b_{\min} = \left(\frac{4}{8} \frac{qL_b^2}{\sigma_a} \right)^{\frac{1}{3}} = 141.2 \text{ mm} \quad A_b = 1.5b_{\min}^2 = 2.989 \times 10^4 \text{ mm}^2$$

$$\frac{A_b}{A_a} = 1.145$$

Problem 5.6-7 A floor system in a small building consists of wood planks supported by 2-in. (nominal width) joists spaced at distance s and measured from center to center (see figure). The span length L of each joist is 12 ft, the spacing s of the joists is 16 in., and the allowable bending stress in the wood is 1250 psi. The uniform floor load is 120 lb/ft², which includes an allowance for the weight of the floor system itself.



- Calculate the required section modulus S for the joists, and then select a suitable joist size (surfaced lumber) from Appendix G, assuming that each joist may be represented as a simple beam carrying a uniform load.
- What is the maximum floor load that can be applied to your final beam selection in part (a)?

Probs. 5.6-7 and 5.6-8

Solution 5.6-7 Floor joists

$$(a) \sigma_a = 1250 \text{ psi} \quad L = 12 \text{ ft} \quad w = 120 \frac{\text{lb}}{\text{ft}^2} \quad s = 16 \text{ in.} \quad q = ws = 160 \frac{\text{lb}}{\text{ft}}$$

$$M_{\max} = \frac{qL^2}{8} = 2880 \text{ lb-ft} \quad S_{\text{reqd}} = \frac{M_{\max}}{\sigma_a} = 27.648 \text{ in.}^3 \quad \boxed{< \text{use } 2 \times 12, S_{\text{act}} = 31.64 \text{ in.}^3}$$

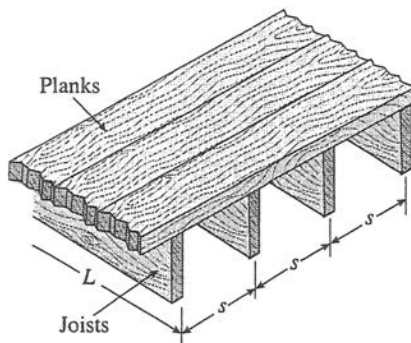
$$S_{\text{act}} = 31.64 \text{ in.}^3$$

$$(b) M_{\text{allow}} = \sigma_a S_{\text{act}} = 3295.833 \text{ lb-ft}$$

$$q_{\max} = M_{\text{allow}} \left(\frac{8}{L^2} \right) = 183.102 \frac{\text{lb}}{\text{ft}} \quad \boxed{w_{\max} = \frac{q_{\max}}{s} = 137.3 \frac{\text{lb}}{\text{ft}^2}}$$

Problem 5.6-8 The wood joists supporting a plank floor (see figure) are 38 mm $\times$ 220 mm in cross section (actual dimensions) and have a span length $L = 4.0$ m. The floor load is 5.0 kPa, which includes the weight of the joists and the floor.

- (a) Calculate the maximum permissible spacing s of the joists if the allowable bending stress is 14 MPa. (Assume that each joist may be represented as a simple beam carrying a uniform load.)
 (b) If spacing $s = 406$ mm, what is the required depth h of the joist? Assume all other variables remain unchanged.

Solution 5.6-8 Spacing of floor joists

$$(a) \sigma_a = 14 \text{ MPa} \quad L = 4 \text{ m} \quad w = 5 \text{ kPa} \quad q = ws \quad b = 38 \text{ mm} \quad h = 220 \text{ mm}$$

$$M_{\max} = \frac{qL^2}{8} \quad \sigma_a S_{\text{act}} = \frac{(ws)L^2}{8} \quad \text{so} \quad s_{\max} = \frac{8\sigma_a}{wL^2} \left(\frac{bh^2}{6} \right) \quad s_{\max} = \frac{4}{3} \frac{bh^2\sigma_a}{wL^2}$$

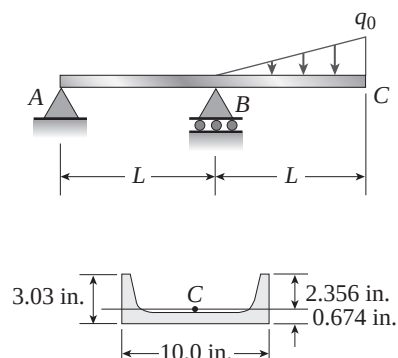
$$S_{\text{act}} = \frac{bh^2}{6} \quad \boxed{s_{\max} = \frac{4}{3} \frac{bh^2\sigma_a}{wL^2} = 429 \text{ mm}}$$

$$(b) s_{\text{act}} = 406 \text{ mm}$$

$$\boxed{h_{\min} = \sqrt{\frac{3wL^2s_{\text{act}}}{4b\sigma_a}} = 214 \text{ mm}}$$

Problem 5.6-9 A beam ABC with an overhang from B to C is constructed of a $C 10 \times 30$ channel section with flanges facing upward (see figure). The beam supports its own weight (30 lb/ft) plus a *triangular* load of maximum intensity q_0 acting on the overhang. The allowable stresses in tension and compression are 18 ksi and 12 ksi, respectively.

- Determine the allowable *triangular* load intensity $q_{0,\text{allow}}$ if the distance L equals 4 ft.
- What is the allowable triangular load intensity $q_{0,\text{allow}}$ if the beam is rotated 180° about its longitudinal centroidal axis so that the flanges are downward?



Solution 5.6-9

NUMERICAL VALUES $C10 \times 30$ $I_{22} = 3.93 \text{ in.}^4$

$$c_2 = 0.674 \text{ in.} \quad c_1 = 3.03 \text{ in.} \quad -c_2 = 2.356 \text{ in.} \quad w = 30 \frac{\text{lb}}{\text{ft}}$$

$$\sigma_{at} = 18 \text{ ksi} \quad \sigma_{ac} = 12 \text{ ksi} \quad L = 4 \text{ ft}$$

$$(a) \quad M_{\text{max}} = M_B \quad M_B = wL\left(\frac{L}{2}\right) + \frac{1}{2} q_0 L\left(\frac{2}{3}L\right) \rightarrow \frac{L^2 q_0}{3} + \frac{L^2 w}{2}$$

$$\sigma_{B\text{top}} = \frac{M_B c_1}{I_{22}} < \text{tension (allowable} = 20 \text{ ksi)} \quad \sigma_{B\text{bot}} = \frac{M_B c_2}{I_{22}} < \text{compression (allowable} = 11 \text{ ksi)}$$

$$q_{0\text{allowtop}} = \frac{3}{L^2} \left[\sigma_{at} \left(\frac{I_{22}}{c_1} \right) - \frac{wL^2}{2} \right] \quad q_{0\text{allowbot}} = \frac{3}{L^2} \left[\sigma_{ac} \left(\frac{I_{22}}{c_2} \right) - \frac{wL^2}{2} \right]$$

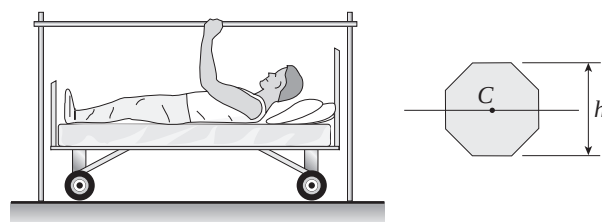
$$q_{0\text{allowtop}} = \frac{3}{L^2} \left[\sigma_{at} \left(\frac{I_{22}}{c_1} \right) - \frac{wL^2}{2} \right] = 424 \frac{\text{lb}}{\text{ft}} \quad q_{0\text{allowbot}} = \frac{3}{L^2} \left[\sigma_{ac} \left(\frac{I_{22}}{c_2} \right) - \frac{wL^2}{2} \right] = 1048 \frac{\text{lb}}{\text{ft}}$$

- If **flanges point downward**, we must switch c_1 and c_2 in above formulas:

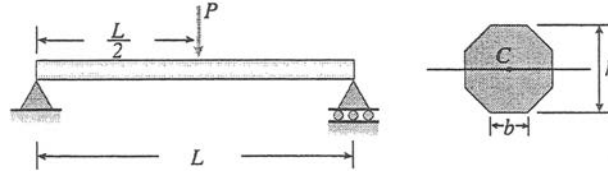
$$q_{0\text{allowtop}} = \frac{3}{L^2} \left[\sigma_{at} \left(\frac{I_{22}}{c_2} \right) - \frac{wL^2}{2} \right] = 1595 \frac{\text{lb}}{\text{ft}} \quad q_{0\text{allowbot}} = \frac{3}{L^2} \left[\sigma_{ac} \left(\frac{I_{22}}{c_1} \right) - \frac{wL^2}{2} \right] = 268 \frac{\text{lb}}{\text{ft}}$$

Problem 5.6-10 A so-called “trapeze bar” in a hospital room provides a means for patients to exercise while in bed (see figure). The bar is 2.1 m long and has a cross section in the shape of a regular octagon. The design load is 1.2 kN applied at the midpoint of the bar, and the allowable bending stress is 200 MPa.

Determine the minimum height h of the bar. (Assume that the ends of the bar are simply supported and that the weight of the bar is negligible.)



Solution 5.6-10 Trapeze bar (regular octagon)



$$P = 1.2 \text{ kN} \quad L = 2.1 \text{ m} \quad \sigma_{\text{allow}} = 200 \text{ MPa}$$

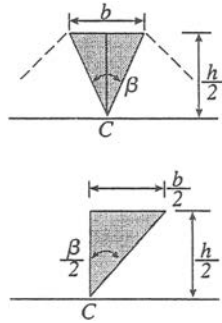
Determine minimum height h .

MAXIMUM BENDING MOMENT

$$M_{\text{max}} = \frac{PL}{4} = \frac{(1.2 \text{ kN})(2.1 \text{ m})}{4} = 630 \text{ N} \cdot \text{m}$$

PROPERTIES OF THE CROSS SECTION

Use Appendix E, Case 25, with $n = 8$.



b = length of one side

$$\beta = \frac{360^\circ}{n} = \frac{360^\circ}{8} = 45^\circ$$

$$\tan \frac{\beta}{2} = \frac{b}{h} \quad (\text{from triangle})$$

$$\cot \frac{\beta}{2} = \frac{h}{b}$$

$$\begin{aligned} \text{For } \beta = 45^\circ: \quad \frac{b}{h} &= \tan \frac{45^\circ}{2} = 0.41421 \\ \frac{h}{b} &= \cot \frac{45^\circ}{2} = 2.41421 \end{aligned}$$

MOMENT OF INERTIA

$$I_c = \frac{nb^4}{192} \left(\cot \frac{\beta}{2} \right) \left(3 \cot^2 \frac{\beta}{2} + 1 \right)$$

$$I_c = \frac{8b^4}{192} (2.41421) [3(2.41421)^2 + 1] = 1.85948b^4$$

$$b = 0.41421h \quad \therefore I_c = 1.85948(0.41421h)^4 = 0.054738h^4$$

SECTION MODULUS

$$S = \frac{I_c}{h/2} = \frac{0.054738h^4}{h/2} = 0.109476h^3$$

MINIMUM HEIGHT h

$$\sigma = \frac{M}{S} \quad S = \frac{M}{\sigma}$$

$$0.109476h^3 = \frac{630 \text{ N} \cdot \text{m}}{200 \text{ MPa}} = 3.15 \times 10^{-6} \text{ m}^3$$

$$h^3 = 28.7735 \times 10^{-6} \text{ m}^3 \quad h = 0.030643 \text{ m}$$

$$\therefore h_{\text{min}} = 30.6 \text{ mm} \quad \leftarrow$$

ALTERNATIVE SOLUTION ($n = 8$)

$$M = \frac{PL}{4} \quad \beta = 45^\circ \quad \tan \frac{\beta}{2} = \sqrt{2} - 1 \quad \cot \frac{\beta}{2} = \sqrt{2} + 1$$

$$b = (\sqrt{2} - 1)h \quad h = (\sqrt{2} + 1)b$$

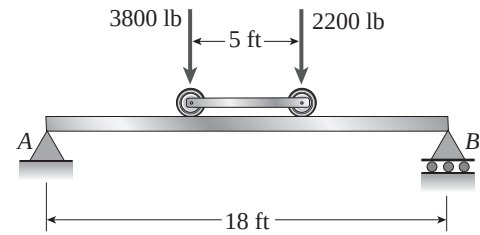
$$I_c = \left(\frac{11 + 8\sqrt{2}}{12} \right) b^4 = \left(\frac{4\sqrt{2} - 5}{12} \right) h^4$$

$$S = \left(\frac{4\sqrt{2} - 5}{6} \right) h^3 \quad h^3 = \frac{3PL}{2(4\sqrt{2} - 5)\sigma_{\text{allow}}} \quad \leftarrow$$

Substitute numerical values:

$$h^3 = 28.7735 \times 10^{-6} \text{ m}^3 \quad h_{\text{min}} = 30.643 \text{ mm} \quad \leftarrow$$

Problem 5.6-11 A two-axle carriage that is part of an overhead traveling crane in a testing laboratory moves slowly across a simple beam AB (see figure). The load transmitted to the beam from the front axle is 2200 lb and from the rear axle is 3800 lb. The weight of the beam itself may be disregarded.



- Determine the minimum required section modulus S for the beam if the allowable bending stress is 17.0 ksi, the length of the beam is 18 ft, and the wheelbase of the carriage is 5 ft.
- Select the most economical I-beam (S shape) from Table F-2(a), Appendix F.

Solution 5.6-11

NUMERICAL DATA

$$L = 18 \text{ ft} \quad P_1 = 2200 \text{ lb}$$

$$P_2 = 3800 \text{ lb} \quad d = 5 \text{ ft}$$

$$\sigma_a = 17 \text{ ksi}$$

- FIND REACTION R_A , THEN AN EXPRESSION FOR MOMENT UNDER LARGER LOAD P_2 ; LET x = DIST. FROM A TO LOAD P_2

$$R_A = P_2 \left(\frac{L - x}{L} \right) + P_1 \left[\frac{L - (x + d)}{L} \right]$$

$$M_2 = R_A x$$

$$M_2 = x \left[P_2 \left(\frac{L - x}{L} \right) + P_1 \left[\frac{L - (x + d)}{L} \right] \right]$$

$$M_2 = \frac{xP_2L - P_2x^2 + xP_1L - P_1x^2 - xP_1d}{L}$$

Take derivative of M_A & set to zero to find max. bending moment at $x = x_m$

$$\begin{aligned} \frac{d}{dx} \left(\frac{xP_2L - P_2x^2 + xP_1L - P_1x^2 - xP_1d}{L} \right) \\ = \frac{P_2L - 2P_2x + P_1L - 2P_1x - P_1d}{L} \end{aligned}$$

$$P_2L - 2P_2x + P_1L - 2P_1x - P_1d = 0$$

$$x_m = \frac{(P_1 + P_2)L - P_1d}{2(P_1 + P_2)} \quad x_m = 8.083 \text{ ft}$$

$$R_A = P_2 \left(\frac{L - x_m}{L} \right) + P_1 \left[\frac{L - (x_m + d)}{L} \right]$$

$$R_A = 2694 \text{ lb}$$

$$M_{\max} = x_m \left[P_2 \left(\frac{L - x_m}{L} \right) + P_1 \left[\frac{L - (x_m + d)}{L} \right] \right]$$

$$M_{\max} = 21,780 \text{ ft-lb}$$

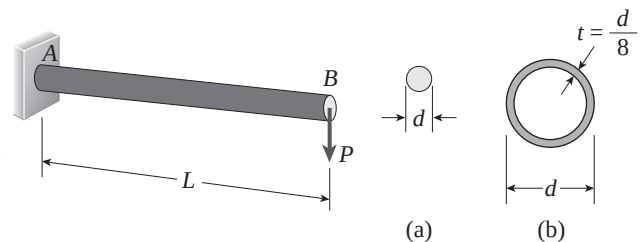
$$S_{\text{reqd}} = \frac{M_{\max}}{\sigma_a} \quad S_{\text{reqd}} = 15.37 \text{ in.}^3 \quad \leftarrow$$

- SELECT MOST ECONOMICAL S SHAPE FROM TABLE F-2(a)

$$\text{select } S 8 \times 23 \quad \leftarrow \quad S_{\text{act}} = 16.2 \text{ in.}^3$$

Problem 5.6-12 A cantilever beam AB of circular cross section and length $L = 750 \text{ mm}$ supports a load $P = 800 \text{ N}$ acting at the free end (see figure). The beam is made of steel with an allowable bending stress of 120 MPa.

- Determine the required diameter $d_{\min}$ (figure part a) of the beam, considering the effect of the beam's own weight.
- Repeat part (a) if the beam is hollow with wall thickness $t = d/8$; (figure part b) compare the cross-sectional areas of the two designs.



Solution 5.6-12 Cantilever beam

NUMERICAL DATA

$$L = 750 \text{ mm} \quad P = 800 \text{ N}$$

$$\sigma_a = 120 \text{ MPa} \quad \gamma_s = 77 \frac{\text{kN}}{\text{m}^3}$$

$$d = 50 \text{ mm}$$

$$(a) \quad M_{\max} = PL + wL \frac{L}{2} \quad w = \gamma_s \left(\frac{\pi d^2}{4} \right) \quad M_{\max} = PL + \gamma_s L^2 \left(\frac{\pi d^2}{8} \right) \quad \text{and} \quad M_{\max} = \sigma_a S$$

Equate $M_{\max}$ expressions and then solve for d :

$$S = \frac{\pi d^3}{32} \quad \frac{\pi \sigma_a d^3}{32} - \frac{\pi \gamma_s L^2 d^2}{8} - PL = 0 \quad \text{solving} \quad \boxed{d_{\min} = 37.6 \text{ mm}}$$

$$A_a = \frac{\pi d_{\min}^2}{4} = 1.108 \times 10^{-3} \text{ m}^2$$

$$(b) \quad w = \gamma_s \left[\frac{\pi \left[d^2 - \left(d - 2 \frac{d}{8} \right)^2 \right]}{4} \right] = \frac{7\pi d^2 \gamma_s}{64} \quad S = \frac{\pi \left[d^3 - \left(d - 2 \frac{d}{8} \right)^3 \right]}{32} = \frac{37\pi d^3}{2048}$$

$$M_{\max} = PL + \gamma_s L^2 \left[\frac{\pi \left[d^2 - \left(d - 2 \frac{d}{8} \right)^2 \right]}{4} \right] = \frac{7\pi \gamma_s L^2 d^2}{64} + PL \quad \text{and} \quad M_{\max} = \sigma_a S$$

Equate $M_{\max}$ expressions and then solve for d :

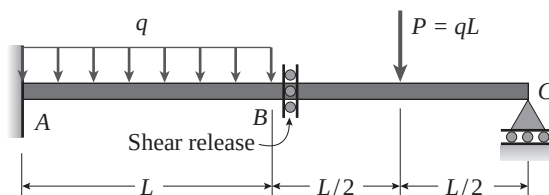
$$\sigma_a \left(\frac{37\pi d^3}{2048} \right) - \left(\frac{7\pi \gamma_s L^2 d^2}{64} + PL \right) = 0$$

$$\boxed{d_{\min} = 45.2 \text{ mm}} \quad A_b = \frac{\pi}{4} \left[d_{\min}^2 - \left(\frac{3}{4} d_{\min} \right)^2 \right] = 7.031 \times 10^{-4} \text{ m}^2$$

$$\boxed{\frac{A_b}{A_a} = 0.635}$$

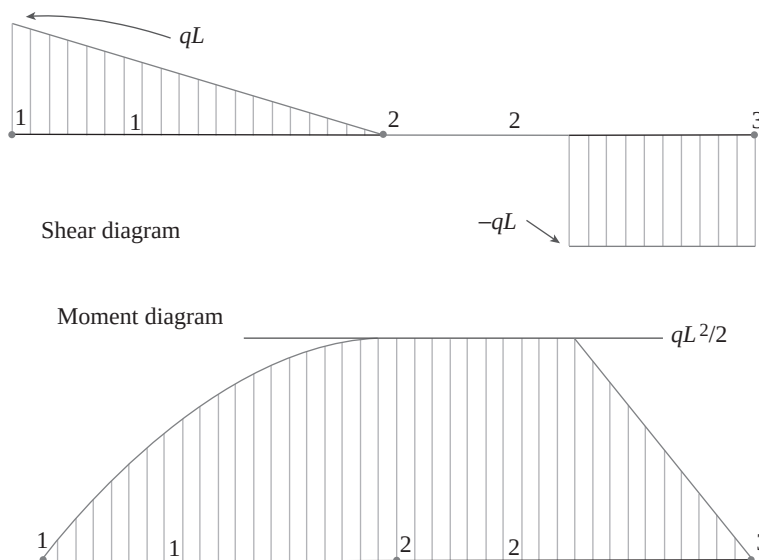
Problem 5.6-13 A propped cantilever beam ABC (see figure) has a shear release just right of the mid-span.

- (a) Select the most economical wood beam from the table in Appendix G; assume $q = 55 \text{ lb/ft}$, $L = 16 \text{ ft}$, $\sigma_{aw} = 1750 \text{ psi}$, $\tau_{aw} = 375 \text{ psi}$; include the self-weight of the beam in your design.
- (b) If a $C10 \times 25$ steel beam is now used for beam ABC , what is the maximum permissible value of load variable q ? Assume $\sigma_{as} = 16 \text{ ksi}$ and $L = 10 \text{ ft}$. Include the self-weight of the beam in your analysis.



Solution 5.6-13

First, plot shear (V) and moment (M) diagrams; label all critical ordinates.



(a) SELECT THE MOST ECONOMICAL WOOD BEAM FROM THE TABLE IN APP. G;

$$q = 55 \frac{\text{lb}}{\text{ft}} \quad L = 16 \text{ ft} \quad \sigma_{aw} = 1750 \text{ psi} \quad \tau_{aw} = 375 \text{ psi}$$

$$M_{\max} = \frac{qL^2}{2} = 7040 \text{ lb-ft} \quad S_{\text{reqd1}} = \frac{M_{\max}}{\sigma_{aw}} = 48.274 \text{ in.}^3 < \text{TRY } 3 \times 12 (S_{11} = 52.73 \text{ in.}^3, w = 6.8 \text{ lb/ft})$$

$$w = 6.8 \frac{\text{lb}}{\text{ft}} \quad M_{\max2} = \frac{qL^2}{2} + \frac{wL^2}{2} = 7910.4 \text{ lb-ft} \quad S_{\text{reqd2}} = \frac{M_{\max2}}{\sigma_{aw}} = 54.243 \text{ in.}^3 < 3 \times 12 \text{ will not work}$$

$$\text{TRIAL \#2: } 4 \times 12 (S_{11} = 73.83 \text{ in.}^3, w = 9.6 \text{ lb/ft})$$

$$w = 9.6 \frac{\text{lb}}{\text{ft}} \quad M_{\max2} = \frac{qL^2}{2} + \frac{wL^2}{2} = 8268.8 \text{ lb-ft} \quad S_{\text{reqd2}} = \frac{M_{\max2}}{\sigma_{aw}} = 56.7 \text{ in.}^3 \quad \boxed{< 4 \times 12 \text{ will work}}$$

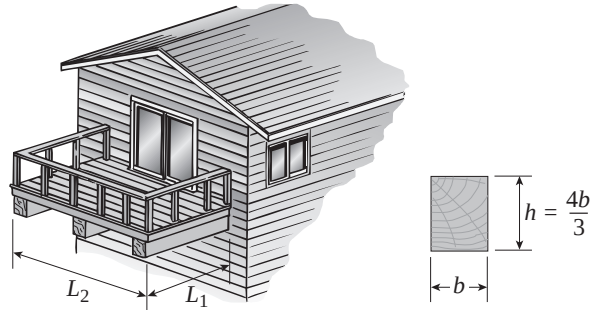
$$(b) S_{22} = 1.47 \text{ in.}^3 \quad \sigma_{as} = 16 \text{ ksi} \quad L = 10 \text{ ft} \quad w = 25 \frac{\text{lb}}{\text{ft}}$$

$$M_{\max} = \frac{wL^2}{2} + \frac{qL^2}{2} \quad \text{and} \quad M_{\max} = \sigma_{as} S_{22}$$

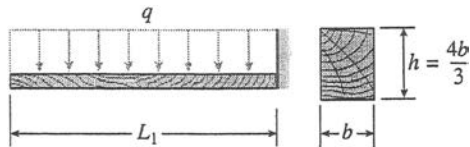
$$\boxed{q_{\max} = \frac{2}{L^2} \left(\sigma_{as} S_{22} - \frac{wL^2}{2} \right) = 14.2 \frac{\text{lb}}{\text{ft}}}$$

Problem 5.6-14 A small balcony constructed of wood is supported by three identical cantilever beams (see figure). Each beam has length $L_1 = 2.1$ m, width b , and height $h = 4b/3$. The dimensions of the balcony floor are $L_1 \times L_2$, with $L_2 = 2.5$ m. The design load is 5.5 kPa acting over the entire floor area. (This load accounts for all loads except the weights of the cantilever beams, which have a weight density $\gamma = 5.5$ kN/m³.) The allowable bending stress in the cantilevers is 15 MPa.

Assuming that the middle cantilever supports 50% of the load and each outer cantilever supports 25% of the load, determine the required dimensions b and h .



Solution 5.6-14 Compound beam



$L_1 = 2.1$ m $L_2 = 2.5$ m Floor dimensions: $L_1 \times L_2$
 Design load = $w = 5.5$ kPa
 $\gamma = 5.5$ kN/m³ (weight density of wood beam)
 $\sigma_{\text{allow}} = 15$ MPa

MIDDLE BEAM SUPPORTS 50% OF THE LOAD.

$$\therefore q = w \left(\frac{L_2}{2} \right) = (5.5 \text{ kPa}) \left(\frac{2.5 \text{ m}}{2} \right) = 6875 \text{ N/m}$$

WEIGHT OF BEAM

$$\begin{aligned} q_0 &= \gamma b h = \frac{4\gamma b^2}{3} = \frac{4}{3} (5.5 \text{ kN/m}^2) b^2 \\ &= 7333b^2 \text{ (N/m)} \quad (b = \text{meters}) \end{aligned}$$

MAXIMUM BENDING MOMENT

$$\begin{aligned} M_{\text{max}} &= \frac{(q + q_0)L_1^2}{2} = \frac{1}{2} (6875 \text{ N/m} + 7333b^2)(2.1 \text{ m})^2 \\ &= 15,159 + 16,170b^2 \text{ (N} \cdot \text{m)} \end{aligned}$$

$$S = \frac{bh^2}{6} = \frac{8b^3}{27}$$

$$M_{\text{max}} = \sigma_{\text{allow}} S$$

$$15,159 + 16,170b^2 = (15 \times 10^6 \text{ N/m}^2) \left(\frac{8b^3}{27} \right)$$

Rearrange the equation:

$$(120 \times 10^6)b^3 - 436,590b^2 - 409,300 = 0$$

SOLVE NUMERICALLY FOR DIMENSION b

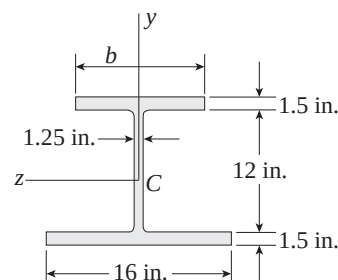
$$b = 0.1517 \text{ m} \quad h = \frac{4b}{3} = 0.2023 \text{ m}$$

REQUIRED DIMENSIONS

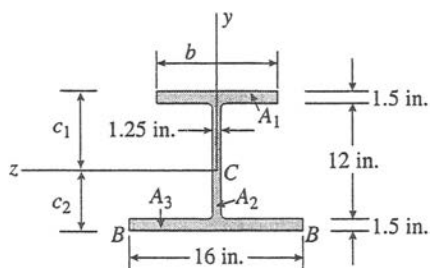
$$b = 152 \text{ mm} \quad h = 202 \text{ mm} \quad \leftarrow$$

Problem 5.6-15 A beam having a cross section in the form of an unsymmetric wide-flange shape (see figure) is subjected to a negative bending moment acting about the z axis.

Determine the width b of the top flange in order that the stresses at the top and bottom of the beam will be in the ratio 4:3, respectively.



Solution 5.6-15 Unsymmetric wide-flange beam



Stresses at top and bottom are in the ratio 4:3.
Find b (inches)

$h = \text{height of beam} = 15 \text{ in.}$

LOCATE CENTROID

$$\frac{\sigma_{\text{top}}}{\sigma_{\text{bottom}}} = \frac{c_1}{c_2} = \frac{4}{3}$$

$$c_1 = \frac{4}{7}h = \frac{60}{7} = 8.57143 \text{ in.}$$

$$c_2 = \frac{3}{7}h = \frac{45}{7} = 6.42857 \text{ in.}$$

AREAS OF THE CROSS SECTION (in.^2)

$$A_1 = 1.5b \quad A_2 = (12)(1.25) = 15 \text{ in.}^2$$

$$A_3 = (16)(1.5) = 24 \text{ in.}^2$$

$$A = A_1 + A_2 + A_3 = 39 + 1.5b \text{ (in.}^2\text{)}$$

FIRST MOMENT OF THE CROSS-SECTIONAL
AREA ABOUT THE LOWER EDGE $B-B$

$$Q_{BB} = \sum \bar{y}_i A_i = (14.25)(1.5b) + (7.5)(15) + (0.75)(24) \\ = 130.5 + 21.375b \text{ (in.}^3\text{)}$$

DISTANCE c_2 FROM LINE $B-B$ TO THE CENTROID C

$$c_2 = \frac{Q_{BB}}{A} = \frac{130.5 + 21.375b}{39 + 1.5b} = \frac{45}{7} \text{ in.}$$

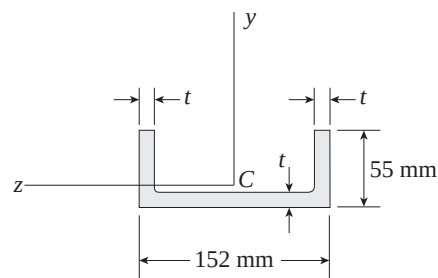
SOLVE FOR b

$$(39 + 1.5b)(45) = (130.5 + 21.375b)(7)$$

$$82.125b = 841.5 \quad b = 10.25 \text{ in.} \quad \leftarrow$$

Problem 5.6-16 A beam having a cross section in the form of a channel (see figure) is subjected to a bending moment acting about the z axis.

Calculate the thickness t of the channel in order that the bending stresses at the top and bottom of the beam will be in the ratio 7:3, respectively.



Solution 5.6-16

NUMERICAL DATA

$$h = 152 \text{ mm} \quad b = 55 \text{ mm}$$

Take 1st moments to find distances c_1 & c_2 .

1st moments about base.

$$c_2 = \frac{\frac{t}{2}(h-2t)(t) + 2bt\left(\frac{b}{2}\right)}{2bt + t(h-2t)}$$

$$c_1 = b - c_2$$

$$c_2 = \frac{\frac{t}{2}(152-2t)(t) + 2.55t\left(\frac{55}{2}\right)}{2.55t + t(152-2t)}$$

$$c_1 = 55 - \frac{\frac{t}{2}(152-2t)(t) + 2.55t\left(\frac{55}{2}\right)}{2.55t + t(152-2t)}$$

$$c_1 = \frac{-1}{2} \frac{11,385 - 186t + t^2}{-131 + t}$$

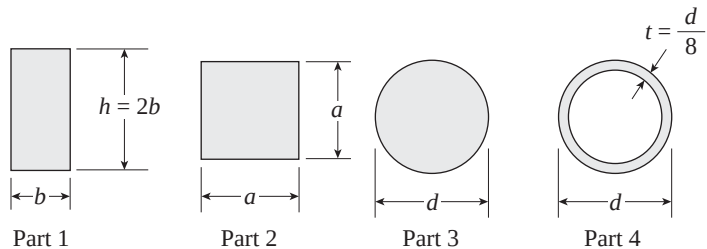
$$\text{ratio of top to bottom stresses} = c_1/c_2 = 7/3$$

$$\begin{aligned} & \frac{-1}{2} \frac{11,385 - 186t + t^2}{-131 + t} \\ & \left[\frac{\frac{t}{2}(152-2t)(t) + 2.55t\left(\frac{55}{2}\right)}{2.55t + t(152-2t)} \right] \\ & = \frac{-(11,385 - 186t + t^2)}{-76t + t^2 - 3025} = 7/3 \\ & \left[3 \left[-\left(11,385 - 186t + t^2\right) \right] \right. \\ & \quad \left. - 7 \left(-76t + t^2 - 3025 \right) \right] = 0 \end{aligned}$$

$$t^2 - 109t + 1298 = 0$$

$$t = \frac{109 - \sqrt{109^2 - 4(1298)}}{2} \quad t = 13.61 \text{ mm} \quad \leftarrow$$

Problem 5.6-17 Determine the ratios of the weights of four beams that have the same length, are made of the same material, are subjected to the same maximum bending moment, and have the same maximum bending stress if their cross sections are (1) a rectangle with height equal to twice the width, (2) a square, (3) a circle, and (4) a hollow pipe with outer diameter d and wall thickness $t = d/8$ (see figures).

**Solution 5.6-17 Ratio of weights of three beams**Since M and σ are the same, the section moduli must be the same.

$$(1) \text{ RECTANGLE: } S = \frac{bh^2}{6} = \frac{2b^3}{3} \quad b = \left(\frac{3S}{2}\right)^{1/3}$$

$$A_1 = 2b^2 = 2\left(\frac{3S}{2}\right)^{2/3} = 2.6207S^{2/3}$$

$$(2) \text{ SQUARE: } S = \frac{a^3}{6} \quad a = (6S)^{1/3}$$

$$A_2 = a^2 = (6S)^{2/3} = 3.3019S^{2/3}$$

$$(3) \text{ CIRCLE: } S = \frac{\pi d^3}{32} \quad d = \left(\frac{32S}{\pi}\right)^{1/3}$$

$$A_3 = \frac{\pi d^2}{4} = \frac{\pi}{4} \left(\frac{32S}{\pi}\right)^{2/3} = 3.6905S^{2/3}$$

$$(4) \text{ HOLLOW PIPE: } S = \frac{\pi}{32} \left[d^3 - \left(d - 2\frac{d}{8} \right)^3 \right] = \frac{37\pi d^3}{2048}$$

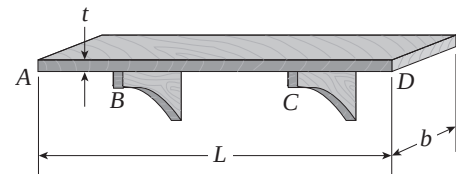
$$A_4 = \frac{\pi}{4} \left[d^2 - \left(\frac{3}{4}d \right)^2 \right] = \frac{7\pi d^2}{64} \quad A_4 = \frac{7\pi \left[\left(\frac{2048}{37\pi} S \right)^{\frac{1}{3}} \right]^2}{64} = \frac{7\pi (d^3)^{\frac{2}{3}}}{64} = \frac{7}{64} \pi \left(\frac{2048}{37\pi} \right)^{\frac{2}{3}} = 2.327$$

$$W_1 : W_2 : W_3 : W_4 = 1 : 1.260 : 1.408 : 0.888$$

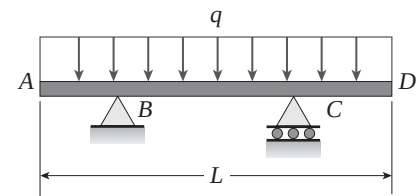
$$\text{denom.} = 2.6207 \quad \frac{3.3019}{\text{denom.}} = 1.26 \quad \frac{3.6905}{\text{denom.}} = 1.408 \quad \frac{2.327}{\text{denom.}} = 0.888$$

Problem 5.6-18 A horizontal shelf AD of length $L = 1215$ mm, width $b = 305$ mm, and thickness $t = 22$ mm is supported by brackets at B and C (see part a of the figure). The brackets are adjustable and may be placed in any desired positions between the ends of the shelf. A uniform load of intensity q , which includes the weight of the shelf itself, acts on the shelf (see part b of the figure).

- Determine the maximum permissible value of the load q if the allowable bending stress in the shelf is $\sigma_{\text{allow}} = 8.5$ MPa and the position of the supports is adjusted for maximum load carrying capacity.
- The bookshelf owner decides to reinforce the shelf with a bottom wood plate $b/2 \times t/2$ along its entire length (see figure part c). Find the new maximum permissible value of the load q if the allowable bending stress in the shelf remains at $\sigma_{\text{allow}} = 8.5$ MPa.



(a)



(b)

Solution 5.6-18

- (a) FOR MAXIMUM LOAD-CARRYING CAPACITY, PLACE THE SUPPORTS SO THAT $M_1 = |M_2|$.

Let x = length of overhang.

$$M_1 = \frac{qL}{8} (L - 4x) \quad |M_2| = \frac{qx^2}{2}$$

$$\therefore \frac{qL}{8} (L - 4x) = \frac{qx^2}{2}$$

$$\text{Solve for } x: x = \frac{L}{2} (\sqrt{2} - 1)$$

$$\text{NUMERICAL VALUES} \quad L = 1215 \text{ mm} \quad b = 305 \text{ mm} \quad t = 22 \text{ mm} \quad \sigma_{\text{allow}} = 8.5 \text{ MPa}$$

$$x = \frac{L}{2} (\sqrt{2} - 1) \quad \text{Solve for } q$$

$$q_{\text{max}} = \frac{4bt^2\sigma_{\text{allow}}}{3L^2(3 - 2\sqrt{2})} = 6.61 \frac{\text{kN}}{\text{m}}$$

$$S = \frac{bt^2}{6} = 2.46 \times 10^4 \text{ mm}^3$$

$$I_a = S \frac{t}{2} = 2.706 \times 10^5 \text{ mm}^4$$

$$(b) \alpha = \frac{b}{2t} \quad b = 0.305 \text{ m} \quad t = 0.022 \text{ m}$$

$$A = bt + \frac{b}{2} \frac{t}{2} \quad \alpha = 6.932$$

$$c_2 = \frac{\frac{b}{2} \frac{t}{2} \frac{t}{4} + bt \left(\frac{t}{2} + \frac{t}{2} \right)}{A} = \frac{17t}{20}$$

$$c_1 = \frac{3}{2}t - c_2 = \frac{13t}{20} \quad c_1 + c_2 = 33 \text{ mm}$$

$$I = \frac{\frac{b}{2} \left(\frac{t}{2} \right)^3}{12} + \frac{b}{2} \frac{t}{2} \left(c_2 - \frac{t}{4} \right)^2 + \frac{bt^3}{12} + bt \left(c_1 - \frac{t}{2} \right)^2 = \frac{193bt^3}{960}$$

$$S_{\text{top}} = \frac{I}{c_1} = \frac{193bt^2}{624} \quad S_{\text{bot}} = \frac{I}{c_2} = \frac{193bt^2}{816}$$

$$S_{\text{top}} = 4.566 \times 10^4 \text{ mm}^3 \quad S_{\text{bot}} = 3.492 \times 10^4 \text{ mm}^3$$

$$\% \text{ increase in smaller section modulus: } \boxed{\frac{S_{\text{top}} - S}{S} = 85.6\%} \quad \frac{S_{\text{bot}} - S}{S} = 41.912\%$$

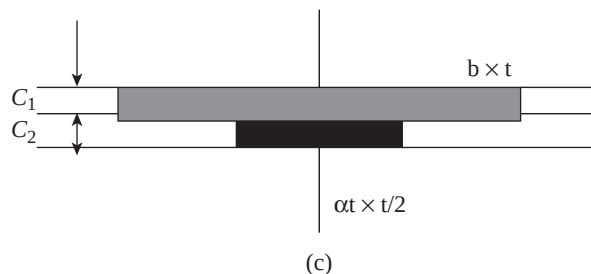
$$M_{\text{max}} = \frac{qL^2}{8} (3 - 2\sqrt{2}) \quad \text{and} \quad M_{\text{max}} = \sigma_{\text{allow}} S_{\text{top}}$$

Check if S_{top} controls:

$$q_{\text{max}} = \frac{8\sigma_{\text{allow}}}{(3 - 2\sqrt{2})L^2} (S_{\text{top}}) = 12.26 \frac{\text{kN}}{\text{m}}$$

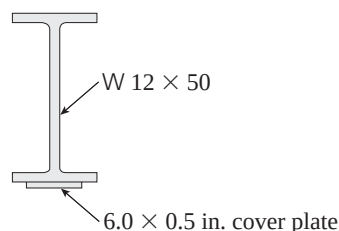
Check if S_{bot} controls:

$$\boxed{q_{\text{max}} = \frac{8\sigma_{\text{allow}}}{(3 - 2\sqrt{2})L^2} (S_{\text{bot}}) = 9.37 \frac{\text{kN}}{\text{m}}}$$



Problem 5.6-19 A steel plate (called a *cover plate*) having cross sectional dimensions 6.0 in. $\times$ 0.5 in. is welded along the full length of the bottom flange of a W 12 $\times$ 50 wide-flange beam (see figure, which shows the beam cross section).

What is the percent increase in the smaller section modulus (as compared to the wide-flange beam alone)?



Solution 5.6-19

NUMERICAL PROPERTIES FOR W 12 $\times$ 50
(FROM TABLE F-1(a))

$$A = 14.6 \text{ in.}^2 \quad d = 12.2 \text{ in.}$$

$$c_1 = c_2 \quad c_1 = \frac{d}{2}$$

$$I = 391 \text{ in.}^4 \quad S = 64.2 \text{ in.}^3$$

FIND CENTROID OF BEAM WITH COVER PLATE (TAKE
1ST MOMENTS ABOUT TOP TO FIND $c_1 > c_2$)

$$c_1 = \frac{A \frac{d}{2} + (6)(0.5) \left(d + \frac{0.5}{2} \right)}{A + (6)(0.5)} \quad c_1 = 7.182 \text{ in.}$$

$$c_2 = (d + 0.5) - c_1 \quad c_2 = 5.518 \text{ in.}$$

FIND I ABOUT HORIZ. CENTROIDAL AXIS

$$I_h = I + A \left(c_1 - \frac{d}{2} \right)^2 + \frac{1}{12} (6)(0.5)^3 + (6)(0.5) \left(c_2 - \frac{0.5}{2} \right)^2$$

$$I_h = 491.411 \text{ in.}^4$$

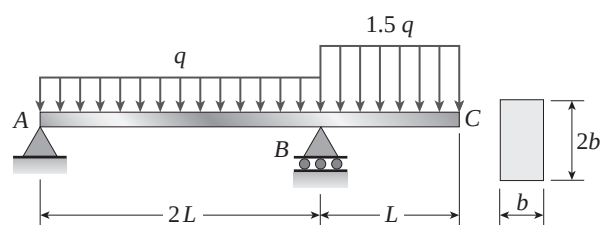
FIND SMALLER SECTION MODULUS

$$S_{\text{top}} = \frac{I_h}{c_1} \quad S_{\text{top}} = 68.419 \text{ in.}^3$$

% increase in smaller section modulus

$$\frac{S_{\text{top}} - S}{S} (100) = 6.57\% \quad \leftarrow$$

Problem 5.6-20 A steel beam ABC is simply supported at A and B and has an overhang BC of length $L = 150 \text{ mm}$ (see figure). The beam supports a uniform load of intensity $q = 4.0 \text{ kN/m}$ over its entire span AB and $1.5q$ over BC . The cross section of the beam is rectangular with width b and height $2b$. The allowable bending stress in the steel is $\sigma_{\text{allow}} = 60 \text{ MPa}$, and its weight density is $\gamma = 77.0 \text{ kN/m}^3$.



- Disregarding the weight of the beam, calculate the required width b of the rectangular cross section.
- Taking into account the weight of the beam, calculate the required width b .

Solution 5.6-20

NUMERICAL DATA

$$L = 150 \text{ mm} \quad q = 4 \frac{\text{kN}}{\text{m}}$$

$$\sigma_a = 60 \text{ MPa} \quad \gamma = 77 \frac{\text{kN}}{\text{m}^3}$$

(a) IGNORE BEAM SELF-WEIGHT; FIND $b_{\min}$

$$M_{\max 1} = 1.5q \frac{L^2}{2} \quad \text{at } B$$

$$\text{and} \quad M_{\max 2} = \sigma_a S \quad S = \frac{2}{3} b^3$$

Equate $M_{\max 1}$ to $M_{\max 2}$ & solve for $b_{\min}$

$$b_{\min} = \left(\frac{9}{8} \frac{qL^2}{\sigma_a} \right)^{\frac{1}{3}}$$

$$b_{\min} = 11.91 \text{ mm} \quad \leftarrow$$

(b) NOW MODIFY; INCLUDE BEAM WEIGHT

$$w = \gamma A \quad w = \gamma(2b^2)$$

$$M_{\max} = (1.5q + w) \frac{L^2}{2}$$

$$\text{and} \quad M_{\max} = \sigma_a \left(\frac{2}{3} b^3 \right)$$

Equate $M_{\max 1}$ to $M_{\max 2}$ & solve for $b_{\min}$

$$\left(\frac{2}{3} \sigma_a \right) b^3 - (\gamma L^2) b^2 - \frac{3}{4} q L^2 = 0$$

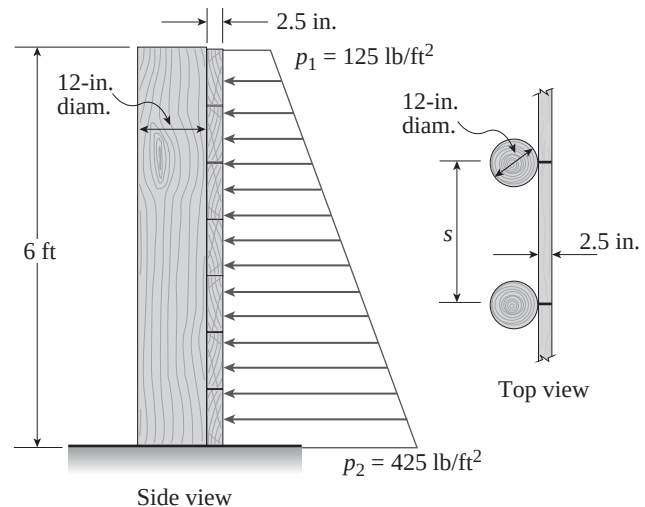
Insert numerical values, then solve for b

$$b_{\min} = 11.92 \text{ mm} \quad \leftarrow$$

Problem 5.6-21 A retaining wall 6 ft high is constructed of horizontal wood planks 2.5 in. thick (actual dimension) that are supported by vertical wood piles of 12-in. diameter (actual dimension), as shown in the figure. The lateral earth pressure is $p_1 = 125 \text{ lb/ft}^2$ at the top of the wall and $p_2 = 425 \text{ lb/ft}^2$ at the bottom.

- Assuming that the allowable stress in the wood is 1175 psi, calculate the maximum permissible spacing s of the piles.
- Find the required diameter of the wood piles so that piles and planks ($t = 2.5 \text{ in.}$) reach the allowable stress at the same time.

(Hint: Observe that the spacing of the piles may be governed by the load-carrying capacity of either the planks or the piles. Consider the piles to act as cantilever beams subjected to a trapezoidal distribution of load, and consider the planks to act as simple beams between the piles. To be on the safe side, assume that the pressure on the bottom plank is uniform and equal to the maximum pressure.)



Solution 5.6-21 Retaining wall

$$(a) \quad \sigma_a = 1175 \text{ psi} \quad t = 2.5 \text{ in.} \quad p_2 = 425 \frac{\text{lb}}{\text{ft}^2} \quad p_1 = 125 \frac{\text{lb}}{\text{ft}^2} \quad h = 6 \text{ ft} \quad d = 12 \text{ in.}$$

$$s_{\text{plank}} = \sqrt{\frac{4\sigma_a t^2}{3p_2}} = 57.599 \text{ in.} \quad s_{\text{pole}} = \frac{3\pi\sigma_a d^3}{16h^2(2p_1 + p_2)} = 49.2 \text{ in.} \quad < \text{controls}$$

$$s_{\text{max}} = s_{\text{pole}}$$

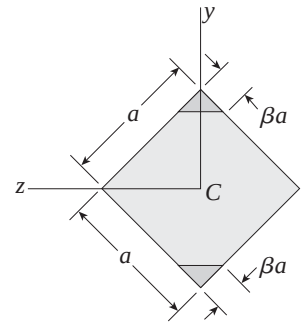
$$s_{\text{max}} = 49.2 \text{ in.}$$

(b) EQUAL SPACING IF $t = 2.14 \text{ in.}$ OR $d = 12.65 \text{ in.}$; HERE t IS SET AT 2.5 in., SO FIND d

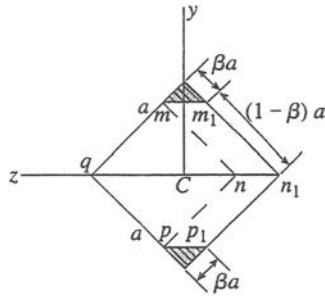
$$\frac{3\pi\sigma_a d^3}{16h^2(2p_1 + p_2)} = \sqrt{\frac{4\sigma_a t^2}{3p_2}} \quad \text{solving for } d \quad \boxed{d = 12.65 \text{ in.}}$$

Problem 5.6-22 A beam of square cross section (a = length of each side) is bent in the plane of a diagonal (see figure). By removing a small amount of material at the top and bottom corners, as shown by the shaded triangles in the figure, we can increase the section modulus and obtain a stronger beam, even though the area of the cross section is reduced.

- Determine the ratio β defining the areas that should be removed in order to obtain the strongest cross section in bending.
- By what percent is the section modulus increased when the areas are removed?



Solution 5.6-22 Beam of square cross section with corners removed



a = length of each side
 βa = amount removed
 Beam is bent about the z axis.

ENTIRE CROSS SECTION (AREA 0)

$$I_0 = \frac{a^4}{12} \quad c_0 = \frac{a}{\sqrt{2}} \quad S_0 = \frac{I_0}{c_0} = \frac{a^3\sqrt{2}}{12}$$

SQUARE $mnpq$ (AREA 1)

$$I_1 = \frac{(1 - \beta)^4 a^4}{12}$$

PARALLELOGRAM mm, n, n (AREA 2)

$$I_2 = \frac{1}{3}(\text{base})(\text{height})^3$$

$$I_2 = \frac{1}{3}(\beta a\sqrt{2})\left[\frac{(1 - \beta)a}{\sqrt{2}}\right]^3 = \frac{\beta a^4}{6}(1 - \beta)^3$$

REDUCED CROSS SECTION (AREA qmm, n, p, pq)

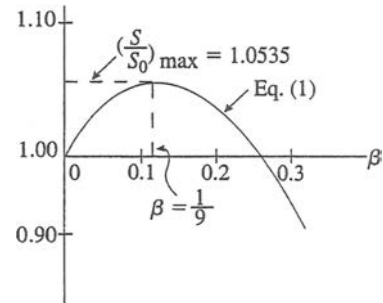
$$I = I_1 + 2I_2 = \frac{a^4}{12}(1 + 3\beta)(1 - \beta)^3$$

$$c = \frac{(1 - \beta)a}{\sqrt{2}} \quad S = \frac{I}{c} = \frac{\sqrt{2}a^3}{12}(1 + 3\beta)(1 - \beta)^2$$

RATIO OF SECTION MODULI

$$\frac{S}{S_0} = (1 + 3\beta)(1 - \beta)^2 \quad \text{Eq. (1)}$$

GRAPH OF EQ. (1)



(a) VALUE OF β FOR A MAXIMUM VALUE OF S/S_0

$$\frac{d}{d\beta}\left(\frac{S}{S_0}\right) = 0$$

Take the derivative and solve this equation for β .

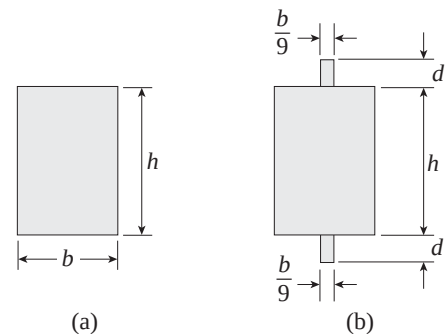
$$\beta = \frac{1}{9} \quad \leftarrow$$

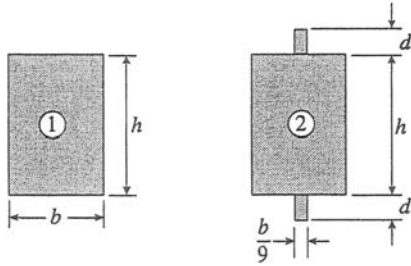
(b) MAXIMUM VALUE OF S/S_0

Substitute $\beta = 1/9$ into Eq. (1). $(S/S_0)_{\max} = 1.0535$
 The section modulus is increased by 5.35% when the triangular areas are removed. $\leftarrow$

Problem 5.6-23 The cross section of a rectangular beam having width b and height h is shown in part a of the figure. For reasons unknown to the beam designer, it is planned to add structural projections of width $b/9$ and height d to the top and bottom of the beam (see part b of the figure).

For what values of d is the bending-moment capacity of the beam increased? For what values is it decreased?



Solution 5.6-23 Beam with projections

(1) ORIGINAL BEAM

$$I_1 = \frac{bh^3}{12} \quad c_1 = \frac{h}{2} \quad S_1 = \frac{I_1}{c_1} = \frac{bh^2}{6}$$

(2) BEAM WITH PROJECTIONS

$$I_2 = \frac{1}{12} \left(\frac{8b}{9} \right) h^3 + \frac{1}{12} \left(\frac{b}{9} \right) (h + 2d)^3$$

$$= \frac{b}{108} [8h^3 + (h + 2d)^3]$$

$$c_2 = \frac{h}{2} + d = \frac{1}{2} (h + 2d)$$

$$S_2 = \frac{I_2}{c_2} = \frac{b[8h^3 + (h + 2d)^3]}{54(h + 2d)}$$

RATIO OF SECTION MODULI

$$\frac{S_2}{S_1} = \frac{b[8h^3 + (h + 2d)^3]}{9(h + 2d)(bh^2)} = \frac{8 + \left(1 + \frac{2d}{h}\right)^3}{9\left(1 + \frac{2d}{h}\right)}$$

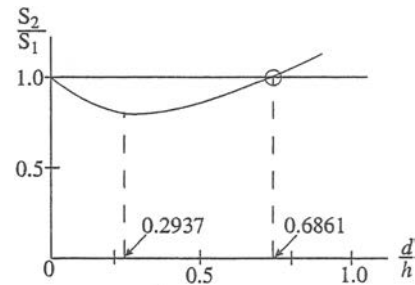
EQUAL SECTION MODULI

Set $\frac{S_2}{S_1} = 1$ and solve numerically for $\frac{d}{h}$.

$$\frac{d}{h} = 0.6861 \quad \text{and} \quad \frac{d}{h} = 0$$

Graph of $\frac{S_2}{S_1}$ versus $\frac{d}{h}$

$\frac{d}{h}$	$\frac{S_2}{S_1}$
0	1.000
0.25	0.8426
0.50	0.8889
0.75	1.0500
1.00	1.2963



Moment capacity is increased when

$$\frac{d}{h} > 0.6861 \quad \leftarrow$$

Moment capacity is decreased when

$$\frac{d}{h} < 0.6861 \quad \leftarrow$$

NOTES:

$$\frac{S_2}{S_1} = 1 \quad \text{when} \quad \left(1 + \frac{2d}{h}\right)^3 - 9\left(1 + \frac{2d}{h}\right) + 8 = 0$$

$$\text{or} \quad \frac{d}{h} = 0.6861 \text{ and } 0$$

$$\frac{S_2}{S_1} \text{ is minimum when } \frac{d}{h} = \frac{\sqrt[3]{4} - 1}{2} = 0.2937$$

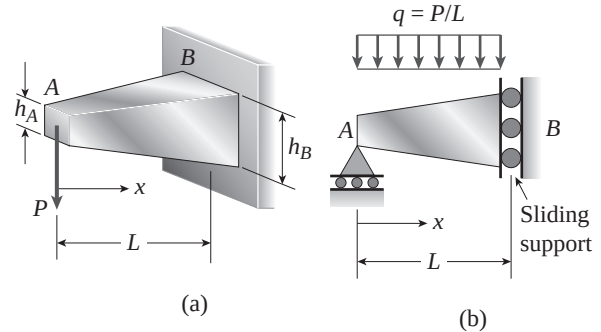
$$\left(\frac{S_2}{S_1}\right)_{\min} = 0.8399$$

Nonprismatic Beams

Problem 5.7-1 A tapered cantilever beam AB of length L has square cross sections and supports a concentrated load P at the free end (see figure part a). The width and height of the beam vary linearly from h_A at the free end to h_B at the fixed end.

Determine the distance x from the free end A to the cross section of maximum bending stress if $h_B = 3h_A$.

- What is the magnitude $\sigma_{\max}$ of the maximum bending stress? What is the ratio of the maximum stress to the largest stress B at the support?
- Repeat part (a) if load P is now applied as a uniform load of intensity $q = P/L$ over the entire beam, A is restrained by a roller support and B is a sliding support (see figure part b).



Solution 5.7-1

- (a) FIND MAX. BENDING STRESS FOR TAPERED CANTILEVER

$$h(x) = h_A \left(1 + \frac{2x}{L} \right) \quad S(x) = \frac{h(x)^3}{6}$$

$$\sigma(x) = \frac{M(x)}{S(x)} \quad \sigma(x) = \frac{6(P)(x)}{\left[h_A \left(1 + \frac{2x}{L} \right) \right]^3}$$

$$\sigma(x) = \frac{6PxL^3}{h_A^3(L + 2x)^3}$$

$$\frac{d}{dx} \sigma(x) = 0 \quad \text{then solve for } x_{\max}$$

$$\frac{d}{dx} \left[\frac{6PxL^3}{h_A^3(L + 2x)^3} \right] = 0$$

$$\left[6P \frac{L^3}{h_A^3(L + 2x)^3} - 36Px \frac{L^3}{h_A^3(L + 2x)^4} \right] = 0$$

$$\frac{-L + 4x}{h_A^3(L + 2x)^4} = 0 \quad \text{so } x = \frac{L}{4}$$

$$\sigma_{\max} = \sigma\left(\frac{L}{4}\right) \quad \sigma_{\max} = \frac{6P \frac{L}{4} L^3}{h_A^3 \left(L + 2 \frac{L}{4} \right)^3}$$

$$\sigma_{\max} = \frac{4PL}{9h_A^3} \quad \leftarrow$$

$$\sigma_B = \sigma(L) \quad \sigma_B = \frac{2PL}{9h_A^3}$$

$$\frac{\sigma_{\max}}{\sigma_B} = \frac{\frac{4PL}{9h_A^3}}{\frac{2PL}{9h_A^3}} \quad \frac{\sigma_{\max}}{\sigma_B} = 2 \quad \leftarrow$$

- (b) REPEAT (A), BUT NOW FOR DISTRIBUTED UNIFORM LOAD OF P/L OVER ENTIRE BEAM

$$\sum F_v = 0 \quad R_A = P$$

$$M(x) = \left[\left[R_A x - \frac{P}{L} x \left(\frac{x}{2} \right) \right] \right]$$

$$M(x) = Px - \frac{1}{2} x^2 \frac{P}{L}$$

$$\sigma(x) = \frac{M(x)}{S(x)} \quad \sigma(x) = \frac{Px - \frac{1}{2} x^2 \frac{P}{L}}{\left[h_A \left(1 + \frac{2x}{L} \right) \right]^3}$$

$$\sigma(x) = -3xP(-2L + x) \frac{L^2}{h_A^3(L + 2x)^3}$$

$$\frac{d}{dx} \sigma(x) = 0 \quad \text{then solve for } x_{\max}$$

$$\frac{d}{dx} \left[-3xP(-2L+x) \frac{L^2}{h_A^3(L+2x)^3} \right] = 0$$

$$\left[-3P(-2L+x) \frac{L^2}{h_A^3(L+2x)^3} - 3xP \frac{L^2}{h_A^3(L+2x)^3} + 18xP(-2L+x) \frac{L^2}{h_A^3(L+2x)^4} \right] = 0$$

Simplifying:

$$L^2 - 5xL + x^2 = 0 \quad \text{so}$$

$$\frac{x_{\max}}{L} = \frac{5 - \sqrt{5^2 - 4}}{2}$$

$$x_{\max} = 0.20871L$$

$$\sigma_{\max} = \sigma(0.20871L)$$

$$\sigma_{\max} = 0.394 \frac{PL}{h_A^3} \quad \leftarrow$$

$$\sigma_B = \sigma(L) \quad \text{so}$$

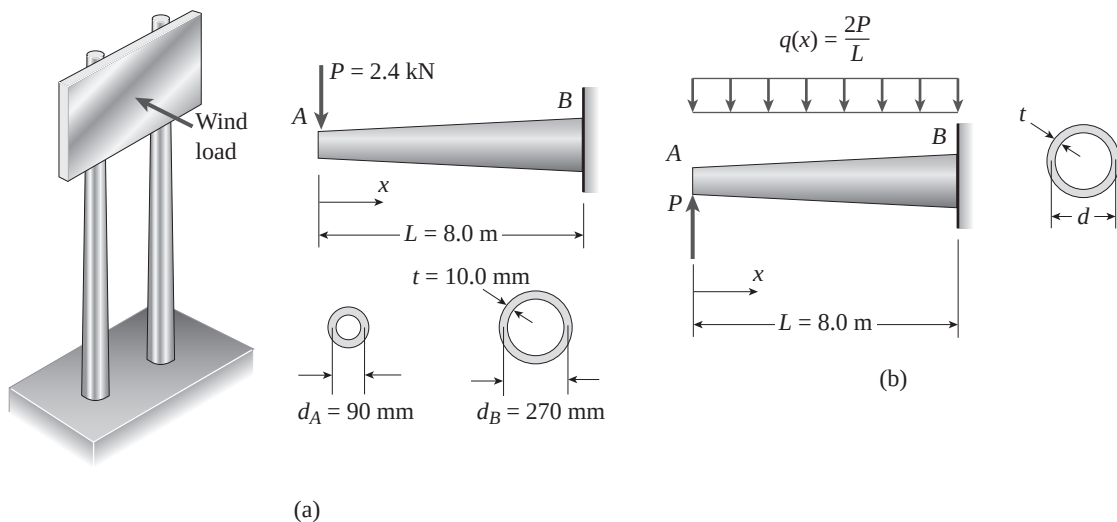
$$\sigma_B = \frac{PL}{9h_A^3} \quad \frac{\sigma_{\max}}{\sigma_B} = \frac{\left(0.39385 \frac{PL}{h_A^3}\right)}{\frac{PL}{9h_A^3}}$$

$$\frac{\sigma_{\max}}{\sigma_B} = 3.54 \quad \leftarrow$$

Problem 5.7-2 A tall signboard is supported by two vertical beams consisting of thin-walled, tapered circular tubes (see figure part a). For purposes of this analysis, each beam may be represented as a cantilever AB of length $L = 8.0$ m subjected to a lateral load $P = 2.4$ kN at the free end. The tubes have constant thickness $t = 10.0$ mm and average diameters $d_A = 90$ mm and $d_B = 270$ mm at ends A and B , respectively.

Because the thickness is small compared to the diameters, the moment of inertia at any cross section may be obtained from the formula $I = \pi d^3 t / 8$ (see Case 22, Appendix E), and therefore, the section modulus may be obtained from the formula $S = \pi d^2 t / 4$.

- At what distance x from the free end does the maximum bending stress occur? What is the magnitude $\sigma_{\max}$ of the maximum bending stress? What is the ratio of the maximum stress to the largest stress σ_B at the support?
- Repeat part (a) if concentrated load P is applied upward at A and downward uniform load $q(x) = 2P/L$ is applied over the entire beam as shown. What is the ratio of the maximum stress to the stress at the location of maximum moment?



Solution 5.7-2

(a) FIND MAX. BENDING STRESS FOR TAPERED CANTILEVER

$$d(x) = d_A \left(1 + \frac{2x}{L} \right) \quad S(x) = \frac{\pi d(x)^2 t}{4}$$

$$P = 2.4 \text{ kN}$$

$$L = 8 \text{ m} \quad t = 10 \text{ mm}$$

$$d_A = 90 \text{ mm}$$

$$d_B = 270 \text{ mm}$$

$$\sigma(x) = \frac{M(x)}{S(x)} \quad \sigma(x) = \frac{4P}{\pi t} \left[\frac{x}{d_A \left(1 + \frac{2x}{L} \right)^2} \right]$$

$$\sigma(x) = \frac{4P}{\pi t} \left[\frac{xL^2}{d_A^2 (L + 2x)^2} \right]$$

$$\frac{d}{dx} \sigma(x) = 0 \quad \text{then solve for } x_{\max}$$

$$\frac{d}{dx} \left[\frac{4P}{\pi t} \left[\frac{xL^2}{d_A^2 (L + 2x)^2} \right] \right] = 0$$

$$\left[4 \frac{P}{\pi t} \frac{L^2}{d_A^2 (L + 2x)^2} - 16 \frac{P}{\pi t} \frac{xL^2}{d_A^2 (L + 2x)^3} \right] = 0$$

$$\text{or} \quad \left[-4PL^2 \frac{-L + 2x}{\pi t d_A^2 (L + 2x)^3} \right] = 0$$

$$\text{so } x_{\max} = \frac{L}{2} = 4 \text{ m} \quad \leftarrow$$

$$\sigma_{\max} = \sigma \left(\frac{L}{2} \right)$$

$$\sigma_{\max} = \frac{4P}{\pi t} \left[\frac{\frac{L}{2} L^2}{d_A^2 \left(L + 2 \frac{L}{2} \right)^2} \right]$$

$$\sigma_{\max} = \frac{PL}{2\pi t d_A^2}$$

$$\text{Stress at support } \sigma_B = \sigma(L)$$

$$\sigma_B = \frac{4}{9} \frac{P}{\pi t} \frac{L}{d_A^2}$$

$$\frac{\sigma_{\max}}{\sigma_B} = \frac{\frac{PL}{2\pi t d_A^2}}{\left(\frac{4}{9} \frac{P}{\pi t} \frac{L}{d_A^2} \right)}$$

$$\frac{\sigma_{\max}}{\sigma_B} = \frac{9}{8} \quad \leftarrow$$

Evaluate using numerical data:

$$\sigma_{\max} = \frac{(2400)(8)}{2\pi(0.010)(0.090)^2}$$

$$\sigma_{\max} = 37.7 \text{ MPa} \quad \leftarrow$$

(b) REPEAT (A), BUT NOW ADD DISTRIBUTED LOAD

$$M(x) = \left(Px - 2 \frac{P}{L} x \frac{x}{2} \right)$$

$$M(x) = -Px \left(\frac{-L + x}{L} \right)$$

$$\sigma(x) = \frac{M(x)}{S(x)} \quad \sigma(x) = \frac{-Px \left(\frac{-L + x}{L} \right)}{\frac{\pi t}{4} \left[d_A \left(1 + \frac{2x}{L} \right) \right]^2}$$

$$\sigma(x) = -4Px(-L+x) \frac{L}{\pi t d_A^2 (L+2x)^2}$$

tension on top, compression on bottom of beam

$$\frac{d}{dx} \sigma(x) = 0 \quad \text{then solve for } x_{\max}$$

$$\frac{d}{dx} \left[-4Px(-L+x) \frac{L}{\pi t d_A^2 (L+2x)^2} \right] = 0$$

$$\left[-4P(-L+x) \frac{L}{\pi t d_A^2 (L+2x)^2} - 4Px \frac{L}{\pi t d_A^2 (L+2x)^2} + 16Px(-L+x) \frac{L}{\pi t d_A^2 (L+2x)^3} \right] = 0$$

$$\text{OR simplifying} \quad \left[-4PL^2 \frac{-L+4x}{\pi t d_A^2 (L+2x)^3} \right] = 0$$

$$\text{so } x_{\max} = \frac{L}{4}$$

$$x_{\max} = 2 \text{ m} \quad \leftarrow$$

$$\sigma_{\max} = \sigma\left(\frac{L}{4}\right)$$

$$\sigma_{\max} = \left[-4P \frac{L}{4} \left(-L + \frac{L}{4} \right) \frac{L}{\pi t d_A^2 \left(L + 2 \frac{L}{4} \right)^2} \right]$$

$$\sigma_{\max} = \frac{PL}{3\pi t d_A^2}$$

Evaluate using numerical data:

$$P = 2.4 \text{ kN} \quad L = 8 \text{ m}$$

$$t = 10 \text{ mm} \quad d_A = 90 \text{ mm}$$

$$d_B = 270 \text{ mm}$$

$$\sigma_{\max} = \frac{(2400)(8)}{3\pi(0.010)(0.090)^2}$$

$$\sigma_{\max} = 25.2 \text{ MPa} \quad \leftarrow$$

Stress at support:

$$\sigma_B = \sigma(L)$$

$$\sigma_B = -4PL(-L + L) \frac{L}{\pi t d_A^2 (L + 2L^2)}$$

$$\sigma_B = 0 \quad \text{so no ratio of } \sigma_{\max}/\sigma_B \text{ is possible.}$$

MAX. MOMENT AT $L/2$, SO COMPARE

Stress at location of max. moment.

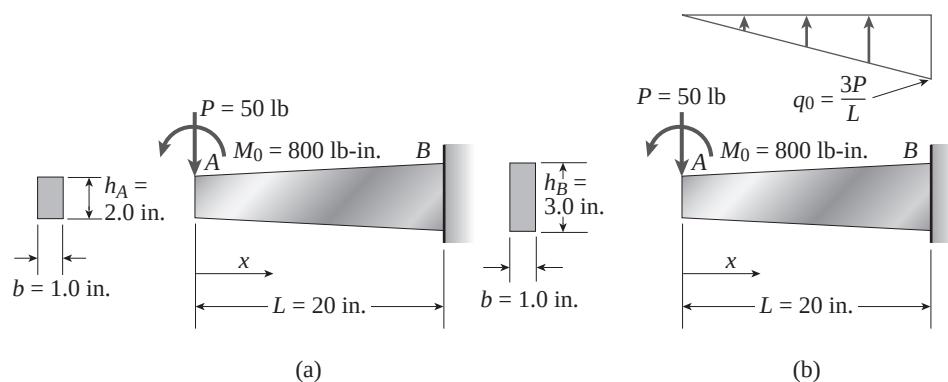
$$\sigma\left(\frac{L}{2}\right) = -4P \frac{L}{2} \left(-L + \frac{L}{2} \right) \frac{L}{\pi t d_A^2 \left(L + 2 \frac{L}{2} \right)^2}$$

$$\sigma\left(\frac{L}{2}\right) = \frac{1}{4} P \frac{L}{\pi t d_A^2}$$

$$\sigma_{\max}/\sigma(L/2) = \frac{\frac{PL}{3\pi t d_A^2}}{\left(\frac{1}{4} P \frac{L}{\pi t d_A^2} \right)} = \frac{4}{3} \quad \leftarrow$$

Problem 5.7-3 A tapered cantilever beam AB having rectangular cross sections is subjected to a concentrated load $P = 50 \text{ lb}$ and a couple $M_0 = 800 \text{ lb-in.}$ acting at the free end (see figure part a). The width b of the beam is constant and equal to 1.0 in. , but the height varies linearly from $h_A = 2.0 \text{ in.}$ at the loaded end to $h_B = 3.0 \text{ in.}$ at the support.

- At what distance x from the free end does the maximum bending stress $\sigma_{\max}$ occur? What is the magnitude $\sigma_{\max}$ of the maximum bending stress? What is the ratio of the maximum stress to the largest stress σ_B at the support?
- Repeat (a) if, in addition to P and M_0 , a triangular distributed load with peak intensity $q_0 = 3P/L$ acts upward over the entire beam as shown in the figure part b. What is the ratio of the maximum stress to the stress at the location of maximum moment?



Solution 5.7-3

- (a) FIND MAX. BENDING STRESS FOR TAPERED CANTILEVER
FIG. (A)

$$h(x) = h_A \left(1 + \frac{x}{2L} \right)$$

NUMERICAL DATA

$$P = 50 \text{ lb} \quad L = 20 \text{ in.}$$

$$h_A = 2 \text{ in.} \quad h_B = 3 \text{ in.} \quad b = 1 \text{ in.}$$

$$M_0 = \frac{4}{5} PL \quad M_0 = 800 \text{ in.-lb}$$

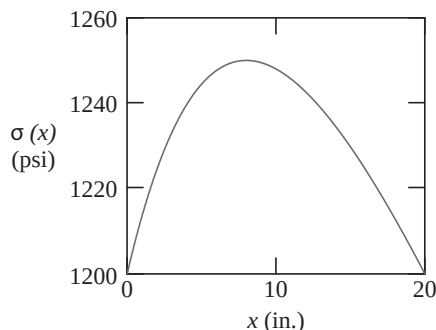
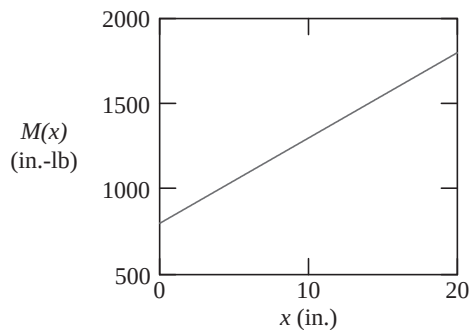
$$I(x) = \frac{bh(x)^3}{12} \quad S(x) = \frac{I(x)}{\frac{h(x)}{2}}$$

$$S(x) = \frac{bh(x)^2}{6}$$

$$S(x) = \frac{b \left[h_A \left(1 + \frac{x}{2L} \right) \right]^2}{6}$$

$$M(x) = Px + M_0$$

$$\sigma(x) = \frac{M(x)}{S(x)}$$



$$\sigma(x) = \frac{Px + M_0}{\frac{b \left[h_A \left(1 + \frac{x}{2L} \right) \right]^2}{6}}$$

$$\sigma(x) = 24 (Px + M_0) \frac{L^2}{bh_A^2 (2L + x)^2}$$

$$\frac{d}{dx} \sigma(x) = 0 \quad \text{then solve for } x_{\max}$$

$$\frac{d}{dx} \left[24 (Px + M_0) \frac{L^2}{bh_A^2 (2L + x)^2} \right] = 0$$

$$24P \frac{L^2}{bh_A^2 (2L + x)^2}$$

$$- 2(24Px + 24M_0) \frac{L^2}{bh_A^2 (2L + x)^3} = 0$$

$$\text{OR simplifying } \left[-24L^2 \frac{-2PL + Px + 2M_0}{bh_A^2 (2L + x)^3} \right] = 0$$

$$\text{so } x = \frac{2(PL - M_0)}{P}$$

$$x_{\max} = 8 \text{ in.} \quad \leftarrow$$

agrees with plot at left

Evaluate max. stress & stress at B using numerical data:

$$\sigma_{\max} = \sigma(8) \quad \sigma_{\max} = 1250 \text{ psi} \quad \leftarrow$$

$$\sigma_B = \sigma(20) \quad \sigma_B = 1200 \text{ psi}$$

$$\frac{\sigma_{\max}}{\sigma_B} = 1.042 \quad \leftarrow$$

- (b) FIND MAX. BENDING STRESS FOR TAPERED CANTILEVER, FIG. (B)

$$h(x) = h_A \left(1 + \frac{x}{2L} \right)$$

$$M_0 = \frac{4}{5} PL \quad M_0 = 800 \text{ in.-lb}$$

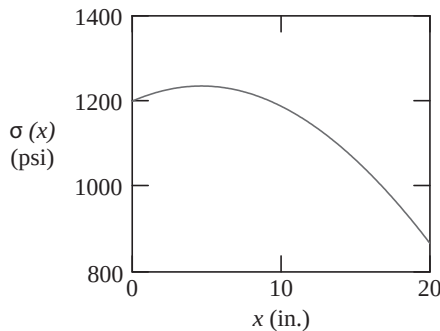
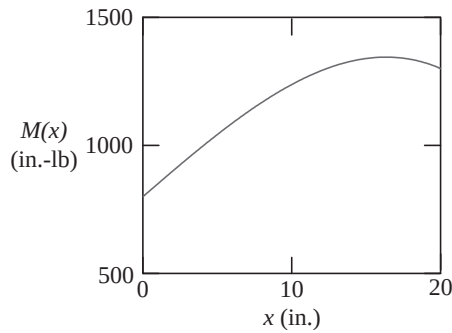
$$q_0 = 3 \frac{P}{L}$$

$$I(x) = \frac{bh(x)^3}{12} \quad S(x) = \frac{I(x)}{\frac{h(x)}{2}} \quad S(x) = \frac{bh(x)^2}{6}$$

$$S(x) = \frac{b \left[h_A \left(1 + \frac{x}{2L} \right) \right]^2}{6}$$

$$M(x) = Px + M_0 + \frac{-1}{2} \left(\frac{x}{L} q_0 \right) x \frac{x}{3}$$

$$\sigma(x) = \frac{M(x)}{S(x)}$$



$$\sigma(x) = \frac{Px + M_0 - \frac{q_0 x^3}{6L}}{\frac{b \left[h_A \left(1 + \frac{x}{2L} \right) \right]^2}{6}}$$

$$\sigma(x) = -4 \left(-6PxL - 6M_0L + x^3q_0 \right) \times \frac{L}{bh_A^2(2L+x)^2}$$

$$\frac{d}{dx} \sigma(x) = 0 \quad \text{then solve for } x_{\max}$$

$$\frac{d}{dx} \sigma(x) = \left[(24PL - 12x^2q_0) \frac{L}{bh_A^2(2L+x)^2} - 2(24PxL + 24M_0L - 4x^3q_0) \times \frac{L}{bh_A^2(2L+x)^3} \right] = 0$$

Simplifying:

$$-12PL^2 + 6PxL + 6x^2q_0L + x^3q_0 + 12M_0L = 0$$

Solve for $x_{\max}$:

$$x_{\max} = 4.642 \text{ in.} \quad \leftarrow$$

Max. stress & stress at B:

$$\sigma_{\max} = \sigma(x_{\max})$$

$$\sigma_{\max} = 1235 \text{ psi} \quad \leftarrow$$

$$\sigma_B = \sigma(20) \quad \sigma_B = 867 \text{ psi}$$

FIND MAX. MOMENT AND STRESS AT LOCATION OF MAX. MOMENT

$$\frac{d}{dx} M(x) = 0 \quad \frac{d}{dx} \left(Px + M_0 - \frac{q_0 x^3}{6L} \right) = 0$$

$$x_m = \sqrt{\frac{P(2L)}{q_0}} \quad x_m = 16.33 \text{ in.}$$

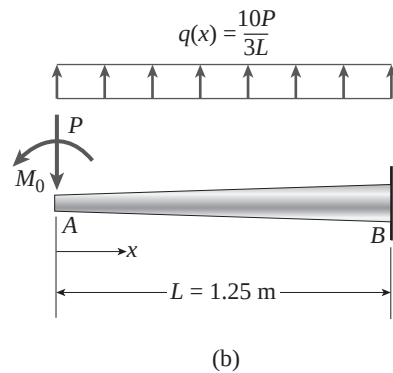
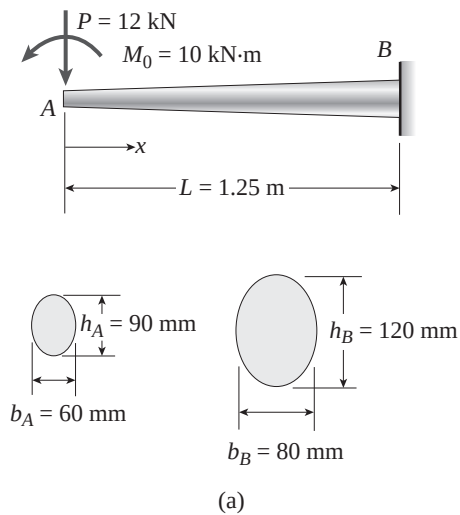
$$\sigma_m = \sigma(x_m) \quad \sigma_m = 1017 \text{ psi}$$

$$\frac{\sigma_{\max}}{\sigma_m} = 1.215 \quad \leftarrow$$

Problem 5.7-4 The spokes in a large flywheel are modeled as beams fixed at one end and loaded by a force P and a couple M_0 at the other (see figure). The cross sections of the spokes are elliptical with major and minor axes (height and width, respectively) having the lengths shown in the figure part a. The cross-sectional dimensions vary linearly from end A to end B.

Considering only the effects of bending due to the loads P and M_0 , determine the following quantities.

- The largest bending stress σ_A at end A.
- The largest bending stress σ_B at end B.
- The distance x to the cross section of maximum bending stress.
- The magnitude $\sigma_{\max}$ of the maximum bending stress.
- Repeat part (d) if uniform load $q(x) = 10P/3L$ is added to loadings P and M_0 , as shown in the figure part b.



Solution 5.7-4

(a-d) FIND MAX. BENDING STRESS FOR TAPERED CANTILEVER

NUMERICAL DATA

$$L = 1.25 \text{ m} \quad b_A = 60 \text{ mm} \quad h_A = 90 \text{ mm}$$

$$b_B = 80 \text{ mm} \quad h_B = 120 \text{ mm}$$

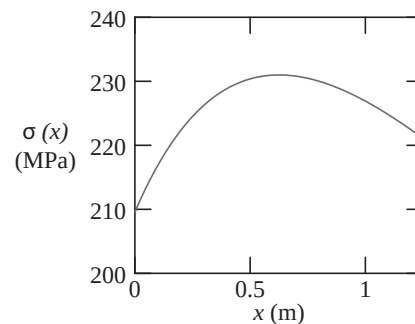
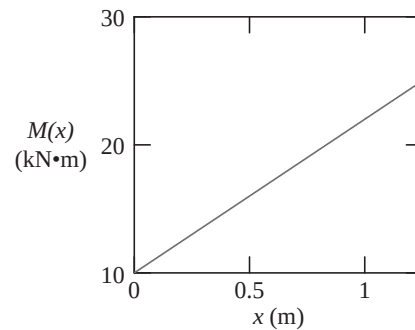
$$P = 12 \text{ kN} \quad M_0 = 10 \text{ kN} \cdot \text{m}$$

$$h(x) = h_A \left(1 + \frac{x}{3L} \right) \quad b(x) = b_A \left(1 + \frac{x}{3L} \right)$$

$$I(x) = \frac{\pi b(x)h(x)^3}{64} \quad S(x) = \frac{I(x)}{\frac{h(x)}{2}}$$

$$S(x) = \frac{\pi b(x)h(x)^2}{32}$$

$$S(x) = \frac{\pi b_A h_A^2 \left(1 + \frac{x}{3L} \right)^3}{32}$$



$$M(x) = Px + M_0 \quad \sigma(x) = \frac{M(x)}{S(x)}$$

$$\sigma(x) = \frac{Px + M_0}{\frac{\pi b_A h_A^2 \left(1 + \frac{x}{3L}\right)^3}{32}}$$

$$\sigma(x) = 864 \left(\frac{Px + M_0}{\pi b_A h_A^2} \right) \left(\frac{L^3}{(3L + x)^3} \right)$$

$$\frac{d}{dx} \sigma(x) = 0 \quad \text{then solve for } x_{\max}$$

$$\frac{d}{dx} \left[864 \frac{Px + M_0}{\pi b_A h_A^2} \frac{L^3}{(3L + x)^3} \right] = 0$$

$$864 \frac{P}{\pi b_A h_A^2} \frac{L^3}{(3L + x)^3} - 2592 \frac{Px + M_0}{\pi b_A h_A^2} \frac{L^3}{(3L + x)^4} = 0$$

OR simplifying:

$$\left[-864L^3 \frac{-3PL + 2Px + 3M_0}{\pi b_A h_A^2 (3L + x)^4} \right] = 0$$

$$\text{so } x_{\max} = \frac{3(PL - M_0)}{2P}$$

$$x_{\max} = 0.625 \text{ m} \quad \leftarrow$$

agrees with plot above

Evaluate using numerical data:

$$\sigma_{\max} = \sigma(x_{\max}) \quad \sigma_{\max} = 231 \text{ MPa} \quad \leftarrow$$

$$\sigma_A = \sigma(0) \quad \sigma_A = 210 \text{ MPa} \quad \leftarrow$$

$$\sigma_B = \sigma(L) \quad \sigma_B = 221 \text{ MPa} \quad \leftarrow$$

$$\frac{\sigma_{\max}}{\sigma_B} = 1.045$$

(e) FIND MAX. BENDING STRESS INCLUDING UNIFORM LOAD

$$L = 1.25 \text{ m} \quad b_A = 60 \text{ mm} \quad h_A = 90 \text{ mm}$$

$$b_B = 80 \text{ mm} \quad h_B = 120 \text{ mm}$$

$$P = 12 \text{ kN} \quad M_0 = 10 \text{ kN} \cdot \text{m}$$

$$h(x) = h_A \left(1 + \frac{x}{3L} \right)$$

$$b(x) = b_A \left(1 + \frac{x}{3L} \right)$$

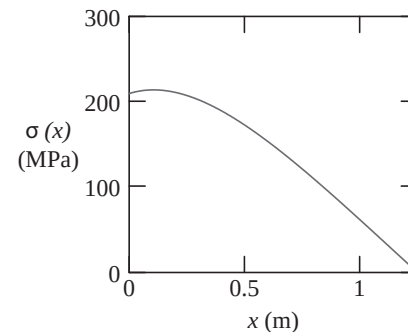
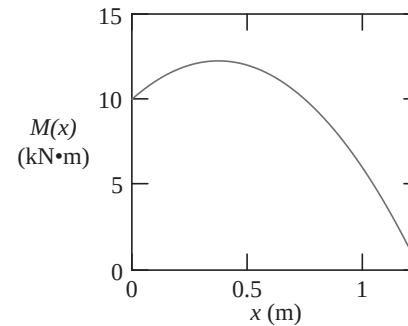
$$I(x) = \frac{\pi b(x) h(x)^3}{64} \quad S(x) = \frac{I(x)}{\frac{h(x)}{2}}$$

$$S(x) = \frac{\pi b(x) h(x)^2}{32}$$

$$S(x) = \frac{\pi b_A h_A^2 \left(1 + \frac{x}{3L} \right)^3}{32}$$

$$M(x) = Px + M_0 - \frac{10}{3} \frac{P}{L} \frac{x^2}{2}$$

$$\sigma(x) = \frac{M(x)}{S(x)}$$



$$\sigma(x) = \frac{Px + M_0 - \frac{10}{3} \frac{P}{L} \frac{x^2}{2}}{\left[\frac{\pi b_A h_A^2 \left(1 + \frac{x}{3L} \right)^3}{32} \right]}$$

$$\sigma(x) = -288 \left(-3PxL - 3M_0L + 5Px^2 \right) \frac{L^2}{\pi b_A h_A^2 (3L + x)^3}$$

$$\frac{d}{dx} \sigma(x) = 0 \quad \text{then solve for } x_{\max}$$

$$\frac{d}{dx} \left[-288 (-3PxL - 3M_0L + 5Px^2) \times \frac{L^2}{\pi b_A h_A^2 (3L + x)^3} \right] = 0$$

$$\frac{d}{dx} \sigma(x) = \left[(864PL - 2880Px) \frac{L^2}{\pi b_A h_A^2 (3L + x)^3} - 3(864PxL + 864M_0L - 1440Px^2) \times \frac{L^2}{\pi b_A h_A^2 (3L + x)^4} \right] = 0$$

OR simplifying:

$$(288L^2) \frac{[9PL^2 - 36PxL + 5Px^2 - 9M_0L]}{[\pi b_A h_A^2 (3L + x)^4]} = 0$$

OR

$$9PL^2 - 36PxL + 5Px^2 - 9M_0L = 0$$

Solving for $x_{\max}$: $x_{\max} = 0.105 \text{ m}$

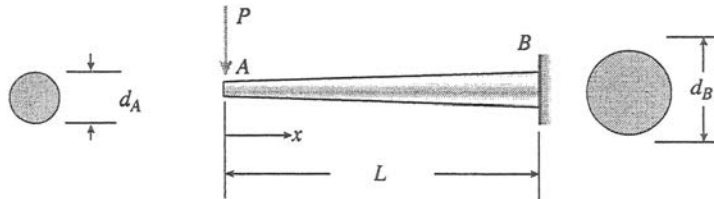
Solution agrees with plot above; evaluate using numerical data.

$$\begin{array}{lll} \sigma_{\max} = \sigma(x_{\max}) & \sigma_{\max} = 214 \text{ MPa} & \leftarrow \\ \sigma_A = \sigma(0) & \sigma_A = 210 \text{ MPa} & \leftarrow \\ \sigma_B = \sigma(L) & \sigma_B = 0 \text{ MPa} & \leftarrow \end{array}$$

Problem 5.7-5 Refer to the tapered cantilever beam of solid circular cross section shown in Fig. 5-24 of Example 5-9.

- Considering only the bending stresses due to the load P , determine the range of values of the ratio d_B/d_A for which the maximum normal stress occurs at the support.
- What is the maximum stress for this range of values?

Solution 5.7-5 Tapered cantilever beam



FROM EQ. (5-33), EXAMPLE 5-9

$$\sigma_1 = \frac{32Px}{\pi \left[d_A + (d_B - d_A) \left(\frac{x}{L} \right) \right]^3} \quad \text{Eq. (1)}$$

FIND THE VALUE OF x THAT MAKES σ_1 A MAXIMUM

$$\text{Let } \sigma_1 = \frac{u}{v} \quad \frac{d\sigma_1}{dx} = \frac{v \left(\frac{du}{dx} \right) - u \left(\frac{dv}{dx} \right)}{v^2} = \frac{N}{D}$$

$$N = \pi \left[d_A + (d_B - d_A) \left(\frac{x}{L} \right) \right]^3 [32P]$$

$$- [32Px] [\pi] [3] \left[d_A + (d_B - d_A) \left(\frac{x}{L} \right) \right]^2 \left[\frac{1}{L} (d_B - d_A) \right]$$

After simplification:

$$N = 32\pi P \left[d_A + (d_B - d_A) \left(\frac{x}{L} \right) \right]^2 \left[d_A - 2(d_B - d_A) \frac{x}{L} \right]$$

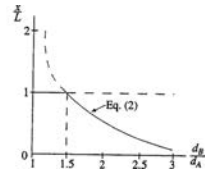
$$D = \pi^2 \left[d_A + (d_B - d_A) \frac{x}{L} \right]^6$$

$$\frac{d\sigma_1}{dx} = \frac{N}{D} = \frac{32P \left[d_A - 2(d_B - d_A) \frac{x}{L} \right]}{\pi \left[d_A + (d_B - d_A) \left(\frac{x}{L} \right) \right]^4}$$

$$\frac{d\sigma_1}{dx} = 0 \quad d_A - 2(d_B - d_A)\left(\frac{x}{L}\right) = 0$$

$$\therefore \frac{x}{L} = \frac{d_A}{2(d_B - d_A)} = \frac{1}{2\left(\frac{d_B}{d_A} - 1\right)} \quad \text{Eq. (2)}$$

(a) GRAPH OF x/L VERSUS d_B/d_A (EQ. 2)



Maximum bending stress occurs at the support when

$$1 \leq \frac{d_B}{d_A} \leq 1.5 \quad \leftarrow$$

(b) MAXIMUM STRESS (AT SUPPORT B)
Substitute $x/L = 1$ into Eq. (1):

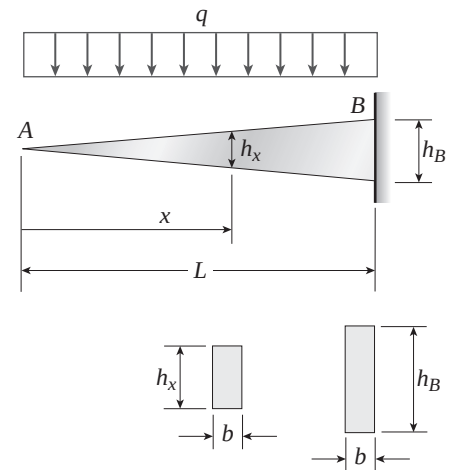
$$\sigma_{\max} = \frac{32PL}{\pi d_B^3} \quad \leftarrow$$

Fully Stressed Beams

Problems 5.7-6 to 5.7-8 pertain to fully stressed beams of rectangular cross section. Consider only the bending stresses obtained from the flexure formula and disregard the weights of the beams.

Problem 5.7-6 A cantilever beam AB having rectangular cross sections with constant width b and varying height h_x is subjected to a uniform load of intensity q (see figure).

How should the height h_x vary as a function of x (measured from the free end of the beam) in order to have a fully stressed beam? (Express h_x in terms of the height h_B at the fixed end of the beam.)



Solution 5.7-6 Fully stressed beam with constant width and varying height

h_x = height at distance x

h_B = height at end B

b = width (constant)

$$\text{AT DISTANCE } x: M = \frac{qx^2}{2} \quad S = \frac{bh_x^2}{6}$$

$$\sigma_{\text{allow}} = \frac{M}{S} = \frac{3qx^2}{bh_x^2}$$

$$h_x = x \sqrt{\frac{3q}{b\sigma_{\text{allow}}}}$$

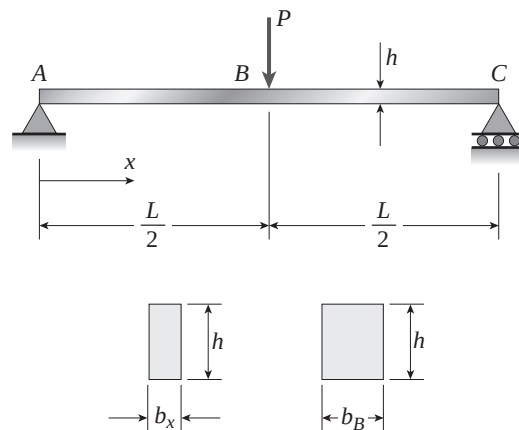
AT THE FIXED END ($x = L$):

$$h_B = L \sqrt{\frac{3q}{b\sigma_{\text{allow}}}}$$

$$\text{Therefore, } \frac{h_x}{h_B} = \frac{x}{L} \quad h_x = \frac{h_B x}{L} \quad \leftarrow$$

Problem 5.7-7 A simple beam ABC having rectangular cross sections with constant height h and varying width b_x supports a concentrated load P acting at the midpoint (see figure).

How should the width b_x vary as a function of x in order to have a fully stressed beam? (Express b_x in terms of the width b_B at the midpoint of the beam.)



Solution 5.7-7 Fully stressed beam with constant height and varying width

h = height of beam (constant)

b_x = width at distance x from end A ($0 \leq x \leq \frac{L}{2}$)

b_B = width at midpoint B ($x = L/2$)

AT DISTANCE x $M = \frac{Px}{2}$ $S = \frac{1}{6}b_x h^2$

$$\sigma_{\text{allow}} = \frac{M}{S} = \frac{3Px}{b_x h^2} \quad b_x = \frac{3Px}{\sigma_{\text{allow}} h^2}$$

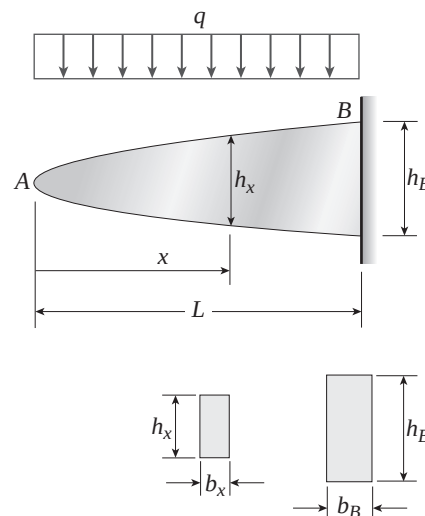
AT MIDPOINT B ($x = L/2$)

$$b_B = \frac{3PL}{2\sigma_{\text{allow}} h^2}$$

Therefore, $\frac{b_x}{b_B} = \frac{2x}{L}$ and $b_x = \frac{2b_B x}{L}$ ←

NOTE: The equation is valid for $0 \leq x \leq \frac{L}{2}$ and the beam is symmetrical about the midpoint.

Problem 5.7-8 A cantilever beam AB having rectangular cross sections with varying width b_x and varying height h_x is subjected to a uniform load of intensity q (see figure). If the width varies linearly with x according to the equation $b_x = b_B x/L$, how should the height h_x vary as a function of x in order to have a fully stressed beam? (Express h_x in terms of the height h_B at the fixed end of the beam.)



Solution 5.7-8 Fully stressed beam with varying width and varying height h_x = height at distance x h_B = height at end B b_x = width at distance x b_B = width at end B

$$h_x = \sqrt{\frac{3qLx}{b_B\sigma_{\text{allow}}}}$$

AT THE FIXED END ($x = L$)

$$h_B = \sqrt{\frac{3qL^2}{b_B\sigma_{\text{allow}}}}$$

$$\text{Therefore, } \frac{h_x}{h_B} = \sqrt{\frac{x}{L}} \quad h_x = h_B\sqrt{\frac{x}{L}} \quad \leftarrow$$

$$b_x = b_B\left(\frac{x}{L}\right)$$

AT DISTANCE x

$$M = \frac{qx^2}{2} \quad S = \frac{b_x h_x^2}{6} = \frac{b_B x}{6L} (h_x)^2$$

$$\sigma_{\text{allow}} = \frac{M}{S} = \frac{3qLx}{b_B h_x^2}$$

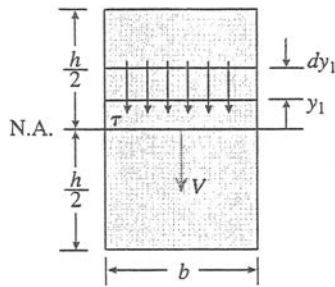
Shear Stresses in Rectangular Beams

Problem 5.8-1 The shear stresses τ in a rectangular beam are given by Eq. (5-43):

$$\tau = \frac{V}{2I} \left(\frac{h^2}{4} - y_1^2 \right)$$

in which V is the shear force, I is the moment of inertia of the cross-sectional area, h is the height of the beam, and y_1 is the distance from the neutral axis to the point where the shear stress is being determined (Fig. 5-30).

By integrating over the cross-sectional area, show that the resultant of the shear stresses is equal to the shear force V .

Solution 5.8-1 Resultant of the shear stresses

$$I = \frac{bh^3}{12}$$

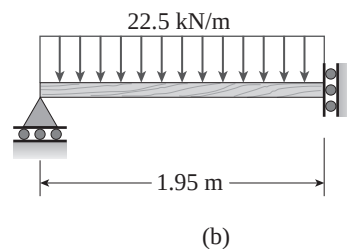
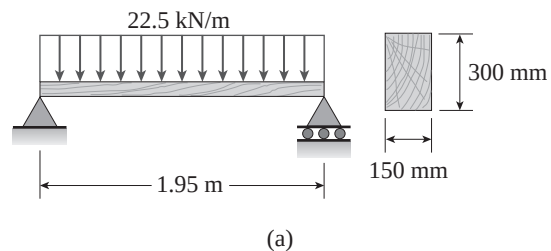
$$\tau = \frac{V}{2I} \left(\frac{h^2}{4} - y_1^2 \right)$$

 V = shear force acting on the cross section R = resultant of shear stresses τ

$$\begin{aligned} R &= \int_{-h/2}^{h/2} \tau b dy_1 = 2 \int_0^{h/2} \frac{V}{2I} \left(\frac{h^2}{4} - y_1^2 \right) b dy_1 \\ &= \frac{12V}{bh^3} (b) \int_0^{h/2} \left(\frac{h^2}{4} - y_1^2 \right) dy_1 \\ &= \frac{12V}{h^3} \left(\frac{2h^3}{24} \right) = V \end{aligned}$$

 $\therefore R = V$ Q.E.D. $\leftarrow$

Problem 5.8-2 Calculate the maximum shear stress $\tau_{\max}$ and the maximum bending stress $\sigma_{\max}$ in a wood beam (see figure) carrying a uniform load of 22.5 kN/m (which includes the weight of the beam) if the length is 1.95 m and the cross section is rectangular with width 150 mm and height 300 mm, and the beam is either (a) simply supported as in the figure part a, or (b) has a sliding support at right as in the figure part b.



Solution 5.8-2

$$q = 22.5 \text{ kN/m} \quad b = 150 \text{ mm}$$

$$h = 300 \text{ mm} \quad L = 1.95 \text{ m}$$

(a) MAXIMUM SHEAR STRESS

$$V = \frac{qL}{2} \quad A = bh$$

$$\tau_{\max} = \frac{3V}{2A} \quad \tau_{\max} = 731 \text{ kPa} \quad \leftarrow$$

MAXIMUM BENDING STRESS

$$M = \frac{qL^2}{8} \quad S = \frac{bh^2}{6}$$

$$\sigma_{\max} = \frac{M}{S} \quad \sigma_{\max} = 4.75 \text{ MPa} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESS

$$V = qL$$

$$\tau_{\max} = \frac{3V}{2A} \quad \tau_{\max} = 1462 \text{ kPa} \quad \leftarrow$$

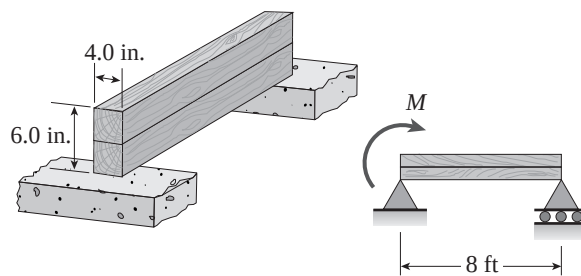
MAXIMUM BENDING STRESS

$$M = \frac{qL^2}{2}$$

$$\sigma_{\max} = \frac{M}{S} \quad \sigma_{\max} = 19.01 \text{ MPa} \quad \leftarrow$$

Problem 5.8-3 Two wood beams, each of rectangular cross section (3.0 in. $\times$ 4.0 in., actual dimensions) are glued together to form a solid beam of dimensions 6.0 in. $\times$ 4.0 in. (see figure). The beam is simply supported with a span of 8 ft.

- What is the maximum moment $M_{\max}$ that may be applied at the left support if the allowable shear stress in the glued joint is 200 psi? (Include the effects of the beam's own weight, assuming that the wood weighs 35 lb/ft³.)
- Repeat part (a) if $M_{\max}$ is based on allowable bending stress of 2500 psi.



Solution 5.8-3

NUMERICAL DATA

$$L = 8 \text{ ft} \quad b = 4 \text{ in.} \quad h = 6 \text{ in.} \quad \tau_{\text{allow}} = 200 \text{ psi} \quad \sigma_a = 2500 \text{ psi}$$

$$A = bh \quad \gamma = 35 \frac{\text{lb}}{\text{ft}^3} \quad q = \gamma A \quad \text{weight of beam per unit distance} \quad q = 5.833 \frac{\text{lb}}{\text{ft}}$$

(a) MAXIMUM LOAD M_{max} BASED ON SHEAR ALONE

$$V = \frac{M}{L} + \frac{qL}{2} \quad \tau_{\text{max}} = \frac{3V}{2A} = \frac{3}{2A} \left(\frac{M}{L} + \frac{qL}{2} \right) \quad M = \frac{2AL}{3} \tau_{\text{max}} - \frac{qL^2}{2}$$

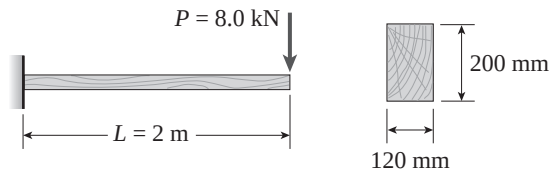
$$M_{\text{max}} = \frac{2AL}{3} \tau_{\text{allow}} - \frac{qL^2}{2} \quad \boxed{M_{\text{max}} = 25.4 \text{ k-ft}} < \text{based on allowable shear}$$

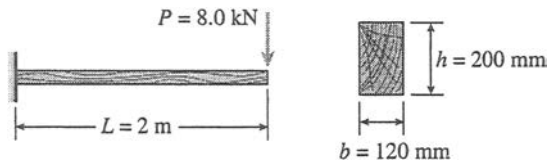
(b) MAXIMUM LOAD M_{max} BASED ON BENDING STRESS $M_m = \sigma_a S \quad S = \frac{bh^2}{6} \quad M_m = M + \frac{qL^2}{8}$

$$M_{\text{max}} = \sigma_a S - \frac{qL^2}{8} = 4.95 \text{ kip-ft} \quad \boxed{M_{\text{max}} = 4.95 \text{ kip-ft}} < \text{bending controls for this beam}$$

Problem 5.8-4 A cantilever beam of length $L = 2 \text{ m}$ supports a load $P = 8.0 \text{ kN}$ (see figure). The beam is made of wood with cross-sectional dimensions $120 \text{ mm} \times 200 \text{ mm}$.

Calculate the shear stresses due to the load P at points located 25 mm, 50 mm, 75 mm, and 100 mm from the top surface of the beam. From these results, plot a graph showing the distribution of shear stresses from top to bottom of the beam.



Solution 5.8-4 Shear stresses in a cantilever beam

$$\text{Eq. (5-43): } \tau = \frac{V}{2I} \left(\frac{h^2}{4} - y_1^2 \right)$$

$$V = P = 8.0 \text{ kN} = 8,000 \text{ N}$$

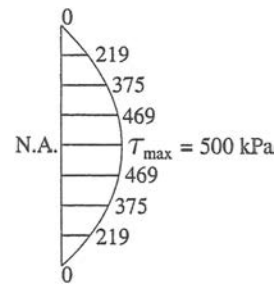
$$I = \frac{bh^3}{12} = 80 \times 10^6 \text{ mm}^4$$

$$h = 200 \text{ mm} \quad (y_1 = \text{mm})$$

$$\tau = \frac{8,000}{2(80 \times 10^6)} \left[\frac{(200)^2}{4} - y_1^2 \right] \quad (\tau = \text{N/mm}^2 = \text{MPa})$$

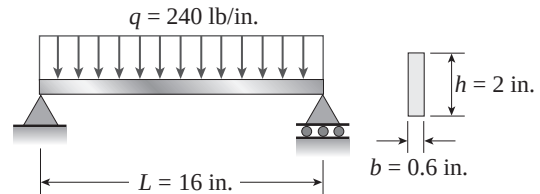
$$\tau = 50 \times 10^{-6} (10,000 - y_1^2) \quad (y_1 = \text{mm}; \tau = \text{MPa})$$

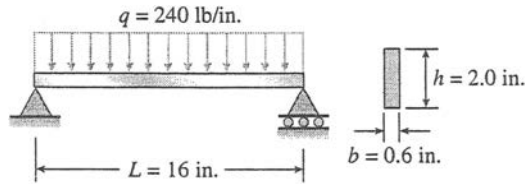
Distance from the top surface (mm)	y_1 (mm)	τ (MPa)	τ (kPa)
0	100	0	0
25	75	0.219	219
50	50	0.375	375
75	25	0.469	469
100 (N.A.)	0	0.500	500

GRAPH OF SHEAR STRESS τ 

Problem 5.8-5 A steel beam of length $L = 16$ in. and cross-sectional dimensions $b = 0.6$ in. and $h = 2$ in. (see figure) supports a uniform load of intensity $q = 240$ lb/in., which includes the weight of the beam.

Calculate the shear stresses in the beam (at the cross section of maximum shear force) at points located $1/4$ in., $1/2$ in., $3/4$ in., and 1 in. from the top surface of the beam. From these calculations, plot a graph showing the distribution of shear stresses from top to bottom of the beam.



Solution 5.8-5 Shear stresses in a simple beam

$$\text{Eq. (5-43): } \tau = \frac{V}{2I} \left(\frac{h^2}{4} - y_1^2 \right)$$

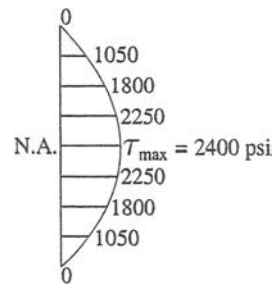
$$V = \frac{qL}{2} = 1920 \text{ lb} \quad I = \frac{bh^3}{12} = 0.4 \text{ in.}^4$$

UNITS: POUNDS AND INCHES

$$\tau = \frac{1920}{2(0.4)} \left[\frac{(2)^2}{4} - y_1^2 \right] = (2400)(1 - y_1^2)$$

$$(\tau = \text{psi}; y_1 = \text{in.})$$

Distance from the top surface (in.)	y_1 (in.)	τ (psi)
0	1.00	0
0.25	0.75	1050
0.50	0.50	1800
0.75	0.25	2250
1.00 (N.A.)	0	2400

GRAPH OF SHEAR STRESS τ 

Problem 5.8-6 A beam of rectangular cross section (width b and height h) supports a uniformly distributed load along its entire length L . The allowable stresses in bending and shear are σ_{allow} and τ_{allow} , respectively.

- If the beam is simply supported, what is the span length L_0 below which the shear stress governs the allowable load and above which the bending stress governs?
- If the beam is supported as a cantilever, what is the length L_0 below which the shear stress governs the allowable load and above which the bending stress governs?

Solution 5.8-6 Beam of rectangular cross section b = width h = height L = lengthUniform load q = intensity of loadALLOWABLE STRESSES σ_{allow} and τ_{allow}

(a) SIMPLE BEAM

BENDING

$$M_{\text{max}} = \frac{qL^2}{8} \quad S = \frac{bh^2}{6}$$

$$\sigma_{\text{max}} = \frac{M_{\text{max}}}{S} = \frac{3qL^2}{4bh^2}$$

$$q_{\text{allow}} = \frac{4\sigma_{\text{allow}} bh^2}{3L^2} \quad (1)$$

SHEAR

$$V_{\text{max}} = \frac{qL}{2} \quad A = bh$$

$$\tau_{\text{max}} = \frac{3V}{2A} = \frac{3qL}{4bh}$$

$$q_{\text{allow}} = \frac{4\tau_{\text{allow}} bh}{3L} \quad (2)$$

Equate (1) and (2) and solve for L_0 :

$$L_0 = h \left(\frac{\sigma_{\text{allow}}}{\tau_{\text{allow}}} \right) \leftarrow$$

(b) CANTILEVER BEAM

BENDING

$$M_{\max} = \frac{qL^2}{2} \quad S = \frac{bh^2}{6}$$

$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{3qL^2}{bh^2}$$

$$q_{\text{allow}} = \frac{\sigma_{\text{allow}} bh^2}{3L^2} \quad (3)$$

SHEAR

$$V_{\max} = qL \quad A = bh$$

$$\tau_{\max} = \frac{3V}{2A} = \frac{3qL}{2bh}$$

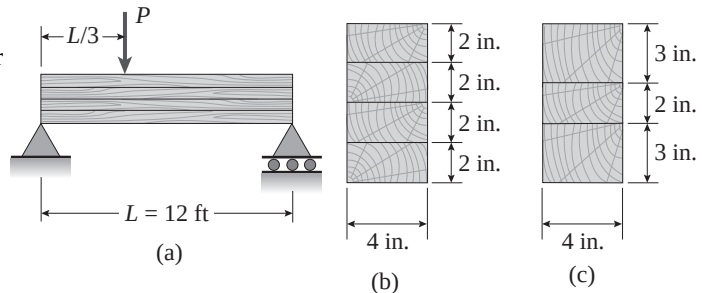
$$q_{\text{allow}} = \frac{2\tau_{\text{allow}} bh}{3L} \quad (4)$$

Equate (3) and (4) and solve for L_0 :

$$L_0 = \frac{h}{2} \left(\frac{\sigma_{\text{allow}}}{\tau_{\text{allow}}} \right) \quad \leftarrow$$

NOTE: If the actual length is less than L_0 , the shear stress governs the design. If the length is greater than L_0 , the bending stress governs.

Problem 5.8-7 A laminated wood beam on simple supports (figure part a) is built up by gluing together four 2 in. $\times$ 4 in. boards (actual dimensions) to form a solid beam 4 in. $\times$ 8 in. in cross section, as shown in figure part b. The allowable shear stress in the glued joints is 62 psi, the allowable shear stress in the wood is 175 psi, and the allowable bending stress in the wood is 1650 psi.



- (a) If the beam is 12 ft long, what is the allowable load P acting at the one-third point along the beam, as shown? (Include the effects of the beam's own weight, assuming that the wood weighs 35 lb/ft³.)
- (b) Repeat part (a) if the beam is assembled by gluing together two 3 in. $\times$ 4 in. boards and a 2 in. $\times$ 4 in. board (see figure part c).

Solution 5.8-7

(a) $L = 12 \text{ ft}$ $b = 4 \text{ in.}$ $h = (2 \text{ in.})4$ $\tau_{ag} = 62 \text{ psi}$ $\tau_{aw} = 175 \text{ psi}$ $\sigma_{aw} = 1650 \text{ psi}$

$$\gamma_w = 35 \frac{\text{lb}}{\text{ft}^3} \quad A = bh = 32 \text{ in.}^2$$

$P_{\max}$ BASED ON ALLOWABLE SHEAR IN GLUE AT NA (CONTROLS OVER SHEAR IN ADJACENT WOOD)

$$V_{\max} = \frac{2}{3}P + \gamma_w A \frac{L}{2} \quad \text{and} \quad \tau_{\max} = \frac{3V_{\max}}{2A} \quad \text{so} \quad V_{\max} = \frac{2A}{3} \tau_{ag}$$

$$P_{\max} = \frac{3}{2} \left(\frac{2A}{3} \tau_{ag} - \gamma_w A \frac{L}{2} \right) = A \tau_{ag} - \frac{3AL\gamma_w}{4} \quad \boxed{P_{\max} = A \tau_{ag} - \frac{3AL\gamma_w}{4} = 1.914 \text{ kip}}$$

$P_{\max}$ BASED ON ALLOWABLE BENDING STRESS IN WOOD (AT TOP OR BOTTOM SURFACE); MAX. MOMENT IS AT LOCATION OF LOAD P

$$M_{\max} = \frac{2}{3}P \frac{L}{3} + \gamma_w A \frac{L}{2} \frac{L}{3} - \gamma_w A \frac{L}{3} \left(\frac{1}{2} \frac{L}{3} \right) \quad \text{and} \quad M_{\max} = \sigma_{aw} S \quad S = \frac{b(h)^2}{6}$$

$$P_{\max} = \frac{9}{2L} \left[\sigma_{aw} S - \left[\gamma_w A \frac{L}{2} \frac{L}{3} - \gamma_w A \frac{L}{3} \left(\frac{1}{2} \frac{L}{3} \right) \right] \right] = 2.15 \text{ kip} < \text{so } P_{\max} \text{ based on shear in glued joint at NA controls}$$

(b) $P_{\max}$ BASED ON ALLOWABLE SHEAR IN GLUE AT 1 IN. ABOVE NA $b = 4$ in. $h = (3 \text{ in.})2 + 2 \text{ in.} = 8$ in. $A = bh$

$$Q = (3 \text{ in.} \cdot 4 \text{ in.}) \left(1 \text{ in.} + \frac{3}{2} \text{ in.} \right) = 30 \text{ in.}^3 \quad I = \frac{bh^3}{12} = 170.667 \text{ in.}^4 \quad \tau_{\max} = \frac{V_{\max} Q}{Ib}$$

$$V_{\max} = \tau_{ag} \frac{Ib}{Q} \quad \text{and} \quad V_{\max} = \frac{2}{3} P + \gamma_w A \frac{L}{2}$$

$$P_{\max} = \frac{3}{2} \left(\tau_{ag} \frac{Ib}{Q} - \gamma_w A \frac{L}{2} \right) = 2.05 \text{ kip} < \text{controls}$$

$$P_{\max} \text{ BASED ON ALLOWABLE SHEAR IN WOOD AT NA} \quad \tau_{\max} = \frac{3}{2} \frac{V_{\max}}{A}$$

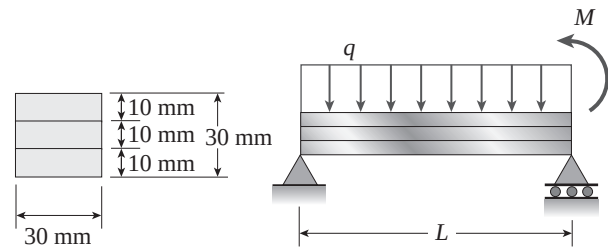
$$P_{\max} = \frac{3}{2} \left(\frac{2 \tau_{aw} A}{3} - \gamma_w A \frac{L}{2} \right) = 5.53 \text{ kip}$$

$P_{\max}$ BASED ON ALLOWABLE BENDING STRESS IN WOOD (AT TOP OR BOTTOM SURFACE); MAX. MOMENT IS AT LOCATION OF LOAD P

Same as in PART (A): $P_{\max} = 2.15 \text{ kip}$

Problem 5.8-8 A laminated plastic beam of square cross section is built up by gluing together three strips, each $10 \text{ mm} \times 30 \text{ mm}$ in cross section (see figure). The beam has a total weight of 3.6 N and is simply supported with span length $L = 360 \text{ mm}$.

Considering the weight of the beam (q) calculate the maximum permissible CCW moment M that may be placed at the right support.



- If the allowable shear stress in the glued joints is 0.3 MPa .
- If the allowable bending stress in the plastic is 8 MPa .

Solution 5.8-8

(a) FIND M BASED ON ALLOWABLE SHEAR STRESS IN GLUED JOINT

$$b = 30 \text{ mm} \quad h = 30 \text{ mm} \quad \tau_a = 0.3 \text{ MPa}$$

$$W = 3.6 \text{ N} \quad L = 360 \text{ mm}$$

$$q = \frac{W}{L}$$

$$q = 10 \frac{\text{N}}{\text{m}} \quad \text{beam distributed weight}$$

MAX. SHEAR AT LEFT SUPPORT

$$V_m = \frac{qL}{2} + \frac{M}{L} \quad \text{and} \quad V_m = \tau_a \left(\frac{Ib}{Q} \right)$$

$$\tau_a = \frac{V_m Q}{Ib} \quad I = \frac{bh^3}{12} \quad Ib = \frac{b^2 h^3}{12}$$

$$Q = \frac{bh}{3} \frac{h}{3} \quad Q = \frac{bh^2}{9} \quad \frac{Q}{Ib} = \frac{\frac{bh^2}{9}}{\frac{b^2 h^3}{12}}$$

$$\frac{Q}{Ib} = \frac{4}{3bh}$$

$$M = L \left[\tau_a \left(\frac{Ib}{Q} \right) - \frac{qL}{2} \right]$$

$$M = L \left[\tau_a \left(\frac{3bh}{4} \right) - \frac{qL}{2} \right]$$

$$M_{\max} = 72.2 \text{ N} \cdot \text{m} \quad \leftarrow$$

- (b) FIND M BASED ON ALLOWABLE BENDING STRESS AT $h/2$ FROM NA AT LOCATION (x_m) OF MAX. BENDING MOMENT, M_m

$$M(x) = \left(\frac{qL}{2} + \frac{M}{L} \right) x - \frac{qx^2}{2} \quad \frac{d}{dx} M(x) = 0$$

Use to find location of zero shear where max. moment occurs:

$$\begin{aligned} \frac{d}{dx} \left[\left(\frac{qL}{2} + \frac{M}{L} \right) x - \frac{qx^2}{2} \right] \\ = \frac{1}{2} qL + \frac{M}{L} - qx = 0 \end{aligned}$$

$$x_m = \frac{L}{2} + \frac{M}{qL}$$

MAX. MOMENT M_m

$$M_m = \left(\frac{qL}{2} + \frac{M}{L} \right) x_m - \frac{qx_m^2}{2}$$

$$\begin{aligned} M_m &= \left(\frac{qL}{2} + \frac{M}{L} \right) \left(\frac{L}{2} + \frac{M}{qL} \right) \\ &\quad - \frac{q \left(\frac{L}{2} + \frac{M}{qL} \right)^2}{2} \end{aligned}$$

Simplifying:

$$M_m = \frac{1}{8q} \frac{(qL^2 + 2M)^2}{L^2}$$

$$\text{Also } M_m = \sigma_a S \quad M_m = \sigma_a \left(\frac{bh^2}{6} \right)$$

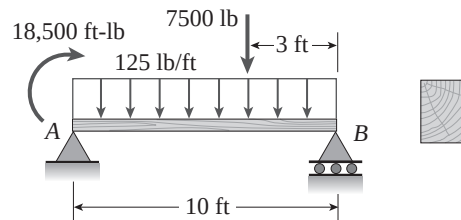
Equating both M_m expressions & solving for M where $\sigma_a = 8 \text{ MPa}$

$$M = \frac{\sqrt{\sigma_a \left(\frac{bh^2}{6} \right) (8qL^2)} - qL^2}{2}$$

$$M_{\max} = 9.01 \text{ N} \cdot \text{m} \quad \leftarrow$$

Problem 5.8-9 A wood beam AB on simple supports with span length equal to 10 ft is subjected to a uniform load of intensity 125 lb/ft acting along the entire length of the beam, a concentrated load of magnitude 7500 lb acting at a point 3 ft from the right-hand support, and a moment at A of 18,500 ft-lb (see figure). The allowable stresses in bending and shear, respectively, are 2250 psi and 160 psi.

- (a) From the table in Appendix G, select the lightest beam that will support the loads (disregard the weight of the beam).
 (b) Taking into account the weight of the beam (weight density = 35 lb/ft³), verify that the selected beam is satisfactory, or if it is not, select a new beam.



Solution 5.8-9

(a) $q = 125 \frac{\text{lb}}{\text{ft}} \quad P = 7500 \text{ lb} \quad M = 18,500 \text{ ft-lb}$

$$L = 10 \text{ ft} \quad d = 3 \text{ ft}$$

$$\sigma_{\text{allow}} = 2250 \text{ psi} \quad \tau_{\text{allow}} = 160 \text{ psi}$$

$$R_A = \frac{qL}{2} + P \frac{d}{L} - \frac{M}{L}$$

$$R_A = 1.025 \times 10^3 \text{ lb}$$

$$R_B = \frac{qL}{2} + P \frac{L-d}{L} + \frac{M}{L}$$

$$R_B = 7.725 \times 10^3 \text{ lb}$$

$$V_{\max} = R_B \quad V_{\max} = 7.725 \times 10^3 \text{ lb}$$

$$M_{\max} = R_B d - \frac{qd^2}{2}$$

$$M_{\max} = 2.261 \times 10^4 \text{ lb-ft}$$

$$\tau_{\max} = \frac{3V}{2A} \quad A_{\text{req}} = \frac{3V_{\max}}{2\tau_{\text{allow}}}$$

$$A_{\text{req}} = 72.422 \text{ in.}^2$$

$$\sigma_{\max} = \frac{M}{S} \quad S_{\text{req}} = \frac{M_{\max}}{\sigma_{\text{allow}}} \quad S_{\text{req}} = 120.6 \text{ in.}^3$$

From Appendix G: Select 8 × 12-in. beam (nominal dimensions). ←

$$A = 86.25 \text{ in.}^2 \quad S = 165.3 \text{ in.}^3$$

(b) REPEAT (A) CONSIDERING THE WEIGHT OF THE BEAM

$$\gamma = 35 \frac{\text{lb}}{\text{ft}^3} \quad q_{\text{beam}} = \gamma A$$

$$q_{\text{beam}} = 20.964 \frac{\text{lb}}{\text{ft}}$$

$$R_B = 7.725 \times 10^3 \text{ lb} + \frac{q_{\text{beam}} L}{2}$$

$$R_B = 7.83 \times 10^3 \text{ lb}$$

$$V_{\max} = R_B \quad A_{\text{req}} = \frac{3V_{\max}}{2\tau_{\text{allow}}}$$

$$A_{\text{req}} = 73.405 \text{ in.}^2 < A$$

8 × 12 beam is still satisfactory for shear.

$$q_{\text{total}} = q + q_{\text{beam}} \quad q_{\text{total}} = 145.964 \frac{\text{lb}}{\text{ft}}$$

$$M_{\max} = R_B d - \frac{q d^2}{2}$$

$$M_{\max} = 2.293 \times 10^4 \text{ lb-ft}$$

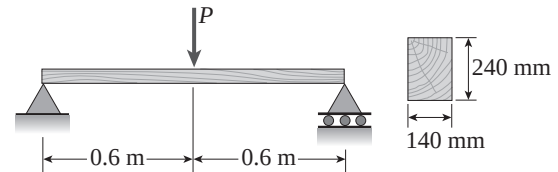
$$S_{\text{req}} = \frac{M_{\max}}{\sigma_{\text{allow}}} \quad S_{\text{req}} = 122.3 \text{ in.}^3 < S$$

8 × 12 beam is still satisfactory for moment.

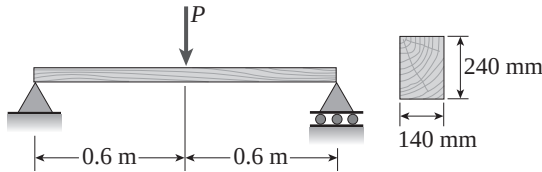
Use 8 × 12-in. beam ←

Problem 5.8-10 A simply supported wood beam of rectangular cross section and span length 1.2 m carries a concentrated load P at midspan in addition to its own weight (see figure). The cross section has width 140 mm and height 240 mm. The weight density of the wood is 5.4 kN/m³.

Calculate the maximum permissible value of the load P if (a) the allowable bending stress is 8.5 MPa, and (b) the allowable shear stress is 0.8 MPa.



Solution 5.8-10 Simply supported wood beam



$$b = 140 \text{ mm} \quad h = 240 \text{ mm}$$

$$A = bh = 33,600 \text{ mm}^2$$

$$S = \frac{bh^2}{6} = 1344 \times 10^3 \text{ mm}^3$$

$$\gamma = 5.4 \text{ kN/m}^3$$

$$L = 1.2 \text{ m} \quad q = \gamma bh = 181.44 \text{ N/m}$$

(a) ALLOWABLE P BASED UPON BENDING STRESS

$$\sigma_{\text{allow}} = 8.5 \text{ MPa} \quad \sigma = \frac{M_{\max}}{S}$$

$$M_{\max} = \frac{PL}{4} + \frac{qL^2}{8} = \frac{P(1.2 \text{ m})}{4} + \frac{(181.44 \text{ N/m})(1.2 \text{ m})^2}{8}$$

$$= 0.3P + 32.66 \text{ N} \cdot \text{m}$$

(P = newtons; M = N · m)

$$M_{\max} = S\sigma_{\text{allow}} = (1344 \times 10^3 \text{ mm}^3)(8.5 \text{ MPa}) = 11,424 \text{ N} \cdot \text{m}$$

Equate values of $M_{\max}$ and solve for P :

$$0.3P + 32.66 = 11,424 \quad P = 37,970 \text{ N}$$

$$\text{or } P = 38.0 \text{ kN} \quad \leftarrow$$

(b) ALLOWABLE LOAD P BASED UPON SHEAR STRESS

$$\tau_{\text{allow}} = 0.8 \text{ MPa} \quad \tau = \frac{3V}{2A}$$

$$V = \frac{P}{2} + \frac{qL}{2} = \frac{P}{2} + \frac{(181.44 \text{ N/m})(1.2 \text{ m})}{2}$$

$$= \frac{P}{2} + 108.86 \text{ (N)}$$

$$V = \frac{2A\tau}{3} = \frac{2}{3} (33,600 \text{ mm}^2)(0.8 \text{ MPa}) = 17,920 \text{ N}$$

Equate values of V and solve for P :

$$\frac{P}{2} + 108.86 = 17,920 \quad P = 35,622 \text{ N}$$

$$\text{or } P = 35.6 \text{ kN} \quad \leftarrow$$

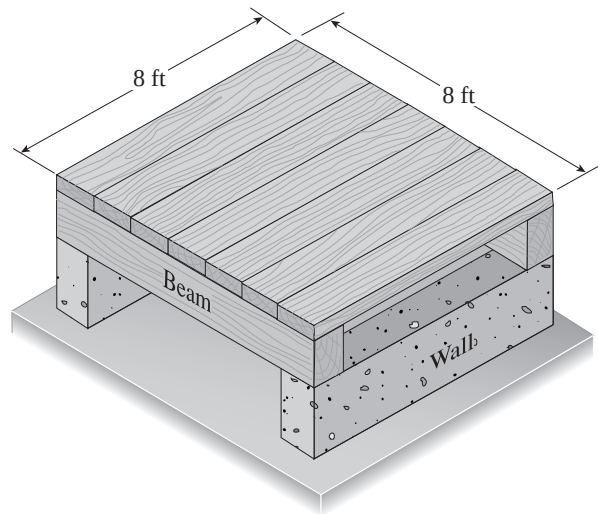
NOTE: The shear stress governs and
 $P_{\text{allow}} = 35.6 \text{ kN}$

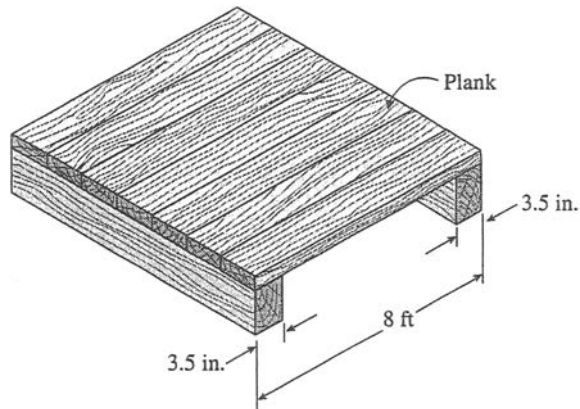
Problem 5.8-11 A square wood platform, $8 \text{ ft} \times 8 \text{ ft}$ in area, rests on masonry walls (see figure). The deck of the platform is constructed of 2-in. nominal thickness tongue-and-groove planks (actual thickness 1.5 in.; see Appendix G) supported on two 8-ft long beams. The beams have 4 in. $\times$ 6 in. nominal dimensions (actual dimensions 3.5 in. $\times$ 5.5 in.).

The planks are designed to support a uniformly distributed load w (lb/ft²) acting over the entire top surface of the platform. The allowable bending stress for the planks is 2400 psi and the allowable shear stress is 100 psi. When analyzing the planks, disregard their weights and assume that their reactions are uniformly distributed over the top surfaces of the supporting beams.

- Determine the allowable platform load w_1 (lb/ft²) based upon the bending stress in the planks.
- Determine the allowable platform load w_2 (lb/ft²) based upon the shear stress in the planks.
- Which of the preceding values becomes the allowable load w_{allow} on the platform?

(Hints: Use care in constructing the loading diagram for the planks, noting especially that the reactions are distributed loads instead of concentrated loads. Also, note that the maximum shear forces occur at the inside faces of the supporting beams.)



Solution 5.8-11 Wood platform with a plank deck

Platform: 8 ft $\times$ 8 ft

t = thickness of planks

= 1.5 in.

w = uniform load on the deck (lb/ft²)

$\sigma_{\text{allow}} = 2400$ psi

$\tau_{\text{allow}} = 100$ psi

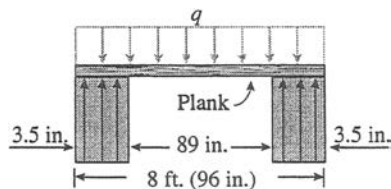
Find w_{allow} (lb/ft²).

- (a) ALLOWABLE LOAD BASED UPON BENDING STRESS IN THE PLANKS

Let b = width of one plank (in.)

$$\begin{aligned}
 A &= 1.5b \text{ (in.}^2\text{)} \\
 S &= \frac{b}{6} (1.5 \text{ in.})^2 \\
 &= 0.375b \text{ (in.}^3\text{)}
 \end{aligned}$$

Free-body diagram of one plank supported on the beams:



Load on one plank:

$$q = \left[\frac{w \text{ (lb/ft}^2\text{)}}{144 \text{ in.}^2/\text{ft}^2} \right] (b \text{ in.}) = \frac{wb}{144} \text{ (lb/in.)}$$

$$\begin{aligned}
 \text{Reaction } R &= q \left(\frac{96 \text{ in.}}{2} \right) = \left(\frac{wb}{144} \right) (48) = \frac{wb}{3} \\
 (R &= \text{lb}; w = \text{lb/ft}^2; b = \text{in.})
 \end{aligned}$$

M_{max} occurs at midspan.

$$\begin{aligned}
 M_{\text{max}} &= R \left(\frac{3.5 \text{ in.}}{2} + \frac{89 \text{ in.}}{2} \right) - \frac{q(48 \text{ in.})^2}{3} \\
 &= \frac{wb}{3} (46.25) - \frac{wb}{144} (1152) = \frac{89}{12} wb \\
 (M &= \text{lb-in.}; w = \text{lb/ft}^2; b = \text{in.})
 \end{aligned}$$

Allowable bending moment:

$$\begin{aligned}
 M_{\text{allow}} &= \sigma_{\text{allow}} S = (2400 \text{ psi})(0.375b) \\
 &= 900b \text{ (lb-in.)}
 \end{aligned}$$

Equate M_{max} and M_{allow} and solve for w :

$$\frac{89}{12} wb = 900b \quad w_1 = 121 \text{ lb/ft}^2 \quad \leftarrow$$

- (b) ALLOWABLE LOAD BASED UPON SHEAR STRESS IN THE PLANKS

See the free-body diagram in part (a).

V_{max} occurs at the inside face of the support.

$$\begin{aligned}
 V_{\text{max}} &= q \left(\frac{89 \text{ in.}}{2} \right) = 44.5q \\
 &= (44.5) \left(\frac{wb}{144} \right) = \frac{89wb}{288} \\
 (V &= \text{lb}; w = \text{lb/ft}^2; b = \text{in.})
 \end{aligned}$$

Allowable shear force:

$$\begin{aligned}
 \tau &= \frac{3V}{2A} \quad V_{\text{allow}} = \frac{2A\tau_{\text{allow}}}{3} \\
 &= \frac{2(1.5b)(100 \text{ psi})}{3} = 100b \text{ (lb)}
 \end{aligned}$$

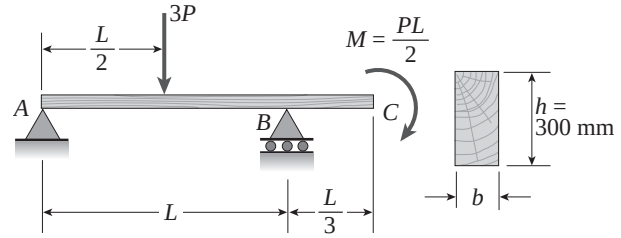
Equate V_{max} and V_{allow} and solve for w :

$$\frac{89wb}{288} = 100b \quad w_2 = 324 \text{ lb/ft}^2 \quad \leftarrow$$

- (c) ALLOWABLE LOAD

Bending stress governs. $w_{\text{allow}} = 121 \text{ lb/ft}^2 \quad \leftarrow$

Problem 5.8-12 A wood beam ABC with simple supports at A and B and an overhang BC has height $h = 300$ mm (see figure). The length of the main span of the beam is $L = 3.6$ m and the length of the overhang is $L/3 = 1.2$ m. The beam supports a concentrated load $3P = 18$ kN at the midpoint of the main span and a moment $PL/2 = 10.8$ kN · m at the free end of the overhang. The wood has weight density $\gamma = 5.5$ kN/m³.



- Determine the required width b of the beam based upon an allowable bending stress of 8.2 MPa.
- Determine the required width based upon an allowable shear stress of 0.7 MPa.

Solution 5.8-12

Numerical data:

$$L = 3.6 \text{ m} \quad h = 300 \text{ mm}$$

$$A = bh \quad P = 6 \text{ kN} \quad M = \frac{PL}{2}$$

$$\gamma = 5.5 \frac{\text{kN}}{\text{m}^3} \quad q_{\text{beam}} = \gamma A$$

Reactions, max. shear and moment equations:

$$R_A = \frac{3P}{2} - \frac{M}{L} + \frac{4}{9} q_{\text{beam}} L = P + \frac{4}{9} q_{\text{beam}} L$$

$$R_B = \frac{3P}{2} + \frac{M}{L} + \frac{5}{9} q_{\text{beam}} L = 2P + \frac{5}{9} q_{\text{beam}} L$$

$$V_{\text{max}} = R_B = 2P + \frac{5}{9} q_{\text{beam}} L$$

$$M_D = R_A \frac{L}{2} - q_{\text{beam}} \frac{L^2}{2} = \frac{PL}{2} + \frac{7}{72} q_{\text{beam}} L^2$$

(a) REQUIRED WIDTH b BASED UPON BENDING STRESS

$$\sigma_{\text{allow}} = 8.2 \text{ MPa}$$

$$M_{\text{max}} = M_D = \frac{PL}{2} + \frac{7}{72} q_{\text{beam}} L^2$$

$$\sigma = \frac{M_{\text{max}}}{S} = \frac{6M_{\text{max}}}{bh^2}$$

$$b = \frac{\frac{PL}{2}}{\left(\frac{-7}{72} \gamma h L^2 + \frac{\sigma_{\text{allow}} h^2}{6} \right)} \quad b = 89.3 \text{ mm}$$

(b) REQUIRED WIDTH b BASED UPON SHEAR STRESS

$$\tau_{\text{allow}} = 0.7 \text{ MPa}$$

$$V_{\text{max}} = 2P + \frac{5}{9} q_{\text{beam}} L$$

$$\tau = \frac{3V_{\text{max}}}{2A} = \frac{3V_{\text{max}}}{2bh}$$

$$= \frac{3}{2bh} \left(2P + \frac{5}{9} q_{\text{beam}} L \right) = \frac{3P}{bh} + \frac{5}{6} \gamma L$$

$$b = \frac{3P}{h \left(\tau_{\text{allow}} - \frac{5}{6} \gamma L \right)}$$

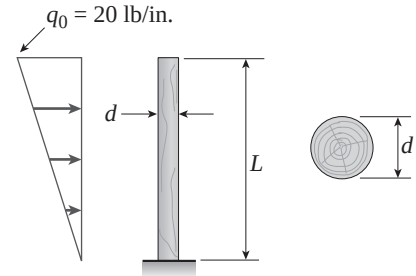
Shear stress governs.

$$b = 87.8 \text{ mm} \quad \leftarrow \text{(governs)}$$

Shear Stresses in Circular Beams

Problem 5.9-1 A wood pole of solid circular cross section (d = diameter) is subjected to a triangular distributed horizontal force of peak intensity $q_0 = 20$ lb/in (see figure). The length of the pole is $L = 6$ ft, and the allowable stresses in the wood are 1900 psi in bending and 120 psi in shear.

Determine the minimum required diameter of the pole based upon (a) the allowable bending stress, and (b) the allowable shear stress.



Solution 5.9-1

$$q = 20 \frac{\text{lb}}{\text{in.}} \quad L = 6 \text{ ft}$$

$$\sigma_{\text{allow}} = 1900 \text{ psi}$$

$$\tau_{\text{allow}} = 120 \text{ psi}$$

$$V_{\text{max}} = \frac{qL}{2} \quad V_{\text{max}} = 720 \text{ lb}$$

$$M_{\text{max}} = \frac{qL}{2} \frac{2L}{3} \quad M_{\text{max}} = 2.88 \times 10^3 \text{ lb-ft}$$

(a) BASED UPON BENDING STRESS

$$\sigma = \frac{M}{S} = \frac{32M}{\pi d^3}$$

$$d_{\text{min}} = \sqrt[3]{\frac{32M_{\text{max}}}{\pi \sigma_{\text{allow}}}}$$

$$d_{\text{min}} = 5.701 \text{ in.}$$

(b) BASED UPON SHEAR STRESS

$$\tau = \frac{4V}{3A} = \frac{16V}{3\pi d^2}$$

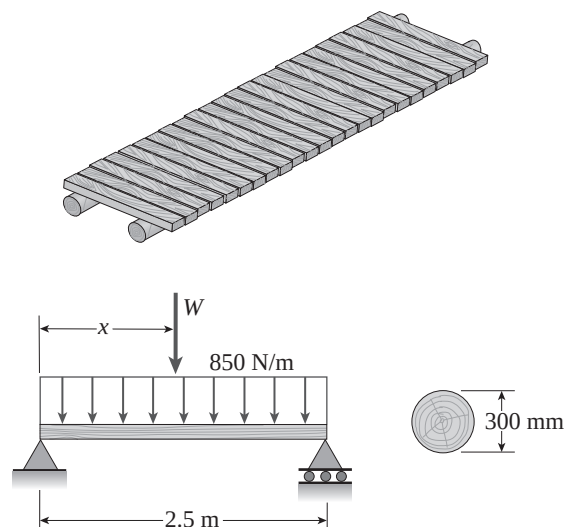
$$d_{\text{min}} = \sqrt{\frac{16V_{\text{max}}}{3\pi \tau_{\text{allow}}}} \quad d_{\text{min}} = 3.192 \text{ in.}$$

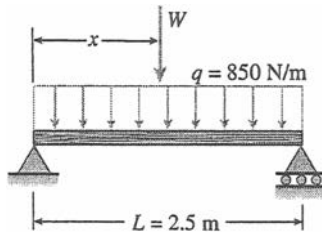
Bending stress governs. $d_{\text{min}} = 5.70 \text{ in.}$ ←

Problem 5.9-2 A simple log bridge in a remote area consists of two parallel logs with planks across them (see figure). The logs are Douglas fir with average diameter 300 mm. A truck moves slowly across the bridge, which spans 2.5 m. Assume that the weight of the truck is equally distributed between the two logs.

Because the wheelbase of the truck is greater than 2.5 m, only one set of wheels is on the bridge at a time. Thus, the wheel load on one log is equivalent to a concentrated load W acting at any position along the span. In addition, the weight of one log and the planks it supports is equivalent to a uniform load of 850 N/m acting on the log.

Determine the maximum permissible wheel load W based upon (a) an allowable bending stress of 7.0 MPa, and (b) an allowable shear stress of 0.75 MPa.



Solution 5.9-2 Log bridgeDiameter $d = 300$ mm $\sigma_{\text{allow}} = 7.0$ MPa $\tau_{\text{allow}} = 0.75$ MPaFind allowable load W .**(a) BASED UPON BENDING STRESS**

Maximum moment occurs when wheel is at midspan ($x = L/2$).

$$\begin{aligned}
 M_{\text{max}} &= \frac{WL}{4} + \frac{qL^2}{8} \\
 &= \frac{W}{4}(2.5 \text{ m}) + \frac{1}{8}(850 \text{ N/m})(2.5 \text{ m})^2 \\
 &= 0.625W + 664.1 \text{ (N} \cdot \text{m)} \quad (W = \text{newtons}) \\
 S &= \frac{\pi d^3}{32} = 2.651 \times 10^{-3} \text{ m}^3 \\
 M_{\text{max}} &= S \cdot \sigma_{\text{allow}} = (2.651 \times 10^{-3} \text{ m}^3)(7.0 \text{ MPa}) \\
 &= 18,560 \text{ N} \cdot \text{m} \\
 \therefore 0.625W + 664.1 &= 18,560 \\
 W &= 28,600 \text{ N} = 28.6 \text{ kN} \quad \leftarrow
 \end{aligned}$$

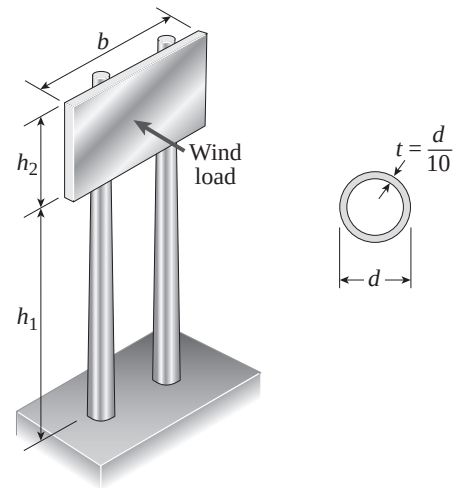
(b) BASED UPON SHEAR STRESS

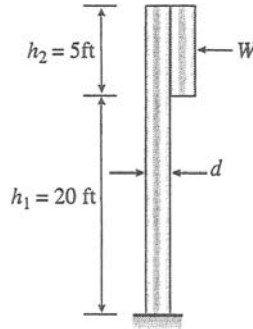
Maximum shear force occurs when wheel is adjacent to support ($x = 0$).

$$\begin{aligned}
 V_{\text{max}} &= W + \frac{qL}{2} = W + \frac{1}{2}(850 \text{ N/m})(2.5 \text{ m}) \\
 &= W + 1062.5 \text{ N} \quad (W = \text{newtons}) \\
 A &= \frac{\pi d^2}{4} = 0.070686 \text{ m}^2 \\
 \tau_{\text{max}} &= \frac{4V_{\text{max}}}{3A} \\
 V_{\text{max}} &= \frac{3A\tau_{\text{allow}}}{4} = \frac{3}{4}(0.070686 \text{ m}^2)(0.75 \text{ MPa}) \\
 &= 39,760 \text{ N} \\
 \therefore W + 1062.5 \text{ N} &= 39,760 \text{ N} \\
 W &= 38,700 \text{ N} = 38.7 \text{ kN} \quad \leftarrow
 \end{aligned}$$

Problem 5.9-3 A sign for an automobile service station is supported by two aluminum poles of hollow circular cross section, as shown in the figure. The poles are being designed to resist a wind pressure of 75 lb/ft^2 against the full area of the sign. The dimensions of the poles and sign are $h_1 = 20$ ft, $h_2 = 5$ ft, and $b = 10$ ft. To prevent buckling of the walls of the poles, the thickness t is specified as one-tenth the outside diameter d .

- Determine the minimum required diameter of the poles based upon an allowable bending stress of 7500 psi in the aluminum.
- Determine the minimum required diameter based upon an allowable shear stress of 2000 psi .



Solution 5.9-3 Wind load on a sign

$$\begin{aligned}
 b &= \text{width of sign} \\
 b &= 10 \text{ ft} \\
 p &= 75 \text{ lb/ft}^2 \\
 \sigma_{\text{allow}} &= 7500 \text{ psi} \\
 \tau_{\text{allow}} &= 2000 \text{ psi} \\
 d &= \text{diameter} \quad W = \text{wind force on one pole} \\
 t &= \frac{d}{10} \quad W = ph_2 \left(\frac{b}{2} \right) = 1875 \text{ lb}
 \end{aligned}$$

(a) REQUIRED DIAMETER BASED UPON BENDING STRESS

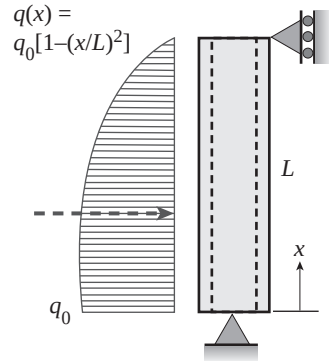
$$\begin{aligned}
 M_{\text{max}} &= W \left(h_1 + \frac{h_2}{2} \right) = 506,250 \text{ lb-in.} \\
 I &= \frac{\pi}{64} (d_2^4 - d_1^4) \quad d_2 = d \quad d_1 = d - 2t = \frac{4}{5}d \\
 I &= \frac{\pi}{64} \left[d^4 - \left(\frac{4d}{5} \right)^4 \right] = \frac{\pi d^4}{64} \left(\frac{369}{625} \right) \\
 &= \frac{369\pi d^4}{40,000} \text{ (in.}^4\text{)} \\
 c &= \frac{d}{2} \quad (d = \text{inches}) \\
 \sigma &= \frac{Mc}{I} = \frac{M(d/2)}{369\pi d^4/40,000} = \frac{17.253 M}{d^3} \\
 d^3 &= \frac{17.253 M_{\text{max}}}{\sigma_{\text{allow}}} = \frac{(17.253)(506,250 \text{ lb-in.})}{7500 \text{ psi}} \\
 &= 1164.6 \text{ in.}^3 \quad d = 10.52 \text{ in.} \quad \leftarrow
 \end{aligned}$$

(b) REQUIRED DIAMETER BASED UPON SHEAR STRESS

$$\begin{aligned}
 V_{\text{max}} &= W = 1875 \text{ lb} \\
 \tau &= \frac{4V}{3A} \left(\frac{r_2^2 + r_2r_1 + r_1^2}{r_2^2 + r_1^2} \right) \quad r_2 = \frac{d}{2} \\
 r_1 &= \frac{d}{2} - t = \frac{d}{2} - \frac{d}{10} = \frac{2d}{5} \\
 &\frac{r_2^2 + r_2r_1 + r_1^2}{r_2^2 + r_1^2} \\
 &= \frac{\left(\frac{d}{2} \right)^2 + \left(\frac{d}{2} \right) \left(\frac{2d}{5} \right) + \left(\frac{2d}{5} \right)^2}{\left(\frac{d}{2} \right)^2 + \left(\frac{2d}{5} \right)^2} = \frac{61}{41} \\
 A &= \frac{\pi}{4} (d_2^2 - d_1^2) = \frac{\pi}{4} \left[d^2 - \left(\frac{4d}{5} \right)^2 \right] = \frac{9\pi d^2}{100} \\
 \tau &= \frac{4V}{3} \left(\frac{61}{41} \right) \left(\frac{100}{9\pi d^2} \right) = 7.0160 \frac{V}{d^2} \\
 d^2 &= \frac{7.0160 V_{\text{max}}}{\tau_{\text{allow}}} \\
 &= \frac{(7.0160)(1875 \text{ lb})}{2000 \text{ psi}} = 6.5775 \text{ in.}^2 \\
 d &= 2.56 \text{ in.} \quad \leftarrow \\
 &\text{(Bending stress governs.)}
 \end{aligned}$$

Problem 5.9-4 A steel pipe is subjected to a quadratic distributed load over its height, with the peak intensity q_0 at the base (see figure). Assume the following pipe properties and dimensions: height L , outside diameter $d = 200$ mm, wall thickness $t = 10$ mm. Allowable stresses for flexure and shear are $\sigma_a = 125$ MPa and $\tau_a = 30$ MPa.

- (a) If $L = 2.6$ m, find $q_{0,\max}$ (kN/m), assuming that allowable flexure and shear stresses in the pipe are not to be exceeded.
 (b) If $q_0 = 60$ kN/m, find the maximum height $L_{\max}$ (m) of the pipe if the allowable flexure and shear stresses in the pipe are not to be exceeded.



Solution 5.9-4

(a) SUM MOMENTS ABOUT A TO GET R_B

$$R_B = \frac{1}{L} \left(\int_0^L q(x) x \, dx \right) \quad R_B = \frac{1}{L} \left[\int_0^L q_0 \left[1 - \left(\frac{x}{L} \right)^2 \right] x \, dx \right] \rightarrow \frac{L q_0}{4} \quad R_B = \frac{q_0 L}{4}$$

$$R_A = -R_B + \int_0^L q(x) \, dx \quad R_A = -R_B + \int_0^L q_0 \left[1 - \left(\frac{x}{L} \right)^2 \right] dx \rightarrow \frac{5L q_0}{12} < \text{this is } V_{\max}$$

Find x so that shear = 0 to find $M_{\max}$:

$$V(x) = R_A - \int_0^x q_0 \left[1 - \left(\frac{\zeta}{L} \right)^2 \right] d\zeta \rightarrow \frac{5L q_0}{12} - q_0 x + \frac{q_0 x^3}{3L^2} \quad V(x) = \frac{5L q_0}{12} - q_0 x + \frac{q_0 x^3}{3L^2}$$

$$\frac{5L q_0}{12} - q_0 x + \frac{q_0 x^3}{3L^2} = 0 \quad M_{\max} \text{ at } x = 0.446298 L$$

FIND EXPRESSION FOR $M(x)$

$$M(x) = R_A x - \int_0^x q_0 \left[1 - \left(\frac{\zeta}{L} \right)^2 \right] (x - \zeta) d\zeta \rightarrow \frac{q_0 x^4}{12L^2} - \frac{q_0 x^4}{2} + \frac{5L q_0 x}{12} \quad M(x) = \frac{q_0 x^4}{12L^2} - \frac{q_0 x^2}{2} + \frac{5L q_0 x}{12}$$

$$M_{\max} = M(0.446298L) = 0.089673L^2 q_0 < \text{this is } M_{\max}$$

FIND $q_{0,\max}$ BASED ON ALLOWABLE SHEAR STRESS $\tau_{\max} = \frac{4}{3} \frac{V_{\max}}{A} \left(\frac{r_1^2 + r_1 r_2 + r_2^2}{r_1^2 + r_2^2} \right)$

$$\tau_a = 30 \text{ MPa} \quad A = 5.969 \times 10^3 \text{ mm}^2 \quad L = 2.6 \text{ m}$$

$$q_1 = \tau_a \left(\frac{3A}{4} \right) \left(\frac{12}{5L} \right) \left(\frac{r_1^2 + r_2^2}{r_1^2 + r_1 r_2 + r_2^2} \right) = 82.801 \frac{\text{kN}}{\text{m}}$$

FIND $q_{0,\max}$ BASED ON ALLOWABLE FLEXURAL STRESS $\sigma_{\max} = \frac{M_{\max}}{S_z}$

$$q_2 = \frac{\sigma_a S_z}{0.089673L^2} = 55.7 \frac{\text{kN}}{\text{m}}$$

$$(b) \quad q_0 = 60 \frac{\text{kN}}{\text{m}}$$

FIND $L_{\max}$ BASED ON ALLOWABLE SHEAR STRESS

$$\tau_{\max} = \frac{4}{3A} \left(\frac{5Lq_0}{12} \right) \left(\frac{r_1^2 + r_1r_2 + r_2^2}{r_1^2 + r_2^2} \right) \quad \text{so} \quad L_1 = \tau_a \left(\frac{3A}{4} \right) \left(\frac{12}{5q_0} \right) \frac{r_1^2 + r_2^2}{r_1^2 + r_1r_2 + r_2^2} = 3.588 \text{ m}$$

FIND $L_{\max}$ BASED ON ALLOWABLE FLEXURAL STRESS

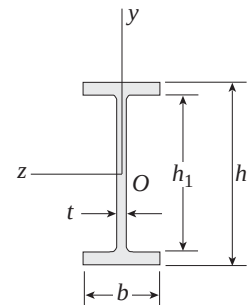
$$\sigma_{\max} = \frac{0.089673L^2q_0}{S_z} \quad \text{so} \quad L_2 = \sqrt{\frac{\sigma_a S_z}{0.089673q_0}} = 2.51 \text{ m}$$

Shear Stresses in Beams with Flanges

Problem 5.10-1 through 5.10-6 A wide-flange beam (see figure) having the cross section described below is subjected to a shear force V . Using the dimensions of the cross section, calculate the moment of inertia and then determine the following quantities:

- The maximum shear stress $\tau_{\max}$ in the web.
- The minimum shear stress $\tau_{\min}$ in the web.
- The average shear stress τ_{aver} (obtained by dividing the shear force by the area of the web) and the ratio $\tau_{\max}/\tau_{\text{aver}}$.
- The shear force V_{web} carried in the web and the ratio V_{web}/V .

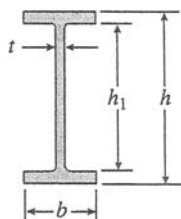
(Note: Disregard the fillets at the junctions of the web and flanges and determine all quantities, including the moment of inertia, by considering the cross section to consist of three rectangles.)



Probs 5.10-1 through 5.10-6

Problem 5.10-1 Dimensions of cross section: $b = 6 \text{ in.}$, $t = 0.5 \text{ in.}$, $h = 12 \text{ in.}$, $h_1 = 10.5 \text{ in.}$, and $V = 30 \text{ k.}$

Solution 5.10-1 Wide-flange beam



$$\begin{aligned} b &= 6.0 \text{ in.} \\ t &= 0.5 \text{ in.} \\ h &= 12.0 \text{ in.} \\ h_1 &= 10.5 \text{ in.} \\ V &= 30 \text{ k} \end{aligned}$$

MOMENT OF INERTIA (Eq. 5-52)

$$I = \frac{1}{12} (bh^3 - bh_1^3 + th_1^3) = 333.4 \text{ in.}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-53a)

$$\tau_{\max} = \frac{V}{8It} (bh^2 - bh_1^2 + th_1^2) = 5795 \text{ psi} \quad \leftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-53b)

$$\tau_{\min} = \frac{Vb}{8It} (h^2 - h_1^2) = 4555 \text{ psi} \quad \leftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-55)

$$\tau_{\text{aver}} = \frac{V}{th_1} = 5714 \text{ psi} \quad \leftarrow$$

$$\frac{\tau_{\max}}{\tau_{\text{aver}}} = 1.014 \quad \leftarrow$$

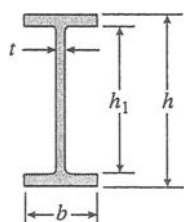
(d) SHEAR FORCE IN THE WEB (Eq. 5-54)

$$V_{\text{web}} = \frac{th_1}{3} (2\tau_{\max} + \tau_{\min}) = 28.25 \text{ k} \quad \leftarrow$$

$$\frac{V_{\text{web}}}{V} = 0.942 \quad \leftarrow$$

Problem 5.10-2 Dimensions of cross section: $b = 180$ mm, $t = 12$ mm, $h = 420$ mm, $h_1 = 380$ mm, and $V = 125$ kN.

Solution 5.10-2 Wide-flange beam



$b = 180$ mm
 $t = 12$ mm
 $h = 420$ mm
 $h_1 = 380$ mm
 $V = 125$ kN

MOMENT OF INERTIA (Eq. 5-52)

$$I = \frac{1}{12} (bh^3 - bh_1^3 + th_1^3) = 343.1 \times 10^6 \text{ mm}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-53a)

$$\tau_{\max} = \frac{V}{8It} (bh^2 - bh_1^2 + th_1^2) = 28.43 \text{ MPa} \quad \leftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-53b)

$$\tau_{\min} = \frac{Vb}{8It} (h^2 - h_1^2) = 21.86 \text{ MPa} \quad \leftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-55)

$$\tau_{\text{aver}} = \frac{V}{th_1} = 27.41 \text{ MPa} \quad \leftarrow$$

$$\frac{\tau_{\max}}{\tau_{\text{aver}}} = 1.037 \quad \leftarrow$$

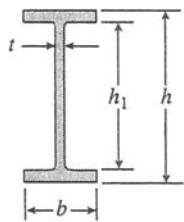
(d) SHEAR FORCE IN THE WEB (Eq. 5-54)

$$V_{\text{web}} = \frac{th_1}{3} (2\tau_{\max} + \tau_{\min}) = 119.7 \text{ kN} \quad \leftarrow$$

$$\frac{V_{\text{web}}}{V} = 0.957 \quad \leftarrow$$

Problem 5.10-3 Wide-flange shape, W 8 $\times$ 28 (see Table F-1(a), Appendix F); $V = 10$ k.

Solution 5.10-3 Wide-flange beam



W 8 $\times$ 28
 $b = 6.535$ in.
 $t = 0.285$ in.
 $h = 8.06$ in.
 $h_1 = 7.13$ in.
 $V = 10$ k

MOMENT OF INERTIA (Eq. 5-52)

$$I = \frac{1}{12} (bh^3 - bh_1^3 + th_1^3) = 96.36 \text{ in.}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-53a)

$$\tau_{\max} = \frac{V}{8It} (bh^2 - bh_1^2 + th_1^2) = 4861 \text{ psi} \quad \leftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-53b)

$$\tau_{\min} = \frac{Vb}{8It} (h^2 - h_1^2) = 4202 \text{ psi} \quad \leftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-55)

$$\tau_{\text{aver}} = \frac{V}{th_1} = 4921 \text{ psi} \quad \leftarrow$$

$$\frac{\tau_{\max}}{\tau_{\text{aver}}} = 0.988 \quad \leftarrow$$

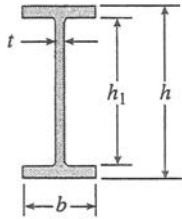
(d) SHEAR FORCE IN THE WEB (Eq. 5-54)

$$V_{\text{web}} = \frac{th_1}{3} (2\tau_{\max} + \tau_{\min}) = 9.432 \text{ k} \quad \leftarrow$$

$$\frac{V_{\text{web}}}{V} = 0.943 \quad \leftarrow$$

Problem 5.10-4 Dimensions of cross section: $b = 220$ mm, $t = 12$ mm, $h = 600$ mm, $h_1 = 570$ mm, and $V = 200$ kN.

Solution 5.10-4 Wide-flange beam



$$\begin{aligned} b &= 220 \text{ mm} \\ t &= 12 \text{ mm} \\ h &= 600 \text{ mm} \\ h_1 &= 570 \text{ mm} \\ V &= 200 \text{ kN} \end{aligned}$$

MOMENT OF INERTIA (Eq. 5-52)

$$I = \frac{1}{12} (bh^3 - bh_1^3 + th_1^3) = 750.0 \times 10^6 \text{ mm}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-53a)

$$\tau_{\max} = \frac{V}{8It} (bh^2 - bh_1^2 + th_1^2) = 32.28 \text{ MPa} \quad \leftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-53b)

$$\tau_{\min} = \frac{Vb}{8It} (h^2 - h_1^2) = 21.45 \text{ MPa} \quad \leftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-55)

$$\tau_{\text{aver}} = \frac{V}{th_1} = 29.24 \text{ MPa} \quad \leftarrow$$

$$\frac{\tau_{\max}}{\tau_{\text{aver}}} = 1.104$$

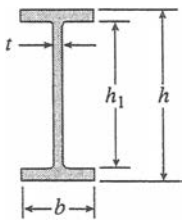
(d) SHEAR FORCE IN THE WEB (Eq. 5-54)

$$V_{\text{web}} = \frac{th_1}{3} (2\tau_{\max} + \tau_{\min}) = 196.1 \text{ kN} \quad \leftarrow$$

$$\frac{V_{\text{web}}}{V} = 0.981 \quad \leftarrow$$

Problem 5.10-5 Wide-flange shape, W 18 $\times$ 71 (see Table F-1(a), Appendix F); $V = 21$ k.

Solution 5.10-5 Wide-flange beam



$$\begin{aligned} \text{W } 18 \times 71 \\ b &= 7.635 \text{ in.} \\ t &= 0.495 \text{ in.} \\ h &= 18.47 \text{ in.} \\ h_1 &= 16.85 \text{ in.} \\ V &= 21 \text{ k} \end{aligned}$$

MOMENT OF INERTIA (Eq. 5-52)

$$I = \frac{1}{12} (bh^3 - bh_1^3 + th_1^3) = 1162 \text{ in.}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-53a)

$$\tau_{\max} = \frac{V}{8It} (bh^2 - bh_1^2 + th_1^2) = 2634 \text{ psi} \quad \leftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-53b)

$$\tau_{\min} = \frac{Vb}{8It} (h^2 - h_1^2) = 1993 \text{ psi} \quad \leftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-55)

$$\tau_{\text{aver}} = \frac{V}{th_1} = 2518 \text{ psi} \quad \leftarrow$$

$$\frac{\tau_{\max}}{\tau_{\text{aver}}} = 1.046 \quad \leftarrow$$

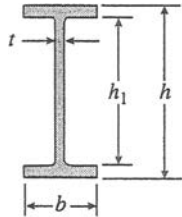
(d) SHEAR FORCE IN THE WEB (Eq. 5-54)

$$V_{\text{web}} = \frac{th_1}{3} (2\tau_{\max} + \tau_{\min}) = 20.19 \text{ k} \quad \leftarrow$$

$$\frac{V_{\text{web}}}{V} = 0.961 \quad \leftarrow$$

Problem 5.10-6 Dimensions of cross section: $b = 120$ mm, $t = 7$ mm, $h = 350$ mm, $h_1 = 330$ mm, and $V = 60$ kN.

Solution 5.10-6 Wide-flange beam



$$b = 120 \text{ mm}$$

$$t = 7 \text{ mm}$$

$$h = 350 \text{ mm}$$

$$h_1 = 330 \text{ mm}$$

$$V = 60 \text{ kN}$$

MOMENT OF INERTIA (Eq. 5-52)

$$I = \frac{1}{12} (bh^3 - bh_1^3 + th_1^3) = 90.34 \times 10^6 \text{ mm}^4$$

(a) MAXIMUM SHEAR STRESS IN THE WEB (Eq. 5-53a)

$$\tau_{\max} = \frac{V}{8It} (bh^2 - bh_1^2 + th_1^2) = 28.40 \text{ MPa} \quad \leftarrow$$

(b) MINIMUM SHEAR STRESS IN THE WEB (Eq. 5-53b)

$$\tau_{\min} = \frac{Vb}{8It} (h^2 - h_1^2) = 19.35 \text{ MPa} \quad \leftarrow$$

(c) AVERAGE SHEAR STRESS IN THE WEB (Eq. 5-55)

$$\tau_{\text{aver}} = \frac{V}{th_1} = 25.97 \text{ MPa} \quad \leftarrow$$

$$\frac{\tau_{\max}}{\tau_{\text{aver}}} = 1.093 \quad \leftarrow$$

(d) SHEAR FORCE IN THE WEB (Eq. 5-54)

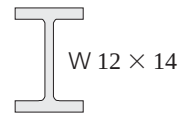
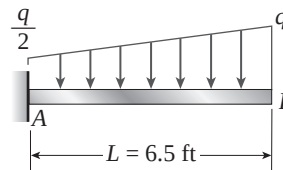
$$V_{\text{web}} = \frac{th_1}{3} (2\tau_{\max} + \tau_{\min}) = 58.63 \text{ kN} \quad \leftarrow$$

$$\frac{V_{\text{web}}}{V} = 0.977 \quad \leftarrow$$

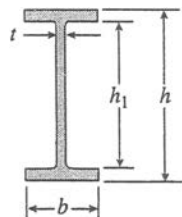
Problem 5.10-7 A cantilever beam AB of length $L = 6.5$ ft supports a trapezoidal distributed load of peak intensity q , and minimum intensity $q/2$, that includes the weight of the beam (see figure). The beam is a steel W 12 $\times$ 14 wide-flange shape (see Table F-1(a), Appendix F).

Calculate the maximum permissible load q based upon

(a) an allowable bending stress $\sigma_{\text{allow}} = 18$ ksi and (b) an allowable shear stress $\tau_{\text{allow}} = 7.5$ ksi. (Note: Obtain the moment of inertia and section modulus of the beam from Table F-1(a).)



Solution 5.10-7



$$b = 3.97 \text{ in.} \quad I = 88.6 \text{ in.}^4$$

$$t = 0.2 \text{ in.}$$

$$t_f = 0.225 \text{ in.}$$

$$S = 14.9 \text{ in.}^3$$

$$h = 11.9 \text{ in.}$$

$$h_1 = h - 2t_f$$

$$h_1 = 11.45 \text{ in.}$$

$$L = 6.5 \text{ ft} \quad \sigma_{\text{allow}} = 18 \text{ ksi} \quad \tau_{\text{allow}} = 7.5 \text{ ksi}$$

$$V_{\max} = \frac{\left(\frac{q}{2} + q\right)L}{2} \quad V_{\max} = \frac{3}{4} qL$$

$$M_{\max} = \frac{1}{2} \frac{q}{2} L^2 + \frac{1}{2} \frac{q}{2} L \frac{2L}{3}$$

$$M_{\max} = \frac{5}{12} qL^2$$

(a) MAXIMUM LOAD BASED UPON BENDING STRESS

$$\sigma = \frac{M}{S} = \frac{\frac{5}{12}qL^2}{S} \quad q = \frac{12S\sigma_{\text{allow}}}{5L^2}$$

$$q = 1270 \text{ lb/ft}$$

(b) MAXIMUM LOAD UPON SHEAR STRESS

$$\tau_{\text{max}} = \frac{V_{\text{max}}}{8It} (bh^2 - bh_1^2 + th_1^2)$$

$$= \frac{3qL}{32It} (bh^2 - bh_1^2 + th_1^2)$$

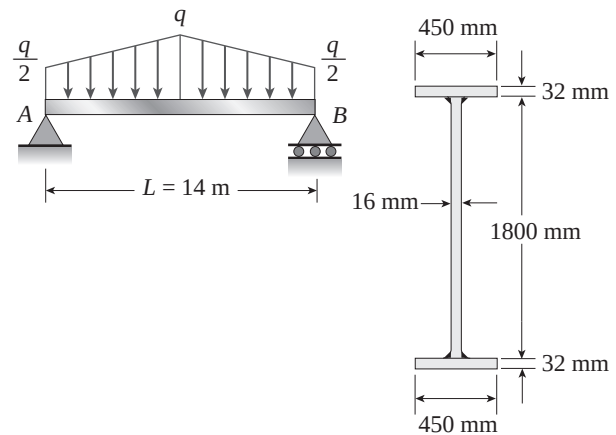
$$q = \frac{\tau_{\text{allow}} 32It}{3L(bh^2 - bh_1^2 + th_1^2)}$$

$$q = 3210 \frac{\text{lb}}{\text{ft}}$$

Shear stress governs. $q = 1270 \text{ lb/ft}$ ←

Problem 5.10-8 A bridge girder AB on a simple span of length $L = 14 \text{ m}$ supports a distributed load of maximum intensity q at mid-span and minimum intensity $q/2$ at supports A and B that includes the weight of the girder (see figure). The girder is constructed of three plates welded to form the cross section shown.

Determine the maximum permissible load q based upon
(a) an allowable bending stress $\sigma_{\text{allow}} = 110 \text{ MPa}$, and
(b) an allowable shear stress $\tau_{\text{allow}} = 50 \text{ MPa}$.

**Solution 5.10-8**

$$L = 14 \text{ m}$$

$$h = 1864 \text{ mm} \quad h_1 = 1800 \text{ mm}$$

$$b = 450 \text{ mm} \quad t_f = 32 \text{ mm} \quad t_w = 16 \text{ mm}$$

$$I = \frac{1}{12} (bh^3 - bh_1^3 + t_w h_1^3)$$

$$I = 3.194 \times 10^{10} \text{ mm}^4$$

$$S = \frac{2I}{h} \quad S = 3.427 \times 10^7 \text{ mm}^3$$

$$R_A = R_B = \frac{q}{2} \frac{L}{2} + \frac{q}{4} \frac{L}{2} = \frac{3}{8} qL$$

(a) MAXIMUM LOAD BASED UPON BENDING STRESS

$$\sigma_{\text{allow}} = 110 \text{ MPa}$$

$$M_{\text{max}} = \frac{3}{8} qL \frac{L}{2} - \frac{q}{2} \frac{L}{2} \frac{L}{4} - \frac{q}{2} \frac{L}{4} \frac{L}{6}$$

$$= \frac{5}{48} qL^2$$

$$\sigma = \frac{M_{\text{max}}}{S} = \frac{\frac{5}{48} qL^2}{S}$$

$$q_{\text{max}} = \frac{\sigma_{\text{allow}} S}{\frac{5}{48} L^2} \quad \leftarrow$$

$$q_{\text{max}} = 184.7 \frac{\text{kN}}{\text{m}} \quad \leftarrow$$

(b) MAXIMUM LOAD BASED UPON SHEAR STRESS

$$\tau_{\text{allow}} = 50 \text{ MPa}$$

$$V_{\text{max}} = R_A = \frac{3}{8} qL$$

$$\tau_{\text{max}} = \frac{V_{\text{max}}}{8It} (bh^2 - bh_1^2 + th_1^2)$$

$$= \frac{3qL}{64It} (bh^2 - bh_1^2 + th_1^2)$$

$$q_{\text{max}} = \frac{64 \tau_{\text{allow}} It_w}{3L(bh^2 - bh_1^2 + t_w h_1^2)}$$

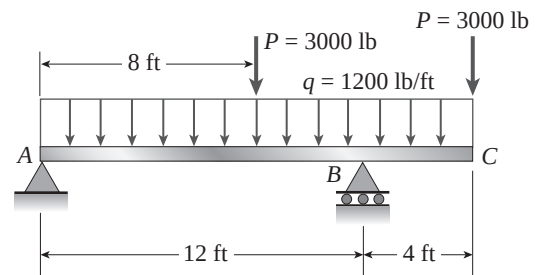
$$q_{\text{max}} = 247 \text{ kN/m} \quad \leftarrow$$

$$\therefore \text{Bending stress governs: } q_{\text{max}} = 184.7 \text{ kN/m} \quad \leftarrow$$

Problem 5.10-9 A simple beam with an overhang supports a uniform load of intensity $q = 1200 \text{ lb/ft}$ and a concentrated load $P = 3000 \text{ lb}$ at 8 ft to the right of A and also at C (see figure). The uniform load includes an allowance for the weight of the beam. The allowable stresses in bending and shear are 18 ksi and 11 ksi, respectively.

Select from Table F-2 (a), Appendix F, the lightest I-beam (S shape) that will support the given loads.

(Hint: Select a beam based upon the bending stress and then calculate the maximum shear stress. If the beam is overstressed in shear, select a heavier beam and repeat.)



Solution 5.10-9 Beam with an overhang

$$\sigma_{\text{allow}} = 18 \text{ ksi} \quad \tau_{\text{allow}} = 11 \text{ ksi} \quad L = 12 \text{ ft}$$

$$q = 1200 \frac{\text{lb}}{\text{ft}} \quad P = 3000 \text{ lb}$$

Sum moments about A & solve for R_B :

$$R_B = \frac{q\left(\frac{4}{3}L\right)^2 \frac{1}{2} + P(8 \text{ ft} + 16 \text{ ft})}{12 \text{ ft}}$$

$$R_B = 1.88 \times 10^4 \text{ lb}$$

Sum forces in vertical direction:

$$R_A = q(16 \text{ ft}) + 2P - R_B$$

$$R_A = 6.4 \times 10^3 \text{ lb}$$

$$V_{\text{max}} = R_B - (P + q4 \text{ ft})$$

$$V_{\text{max}} = 1.1 \times 10^4 \text{ lb} \quad \text{at B}$$

$$M_B = -P(4 \text{ ft}) - q \frac{(4 \text{ ft})^2}{2}$$

$$M_B = -2.16 \times 10^4 \text{ lb-ft}$$

Find moment at D (at Load P between A and B):

$$M_D = R_A(8 \text{ ft}) - q \frac{(8 \text{ ft})^2}{2}$$

$$M_D = 1.28 \times 10^4 \text{ lb-ft}$$

$$M_{\text{max}} = |M_B| \quad M_{\text{max}} = 2.16 \times 10^4 \text{ lb-ft}$$

Required section modulus:

$$S = \frac{M_{\text{max}}}{\sigma_{\text{allow}}} \quad S = 14.4 \text{ in.}^3$$

Lightest beam is S 8 × 23 (from Table F-2(a)).

$$I = 64.7 \text{ in.}^4 \quad S = 16.2 \text{ in.}^3$$

$$b = 4.17 \text{ in.} \quad t = 0.441 \text{ in.}$$

$$t_f = 0.425 \text{ in.} \quad h = 8 \text{ in.}$$

$$h_1 = h - 2t_f \quad h_1 = 7.15 \text{ in.}$$

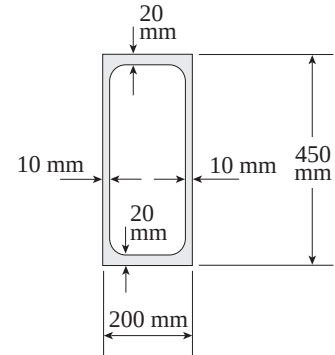
Check max. shear stress.

$$\tau_{\text{max}} = \frac{V_{\text{max}}}{8It} (bh^2 - bh_1^2 + th_1^2)$$

$$\tau_{\text{max}} = 3674 < 11,000 \text{ psi, so ok for shear}$$

Select S 8 × 23 beam. $\leftarrow$

Problem 5.10-10 A hollow steel box beam has the rectangular cross section shown in the figure. Determine the maximum allowable shear force V that may act on the beam if the allowable shear stress is 36 MPa.



Solution 5.10-10 Rectangular box beam

$$\tau_{\text{allow}} = 36 \text{ MPa}$$

Find V_{allow} :

$$\tau = \frac{VQ}{It}$$

$$V_{\text{allow}} = \frac{\tau_{\text{allow}} It}{Q}$$

$$I = \frac{1}{12}(200)(450)^3 - \frac{1}{12}(180)(410)^3$$

$$= 484.9 \times 10^6 \text{ mm}^4$$

$$t = 2(10 \text{ mm}) = 20 \text{ mm}$$

$$Q = (200)\left(\frac{450}{2}\right)\left(\frac{450}{4}\right) - (180)\left(\frac{410}{2}\right)\left(\frac{410}{4}\right)$$

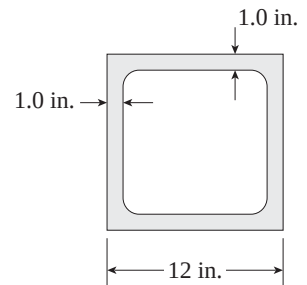
$$= 1.280 \times 10^6 \text{ mm}^3$$

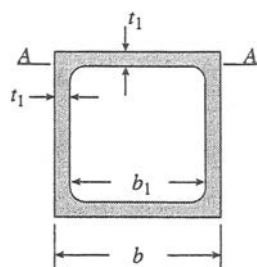
$$V_{\text{allow}} = \frac{\tau_{\text{allow}} It}{Q}$$

$$= \frac{(36 \text{ MPa})(484.9 \times 10^6 \text{ mm}^4)(20 \text{ mm})}{1.280 \times 10^6 \text{ mm}^3}$$

$$= 273 \text{ kN} \quad \leftarrow$$

Problem 5.10-11 A hollow aluminum box beam has the square cross section shown in the figure. Calculate the maximum and minimum shear stresses τ_{max} and τ_{min} in the webs of the beam due to a shear force $V = 28 \text{ k}$.



Solution 5.10-11 Square box beam

$$V = 28 \text{ k} = 28,000 \text{ lb}$$

$$t_1 = 1.0 \text{ in.}$$

$$b = 12 \text{ in.}$$

$$b_1 = 10 \text{ in.}$$

$$\tau = \frac{VQ}{It} \quad t = 2t_1 = 2.0 \text{ in.}$$

MOMENT OF INERTIA

$$I = \frac{1}{12}(b^4 - b_1^4) = 894.67 \text{ in.}^4$$

MAXIMUM SHEAR STRESS IN THE WEB (AT NEUTRAL AXIS)

$$Q = A_1\bar{y}_1 - A_2\bar{y}_2 \quad A_1 = b\left(\frac{b}{2}\right) = \frac{b^2}{2}$$

$$A_2 = b_1\left(\frac{b_1}{2}\right) = \frac{b_1^2}{2}$$

$$\bar{y}_1 = \frac{1}{2}\left(\frac{b}{2}\right) = \frac{b}{4} \quad \bar{y}_2 = \frac{1}{2}\left(\frac{b_1}{2}\right) = \frac{b_1}{4}$$

$$Q = \left(\frac{b^2}{2}\right)\left(\frac{b}{4}\right) - \left(\frac{b_1^2}{2}\right)\left(\frac{b_1}{4}\right)$$

$$= \frac{1}{8}(b^3 - b_1^3) = 91.0 \text{ in.}^3$$

$$\tau_{\max} = \frac{VQ}{It} = \frac{(28,000 \text{ lb})(91.0 \text{ in.}^3)}{(894.67 \text{ in.}^4)(2.0 \text{ in.})} = 1424 \text{ psi}$$

$$= 1.42 \text{ ksi} \quad \leftarrow$$

MINIMUM SHEAR STRESS IN THE WEB (AT LEVEL A-A)

$$Q = A\bar{y} = (bt_1)\left(\frac{b}{2} - \frac{t_1}{2}\right) = \left(\frac{bt_1}{2}\right)(b - t_1)$$

$$t_1 = \frac{b - b_1}{2} \quad Q = \frac{b}{8}(b^2 - b_1^2)$$

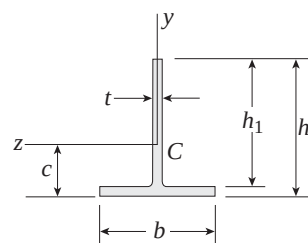
$$Q = \frac{(12 \text{ in.})}{8}[(12 \text{ in.})^2 - (10 \text{ in.})^2] = 66.0 \text{ in.}^3$$

$$\tau_{\min} = \frac{VQ}{It} = \frac{(28,000 \text{ lb})(66.0 \text{ in.}^3)}{(894.67 \text{ in.}^4)(2.0 \text{ in.})} = 1033 \text{ psi}$$

$$= 1.03 \text{ ksi} \quad \leftarrow$$

Problem 5.10-12 The T-beam shown in the figure has cross-sectional dimensions as follows: $b = 210 \text{ mm}$, $t = 16 \text{ mm}$, $h = 300 \text{ mm}$, and $h_1 = 280 \text{ mm}$. The beam is subjected to a shear force $V = 68 \text{ kN}$.

Determine the maximum shear stress $\tau_{\max}$ in the web of the beam.

**Probs 5.10-12 and 5.10-13****Solution 5.10-12**

$$h = 300 \text{ mm} \quad h_1 = 280 \text{ mm}$$

$$b = 210 \text{ mm} \quad t = 16 \text{ mm}$$

$$t_f = h - h_1 \quad V = 68 \text{ kN}$$

$$t_f = 20 \text{ mm}$$

LOCATION OF NEUTRAL AXIS

$$c = \frac{b(h - h_1)\left(\frac{h - h_1}{2}\right) + th_1\left(h - \frac{h_1}{2}\right)}{b(h - h_1) + th_1}$$

$$c = 87.419 \text{ mm}$$

$$c_1 = c \quad c_1 = 87.419 \text{ mm}$$

$$c_2 = h - c \quad c_2 = 212.581 \text{ mm}$$

MOMENT OF INERTIA ABOUT THE Z AXIS

$$I_{\text{web}} = \frac{1}{3} t c_2^3 + \frac{1}{3} t (c_1 - t_f)^3$$

$$I_{\text{web}} = 5.287 \times 10^7 \text{ mm}^4$$

$$I_{\text{flange}} = \frac{1}{12} b t_f^3 + b t_f \left(c_1 - \frac{t_f}{2} \right)^2$$

$$I_{\text{flange}} = 2.531 \times 10^7 \text{ mm}^4$$

$$I = I_{\text{web}} + I_{\text{flange}} \quad I = 7.818 \times 10^7 \text{ mm}^4$$

FIRST MOMENT OF AREA ABOVE THE Z AXIS

$$Q = t c_2 \frac{c_2}{2}$$

$$\tau_{\text{max}} = \frac{VQ}{It} \quad \tau_{\text{max}} = 19.7 \text{ MPa} \quad \leftarrow$$

Problem 5.10-13 Calculate the maximum shear stress τ_{max} in the web of the T-beam shown in the figure if $b = 10$ in., $t = 0.5$ in., $h = 7$ in., $h_1 = 6.2$ in., and the shear force $V = 5300$ lb.

Solution 5.10-13 T-beam

$$h = 7 \text{ in.} \quad h_1 = 6.2 \text{ in.}$$

$$b = 10 \text{ in.} \quad t = 0.5 \text{ in.}$$

$$t_f = h - h_1 \quad t_f = 0.8 \text{ in.}$$

$$V = 5300 \text{ lb}$$

LOCATION OF NEUTRAL AXIS

$$c = \frac{b(h - h_1) \left(\frac{h - h_1}{2} \right) + t h_1 \left(h - \frac{h_1}{2} \right)}{b(h - h_1) + t h_1}$$

$$c = 1.377 \text{ in.}$$

$$c_1 = c \quad c_1 = 1.377 \text{ in.}$$

$$c_2 = h - c \quad c_2 = 5.623 \text{ in.}$$

MOMENT OF INERTIA ABOUT THE Z AXIS

$$I_{\text{web}} = \frac{1}{3} t c_2^3 + \frac{1}{3} t (c_1 - t_f)^3$$

$$I_{\text{web}} = 29.656 \text{ in.}^4$$

$$I_{\text{flange}} = \frac{1}{12} b t_f^3 + b t_f \left(c_1 - \frac{t_f}{2} \right)^2$$

$$I_{\text{flange}} = 8.07 \text{ in.}^4$$

$$I = I_{\text{web}} + I_{\text{flange}} \quad I = 37.726 \text{ in.}^4$$

FIRST MOMENT OF AREA ABOVE THE Z AXIS

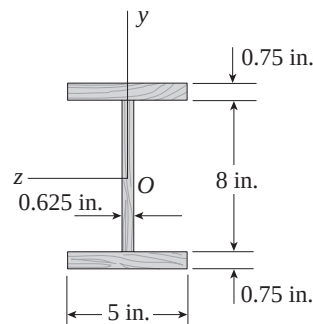
$$Q = t c_2 \frac{c_2}{2}$$

$$\tau_{\text{max}} = \frac{VQ}{It} \quad \tau_{\text{max}} = 2221 \text{ psi} \quad \leftarrow$$

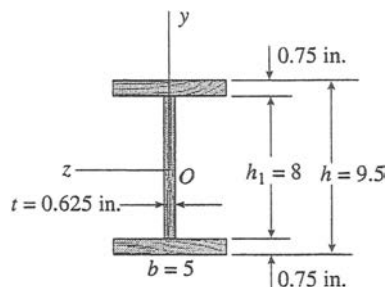
Built-Up Beams

Problem 5.11-1 A prefabricated wood I-beam serving as a floor joist has the cross section shown in the figure. The allowable load in shear for the glued joints between the web and the flanges is 65 lb/in. in the longitudinal direction.

Determine the maximum allowable shear force $V_{\max}$ for the beam.



Solution 5.11-1 Wood I-beam



All dimensions in inches.

Find $V_{\max}$ based upon shear in the glued joints.

Allowable load in shear for the glued joints is 65 lb/in.

$$\therefore f_{\text{allow}} = 65 \text{ lb/in.}$$

$$f = \frac{VQ}{I} \quad V_{\max} = \frac{f_{\text{allow}} I}{Q} \quad \leftarrow$$

$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12}$$

$$= \frac{1}{12} (5)(9.5)^3 - \frac{1}{12} (4.375)(8)^3 = 170.57 \text{ in.}^4$$

$$Q = Q_{\text{flange}} = A_f d_f$$

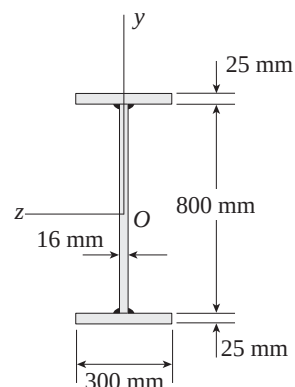
$$= (5)(0.75)(4.375) = 16.406 \text{ in.}^3$$

$$V_{\max} = \frac{f_{\text{allow}} I}{Q}$$

$$= \frac{(65 \text{ lb/in.})(170.57 \text{ in.}^4)}{16.406 \text{ in.}^3} = 676 \text{ lb} \quad \leftarrow$$

Problem 5.11-2 A welded steel girder having the cross section shown in the figure is fabricated of two 300 mm × 25 mm flange plates and a 800 mm × 16 mm web plate. The plates are joined by four fillet welds that run continuously for the length of the girder. Each weld has an allowable load in shear of 920 kN/m.

Calculate the maximum allowable shear force $V_{\max}$ for the girder.



Solution 5.11-2

$$h = 850 \text{ mm} \quad h_1 = 800 \text{ mm}$$

$$b = 300 \text{ mm} \quad t = 16 \text{ mm}$$

$$t_f = 25 \text{ mm}$$

$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12}$$

$$I = 3.236 \times 10^9 \text{ mm}^4$$

$$Q_{\text{flange}} = A_f d_f \quad Q_{\text{flange}} = bt_f \left(\frac{h-t_f}{2} \right)$$

$$Q_{\text{flange}} = 3.094 \times 10^6 \text{ mm}^3$$

$$f_{\text{allow}} = 920 \frac{\text{kN}}{\text{m}} \quad f = 2 f_{\text{allow}}$$

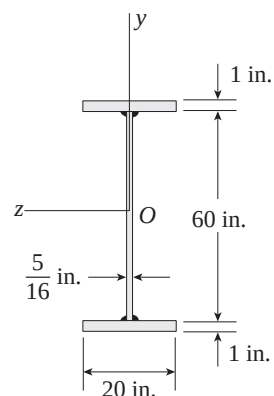
(2 welds, one either side of web)

$$f = \frac{VQ}{I} \quad V_{\max} = \frac{fI}{Q_{\text{flange}}}$$

$$V_{\max} = 1.924 \text{ MN} \quad \leftarrow$$

Problem 5.11-3 A welded steel girder having the cross section shown in the figure is fabricated of two 20 in. × 1 in. flange plates and a 60 in. × 5/16 in. web plate. The plates are joined by four longitudinal fillet welds that run continuously throughout the length of the girder.

If the girder is subjected to a shear force of 280 kips, what force F (per inch of length of weld) must be resisted by each weld?



Solution 5.11-3

$$h = 62 \text{ in.} \quad h_1 = 60 \text{ in.}$$

$$b = 20 \text{ in.} \quad t = \frac{5}{16} \text{ in.}$$

$$t_f = 1 \text{ in.}$$

$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12}$$

$$I = 4.284 \times 10^4 \text{ in.}^4$$

$$Q_{\text{flange}} = A_f d_f$$

$$Q_{\text{flange}} = bt_f \left(\frac{h-t_f}{2} \right)$$

$$Q_{\text{flange}} = 610 \text{ in.}^3$$

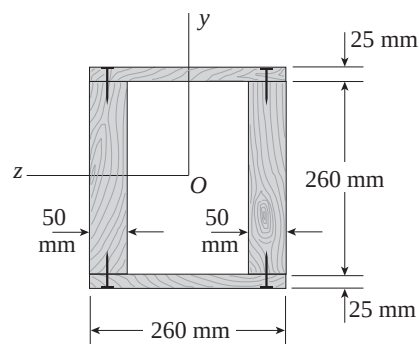
$$V = 280 \text{ k} \quad f = 2F = \frac{VQ}{I}$$

$$F = \frac{VQ_{\text{flange}}}{21} \quad F = 1994 \times 10^3 \text{ lb-in.}$$

$$F = 1994 \text{ lb/in.} \quad \leftarrow$$

Problem 5.11-4 A box beam of wood is constructed of two 260 mm × 50 mm boards and two 260 mm × 25 mm boards (see figure). The boards are nailed at a longitudinal spacing $s = 100 \text{ mm}$.

If each nail has an allowable shear force $F = 1200 \text{ N}$, what is the maximum allowable shear force V_{max} ?

**Solution 5.11-4 Wood box beam**

All dimensions in millimeters.

$$b = 260 \quad b_1 = 260 - 2(50) = 160$$

$$h = 310 \quad h_1 = 260$$

$$s = \text{nail spacing} = 100 \text{ mm}$$

$$F = \text{allowable shear force for one nail} = 1200 \text{ N}$$

$$f = \text{shear flow between one flange and both webs}$$

$$f_{\text{allow}} = \frac{2F}{s} = \frac{2(1200 \text{ N})}{100 \text{ mm}} = 24 \text{ kN/m}$$

$$f = \frac{VQ}{I} \quad V_{\text{max}} = \frac{f_{\text{allow}} I}{Q}$$

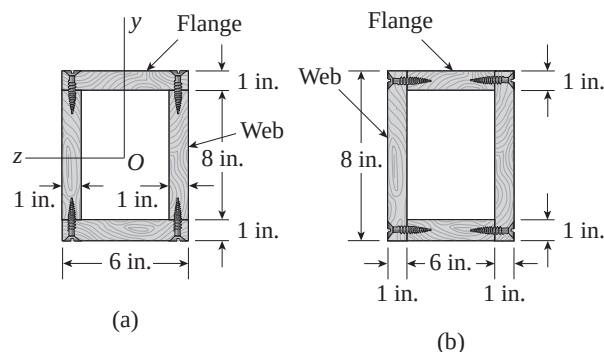
$$I = \frac{1}{12}(bh^3 - b_1h_1^3) = 411.125 \times 10^6 \text{ mm}^4$$

$$Q = Q_{\text{flange}} = A_f d_f = (260)(25)(142.5) \\ = 926.25 \times 10^3 \text{ mm}^4$$

$$V_{\text{max}} = \frac{f_{\text{allow}} I}{Q} = \frac{(24 \text{ kN/m})(411.25 \times 10^6 \text{ mm}^4)}{926.25 \times 10^3 \text{ mm}^3} \\ = 10.7 \text{ kN} \quad \leftarrow$$

Problem 5.11-5 A box beam is constructed of four wood boards as shown in the figure part a. The webs are 8 in. $\times$ 1 in. and the flanges are 6 in. $\times$ 1 in. boards (actual dimensions), joined by screws for which the allowable load in shear is $F = 250$ lb per screw.

- Calculate the maximum permissible longitudinal spacing $S_{\max}$ of the screws if the shear force V is 1200 lb.
- Repeat part (a) if the flanges are attached to the webs using a *horizontal* arrangement of screws as shown in the figure part b.



Solution 5.11-5 Wood box beam

$$V = 1200 \text{ lb} \quad F = 250 \text{ lb}$$

(a) Vertical screws:

$$h = 10 \text{ in.} \quad h_1 = 8 \text{ in.}$$

$$b = 6 \text{ in.} \quad t = 1 \text{ in.}$$

$$I = \frac{bh^3}{12} - \frac{(b-2t)h_1^3}{12} \quad I = 329.333 \text{ in.}^4$$

$$Q_a = bt(4.5 \text{ in.}) \quad Q_a = 27 \text{ in.}^3$$

$$f = \frac{VQ}{I} = \frac{2F}{S}$$

$$S_{\max} = \frac{2FI}{VQ_a}$$

$$S_{\max} = 5.08 \text{ in.} \quad \leftarrow$$

(b) Horizontal screws:

$$h = 8 \text{ in.} \quad h_1 = 6 \text{ in.}$$

$$b = 8 \text{ in.} \quad t = 1 \text{ in.}$$

$$I = \frac{bh^3}{12} - \frac{(b-2t)h_1^3}{12} \quad I = 233.333 \text{ in.}^4$$

$$Q_b = (b-2t)t(3.5 \text{ in.}) \quad Q_b = 21 \text{ in.}^3$$

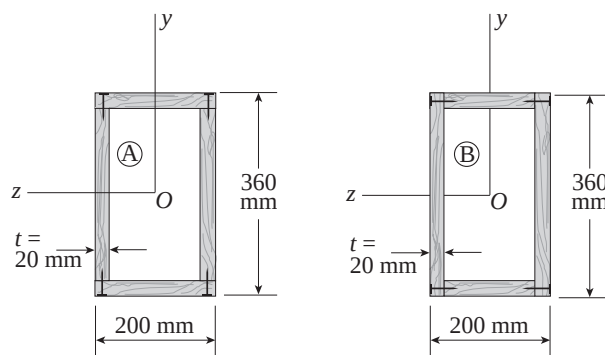
$$f = \frac{VQ}{I} = \frac{2F}{s}$$

$$S_{\max} = \frac{2FI}{VQ_b}$$

$$S_{\max} = 4.63 \text{ in.} \quad \leftarrow$$

Problem 5.11-6 Two wood box beams (beams A and B) have the same outside dimensions (200 mm $\times$ 360 mm) and the same thickness ($t = 20$ mm) throughout, as shown in the figure. Both beams are formed by nailing, with each nail having an allowable shear load of 250 N. The beams are designed for a shear force $V = 3.2$ kN.

- What is the maximum longitudinal spacing s_A for the nails in beam A?
- What is the maximum longitudinal spacing s_B for the nails in beam B?
- Which beam is more efficient in resisting the shear force?



Solution 5.11-6 Two wood box beams

Cross-sectional dimensions are the same.

All dimensions in millimeters.

$$b = 200 \quad b_1 = 200 - 2(20) = 160$$

$$h = 360 \quad h_1 = 360 - 2(20) = 320$$

$$t = 20$$

$$F = \text{allowable load per nail} = 250 \text{ N}$$

$$V = \text{shear force} = 3.2 \text{ kN}$$

$$I = \frac{1}{12}(bh^3 - b_1h_1^3) = 340.69 \times 10^6 \text{ mm}^4$$

s = longitudinal spacing of the nails

f = shear flow between one flange and both webs

$$f = \frac{2F}{s} = \frac{VQ}{I} \quad \therefore s_{\max} = \frac{2FI}{VQ}$$

(a) BEAM A

$$\begin{aligned} Q &= A_f d_f = (bt) \left(\frac{h-t}{2} \right) = (200)(20) \left(\frac{1}{2} \right) (340) \\ &= 680 \times 10^3 \text{ mm}^3 \\ s_A &= \frac{2FI}{VQ} = \frac{(2)(250 \text{ N})(340.7 \times 10^6 \text{ mm}^4)}{(3.2 \text{ kN})(680 \times 10^3 \text{ mm}^3)} \\ &= 78.3 \text{ mm} \quad \leftarrow \end{aligned}$$

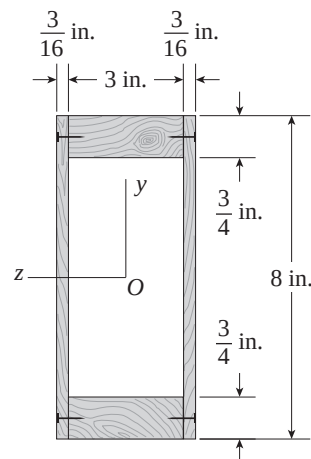
(b) BEAM B

$$\begin{aligned} Q &= A_f d_f = (b - 2t)(t) \left(\frac{h-t}{2} \right) \\ &= (160)(20) \left(\frac{1}{2} \right) (340) \\ &= 544 \times 10^3 \text{ mm}^3 \\ s_B &= \frac{2FI}{VQ} = \frac{(2)(250 \text{ N})(340.7 \times 10^6 \text{ mm}^4)}{(3.2 \text{ kN})(544 \times 10^3 \text{ mm}^3)} \\ &= 97.9 \text{ mm} \quad \leftarrow \end{aligned}$$

(c) BEAM B IS MORE EFFICIENT because the shear flow on the contact surfaces is smaller and therefore fewer nails are needed. $\leftarrow$

Problem 5.11-7 A hollow wood beam with plywood webs has the cross-sectional dimensions shown in the figure. The plywood is attached to the flanges by means of small nails. Each nail has an allowable load in shear of 30 lb.

Find the maximum allowable spacing s of the nails at cross sections where the shear force V is equal to (a) 200 lb, and (b) 300 lb.



Solution 5.11-7 Wood beam with plywood webs

All dimensions in inches.

$$b = 3.375 \quad b_1 = 3.0$$

$$h = 8.0 \quad h_1 = 6.5$$

F = allowable shear force for one nail = 30 lb

s = longitudinal spacing of the nails

f = shear flow between one flange and both webs

$$f = \frac{VQ}{I} = \frac{2F}{s} \quad \therefore s_{\max} = \frac{2FI}{VQ}$$

$$I = \frac{1}{12} (bh^3 - b_1h_1^3) = 75.3438 \text{ in.}^4$$

$$Q = Q_{\text{flange}} = A_f d_f = (3.0)(0.75)(3.625) = 8.1563 \text{ in.}^3$$

$$(a) \quad V = 200 \text{ lb}$$

$$s_{\max} = \frac{2FI}{VQ} = \frac{2(30 \text{ lb})(75.344 \text{ in.}^4)}{(200 \text{ lb})(8.1563 \text{ in.}^3)} \\ = 2.77 \text{ in.} \quad \leftarrow$$

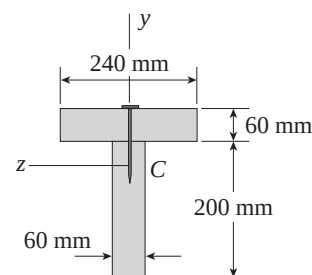
$$(b) \quad V = 300 \text{ lb}$$

By proportion,

$$s_{\max} = (2.77 \text{ in.}) \left(\frac{200}{300} \right) = 1.85 \text{ in.} \quad \leftarrow$$

Problem 5.11-8 A beam of T cross section is formed by nailing together two boards having the dimensions shown in the figure.

If the total shear force V acting on the cross section is 1500 N and each nail may carry 760 N in shear, what is the maximum allowable nail spacing s ?

**Solution 5.11-8**

$$V = 1500 \text{ N} \quad F_{\text{allow}} = 760 \text{ N}$$

$$h_1 = 200 \text{ mm} \quad b = 240 \text{ mm}$$

$$t = 60 \text{ mm} \quad h = 260 \text{ mm}$$

$$A = bt + h_1 t \quad A = 2.64 \times 10^4 \text{ mm}^2$$

LOCATION OF NEUTRAL AXIS (Z AXIS)

$$c_2 = \frac{bt \left(h_1 + \frac{t}{2} \right) + th_1 \frac{h_1}{2}}{A}$$

$$c_2 = 170.909 \text{ mm}$$

$$c_1 = h - c_2$$

$$c_1 = 89.091 \text{ mm}$$

MOMENT OF INERTIA ABOUT THE NEUTRAL AXIS

$$I = \frac{1}{3} tc_2^3 + \frac{1}{3} t(h_1 - c_2)^3 \\ + \frac{1}{12} bt^3 + bt \left(c_1 - \frac{t}{2} \right)^2$$

$$I = 1.549 \times 10^8 \text{ mm}^4$$

FIRST MOMENT OF AREA OF FLANGE

$$Q = bt \left(c_1 - \frac{t}{2} \right)$$

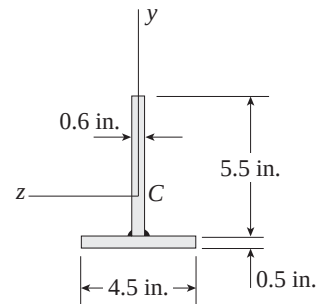
$$Q = 8.509 \times 10^5 \text{ mm}^3$$

MAXIMUM ALLOWABLE SPACING OF NAILS

$$f = \frac{VQ}{I} = \frac{F}{s}$$

$$s_{\max} = \frac{F_{\text{allow}} I}{VQ} \quad s_{\max} = 92.3 \text{ mm} \quad \leftarrow$$

Problem 5.11-9 The T-beam shown in the figure is fabricated by welding together two steel plates. If the allowable load for each weld is 1.8 k/in. in the longitudinal direction, what is the maximum allowable shear force V ?



Solution 5.11-9 T-beam (welded)

$$F_{\text{allow}} = 1.8 \frac{\text{k}}{\text{in.}}$$

$$h_1 = 5.5 \text{ in.} \quad b = 4.5 \text{ in.}$$

$$t_1 = 0.6 \text{ in.} \quad t_2 = 0.5 \text{ in.}$$

$$h = 6 \text{ in.}$$

$$A = bt_2 + h_1t_1 \quad A = 5.55 \text{ in.}^2$$

LOCATION OF NEUTRAL AXIS (z AXIS)

$$c_2 = \frac{bt_2 \frac{t_2}{2} + t_1h_1 \left(\frac{h_1}{2} + t_2 \right)}{A}$$

$$c_2 = 2.034 \text{ in.} \quad c_1 = h - c_2$$

$$c_1 = 3.966 \text{ in.}$$

MOMENT OF INERTIA ABOUT THE NEUTRAL AXIS

$$I = \frac{1}{3}t_1c_1^3 + \frac{1}{3}t_1(c_2 - t_2)^3 + \frac{1}{12}bt_2^3 + bt_2\left(c_2 - \frac{t_2}{2}\right)^2$$

$$I = 20.406 \text{ in.}^4$$

FIRST MOMENT OF AREA OF FLANGE

$$Q = bt_2\left(c_2 - \frac{t_2}{2}\right) \quad Q = 4.014 \text{ in.}^3$$

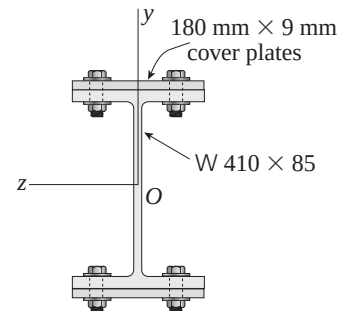
MAXIMUM ALLOWABLE SHEAR FORCE

$$f = \frac{VQ}{I} = 2F$$

$$V_{\text{max}} = \frac{2F_{\text{allow}}I}{Q}$$

$$V_{\text{max}} = 18.30 \text{ k} \quad \leftarrow$$

Problem 5.11-10 A steel beam is built up from a W 410 × 85 wide flange beam and two 180 mm × 9 mm cover plates (see figure). The allowable load in shear on each bolt is 9.8 kN. What is the required bolt spacing s in the longitudinal direction if the shear force $V = 110 \text{ kN}$ (Note: Obtain the dimensions and moment of inertia of the W shape from Table F-1(b).)



Solution 5.11-10

$$V = 110 \text{ kN} \quad F_{\text{allow}} = 9.8 \text{ kN}$$

$$W 410 \times 85$$

$$A_w = 10,800 \text{ mm}^2 \quad h_w = 417 \text{ mm}$$

$$I_w = 310 \times 10^6 \text{ mm}^4$$

$$A_{cp} = (180)(9)(2) \text{ mm}^2 \text{ for two plates}$$

$$h = h_w + (9 \text{ mm})(2)$$

$$A = A_w + A_{cp} \quad A = 1.404 \times 10^4 \text{ mm}^2$$

LOCATION OF NEUTRAL AXIS (z AXIS)

$$c = \frac{h}{2} \quad c = 217.5 \text{ mm}$$

Moment of inertia about the neutral axis:

$$I = I_w + \frac{180 \text{ mm} (9 \text{ mm})^3}{12} (2)$$

$$+ A_{cp} \left(c - \frac{9 \text{ mm}}{2} \right)^2$$

$$I = 4.57 \times 10^8 \text{ mm}^4$$

First moment of area of one flange:

$$Q = 180 \text{ mm} (9 \text{ mm}) \left(c - \frac{9 \text{ mm}}{2} \right)$$

$$Q = 3.451 \times 10^5 \text{ mm}^3$$

Maximum allowable spacing of nails:

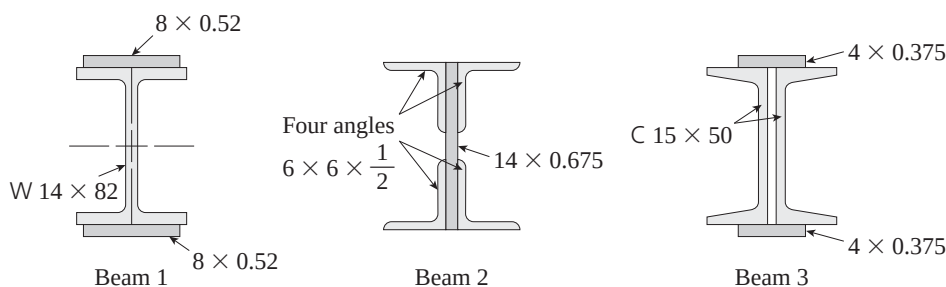
$$f = \frac{VQ}{I} = \frac{2F}{s}$$

$$s_{\text{max}} = \frac{2F_{\text{allow}} I}{VQ} \quad s_{\text{max}} = 236 \text{ mm} \quad \leftarrow$$

Problem 5.11-11 The three beams shown have approximately the same cross-sectional area. Beam 1 is a W 14 × 82 with flange plates; Beam 2 consists of a web plate with four angles; and Beam 3 is constructed of 2 C shapes with flange plates.

- Which design has the largest moment capacity?
- Which has the largest shear capacity?
- Which is the most economical in bending?
- Which is the most economical in shear?

Assume allowable stress values are: $\sigma_a = 18 \text{ ksi}$ and $\tau_a = 11 \text{ ksi}$. The most economical beam is that having the largest capacity-to-weight ratio. Neglect fabrication costs in answering (c) and (d) above. (Note: Obtain the dimensions and properties of all rolled shapes from tables in Appendix F.)

**Solution 5.11-11 Built-up steel beam**

Beam 1: properties and dimensions for W14 × 82 with flange plates

$$A_w = 24 \text{ in.}^2 \quad h_w = 14.3 \text{ in.} \quad I_w = 881 \text{ in.}^4$$

$$b_1 = 8 \text{ in.} \quad t_1 = 0.52 \text{ in.}$$

$$h_1 = h_w + 2t_1 \quad b_{f1} = 10.1 \text{ in.}$$

$$t_{f1} = 0.855 \text{ in.} \quad t_{w1} = 0.51 \text{ in.}$$

$$A_1 = A_w + 2b_1t_1 \quad A_1 = 32.32 \text{ in.}^2$$

$$I_1 = I_w + \frac{b_1 \times t_1^3}{12} 2 + b_1 t_1 \left(\frac{h_w}{2} + \frac{t_1}{2} \right)^2 2$$

$$I_1 = 1.338 \times 10^3 \text{ in.}^4$$

Beam 2: properties and dimensions for L6 × 6 × 1/2 angles with web plate

$$A_a = 5.77 \text{ in.}^2 \quad c_a = 1.67 \text{ in.} \quad h_a = 6 \text{ in.}$$

$$I_a = 19.9 \text{ in.}^4 \quad b_2 = 14 \text{ in.}$$

$$t_2 = 0.675 \text{ in.} \quad h_2 = b_2$$

$$A_2 = 4A_a + b_2 t_2 \quad A_2 = 32.53 \text{ in.}^2$$

$$I_2 = 4I_a + A_a \left(\frac{b_2}{2} - c_a \right)^2 4 + \frac{t_2 b_2^3}{12}$$

$$I_2 = 889.627 \text{ in.}^4$$

Beam 3: properties and dimensions for C15 × 50 with flange plates

$$A_c = 14.7 \text{ in.}^2 \quad h_c = 15 \text{ in.} \quad I_c = 404 \text{ in.}^4$$

$$b_3 = 4 \text{ in.} \quad t_3 = 0.375 \text{ in.} \quad h_3 = h_c + 2t_3$$

$$b_{f3} = 3.72 \text{ in.} \quad t_{f3} = 0.65 \text{ in.} \quad t_{w3} = 0.716 \text{ in.}$$

$$A_3 = 2A_c + 2b_3 t_3 \quad A_3 = 32.4 \text{ in.}^2$$

$$I_3 = I_c 2 + \frac{b_3 t_3^3}{12} 2 + b_3 t_3 \left(\frac{h_c}{2} + \frac{t_3}{2} \right)^2 2$$

$$I_3 = 985.328 \text{ in.}^4$$

- (a) Beam with largest moment capacity; largest section modulus controls.

$$M_{\max} = \sigma_{\text{allow}} S$$

$$S_1 = \frac{2I_1}{h_1} \quad S_1 = 174.449 \text{ in.}^3 \quad \text{largest value}$$

$$S_2 = \frac{2I_2}{h_2} \quad S_2 = 127.09 \text{ in.}^3$$

$$S_3 = \frac{2I_3}{h_3} \quad S_3 = 125.121 \text{ in.}^3$$

Case (1) with maximum S has the largest moment capacity. ←

- (b) BEAM WITH LARGEST SHEAR CAPACITY: LARGEST $I_t w/Q$ RATIO CONTROLS

$$V_{\max} = \frac{\tau_{\text{allow}} I_t w}{Q}$$

$$Q_1 = b_1 t_1 \left(\frac{h_1}{2} - \frac{t_1}{2} \right) + b_{f1} t_{f1} \left(\frac{h_w}{2} - \frac{t_{f1}}{2} \right) + t_{w1} \frac{\left(\frac{h_w}{2} - t_{f1} \right)^2}{2} \quad Q_1 = 98.983 \text{ in.}^3$$

$$Q_2 = 2A_a \left(\frac{h_2}{2} - c_a \right) + t_2 \frac{\left(\frac{b_2}{2} \right)^2}{2}$$

$$Q_2 = 78.046 \text{ in.}^3$$

$$Q_3 = b_3 t_3 \left(\frac{h_3}{2} - \frac{t_3}{2} \right) + 2b_{f3} t_{f3} \left(\frac{h_c}{2} - \frac{t_{f3}}{2} \right) + 2t_{w3} \frac{\left(\frac{h_c}{2} - t_{f3} \right)^2}{2} \quad Q_3 = 79.826 \text{ in.}^3$$

$$\frac{I_1 t_{w1}}{Q_1} = 6.894 \text{ in.}^2$$

$$\frac{I_2 t_2}{Q_2} = 7.694 \text{ in.}^2$$

$$\frac{I_3 2t_{w3}}{Q_3} = 17.676 \text{ in.}^2 \quad \text{largest value}$$

Case (3) with maximum $\frac{I_t w}{Q}$ has the largest shear capacity. ←

- (c) MOST ECONOMICAL BEAM IN BENDING HAS LARGEST BENDING CAPACITY-TO-WEIGHT RATIO

$$\frac{S_3}{A_3} = 3.862 \text{ in.} < \frac{S_2}{A_2} = 3.907 \text{ in.} <$$

$$\frac{S_1}{A_1} = 5.398 \text{ in.}$$

Case (1) is the most economical in bending. ←

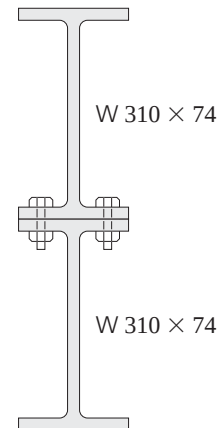
- (d) MOST ECONOMICAL BEAM IN SHEAR HAS LARGEST SHEAR CAPACITY-TO-WEIGHT RATIO

$$\frac{I_1 t_{w1}}{Q_1 A_1} = 0.213 < \frac{I_2 t_2}{Q_2 A_2} = 0.237$$

$$< \frac{I_3 2t_{w3}}{Q_3 A_3} = 0.273$$

Case (3) is the most economical in shear. ←

Problem 5.11-12 Two W 310 × 74 steel wide-flange beams are bolted together to form a built-up beam as shown in the figure. What is the maximum permissible bolt spacing s if the shear force $V = 80$ kN and the allowable load in shear on each bolt is $F = 13.5$ kN [Note: Obtain the dimensions and properties of the W shapes from Table F-1(b).]



Solution 5.11-12

$$V = 80 \text{ kN} \quad \text{W } 310 \times 74$$

$$F_{\text{allow}} = 13.5 \text{ kN} \quad A_w = 9420 \text{ mm}^2$$

$$h_w = 310 \text{ mm} \quad I_w = 163 \times 10^6 \text{ mm}^4$$

Location of neutral axis (z axis):

$$c = h_w \quad c = 310 \text{ mm}$$

MOMENT OF INERTIA ABOUT THE NEUTRAL AXIS

$$I = \left[I_w + A_w \left(\frac{h_w}{2} \right)^2 \right] (2)$$

$$I = 7.786 \times 10^8 \text{ mm}^4$$

FIRST MOMENT OF AREA OF FLANGE

$$Q = A_w \frac{h_w}{2} \quad Q = 1.46 \times 10^6 \text{ mm}^3$$

MAXIMUM ALLOWABLE SPACING OF BOLTS

$$f = \frac{VQ}{I} = \frac{2F}{s}$$

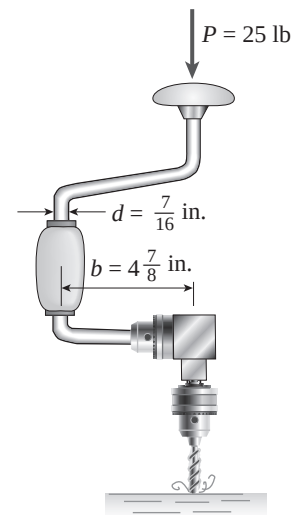
$$s_{\text{max}} = \frac{2F_{\text{allow}} I}{VQ} \quad s_{\text{max}} = 180 \text{ mm} \quad \leftarrow$$

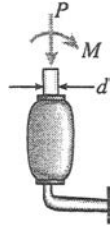
Beams with Axial Loads

When solving the problems for Section 5.12, assume that the bending moments are not affected by the presence of lateral deflections.

Problem 5.12-1 While drilling a hole with a brace and bit, you exert a downward force $P = 25$ lb on the handle of the brace (see figure). The diameter of the crank arm is $d = 7/16$ in. and its lateral offset is $b = 4\frac{7}{8}$ in.

Determine the maximum tensile and compressive stresses σ_t and σ_c , respectively, in the crank.



Solution 5.12-1 Brace and bit

$$P = 25 \text{ lb (compression)}$$

$$M = Pb = (25 \text{ lb})(4\text{--}7/8 \text{ in.}) \\ = 121.9 \text{ lb-in.}$$

$$d = \text{diameter}$$

$$d = 7/16 \text{ in.}$$

$$A = \frac{\pi d^2}{4} = 0.1503 \text{ in.}^2$$

$$S = \frac{\pi d^3}{32} = 0.008221 \text{ in.}^3$$

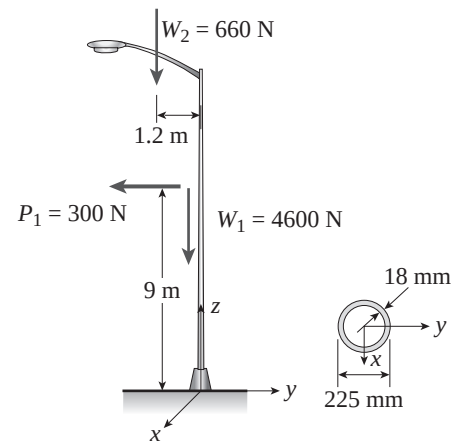
MAXIMUM STRESSES

$$\sigma_t = -\frac{P}{A} + \frac{M}{S} = -\frac{25 \text{ lb}}{0.1503 \text{ in.}^2} + \frac{121.9 \text{ lb-in.}}{0.008221 \text{ in.}^3} \\ = -166 \text{ psi} + 14,828 \text{ psi} = 14,660 \text{ psi} \quad \leftarrow$$

$$\sigma_c = -\frac{P}{A} - \frac{M}{S} = -166 \text{ psi} - 14,828 \text{ psi} \\ = -14,990 \text{ psi} \quad \leftarrow$$

Problem 5.12-2 An aluminum pole for a street light weighs 4600 N and supports an arm that weighs 660 N (see figure). The center of gravity of the arm is 1.2 m from the axis of the pole. A wind force of 300 N also acts in the $(-y)$ direction at 9 m above the base. The outside diameter of the pole (at its base) is 225 mm, and its thickness is 18 mm.

Determine the maximum tensile and compressive stresses σ_t and σ_c , respectively, in the pole (at its base) due to the weights and the wind force.

**Solution 5.12-2**

$$W_1 = 4600 \text{ N} \quad b = 1.2 \text{ m}$$

$$W_2 = 660 \text{ N} \quad h = 9 \text{ m}$$

$$P_1 = 300 \text{ N} \quad d_1 = 225 \text{ mm} \quad t = 18 \text{ mm}$$

$$d_2 = d_1 - 2t$$

$$A = \frac{\pi}{4} (d_1^2 - d_2^2) \quad I = \frac{\pi}{64} (d_1^4 - d_2^4)$$

$$A = 1.171 \times 10^4 \text{ mm}^2 \quad I = 6.317 \times 10^7 \text{ mm}^4$$

AT BASE OF POLE

$$P_z = W_1 + W_2$$

$$P_z = 5.26 \times 10^3 \text{ N} \quad (\text{Axial force})$$

$$V_y = P_1 \quad V_y = 300 \text{ N} \quad (\text{Shear force})$$

$$M_x = W_2 b + P_1 h$$

$$M_x = 3.492 \times 10^3 \text{ N} \cdot \text{m} \quad (\text{Moment})$$

MAXIMUM STRESS

$$\sigma_t = \left(-\frac{P_z}{A} + \frac{M_x}{I} \frac{d_1}{2} \right)$$

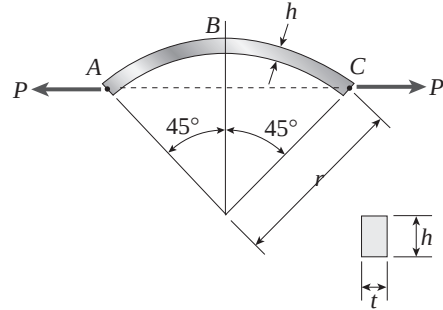
$$\sigma_t = 5.77 \times 10^3 \text{ kPa} \\ = 5770 \text{ kPa} \quad \leftarrow$$

$$\sigma_c = \left(-\frac{P_z}{A} - \frac{M_x}{I} \frac{d_1}{2} \right) \quad \leftarrow$$

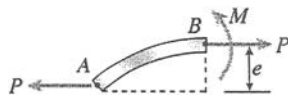
$$\sigma_c = -6.668 \times 10^3 \\ = -6668 \text{ kPa} \quad \leftarrow$$

Problem 5.12-3 A curved bar ABC having a circular axis (radius $r = 12$ in.) is loaded by forces $P = 400$ lb (see figure). The cross section of the bar is rectangular with height h and thickness t .

If the allowable tensile stress in the bar is 12,000 psi and the height $h = 1.25$ in., what is the minimum required thickness $t_{\min}$?



Solution 5.12-3 Curved bar



r = radius of curved bar

$e = r - r \cos 45^\circ$

$$= r \left(1 - \frac{1}{\sqrt{2}} \right)$$

$$M = Pe = \frac{Pr}{2} (2 - \sqrt{2})$$

CROSS SECTION

$$h = \text{height} \quad t = \text{thickness} \quad A = ht \quad S = \frac{1}{6} th^2$$

TENSILE STRESS

$$\begin{aligned} \sigma_t &= \frac{P}{A} + \frac{M}{S} = \frac{P}{ht} + \frac{3Pr(2 - \sqrt{2})}{th^2} \\ &= \frac{P}{ht} \left[1 + 3(2 - \sqrt{2}) \frac{r}{h} \right] \end{aligned}$$

MINIMUM THICKNESS

$$t_{\min} = \frac{P}{h\sigma_{\text{allow}}} \left[1 + 3(2 - \sqrt{2}) \frac{r}{h} \right]$$

SUBSTITUTE NUMERICAL VALUES:

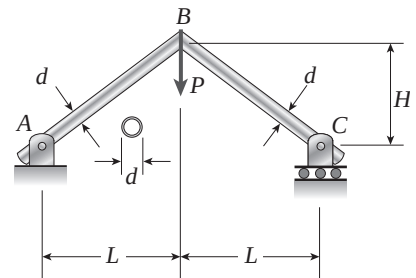
$$P = 400 \text{ lb} \quad \sigma_{\text{allow}} = 12,000 \text{ psi}$$

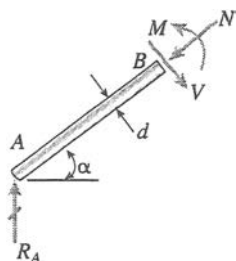
$$r = 12 \text{ in.} \quad h = 1.25 \text{ in.}$$

$$t_{\min} = 0.477 \text{ in.} \quad \leftarrow$$

Problem 5.12-4 A rigid frame ABC is formed by welding two steel pipes at B (see figure). Each pipe has cross-sectional area $A = 11.31 \times 10^3 \text{ mm}^2$, moment of inertia $I = 46.37 \times 10^6 \text{ mm}^4$, and outside diameter $d = 200$ mm.

Find the maximum tensile and compressive stresses σ_t and σ_c , respectively, in the frame due to the load $P = 8.0$ kN if $L = H = 1.4$ m.



Solution 5.12-4 Rigid frame

Load P at midpoint B :

$$\text{REACTIONS: } R_A = R_C = \frac{P}{2}$$

BAR AB :

$$\tan \alpha = \frac{H}{L}$$

$$\sin \alpha = \frac{H}{\sqrt{H^2 + L^2}}$$

d = diameter

$c = d/2$

$$\text{AXIAL FORCE: } N = R_A \sin \alpha = \frac{P}{2} \sin \alpha$$

$$\text{BENDING MOMENT: } M = R_A L = \frac{PL}{2}$$

TENSILE STRESS

$$\sigma_t = -\frac{N}{A} + \frac{Mc}{I} = -\frac{P \sin \alpha}{2A} + \frac{PLd}{4I}$$

SUBSTITUTE NUMERICAL VALUES

$$P = 8.0 \text{ kN} \quad L = H = 1.4 \text{ m} \quad \alpha = 45^\circ$$

$$\sin \alpha = 1/\sqrt{2} \quad d = 200 \text{ mm}$$

$$A = 11.31 \times 10^3 \text{ mm}^2 \quad I = 46.37 \times 10^6 \text{ mm}^4$$

$$\begin{aligned} \sigma_t &= -\frac{(8.0 \text{ kN})(1/\sqrt{2})}{2(11.31 \times 10^3 \text{ mm}^2)} \\ &\quad + \frac{(8.0 \text{ kN})(1.4 \text{ m})(200 \text{ mm})}{4(46.37 \times 10^6 \text{ mm}^4)} \end{aligned}$$

$$= -0.250 \text{ MPa} + 12.08 \text{ MPa}$$

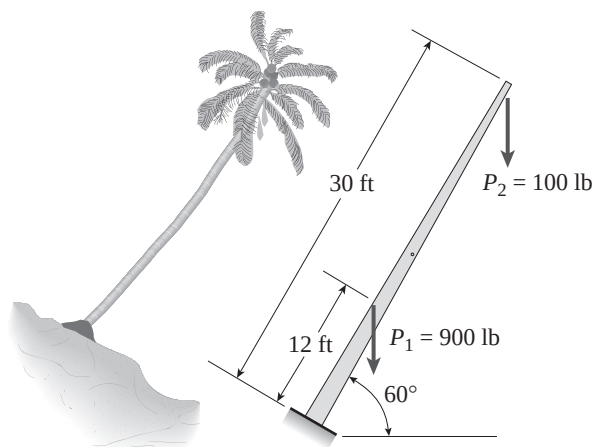
$$= 11.83 \text{ MPa (tension)} \quad \leftarrow$$

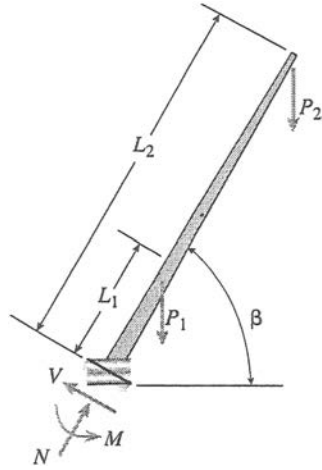
$$\sigma_c = -\frac{N}{A} - \frac{Mc}{I} = -0.250 \text{ MPa} - 12.08 \text{ MPa}$$

$$= -12.33 \text{ MPa (compression)} \quad \leftarrow$$

Problem 5.12-5 A palm tree weighing 1000 lb is inclined at an angle of 60° (see figure). The weight of the tree may be resolved into two resultant forces, a force $P_1 = 900 \text{ lb}$ acting at a point 12 ft from the base and a force $P_2 = 100 \text{ lb}$ acting at the top of the tree, which is 30 ft long. The diameter at the base of the tree is 14 in.

Calculate the maximum tensile and compressive stresses σ_t and σ_c , respectively, at the base of the tree due to its weight.



Solution 5.12-5 Palm tree**FREE-BODY DIAGRAM**

$$P_1 = 900 \text{ lb}$$

$$P_2 = 100 \text{ lb}$$

$$L_1 = 12 \text{ ft} = 144 \text{ in.}$$

$$L_2 = 30 \text{ ft} = 360 \text{ in.}$$

$$d = 14 \text{ in.}$$

$$A = \frac{\pi d^2}{4} = 153.94 \text{ in.}^2$$

$$S = \frac{\pi d^3}{32} = 269.39 \text{ in.}^3$$

$$M = P_1 L_1 \cos 60^\circ + P_2 L_2 \cos 60^\circ$$

$$= [(900 \text{ lb})(144 \text{ in.}) + (100 \text{ lb})(360 \text{ in.})] \cos 60^\circ$$

$$= 82,800 \text{ lb-in.}$$

$$N = (P_1 + P_2) \sin 60^\circ = (1000 \text{ lb}) \sin 60^\circ = 866 \text{ lb}$$

MAXIMUM TENSILE STRESS

$$\sigma_t = -\frac{N}{A} + \frac{M}{S} = -\frac{866 \text{ lb}}{153.94 \text{ in.}^2} + \frac{82,800 \text{ lb-in.}}{269.39 \text{ in.}^3}$$

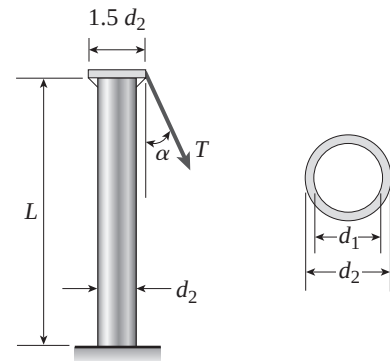
$$= -5.6 \text{ psi} + 307.4 \text{ psi} = 302 \text{ psi} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS

$$\sigma_c = -5.6 \text{ psi} - 307.4 \text{ psi} = -313 \text{ psi} \quad \leftarrow$$

Problem 5.12-6 A vertical pole of aluminum is fixed at the base and pulled at the top by a cable having a tensile force T (see figure). The cable is attached at the outer edge of a stiffened cover plate on top of the pole and makes an angle $\alpha = 20^\circ$ at the point of attachment. The pole has length $L = 2.5 \text{ m}$ and a hollow circular cross section with outer diameter $d_2 = 280 \text{ mm}$ and inner diameter $d_1 = 220 \text{ mm}$. The circular cover plate has diameter $1.5d_2$.

Determine the allowable tensile force T_{allow} in the cable if the allowable compressive stress in the aluminum pole is 90 MPa .

**Solution 5.12-6**

$$\sigma_{\text{allow}} = 90 \text{ MPa} \quad d_1 = 220 \text{ mm}$$

$$d_2 = 280 \text{ mm}$$

$$t = \frac{d_2 - d_1}{2} \quad \alpha = 20^\circ \quad L = 2.5 \text{ m}$$

$$A = \frac{\pi}{4} (d_2^2 - d_1^2) \quad I = \frac{\pi}{64} (d_2^4 - d_1^4)$$

$$A = 2.356 \times 10^4 \text{ mm}^2 \quad I = 1.867 \times 10^8 \text{ mm}^4$$

$$P_N = T \cos(\alpha) \quad (\text{Axial force})$$

$$V = T \sin(\alpha) \quad (\text{Shear force})$$

$$M = VL + P_N \left(\frac{1.5d_2}{2} \right) \quad (\text{Moment})$$

ALLOWABLE TENSILE FORCE

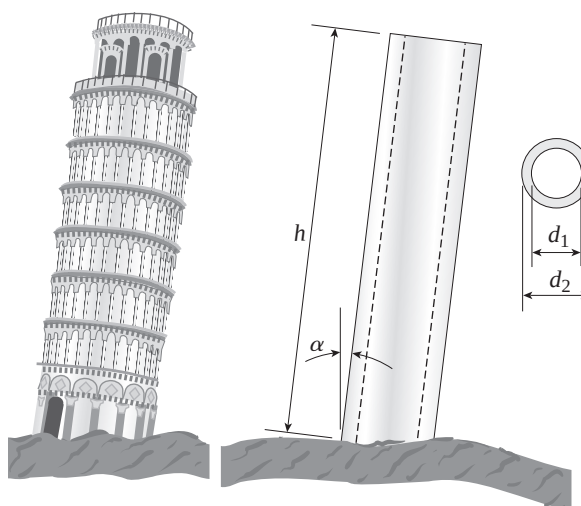
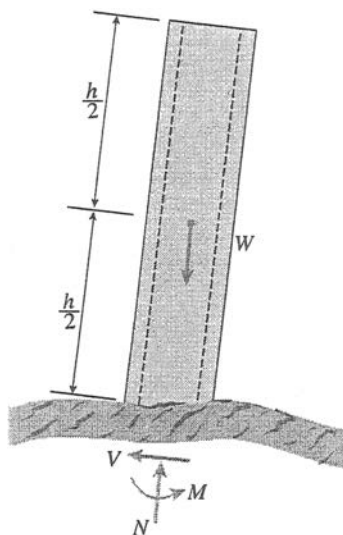
$$\sigma_c = -\frac{P_N}{A} - \frac{M}{I} \frac{d_2}{2} = -\frac{T \cos(\alpha)}{A} - \frac{T \sin(\alpha)L + T \cos(\alpha) \left(\frac{1.5d_2}{2} \right) \frac{d_2}{2}}{I}$$

$$T_{\text{allow}} = \frac{\sigma_{\text{allow}}}{\frac{\cos(\alpha)}{A} + \frac{\sin(\alpha)L + \cos(\alpha) \left(\frac{1.5d_2}{2} \right) \frac{d_2}{2}}{I}}$$

$$T_{\text{allow}} = 108.6 \text{ kN} \quad \leftarrow$$

Problem 5.12-7 Because of foundation settlement, a circular tower is leaning at an angle α to the vertical (see figure). The structural core of the tower is a circular cylinder of height h , outer diameter d_2 , and inner diameter d_1 . For simplicity in the analysis, assume that the weight of the tower is uniformly distributed along the height.

Obtain a formula for the maximum permissible angle α if there is to be no tensile stress in the tower.

**Solution 5.12-7 Leaning tower**

W = weight of tower
 α = angle of tilt

CROSS SECTION

$$A = \frac{\pi}{4}(d_2^2 - d_1^2)$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4)$$

$$= \frac{\pi}{64}(d_2^2 - d_1^2)(d_2^2 + d_1^2)$$

$$\frac{I}{A} = \frac{d_2^2 + d_1^2}{16}$$

$$c = \frac{d_2}{2}$$

AT THE BASE OF THE TOWER

$$N = W \cos \alpha \quad M = W \left(\frac{h}{2} \right) \sin \alpha$$

TENSILE STRESS (EQUAL TO ZERO)

$$\sigma_t = -\frac{N}{A} + \frac{Mc}{I} = -\frac{W \cos \alpha}{A} + \frac{W \left(\frac{h}{2} \sin \alpha \right) \left(\frac{d_2}{2} \right)}{I} = 0$$

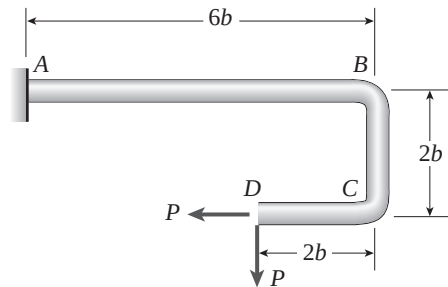
$$\therefore \frac{\cos \alpha}{A} = \frac{hd_2 \sin \alpha}{4I} \quad \tan \alpha = \frac{4I}{hd_2 A} = \frac{d_2^2 + d_1^2}{4hd_2}$$

MAXIMUM ANGLE α

$$\alpha = \arctan \frac{d_2^2 + d_1^2}{4hd_2} \quad \leftarrow$$

Problem 5.12-8 A steel bracket of solid circular cross section is subjected to two loads, each of which is $P = 4.5 \text{ kN}$ at D (see figure). Let the dimension variable $b = 240 \text{ mm}$.

- Find the minimum permissible diameter $d_{\min}$ of the bracket if the allowable normal stress is 110 MPa .
- Repeat part (a), including the weight of the bracket. The weight density of steel is 77.0 kN/m^3 .

**Solution 5.12-8**

(a) $P = 4.5 \text{ kN}$ $b = 240 \text{ mm}$ $\sigma_a = 110 \text{ MPa}$ $\gamma_s = 77 \frac{\text{kN}}{\text{m}^3}$

Reactions at support A:

$$N_A = P \quad M_A = P(2b + 4b) = 6.5 \text{ kN} \cdot \text{m}$$

(compression) (compression on bottom)

Max. normal stress at top of cross section at A (compression): $N/A + My/I$

$$\sigma_{\max} = \frac{P}{\left(\frac{\pi d^2}{4} \right)} + \frac{(6Pb) \left(\frac{d}{2} \right)}{\left(\frac{\pi d^4}{64} \right)} = \frac{4P}{\pi d^2} + \frac{192Pb}{\pi d^3} \quad \sigma_{\max} = \frac{4P(48b + d)}{\pi d^3}$$

Set $\sigma_{\max} = \sigma_a$ and solve for required diameter d :

$$(\pi \sigma_a) d^3 - (4P)d - 192Pb = 0 \quad \leftarrow \text{solve numerically or by trial \& error to find } d_{\text{reqd}} = 8.46 \text{ cm}$$

(b) DISTRIBUTED WEIGHT OF BRACKET = $w = \gamma A$; AFFECTS MOMENT M AT A, NOT AXIAL FORCE N $w = \gamma_s \left(\frac{\pi d^2}{4} \right)$

$$M_A = P(2b) + P(4b) + w(6b)(3b) + w(2b)(6b) + w(2b)(5b) \rightarrow 40wb^2 + 6Pb$$

OR $M_A = 2Pb + 40wb^2$ < compression on bottom

$$\text{so } \sigma_{\max} = \frac{P}{\left(\frac{\pi d^2}{4} \right)} + \frac{(6Pb + 40wb^2) \left(\frac{d}{2} \right)}{\left(\frac{\pi d^4}{64} \right)}$$

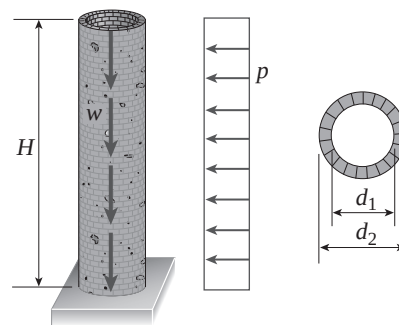
or

$$\pi \sigma_a d^3 - (320\pi \gamma_s b^2 d^2 + 192Pb + 4Pd) = 0 \quad \leftarrow \text{solve numerically or by trial \& error to find } d_{\text{reqd}} = 8.91 \text{ cm}$$

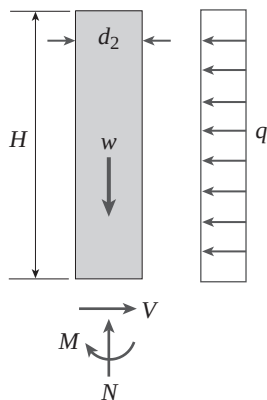
$$\frac{8.91 - 8.46}{8.46} = 5.3\% \quad \text{increase in diameter}$$

Problem 5.12-9 A cylindrical brick chimney of height H weighs $w = 825$ lb/ft of height (see figure). The inner and outer diameters are $d_1 = 3$ ft and $d_2 = 4$ ft, respectively. The wind pressure against the side of the chimney is $p = 10$ lb/ft² of projected area.

Determine the maximum height H if there is to be no tension in the brickwork.



Solution 5.12-9 Brick Chimney



p = wind pressure

q = intensity of load $= pd_2$

d_2 = outer diameter

d_1 = inner diameter

W = total weight of chimney $= wH$

CROSS SECTION

$$A = \frac{\pi}{4} (d_2^2 - d_1^2)$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) = \frac{\pi}{64} (d_2^2 - d_1^2)(d_2^2 + d_1^2)$$

$$\frac{I}{A} = \frac{1}{16} (d_2^2 + d_1^2) \quad c = \frac{d_2}{2}$$

AT BASE OF CHIMNEY

$$N = W = wH \quad M = qH \left(\frac{H}{2} \right) = \frac{1}{2} pd_2 H^2$$

TENSILE STRESS (EQUAL TO ZERO)

$$\sigma_1 = -\frac{N}{A} + \frac{Md_2}{2I} = 0 \quad \text{or} \quad \frac{M}{N} = \frac{2I}{Ad_2}$$

$$\frac{pd_2 H^2}{2wH} = \frac{d_2^2 + d_1^2}{8d_2}$$

$$\text{SOLVE FOR } H \quad H = \frac{w(d_2^2 + d_1^2)}{4pd_2^2} \quad \leftarrow$$

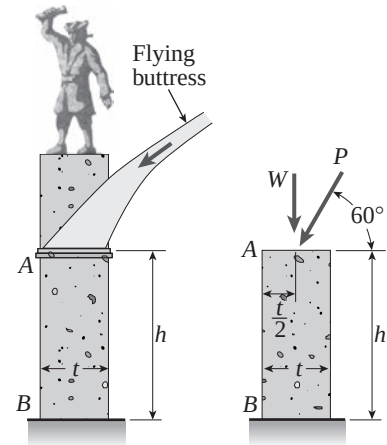
SUBSTITUTE NUMERICAL VALUES

$$w = 825 \text{ lb/ft} \quad d_2 = 4 \text{ ft} \quad d_1 = 3 \text{ ft}$$

$$q = 10 \text{ lb/ft}^2 \quad H_{\max} = 32.2 \text{ ft} \quad \leftarrow$$

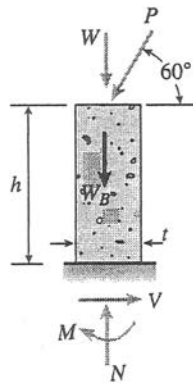
Problem 5.12-10 A flying buttress transmits a load $P = 25$ kN, acting at an angle of 60° to the horizontal, to the top of a vertical buttress AB (see figure). The vertical buttress has height $h = 5.0$ m and rectangular cross section of thickness $t = 1.5$ m and width $b = 1.0$ m (perpendicular to the plane of the figure). The stone used in the construction weighs $\gamma = 26$ kN/m³.

What is the required weight W of the pedestal and statue above the vertical buttress (that is, above section A) to avoid any tensile stresses in the vertical buttress?



Solution 5.12-10 Flying buttress

FREE-BODY DIAGRAM OF VERTICAL BUTTRESS



$$P = 25 \text{ kN}$$

$$h = 5.0 \text{ m}$$

$$t = 1.5 \text{ m}$$

$$b = \text{width of buttress perpendicular to the figure}$$

$$b = 1.0 \text{ m}$$

$$\gamma = 26 \text{ kN/m}^3$$

$$W_B = \text{weight of vertical buttress}$$

$$= bth\gamma$$

$$= 195 \text{ kN}$$

CROSS SECTION

$$A = bt = (1.0 \text{ m})(1.5 \text{ m}) = 1.5 \text{ m}^2$$

$$S = \frac{1}{6}bt^2 = \frac{1}{6}(1.0 \text{ m})(1.5 \text{ m})^2 = 0.375 \text{ m}^3$$

AT THE BASE

$$\begin{aligned} N &= W + W_B + P \sin 60^\circ \\ &= W + 195 \text{ kN} + (25 \text{ kN}) \sin 60^\circ \\ &= W + 216.651 \text{ kN} \end{aligned}$$

$$\begin{aligned} M &= (P \cos 60^\circ) h = (25 \text{ kN})(\cos 60^\circ)(5.0 \text{ m}) \\ &= 62.5 \text{ kN} \cdot \text{m} \end{aligned}$$

TENSILE STRESS (EQUAL TO ZERO)

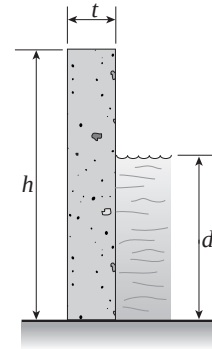
$$\begin{aligned} \sigma_t &= -\frac{N}{A} + \frac{M}{S} \\ &= -\frac{W + 216.651 \text{ kN}}{1.5 \text{ m}^2} + \frac{62.5 \text{ kN} \cdot \text{m}}{0.375 \text{ m}^3} = 0 \end{aligned}$$

$$\text{or } -W - 216.651 \text{ kN} + 250 \text{ kN} = 0$$

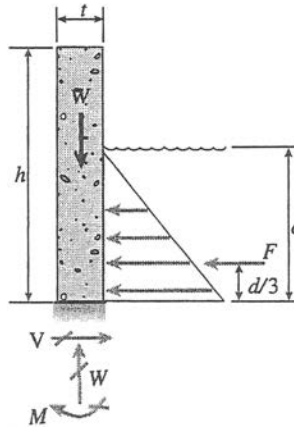
$$W = 33.3 \text{ kN} \quad \leftarrow$$

Problem 5.12-11 A plain concrete wall (i.e., a wall with no steel reinforcement) rests on a secure foundation and serves as a small dam on a creek (see figure). The height of the wall is $h = 6.0$ ft and the thickness of the wall is $t = 1.0$ ft.

- Determine the maximum tensile and compressive stresses σ_t and σ_c , respectively, at the base of the wall when the water level reaches the top ($d = h$). Assume plain concrete has weight density $\gamma_c = 145$ lb/ft³.
- Determine the maximum permissible depth $d_{\max}$ of the water if there is to be no tension in the concrete.



Solution 5.12-11 Concrete wall



h = height of wall

t = thickness of wall

b = width of wall (perpendicular to the figure)

γ_c = weight density of concrete

γ_w = weight density of water

d = depth of water

W = weight of wall

$W = bht\gamma_c$

F = resultant force for the water pressure

MAXIMUM WATER PRESSURE = $\gamma_w d$

$$F = \frac{1}{2}(d)(\gamma_w d)(b) = \frac{1}{2}bd^2\gamma_w$$

$$M = F\left(\frac{d}{3}\right) = \frac{1}{6}bd^3\gamma_w$$

$$A = bt \quad S = \frac{1}{6}bt^2$$

STRESSES AT THE BASE OF THE WALL
(d = DEPTH OF WATER)

$$\sigma_t = -\frac{W}{A} + \frac{M}{S} = -h\gamma_c + \frac{d^3\gamma_w}{t^2} \quad \text{Eq. (1)}$$

$$\sigma_c = -\frac{W}{A} - \frac{M}{S} = -h\gamma_c - \frac{d^3\gamma_w}{t^2} \quad \text{Eq. (2)}$$

(a) STRESSES AT THE BASE WHEN $d = h$

$$h = 6.0 \text{ ft} = 72 \text{ in.} \quad d = 72 \text{ in.}$$

$$t = 1.0 \text{ ft} = 12 \text{ in.}$$

$$\gamma_c = 145 \text{ lb/ft}^3 = \frac{145}{1728} \text{ lb/in.}^3$$

$$\gamma_w = 62.4 \text{ lb/ft}^3 = \frac{62.4}{1728} \text{ lb/in.}^3$$

Substitute numerical values into Eqs. (1) and (2):

$$\sigma_t = -6.042 \text{ psi} + 93.600 \text{ psi} = 87.6 \text{ psi} \quad \leftarrow$$

$$\sigma_c = -6.042 \text{ psi} - 93.600 \text{ psi} = -99.6 \text{ psi} \quad \leftarrow$$

$$d^3 = (72 \text{ in.})(12 \text{ in.})^2 \left(\frac{145}{62.4} \right) = 24,092 \text{ in.}^3$$

$$d_{\max} = 28.9 \text{ in.} \quad \leftarrow$$

(b) MAXIMUM DEPTH FOR NO TENSION

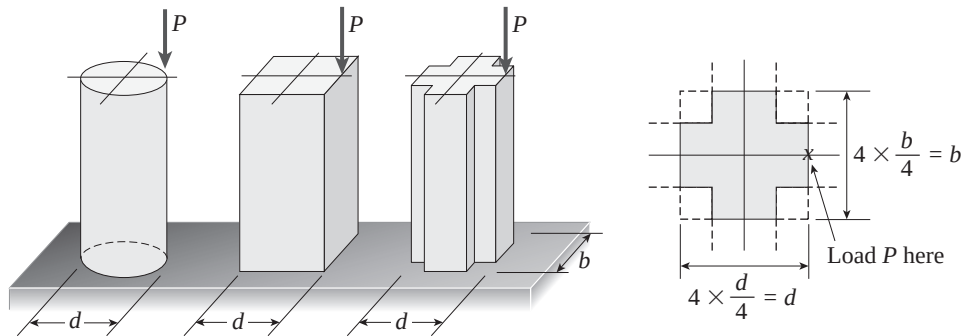
Set $\sigma_t = 0$ in Eq. (1):

$$-h\gamma_c + \frac{d^3\gamma_w}{t^2} = 0 \quad d^3 = ht^2 \left(\frac{\gamma_c}{\gamma_w} \right)$$

Eccentric Axial Loads

Problem 5.12-12 A circular post, a rectangular post, and a post of cruciform cross section are each compressed by loads that produce a resultant force P acting at the edge of the cross section (see figure). The diameter of the circular post and the depths of the rectangular and cruciform posts are the same.

- For what width b of the rectangular post will the maximum tensile stresses be the same in the circular and rectangular posts?
- Repeat part (a) for the post with cruciform cross section.
- Under the conditions described in parts (a) and (b), which post has the largest compressive stress?



Solution 5.12-12

(a) EQUAL MAXIMUM TENSILE STRESSES

CIRCULAR POST

$$A = \frac{\pi}{4} d^2 \quad S = \frac{\pi}{32} d^3 \quad M = \frac{Pd}{2}$$

Tension

$$\sigma_t = -\frac{P}{A} + \frac{M}{S} = -\frac{4P}{\pi d^2} + \frac{16P}{\pi d^2} = \frac{12P}{\pi d^2}$$

$$\begin{aligned} \text{COMPRESSION } \sigma_c &= -\frac{P}{A} - \frac{M}{S} \\ &= -\frac{4P}{\pi d^2} - \frac{16P}{\pi d^2} = -\frac{20P}{\pi d^2} \end{aligned}$$

RECTANGULAR POST

$$A = bd \quad S = \frac{bd^2}{6} \quad M = \frac{Pd}{2}$$

$$\text{TENSION } \sigma_t = -\frac{P}{A} + \frac{M}{S} = -\frac{P}{bd} + \frac{3P}{bd} = \frac{2P}{bd}$$

$$\text{COMPRESSION } \sigma_c = -\frac{P}{A} - \frac{M}{S} = -\frac{P}{bd} - \frac{3P}{bd} = -\frac{4P}{bd}$$

Equate tensile stress expressions; solve for b :

$$\frac{12P}{\pi d^2} = \frac{2P}{bd} \quad \frac{6}{\pi d} = \frac{1}{b} \quad b = \frac{\pi d}{6} \quad \leftarrow$$

(b) CRUCIFORM CROSS SECTION

$$A = \left[bd - \left(\frac{b}{2} \frac{d}{2} \right) \right]$$

$$S = \left[\frac{b}{2} \frac{d^3}{12} + \frac{b}{2} \left(\frac{d}{2} \right)^2 \frac{1}{12} \right] \frac{2}{d} = \frac{3}{32} bd^2$$

$$M = \frac{Pd}{2 \left(\frac{3}{32} bd^2 \right)} = \frac{16P}{3bd}$$

$$\begin{aligned} \text{TENSION } \sigma_t &= -\frac{P}{A} + \frac{M}{S} \\ &= -\frac{4P}{3bd} + \frac{16P}{3bd} = \frac{12P}{3bd} \end{aligned}$$

$$\begin{aligned} \text{COMPRESSION } \sigma_c &= -\frac{P}{A} - \frac{M}{S} \\ &= -\frac{4P}{3bd} - \frac{16P}{3bd} = -\frac{20P}{3bd} \end{aligned}$$

Equate compressive stresses & solve for b :

$$\frac{12P}{\pi d^2} = \frac{20P}{3bd} \quad \frac{3}{\pi d} = \frac{1}{b} \quad b = \frac{\pi d}{3} \quad \leftarrow$$

(c) THE LARGEST COMPRESSIVE STRESS

Substitute expressions for b above & compare compressive stresses:

CIRCULAR POST

$$\sigma_c = -\frac{20P}{\pi d^2}$$

RECTANGULAR POST

$$\sigma_c = -\frac{4P}{\left(\frac{\pi d}{6} \right) d} = -\frac{24P}{\pi d^2}$$

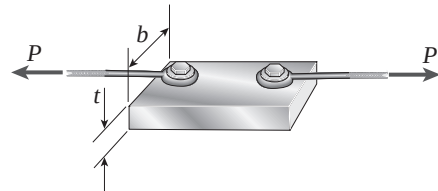
CRUCIFORM POST

$$\sigma_c = -\frac{20P}{3 \frac{\pi d}{3} d} = -\frac{20P}{\pi d^2}$$

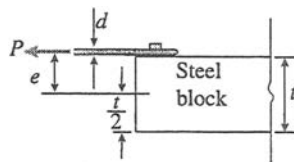
Rectangular post has the largest compressive stress. $\leftarrow$

Problem 5.12-13 Two cables, each carrying a tensile force $P = 1200$ lb, are bolted to a block of steel (see figure). The block has thickness $t = 1$ in. and width $b = 3$ in.

- If the diameter d of the cable is 0.25 in., what are the maximum tensile and compressive stresses σ_t and σ_c , respectively, in the block?
- If the diameter of the cable is increased (without changing the force P), what happens to the maximum tensile and compressive stresses?



Solution 5.12-13 Steel block loaded by cables



$$P = 1200 \text{ lb} \quad d = 0.25 \text{ in.}$$

$$t = 1.0 \text{ in.} \quad e = \frac{t}{2} + \frac{d}{2} = 0.625 \text{ in.}$$

$$\begin{aligned} b &= \text{width of block} \\ &= 3.0 \text{ in.} \end{aligned}$$

CROSS SECTION OF BLOCK

$$A = bt = 30 \text{ in.}^2 \quad I = \frac{1}{12}bt^3 = 0.25 \text{ in.}^4$$

(a) MAXIMUM TENSILE STRESS (AT TOP OF BLOCK)

$$y = \frac{t}{2} = 0.5 \text{ in.}$$

$$\begin{aligned} \sigma_t &= \frac{P}{A} + \frac{Pey}{I} \\ &= \frac{1200 \text{ lb}}{3 \text{ in.}^2} + \frac{(1200 \text{ lb})(0.625 \text{ in.})(0.5 \text{ in.})}{0.25 \text{ in.}^4} \\ &= 400 \text{ psi} + 1500 \text{ psi} = 1900 \text{ psi} \quad \leftarrow \end{aligned}$$

MAXIMUM COMPRESSIVE STRESS (AT BOTTOM OF BLOCK)

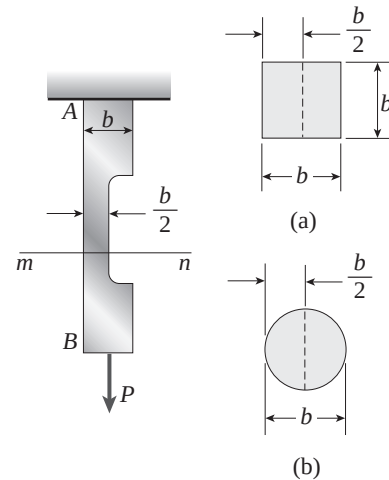
$$y = -\frac{t}{2} = -0.5 \text{ in.}$$

$$\begin{aligned} \sigma_c &= \frac{P}{A} + \frac{Pey}{I} \\ &= \frac{1200 \text{ lb}}{3 \text{ in.}^2} + \frac{(1200 \text{ lb})(0.625 \text{ in.})(-0.5 \text{ in.})}{0.25 \text{ in.}^4} \\ &= 400 \text{ psi} - 1500 \text{ psi} = -1100 \text{ psi} \quad \leftarrow \end{aligned}$$

(b) If d is INCREASED, the eccentricity e increases and both stresses increase in magnitude.

Problem 5.12-14 A bar AB supports a load P acting at the centroid of the end cross section (see figure). In the middle region of the bar the cross-sectional area is reduced by removing one-half of the bar.

- (a) If the end cross sections of the bar are square with sides of length b , what are the maximum tensile and compressive stresses σ_t and σ_c , respectively, at cross section mn within the reduced region?
- (b) If the end cross sections are circular with diameter b , what are the maximum stresses σ_t and σ_c ?



Solution 5.12-14 Bar with reduced cross section

(a) SQUARE BAR

Cross section mn is a rectangle.

$$A = (b)\left(\frac{b}{2}\right) = \frac{b^2}{2} \quad I = \frac{1}{12}(b)\left(\frac{b}{2}\right)^3 = \frac{b^4}{96}$$

$$M = P\left(\frac{b}{4}\right) \quad c = \frac{b}{4}$$

STRESSES

$$\sigma_t = \frac{P}{A} + \frac{Mc}{I} = \frac{2P}{b^2} + \frac{6P}{b^2} = \frac{8P}{b^2} \quad \leftarrow$$

$$\sigma_c = \frac{P}{A} - \frac{Mc}{I} = \frac{2P}{b^2} - \frac{6P}{b^2} = -\frac{4P}{b^2} \quad \leftarrow$$

(b) CIRCULAR BAR

Cross section mn is a semicircle

$$A = \frac{1}{2}\left(\frac{\pi b^2}{4}\right) = \frac{\pi b^2}{8} = 0.3927b^2$$

From Appendix E, Case 10:

$$I = 0.1098\left(\frac{b}{2}\right)^4 = 0.006860b^4$$

$$M = P\left(\frac{2b}{3\pi}\right) = 0.2122Pb$$

FOR TENSION:

$$c_t = \frac{4r}{3\pi} = \frac{2b}{3\pi} = 0.2122b$$

FOR COMPRESSION:

$$c_c = r - c_t = \frac{b}{2} - \frac{2b}{3\pi} = 0.2878b$$

STRESSES

$$\sigma_t = \frac{P}{A} + \frac{Mc_t}{I} = \frac{P}{0.3927b^2} + \frac{(0.2122Pb)(0.2122b)}{0.006860b^4}$$

$$= 2.546 \frac{P}{b^2} + 6.564 \frac{P}{b^2} = 9.11 \frac{P}{b^2} \quad \leftarrow$$

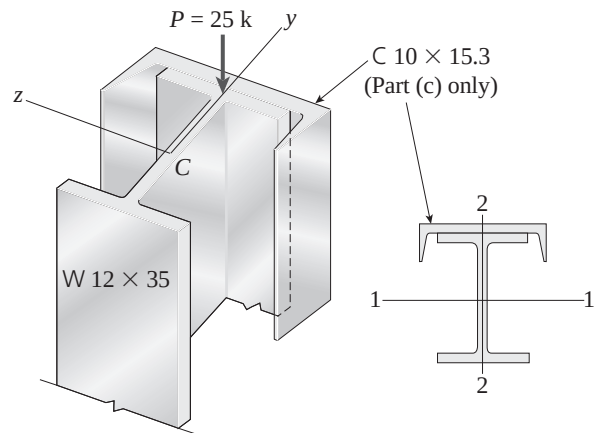
$$\sigma_c = \frac{P}{A} - \frac{Mc_c}{I}$$

$$= \frac{P}{0.3927b^2} - \frac{(0.2122Pb)(0.2878b)}{0.006860b^4}$$

$$= 2.546 \frac{P}{b^2} - 8.903 \frac{P}{b^2} = -6.36 \frac{P}{b^2} \quad \leftarrow$$

Problem 5.12-15 A short column constructed of a W 12 × 35 wide-flange shape is subjected to a resultant compressive load $P = 12$ k having its line of action at the midpoint of one flange (see figure).

- Determine the maximum tensile and compressive stresses σ_t and σ_c , respectively, in the column.
- Locate the neutral axis under this loading condition.
- Recompute maximum tensile and compressive stresses if a C 10 × 15.3 is attached to one flange, as shown.



Solution 5.12-15 Column of wide-flange shape

PROPERTIES OF EACH SHAPE:

W 12 × 35 C 10 × 15.3

 $A_w = 10.3 \text{ in.}^2$ $A_c = 4.48 \text{ in.}^2$ $h_w = 12.5 \text{ in.}$ $t_{wc} = 0.24 \text{ in.}$ $t_f = 0.52 \text{ in.}$ $x_p = 0.634 \text{ in.}$ $I_w = 285 \text{ in.}^4$ $I_c = 2.27 \text{ in.}^4$ (2-2 axis)

- (a) THE MAXIMUM TENSILE AND COMPRESSIVE STRESSES
LOCATION OF CENTROID FOR W 12 × 35 ALONE

$$c_w = \frac{h_w}{2} \quad c_w = 6.25 \text{ in.}$$

$$P = 25 \text{ k} \quad e_w = \frac{h_w}{2} - \frac{t_f}{2} \quad e_w = 5.99 \text{ in.}$$

$$\sigma_t = -\frac{P}{A_w} + \frac{Pe_w}{I_w} c_w \quad \sigma_t = 857 \text{ psi} \quad \leftarrow$$

$$\sigma_c = -\frac{P}{A_w} - \frac{Pe_w}{I_w} c_w \quad \sigma_c = -5711 \text{ psi} \quad \leftarrow$$

- (b) NEUTRAL AXIS (W SHAPE ALONE)

$$y_0 = -\frac{I_w}{A_w e_w} \quad y_0 = -4.62 \text{ in.} \quad \leftarrow$$

- (c) COMBINED COLUMN, W 12 × 35 with C 10 × 15.3

$$h = h_w + t_{wc}$$

$$h = 12.74 \text{ in.}$$

$$A = A_w + A_c \quad A = 14.78 \text{ in.}^2$$

LOCATION OF CENTROID OF COMBINED SHAPE

$$c = \frac{A_w \left(\frac{h_w}{2} \right) + A_c (h - x_p)}{A} \quad c = 8.025 \text{ in.}$$

$$I = I_w + A_w \left(c - \frac{h_w}{2} \right)^2 + I_c + A_c (h - x_p - c)^2$$

$$I = 394.334 \text{ in.}^4$$

$$P = 25 \text{ k} \quad e = h_w - \frac{t_f}{2} - c \quad e = 4.215 \text{ in.}$$

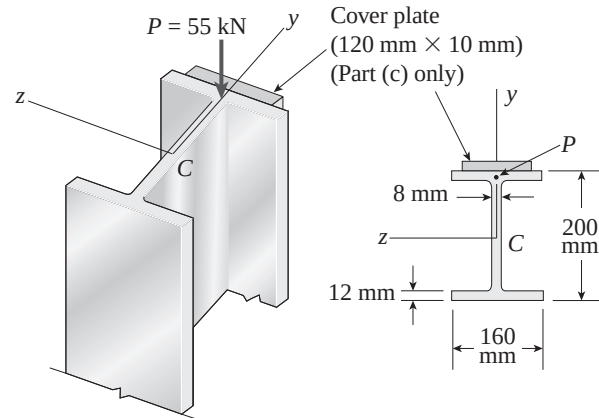
$$\sigma_t = -\frac{P}{A} + \frac{Pe}{I} c \quad \sigma_t = 453 \text{ psi} \quad \leftarrow$$

$$\sigma_c = -\frac{P}{A} - \frac{Pe}{I} (h - c) \quad \sigma_c = -2951 \text{ psi} \quad \leftarrow$$

$$y_0 = -\frac{I}{Ae} \quad y_0 = -6.33 \text{ in. (from centroid)} \quad \leftarrow$$

Problem 5.12-16 A short column of wide-flange shape is subjected to a compressive load that produces a resultant force $P = 55 \text{ kN}$ acting at the midpoint of one flange (see figure).

- Determine the maximum tensile and compressive stresses σ_t and σ_c , respectively, in the column.
- Locate the neutral axis under this loading condition.
- Recompute maximum tensile and compressive stresses if a $120 \text{ mm} \times 10 \text{ mm}$ cover plate is added to one flange as shown.

**Solution 5.12-16**

$$P = 55 \text{ kN}$$

- (a) MAXIMUM TENSILE AND COMPRESSIVE STRESSES FOR W SHAPE ALONE

PROPERTIES AND DIMENSIONS FOR W SHAPE

$$b = 160 \text{ mm} \quad d = 200 \text{ mm}$$

$$t_f = 12 \text{ mm} \quad t_w = 8 \text{ mm}$$

$$A_w = bd - (b - t_w)(d - 2t_f)$$

$$A_w = 5.248 \times 10^3 \text{ mm}^2$$

$$I_w = \frac{bd^3}{12} - \frac{(b - t_w)(d - 2t_f)^3}{12}$$

$$I_w = 3.761 \times 10^7 \text{ mm}^4$$

$$e = \frac{d}{2} - \frac{t_f}{2} \quad e = 94 \text{ mm}$$

$$\sigma_t = -\frac{P}{A_w} + \frac{Pe}{I_w} \frac{d}{2} \quad \sigma_t = 3.27 \text{ MPa} \quad \leftarrow$$

$$\sigma_c = -\frac{P}{A_w} - \frac{Pe}{I_w} \frac{d}{2} \quad \sigma_c = -24.2 \text{ MPa} \quad \leftarrow$$

- (b) NEUTRAL AXIS (W SHAPE ALONE)

$$y_0 = -\frac{I_w}{A_w e} \quad y_0 = -76.2 \text{ mm} \quad \leftarrow$$

- (c) COMBINED COLUMN: W SHAPE & COVER PLATE

$$b_p = 120 \text{ mm} \quad t_p = 10 \text{ mm}$$

$$h = d + t_p$$

$$h = 210 \text{ mm}$$

$$A = A_w + b_p t_p \quad A = 6.448 \times 10^3 \text{ mm}^2$$

CENTROID OF COMPOSITE SECTION

$$c = \frac{A_w \frac{d}{2} + b_p t_p \left(d + \frac{t_p}{2} \right)}{A}$$

$$c = 119.541 \text{ mm}$$

$$I = I_w + A_w \left(c - \frac{d}{2} \right)^2 + \frac{b_p t_p^3}{12} + b_p t_p \left(d + \frac{t_p}{2} - c \right)^2$$

$$I = 4.839 \times 10^7 \text{ mm}^4$$

$$e = d - \frac{t_f}{2} - c \quad e = 74.459 \text{ mm}$$

$$\sigma_t = -\frac{P}{A} + \frac{Pe}{I} c \quad \sigma_t = 1.587 \text{ MPa} \quad \leftarrow$$

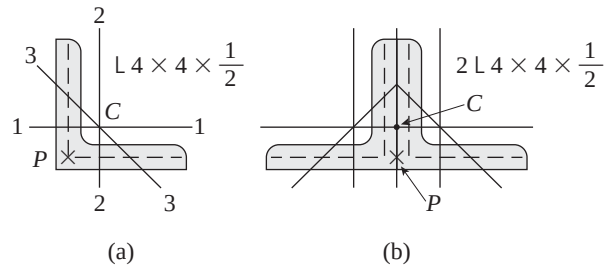
$$\sigma_c = -\frac{P}{A_w} - \frac{Pe}{I_w} (h - c) \quad \sigma_c = -20.3 \text{ MPa} \quad \leftarrow$$

NEUTRAL AXIS

$$y_0 = -\frac{I}{Ae} \quad y_0 = -100.8 \text{ mm (from centroid)}$$

Problem 5.12-17 A tension member constructed of an $L 4 \times 4 \times \frac{1}{2}$ inch angle section (see Table F-4(a) in Appendix F) is subjected to a tensile load $P = 12.5$ kips that acts through the point where the mid-lines of the legs intersect (see figure part a).

- Determine the maximum tensile stress σ_t in the angle section.
- Recompute the maximum tensile stress if two angles are used and P is applied as shown in the figure part b.



Solution 5.12-17 Angle section in tension

(a) ONE ANGLE: $L 4 \times 4 \times \frac{1}{2}$

$$A_L = 3.75 \text{ in.}^2 \quad r_{\min} = 0.776 \text{ in.}$$

$$t = 0.5 \text{ in.}$$

$$c = 1.18 \text{ in.}$$

$$e = \left(c - \frac{t}{2} \right) \sqrt{2} \quad e = 1.315 \text{ in.}$$

$$P = 12.5 \text{ k}$$

$$c_1 = c \sqrt{2} \quad c_1 = 1.699 \text{ in.}$$

$$I_3 = A_L r_{\min}^2 \quad I_3 = 2.258 \text{ in.}^4$$

$$M = Pe \quad M = 16.44 \text{ k-in.}$$

MAXIMUM TENSILE STRESS OCCURS AT CORNER

$$\sigma_t = \frac{P}{A_L} + \frac{Mc_1}{I_3} \quad \sigma_t = 15.48 \text{ ksi} \quad \leftarrow$$

(b) TWO ANGLES: $L 4 \times 4 \times \frac{1}{2}$

$$A = 2A_L$$

$$t = 0.5 \text{ in.}$$

$$c = 1.18 \text{ in.}$$

$$I_L = 5.52 \text{ in.}^4 \text{ (2-2 axis)}$$

$$e = \left(c - \frac{t}{2} \right) \quad e = 0.93 \text{ in.}$$

$$P = 12.5 \text{ k}$$

$$I = 2I_L \quad I = 11.04 \text{ in.}^4$$

$$M = Pe \quad M = 11.625 \text{ k-in.}$$

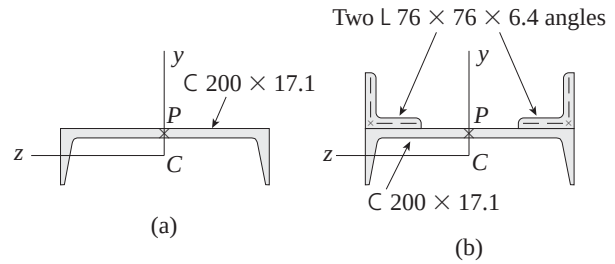
MAXIMUM TENSILE STRESS OCCURS AT THE LOWER EDGE

$$\sigma_t = \frac{P}{A} + \frac{Mc}{I} \quad \sigma_t = 2.91 \text{ ksi} \quad \leftarrow$$

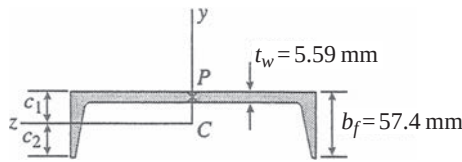
Problem 5.12-18 A short length of a C 200 × 17.1 channel is subjected to an axial compressive force P that has its line of action through the midpoint of the web of the channel (see figure part a).

- Determine the equation of the neutral axis under this loading condition.
- If the allowable stresses in tension and compression are 76 MPa and 52 MPa respectively, find the maximum permissible load $P_{\max}$.
- Repeat parts (a) and (b) if two L 76 × 76 × 6.4 angles are added to the channel as shown in the figure part b.

See Table F-3(b) in Appendix F for channel properties and Table F-4(b) for angle properties.



Solution 5.12-18



C 200 × 17.1

$$A_c = 2170 \text{ mm}^2 \quad d_c = 203 \text{ mm} \quad c_1 = 14.5 \text{ mm}$$

$$I_c = 0.545 \times 10^6 \text{ mm}^4 \quad (z\text{-axis})$$

$$c_2 = b_f - c_1 \quad c_2 = 42.9 \text{ mm}$$

ALLOWABLE STRESSES

$$\sigma_t = 76 \text{ MPa} \quad \sigma_c = -52 \text{ MPa}$$

ECCENTRICITY OF THE LOAD

$$e = c_1 - \frac{t_w}{2} \quad e = 11.705 \text{ mm}$$

(a) LOCATION OF THE NEUTRAL AXIS (CHANNEL ALONE)

$$y_0 = \frac{-I_c}{A_c \cdot e} \quad y_0 = -21.5 \text{ mm} \quad \leftarrow$$

(b) FIND $P_{\max}$

$$\sigma_t = -\frac{P}{A} + \frac{Pe}{I} c_2 \quad P = \frac{\sigma_t}{-\frac{1}{A_c} + \frac{e}{I} c_2^2}$$

$$P = 165.025 \text{ kN}$$

$$\sigma_c = -\frac{P}{A} - \frac{Pe}{I} c_1$$

$$P = \frac{\sigma_c}{-\frac{1}{A_c} - \frac{e}{I} c_1}$$

$$P_{\max} = 67.3 \text{ kN} \quad \leftarrow$$

(c) COMBINED COLUMN WITH 2 ANGLES

L 76 × 76 × 6.4

$$A_L = 929 \text{ mm}^2 \quad I_L = 0.512 \times 10^6 \text{ mm}^4$$

$$c_L = 21.2 \text{ mm}$$

COMPOSITE SECTION

$$A = A_c + 2A_L \quad A = 4.028 \times 10^3 \text{ mm}^2$$

$$h = b_f + 76 \text{ mm} \quad h = 133.4 \text{ mm}$$

CENTROID OF COMPOSITE SECTION

$$c = \frac{A_c(b_f - c_1) + 2A_L(b_f + c_L)}{A}$$

$$c = 59.367 \text{ mm}$$

$$I = I_c + A_c(b_f - c_1 - c)^2 + 2I_L + 2A_L(b_f + c_L - c)^2$$

$$I = 2.845 \times 10^6 \text{ mm}^4$$

$$e = b_f - \frac{t_w}{2} - c \quad e = -4.762 \text{ mm}$$

$$b_f = 57.4 \text{ mm}$$

LOCATION OF THE NEUTRAL AXIS

$$y_0 = -\frac{I}{Ae} \quad y_0 = 148.3 \text{ mm} \quad \leftarrow$$

$$y_0 = 148.3 \text{ mm} > h = 133.4 \text{ mm} \quad \leftarrow$$

Thus, this composite section has no tensile stress.

$$\sigma_c = -\frac{P}{A} + \frac{Pe}{I}c$$

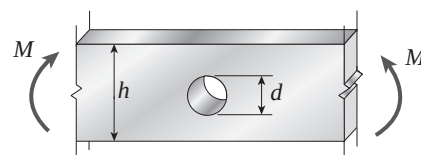
$$P = \frac{\sigma_c}{-\frac{1}{A} + \frac{e}{I}c} \quad P_{\max} = 149.6 \text{ kN} \quad \leftarrow$$

Stress Concentrations

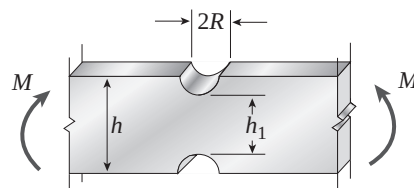
The problems for Section 5.13 are to be solved considering the stress-concentration factors.

Problem 5.13-1 The beams shown in the figure are subjected to bending moments $M = 2100 \text{ lb-in.}$ Each beam has a rectangular cross section with height $h = 1.5 \text{ in.}$ and width $b = 0.375 \text{ in.}$ (perpendicular to the plane of the figure).

- For the beam with a hole at midheight, determine the maximum stresses for hole diameters $d = 0.25, 0.50, 0.75,$ and 1.00 in.
- For the beam with two identical notches (inside height $h_1 = 1.25 \text{ in.}$), determine the maximum stresses for notch radii $R = 0.05, 0.10, 0.15,$ and 0.20 in.



(a)



(b)

Probs. 5.13-1 through 5.13-4

Solution 5.13-1

$$M = 2100 \text{ lb-in.} \quad h = 1.5 \text{ in.} \quad b = 0.375 \text{ in.}$$

(a) BEAM WITH A HOLE

$$\frac{d}{h} \leq \frac{1}{2} \quad \text{Eq. (5-62):} \quad \sigma_c = \frac{6Mh}{b(h^3 - d^3)} = \frac{50,400}{3.375 - d^3} \quad (1)$$

$$\frac{d}{h} \geq \frac{1}{2} \quad \text{Eq. (5-61):} \quad \sigma_B = \frac{12Md}{b(h^3 - d^3)} = \frac{67,200 d}{3.375 - d^3} \quad (2)$$

$d \text{ (in.)}$	$\frac{d}{h}$	σ_c Eq. (1) (psi)	σ_B Eq. (2) (psi)	$\sigma_{\max}$ (psi)
0.25	0.1667	15,000	—	15,000
0.50	0.3333	15,500	—	15,500
0.75	0.5000	17,100	17,100	17,100
1.00	0.6667	—	28,300	28,300

NOTE: The larger the hole, the larger the stress.

(b) BEAM WITH NOTCHES

$$h_1 = 1.25 \text{ in.} \quad \frac{h}{h_1} = \frac{1.5 \text{ in.}}{1.25 \text{ in.}} = 1.2$$

Eq. (5-63):

$$\sigma_{\text{nom}} = \frac{6M}{bh_1^2} = 21,500 \text{ psi}$$

$R \text{ (in.)}$	$\frac{R}{h_1}$	K (Fig. 5-50)	$\sigma_{\max} = K\sigma_{\text{nom}}$ $\sigma_{\max} \text{ (psi)}$
0.05	0.04	3.0	65,000
0.10	0.08	2.3	49,000
0.15	0.12	2.1	45,000
0.20	0.16	1.9	41,000

NOTE: The larger the notch radius, the smaller the stress.

Problem 5.13-2 The beams shown in the figure are subjected to bending moments $M = 250 \text{ N} \cdot \text{m}$. Each beam has a rectangular cross section with height $h = 44 \text{ mm}$ and width $b = 10 \text{ mm}$ (perpendicular to the plane of the figure).

- For the beam with a hole at midheight, determine the maximum stresses for hole diameters $d = 10, 16, 22$, and 28 mm .
- For the beam with two identical notches (inside height $h_1 = 40 \text{ mm}$), determine the maximum stresses for notch radii $R = 2, 4, 6$, and 8 mm .

Solution 5.13-2

$$M = 250 \text{ N} \cdot \text{m} \quad h = 44 \text{ mm} \quad b = 10 \text{ mm}$$

(a) BEAM WITH A HOLE

$$\frac{d}{h} \leq \frac{1}{2} \quad \text{Eq. (5-62):}$$

$$\sigma_c = \frac{6Mh}{b(h^3 - d^3)} = \frac{6.6 \times 10^6}{85,180 - d^3} \text{ MPa} \quad (1)$$

$$\frac{d}{h} \geq \frac{1}{2} \quad \text{Eq. (5-61):}$$

$$\sigma_B = \frac{12Md}{b(h^3 - d^3)} = \frac{300 \times 10^3 d}{85,180 - d^3} \text{ MPa} \quad (2)$$

$d \text{ (mm)}$	$\frac{d}{h}$	σ_c Eq. (1) (MPa)	σ_B Eq. (2) (MPa)	$\sigma_{\max}$ (MPa)
10	0.227	78	—	78
16	0.364	81	—	81
22	0.500	89	89	89
28	0.636	—	133	133

NOTE: The larger the hole, the larger the stress.

(b) BEAM WITH NOTCHES

$$h_1 = 40 \text{ mm} \quad \frac{h}{h_1} = \frac{44 \text{ mm}}{40 \text{ mm}} = 1.1$$

$$\text{Eq. (5-63): } \sigma_{\text{nom}} = \frac{6M}{bh_1^2} = 93.8 \text{ MPa}$$

R (mm)	$\frac{R}{h_1}$	K (Fig. 5-50)	$\sigma_{\max} = K\sigma_{\text{nom}}$ $\sigma_{\max}$ (MPa)
2	0.05	2.6	240
4	0.10	2.1	200
6	0.15	1.8	170
8	0.20	1.7	160

NOTE: The larger the notch radius, the smaller the stress.

Problem 5.13-3 A rectangular beam with semicircular notches, as shown in part b of the figure, has dimensions $h = 0.88 \text{ in.}$ and $h_1 = 0.80 \text{ in.}$ The maximum allowable bending stress in the metal beam is $\sigma_{\max} = 60 \text{ ksi}$, and the bending moment is $M = 600 \text{ lb-in.}$

Determine the minimum permissible width $b_{\min}$ of the beam.

Solution 5.13-3 Beam with semicircular notches

$$h = 0.88 \text{ in.} \quad h_1 = 0.80 \text{ in.}$$

$$\sigma_{\max} = 60 \text{ ksi} \quad M = 600 \text{ lb-in.}$$

$$h = h_1 + 2R \quad R = \frac{1}{2}(h - h_1) = 0.04 \text{ in.}$$

$$\frac{R}{h_1} = \frac{0.04 \text{ in.}}{0.80 \text{ in.}} = 0.05$$

From Fig. 5-50: $K \approx 2.57$

$$\sigma_{\max} = K\sigma_{\text{nom}} = K\left(\frac{6M}{bh_1^2}\right)$$

$$60 \text{ ksi} = 2.57 \left[\frac{6(600 \text{ lb-in.})}{b(0.80 \text{ in.})^2} \right]$$

Solve for b :

$$b_{\min} \approx 0.24 \text{ in.} \quad \leftarrow$$

Problem 5.13-4 A rectangular beam with semicircular notches, as shown in part b of the figure, has dimension $h = 120 \text{ mm}$ and $h_1 = 100 \text{ mm}$. The maximum allowable bending stress in the plastic beam is $\sigma_{\max} = 6 \text{ MPa}$, and the bending moment is $M = 150 \text{ N} \cdot \text{m}$. Determine the minimum permissible width $b_{\min}$ of the beam.

Solution 5.13-4 Beam with semicircular notches

$$h = 120 \text{ mm} \quad h_1 = 100 \text{ mm}$$

$$\sigma_{\max} = 6 \text{ MPa} \quad M = 150 \text{ N} \cdot \text{m}$$

$$h = h_1 + 2R \quad R = \frac{1}{2}(h - h_1) = 10 \text{ mm}$$

$$\frac{R}{h_1} = \frac{10 \text{ mm}}{100 \text{ mm}} = 0.10$$

From Fig. 5-50: $K \approx 2.20$

$$\sigma_{\max} = K\sigma_{\text{nom}} = K\left(\frac{6M}{bh_1^2}\right)$$

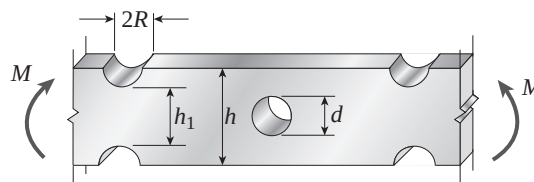
$$6 \text{ MPa} = 2.20 \left[\frac{6(150 \text{ N} \cdot \text{m})}{b(100 \text{ mm})^2} \right]$$

Solve for b :

$$b_{\min} \approx 33 \text{ mm} \quad \leftarrow$$

Problem 5.13-5 A rectangular beam with notches and a hole (see figure) has dimensions $h = 5.5 \text{ in.}$, $h_1 = 5 \text{ in.}$, and width $b = 1.6 \text{ in.}$ The beam is subjected to a bending moment $M = 130 \text{ k-in.}$, and the maximum allowable bending stress in the material (steel) is $\sigma_{\max} = 42,000 \text{ psi}$.

- What is the smallest radius $R_{\min}$ that should be used in the notches?
- What is the diameter $d_{\max}$ of the largest hole that should be drilled at the midheight of the beam?



Solution 5.13-5 Beam with notches and a hole

$$h = 5.5 \text{ in.} \quad h_1 = 5 \text{ in.} \quad b = 1.6 \text{ in.}$$

$$M = 130 \text{ k-in.} \quad \sigma_{\max} = 42,000 \text{ psi}$$

(a) MINIMUM NOTCH RADIUS

$$\frac{h}{h_1} = \frac{5.5 \text{ in.}}{5 \text{ in.}} = 1.1$$

$$\sigma_{\text{nom}} = \frac{6M}{bh_1^2} = 19,500 \text{ psi}$$

$$K = \frac{\sigma_{\max}}{\sigma_{\text{nom}}} = \frac{42,000 \text{ psi}}{19,500 \text{ psi}} = 2.15$$

From Fig. 5-50, with $K = 2.15$ and $\frac{h}{h_1} = 1.1$, we get

$$\frac{R}{h_1} \approx 0.090$$

$$\therefore R_{\min} \approx 0.090h_1 = 0.45 \text{ in.} \quad \leftarrow$$

(b) LARGEST HOLE DIAMETER

Assume $\frac{d}{h} > \frac{1}{2}$ and use Eq. (5-61).

$$\sigma_B = \frac{12Md}{b(h^3 - d^3)}$$

$$42,000 \text{ psi} = \frac{12(130 \text{ k-in.})d}{(1.6 \text{ in.})[(5.5 \text{ in.})^3 - d^3]} \quad \text{or}$$

$$d^3 + 23.21d - 166.4 = 0$$

Solve numerically:

$$d_{\max} = 4.13 \text{ in.} \quad \leftarrow$$

6

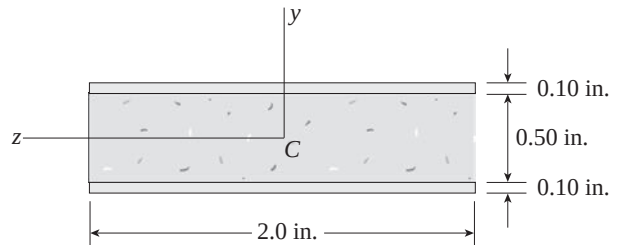
Stresses in Beams (Advanced Topics)

Composite Beams

When solving the problems for Section 6.2, assume that the component parts of the beams are securely bonded by adhesives or connected by fasteners. Also, be sure to use the general theory for composite beams described in Sect. 6.2.

Problem 6.2-1 A composite beam consisting of fiberglass faces and a core of particle board has the cross section shown in the figure. The width of the beam is 2.0 in., the thickness of the faces is 0.10 in., and the thickness of the core is 0.50 in. The beam is subjected to a bending moment of 250 lb-in. acting about the z axis.

Find the maximum bending stresses σ_{face} and σ_{core} in the faces and the core, respectively, if their respective moduli of elasticity are 4×10^6 psi and 1.5×10^6 psi.



Solution 6.2-1 Composite beam

$$b = 2.0 \text{ in.} \quad h = 0.7 \text{ in.}$$

$$h_c = 0.5 \text{ in.} \quad M = 250 \text{ lb-in.}$$

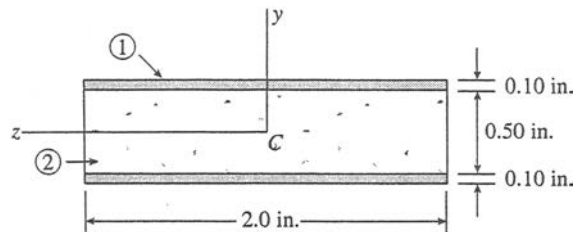
$$E_1 = 4 \times 10^6 \text{ psi} \quad E_2 = 1.5 \times 10^6 \text{ psi}$$

$$I_1 = \frac{b}{12} (h^3 - h_c^3) = 0.03633 \text{ in.}^4$$

$$I_2 = \frac{bh_c^3}{12} = 0.02083 \text{ in.}^4$$

$$E_1 I_1 + E_2 I_2 = 176,600 \text{ lb-in.}^2$$

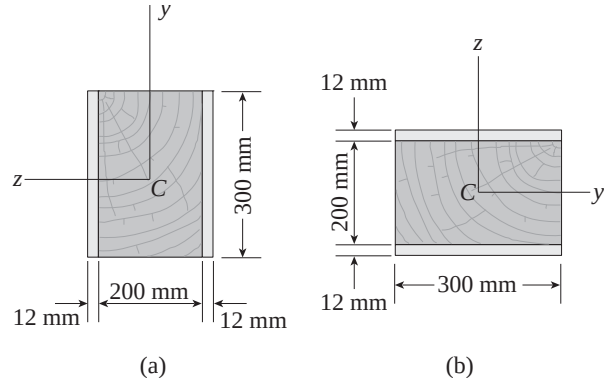
$$\begin{aligned} \text{From Eq. (6-6a): } \sigma_{\text{face}} &= \pm \frac{M(h/2)E_1}{E_1 I_1 + E_2 I_2} \\ &= \pm 1980 \text{ psi} \quad \leftarrow \end{aligned}$$



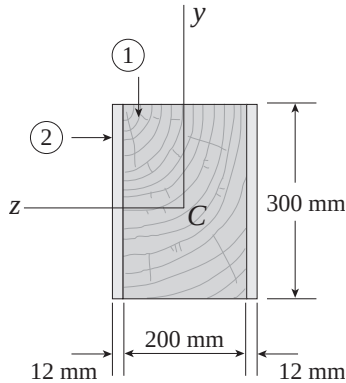
$$\begin{aligned} \text{From Eq. (6-6b): } \sigma_{\text{core}} &= \pm \frac{M(h_c/2)E_2}{E_1 I_1 + E_2 I_2} \\ &= \pm 531 \text{ psi} \quad \leftarrow \end{aligned}$$

Problem 6.2-2 A wood beam with cross-sectional dimensions $200 \text{ mm} \times 300 \text{ mm}$ is reinforced on its sides by steel plates 12 mm thick (see figure). The moduli of elasticity for the steel and wood are $E_s = 190 \text{ GPa}$ and $E_w = 11 \text{ GPa}$, respectively. Also, the corresponding allowable stresses are $\sigma_s = 110 \text{ MPa}$ and $\sigma_w = 7.5 \text{ MPa}$.

- Calculate the maximum permissible bending moment $M_{\max}$ when the beam is bent about the z axis.
- Repeat part (a) if the beam is now bent about its y axis.
- Find the required thickness of the steel plates on the beam bent about the y axis so that $M_{\max}$ is the same for both beam orientations.



Solution 6.2-2



- (a) BENT ABOUT THE z AXIS

$$b = 200 \text{ mm} \quad t = 12 \text{ mm} \quad h = 300 \text{ mm}$$

$$E_w = 11 \text{ GPa} \quad E_s = 190 \text{ GPa}$$

$$\sigma_{\text{allow}_w} = 7.5 \text{ MPa} \quad \sigma_{\text{allow}_s} = 110 \text{ MPa}$$

$$I_w = \frac{bh^3}{12} \quad I_w = 4.50 \times 10^8 \text{ mm}^4$$

$$I_s = \frac{2th^3}{12} \quad I_s = 5.40 \times 10^7 \text{ mm}^4$$

$$E_w I_w + E_s I_s = 1.52 \times 10^7 \text{ N} \cdot \text{m}^2$$

MAXIMUM MOMENT BASED UPON THE WOOD

$$M_{\max_w} = \sigma_{\text{allow}_w} \left[\frac{E_w I_w + E_s I_s}{\left(\frac{h}{2}\right) E_w} \right]$$

$$M_{\max_w} = 69.1 \text{ kN} \cdot \text{m}$$

MAXIMUM MOMENT BASED UPON THE STEEL

$$M_{\max_s} = \sigma_{\text{allow}_s} \left[\frac{E_w I_w + E_s I_s}{\left(\frac{h}{2}\right) E_s} \right]$$

$$M_{\max_s} = 58.7 \text{ kN} \cdot \text{m}$$

$$M_{\max} = \min(M_{\max_w}, M_{\max_s})$$

$$\text{STEEL GOVERNS.} \quad M_{\max} = 58.7 \text{ kN} \cdot \text{m} \quad \leftarrow$$

- (b) BENT ABOUT THE y AXIS

$$I_w = \frac{b^3 h}{12} \quad I_w = 2.00 \times 10^8 \text{ mm}^4$$

$$I_s = 2 \left[\frac{t^3 h}{12} + th \left(\frac{b+t}{2} \right)^2 \right]$$

$$I_s = 8.10 \times 10^7 \text{ mm}^4$$

$$E_w I_w + E_s I_s = 1.76 \times 10^7 \text{ N} \cdot \text{m}^2$$

MAXIMUM MOMENT BASED UPON THE WOOD

$$M_{\max_w} = \sigma_{\text{allow}_w} \left[\frac{E_w I_w + E_s I_s}{\left(\frac{b}{2}\right) E_w} \right]$$

$$M_{\max_w} = 119.9 \text{ kN} \cdot \text{m}$$

MAXIMUM MOMENT BASED UPON THE STEEL

$$M_{\max_s} = \sigma_{\text{allow}_s} \left[\frac{E_w I_w + E_s I_s}{\left(\frac{b}{2} + t\right) E_s} \right]$$

$$M_{\max_s} = 90.9 \text{ kN} \cdot \text{m} \quad \leftarrow$$

$$M_{\max} = \min(M_{\max_w}, M_{\max_s})$$

$$\text{STEEL GOVERNS.} \quad M_{\max} = 90.9 \text{ kN} \cdot \text{m} \quad \leftarrow$$

(c) $M_{\max} = 58.7053 \text{ kN} \cdot \text{m}$

Beam bent about y-axis (Fig. b): $I_w = \frac{hb^3}{12} = 2 \times 10^8 \text{ mm}^4$ $I_s = 2 \frac{ht^3}{12} + 2ht \left(\frac{b}{2} + \frac{t}{2} \right)^2$

$$M_{\max Y} = \sigma_{as} \left[\frac{E_w I_w + E_s I_s}{E_s \left(\frac{b}{2} + t \right)} \right] = 58.696 \text{ kN} \cdot \text{m} \quad \sigma_{as} \left[\frac{E_w I_w + E_s I_s}{E_s \left(\frac{b}{2} + t \right)} \right] = M_{\max X}$$

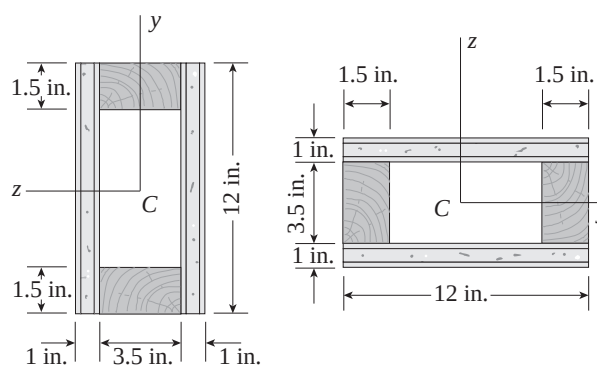
Find $(t) = 7.062 \text{ mm}$

$t = 7.08 \text{ mm}$

$$\sigma_{aw} \left[\frac{E_w I_w + E_s I_s}{E_w \left(\frac{b}{2} \right)} \right] = 74.019 \text{ kN} \cdot \text{m}$$

Problem 6.2-3 A hollow box beam is constructed with webs of Douglas-fir plywood and flanges of pine, as shown in the figure in a cross-sectional view. The plywood is 1 in. thick and 12 in. wide; the flanges are 2 in. $\times$ 4 in. (nominal size). The modulus of elasticity for the plywood is 1,800,000 psi and for the pine is 1,400,000 psi.

- If the allowable stresses are 2000 psi for the plywood and 1750 psi for the pine, find the allowable bending moment $M_{\max}$ when the beam is bent about the z axis.
- Repeat part (a) if the beam is now bent about its y axis.

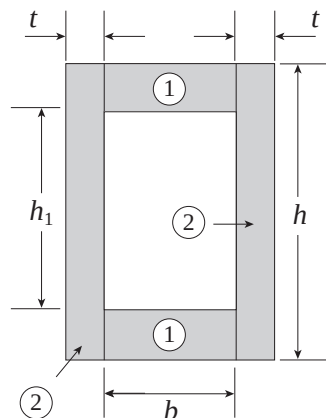


(a)

(b)

Solution 6.2-3

(a) BENT ABOUT THE Z AXIS



$$b = 3.5 \text{ in.} \quad t = 1 \text{ in.} \quad h = 12 \text{ in.} \quad h_1 = 9 \text{ in.}$$

$$E_1 = 1.4 \times 10^6 \text{ psi} \quad E_2 = 1.8 \times 10^6 \text{ psi}$$

$$\sigma_{\text{allow}_1} = 1750 \text{ psi} \quad \sigma_{\text{allow}_2} = 2000 \text{ psi}$$

$$I_1 = \frac{b(h^3 - h_1^3)}{12} \quad I_1 = 291 \text{ in.}^4$$

$$I_2 = \frac{2th^3}{12} \quad I_2 = 288 \text{ in.}^4$$

$$E_1 I_1 + E_2 I_2 = 9.26 \times 10^8 \text{ lb} \cdot \text{in.}^2$$

MAXIMUM MOMENT BASED UPON THE WOOD

$$M_{\max_1} = \sigma_{\text{allow}_1} \left[\frac{E_1 I_1 + E_2 I_2}{\left(\frac{h}{2}\right) E_1} \right]$$

$$M_{\max_1} = 193 \text{ k-in.} \quad \leftarrow$$

MAXIMUM MOMENT BASED UPON THE PLYWOOD

$$M_{\max_2} = \sigma_{\text{allow}_2} \left[\frac{E_1 I_1 + E_2 I_2}{\left(\frac{h}{2}\right) E_2} \right]$$

$$M_{\max_2} = 172 \text{ k-in.} \quad \leftarrow$$

$$M_{\max} = \min(M_{\max_1}, M_{\max_2})$$

$$\text{PLYWOOD GOVERNS.} \quad M_{\max} = 172 \text{ k-in.} \quad \leftarrow$$

MAXIMUM MOMENT BASED UPON THE WOOD

$$M_{\max_1} = \sigma_{\text{allow}_1} \left[\frac{E_1 I_1 + E_2 I_2}{\left(\frac{b}{2}\right) E_1} \right]$$

$$M_{\max_1} = 170 \text{ k-in.}$$

MAXIMUM MOMENT BASED UPON THE PLYWOOD

$$M_{\max_2} = \sigma_{\text{allow}_2} \left[\frac{E_1 I_1 + E_2 I_2}{\left(\frac{b}{2} + t\right) E_2} \right]$$

$$M_{\max_2} = 96 \text{ k-in.}$$

$$M_{\max} = \min(M_{\max_1}, M_{\max_2})$$

$$\text{PLYWOOD GOVERNS.} \quad M_{\max} = 96 \text{ k-in.} \quad \leftarrow$$

(b) BENT ABOUT THE Y AXIS

$$I_1 = \frac{b^3(h - h_1)}{12} \quad I_1 = 11 \text{ in.}^4$$

$$I_2 = 2 \left[\frac{t^3 h}{12} + th \left(\frac{b + t}{2} \right)^2 \right] \quad I_2 = 123 \text{ in.}^4$$

$$E_1 I_1 + E_2 I_2 = 2.37 \times 10^8 \text{ lb} \cdot \text{in.}^2$$

Problem 6.2-4 A round steel tube of outside diameter d_2 and a brass core of diameter d_1 are bonded to form a composite beam, as shown in the figure.

- Derive formulas for the allowable bending moment M that can be carried by the beam based upon an allowable stress σ_s in the steel and an allowable stress σ_B in the brass. (Assume that the moduli of elasticity for the steel and brass are E_s and E_B , respectively.)
- If $d_2 = 50 \text{ mm}$, $d_1 = 40 \text{ mm}$, $E_s = 210 \text{ GPa}$, $E_B = 110 \text{ GPa}$, $\sigma_s = 150 \text{ MPa}$, $\sigma_B = 100 \text{ MPa}$, what is the maximum bending moment M ?
- What new value of brass diameter d_1 will result in a balanced design? (Note that a balanced design is that in which steel and brass reach allowable stress values at the same time).

Solution 6.2-4

(a) DERIVATION OF FORMULAS

$$I_s = \frac{\pi}{64}(d_2^4 - d_1^4) \quad I_B = \frac{\pi}{64}(d_1^4)$$

$$M_{\text{allowSteel}} = \frac{\sigma_s(E_s I_s + E_B I_B)}{E_s \left(\frac{d_2}{2}\right)} = \frac{\pi \sigma_s (E_B d_1^4 - E_s d_1^4 + E_s d_2^4)}{32 E_s d_2}$$

$$M_{\text{allowBrass}} = \sigma_B \frac{(E_s I_s + E_B I_B)}{E_s \left(\frac{d_1}{2}\right)} = \frac{\pi \sigma_B (E_B d_1^4 - E_s d_1^4 + E_s d_2^4)}{32 E_s d_1}$$

(b) MAXIMUM BENDING MOMENT

$$d_2 = 50 \text{ mm} \quad d_1 = 40 \text{ mm} \quad E_s = 210 \text{ GPa} \quad E_B = 110 \text{ GPa} \quad \sigma_s = 150 \text{ MPa} \quad \sigma_B = 100 \text{ MPa}$$

$$M_{\text{maxS}} = \frac{\pi \sigma_s (E_B d_1^4 - E_s d_1^4 + E_s d_2^4)}{32 E_s d_2} = 1482 \text{ N} \cdot \text{m}$$

$$M_{\text{maxB}} = \frac{\pi \sigma_B (E_B d_1^4 - E_s d_1^4 + E_s d_2^4)}{32 E_s d_1} = 1235 \text{ N} \cdot \text{m}$$

(c) BALANCED DESIGN: equate allowable moments, then solve for d_1 .

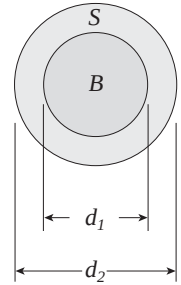
$$I_s = \frac{\pi}{64}(d_2^4 - d_1^4) \quad I_B = \frac{\pi}{64}(d_1^4)$$

$$M_{as} = \frac{\sigma_s (E_s I_s + E_B I_B)}{E_s \left(\frac{d_2}{2}\right)} = 1481.738 \text{ N} \cdot \text{m}$$

$$M_{aB} = \sigma_B \frac{(E_s I_s + E_B I_B)}{E_s \left(\frac{d_1}{2}\right)} = 1234.781 \text{ N} \cdot \text{m}$$

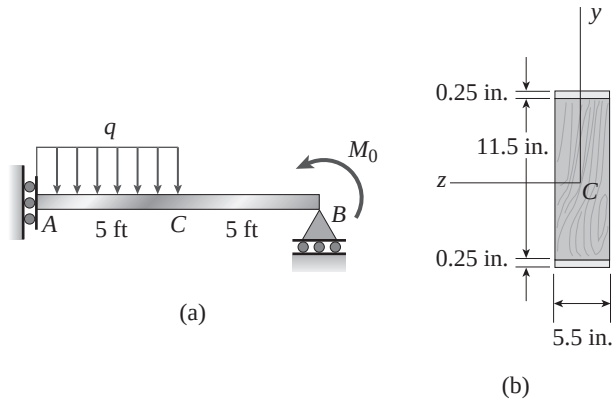
$$\sigma_B \frac{(E_s I_s + E_B I_B)}{E_s \left(\frac{d_1}{2}\right)} = \sigma_s \frac{(E_s I_s + E_B I_B)}{E_s \left(\frac{d_2}{2}\right)} \quad \text{or} \quad d_1 = \frac{\sigma_B}{\sigma_s}(d_2) = 33.333 \text{ mm}$$

$$d_1 = 33.3 \text{ mm}$$



Problem 6.2-5 A beam with a guided support and 10-ft span supports a distributed load of intensity $q = 660 \text{ lb/ft}$ over its first half (see figure part a) and a moment $M_0 = 300 \text{ ft}\cdot\text{lb}$ at joint B. The beam consists of a wood member (nominal dimensions $6 \text{ in.} \times 12 \text{ in.}$, actual dimensions $5.5 \text{ in.} \times 11.5 \text{ in.}$ in cross section, as shown in the figure part b) that is reinforced by 0.25-in.-thick steel plates on top and bottom. The moduli of elasticity for the steel and wood are $E_s = 30 \times 10^6 \text{ psi}$ and $E_w = 1.5 \times 10^6 \text{ psi}$, respectively.

- Calculate the maximum bending stresses σ_s in the steel plates and σ_w in the wood member due to the applied loads.
- If the allowable bending stress in the steel plates is $\sigma_{as} = 14,000 \text{ psi}$ and that in the wood is $\sigma_{aw} = 900 \text{ psi}$, find $q_{\max}$. (Assume that the moment at B, M_0 , remains at $300 \text{ ft}\cdot\text{lb}$.)
- If $q = 660 \text{ lb/ft}$ and allowable stress values in (b) apply, what is $M_{0,\max}$ at B?

**Solution 6.2-5**

$$q = 660 \text{ lb/in.} \quad M_0 = 300 \text{ lb}\cdot\text{ft} \quad L = 10 \text{ ft}$$

(a) MAXIMUM BENDING STRESSES

$$M_{\max} = q\left(\frac{L}{2}\right)\left(\frac{3L}{4}\right) + M_0$$

$$M_{\max} = 25,050 \text{ lb}\cdot\text{ft}$$

$$\text{Wood (1):} \quad b = 5.5 \text{ in.} \quad h_1 = 11.5 \text{ in.}$$

$$E_w = 1.5 \times 10^6 \text{ psi}$$

$$I_1 = \frac{bh_1^3}{12} \quad I_1 = 697.07 \text{ in.}^4$$

$$\text{Plate (2):} \quad b = 5.5 \text{ in.} \quad t = 0.25 \text{ in.} \\ h = 12 \text{ in.} \quad E_s = 30 \times 10^6 \text{ psi}$$

$$I_2 = \frac{b}{12}(h^3 - h_1^3) \quad I_2 = 94.93 \text{ in.}^4$$

$$E_w I_1 + E_s I_2 = 3.894 \times 10^9 \text{ lb}\cdot\text{in.}^2$$

$$\sigma_w = \frac{M_{\max} \left(\frac{h_1}{2}\right) E_w}{E_w I_1 + E_s I_2} \quad \sigma_w = 666 \text{ psi} \quad \leftarrow$$

$$\sigma_s = \frac{M_{\max} \left(\frac{h}{2}\right) E_s}{E_w I_1 + E_s I_2} \quad \sigma_s = 13,897 \text{ psi} \quad \leftarrow$$

(b) MAXIMUM UNIFORM DISTRIBUTED LOAD**MAXIMUM MOMENT BASED UPON WOOD**

$$\sigma_{\text{allow}_w} = 900 \text{ psi}$$

$$\text{From } \sigma_{\text{allow}_w} = \frac{M_{\text{allow}_w} \left(\frac{h_1}{2}\right) E_w}{E_w I_1 + E_s I_2}$$

$$M_{\text{allow}_w} = 33,857 \text{ lb}\cdot\text{ft}$$

MAXIMUM MOMENT BASED UPON STEEL PLATE

$$\sigma_{\text{allow}_s} = 14,000 \text{ psi}$$

$$\text{From } \sigma_{\text{allow}_s} = \frac{M_{\text{allow}_s} \left(\frac{h}{2}\right) E_s}{E_w I_1 + E_s I_2}$$

$$M_{\text{allow}_s} = 25,236 \text{ lb}\cdot\text{ft}$$

MAXIMUM ALLOWABLE MOMENT

$$M_{\text{allow}} = \min(M_{\text{allow}_s}, M_{\text{allow}_w})$$

STEEL PLATES GOVERN

$$M_{\text{allow}} = 25,236 \text{ lb}\cdot\text{ft} \quad \leftarrow$$

MAXIMUM UNIFORM DISTRIBUTED LOAD

$$\text{From } M_{\text{allow}} = q_{\text{max}} \left(\frac{L}{2} \right) \left(\frac{3L}{4} \right) + M_0$$

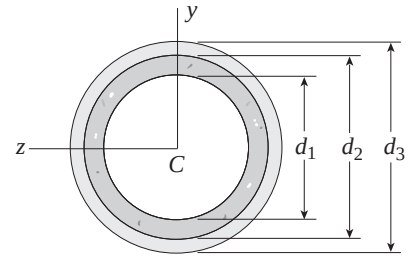
$$q_{\text{max}} = 665 \text{ lb/ft} \quad \leftarrow$$

(c) MAXIMUM APPLIED MOMENT

$$\text{From } M_{\text{allow}} = q \left(\frac{L}{2} \right) \left(\frac{3L}{4} \right) + M_{0_{\text{max}}}$$

$$M_{0_{\text{max}}} = 486 \text{ lb-ft} \quad \leftarrow$$

Problem 6.2-6 A plastic-lined steel pipe has the cross-sectional shape shown in the figure. The steel pipe has outer diameter $d_3 = 100 \text{ mm}$ and inner diameter $d_2 = 94 \text{ mm}$. The plastic liner has inner diameter $d_1 = 82 \text{ mm}$. The modulus of elasticity of the steel is 75 times the modulus of the plastic.



- Determine the allowable bending moment M_{allow} if the allowable stress in the steel is 35 MPa and in the plastic is 600 kPa.
- If pipe and liner diameters remain unchanged, what new value of allowable stress for the steel pipe will result in the steel pipe and plastic liner reaching their allowable stress values under the same maximum moment (i.e., a balanced design)? What is the new maximum moment?

Solution 6.2-6

$$d_3 = 100 \text{ mm} \quad d_2 = 94 \text{ mm} \quad d_1 = 82 \text{ mm}$$

$$n = 75 < E_s/E_p$$

$$\sigma_{sa} = 35 \text{ MPa} \quad \sigma_{pa} = 600 \text{ kPa}$$

- FIND ALLOWABLE MOMENT BASED ON ALLOWABLE STRESSES IN STEEL AND PLASTIC

CROSS-SECTIONAL PROPERTIES

$$I_s = \frac{\pi}{64} (d_3^4 - d_2^4) = 1.076 \times 10^{-6} \text{ m}^4 \quad I_p = \frac{\pi}{64} (d_2^4 - d_1^4) = 1.613 \times 10^{-6} \text{ m}^4$$

MAXIMUM MOMENT BASED ON ALLOWABLE STRESS IN **STEEL**

$$M_{s\text{max}} = \frac{\sigma_{sa}(E_s I_s + E_p I_p)}{E_s \frac{d_3}{2}} = \frac{\sigma_{sa}(n I_s + I_p)}{n \left(\frac{d_3}{2} \right)} = 768.428 \text{ N} \cdot \text{m} < \text{steel governs.} \quad \boxed{M_{\text{allow}} = 768 \text{ N} \cdot \text{m}}$$

MAXIMUM MOMENT BASED ON ALLOWABLE STRESS IN **PLASTIC**

$$M_{p\text{max}} = \frac{\sigma_{pa}(E_s I_s + E_p I_p)}{E_p \frac{d_2}{2}} = \frac{\sigma_{pa}(n I_s + I_p)}{\left(\frac{d_2}{2} \right)} = 1051.042 \text{ N} \cdot \text{m}$$

- BALANCED DESIGN - must equate expressions for M_{max} , then solve for required σ_{sa} :

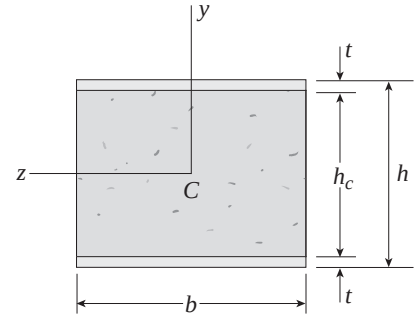
$$\frac{\sigma_{sa}}{n d_3} = \frac{\sigma_{pa}}{d_2} \quad \sigma_{sa} = n \sigma_{pa} \left(\frac{d_3}{d_2} \right) = 47.87 \text{ MPa} \quad \boxed{\sigma_{sa} = 47.9 \text{ MPa}}$$

$$M_{s\text{max}} = \frac{\sigma_{sa}(n I_s + I_p)}{n \left(\frac{d_3}{2} \right)} = 1052 \text{ N} \cdot \text{m} \quad M_{p\text{max}} = \frac{\sigma_{pa}(n I_s + I_p)}{\left(\frac{d_2}{2} \right)} = 1051 \text{ N} \cdot \text{m} < \text{same as above}$$

$\wedge$ increased due to increased σ_{sa}

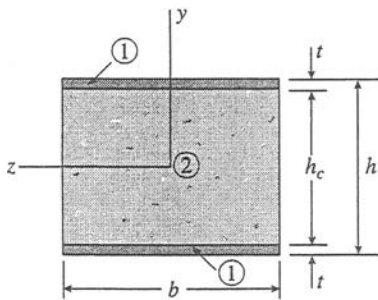
Problem 6.2-7 The cross section of a sandwich beam consisting of aluminum alloy faces and a foam core is shown in the figure. The width b of the beam is 8.0 in., the thickness t of the faces is 0.25 in., and the height h_c of the core is 5.5 in. (total height $h = 6.0$ in.). The moduli of elasticity are 10.5×10^6 psi for the aluminum faces and 12,000 psi for the foam core. A bending moment $M = 40$ k-in. acts about the z axis.

Determine the maximum stresses in the faces and the core using (a) the general theory for composite beams, and (b) the approximate theory for sandwich beams.



Probs. 6.2-7 and 6.2-8

Solution 6.2-7 Sandwich beam



(1) ALUMINUM FACES:

$$b = 8.0 \text{ in.} \quad t = 0.25 \text{ in.} \quad h = 6.0 \text{ in.}$$

$$E_1 = 10.5 \times 10^6 \text{ psi}$$

$$I_1 = \frac{b}{12} (h^3 - h_c^3) = 33.08 \text{ in.}^4$$

(2) Foam core:

$$b = 8.0 \text{ in.} \quad h_c = 5.5 \text{ in.} \quad E_2 = 12,000 \text{ psi}$$

$$I_2 = \frac{bh_c^3}{12} = 110.92 \text{ in.}^4$$

$$M = 40 \text{ k-in.} \quad E_1 I_1 + E_2 I_2 = 348.7 \times 10^6 \text{ lb-in.}^2$$

(a) GENERAL THEORY (EQS. 6-7a AND b)

$$\sigma_{\text{face}} = \sigma_1 = \frac{M(h/2)E_1}{E_1 I_1 + E_2 I_2} = 3610 \text{ psi} \quad \leftarrow$$

$$\sigma_{\text{core}} = \sigma_2 = \frac{M(h_c/2)E_2}{E_1 I_1 + E_2 I_2} = 4 \text{ psi} \quad \leftarrow$$

(b) APPROXIMATE THEORY (EQS. 6-9 AND 6-10)

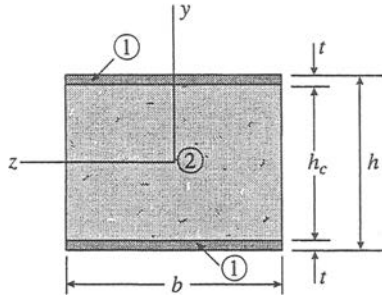
$$I_1 = \frac{b}{12} (h^3 - h_c^3) = 33.08 \text{ in.}^4$$

$$\sigma_{\text{face}} = \frac{Mh}{2I_1} = 3630 \text{ psi} \quad \leftarrow$$

$$\sigma_{\text{core}} = 0 \quad \leftarrow$$

Problem 6.2-8 The cross section of a sandwich beam consisting of fiberglass faces and a lightweight plastic core is shown in the figure. The width b of the beam is 50 mm, the thickness t of the faces is 4 mm, and the height h_c of the core is 92 mm (total height $h = 100$ mm). The moduli of elasticity are 75 GPa for the fiberglass and 1.2 GPa for the plastic. A bending moment $M = 275 \text{ N} \cdot \text{m}$ acts about the z axis.

Determine the maximum stresses in the faces and the core using (a) the general theory for composite beams, and (b) the approximate theory for sandwich beams.

Solution 6.2-8 Sandwich beam

(1) Fiber glass faces:

$$b = 50 \text{ mm} \quad t = 4 \text{ mm} \quad h = 100 \text{ mm}$$

$$E_1 = 75 \text{ GPa}$$

$$I_1 = \frac{b}{12} (h^3 - h_c^3) = 0.9221 \times 10^{-6} \text{ m}^4$$

(2) Plastic core:

$$b = 50 \text{ mm} \quad h_c = 92 \text{ mm} \quad E_2 = 1.2 \text{ GPa}$$

$$I_2 = \frac{bh_c^3}{12} = 3.245 \times 10^{-6} \text{ m}^4$$

$$M = 275 \text{ N} \cdot \text{m} \quad E_1 I_1 + E_2 I_2 = 73,050 \text{ N} \cdot \text{m}^2$$

(a) GENERAL THEORY (EQS. 6-7a AND b)

$$\sigma_{\text{face}} = \sigma_1 = \frac{M(h/2)E_1}{E_1 I_1 + E_2 I_2} = 14.1 \text{ MPa} \quad \leftarrow$$

$$\sigma_{\text{core}} = \sigma_2 = \frac{M(h_c/2)E_2}{E_1 I_1 + E_2 I_2} = 0.21 \text{ MPa} \quad \leftarrow$$

(b) APPROXIMATE THEORY (EQS. 6-9 AND 6-10)

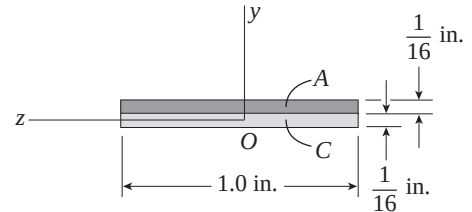
$$I_1 = \frac{b}{12} (h^3 - h_c^3) = 0.9221 \times 10^{-6} \text{ m}^4$$

$$\sigma_{\text{face}} = \frac{Mh}{2I_1} = 14.9 \text{ MPa} \quad \leftarrow$$

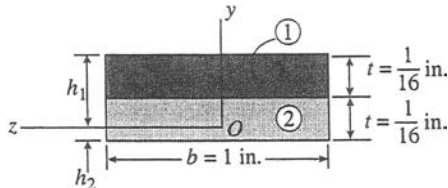
$$\sigma_{\text{core}} = 0 \quad \leftarrow$$

Problem 6.2-9 A bimetallic beam used in a temperature-control switch consists of strips of aluminum and copper bonded together as shown in the figure, which is a cross-sectional view. The width of the beam is 1.0 in., and each strip has a thickness of 1/16 in.

Under the action of a bending moment $M = 12 \text{ lb} \cdot \text{in.}$ acting about the z axis, what are the maximum stresses σ_a and σ_c in the aluminum and copper, respectively? (Assume $E_a = 10.5 \times 10^6 \text{ psi}$ and $E_c = 16.8 \times 10^6 \text{ psi}$.)

**Solution 6.2-9 Bimetallic beam**

CROSS SECTION

(1) Aluminum: $E_1 = E_a = 10.5 \times 10^6 \text{ psi}$ (2) Copper: $E_2 = E_c = 16.8 \times 10^6 \text{ psi}$

$$M = 12 \text{ lb} \cdot \text{in.}$$

NEUTRAL AXIS (EQ. 6-3)

$$\begin{aligned} \int_1 y dA &= \bar{y}_1 A_1 = (h_1 - t/2)(bt) \\ &= (h_1 - 1/32)(1)(1/16) \text{ in.}^3 \end{aligned}$$

$$\begin{aligned} \int_2 y dA &= \bar{y}_2 A_2 = (h_1 - t - t/2)(bt) \\ &= (h_1 - 3/32)(1)(1/16) \text{ in.}^3 \end{aligned}$$

$$\text{Eq. (6-4): } E_1 \int_1 y dA + E_2 \int_2 y dA = 0$$

$$(10.5 \times 10^6)(h_1 - 1/32)(1/16) + (16.8 \times 10^6)(h_1 - 3/32)(1/16) = 0$$

Solve for h_1 : $h_1 = 0.06971$ in.

$$h_2 = 2(1/16 \text{ in.}) - h_1 = 0.05529 \text{ in.}$$

MOMENTS OF INERTIA (FROM PARALLEL-AXIS THEOREM)

$$I_1 = \frac{bt^3}{12} + bt(h_1 - t/2)^2 = 0.0001128 \text{ in.}^4$$

$$I_2 = \frac{bt^3}{12} + bt(h_2 - t/2)^2 = 0.00005647 \text{ in.}^4$$

$$E_1 I_1 + E_2 I_2 = 2133 \text{ lb-in.}^2$$

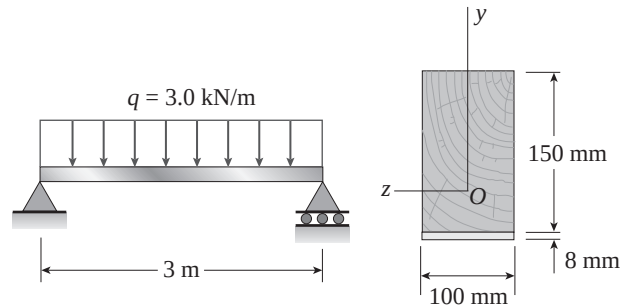
MAXIMUM STRESSES (EQS. 6-7a AND b)

$$\sigma_a = \sigma_1 = \frac{M h_1 E_1}{E_1 I_1 + E_2 I_2} = 4120 \text{ psi} \quad \leftarrow$$

$$\sigma_c = \sigma_2 = \frac{M h_2 E_2}{E_1 I_1 + E_2 I_2} = 5230 \text{ psi} \quad \leftarrow$$

Problem 6.2-10 A simply supported composite beam 3 m long carries a uniformly distributed load of intensity $q = 3.0 \text{ kN/m}$ (see figure). The beam is constructed of a wood member, 100 mm wide by 150 mm deep, reinforced on its lower side by a steel plate 8 mm thick and 100 mm wide.

- Find the maximum bending stresses σ_w and σ_s in the wood and steel, respectively, due to the uniform load if the moduli of elasticity are $E_w = 10 \text{ GPa}$ for the wood and $E_s = 210 \text{ GPa}$ for the steel.
- Find the required thickness of the steel plate so that the steel plate and wood reach their allowable stress values, $\sigma_{as} = 100 \text{ MPa}$ and $\sigma_{aw} = 8.5 \text{ MPa}$, simultaneously under the maximum moment.

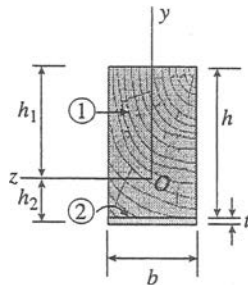


Solution 6.2-10 Simply supported composite beam

- (a) BEAM: $L = 3 \text{ m}$ $q = 3.0 \text{ kN/m}$

$$M_{\max} = \frac{qL^2}{8} = 3375 \text{ N} \cdot \text{m}$$

CROSS SECTION



$$b = 100 \text{ mm} \quad h = 150 \text{ mm} \quad t = 8 \text{ mm}$$

- (1) Wood: $E_1 = E_w = 10 \text{ GPa}$

- (2) Steel: $E_2 = E_s = 210 \text{ GPa}$

NEUTRAL AXIS

$$\begin{aligned} \int_1 y dA &= \bar{y}_1 A_1 = (h_1 - h/2)(bh) \\ &= (h_1 - 75)(100)(150) \text{ mm}^3 \end{aligned}$$

$$\begin{aligned} \int_2 y dA &= \bar{y}_2 A_2 = -(h + t/2 - h_1)(bt) \\ &= -(154 - h_1)(100)(18) \text{ mm}^3 \end{aligned}$$

$$\text{Eq. (6-4): } E_1 \int_1 y dA + E_2 \int_2 y dA = 0$$

$$(10 \text{ GPa})(h_1 - 75)(100)(150)(10^{-9}) \\ + (210 \text{ GPa})(h_1 - 154)(100)(8)(10^{-9}) = 0$$

Solve for h_1 : $h_1 = 116.74 \text{ mm}$

$$h_2 = h + t - h_1 = 41.26 \text{ mm}$$

MOMENTS OF INERTIA (FROM PARALLEL-AXIS THEOREM)

$$I_1 = \frac{bh^3}{12} + bh(h_1 - h/2)^2 = 54.26 \times 10^6 \text{ mm}^4$$

$$I_2 = \frac{bt^3}{12} + bt(h_2 - t/2)^2 = 1.115 \times 10^6 \text{ mm}^4$$

$$E_1 I_1 + E_2 I_2 = 776,750 \text{ N} \cdot \text{m}^2$$

MAXIMUM STRESSES (EQS. 6-7a AND b)

$$\sigma_w = \sigma_1 = \frac{M h_1 E_1}{E_1 I_1 + E_2 I_2} \\ = 5.1 \text{ MPa (Compression)} \quad \leftarrow$$

$$\sigma_s = \sigma_2 = \frac{M h_2 E_2}{E_1 I_1 + E_2 I_2} \\ = 37.6 \text{ MPa (Tension)} \quad \leftarrow$$

$$(b) \quad \sigma_{as} = 100 \text{ MPa} \quad \sigma_{aw} = 8.5 \text{ MPa} \quad E_w = 10 \text{ GPa} \quad E_s = 210 \text{ GPa}$$

$$b = 100 \text{ mm} \quad h = 150 \text{ mm}$$

$$M = 3375 \text{ N} \cdot \text{m}$$

$$\sigma_s = \frac{M h_2 E_s}{E_s I_s + E_w I_w} \quad \sigma_w = \frac{M h_1 E_w}{E_s I_s + E_w I_w}$$

$$\text{Solving for } M \text{ and equating expressions gives} \quad \frac{\sigma_{aw}}{h_1 E_w} = \frac{\sigma_{as}}{h_2 E_s} \quad \text{so} \quad \frac{h_1}{h_2} = \frac{\sigma_{aw} E_s}{\sigma_{as} E_w}$$

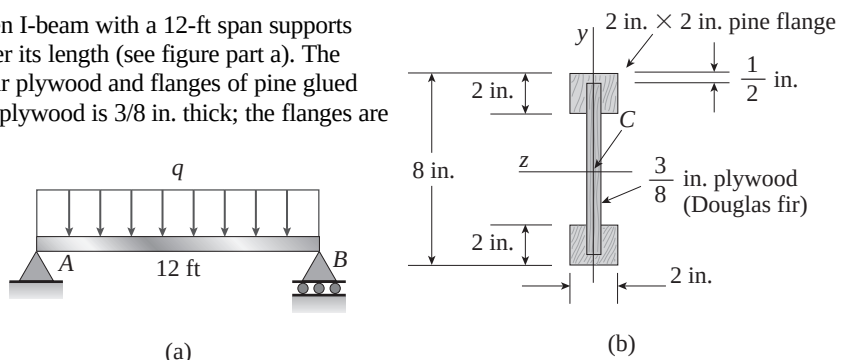
From Eq. 6-3, Eq. g

$$t_s^2 \left[n b_s \left(1 - n \frac{\sigma_{aw}}{\sigma_{as}} \right) \right] + t_s \left[2 h \left[n b_s - b \left(n \frac{\sigma_{aw}}{\sigma_{as}} \right) \right] \right] + b h^2 \left(1 - n \frac{\sigma_{aw}}{\sigma_{as}} \right) = 0 \quad (g)$$

$$\text{Solving for } t_s: \quad t_s = 3.09 \text{ mm}$$

Problem 6.2-11 A simply supported wooden I-beam with a 12-ft span supports a distributed load of intensity $q = 90 \text{ lb/ft}$ over its length (see figure part a). The beam is constructed with a web of Douglas-fir plywood and flanges of pine glued to the web as shown in the figure part b. The plywood is $3/8 \text{ in.}$ thick; the flanges are $2 \text{ in.} \times 2 \text{ in.}$ (actual size). The modulus of elasticity for the plywood is $1,600,000 \text{ psi}$ and for the pine is $1,200,000 \text{ psi}$.

- Calculate the maximum bending stresses in the pine flanges and in the plywood web.
- What is $q_{\max}$ if allowable stresses are 1600 psi in the flanges and 1200 psi in the web?



Solution 6.2-11

$$q = 90 \text{ lb/ft} \quad L = 12 \text{ ft}$$

(a) MAXIMUM BENDING STRESSES

$$M_{\max} = \frac{qL^2}{8} \quad M_{\max} = 1620 \text{ lb-ft}$$

$$\text{Plywood (1):} \quad t = \frac{3}{8} \text{ in.} \quad h_1 = 7 \text{ in.}$$

$$E_{\text{plywood}} = 1.6 \times 10^6 \text{ psi}$$

$$I_1 = \frac{th_1^3}{12} \quad I_1 = 10.72 \text{ in.}^4$$

$$\text{Pine (2):} \quad b = 2 \text{ in.} \quad h_2 = 2 \text{ in.} \quad a = \frac{1}{2} \text{ in.}$$

$$E_{\text{pine}} = 1.2 \times 10^6 \text{ psi}$$

$$I_2 = 2 \left[\frac{ba^3}{12} + ba \left(\frac{h_1 + a}{2} \right)^2 + \frac{(b - t)(h_2 - a)^3}{12} \right.$$

$$\left. + (b - t)(h_2 - a) \left(\frac{h_1}{2} - \frac{h_2 - a}{2} \right)^2 \right]$$

$$I_2 = 65.95 \text{ in.}^4 \quad \leftarrow$$

$$E_{\text{plywood}} I_1 + E_{\text{pine}} I_2 = 96.287 \times 10^6 \text{ lb/in.}^2$$

$$\sigma_{\text{plywood}} = \frac{M_{\max} \left(\frac{h_1}{2} \right) E_{\text{plywood}}}{E_{\text{plywood}} I_1 + E_{\text{pine}} I_2}$$

$$\sigma_{\text{plywood}} = 1131 \text{ psi} \quad \leftarrow$$

$$\sigma_{\text{pine}} = \frac{M_{\max} \left(\frac{h_1}{2} + a \right) E_{\text{pine}}}{E_{\text{plywood}} I_1 + E_{\text{pine}} I_2}$$

$$\sigma_{\text{pine}} = 969 \text{ psi} \quad \leftarrow$$

(b) MAXIMUM UNIFORM DISTRIBUTED LOAD**MAXIMUM MOMENT BASED UPON PLYWOOD**

$$\sigma_{\text{allow\_plywood}} = 1200 \text{ psi}$$

$$\text{From } \sigma_{\text{allow\_plywood}} = \frac{M_{\text{allow\_plywood}} \left(\frac{h_1}{2} \right) E_{\text{plywood}}}{E_{\text{plywood}} I_1 + E_{\text{pine}} I_2}$$

$$M_{\text{allow\_plywood}} = 1719 \text{ lb-ft}$$

MAXIMUM MOMENT BASED UPON PINE

$$\sigma_{\text{allow\_pine}} = 1600 \text{ psi}$$

$$\text{From } \sigma_{\text{allow\_pine}} = \frac{M_{\text{allow\_pine}} \left(\frac{h_1}{2} + a \right) E_{\text{pine}}}{E_{\text{plywood}} I_1 + E_{\text{pine}} I_2}$$

$$M_{\text{allow\_pine}} = 2675 \text{ lb-ft}$$

MAXIMUM ALLOWABLE MOMENT

$$M_{\text{allow}} = \min (M_{\text{allow\_plywood}}, M_{\text{allow\_pine}})$$

$$\text{PLYWOOD GOVERNS.} \quad M_{\text{allow}} = 1719 \text{ lb-ft} \quad \leftarrow$$

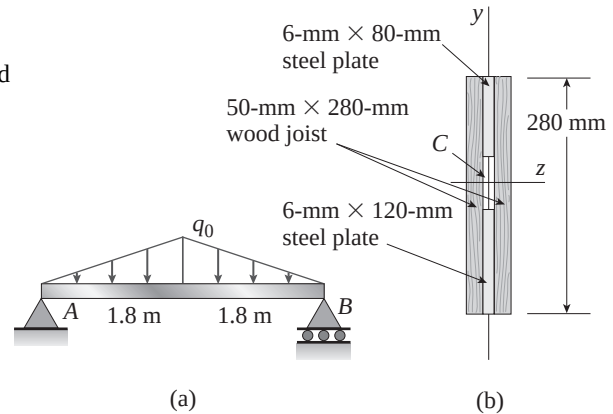
MAXIMUM UNIFORM DISTRIBUTED LOAD

$$\text{From } M_{\text{allow}} = \frac{q_{\max} L^2}{8}$$

$$q_{\max} = 95.5 \text{ lb/ft} \quad \leftarrow$$

Problem 6.2-12 A simply supported composite beam with a 3.6-m span supports a triangularly distributed load of peak intensity q_0 at midspan (see figure part a). The beam is constructed of two wood joists, each $50 \text{ mm} \times 280 \text{ mm}$, fastened to two steel plates, one of dimensions $6 \text{ mm} \times 80 \text{ mm}$ and the lower plate of dimensions $6 \text{ mm} \times 120 \text{ mm}$ (see figure part b). The modulus of elasticity for the wood is 11 GPa and for the steel is 210 GPa .

If the allowable stresses are 7 MPa for the wood and 120 MPa for the steel, find the allowable peak load intensity $q_{0,\max}$ when the beam is bent about the z axis. Neglect the weight of the beam.



Solution 6.2-12

$$L = 3.6 \text{ m}$$

DETERMINE NEUTRAL AXIS

$$\text{WOOD (1): } t_1 = 50 \text{ mm} \quad h = 280 \text{ mm}$$

$$E_w = 11 \text{ GPa}$$

$$\int y_1 dA = \bar{y}_1 A_1 = \left(\frac{h}{2} - h_1 \right) (2t_1 h)$$

$$\text{Steel (2): } t_2 = 6 \text{ mm} \quad b_1 = 80 \text{ mm}$$

$$b_2 = 120 \text{ mm} \quad E_s = 210 \text{ GPa}$$

$$\int y_2 dA = \bar{y}_2 A_2 = \left(h - h_1 - \frac{b_1}{2} \right) (t_2 b_1) - \left(h_1 - \frac{b_2}{2} \right) (t_2 b_2)$$

$$\text{From } E_1 \int y_1 dA + E_2 \int y_2 dA = 0$$

$$E_w \left(\frac{h}{2} - h_1 \right) (2t_1 h) + E_s \left[\left(h - h_1 - \frac{b_1}{2} \right) (t_2 b_1) - \left(h_1 - \frac{b_2}{2} \right) (t_2 b_2) \right] = 0$$

$$h_1 = 136.4 \text{ mm}$$

MOMENT OF INERTIA

$$\text{Wood (1): } I_1 = 2 \left[\frac{t_1 h^3}{12} + (t_1 h) \left(\frac{h}{2} - h_1 \right)^2 \right]$$

$$I_1 = 183.30 \times 10^6 \text{ mm}^4$$

$$\text{Steel (2): } I_2 = \frac{t_2 b_1^3}{12} + t_2 b_1 \left(h - h_1 - \frac{b_1}{2} \right)^2 + \frac{t_2 b_2^3}{12} + t_2 b_2 \left(h_1 - \frac{b_2}{2} \right)^2$$

$$I_2 = 10.47 \times 10^6 \text{ mm}^4$$

$$E_w I_1 + E_s I_2 = 4.22 \times 10^{12} \text{ N} \cdot \text{mm}^2$$

MAXIMUM MOMENT BASED UPON WOOD

$$\sigma_{\text{allow}_w} = 7 \text{ MPa}$$

$$\text{From } \sigma_{\text{allow}_w} = \frac{M_{\text{allow}_w} (h - h_1) E_w}{E_w I_1 + E_s I_2}$$

$$M_{\text{allow}_w} = 18.68 \text{ kN} \cdot \text{m}$$

MAXIMUM MOMENT BASED UPON STEEL

$$\sigma_{\text{allow}_s} = 120 \text{ MPa}$$

$$\text{From } \sigma_{\text{allow}_s} = \frac{M_{\text{allow}_s} (h - h_1) E_s}{E_w I_1 + E_s I_2}$$

$$M_{\text{allow}_s} = 16.78 \text{ kN} \cdot \text{m}$$

MAXIMUM ALLOWABLE MOMENT

$$M_{\text{allow}} = \min (M_{\text{allow}_w}, M_{\text{allow}_s})$$

$$\text{STEEL GOVERNS. } M_{\text{allow}} = 16.78 \text{ kN} \cdot \text{m} \quad \leftarrow$$

MAXIMUM UNIFORM DISTRIBUTED LOAD

$$\text{From } M_{\text{allow}} = \frac{q_{0\max} L^2}{12}$$

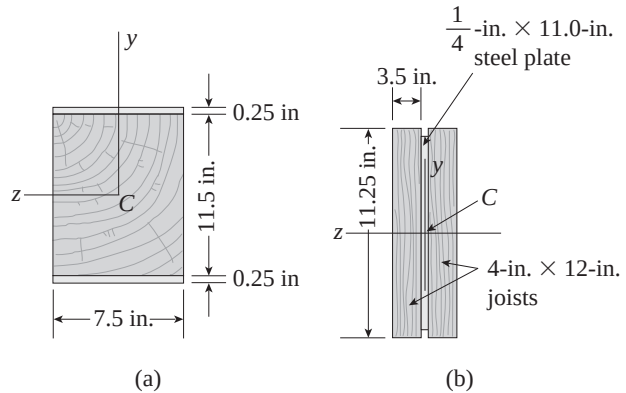
$$q_{0\max} = 15.53 \text{ kN/m} \quad \leftarrow$$

Transformed-Section Method

When solving the problems for Section 6.3, assume that the component parts of the beams are securely bonded by adhesives or connected by fasteners. Also, be sure to use the transformed-section method in the solutions.

Problem 6.3-1 A wood beam 8 in. wide and 12 in. deep (nominal dimensions) is reinforced on top and bottom by 0.25-in.-thick steel plates (see figure part a).

- Find the allowable bending moment $M_{\max}$ about the z axis if the allowable stress in the wood is 1100 psi and in the steel is 15,000 psi. (Assume that the ratio of the moduli of elasticity of steel and wood is 20.)
- Compare the moment capacity of the beam in part a with that shown in the figure part b which has two 4-in. $\times$ 12-in. joists (nominal dimensions) attached to a 1/4-in. $\times$ 11.0-in. steel plate.



Solution 6.3-1

(a) FIND $M_{\max}$:

(1) Wood beam: $b = 7.5$ in. $h_1 = 11.5$ in.

$$\sigma_{\text{allow}_w} = 1100 \text{ psi}$$

(2) Steel plates: $b = 7.5$ in. $h_2 = 12$ in.

$$t = 0.25 \text{ in.}$$

$$\sigma_{\text{allow}_s} = 15,000 \text{ psi}$$

TRANSFORMED SECTION (WOOD)

$$n = 20$$

WIDTH OF STEEL PLATES

$$b_T = nb \quad b_T = 150 \text{ in.}$$

$$I_T = \frac{bh_1^3}{12} + 2 \left[\frac{t^3 b_T}{12} + t b_T \left(\frac{h_2 - t}{2} \right)^2 \right]$$

$$I_T = 3540 \text{ in.}^4$$

MAXIMUM MOMENT BASED UPON THE WOOD (1)

$$M_1 = \frac{\sigma_{\text{allow}_w} I_T}{\frac{h_1}{2}} \quad M_1 = 677 \text{ k-in.}$$

MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$M_2 = \frac{\sigma_{\text{allow}_s} I_T}{\frac{h_2 n}{2}} \quad M_2 = 442 \text{ k-in.}$$

$$M_{\max} = \min(M_1, M_2)$$

$$\text{STEEL GOVERNS} \quad M_{\max} = 422 \text{ k-in.} \quad \leftarrow$$

(b) COMPARE MOMENT CAPACITIES

(1) Wood beam: $b = 3.5$ in. $h_1 = 11.25$ in.

(2) Steel plates: $h_2 = 11$ in. $t = 0.25$ in.

WIDTH OF STEEL PLATES

$$b_T = nt \quad b_T = 5 \text{ in.}$$

$$I_T = 2 \frac{bh_1^3}{12} + \frac{b_T h_2^3}{12} \quad I_T = 1385 \text{ in.}^4$$

MAXIMUM MOMENT BASED UPON THE WOOD (1)

$$M_1 = \frac{\sigma_{\text{allow}_w} I_T}{\frac{h_1}{2}} \quad M_1 = 271 \text{ k-in.}$$

MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$M_2 = \frac{\sigma_{\text{allow}_s} I_T}{\frac{h_2 n}{2}} \quad M_2 = 189 \text{ k-in.}$$

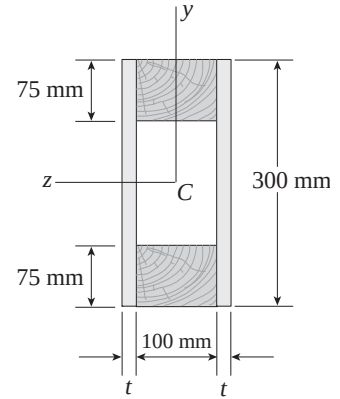
$$M_{\max} = \min(M_1, M_2)$$

$$\text{STEEL GOVERNS.} \quad M_{\max} = 189 \text{ k-in.} \quad \leftarrow$$

THE MOMENT CAPACITY OF THE BEAM IN (a) IS 2.3 TIMES MORE THAN THE BEAM IN (b)

Problem 6.3-2 A simple beam of span length 3.2 m carries a uniform load of intensity 48 kN/m. The cross section of the beam is a hollow box with wood flanges and steel side plates, as shown in the figure. The wood flanges are 75 mm by 100 mm in cross section, and the steel plates are 300 mm deep.

What is the required thickness t of the steel plates if the allowable stresses are 120 MPa for the steel and 6.5 MPa for the wood? (Assume that the moduli of elasticity for the steel and wood are 210 GPa and 10 GPa, respectively, and disregard the weight of the beam.)



Solution 6.3-2 Box beam

$$M_{\max} = \frac{qL^2}{8} = 61.44 \text{ kN} \cdot \text{m}$$

SIMPLE BEAM: $L = 3.2 \text{ m}$ $q = 48 \text{ kN/m}$

(1) Wood flanges: $b = 100 \text{ mm}$ $h = 300 \text{ mm}$

$$h_1 = 150 \text{ mm}$$

$$(\sigma_1)_{\text{allow}} = 6.5 \text{ MPa}$$

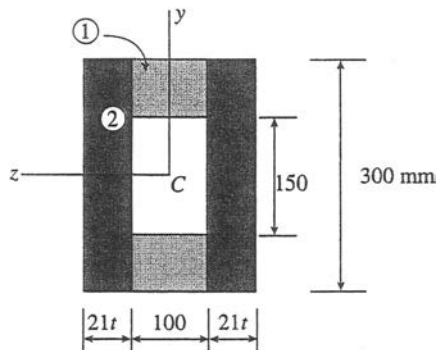
$$E_w = 10 \text{ GPa}$$

(2) Steel plates: $t = \text{thickness}$ $h = 300 \text{ mm}$

$$(\sigma_2)_{\text{allow}} = 120 \text{ MPa}$$

$$E_s = 210 \text{ GPa}$$

TRANSFORMED SECTION (WOOD)



Wood flanges are not changed.

$$n = \frac{E_s}{E_w} = 21$$

Width of steel plates

$$= nt = 21t$$

All dimensions in millimeters.

$$I_T = \frac{1}{12} (100 + 42t)(300)^3 - \frac{1}{12} (100)(150)^3$$

$$= 196.9 \times 10^6 \text{ mm}^4 + 94.5t \times 10^6 \text{ mm}^4$$

REQUIRED THICKNESS BASED UPON THE WOOD (1)
(EQ. 6-16)

$$\sigma_1 = \frac{M(h/2)}{I_T} \quad (I_T)_1 = \frac{M_{\max}(h/2)}{(\sigma_1)_{\text{allow}}}$$

$$= 1.418 \times 10^9 \text{ mm}^4$$

Equate I_T and $(I_T)_1$ and solve for t : $t_1 = 12.92 \text{ mm}$

REQUIRED THICKNESS BASED UPON THE STEEL (2) (EQ. 6-18b)

$$\sigma_2 = \frac{M(h/2)n}{I_T} \quad (I_T)_2 = \frac{M_{\max}(h/2)n}{(\sigma_2)_{\text{allow}}}$$

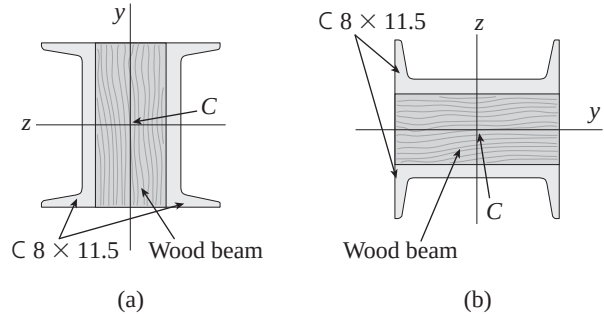
$$= 1.612 \times 10^9 \text{ mm}^4$$

Equate I_T and $(I_T)_2$ and solve for t : $t_2 = 14.97 \text{ mm}$

STEEL GOVERNS. $t_{\min} = 15.0 \text{ mm}$ ←

Problem 6.3-3 A simple beam that is 18 ft long supports a uniform load of intensity q . The beam is constructed of two C 8 × 11.5 sections (channel sections or C shapes) on either side of a 4 × 8 (actual dimensions) wood beam (see the cross section shown in the figure part a). The modulus of elasticity of the steel ($E_s = 30,000$ ksi) is 20 times that of the wood (E_w).

- (a) If the allowable stresses in the steel and wood are 12,000 psi and 900 psi, respectively, what is the allowable load q_{allow} ? (Note: Disregard the weight of the beam, and see Table F-3a, Appendix F for the dimensions and properties of the C-shape beam.)
- (b) If the beam is rotated 90° to bend about its y axis (see figure part b), and uniform load $q = 250$ lb/ft is applied, find the maximum stresses σ_s and σ_w in the steel and wood, respectively. Include the weight of the beam. (Assume weight densities of 35 lb/ft³ and 490 lb/ft³ for the wood and steel, respectively.)



Solution 6.3-3

$$L = 18 \text{ ft}$$

(a) BENT ABOUT THE z AXIS

$$(1) \text{ Wood beam: } b = 4 \text{ in. } h = 8 \text{ in.}$$

$$\sigma_{\text{allow}_w} = 900 \text{ psi}$$

$$(2) \text{ Steel channel: } h = 8.0 \text{ in. } I_z = 32.5 \text{ in.}^4$$

$$I_y = 1.31 \text{ in.}^4 \quad c = 0.572 \text{ in.}$$

$$\sigma_{\text{allow}_s} = 12,000 \text{ psi}$$

TRANSFORMED SECTION (WOOD)

$$n = 20$$

$$I_T = \frac{bh^3}{12} + 2I_z n \quad I_T = 1471 \text{ in.}^4$$

MAXIMUM MOMENT BASED UPON THE WOOD (1)

$$M_1 = \frac{\sigma_{\text{allow}_w} I_T}{h/2} \quad M_1 = 331 \text{ k-in.}$$

MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$M_2 = \frac{\sigma_{\text{allow}_s} I_T}{hn/2} \quad M_2 = 221 \text{ k-in.}$$

$$M_{\text{max}} = \min(M_1, M_2)$$

$$\text{STEEL GOVERNS. } M_{\text{max}} = 221 \text{ k-in.}$$

ALLOWABLE LOAD ON A 18-FT-LONG SIMPLE BEAM

$$\text{From } M_{\text{max}} = \frac{q_{\text{allow}} L^2}{8} \quad q_{\text{allow}} = 454 \text{ lb/ft.} \quad \leftarrow$$

(b) BENT ABOUT THE y AXIS (INCLUDING THE WEIGHT OF THE BEAM) $q = 250$ lb/ft

$$(1) \text{ Wood beam: } \rho_w = 35 \text{ lb/ft}^3 \quad q_w = lbh\rho_w$$

$$q_w = 7.778 \text{ lb/ft}$$

$$(2) \text{ Steel channels: } q_s = 11.5 \text{ lb/ft}$$

$$A_s = 3.37 \text{ in.}^2 \quad b_s = 2.26 \text{ in.}$$

$$q_{\text{total}} = q + q_w + 2q_s \quad q_{\text{total}} = 281 \text{ lb/ft}$$

$$M_{\text{max}} = \frac{q_{\text{total}} L^2}{8} \quad M_{\text{max}} = 11.4 \text{ k-ft}$$

TRANSFORMED SECTION (WOOD)

$$I_T = \frac{b^3 h}{12} + 2n \left[I_y + A_s \left(c + \frac{b}{2} \right)^2 \right]$$

$$I_T = 987 \text{ in.}^4$$

MAXIMUM STRESS IN THE WOOD (1)

$$\sigma_{w_{\text{max}}} = \frac{M_{\text{max}} \left(\frac{b}{2} \right)}{I_T} \quad \sigma_{w_{\text{max}}} = 277 \text{ psi} \quad \leftarrow$$

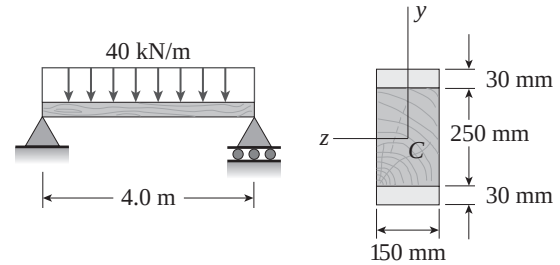
MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$\sigma_{s_{\text{max}}} = \frac{n M_{\text{max}} \left(\frac{b}{2} + b_s \right)}{I_T}$$

$$\sigma_{s_{\text{max}}} = 11,782 \text{ psi} \quad \leftarrow$$

Problem 6.3-4 The composite beam shown in the figure is simply supported and carries a total uniform load of 40 kN/m on a span length of 4.0 m. The beam is built of a southern pine wood member having cross-sectional dimensions 150 mm $\times$ 250 mm and two brass plates of cross-sectional dimensions 30 mm $\times$ 150 mm.

- Determine the maximum stresses σ_B and σ_w in the brass and wood, respectively, if the moduli of elasticity are $E_B = 96$ GPa and $E_w = 14$ GPa. (Disregard the weight of the beam.)
- Find the required thickness of the brass plates so that the plate and wood reach their allowable stress values, $\sigma_{aB} = 70$ MPa and $\sigma_{aw} = 8.5$ MPa, simultaneously under the maximum moment. What is the maximum moment?



Solution 6.3-4

Southern Pine & Brass

$$\sigma_{aB} = 70 \text{ MPa} \quad \sigma_{aw} = 8.5 \text{ MPa} \quad E_w = 14 \text{ GPa} \quad E_B = 96 \text{ GPa} \quad t_B = 30 \text{ mm}$$

$$b = 150 \text{ mm} \quad h = 250 \text{ mm} \quad n = \frac{E_B}{E_w} = 6.857 \quad h_1 = \frac{h}{2} + t_B \quad h_2 = h_1$$

$$I_w = \frac{bh^3}{12} \quad I_B = 2 \left[\frac{bt_B^3}{12} + bt_B \left(h_1 - \frac{t_B}{2} \right)^2 \right]$$

(a) MAXIMUM STRESSES

$$M_{\max B} = \sigma_{aB} \left(\frac{E_B I_B + E_w I_w}{h_2 E_B} \right) = 92.833 \text{ kN} \cdot \text{m} \quad M_{\max w} = \sigma_{aw} \left[\frac{E_B I_B + E_w I_w}{(h_1 - t_B) E_w} \right] = 95.849 \text{ kN} \cdot \text{m}$$

$$M = 80 \text{ kN} \cdot \text{m}$$

$$\sigma_B = \frac{M h_1 E_B}{E_B I_B + E_w I_w} = 60.3 \text{ MPa} \quad \sigma_w = \frac{M (h_1 - t_B) E_w}{E_B I_B + E_w I_w} = 7.09 \text{ MPa}$$

$$\boxed{\sigma_B = 60.3 \text{ MPa}} \quad \boxed{\sigma_w = 7.09 \text{ MPa}}$$

(b) PLATE THICKNESS

$$\text{SOLVING FOR } M \text{ AND EQUATING EXPRESSIONS GIVES } \frac{\sigma_{aw}}{[(h_1 - t_B) E_w]} = \frac{\sigma_{aB}}{h_1 E_B} \quad \text{or} \quad \frac{h_1}{h_1 - t_B} = \frac{\sigma_{aB}}{n \sigma_{aw}}$$

$$\text{or} \quad \frac{\frac{h}{2} + t_B}{\frac{h}{2} + t_B - t_B} = \frac{\sigma_{aB}}{n \sigma_{aw}}$$

$$\text{or} \quad t_B = \frac{\sigma_{aB}}{n \sigma_{aw}} \left(\frac{h}{2} \right) - \frac{h}{2} = 25.123 \text{ mm} \quad \boxed{t_B = 25.1 \text{ mm}}$$

$$I_B = 2 \left[\frac{bt_B^3}{12} + bt_B \left(h_1 - \frac{t_B}{2} \right)^2 \right] \quad n = \frac{E_B}{E_w} = 6.857 \quad h_1 = \frac{h}{2} + t_B \quad h_2 = h_1$$

$$M_{\max B} = \sigma_{aB} \left(\frac{E_B I_B + E_w I_w}{h_2 E_B} \right) = 79.967 \text{ kN} \cdot \text{m} \quad M_{\max w} = \sigma_{aw} \left[\frac{E_B I_B + E_w I_w}{(h_1 - t_B) E_w} \right] = 79.967 \text{ kN} \cdot \text{m}$$

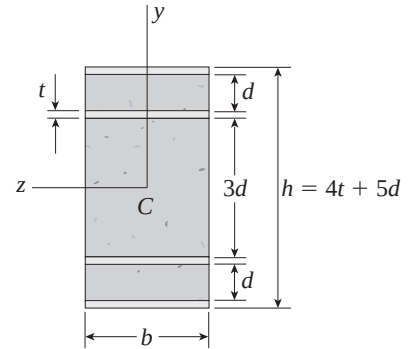
$$M_{\max} = M_{\max B} \quad \boxed{M_{\max} = 80.0 \text{ kN} \cdot \text{m}}$$

$$\sigma_B = \frac{M_{\max} h_1 E_B}{E_B I_B + E_w I_w} = 70 \text{ MPa} \quad \sigma_w = \frac{M_{\max} (h_1 - t_B) E_w}{E_B I_B + E_w I_w} = 8.5 \text{ MPa}$$

Problem 6.3-5 The cross section of a beam made of thin strips of aluminum separated by a lightweight plastic is shown in the figure. The beam has width $b = 3.0$ in., the aluminum strips have thickness $t = 0.1$ in., and the plastic segments have heights $d = 1.2$ in. and $3d = 3.6$ in. The total height of the beam is $h = 6.4$ in.

The moduli of elasticity for the aluminum and plastic are $E_a = 11 \times 10^6$ psi and $E_p = 440 \times 10^3$ psi, respectively.

Determine the maximum stresses σ_a and σ_p in the aluminum and plastic, respectively, due to a bending moment of 6.0 k-in.



Probs. 6.3-5 and 6.3-6

Solution 6.3-5 Plastic beam with aluminum strips

(1) Plastic segments: $b = 3.0$ in. $d = 1.2$ in.

$$3d = 3.6 \text{ in.}$$

$$E_p = 440 \times 10^3 \text{ psi}$$

(2) Aluminum strips: $b = 3.0$ in. $t = 0.1$ in.

$$E_a = 11 \times 10^6 \text{ psi}$$

$$h = 4t + 5d = 6.4 \text{ in.}$$

$$M = 6.0 \text{ k-in.}$$

Width of aluminum strips

$$= nb = (25)(3.0 \text{ in.}) = 75 \text{ in.}$$

All dimensions in inches.

$$\begin{aligned} \text{Plastic: } I_1 &= 2 \left[\frac{1}{12} (3.0)(1.2)^3 + (3.0)(1.2)(2.50)^2 \right] \\ &\quad + \frac{1}{12} (3.0)(3.6)^3 = 57.528 \text{ in.}^4 \end{aligned}$$

Aluminum:

$$\begin{aligned} I_2 &= 2 \left[\frac{1}{12} (75)(0.1)^3 + (75)(0.1)(3.15)^2 \right] \\ &\quad + \frac{1}{12} (75)(0.1)^3 + (75)(0.1)(1.85)^2 \\ &= 200.2 \text{ in.}^4 \end{aligned}$$

$$I_T = I_1 + I_2 = 257.73 \text{ in.}^4$$

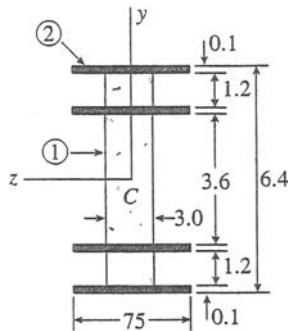
MAXIMUM STRESS IN THE PLASTIC (1) (Eq. 6-16)

$$\sigma_p = \sigma_1 = \frac{M(h/2 - t)}{I_T} = 72 \text{ psi} \quad \leftarrow$$

MAXIMUM STRESS IN THE ALUMINUM (2) (Eq. 6-18b)

$$\sigma_a = \sigma_2 = \frac{M(h/2)n}{I_T} = 1860 \text{ psi} \quad \leftarrow$$

TRANSFORMED SECTION (PLASTIC)

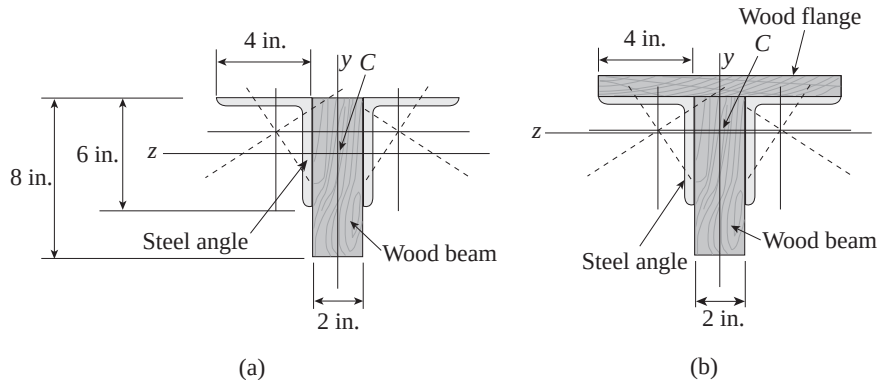


Plastic segments are not changed.

$$n = \frac{E_a}{E_p} = 25$$

Problem 6.3-7 A simple beam that is 18 ft long supports a uniform load of intensity q . The beam is constructed of two angle sections, each L 6 $\times$ 4 $\times$ 1/2, on either side of a 2-in. $\times$ 8-in. (actual dimensions) wood beam (see the cross section shown in the figure part a). The modulus of elasticity of the steel is 20 times that of the wood.

- (a) If the allowable stresses in the steel and wood are 12,000 psi and 900 psi, respectively, what is the allowable load q_{allow} ? (Note: Disregard the weight of the beam, and see Table F-5a of Appendix F for the dimensions and properties of the angles.)
- (b) Repeat part (a) if a 1-in. $\times$ 10-in. wood flange (actual dimensions) is added (see figure part b).



Solution 6.3-7

$$L = 18 \text{ ft}$$

(a) WOOD BEAM AND STEEL ANGLES

(1) Wood beam: $b = 2 \text{ in.}$ $h = 8 \text{ in.}$

$$\sigma_{\text{allow}_w} = 900 \text{ psi}$$

(2) Steel channel: $I_z = 17.3 \text{ in.}^4$ $d = 1.98 \text{ in.}$

$$A_s = 4.75 \text{ in.}^2 \quad h_s = 6 \text{ in.}$$

$$\sigma_{\text{allow}_s} = 12,000 \text{ psi}$$

TRANSFORMED SECTION (WOOD)

$$n = 20$$

NEUTRAL AXIS

$$\text{From } \int y_1 dA + \int ny_2 dA = 0$$

$$bh\left(\frac{h}{2} - h_1\right) - 2nA_s(h_1 - d) = 0$$

$$h_1 = 2.137 \text{ in.}$$

$$I_T = \left[\frac{bh^3}{12} + bh\left(\frac{h}{2} - h_1\right)^2 \right] + 2n[I_z + A_s(h_1 - d)^2] \quad I_T = 838 \text{ in.}^4$$

MAXIMUM MOMENT BASED UPON THE WOOD (1)

$$M_1 = \frac{\sigma_{\text{allow}_w} I_T}{h - h_1} \quad M_1 = 128.6 \text{ k-in.}$$

MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$M_2 = \frac{\sigma_{\text{allow}_s} I_T}{(h_s - h_1)n} \quad M_2 = 130.1 \text{ k-in.}$$

$$M_{\text{max}} = \min(M_1, M_2)$$

$$\text{WOOD GOVERNS} \quad M_{\text{max}} = 128.6 \text{ k-in.} \quad \leftarrow$$

ALLOWABLE LOAD ON A 18-FT-LONG SIMPLE BEAM

$$\text{From } M_{\text{max}} = \frac{q_{\text{allow}} L^2}{8}$$

$$q_{\text{allow}} = 264 \text{ lb/ft} \quad \leftarrow$$

(b) ADDITIONAL WOOD FLANGE $b_f = 1 \text{ in.}$
 $h_f = 10 \text{ in.}$

TRANSFORMED SECTION (WOOD)

NEUTRAL AXIS

$$\text{From } \int y_1 dA + \int ny_2 dA = 0$$

$$(bh + 2nA_s)(h_1 + b_f - h_{1_b})$$

$$- b_f h_f \left(h_{1_b} - \frac{b_f}{2} \right) = 0$$

$$h_{1_b} = 3.015 \text{ in.}$$

$$I_{T_b} = [I_T + (bh + 2nA_s)$$

$$(h_1 + b_f - h_{1_b})^2]$$

$$+ \left[\frac{b_f^3 h_f}{12} + b_f h_f \left(h_{1_b} - \frac{b_f}{2} \right)^2 \right]$$

$$I_{T_b} = 905 \text{ in.}^4$$

MAXIMUM MOMENT BASED UPON THE WOOD (1)

$$M_1 = \frac{\sigma_{\text{allow}_w} I_{T_b}}{h + b_f - h_{1_b}} \quad M_1 = 136.0 \text{ k-in.}$$

MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$M_2 = \frac{\sigma_{\text{allow}_s} I_{T_b}}{(h_s + b_f - h_{1_b})n} \quad M_2 = 136.2 \text{ k-in.}$$

$$M_{\text{max}} = \min(M_1, M_2)$$

WOOD GOVERNS $M_{\text{max}} = 136.0 \text{ k-in.}$ ←

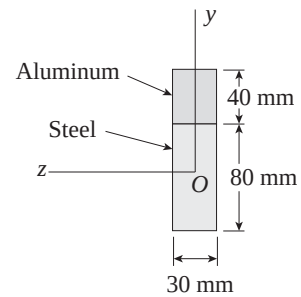
ALLOWABLE LOAD ON A 18-FT-LONG SIMPLE BEAM

$$\text{From } M_{\text{max}} = \frac{q_{\text{allow}} L^2}{8}$$

$$q_{\text{allow}} = 280 \text{ lb/ft} \quad \leftarrow$$

Problem 6.3-8 The cross section of a composite beam made of aluminum and steel is shown in the figure. The moduli of elasticity are $E_a = 75 \text{ GPa}$ and $E_s = 200 \text{ GPa}$.

- Under the action of a bending moment that produces a maximum stress of 50 MPa in the aluminum, what is the maximum stress σ_s in the steel?
- If the height of the beam remains at 120 mm, and allowable stresses in steel and aluminum are defined as 94 MPa and 40 MPa, respectively, what heights h_a and h_s are required so that both steel and aluminum reach their allowable stress values under the maximum moment?



Solution 6.3-8 Composite beam of aluminum and steel

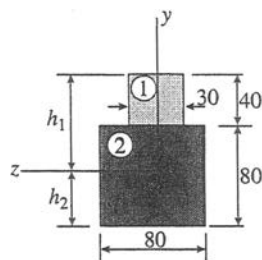
$$E_s = 200 \text{ GPa} \quad E_a = 75 \text{ GPa} \quad n = \frac{E_s}{E_a} = 2.667$$

(a) STRESS IN STEEL

$$(1) \text{ Aluminum: } b = 30 \text{ mm} \quad h_a = 40 \text{ mm} \\ E_a = 75 \text{ GPa} \quad \sigma_a = 50 \text{ MPa}$$

$$(2) \text{ Steel: } b = 30 \text{ mm} \quad h_s = 80 \text{ mm} \\ E_s = 200 \text{ GPa} \quad \sigma_s = ?$$

TRANSFORMED SECTION (ALUMINUM)



Aluminum part is not changed.

$$n = \frac{E_s}{E_a} = \frac{200}{75} = 2.667$$

Width of steel part =

$$nb = (2.667)(30 \text{ mm}) = 80 \text{ mm}$$

All dimensions in millimeters.

Use the base of the cross section as a reference line.

$$h_2 = \frac{\sum y_i A_i}{\sum A_i} = \frac{(40)(80)(80) + (100)(30)(40)}{(80)(80) + (30)(40)} \\ = 49.474 \text{ mm}$$

$$h_1 = 120 - h_2 = 70.526 \text{ mm}$$

MAXIMUM STRESS IN THE ALUMINUM (1) (Eq. 6-16)

$$\sigma_a = \sigma_1 = \frac{M h_1}{I_T}$$

MAXIMUM STRESS IN THE STEEL (2) (Eq. 6-18b)

$$\sigma_s = \sigma_2 = \frac{M h_2 n}{I_T}$$

$$\frac{\sigma_s}{\sigma_a} = \frac{h_2 n}{h_1} = \frac{(49.474)(2.667)}{70.526} = 1.8707$$

$$\sigma_s = 1.8707 (50 \text{ MPa}) = 93.5 \text{ MPa} \quad \leftarrow$$

(b) BALANCED DESIGN $b = 30 \text{ mm}$ $h_a = 115 \text{ mm}$ $h_s = 5 \text{ mm}$

$$\sigma_{aa} = 40 \text{ MPa} \quad \sigma_{as} = 94 \text{ MPa}$$

$$h_s = 5 \text{ mm}$$

$$\frac{bh_a^2 + 2bnh_a h_s + bn h_s^2}{bh_a^2 + 2bnh_a h_s + bn h_s^2} = \frac{n\sigma_{aa}}{\sigma_{as}}$$

$$h_s = \text{Find}(h_s) = 5.08 \text{ mm} \quad h_a = 120 \text{ mm} - h_s = 114.92 \text{ mm}$$

$$h_2 = \frac{nbh_s\left(\frac{h_s}{2}\right) + bh_a\left(h_s + \frac{h_a}{2}\right)}{nbh_s + bh_a} = 56.21 \text{ mm} \quad h_1 = h_a + h_s - h_2 = 63.79 \text{ mm}$$

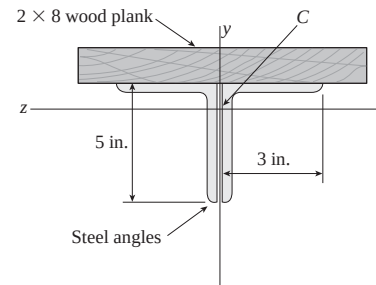
$$\sigma_s = \left(\frac{h_2 n}{h_1}\right) \sigma_{aa} = 93.993 \text{ MPa}$$

$$I_T = \frac{nbh_s^3}{12} + nbh_s\left(h_2 - \frac{h_s}{2}\right)^2 + \frac{bh_a^3}{12} + bh_a\left(h_1 - \frac{h_a}{2}\right)^2 = 5.104 \times 10^{-6} \text{ m}^4$$

$$M_{\max a} = \frac{\sigma_{aa} I_T}{h_1} = 3.201 \text{ kN} \cdot \text{m} \quad M_{\max s} = \frac{\sigma_{as} I_T}{nh_2} = 3.201 \text{ kN} \cdot \text{m}$$

Problem 6.3-9 A beam is constructed of two angle sections, each L $5 \times 3 \times 1/2$, which reinforce a 2×8 (actual dimensions) wood plank (see the cross section shown in the figure). The modulus of elasticity for the wood is $E_w = 1.2 \times 10^6 \text{ psi}$ and for the steel is $E_s = 30 \times 10^6 \text{ psi}$.

Find the allowable bending moment M_{allow} for the beam if the allowable stress in the wood is $\sigma_w = 1100 \text{ psi}$ and in the steel is $\sigma_s = 12,000 \text{ psi}$. (Note: Disregard the weight of the beam, and see Table F-5a of Appendix F for the dimensions and properties of the angles.)



Solution 6.3-9

(1) Wood beam: $b = 2 \text{ in.}$ $h = 8 \text{ in.}$

$$\sigma_{\text{allow}_w} = 1100 \text{ psi} \quad E_w = 1.2 \cdot 10^6 \text{ psi}$$

(2) Steel angle: $I_z = 9.43 \text{ in.}^4$ $d = 1.74 \text{ in.}$

$$A_s = 3.75 \text{ in.}^2 \quad h_s = 5 \text{ in.}$$

$$\sigma_{\text{allow}_s} = 12,000 \text{ psi} \quad E_s = 30 \cdot 10^6 \text{ psi}$$

TRANSFORMED SECTION (WOOD)

$$n = \frac{E_s}{E_w} \quad n = 25$$

NEUTRAL AXIS

$$\text{From } \int y_1 dA + \int ny_2 dA = 0$$

$$bh\left(h_1 - \frac{b}{2}\right) - 2nA_s(b + d - h_1) = 0$$

$$h_1 = 3.525 \text{ in.}$$

$$I_T = \left[\frac{b^3 h}{12} + bh\left(h_1 - \frac{b}{2}\right)^2 \right] + 2n[I_z + A_s(b + d - h_1)^2]$$

$$I_T = 588 \text{ in.}^4$$

MAXIMUM MOMENT BASED UPON THE WOOD (1)

$$M_1 = \frac{\sigma_{\text{allow}_w} I_T}{h_1} \quad M_1 = 183.4 \text{ k-in.}$$

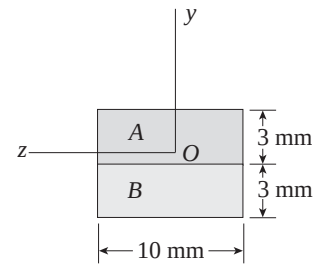
MAXIMUM MOMENT BASED UPON THE STEEL (2)

$$M_2 = \frac{\sigma_{\text{allow}_s} I_T}{(h_s + b - h_1)n} \quad M_2 = 81.1 \text{ k-in.}$$

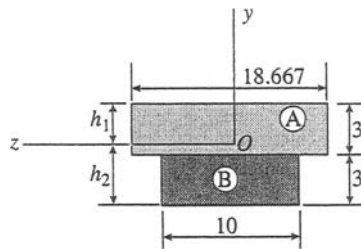
$$M_{\max} = \min(M_1, M_2)$$

$$\text{STEEL GOVERNS} \quad M_{\max} = 81.1 \text{ k-in.} \quad \leftarrow$$

Problem 6.3-10 The cross section of a bimetallic strip is shown in the figure. Assuming that the moduli of elasticity for metals A and B are $E_A = 168 \text{ GPa}$ and $E_B = 90 \text{ GPa}$, respectively, determine the smaller of the two section moduli for the beam. (Recall that section modulus is equal to bending moment divided by maximum bending stress.) In which material does the maximum stress occur?

**Solution 6.3-10 Bimetallic strip**Metal A: $b = 10 \text{ mm}$ $h_A = 3 \text{ mm}$ $E_A = 168 \text{ GPa}$ Metal B: $b = 10 \text{ mm}$ $h_B = 3 \text{ mm}$ $E_B = 90 \text{ GPa}$

TRANSFORMED SECTION



TRANSFORMED SECTION (METAL B)

Metal B does not change.

$$n = \frac{E_A}{E_B} = \frac{168}{90} = 1.8667$$

Width of metal A

$$= nb = (1.8667)(10 \text{ mm}) = 18.667 \text{ mm}$$

All dimensions in millimeters.

Use the base of the cross section as a reference line.

$$h_2 = \frac{\sum y_i A_i}{\sum A_i} = \frac{(1.5)(10)(3) + (4.5)(18.667)(3)}{(10)(3) + (18.667)(3)}$$

$$= 3.4535 \text{ mm}$$

$$h_1 = 6 - h_2 = 2.5465 \text{ mm}$$

$$\begin{aligned} I_T &= \frac{1}{12} (10)(3)^3 + (10)(3)(h_2 - 1.5)^2 \\ &\quad + \frac{1}{12} (18.667)(3)^3 + (18.667)(3)(h_1 - 1.5)^2 \\ &= 240.31 \text{ mm}^4 \end{aligned}$$

MAXIMUM STRESS IN MATERIAL B (EQ. 6-16)

$$\sigma_B = \sigma_1 = \frac{M h_2}{I_T} \quad S_B = \frac{M}{\sigma_B} = \frac{I_T}{h_2} = 69.6 \text{ mm}^3$$

MAXIMUM STRESS IN MATERIAL A (EQ. 6-18b)

$$\begin{aligned} \sigma_A = \sigma_2 &= \frac{M h_1 n}{I_T} \quad S_A = \frac{M}{\sigma_A} = \frac{I_T}{h_1 n} \\ &= 50.6 \text{ mm}^3 \end{aligned}$$

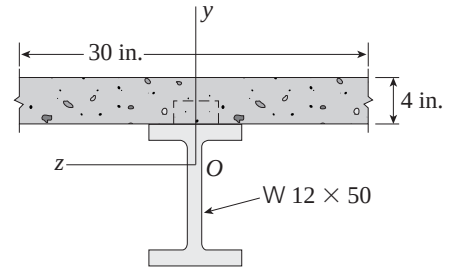
SMALLER SECTION MODULUS

$$S_A = 50.6 \text{ mm}^3 \quad \leftarrow$$

 $\therefore$ Maximum stress occurs in metal A. $\leftarrow$

Problem 6.3-11 A W 12 × 50 steel wide-flange beam and a segment of a 4-inch thick concrete slab (see figure) jointly resist a positive bending moment of 95 k-ft. The beam and slab are joined by shear connectors that are welded to the steel beam. (These connectors resist the horizontal shear at the contact surface.) The moduli of elasticity of the steel and the concrete are in the ratio of 12 to 1.

Determine the maximum stresses σ_s and σ_c in the steel and concrete, respectively. (Note: See Table F-1a of Appendix F for the dimensions and properties of the steel beam.)



Solution 6.3-11 Steel beam and concrete slab

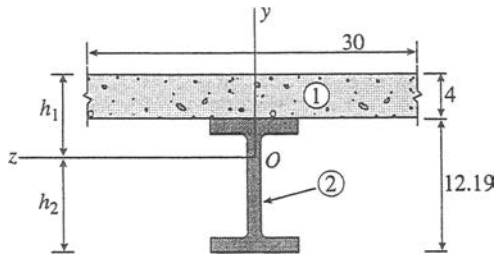
(1) Concrete: $b = 30$ in. $t = 4$ in.

(2) Wide-flange beam: W 12 × 50

$d = 12.19$ in. $I = 394$ in.⁴

$A = 14.7$ in.² $M = 95$ k-ft = 1140 k-in.

TRANSFORMED SECTION (CONCRETE)



No change in dimensions of the concrete.

$$n = \frac{E_s}{E_c} = \frac{E_2}{E_1} = 12$$

Width of steel beam is increased by the factor n to transform to concrete.

All dimensions in inches.

Use the base of the cross section as a reference line.

$$nI = 4728 \text{ in.}^4 \quad nA = 176.4 \text{ in.}^2$$

$$h_2 = \frac{\sum y_i A_i}{\sum A_i} = \frac{(12.19/2)(176.4) + (14.19)(30)(4)}{176.4 + (30)(4)} = 9.372 \text{ in.}$$

$$h_1 = 16.19 - h_2 = 6.818 \text{ in.}$$

$$I_T = \frac{1}{12} (30)(4)^3 + (30)(4)(h_1 - 2)^2 + 4728 + (176.4)(h_2 - 12.19/2)^2 = 9568 \text{ in.}^4$$

MAXIMUM STRESS IN THE CONCRETE (1) (Eq. 6-16)

$$\sigma_c = \sigma_1 = \frac{Mh_1}{I_T} = 812 \text{ psi (Compression)} \quad \leftarrow$$

MAXIMUM STRESS IN THE STEEL (2) (Eq. 6-18b)

$$\sigma_s = \sigma_2 = \frac{Mh_2 n}{I_T} = 13,400 \text{ psi (Tension)} \quad \leftarrow$$

Problem 6.3-12 A reinforced concrete beam (see fig.) is acted on by a positive bending moment of $M = 160 \text{ kN} \cdot \text{m}$. Steel reinforcement consists of 4 bars of 28-mm diameter. The modulus of elasticity for the concrete is $E_c = 25 \text{ GPa}$ while that of the steel is $E_s = 200 \text{ GPa}$.

- Find the maximum stresses in steel and concrete.
- If *allowable* stresses for concrete and steel are $\sigma_{ac} = 9.2 \text{ MPa}$ and $\sigma_{as} = 135 \text{ MPa}$, respectively, what is the maximum permissible positive bending moment?
- What is the required area of steel reinforcement, A_s , if a balanced condition must be achieved? What is the allowable positive bending moment? (Recall that in a balanced design, both steel and concrete reach allowable stress values simultaneously under the design moment).

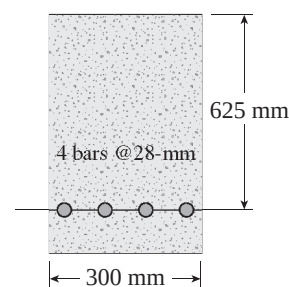
$$E_c = 25 \text{ GPa} \quad E_s = 200 \text{ GPa}$$

$$n = \frac{E_s}{E_c} = 8 \quad M = 160 \text{ kN} \cdot \text{m}$$

$$\sigma_{ac} = 9.2 \text{ MPa} \quad \sigma_{as} = 135 \text{ MPa}$$

$$b = 300 \text{ mm} \quad d = 625 \text{ mm} \quad d_s = 28 \text{ mm}$$

$$A_s = 4 \frac{\pi}{4} d_s^2 = 2.463 \times 10^{-3} \text{ m}^2$$



Solution 6.3-12

- (a) MAXIMUM STRESSES IN STEEL & CONCRETE

Take 1st moments of transformed section to locate nA : $by\frac{y}{2} - nA_s(d - y) = 0$

$$y = \sqrt{\left(\frac{nA_s}{b}\right)^2 + 2d\left(\frac{nA_s}{b}\right)} - \left(\frac{nA_s}{b}\right) = 228.283 \text{ mm}$$

$$\text{Moment of inertia of transformed section: } I_T = \frac{by^3}{3} + nA_s(d - y)^2 = 4.291 \times 10^{-3} \text{ m}^4$$

$$\text{Stresses in concrete \& steel: } \sigma_c = \frac{My}{I_T} = 8.51 \text{ MPa} \quad \sigma_s = n \frac{M(d - y)}{I_T} = 118.3 \text{ MPa}$$

- (b) MAXIMUM PERMISSIBLE POSITIVE BENDING MOMENT $M = 160 \text{ kN} \cdot \text{m}$

$$M_C = \frac{\sigma_{ac}}{y} I_T = 172.9 \text{ kN} \cdot \text{m} \quad M_s = \frac{\sigma_{as}}{n(d - y)} I_T = 182.5 \text{ kN} \cdot \text{m} < \text{Concrete governs.}$$

$$M_{\max} = M_C = 172.9 \text{ kN} \cdot \text{m}$$

- (c) BALANCED DESIGN: REQUIRED AREA OF STEEL $b = 300 \text{ mm} \quad d = 625 \text{ mm}$

Equate moment capacities of concrete & steel, then solve for required y :

$$\frac{\sigma_{ac}}{y} I_T = \frac{\sigma_{as}}{n(d - y)} I_T \quad \text{or} \quad \frac{y}{d - y} = \frac{n\sigma_{ac}}{\sigma_{as}} = 0.54519 \quad \text{gives} \quad y = 220.51774 \text{ mm}$$

Use required y in 1st moments equation to find required A_s : $b = 300 \text{ mm} \quad d = 625 \text{ mm} \quad n = 8$

$$by\frac{y}{2} - nA_s(d - y) = 0$$

$$\text{Solving for } A_s \text{ gives} \quad A_s = 2254 \text{ mm}^2$$

Recompute I_T :

$$I_T = \frac{by^3}{3} + nA_s(d-y)^2 = 4.023 \times 10^{-3} \text{ m}^4$$

Allowable positive bending moments are equal for balanced design.

$$\boxed{M_c = \frac{\sigma_{ac} I_T}{y} = 167.8 \text{ kN} \cdot \text{m}} \quad \boxed{M_s = \frac{\sigma_{as} I_T}{n(d-y)} = 167.8 \text{ kN} \cdot \text{m}} \quad M_{\text{allow}} = M_c$$

$$A_s = 2254 \text{ mm}^2 \quad M_{\text{allow}} = 167.8 \text{ kN} \cdot \text{m}$$

Problem 6.3-13 A reinforced concrete T-beam (see fig.) is acted on by a positive bending moment of $M = 175$ ft-kips. Steel reinforcement consists of four bars of 1.41-inch diameter. The modulus of elasticity for the concrete is $E_c = 3000$ ksi while that of the steel is $E_s = 29,000$ ksi. Let $b = 48$ in., $t_f = 4$ in., $b_w = 15$ in., and $d = 24$ in.

- (a) Find the maximum stresses in steel and concrete.
 (b) If *allowable* stresses for concrete and steel are $\sigma_{ac} = 1400$ psi and $\sigma_{as} = 18$ ksi, respectively, what is the maximum permissible positive bending moment?

$$E_c = 3000 \text{ ksi}$$

$$E_s = 29,000 \text{ ksi}$$

$$n = \frac{E_s}{E_c} = 9.667$$

$$b = 48 \text{ in.} \quad t_f = 4 \text{ in.}$$

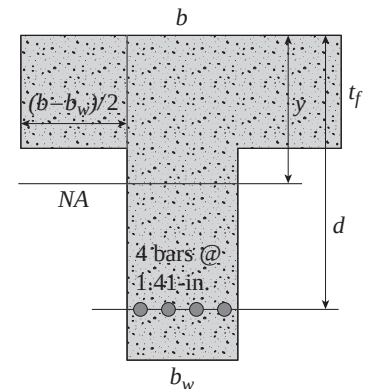
$$b_w = 15 \text{ in.} \quad d = 24 \text{ in.}$$

$$\sigma_{ac} = 1400 \text{ psi}$$

$$\sigma_{as} = 18 \text{ ksi} \quad d_s = 1.41 \text{ in.}$$

$$M = 175 \text{ kip-ft}$$

$$A_s = 4 \frac{\pi}{4} d_s^2 = 6.246 \text{ in.}^2$$



Solution 6.3-13

(a) MAXIMUM STRESSES IN STEEL AND CONCRETE

$$(b - b_w)t_f \left(y - \frac{t_f}{2} \right) + b_w y \frac{y}{2} - nA_s(d - y) = 0$$

$$\text{Solving,} \quad y = 6.996 \text{ in.}$$

$$I_T = \frac{b_w y^3}{3} + \frac{(b - b_w)t_f^3}{12} + (b - b_w)t_f \left(y - \frac{t_f}{2} \right)^2 + nA_s(d - y)^2 = 2.264 \times 10^4 \text{ in.}^4$$

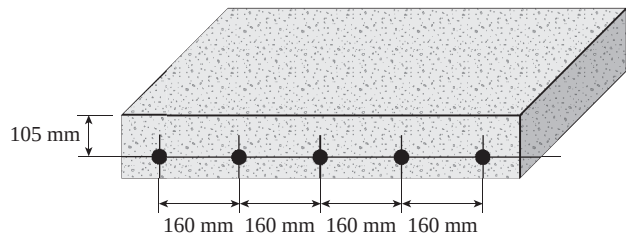
$$\boxed{\sigma_c = \frac{My}{I_T} = 649 \text{ psi}} \quad \boxed{\sigma_s = n \frac{M(d - y)}{I_T} = 15.25 \text{ ksi}} \quad \sigma_c = 649 \text{ psi} \quad \sigma_s = 15,246 \text{ psi}$$

- (b) MAXIMUM ALLOWABLE POSITIVE BENDING MOMENT $\sigma_{ac} = 1.4 \times 10^3 \text{ psi}$ $\sigma_{as} = 1.8 \times 10^4 \text{ psi}$

$$M_c = \frac{\sigma_{ac} I_T}{y} = 378 \text{ kip-ft} \quad \boxed{M_s = \frac{\sigma_{as} I_T}{n(d-y)} = 207 \text{ kip-ft}} < \text{Steel controls.}$$

$$M_{\text{allow}} = M_s = 207 \text{ kip-ft}$$

Problem 6.3-14 A reinforced concrete slab (see fig.) is reinforced with 13-mm bars spaced 160 mm apart at $d = 105 \text{ mm}$ from the top of the slab. The modulus of elasticity for the concrete is $E_c = 25 \text{ GPa}$ while that of the steel is $E_s = 200 \text{ GPa}$. Assume that *allowable* stresses for concrete and steel are $\sigma_{ac} = 9.2 \text{ MPa}$ and $\sigma_{as} = 135 \text{ MPa}$.



- (a) Find the maximum permissible positive bending moment for a 1-m-wide strip of the slab.
 (b) What is the required area of steel reinforcement, A_s , if a *balanced condition* must be achieved? What is the allowable positive bending moment? (Recall that in a balanced design, both steel and concrete reach allowable stress values simultaneously under the design moment).

Solution 6.3-14

$$E_c = 25 \text{ GPa} \quad E_s = 200 \text{ GPa} \quad n = \frac{E_s}{E_c} = 8 \quad s = 160 \text{ mm} \quad \sigma_{ac} = 9.2 \text{ MPa} \quad \sigma_{as} = 135 \text{ MPa}$$

$$b = 1 \text{ m} \quad d = 105 \text{ mm} \quad d_s = 13 \text{ mm}$$

$$A_s = \left(\frac{1 \text{ m}}{s} \right) \frac{\pi}{4} d_s^2 = 829.577 \text{ mm}^2$$

- (a) MAXIMUM PERMISSIBLE POSITIVE BENDING MOMENT

$$\text{Take 1st moments of transformed section to locate } nA: \quad by \frac{y}{2} - nA_s(d-y) = 0$$

$$y = \sqrt{\left(\frac{nA_s}{b} \right)^2 + 2d \left(\frac{nA_s}{b} \right)} - \left(\frac{nA_s}{b} \right) = 31.281 \text{ mm}$$

$$\text{Moment of inertia of transformed section: } I_T = \frac{by^3}{3} + nA_s(d-y)^2 = 4.627 \times 10^{-5} \text{ m}^4$$

$$M_c = \frac{\sigma_{ac}}{y} I_T = 13.6 \text{ kN} \cdot \text{m} \quad \boxed{M_s = \frac{\sigma_{as}}{n(d-y)} I_T = 10.59 \text{ kN} \cdot \text{m}} < \text{Steel governs.}$$

$$M_{\text{max}} = M_s = 10.59 \text{ kN} \cdot \text{m}$$

- (b) BALANCED DESIGN: REQUIRED AREA OF STEEL $b = 1000 \text{ mm}$ $d = 105 \text{ mm}$

Equate moment capacities of concrete & steel, then solve for required y :

$$\frac{\sigma_{ac}}{y} I_T = \frac{\sigma_{as}}{n(d-y)} I_T \quad \text{or} \quad \frac{y}{d-y} = \frac{n\sigma_{ac}}{\sigma_{as}} = 0.54519 \quad \text{gives} \quad y = 37.047 \text{ mm}$$

Use required y in 1st moments equation to find required A_s : $b = 1000 \text{ mm}$ $d = 105 \text{ mm}$ $n = 8$

$$by \frac{y}{2} - nA_s(d - y) = 0$$

Solving for required A_s gives $A_s = 1262 \text{ mm}^2$ $< 52\%$ increase in steel area

$$\text{Solve for new bar size at spacing } s = 160 \text{ mm} \quad d_s = \sqrt{A_s \left(\frac{4}{\pi} \right) \left(\frac{s}{1 \text{ m}} \right)} = 16 \text{ mm}$$

Recompute I_T :

$$I_T = \frac{by^3}{3} + nA_s(d - y)^2 = 6.358 \times 10^{-5} \text{ m}^4$$

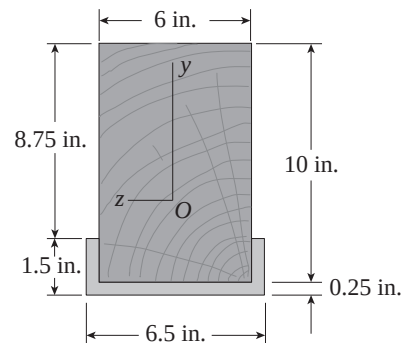
Allowable positive bending moments based on allowable stresses in steel and concrete are equal for balanced design.

$$M_c = \frac{\sigma_{ac} I_T}{y} = 15.79 \text{ kN} \cdot \text{m} \quad M_s = \frac{\sigma_{as} I_T}{n(d - y)} = 15.79 \text{ kN} \cdot \text{m} \quad < 33\% \text{ increase in moment capacity}$$

$$M_{\text{allow}} = 15.79 \text{ kN} \cdot \text{m}$$

Problem 6.3-15 A wood beam reinforced by an aluminum channel section is shown in the figure. The beam has a cross section of dimensions 6 in. by 10 in., and the channel has a uniform thickness of 0.25 in.

If the allowable stresses in the wood and aluminum are 1160 psi and 5500 psi, respectively, and if their moduli of elasticity are in the ratio of 1 to 6, what is the maximum allowable bending moment for the beam?



Solution 6.3-15

$$n = \frac{E_A}{E_W} \quad n = 6$$

$$b = 6 \text{ in.} \quad t = 0.25 \text{ in.}$$

$$b_c = 6.5 \text{ in.} \quad h = 10 \text{ in.}$$

$$\sigma_{aA} = 5500 \text{ psi}$$

$$\sigma_{aW} = 1160 \text{ psi}$$

(1) TAKE 1ST MOMENTS OF TRANSFORMED SECTION AREAS ABOUT BASE TO LOCATE nA (h_1 , h_2)

$$h_2 = \frac{nb_c t \frac{t}{2} + 2nt(1.25 \text{ in.}) \left(t + \frac{1.25 \text{ in.}}{2} \right) + bh \left(t + \frac{h}{2} \right)}{nb_c t + 2nt(1.25 \text{ in.}) + bh} = 4.347 \text{ in.} \quad h_1 = h + t - h_2 = 5.903 \text{ in.}$$

(2) COMPUTE I_T = MOMENT OF INERTIA OF TRANSFORMED SECTION ABOUT nA

$$I_1 = \frac{nb_c t^3}{12} + nb_c t \left(h_2 - \frac{t}{2} \right)^2$$

$$I_1 = 173.842 \text{ in.}^4$$

$$I_2 = \frac{2nt[(1.25 \text{ in.})^3]}{12} + 2nt(1.25 \text{ in.})\left(h_2 - t - \frac{1.25 \text{ in.}}{2}\right)$$

$$I_2 = 45.692 \text{ in.}^4$$

$$I_3 = \frac{bh^3}{12} + bh\left(h_1 - \frac{h}{2}\right)^2$$

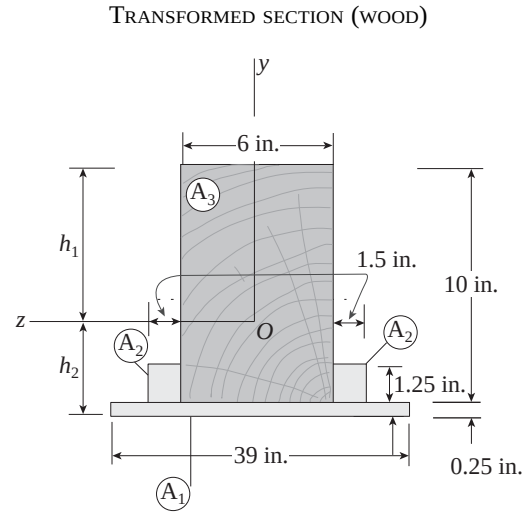
$$I_3 = 548.931 \text{ in.}^4$$

$$I_T = I_1 + I_2 + I_3 = 768.466 \text{ in.}^4$$

- (3) COMPUTE MAXIMUM ALLOWABLE BENDING MOMENTS BASED ON ALLOWABLE STRESSES IN ALUMINUM AND WOOD

$$M_W = \frac{\sigma_{aW} I_T}{h_1} = 12.58 \text{ kip-ft} \quad M_A = \frac{\sigma_{aA} I_T}{nh_2} = 13.5 \text{ kip-ft}$$

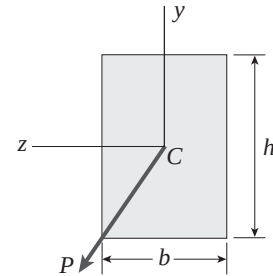
$$M_{\text{allow}} = M_W = 12.58 \text{ kip-ft}$$



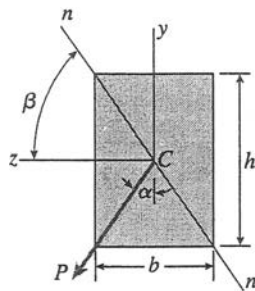
Beams with Inclined Loads

When solving the problems for Section 6.4, be sure to draw a sketch of the cross section showing the orientation of the neutral axis and the locations of the points where the stresses are being found.

Problem 6.4-1 A beam of rectangular cross section supports an inclined load P having its line of action along a diagonal of the cross section (see figure). Show that the neutral axis lies along the other diagonal.



Solution 6.4-1 Location of neutral axis



Load P acts along a diagonal.

$$\tan \alpha = \frac{b/2}{h/2} = \frac{b}{h}$$

$$I_z = \frac{bh^3}{12}$$

$$I_y = \frac{hb^3}{12}$$

$$\frac{I_z}{I_y} = \frac{h^2}{b^2}$$

See Figure 6-19b.

β = angle between the z axis and the neutral axis nn

θ = angle between the y axis and the load P

$$\theta = \alpha + 180^\circ$$

$$\tan \theta = \tan (\alpha + 180^\circ) = \tan \alpha$$

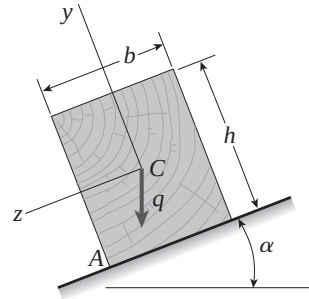
$$\begin{aligned} \text{(Eq. 6-24):} \quad \tan \beta &= \frac{I_z}{I_y} \tan \theta = \frac{h^2}{b^2} \tan \theta \\ &= \left(\frac{h^2}{b^2}\right)\left(\frac{b}{h}\right) = \frac{h}{b} \end{aligned}$$

$\therefore$ The neutral axis lies along the other diagonal. QED $\leftarrow$

Problem 6.4-2 A wood beam of rectangular cross section (see figure) is simply supported on a span of length L . The longitudinal axis of the beam is horizontal, and the cross section is tilted at an angle α . The load on the beam is a vertical uniform load of intensity q acting through the centroid C .

Determine the orientation of the neutral axis and calculate the maximum tensile stress $\sigma_{\max}$ if $b = 80$ mm, $h = 140$ mm, $L = 1.75$ m, $\alpha = 22.5^\circ$, and $q = 7.5$ kN/m.

Probs. 6.4-2 and 6.4-3



Solution 6.4-2

$$L = 1.75 \text{ m} \quad q = 7.5 \text{ kN/m} \quad b = 80 \text{ mm} \\ h = 140 \text{ mm} \quad \alpha = 22.5^\circ$$

BENDING MOMENTS

$$M_y = \frac{q \sin(\alpha) L^2}{8} \quad M_y = 1099 \text{ N} \cdot \text{m}$$

$$M_z = \frac{q \cos(\alpha) L^2}{8} \quad M_z = 2653 \text{ N} \cdot \text{m}$$

MOMENT OF INERTIA

$$I_y = \frac{hb^3}{12} \quad I_y = 5.973 \times 10^6 \text{ mm}^4$$

$$I_z = \frac{bh^3}{12} \quad I_z = 18.293 \times 10^6 \text{ mm}^4$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_z}{I_y} \tan(\alpha)\right) \quad \beta = 51.8^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

$$\sigma_{\max} = \frac{M_y \left(\frac{b}{2}\right)}{I_y} - \frac{M_z \left(\frac{-h}{2}\right)}{I_z} \\ \sigma_{\max} = 17.5 \text{ MPa} \quad \leftarrow$$

Problem 6.4-3 Solve the preceding problem for the following data: $b = 6$ in., $h = 10$ in., $L = 12.0$ ft, $\tan \alpha = 1/3$, and $q = 325$ lb/ft.

Solution 6.4-3

$$L = 12 \text{ ft} \quad q = 325 \text{ lb/ft} \quad b = 6 \text{ in.} \\ h = 10 \text{ in.} \quad \alpha = \arctan\left(\frac{1}{3}\right)$$

BENDING MOMENTS

$$M_y = \frac{q \sin(\alpha) L^2}{8} \quad M_y = 22,199 \text{ lb-in.}$$

$$M_z = \frac{q \cos(\alpha) L^2}{8} \quad M_z = 66,598 \text{ lb-in.}$$

MOMENT OF INERTIA

$$I_y = \frac{hb^3}{12} \quad I_y = 180 \text{ in.}^4$$

$$I_z = \frac{bh^3}{12} \quad I_z = 500 \text{ in.}^4$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_z}{I_y} \tan(\alpha)\right) \quad \beta = 42.8^\circ \quad \leftarrow$$

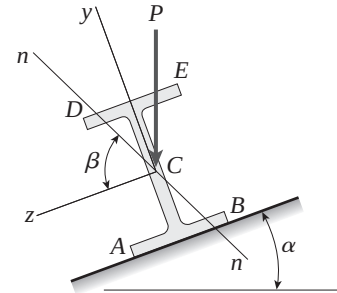
MAXIMUM TENSILE STRESS (AT POINT A)

$$\sigma_{\max} = \frac{M_y\left(\frac{b}{2}\right)}{I_y} - \frac{M_z\left(\frac{-h}{2}\right)}{I_z}$$

$$\sigma_{\max} = 1036 \text{ psi} \quad \leftarrow$$

Problem 6.4-4 A simply supported wide-flange beam of span length L carries a vertical concentrated load P acting through the centroid C at the midpoint of the span (see figure). The beam is attached to supports inclined at an angle α to the horizontal.

Determine the orientation of the neutral axis and calculate the maximum stresses at the outside corners of the cross section (points A , B , D , and E) due to the load P . Data for the beam are as follows: W 250 $\times$ 44.8 section, $L = 3.5$ m, $P = 18$ kN, and $\alpha = 26.57^\circ$. (Note: See Table F-1b of Appendix F for the dimensions and properties of the beam.)



Probs. 6.4-4 and 6.4-5

Solution 6.4-4

$$L = 3.5 \text{ m} \quad P = 18 \text{ kN} \quad \alpha = 26.57^\circ$$

Wide-flange beam: W 250 $\times$ 44.8

$$I_y = 6.95 \times 10^6 \text{ mm}^4 \quad I_z = 70.8 \times 10^6 \text{ mm}^4$$

$$d = 267 \text{ mm} \quad b = 148 \text{ mm}$$

BENDING MOMENTS

$$M_y = \frac{P \sin(\alpha)L}{4} \quad M_y = 7045 \text{ N} \cdot \text{m}$$

$$M_z = \frac{P \cos(\alpha)L}{4} \quad M_z = 14,087 \text{ N} \cdot \text{m}$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_z}{I_y} \tan(\alpha)\right) \quad \beta = 78.9^\circ \quad \leftarrow$$

BENDING STRESSES

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

$$\text{POINT A:} \quad z_A = \frac{b}{2} \quad y_A = \frac{-d}{2} \quad \sigma_A = \sigma_x(z_A, y_A)$$

$$\sigma_A = 102 \text{ MPa} \quad \leftarrow$$

$$\text{POINT B:} \quad z_B = -\frac{b}{2} \quad y_B = \frac{-d}{2}$$

$$\sigma_B = \sigma_x(z_B, y_B) \quad \sigma_B = -48 \text{ MPa} \quad \leftarrow$$

$$\text{POINT D:} \quad \sigma_D = -\sigma_B \quad \sigma_D = 48 \text{ MPa} \quad \leftarrow$$

$$\text{POINT E:} \quad \sigma_E = -\sigma_A \quad \sigma_E = -102 \text{ MPa} \quad \leftarrow$$

Problem 6.4-5 Solve the preceding problem using the following data: W 8 × 21 section, $L = 84$ in., $P = 4.5$ k, and $\alpha = 22.5^\circ$.

Solution 6.4-5

$$L = 84 \text{ in.} \quad P = 4.5 \text{ k} \quad \alpha = 22.5^\circ$$

WIDE-FLANGE BEAM:

$$\text{W } 8 \times 21 \quad I_y = 9.77 \text{ in.}^4 \quad I_z = 75.3 \text{ in.}^4$$

$$d = 8.28 \text{ in.} \quad b = 5.270 \text{ in.}$$

BENDING MOMENTS

$$M_y = \frac{P \sin(\alpha)L}{4} \quad M_y = 36,164 \text{ lb-in.}$$

$$M_z = \frac{P \cos(\alpha)L}{4} \quad M_z = 87,307 \text{ lb-in.}$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_z}{I_y} \tan(\alpha)\right) \quad \beta = 72.6^\circ \quad \leftarrow$$

BENDING STRESSES

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

$$\text{POINT A:} \quad z_A = \frac{b}{2} \quad y_A = \frac{-d}{2}$$

$$\sigma_A = \sigma_x(z_A, y_A)$$

$$\sigma_A = 14,554 \text{ psi} \quad \leftarrow$$

$$\text{POINT B:} \quad z_B = -\frac{b}{2} \quad y_B = \frac{-d}{2}$$

$$\sigma_B = \sigma_x(z_B, y_B)$$

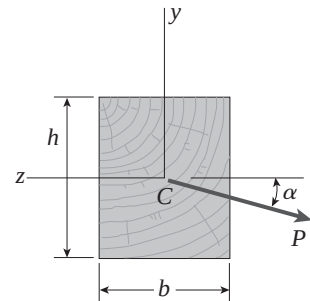
$$\sigma_B = -4953 \text{ psi} \quad \leftarrow$$

$$\text{POINT D:} \quad \sigma_D = -\sigma_B \quad \sigma_D = 4953 \text{ psi} \quad \leftarrow$$

$$\text{POINT E:} \quad \sigma_E = -\sigma_A \quad \sigma_E = -14,554 \text{ psi} \quad \leftarrow$$

Problem 6.4-6 A wood cantilever beam of rectangular cross section and length L supports an inclined load P at its free end (see figure).

Determine the orientation of the neutral axis and calculate the maximum tensile stress $\sigma_{\max}$ due to the load P . Data for the beam are as follows: $b = 80$ mm, $h = 140$ mm, $L = 2.0$ m, $P = 575$ N, and $\alpha = 30^\circ$.



Probs. 6.4-6 and 6.4-7

Solution 6.4-6

$$L = 2.0 \text{ m} \quad P = 575 \text{ N} \quad b = 80 \text{ mm}$$

$$h = 140 \text{ mm} \quad \alpha = 30^\circ$$

BENDING MOMENTS

$$M_y = P \cos(\alpha)L \quad M_y = 996 \text{ N} \cdot \text{m}$$

$$M_z = -P \sin(\alpha)L \quad M_z = -575 \text{ N} \cdot \text{m}$$

MOMENT OF INERTIA

$$I_y = \frac{hb^3}{12} \quad I_y = 5.973 \times 10^6 \text{ mm}^4$$

$$I_z = \frac{bh^3}{12} \quad I_z = 18.293 \times 10^6 \text{ mm}^4$$

NEUTRAL AXIS nn

$$\beta = \arctan \left(\frac{I_z}{I_y} \tan (\alpha + 90^\circ) \right)$$

$$\beta = -79.3^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

$$\sigma_{\max} = \frac{M_y \left(\frac{b}{2} \right)}{I_y} - \frac{M_z \left(\frac{h}{2} \right)}{I_z}$$

$$\sigma_{\max} = 8.87 \text{ MPa} \quad \leftarrow$$

Problem 6.4-7 Solve the preceding problem for a cantilever beam with data as follows: $b = 4$ in., $h = 9$ in., $L = 10.0$ ft, $P = 325$ lb, and $\alpha = 45^\circ$.

Solution 6.4-7

$$L = 10.0 \text{ ft} \quad P = 325 \text{ lb} \quad b = 4 \text{ in.}$$

$$h = 9 \text{ in.} \quad \alpha = 45^\circ$$

BENDING MOMENTS

$$M_y = P \cos (\alpha) L \quad M_y = 27,577 \text{ lb-in.}$$

$$M_z = -P \sin (\alpha) L \quad M_z = -27,577 \text{ lb-in.}$$

MOMENT OF INERTIA

$$I_y = \frac{hb^3}{12} \quad I_y = 48.000 \text{ in.}^4$$

$$I_z = \frac{bh^3}{12} \quad I_z = 243.000 \text{ in.}^4$$

NEUTRAL AXIS nn

$$\beta = \arctan \left(\frac{I_z}{I_y} \tan (\alpha + 90^\circ) \right)$$

$$\beta = -78.8^\circ \quad \leftarrow$$

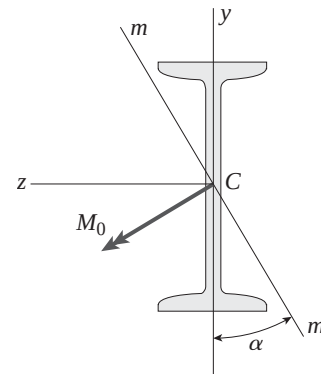
MAXIMUM TENSILE STRESS (AT POINT A)

$$\sigma_{\max} = \frac{M_y \left(\frac{b}{2} \right)}{I_y} - \frac{M_z \left(\frac{h}{2} \right)}{I_z}$$

$$\sigma_{\max} = 1660 \text{ psi} \quad \leftarrow$$

Problem 6.4-8 A steel beam of I-section (see figure) is simply supported at the ends. Two equal and oppositely directed bending moments M_0 act at the ends of the beam, so that the beam is in pure bending. The moments act in plane mm , which is oriented at an angle α to the xy plane.

Determine the orientation of the neutral axis and calculate the maximum tensile stress $\sigma_{\max}$ due to the moments M_0 . Data for the beam are as follows: S 200×27.4 section, $M_0 = 4 \text{ kN} \cdot \text{m}$, and $\alpha = 24^\circ$. (Note: See Table F-2b of Appendix F for the dimensions and properties of the beam.)



Solution 6.4-8

$$M_0 = 4.0 \text{ kN} \cdot \text{m} \quad \alpha = 24^\circ$$

$$S \ 200 \times 27.4 \quad I_y = 1.54 \times 10^6 \text{ mm}^4$$

$$I_z = 23.9 \times 10^6 \text{ mm}^4 \quad d = 203 \text{ mm} \quad b = 102 \text{ mm}$$

BENDING MOMENTS

$$M_y = -M_0 \sin(\alpha) \quad M_y = -1627 \text{ N} \cdot \text{m}$$

$$M_z = M_0 \cos(\alpha) \quad M_z = 3654 \text{ N} \cdot \text{m}$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_z}{I_y} \tan(-\alpha)\right) \quad \beta = -81.8^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

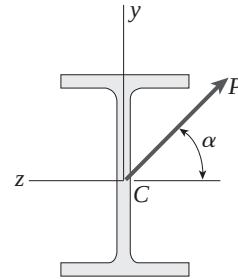
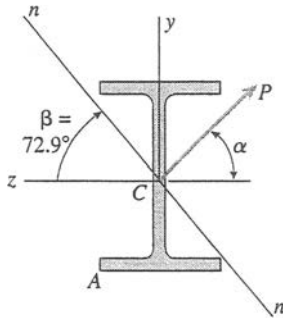
$$\sigma_{\max} = \frac{M_y\left(-\frac{b}{2}\right)}{I_y} - \frac{M_z\left(-\frac{d}{2}\right)}{I_z}$$

$$\sigma_{\max} = 69.4 \text{ MPa} \quad \leftarrow$$

Problem 6.4-9 A cantilever beam of wide-flange cross section and length L supports an inclined load P at its free end (see figure).

Determine the orientation of the neutral axis and calculate the maximum tensile stress $\sigma_{\max}$ due to the load P .

Data for the beam are as follows: W 10 $\times$ 45 section, $L = 8.0$ ft, $P = 1.5$ k, and $\alpha = 55^\circ$. (Note: See Table F-1a of Appendix F for the dimensions and properties of the beam.)

**Probs. 6.4-9 and 6.4-10****Solution 6.4-9 Cantilever beam with inclined load**

$$P = 1.5 \text{ k} = 1500 \text{ lb}$$

$$L = 8.0 \text{ ft} = 96 \text{ in.}$$

$$\alpha = 55^\circ$$

$$\text{W } 10 \times 45$$

$$I_y = 53.4 \text{ in.}^4 \quad I_z = 248 \text{ in.}^4$$

$$d = 10.10 \text{ in.} \quad b = 8.02 \text{ in.}$$

BENDING MOMENTS

$$M_y = (P \cos \alpha)L = 82,595 \text{ lb-in.}$$

$$M_z = (P \sin \alpha)L = 117,960 \text{ lb-in.}$$

NEUTRAL AXIS nn (EQ. 6-24)

$$\theta = 90^\circ - \alpha = 35^\circ \quad (\text{see Fig. 6-19})$$

$$\tan \beta = \frac{I_z}{I_y} \tan \theta = \frac{248}{53.4} \tan 35^\circ = 3.2519$$

$$\beta = 72.91^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS (POINT A) (EQ. 6-19)

$$z_A = b/2 = 4.01 \text{ in.}$$

$$y_A = -d/2 = -5.05 \text{ in.}$$

$$\sigma_{\max} = \sigma_A = \frac{M_y z_A}{I_y} - \frac{M_z y_A}{I_z} = 8600 \text{ psi} \quad \leftarrow$$

Problem 6.4-10 Solve the preceding problem using the following data: W 310 $\times$ 129 section, $L = 1.8$ m, $P = 9.5$ kN, and $\alpha = 60^\circ$. (Note: See Table F-1b of Appendix F for the dimensions and properties of the beam.)

Solution 6.4-10

$$P = 9.5 \text{ kN} \quad L = 1.8 \text{ m} \quad \alpha = 60^\circ$$

$$\text{W } 310 \times 129 \quad I_y = 100 \times 10^6 \text{ mm}^4$$

$$I_z = 308 \times 10^6 \text{ mm}^4 \quad d = 318 \text{ mm} \quad b = 307 \text{ mm}$$

BENDING MOMENTS

$$M_y = P \cos(\alpha)L \quad M_y = 8550 \text{ N} \cdot \text{m}$$

$$M_z = P \sin(\alpha)L \quad M_z = 14,809 \text{ N} \cdot \text{m}$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_z}{I_y} \tan(90^\circ - \alpha)\right) \quad \beta = 60.6^\circ$$

$$\beta = 60.6^\circ \quad \leftarrow$$

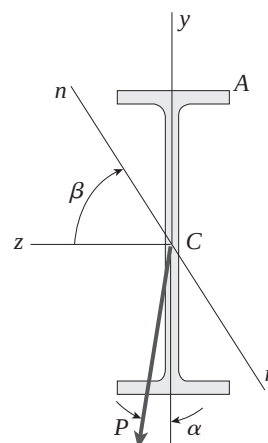
MAXIMUM TENSILE STRESS (AT POINT A)

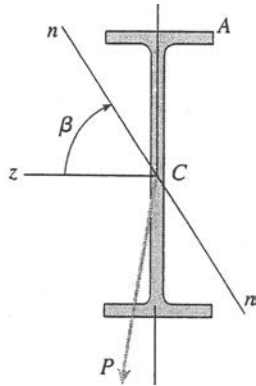
$$\sigma_{\max} = \frac{M_y\left(\frac{b}{2}\right)}{I_y} - \frac{M_z\left(-\frac{d}{2}\right)}{I_z}$$

$$\sigma_{\max} = 20.8 \text{ MPa} \quad \leftarrow$$

Problem 6.4-11 A cantilever beam of W 12 $\times$ 14 section and length $L = 9$ ft supports a slightly inclined load $P = 500$ lb at the free end (see figure).

- Plot a graph of the stress σ_A at point A as a function of the angle of inclination α .
- Plot a graph of the angle β , which locates the neutral axis nn , as a function of the angle α . (When plotting the graphs, let α vary from 0 to 10° .) (Note: See Table F-1a of Appendix F for the dimensions and properties of the beam.)



Solution 6.4-11 Cantilever beam with inclined load

$$\begin{aligned}
 P &= 500 \text{ lb} & L &= 9 \text{ ft} = 108 \text{ in.} & W &12 \times 14 \\
 I_y &= 2.36 \text{ in.}^4 & I_z &= 88.6 \text{ in.}^4 \\
 d &= 11.91 \text{ in.} & b &= 3.970 \text{ in.}
 \end{aligned}$$

BENDING MOMENTS

$$\begin{aligned}
 M_y &= -(P \sin \alpha)L = -54,000 \sin \alpha \\
 M_z &= -(P \cos \alpha)L = -54,000 \cos \alpha
 \end{aligned}$$

(a) STRESS AT POINT A (Eq. 6-19)

$$z_A = -b/2 = -1.985 \text{ in.}$$

$$y_A = d/2 = 5.955 \text{ in.}$$

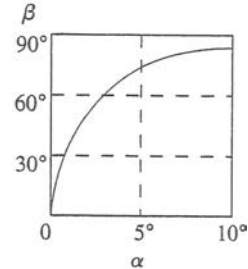
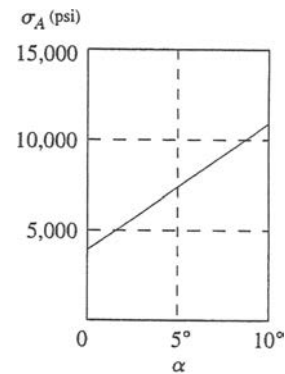
$$\begin{aligned}
 \sigma_A &= \frac{M_y z_A}{I_y} - \frac{M_z y_A}{I_z} = 45,420 \sin \alpha \\
 &\quad + 3629 \cos \alpha \text{ (psi)} \quad \leftarrow
 \end{aligned}$$

(b) NEUTRAL AXIS nn (Eq. 6-24)

$$\theta = 180^\circ + \alpha \quad (\text{see Fig. 6-19})$$

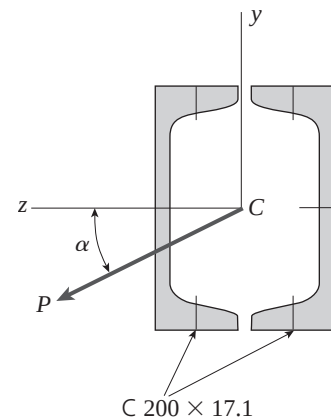
$$\begin{aligned}
 \tan \beta &= \frac{I_z}{I_y} \tan \theta = \frac{I_z}{I_y} \tan (180^\circ + \alpha) \\
 &= \frac{88.6}{2.36} \tan (180^\circ + \alpha) = 37.54 \tan \alpha
 \end{aligned}$$

$$\beta = \arctan (37.54 \tan \alpha) \quad \leftarrow$$



Problem 6.4-12 A cantilever beam built up from two channel shapes, each C 200 $\times$ 17.1 and of length L , supports an inclined load P at its free end (see figure).

Determine the orientation of the neutral axis and calculate the maximum tensile stress $\sigma_{\max}$ due to the load P . Data for the beam are as follows: $L = 4.5 \text{ m}$, $P = 500 \text{ N}$, and $\alpha = 30^\circ$.



Solution 6.4-12

$$L = 4.5 \text{ m} \quad P = 500 \text{ N} \quad \alpha = 30^\circ$$

BUILT UP BEAM: Double C 200 $\times$ 17.1

$$I_{cy} = 0.545 \times 10^6 \text{ mm}^4 \quad I_{cz} = 13.5 \times 10^6 \text{ mm}^4$$

$$c = 14.5 \text{ mm} \quad b_c = 57.4 \text{ mm} \quad A_c = 2170 \text{ mm}^2$$

$$I_y = 2[I_{cy} + A_c(b_c - c)^2] \quad I_y = 9.08 \times 10^6 \text{ mm}^4$$

$$I_z = 2I_{cz} \quad I_z = 27.0 \times 10^6 \text{ mm}^4$$

$$d = 203 \text{ mm} \quad b = 2b_c \quad b = 114.8 \text{ mm}$$

BENDING MOMENTS

$$M_y = -P \cos(\alpha)L \quad M_y = -1949 \text{ N} \cdot \text{m}$$

$$M_z = -P \sin(\alpha)L \quad M_z = -1125 \text{ N} \cdot \text{m}$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_z}{I_y} \tan(90^\circ - \alpha)\right)$$

$$\beta = 79.0^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS

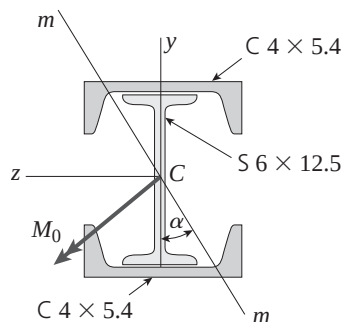
$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

$$\text{POINT A: } z_A = -\frac{b}{2} \quad y_A = \frac{d}{2}$$

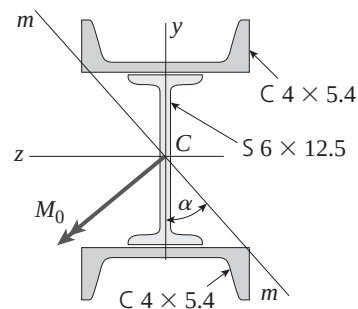
$$\sigma_A = \sigma_x(z_A, y_A) \quad \sigma_A = 16.6 \text{ MPa} \quad \leftarrow$$

Problem 6.4-13 A built-up steel beam of I-section with channels attached to the flanges (see figure part a) is simply supported at the ends. Two equal and oppositely directed bending moments M_0 act at the ends of the beam, so that the beam is in pure bending. The moments act in plane mm , which is oriented at an angle α to the xy plane.

- Determine the orientation of the neutral axis and calculate the maximum tensile stress $\sigma_{\max}$ due to the moments M_0 .
- Repeat part a if the channels are now with their flanges pointing away from the beam flange, as shown in figure part b. Data for the beam are as follows: S 6 $\times$ 12.5 section with C 4 $\times$ 5.4 sections attached to the flanges, $M_0 = 45 \text{ k-in.}$, and $\alpha = 40^\circ$. (Note: See Tables F-2a and F-3a of Appendix F for the dimensions and properties of the S and C shapes.)



(a)



(b)

Solution 6.4-13

$$M_0 = 45 \text{ k-in.} \quad \alpha = 40^\circ$$

$$\text{S } 6 \times 12.5: \quad I_{sy} = 1.80 \text{ in.}^4 \quad I_{sz} = 22.0 \text{ in.}^4$$

$$d_s = 6.0 \text{ in.} \quad b_s = 3.33 \text{ in.} \quad A_s = 3.66 \text{ in.}^2$$

$$\text{C } 4 \times 5.4: \quad I_{cy} = 0.312 \text{ in.}^4 \quad I_{cz} = 3.85 \text{ in.}^4$$

$$d_c = 4.0 \text{ in.} \quad b_c = 1.58 \text{ in.} \quad A_c = 1.58 \text{ in.}^2$$

$$t_{wc} = 0.184 \text{ in.} \quad c = 0.457 \text{ in.}$$

$$(a) \text{ BUILT UP SECTION: } I_y = I_{sy} + 2I_{cz} \quad I_y = 9.50 \text{ in.}^4$$

$$I_z = I_{sz} + 2\left[I_{cy} + A_c\left(\frac{d_s}{2} + t_{wc} - c\right)^2\right]$$

$$I_z = 46.1 \text{ in.}^4$$

$$d = d_s + 2t_{wc} \quad d = 6.368 \text{ in.}$$

$$b = d_c \quad b = 4.0 \text{ in.}$$

BENDING MOMENTS

$$M_y = -M_0 \sin(\alpha) \quad M_y = -28.9 \text{ k-in.}$$

$$M_z = M_0 \cos(\alpha) \quad M_z = 34.5 \text{ k-in.}$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_z}{I_y} \tan(-\alpha)\right)$$

$$\beta = -76.2^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

$$\text{POINT A: } z_A = -\frac{b}{2} \quad y_A = -\frac{d}{2}$$

$$\sigma_A = \sigma_x(z_A, y_A) \quad \sigma_A = 8469 \text{ psi} \quad \leftarrow$$

$$(b) \text{ BUILT UP SECTION: } I_y = I_{sy} + 2I_{cz} \quad I_y = 9.50 \text{ in.}^4$$

$$I_z = I_{sz} + 2\left[I_{cy} + A_c\left(\frac{d_s}{2} + c\right)^2\right]$$

$$I_z = 60.4 \text{ in.}^4$$

$$d = d_s + 2b_c \quad d = 9.160 \text{ in.}$$

$$b = d_c \quad b = 4.000 \text{ in.}$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_z}{I_y} \tan(-\alpha)\right)$$

$$\beta = -79.4^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

$$\text{POINT A: } z_A = -\frac{b}{2} \quad y_A = -\frac{d}{2}$$

$$\sigma_A = \sigma_x(z_A, y_A) \quad \sigma_A = 8704 \text{ psi} \quad \leftarrow$$

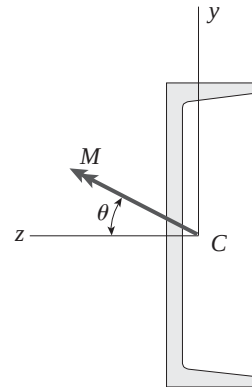
Bending of Unsymmetric Beams

When solving the problems for Section 6.5, be sure to draw a sketch of the cross section showing the orientation of the neutral axis and the locations of the points where the stresses are being found.

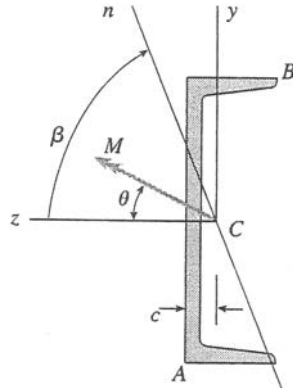
Problem 6.5-1 A beam of channel section is subjected to a bending moment M having its vector at an angle θ to the z axis (see figure).

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam.

Use the following data: C 8 $\times$ 11.5 section, $M = 20$ k-in., $\tan \theta = 1/3$. (Note: See Table F-3a of Appendix F for the dimensions and properties of the channel section.)



Probs. 6.5-1 and 6.5-2

Solution 6.5-1 Channel section

$$\begin{aligned}
 M &= 20 \text{ k-in.} & \tan \theta &= 1/3 & \theta &= 18.435^\circ \\
 &C\ 8 \times 11.5 \\
 c &= 0.571 \text{ in.} & I_y &= 1.32 \text{ in.}^4 & I_z &= 32.6 \text{ in.}^4 \\
 d &= 8.00 \text{ in.} & b &= 2.260 \text{ in.}
 \end{aligned}$$

NEUTRAL AXIS nn (Eq. 6-41)

$$\begin{aligned}
 \tan \beta &= \frac{I_z}{I_y} \tan \theta = \frac{32.6}{1.32} (1/3) = 8.2323 \\
 \beta &= 83.07^\circ \quad \leftarrow
 \end{aligned}$$

MAXIMUM TENSILE STRESS (POINT A) (Eq. 6-39)

$$\begin{aligned}
 z_A &= c = 0.571 \text{ in.} & y_A &= -d/2 = -4.00 \text{ in.} \\
 \sigma_t = \sigma_A &= \frac{(M \sin \theta) z_A}{I_y} - \frac{(M \cos \theta) y_A}{I_z} \\
 &= 5060 \text{ psi} \quad \leftarrow
 \end{aligned}$$

MAXIMUM COMPRESSIVE STRESS (POINT B) (Eq. 6-39)

$$\begin{aligned}
 z_B &= -(b - c) = -(2.260 - 0.571) = -1.689 \text{ in.} \\
 y_B &= d/2 = 4.00 \text{ in.} \\
 \sigma_c = \sigma_B &= \frac{(M \sin \theta) z_B}{I_y} - \frac{(M \cos \theta) y_B}{I_z} \\
 &= -10,420 \text{ psi} \quad \leftarrow
 \end{aligned}$$

Problem 6.5-2 A beam of channel section is subjected to a bending moment M having its vector at an angle θ to the z axis (see figure).

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Use a C 200 $\times$ 20.5 channel section with $M = 0.75 \text{ kN} \cdot \text{m}$ and $\theta = 20^\circ$.

Solution 6.5-2

$$\begin{aligned}
 M &= 0.75 \text{ kN} \cdot \text{m} & \theta &= 20^\circ \\
 &C\ 200 \times 20.5 & I_y &= 0.633 \cdot 10^6 \text{ mm}^4 \\
 I_z &= 15.0 \times 10^6 \text{ mm}^4 & d &= 203 \text{ mm} & b &= 59.4 \text{ mm} \\
 c &= 14.1 \text{ mm}
 \end{aligned}$$

BENDING MOMENTS

$$\begin{aligned}
 M_y &= M \sin(\theta) & M_y &= 257 \text{ N} \cdot \text{m} \\
 M_z &= M \cos(\theta) & M_z &= 705 \text{ N} \cdot \text{m}
 \end{aligned}$$

NEUTRAL AXIS nn

$$\beta = \tan^{-1} \left(\frac{I_z}{I_y} \tan(\theta) \right) \quad \beta = 83.4^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

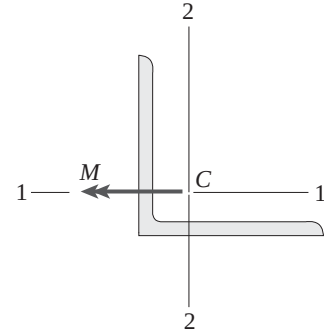
$$\begin{aligned}
 \sigma_{\max} &= \frac{M_y(c)}{I_y} - \frac{M_z \left(-\frac{d}{2} \right)}{I_z} \\
 \sigma_{\max} &= 10.5 \text{ MPa} \quad \leftarrow
 \end{aligned}$$

MAXIMUM TENSILE STRESS (AT POINT B)

$$\begin{aligned}
 \sigma_{\max} &= \frac{M_y(-b + c)}{I_y} - \frac{M_z \left(\frac{d}{2} \right)}{I_z} \\
 \sigma_{\max} &= -23.1 \text{ MPa} \quad \leftarrow
 \end{aligned}$$

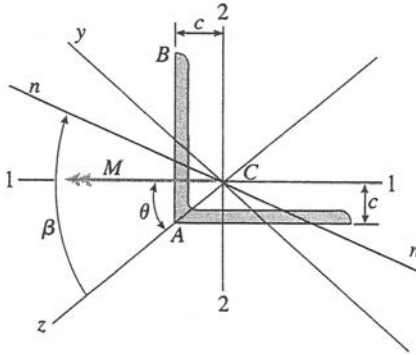
Problem 6.5-3 An angle section with equal legs is subjected to a bending moment M having its vector directed along the 1–1 axis, as shown in the figure.

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c if the angle is an L $6 \times 6 \times 3/4$ section and $M = 20$ k-in. (Note: See Table F-4a of Appendix F for the dimensions and properties of the angle section.)



Probs. 6.5-3 and 6.5-4

Solution 6.5-3 Angle section with equal legs



$$\begin{aligned}
 M &= 20 \text{ k-in.} & \text{L } 6 \times 6 \times 3/4 \text{ in.} \\
 h &= b = 6 \text{ in.} & c = 1.78 \text{ in.} \\
 I_1 &= I_2 = 28.2 \text{ in.}^4 \\
 \theta &= 45^\circ & A = 8.44 \text{ in.}^2 & r_{\min} = 1.17 \text{ in.} \\
 I_y &= Ar_{\min}^2 = 11.55 \text{ in.}^4 \\
 I_z &= I_1 + I_2 - I_y = 44.85 \text{ in.}^4
 \end{aligned}$$

NEUTRAL AXIS nn (EQ. 6-41)

$$\begin{aligned}
 \tan \beta &= \frac{I_z}{I_y} \tan \theta = \frac{44.85}{11.55} \tan 45^\circ = 3.8831 \\
 \beta &= 75.56^\circ \quad \leftarrow
 \end{aligned}$$

MAXIMUM TENSILE STRESS (POINT A) (EQ. 6-39)

$$\begin{aligned}
 z_A &= c\sqrt{2} = 2.517 \text{ in.} & y_A &= 0 \\
 \sigma_t = \sigma_A &= \frac{(M \sin \theta)z_A}{I_y} - \frac{(M \cos \theta)y_A}{I_z} \\
 &= 3080 \text{ psi} \quad \leftarrow
 \end{aligned}$$

MAXIMUM COMPRESSIVE STRESS (POINT B) (EQ. 6-39)

$$\begin{aligned}
 z_B &= c\sqrt{2} - h/\sqrt{2} = -1.725 \text{ in.} \\
 y_B &= h/\sqrt{2} = 4.243 \text{ in.} \\
 \sigma_c = \sigma_B &= \frac{(M \sin \theta)z_B}{I_y} - \frac{(M \cos \theta)y_B}{I_z} \\
 &= -3450 \text{ psi} \quad \leftarrow
 \end{aligned}$$

Problem 6.5-4 An angle section with equal legs is subjected to a bending moment M having its vector directed along the 1–1 axis, as shown in the figure.

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c if the section is an L $152 \times 152 \times 12.7$ section and $M = 2.5 \text{ kN} \cdot \text{m}$. (Note: See Table F-4b of Appendix F for the dimensions and properties of the angle section.)

Solution 6.5-4

$$M = 2.5 \text{ kN} \cdot \text{m} \quad \theta = 45^\circ$$

$$L \ 152 \times 152 \times 12.7 \quad I_1 = 8.28 \times 10^6 \text{ mm}^4 \quad I_2 = I_1$$

$$r_{\min} = 30.0 \text{ mm} \quad A = 3720 \text{ mm}^2$$

$$I_y = Ar_{\min}^2 \quad I_y = 3.348 \times 10^6 \text{ mm}^4$$

$$I_z = I_1 + I_2 - I_y \quad I_z = 13.212 \times 10^6 \text{ mm}^4$$

$$h = 152 \text{ mm} \quad b = h \quad c = 42.4 \text{ mm}^4$$

BENDING MOMENTS

$$M_y = M \sin(\theta) \quad M_y = 1768 \text{ N} \cdot \text{m}$$

$$M_z = M \cos(\theta) \quad M_z = 1768 \text{ N} \cdot \text{m}$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_z}{I_y} \tan(\theta)\right) \quad \beta = 75.8^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

$$\sigma_{\max} = \frac{M_y(c\sqrt{2})}{I_y} - \frac{M_z(0)}{I_z}$$

$$\sigma_{\max} = 31.7 \text{ MPa} \quad \leftarrow$$

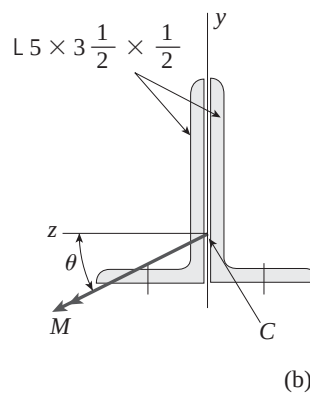
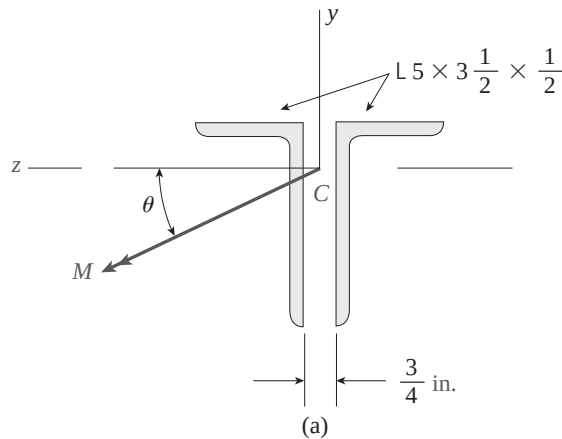
MAXIMUM TENSILE STRESS (AT POINT B)

$$\sigma_{\max} = \frac{M_y\left(c\sqrt{2} - \frac{h}{\sqrt{2}}\right)}{I_y} - \frac{M_z\left(\frac{h}{\sqrt{2}}\right)}{I_z}$$

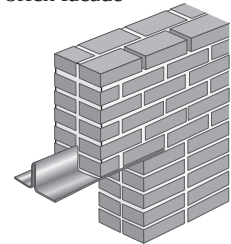
$$\sigma_{\max} = -39.5 \text{ MPa} \quad \leftarrow$$

Problem 6.5-5 A beam made up of two unequal leg angles is subjected to a bending moment M having its vector at an angle θ to the z axis (see figure part a).

- For the position shown in the figure, determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Assume that $\theta = 30^\circ$ and $M = 30 \text{ k-in.}$
- The two angles are now inverted and attached back-to-back to form a lintel beam that supports two courses of brick façade (see figure part b). Find the new orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam using $\theta = 30^\circ$ and $M = 30 \text{ k-in.}$



Lintel beam supporting brick façade



Solution 6.5-5

$$M = 30 \text{ k-in.} \quad \theta = 30^\circ$$

$$L \ 5 \times 3\text{-}1/2 \times 1/2 \quad I_{L1} = 10.0 \text{ in.}^4$$

$$I_{L2} = 4.02 \text{ in.}^4 \quad d = 1.65 \text{ in.} \quad c = 0.901 \text{ in.}$$

$$h_{L1} = 5 \text{ in.} \quad h_{L2} = 3.5 \text{ in.}$$

$$A_L = 4 \text{ in.}^2 \quad \text{gap} = \frac{3}{4} \text{ in.} \quad t = \frac{1}{2} \text{ in.}$$

$$(a) \text{ BUILT UP SECTION: } I_z = 2I_{L1} \quad I_z = 20.000 \text{ in.}^4$$

$$I_y = 2 \left[I_{L2} + A_L \left(\frac{\text{gap}}{2} + c \right)^2 \right]$$

$$I_y = 21.065 \text{ in.}^4$$

$$h = h_{L1} \quad h = 5.000 \text{ in.} \quad b = \text{gap} + 2h_{L2}$$

$$b = 7.750 \text{ in.}$$

$$h_1 = d \quad h_1 = 1.650 \text{ in.}$$

BENDING MOMENTS

$$M_y = -M \sin(\theta) \quad M_y = -1.250 \text{ k-ft}$$

$$M_z = M \cos(\theta) \quad M_z = 2.165 \text{ k-ft}$$

NEUTRAL AXIS nn

$$\beta = \arctan \left(\frac{I_z}{I_y} \tan(-\theta) \right)$$

$$\beta = -28.7^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

$$\text{POINT A: } z_A = -\frac{\text{gap}}{2} + t \quad y_A = -h + h_1$$

$$\sigma_t = \sigma_x(z_A, y_A) \quad \sigma_t = 4263 \text{ psi} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS

$$\text{POINT B: } z_B = \frac{b}{2} \quad y_B = h_1$$

$$\sigma_c = \sigma_x(z_B, y_B) \quad \sigma_c = -4903 \text{ psi} \quad \leftarrow$$

$$(b) \text{ BUILT UP SECTION: } I_z = 2I_{L1} \quad I_z = 20.000 \text{ in.}^4$$

$$I_y = 2(I_{L2} + A_L c^2) \quad I_y = 14.534 \text{ in.}^4$$

$$h = h_{L1} \quad h = 5.000 \text{ in.} \quad b = 2h_{L2}$$

$$b = 7.000 \text{ in.} \quad h_1 = h - d \quad h_1 = 3.350 \text{ in.}$$

NEUTRAL AXIS nn

$$\beta = \arctan \left(\frac{I_z}{I_y} \tan(-\theta) \right)$$

$$\beta = -38.5^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

$$\text{POINT A: } z_A = -\frac{b}{2} \quad y_A = -h + h_1$$

$$\sigma_t = \sigma_x(z_A, y_A) \quad \sigma_t = 5756 \text{ psi} \quad \leftarrow$$

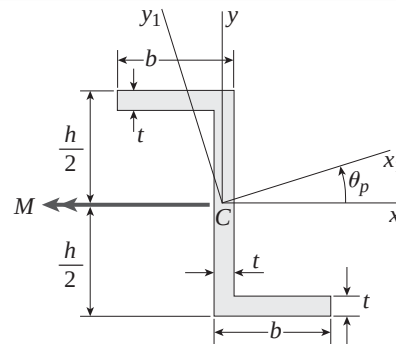
MAXIMUM COMPRESSIVE STRESS

$$\text{POINT B: } z_B = t \quad y_B = h_1$$

$$\sigma_c = \sigma_x(z_B, y_B) \quad \sigma_c = -4868 \text{ psi} \quad \leftarrow$$

Problem 6.5-6 The Z-section of Example 12-7 is subjected to $M = 5 \text{ kN} \cdot \text{m}$, as shown.

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Use the following numerical data: height $h = 200 \text{ mm}$, width $b = 90 \text{ mm}$, constant thickness $t = 15 \text{ mm}$, and $\theta_p = 19.2^\circ$. Use $I_1 = 32.6 \times 10^6 \text{ mm}^4$ and $I_2 = 2.4 \times 10^6 \text{ mm}^4$ from Example 12-7.



Solution 6.5-6

$$M = 5 \text{ kN} \cdot \text{m} \quad \theta = 19.2^\circ$$

Z-SECTION

$$I_{zp} = I_1 \quad I_{zp} = 32.6 \times 10^6 \text{ mm}^4$$

$$I_{yp} = I_2 \quad I_{yp} = 2.4 \times 10^6 \text{ mm}^4$$

$$h = 200 \text{ mm} \quad b = 90 \text{ mm} \quad t = 15 \text{ mm}$$

BENDING MOMENTS

$$M_{yp} = M \sin(\theta) \quad M_{yp} = 1644 \text{ N} \cdot \text{m}$$

$$M_{zp} = M \cos(\theta) \quad M_{zp} = 4722 \text{ N} \cdot \text{m}$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_{zp}}{I_{yp}} \tan(\theta)\right) \quad \beta = 78.1^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

$$zp_A = \left(\frac{t}{2}\right) \cos(\theta) - \left(\frac{-h}{2}\right) \sin(\theta) \quad zp_A = 39.97 \text{ mm}$$

$$yp_A = \left(\frac{t}{2}\right) \sin(\theta) + \left(\frac{-h}{2}\right) \cos(\theta)$$

$$yp_A = -91.97 \text{ mm}$$

$$\sigma_t = \frac{M_{yp}(zp_A)}{I_{yp}} - \frac{M_{zp}(yp_A)}{I_{zp}}$$

$$\sigma_t = 40.7 \text{ MPa} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS (AT POINT B)

$$zp_B = \left(-\frac{t}{2}\right) \cos(\theta) - \left(\frac{h}{2}\right) \sin(\theta)$$

$$zp_B = -39.97 \text{ mm}$$

$$yp_B = \left(-\frac{t}{2}\right) \sin(\theta) + \left(\frac{h}{2}\right) \cos(\theta)$$

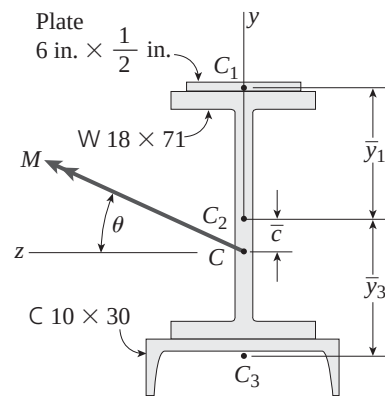
$$yp_B = 91.97 \text{ mm}$$

$$\sigma_c = \frac{M_{yp}(zp_B)}{I_{yp}} - \frac{M_{zp}(yp_B)}{I_{zp}}$$

$$\sigma_c = -40.7 \text{ MPa} \quad \leftarrow$$

Problem 6.5-7 The cross section of a steel beam is constructed of a W 18 × 71 wide-flange section with a 6-in. × 1/2-in. cover plate welded to the top flange and a C 10 × 30 channel section welded to the bottom flange. This beam is subjected to a bending moment M having its vector at an angle θ to the z axis (see figure).

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Assume that $\theta = 30^\circ$ and $M = 75 \text{ k-in.}$ (Note: The cross-sectional properties of this beam were computed in Examples 12-2 and 12-5.)



Solution 6.5-7

$$M = 75 \text{ k-in.} \quad \theta = 30^\circ$$

PLATE: $b_p = \frac{1}{2}$ in. $h_p = 6$ in.

$$I_{\text{plate}} = \frac{b_p h_p^3}{12} \quad I_{\text{plate}} = 9.00 \text{ in.}^4$$

W SECTION: $h_w = 18.47$ in. $b_w = 7.635$ in.
 $I_{wy} = 60.3$ in.⁴

C SECTION: $h_c = 10.0$ in. $b_c = 3.033$ in.
 $I_{cz} = 103$ in.⁴

BUILT-UP SECTION:

$$c_{\text{bar}} = 1.80 \text{ in.}$$

$$I_z = 2200 \text{ in.}^4 \quad I_y = I_{wy} + I_{cz} + I_{\text{plate}}$$

$$I_y = 172.3 \text{ in.}^4$$

BENDING MOMENTS

$$M_y = M \sin(\theta) \quad M_y = 3.125 \text{ k-ft}$$

$$M_z = M \cos (\theta) \quad M_z = 5.413 \text{ k-ft}$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_z}{I_y} \tan(\theta)\right) \quad \beta = 82.3^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS

$$\sigma_x(z, y) = \frac{M_y z}{I_y} - \frac{M_z y}{I_z}$$

$$\text{POINT A: } z_A = \frac{h_c}{2} \quad y_A = -\frac{h_w}{2} - b_c + c_{\text{bar}}$$

$$\sigma_t = \sigma_x(z_A, y_A) \quad \sigma_t = 1397 \text{ psi} \quad \leftarrow$$

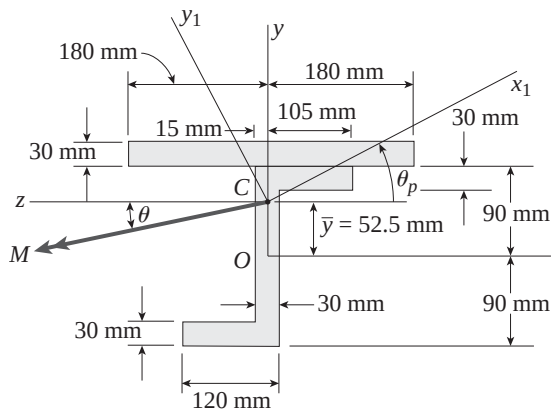
MAXIMUM COMPRESSIVE STRESS

$$\text{POINT B: } z_B = -\frac{b_w}{2} \quad y_B = \frac{h_w}{2} + c_{\text{bar}}$$

$$\sigma_c = \sigma_x(z_B, y_B) \quad \sigma_c = -1157 \text{ psi} \quad \leftarrow$$

Problem 6.5-8 The cross section of a steel beam is shown in the figure. This beam is subjected to a bending moment M having its vector at an angle θ to the z axis.

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Assume that $\theta = 22.5^\circ$ and $M = 4.5 \text{ kN} \cdot \text{m}$. Use cross-sectional properties $I_{x1} = 93.14 \times 10^6 \text{ mm}^4$, $I_{y1} = 152.7 \times 10^6 \text{ mm}^4$, and $\theta_p = 27.3^\circ$.



Solution 6.5-8

$$M = 4.5 \text{ kN} \cdot \text{m} \quad \theta = 22.5^\circ$$

BUILT-UP SECTION:

$$b = 120 \text{ mm} \quad t = 30 \text{ mm}$$

$$h = 180 \text{ mm} \quad b_1 = 360 \text{ mm}$$

$$y_{\text{bar}} = \frac{tb_1 \left(\frac{t+h}{2} \right)}{tb_1 + [2tb + (h-2t)t]}$$

$$y_{\text{bar}} = 52.5 \text{ mm}$$

$$\theta_P = 27.3^\circ$$

$$I_{zp} = I_{x1} \quad I_{yp} = I_{y1}$$

$$I_{zp} = 93.14 \times 10^6 \text{ mm}^4 \quad I_{yp} = 152.7 \times 10^6 \text{ mm}^4$$

BENDING MOMENTS

$$M_{yp} = M \sin(\theta_P - \theta) \quad M_{yp} = 377 \text{ N} \cdot \text{m}$$

$$M_{zp} = M \cos(\theta_P - \theta) \quad M_{zp} = 4484 \text{ N} \cdot \text{m}$$

NEUTRAL AXIS nn

$$\beta = \arctan \left(\frac{I_{zp}}{I_{yp}} \tan(\theta_P - \theta) \right)$$

$$\beta = 2.93^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

$$z_A = -\frac{t}{2} \quad y_A = -\frac{h}{2} - y_{\text{bar}}$$

$$z_{pA} = (z_A) \cos(\theta_P) - (y_A) \sin(\theta_P) \quad z_{pA} = 52.03 \text{ mm}$$

$$y_{pA} = (z_A) \sin(\theta_P) - (y_A) \cos(\theta_P)$$

$$y_{pA} = -133.51 \text{ mm}$$

$$\sigma_t = \frac{M_{yp}(z_{pA})}{I_{yp}} - \frac{M_{zp}(y_{pA})}{I_{zp}}$$

$$\sigma_t = 6.56 \text{ MPa} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS (AT POINT B)

$$z_B = \frac{b_1}{2} \quad y_B = \frac{h}{2} + t - y_{\text{bar}}$$

$$z_{pB} = (z_B) \cos(\theta_P) - (y_B) \sin(\theta_P) \quad z_{pB} = 128.99 \text{ mm}$$

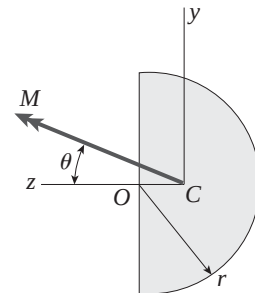
$$y_{pB} = (z_B) \sin(\theta_P) + (y_B) \cos(\theta_P) \quad y_{pB} = 142.5 \text{ mm}$$

$$\sigma_c = \frac{M_{yp}(z_{pB})}{I_{yp}} - \frac{M_{zp}(y_{pB})}{I_{zp}}$$

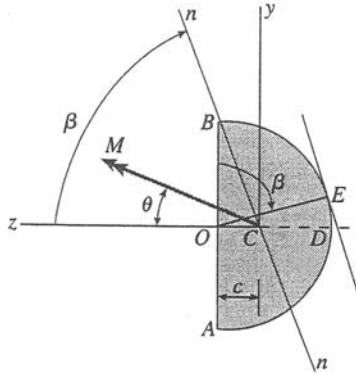
$$\sigma_c = -6.54 \text{ MPa} \quad \leftarrow$$

Problem 6.5-9 A beam of semicircular cross section of radius r is subjected to a bending moment M having its vector at an angle θ to the z axis (see figure).

Derive formulas for the maximum tensile σ_t and the maximum compressive stress σ_c in the beam for $\theta = 0, 45^\circ$, and 90° . (Note: Express the results in the form $\alpha M/r^3$, where α is a numerical value.)



Soulution 6.5.-9 Semicircle



r = radius

$$c = \frac{4r}{3\pi} = 0.42441r$$

$$I_y = \frac{(9\pi^2 - 64)}{72\pi} r^4$$

$$= 0.109757r^4$$

$$I_z = \frac{\pi r^4}{8}$$

σ_t = maximum tensile stress

σ_c = maximum compressive stress

$$\text{FOR } \theta = 0^\circ: \sigma_t = \sigma_A = \frac{Mr}{I_z} = \frac{8M}{\pi r^3}$$

$$= 2.546 \frac{M}{r^3} \quad \leftarrow$$

$$\sigma_c = \sigma_B = -\sigma_A = -\frac{8M}{\pi r^3}$$

$$= -2.546 \frac{M}{r^3} \quad \leftarrow$$

$$\text{FOR } \theta = 90^\circ: \sigma_t = \sigma_o = \frac{Mc}{I_y}$$

$$= 3.867 \frac{M}{r^3} \quad \leftarrow$$

$$\sigma_c = \sigma_D = \frac{M(r - c)}{I_y}$$

$$= -5.244 \frac{M}{r^3} \quad \leftarrow$$

$$\text{FOR } \theta = 45^\circ: \text{Eq. (6-41): } \tan \beta = \frac{I_z}{I_y} \tan \theta$$

$$\tan \beta = \frac{9\pi^2}{9\pi^2 - 64} (1) = 3.577897$$

$$\beta = 74.3847^\circ$$

$$90^\circ - \beta = 15.6153^\circ$$

MAXIMUM TENSILE STRESS for $\theta = 45^\circ$ occurs at point A.

$$z_A = c = 0.42441r \quad y_A = -r$$

From (Eq. 6-39):

$$\sigma_t = \sigma_A = \frac{(M \sin \theta)z_A}{I_y} - \frac{(M \cos \theta)y_A}{I_z}$$

$$= 4.535 \frac{M}{r^3} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS for $\theta = 45^\circ$ occurs at point E, where the tangent to the circle is parallel to the **neutral** axis nn.

$$z_E = c - r \cos (90^\circ - \beta) = -0.53868r$$

$$y_E = r \sin (90^\circ - \beta) = 0.26918r$$

From (Eq. 6-39):

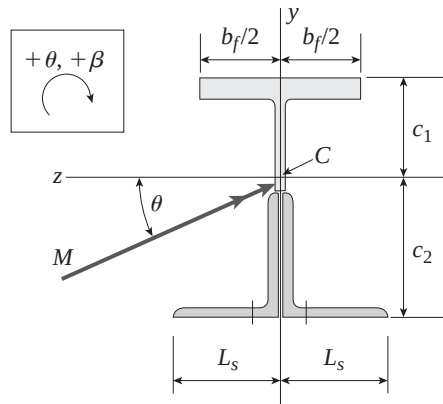
$$\sigma_C = \sigma_E = \frac{(M \sin \theta)z_E}{I_y} - \frac{(M \cos \theta)y_E}{I_z}$$

$$= -3.955 \frac{M}{r^3} \quad \leftarrow$$

Problem 6.5-10 A built-up beam supporting a condominium balcony is made up of a structural T (one half of a W 200 $\times$ 31.3) for the top flange and web and two angles (2 L 102 $\times$ 76 $\times$ 6.4, long legs back-to-back) for the bottom flange and web, as shown. The beam is subjected to a bending moment M having its vector at an angle θ to the z axis (see figure).

Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c in the beam. Assume that $\theta = 30^\circ$ and $M = 15 \text{ kN} \cdot \text{m}$.

Use the following numerical properties: $c_1 = 4.111 \text{ mm}$, $c_2 = 4.169 \text{ mm}$, $b_f = 134 \text{ mm}$, $L_s = 76 \text{ mm}$, $A = 4144 \text{ mm}^2$, $I_y = 3.88 \times 10^6 \text{ mm}^4$, and $I_z = 34.18 \times 10^6 \text{ mm}^4$.



Solution 6.5-10

$$M = 15 \text{ kN} \cdot \text{m} \quad \theta = 30^\circ$$

BUILT-UP SECTION:

$$c_1 = 4.111 \text{ mm} \quad c_2 = 4.169 \text{ mm} \quad b_f = 134 \text{ mm}$$

$$L_s = 76 \text{ mm} \quad A = 4144 \text{ mm}^2$$

$$I_y = 3.88 \times 10^6 \text{ mm}^4 \quad I_z = 34.18 \times 10^6 \text{ mm}^4$$

BENDING MOMENTS

$$M_y = M \sin(180^\circ - \theta) \quad M_y = 7500 \text{ N} \cdot \text{m}$$

$$M_z = M \cos(180^\circ - \theta) \quad M_z = -12,990 \text{ N} \cdot \text{m}$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_z}{I_y} \tan(180^\circ - \theta)\right)$$

$$\beta = -78.9^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

$$z_A = \frac{b_f}{2} \quad y_A = c_1$$

$$\sigma_t = \frac{M_y(z_A)}{I_y} - \frac{M_z(y_A)}{I_z}$$

$$\sigma_t = 131.1 \text{ MPa} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS (AT POINT B)

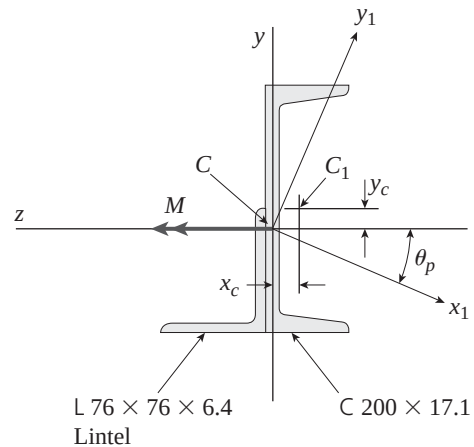
$$z_B = -L_s \quad y_B = -c_2$$

$$\sigma_c = \frac{M_y(z_B)}{I_y} - \frac{M_z(y_B)}{I_z}$$

$$\sigma_c = -148.5 \text{ MPa} \quad \leftarrow$$

Problem 6.5-12 A C 200 × 17.1 channel section has an angle with equal legs attached as shown; the angle serves as a lintel beam. The combined steel section is subjected to a bending moment M having its vector directed along the z axis, as shown in the figure. The centroid C of the combined section is located at distances x_c and y_c from the centroid (C_1) of the channel alone. Principal axes x_1 and y_1 are also shown in the figure and properties I_{x1} , I_{y1} , and θ_p are given below.

Find the orientation of the neutral axis and calculate the maximum tensile stress σ_t and maximum compressive stress σ_c if the angle is an L 76 × 76 × 6.4 section and $M = 3.5 \text{ kN} \cdot \text{m}$. Use the following properties for principal axes for the combined section: $I_{x1} = 18.49 \times 10^6 \text{ mm}^4$, $I_{y1} = 1.602 \times 10^6 \text{ mm}^4$, $\theta_p = 7.448^\circ$ (CW), $x_c = 10.70 \text{ mm}$, $y_c = 24.07 \text{ mm}$.



Solution 6.5-12

$$M = 3.5 \text{ kN} \cdot \text{m} \quad \theta_p = 7.448^\circ$$

$$\text{ANGLE: } c_a = 21.2 \text{ mm} \quad L_a = 76 \text{ mm}$$

$$\text{CHANNEL: } c_c = 14.5 \text{ mm} \quad d_c = 203 \text{ mm} \\ b_c = 57.4 \text{ mm}$$

BUILT-UP SECTION:

$$y_{\text{bar}} = 24.07 \text{ mm} \quad x_{\text{bar}} = 10.70 \text{ mm}$$

$$I_{zp} = I_{x1} \quad I_{yp} = I_{y1}$$

$$I_{zp} = 18.49 \times 10^6 \text{ mm}^4 \quad I_{yp} = 1.602 \times 10^6 \text{ mm}^4$$

BENDING MOMENTS

$$M_{yp} = M \sin(-\theta_p) \quad M_{yp} = -454 \text{ N} \cdot \text{m}$$

$$M_{zp} = M \cos(-\theta_p) \quad M_{zp} = 3470 \text{ N} \cdot \text{m}$$

NEUTRAL AXIS nn

$$\beta = \arctan\left(\frac{I_{zp}}{I_{yp}} \tan(-\theta_p)\right) \quad \beta = -56.5^\circ \quad \leftarrow$$

MAXIMUM TENSILE STRESS (AT POINT A)

$$z_A = -x_{\text{bar}} + c_c - b_c \quad y_A = -\frac{d_c}{2} + y_{\text{bar}}$$

$$z_{pA} = (z_A) \cos(-\theta_p) - (y_A) \sin(-\theta_p)$$

$$z_{pA} = -63.18 \text{ mm}$$

$$y_{pA} = (z_A) \sin(-\theta_p) + (y_A) \cos(-\theta_p)$$

$$y_{pA} = -69.83 \text{ mm}$$

$$\sigma_t = \frac{M_{yp}(z_{pA})}{I_{yp}} - \frac{M_{zp}(y_{pA})}{I_{zp}}$$

$$\sigma_t = 31.0 \text{ MPa} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS (AT POINT B)

$$z_B = -x_{\text{bar}} + c_c \quad y_B = \frac{d_c}{2} + y_{\text{bar}}$$

$$z_{pB} = (z_B) \cos(-\theta_p) - (y_B) \sin(-\theta_p)$$

$$z_{pB} = 20.05 \text{ mm}$$

$$y_{pB} = (z_B) \sin(-\theta_p) + (y_B) \cos(-\theta_p)$$

$$y_{pB} = 124.0 \text{ mm}$$

$$\sigma_c = \frac{M_{yp}(z_{pB})}{I_{yp}} - \frac{M_{zp}(y_{pB})}{I_{zp}}$$

$$\sigma_c = -29.0 \text{ MPa} \quad \leftarrow$$

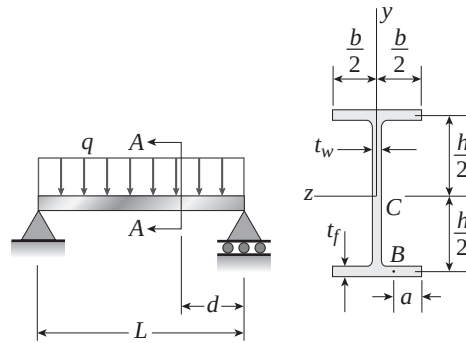
Shear Stresses in Wide-Flange Beams

When solving the problems for Section 6.8, assume the cross sections are thin-walled. Use centerline dimensions for all calculations and derivations, unless otherwise specified

Problem 6.8-1 A simple beam of W 10 × 30 wide-flange cross section supports a uniform load of intensity $q = 3.0$ k/ft on a span of length $L = 12$ ft (see figure). The dimensions of the cross section are $h = 10.5$ in., $b = 5.81$ in., $t_f = 0.510$ in., and $t_w = 0.300$ in.

- Calculate the maximum shear stress $\tau_{\max}$ on cross section A–A located at distance $d = 2.5$ ft from the end of the beam.
- Calculate the shear stress τ at point B on the cross section. Point B is located at a distance $a = 1.5$ in. from the edge of the lower flange.

Probs. 6.8-1 and 6.8-2



Solution 6.8-1

SIMPLE BEAM:

$$q = 3.0 \text{ k/ft} \quad L = 12 \text{ ft}$$

$$R = \frac{qL}{2} \quad R = 18.0 \text{ k} \quad d = 2.5 \text{ ft}$$

$$V = |R - qd| \quad V = 10.5 \text{ k}$$

CROSS SECTION:

$$h = 10.5 \text{ in.} \quad b = 5.81 \text{ in.} \quad t_f = 0.510 \text{ in.}$$

$$t_w = 0.30 \text{ in.}$$

$$I_z = \frac{t_w h^3}{12} + \frac{b t_f h^2}{2} \quad I_z = 192.28 \text{ in.}^4$$

- (a) MAXIMUM SHEAR STRESS

$$\tau_{\max} = \left(\frac{b t_f}{t_w} + \frac{h}{4} \right) \frac{V h}{2 I_z}$$

$$\tau_{\max} = 3584 \text{ psi} \quad \leftarrow$$

- (b) SHEAR STRESS AT POINT B

$$a = 1.5 \text{ in.} \quad \frac{b}{2} = 2.9 \text{ in.}$$

$$\tau_1 = \frac{b h V}{4 I_z} \quad \tau_1 = 832.8 \text{ psi}$$

$$\tau_B = \frac{a}{\frac{b}{2}} (\tau_1) \quad \tau_B = 430 \text{ psi} \quad \leftarrow$$

Problem 6.8-2 Solve the preceding problem for a W 250 × 44.8 wide-flange shape with the following data: $L = 3.5$ m, $q = 45$ kN/m, $h = 267$ mm, $b = 148$ mm, $t_f = 13$ mm, $t_w = 7.62$ mm, $d = 0.5$ m, and $a = 50$ mm.

Solution 6.8-2

SIMPLE BEAM:

$$q = 45 \text{ kN/m} \quad L = 3.5 \text{ m}$$

$$R = \frac{qL}{2} \quad R = 78.8 \text{ kN} \quad d = 0.5 \text{ m}$$

$$V = |R - qd| \quad V = 56.3 \text{ kN}$$

CROSS SECTION:

$$h = 267 \text{ mm} \quad b = 148 \text{ mm}$$

$$t_f = 13 \text{ mm} \quad t_w = 7.62 \text{ mm}$$

$$I_z = \frac{t_w h^3}{12} + \frac{bt_f h^2}{2}$$

$$I_z = 80.667 \times 10^6 \text{ mm}^4$$

(a) MAXIMUM SHEAR STRESS

$$\tau_{\max} = \left(\frac{bt_f}{t_w} + \frac{h}{4} \right) \frac{Vh}{2I_z}$$

$$\tau_{\max} = 29.7 \text{ MPa} \quad \leftarrow$$

(b) SHEAR STRESS AT POINT B

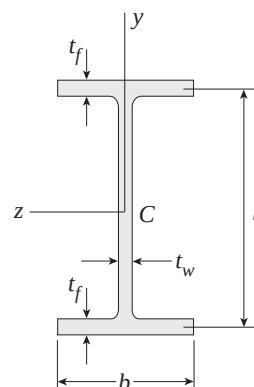
$$a = 50 \text{ mm} \quad b/2 = 74.0 \text{ mm}$$

$$\tau_1 = \frac{bhV}{4I_z} \quad \tau_1 = 999.1 \text{ psi}$$

$$\tau_B = \frac{a}{b/2}(\tau_1) \quad \tau_B = 4.65 \text{ MPa} \quad \leftarrow$$

Problem 6.8-3 A beam of wide-flange shape, W 8 × 28, has the cross section shown in the figure. The dimensions are $b = 6.54 \text{ in.}$, $h = 8.06 \text{ in.}$, $t_w = 0.285 \text{ in.}$, and $t_f = 0.465 \text{ in.}$ The loads on the beam produce a shear force $V = 7.5 \text{ k}$ at the cross section under consideration.

- Using centerline dimensions, calculate the maximum shear stress $\tau_{\max}$ in the web of the beam.
- Using the more exact analysis of Section 5.10 in Chapter 5, calculate the maximum shear stress in the web of the beam and compare it with the stress obtained in part a.

**Probs. 6.8-3 and 6.8-4****Solution 6.8-3**

$$b = 6.54 \text{ in.} \quad h = 8.06 \text{ in.} \quad t_w = 0.285 \text{ in.}$$

$$t_f = 0.465 \text{ in.} \quad V = 7.5 \text{ k}$$

(a) CALCULATIONS BASED ON CENTERLINE DIMENSIONS

Moment of inertia:

$$I_z = \frac{t_w h^3}{12} + \frac{bt_f h^2}{2}$$

$$I_z = 111.216 \text{ in.}^4$$

Maximum shear stress in the web:

$$\tau_{\max} = \left(\frac{bt_f}{t_w} + \frac{h}{4} \right) \frac{Vh}{2I_z}$$

$$\tau_{\max} = 3448 \text{ psi} \quad \leftarrow$$

(b) CALCULATIONS BASED ON MORE EXACT ANALYSIS

$$h_2 = h + t_f \quad h_2 = 8.5 \text{ in.} \quad h_1 = h - t_f$$

$$h_1 = 7.6 \text{ in.}$$

Moment of inertia:

$$I = \frac{1}{12} (bh_2^3 - bh_1^3 + t_w h_1^3)$$

$$I = 109.295 \text{ in.}^4$$

Maximum shear stress in the web:

$$\tau_{\max} = \frac{V}{8It_w} (bh_2^2 - bh_1^2 + t_w h_1^2)$$

$$\tau_{\max} = 3446 \text{ psi} \quad \leftarrow$$

Problem 6.8-4 Solve the preceding problem for a W 200 $\times$ 41.7 shape with the following data: $b = 166 \text{ mm}$, $h = 205 \text{ mm}$, $t_w = 7.24 \text{ mm}$, $t_f = 11.8 \text{ mm}$, and $V = 38 \text{ kN}$.

Solution 6.8-4

$$b = 166 \text{ mm} \quad h = 205 \text{ mm} \quad t_w = 7.24 \text{ mm}$$

$$t_f = 11.8 \text{ mm} \quad V = 38 \text{ kN}$$

(a) CALCULATIONS BASED ON CENTERLINE DIMENSIONS

Moment of inertia:

$$I_z = \frac{t_w h^3}{12} + \frac{bt_f h^2}{2}$$

$$I_z = 46.357 \times 10^6 \text{ mm}^4$$

Maximum shear stress in the web:

$$\tau_{\max} = \left(\frac{bt_f}{t_w} + \frac{h}{4} \right) \frac{Vh}{2I_z}$$

$$\tau_{\max} = 27.04 \text{ MPa} \quad \leftarrow$$

(b) CALCULATIONS BASED ON MORE EXACT ANALYSIS

$$h_2 = h + t_f \quad h_2 = 216.8 \text{ mm} \quad h_1 = h - t_f$$

$$h_1 = 193.2 \text{ mm}$$

Moment of inertia:

$$I = \frac{1}{12} (bh_2^3 - bh_1^3 + t_w h_1^3)$$

$$I = 45.556 \times 10^6 \text{ mm}^4$$

Maximum shear stress in the web:

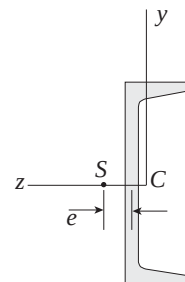
$$\tau_{\max} = \frac{V}{8It_w} (bh_2^2 - bh_1^2 + t_w h_1^2)$$

$$\tau_{\max} = 27.02 \text{ MPa} \quad \leftarrow$$

Shear Centers of Thin-Walled Open Sections

When locating the shear centers in the problems for Section 6.9, assume that the cross sections are thin-walled and use centerline dimensions for all calculations and derivations.

Problem 6.9-1 Calculate the distance e from the centerline of the web of a C 15 $\times$ 40 channel section to the shear center S (see figure). (Note: For purposes of analysis, consider the flanges to be rectangles with thickness t_f equal to the average flange thickness given in Table F-3a of Appendix F.)

**Probs. 6.9-1 and 6.9-2**

Solution 6.9-1

$$C\ 15 \times 40 \quad d = 15.0\text{ in.} \quad t_w = 0.520\text{ in.}$$

$$h = d - t_f \quad h = 14.350\text{ in.}$$

$$b_f = 3.520\text{ in.} \quad t_f = 0.650\text{ in.} \quad b = b_f - \frac{t_w}{2}$$

$$e = \frac{3b^2 t_f}{ht_w + 6bt_f} \quad e = 1.027\text{ in.} \quad \leftarrow$$

$$b = 3.260\text{ in.}$$

Problem 6.9-2 Calculate the distance e from the centerline of the web of a C 310 $\times$ 45 channel section to the shear center S (see figure). (Note: For purposes of analysis, consider the flanges to be rectangles with thickness t_f equal to the average flange thickness given in Table F-3b of Appendix F.)

Solution 6.9-2

$$C\ 310 \times 45 \quad d = 305\text{ mm} \quad t_w = 13.0\text{ mm}$$

$$b = 74.0\text{ mm} \quad h = d - t_f \quad h = 292.3\text{ mm}$$

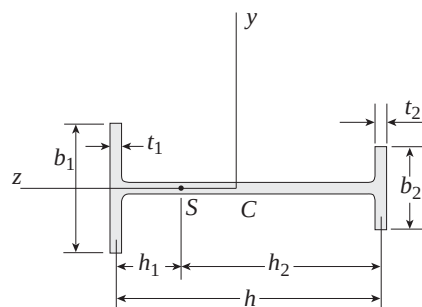
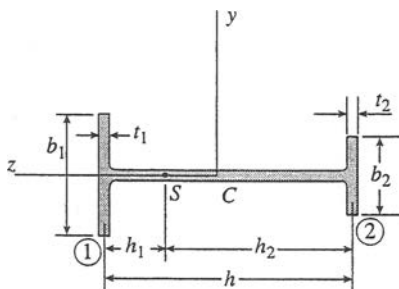
$$b_f = 80.5\text{ mm} \quad t_f = 12.7\text{ mm} \quad b = b_f - \frac{t_w}{2}$$

$$e = \frac{3b^2 t_f}{ht_w + 6bt_f} \quad e = 22.1\text{ mm} \quad \leftarrow$$

Problem 6.9-3 The cross section of an unbalanced wide-flange beam is shown in the figure. Derive the following formula for the distance h_1 from the centerline of one flange to the shear center S :

$$h_1 = \frac{t_2 b_2^3 h}{t_1 b_1^3 + t_2 b_2^3}$$

Also, check the formula for the special cases of a T-beam ($b_2 = t_2 = 0$) and a balanced wide-flange beam ($t_2 = t_1$ and $b_2 = b_1$).

**Solution 6.9-3 Unbalanced wide-flange beam**

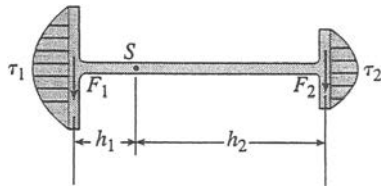
FLANGE 1:

$$\tau_1 = \frac{VQ}{I_z t_1}$$

$$Q = (b_1/2)(t_1)(b_1/4) = \frac{t_1 b_1^2}{8}$$

$$\tau_1 = \frac{V b_1^2}{8 I_z}$$

$$F_1 = \frac{2}{3}(\tau_1)(b_1)(t_1) = \frac{V t_1 b_1^3}{12 I_z}$$



FLANGE 2:

$$F_2 = \frac{V t_2 b_2^3}{12 I_z}$$

 Shear force V acts through the shear center S .

$$\therefore \sum M_S = F_1 h_1 - F_2 h_2 = 0$$

$$\text{or } (t_1 b_1^3) h_1 = (t_2 b_2^3) h_2 \quad (1)$$

$$h_1 + h_2 = h \quad (2)$$

$$\text{Solve Eqs. (1) and (2): } h_1 = \frac{t_2 b_2^3 h}{t_1 b_1^3 + t_2 b_2^3} \quad \leftarrow$$

T-BEAM

$$b_2 = t_2 = 0;$$

$$\therefore h_1 = 0 \quad \leftarrow$$

WIDE-FLANGE BEAM

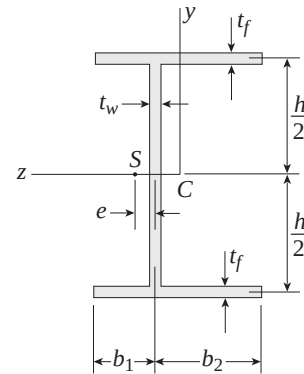
$$t_2 = t_1 \text{ and } b_2 = b_1;$$

$$\therefore h_1 = h/2 \quad \leftarrow$$

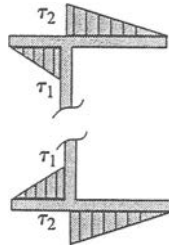
Problem 6.9-4 The cross section of an unbalanced wide-flange beam is shown in the figure. Derive the following formula for the distance e from the centerline of the web to the shear center S :

$$e = \frac{3 t_f (b_2^2 - b_1^2)}{h t_w + 6 t_f (b_1 + b_2)}$$

Also, check the formula for the special cases of a channel section ($b_1 = 0$ and $b_2 = b$) and a doubly symmetric beam ($b_1 = b_2 = b/2$).



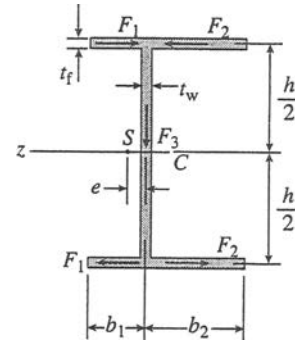
Solution 6.9-4 Unbalanced wide-flange beam



$$\tau_1 = \frac{VQ}{I t_f} = \frac{b_1 h V}{2 I_z} \quad \tau_2 = \frac{b_2 h V}{2 I_z}$$

$$F_1 = \frac{b_1 \tau_1 t_f}{2} = \frac{b_1^2 h t_f V}{4 I_z}$$

$$F_2 = \frac{b_2^2 h t_f V}{4 I_z} \quad F_3 = V$$


 Shear force V acts through the shear center S .

$$\therefore \sum M_S = -F_3 e - F_1 h + F_2 h = 0$$

$$e = \frac{F_2 h - F_1 h}{F_3} = \frac{h^2 t_f}{4 I_z} (b_2^2 - b_1^2)$$

$$\begin{aligned}
 I_z &= \frac{t_w h^3}{12} + 2(b_1 + b_2)(t_f)\left(\frac{h}{2}\right)^2 \\
 &= \frac{h^2}{12} [ht_w + 6t_f(b_1 + b_2)] \\
 e &= \frac{3t_f(b_2^2 - b_1^2)}{ht_w + 6t_f(b_1 + b_2)} \quad \leftarrow
 \end{aligned}$$

CHANNEL SECTION ($b_1 = 0, b_2 = b$)

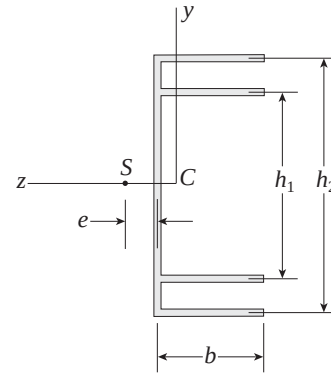
$$e = \frac{3b^2 t_f}{ht_w + 6bt_f} \quad (\text{Eq. 6-83})$$

DOUBLY SYMMETRIC BEAM ($b_1 = b_2 = b/2$) $e = 0$ (Shear center coincides with the centroid.)

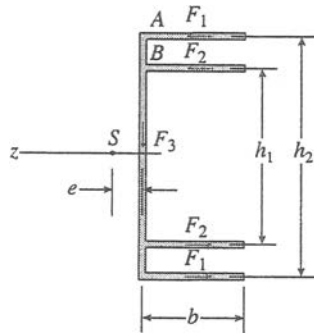
Problem 6.9-5 The cross section of a channel beam with double flanges and constant thickness throughout the section is shown in the figure.

Derive the following formula for the distance e from the centerline of the web of the shear center S :

$$e = \frac{3b^2(h_1^2 + h_2^2)}{h^3 + 6b(h_1^2 + h_2^2)}$$



Solution 6.9-5 Channel beam with double flanges



t = thickness

$$\tau_A = \frac{VQ_A}{I_z t} = \frac{V(bt)\left(\frac{h_2}{2}\right)}{I_z t} = \frac{bh_2 V}{2I_z}$$

$$F_1 = \frac{1}{2} \tau_A bt = \frac{b^2 h_2 t V}{4I_z}$$

$$\tau_B = \frac{bh_1 V}{2I_z} \quad F_2 = \frac{b^2 h_1 t V}{4I_z}$$

$$F_3 = V$$

Shear force V acts through the shear center S .

$$\therefore \sum M_S = -F_3 e + F_1 h_2 + F_2 h_1 = 0$$

$$e = \frac{F_2 h_1 + F_1 h_2}{F_3} = \frac{b^2 t}{4I_z} (h_1^2 + h_2^2)$$

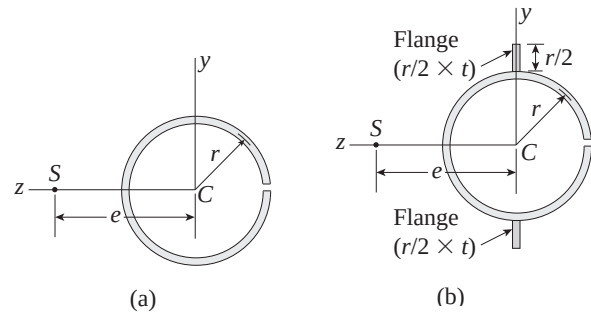
$$I_z = \frac{th^3}{12} + 2[bt(h_2/2)^2 + bt(h_1/2)^2]$$

$$= \frac{t}{12} [h^3 + 6b(h_1^2 + h_2^2)]$$

$$e = \frac{3b^2(h_1^2 + h_2^2)}{h^3 + 6b(h_1^2 + h_2^2)} \quad \leftarrow$$

Problem 6.9-6 The cross section of a slit circular tube of constant thickness is shown in the figure.

- (a) Show that the distance e from the center of the circle to the shear center S is equal to $2r$ in the figure part a.
 (b) Find an expression for e if flanges with the same thickness as that of the tube are added, as shown in the figure part b.



Solution 6.9-6

$$(a) \quad Q_A = \int y dA = \int_0^\theta (rt \sin(\phi)) d\phi$$

$$Q_A = r^2 t (1 - \cos(\theta))$$

$$\tau_A = \frac{VQ_A}{I_z t} = \frac{Vr^2(1 - \cos(\theta))}{I_z}$$

$$I_z = \pi r^3 t$$

$$\tau_A = \frac{V(1 - \cos(\theta))}{\pi r t}$$

At point A: $dA = rt d\theta$

T_C = moment of shear stresses about center C.

$$T_C = \int \tau_A r dA = \int_0^{2\pi} \frac{Vr}{\pi} (1 - \cos(\theta)) d\theta$$

$$(1 - \cos(\theta)) d\theta = 2Vr$$

Shear force V acts through the shear center S .

Moment of the shear force V about any point must be equal to the moment of the shear stresses about that same point.

$$\Sigma M_C = Ve = T_C \quad e = \frac{T_C}{V} = 2r \quad \leftarrow$$

$$(b) \quad I_z = \pi r^3 t + 2 \left[\frac{t \left(\frac{r}{2} \right)^3}{12} + t \frac{r \left(\frac{5r}{4} \right)^2}{2} \right]$$

$$I_z = \pi r^3 t + \frac{19}{12} tr^3 = tr^3 \left(\pi + \frac{19}{12} \right)$$

$$\text{for } 0 \leq \theta < \frac{\pi}{2}$$

$$Q_A = \int y dA = \int_0^\theta (rt \sin(\phi)) d\phi$$

$$Q_A = r^2 t (1 - \cos(\theta))$$

$$\tau_A = \frac{VQ_A}{I_z t} = \frac{Vr^2(1 - \cos(\theta))}{I_z}$$

$$\text{for } \frac{\pi}{2} \leq \theta < \frac{3\pi}{2}$$

$$Q_A = \int y dA = \int_0^\theta (rt \sin(\phi)) d\phi + \frac{r}{2} t \frac{5r}{4}$$

$$Q_A = r^2 t \left(\frac{13}{8} - \cos(\theta) \right)$$

$$\tau_A = \frac{Vr^2 \left(\frac{13}{8} - \cos(\theta) \right)}{I_z}$$

At point A: $dA = rt d\theta$

T_C = moment of shear stresses about center C

$$T_C = \int \tau_A r dA = 2 \int_0^{\frac{\pi}{2}} \frac{Vr^4 t}{I_z} (1 - \cos(\theta)) d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{Vr^4 t}{I_z} \left(\frac{13}{8} - \cos(\theta) \right) d\theta$$

$$(1 - \cos(\theta)) d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{Vr^4 t}{I_z} \left(\frac{13}{8} - \cos(\theta) \right) d\theta$$

$$\left(\frac{13}{8} - \cos(\theta) \right) d\theta$$

$$\begin{aligned}
 T_C &= \int \tau_{Ar} dA = Vr^4 t \frac{\pi-2}{I_z} \\
 &\quad + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{Vr^4 t}{I_z} \left(\frac{13}{8} - \cos(\theta) \right) d\theta \\
 T_C &= Vr^4 t \frac{\pi-2}{I_z} + \frac{1}{8} Vr^4 t \frac{13\pi+16}{I_z} \\
 &= \frac{21Vr^4 t \pi}{8I_z}
 \end{aligned}$$

Shear force V acts through the shear center S . Moment of the shear force V about any point must be equal to the moment of the shear stresses about that same point.

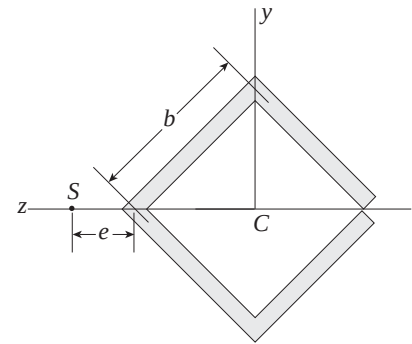
$$\Sigma M_C = Ve = T_C$$

$$e = \frac{T_C}{V} = \frac{21r^4 t \pi}{8I_z} = \frac{21r^4 t \pi}{8 \left[tr^3 \left(\pi + \frac{19}{12} \right) \right]}$$

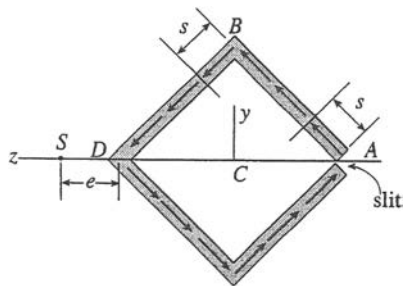
$$e = \frac{63\pi r}{24\pi + 38} = 1.745r \quad \leftarrow$$

Problem 6.9-7 The cross section of a slit square tube of constant thickness is shown in the figure. Derive the following formula for the distance e from the corner of the cross section to the shear center S :

$$e = \frac{b}{2\sqrt{2}}$$



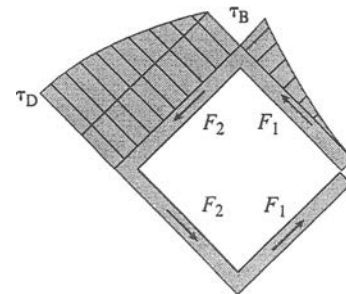
Solution 6.9-7 Slit square tube



b = length of each side

t = thickness

$$\tau = \frac{VQ}{I_z t}$$



FROM A TO B:

$$Q = \frac{ts^2}{2\sqrt{2}}$$

At A: $Q = 0$ $\tau_A = 0$

$$\text{At } B: Q = \frac{tb^2}{2\sqrt{2}}$$

$$\tau_B = \frac{b^2V}{2\sqrt{2}I_z}$$

$$F_1 = \frac{\tau_B bt}{3} = \frac{b^3 t V}{6\sqrt{2}I_z}$$

FROM B TO D:

$$Q = bt\left(\frac{b}{2\sqrt{2}}\right) + st\left(\frac{b}{\sqrt{2}} - \frac{s}{2\sqrt{2}}\right)$$

$$= \frac{tb^2}{2\sqrt{2}} + \frac{ts}{2\sqrt{2}}(2b - s)$$

$$\tau = \frac{VQ}{I_z t} = \frac{V}{I_z} \left[\frac{b^2}{2\sqrt{2}} + \frac{s}{2\sqrt{2}}(2b - s) \right]$$

$$\text{At } B: \tau_B = \frac{b^2V}{2\sqrt{2}I_z} \quad \text{At } D: \tau_D = \frac{b^2V}{\sqrt{2}I_z}$$

$$F_2 = \tau_B bt + \frac{2}{3}(\tau_D - \tau_B)bt = \frac{5tb^3V}{6\sqrt{2}I_z}$$

Shear force V acts through the shear center S .

$$\therefore \sum M_S = 0$$

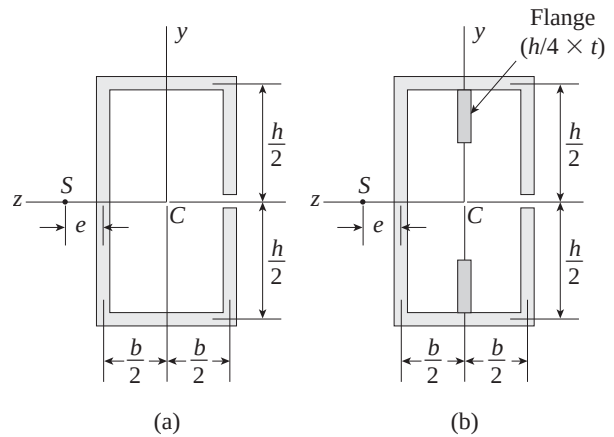
$$2(F_1/\sqrt{2})(b\sqrt{2} + e) + 2(F_2/\sqrt{2})(e) = 0$$

Substitute for F_1 and F_2 and solve for e :

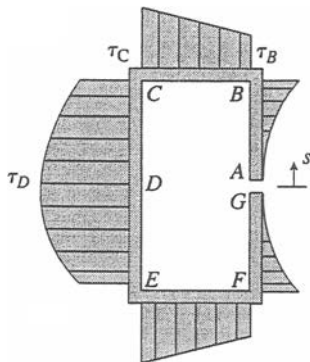
$$e = \frac{b}{2\sqrt{2}} \quad \leftarrow$$

Problem 6.9-8 The cross section of a slit rectangular tube of constant thickness is shown in the figures.

- Derive the following formula for the distance e from the centerline of the wall of the tube in the figure part a to the shear center S :
$$e = \frac{b(2h + 3b)}{2(h + 3b)}$$
- Find an expression for e if flanges with the same thickness as that of the tube are added as shown in the figure part b.



Solution 6.9-8



$$\text{(a) FROM A TO B: } Q = \frac{ts^2}{2} \quad \tau = \frac{VQ}{I_z t} = \frac{s^2V}{2I_z}$$

$$\tau_A = 0 \quad \tau_B = \frac{h^2V}{8I_z}$$

$$F_1 = \frac{\tau_B t}{3} \left(\frac{h}{2} \right) = \frac{th^3V}{48I_z}$$

$$\text{FROM B TO C: } \tau_B = \frac{h^2V}{8I_z}$$

$$Q_C = \frac{th}{2} \left(\frac{h}{4} \right) + bt \left(\frac{h}{2} \right) = \frac{th}{8} (h + 4b)$$

$$\tau_C = \frac{h(h + 4b)V}{8I_z}$$

$$F_2 = \frac{1}{2}(\tau_B + \tau_C)bt = \frac{bht(h + 2b)V}{8I_z}$$

$$\Sigma F_{\text{VERT}} = V \quad F_3 - 2F_1 = V$$

$$F_3 = V \left(1 + \frac{th^3}{24I_z} \right)$$

Shear force V acts through the shear center S .

$$\Sigma M_S = 0 \quad -F_3e + F_2h + 2F_1(b + e) = 0$$

$$\text{Solve for } e = \frac{bht(2h + 3b)}{12I_z}$$

$$I_z = 2 \left[\frac{1}{12}th^3 + bt \left(\frac{h}{2} \right)^2 \right] = \frac{th^2}{6}(h + 3b)$$

$$\text{Therefore, } e = \frac{b}{2} \left(\frac{2h + 3b}{h + 3b} \right) \quad \leftarrow$$

$$(b) \text{ FROM A TO B: } Q = \frac{ts^2}{2} \quad \tau = \frac{VQ}{I_z t} = \frac{s^2 V}{2I_z}$$

$$\tau_A = 0 \quad \tau_B = \frac{h^2 V}{8I_z}$$

$$F_1 = \frac{\tau_B t \left(\frac{h}{2} \right)}{3} = \frac{th^3 V}{48I_z}$$

$$\text{FROM B TO C: } \tau_B = \frac{h^2 \cdot V}{8I_z}$$

$$Q_C = \frac{th}{2} \left(\frac{h}{4} \right) + \frac{b}{2} t \left(\frac{h}{2} \right) = \frac{th}{8} (h + 2b)$$

$$\tau_C = \frac{h(h + 2b)V}{8I_z}$$

$$F_{BC} = \frac{1}{2}(\tau_B + \tau_C) \frac{b}{2} t = \frac{bht(h + b)V}{16I_z}$$

In flange:

$$Q_{C\_flange} = ts \left(\frac{s}{2} + \frac{h}{4} \right) = \frac{ts}{4} (2s + h)$$

$$F_{flange} = \int_0^{\frac{h}{4}} \left[\frac{s}{4} (2s + h) \right] \frac{tV}{I_z} ds = \frac{h^3 V t}{96I_z}$$

FROM C TO D:

$$Q = \frac{th^2}{8} + t \frac{b}{2} \left(\frac{h}{2} \right) + t \frac{h}{4} \left(\frac{3h}{8} \right) + t \frac{h}{2} s$$

$$F_{CD} = \int_0^{\frac{b}{2}} \left[\frac{th^2}{8} + t \frac{b}{2} \left(\frac{h}{2} \right) + t \frac{h}{4} \left(\frac{3h}{8} \right) + t \frac{h}{2} s \right] \frac{Vt}{I_z} ds = \frac{Vthb}{64I_z} (7h + 12b)$$

$$\Sigma F_{\text{VERT}} = V \quad F_3 - 2F_1 - 2F_{flange} = V$$

$$F_3 = V \left(1 + \frac{th^3}{16I_z} \right)$$

Shear force V acts through the shear center S .

$$\Sigma M_S = 0 \quad -F_3e + (F_{BC} + F_{CD})h + 2F_1(b + e) + 2F_{flange} \left(\frac{b}{2} + e \right) = 0$$

$$e = \frac{61bth^3}{192I_z} + \frac{b^2h^2t}{4I_z}$$

$$e = \frac{th^2b}{192I_z} (43h + 48b)$$

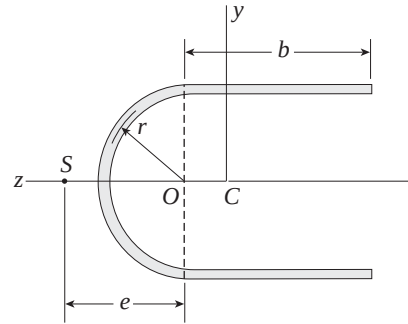
$$I_z = 2 \left[\frac{1}{12}th^3 + bt \left(\frac{h}{2} \right)^2 + \frac{1}{12}t \left(\frac{h}{4} \right)^3 + t \left(\frac{h}{4} \right) \left(\frac{3h}{8} \right)^2 \right] = \frac{th^2}{96} (23h + 48b)$$

$$e = \frac{b}{2} \left(\frac{43h + 48b}{23h + 48b} \right) \quad \leftarrow$$

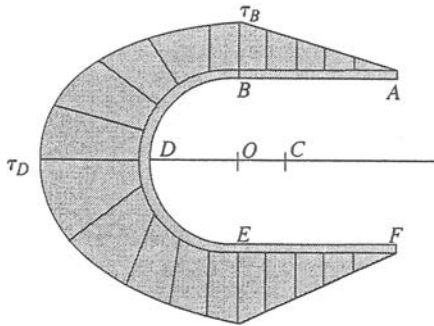
Problem 6.9-9 A U-shaped cross section of constant thickness is shown in the figure. Derive the following formula for the distance e from the center of the semicircle to the shear center S :

$$e = \frac{2(2r^2 + b^2 + \pi br)}{4b + \pi r}$$

Also, plot a graph showing how the distance e (expressed as the nondimensional ratio e/r) varies as a function of the ratio b/r . (Let b/r range from 0 to 2.)



Solution 6.9-9 U-shaped cross section



r = radius F_1 = force in AB

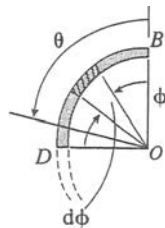
t = thickness F_2 = force in EF

T_0 = moment in BDE

$$\text{FROM A TO B: } \tau_A = 0 \quad \tau_B = \frac{VQ}{I_z t} = \frac{V(btr)}{I_z t} = \frac{Vbr}{I_z}$$

$$F_1 = \frac{bt\tau_B}{2} = \frac{Vb^2rt}{2I_z}$$

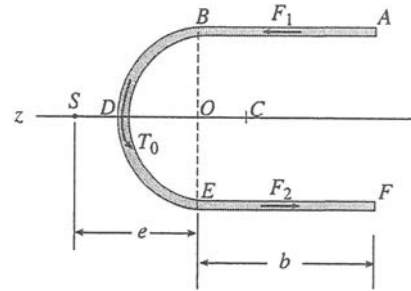
$$\text{FROM B TO E: } Q_1 = \int y dA = \int_0^\theta (r \cos \phi) rt d\phi \\ = r^2 t \sin \theta$$



$$Q_B = btr$$

$$Q_\theta = Q_B + Q_1 = btr + r^2 t \sin \theta$$

$$\tau_\theta = \frac{VQ_B}{I_z t} = \frac{Vr(b + r \sin \theta)}{I_z}$$



At angle θ : $dA = rt d\theta$

$$T_0 = \int \tau r dA = \int_0^\pi \tau r^2 t d\theta$$

$$= \int_0^\pi \frac{Vr^3 t (b + r \sin \theta) d\theta}{I_z}$$

$$= \frac{Vr^3 t}{I_z} (\pi b + 2r)$$

Shear force V acts through the shear center S . Moment of the shear force V about any point must be equal to the moment of the shear stresses about that same point.

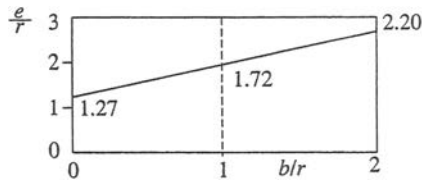
$$\therefore \sum M_0 = Ve = T_0 + F_1(2r)$$

$$e = \frac{T_0 + 2F_1r}{V} = \frac{r^2 t}{I_z} (\pi br + 2r^2 + b^2)$$

$$I_z = \frac{\pi r^3 t}{2} + 2(btr^2) \quad e = \frac{2(2r^2 + b^2 + \pi br)}{4b + \pi r} \quad \leftarrow$$

GRAPH

$$\frac{e}{r} = \frac{2(2 + b^2/r^2 + \pi b/r)}{4b/r + \pi}$$

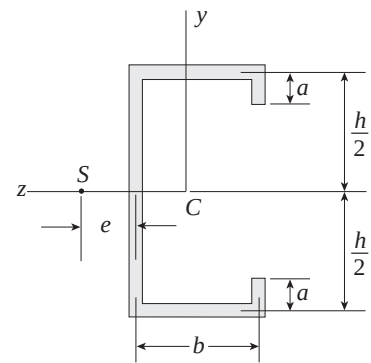
NOTE: When $b/r = 0$,

$$e/r = \frac{4}{\pi} \quad (\text{Eq. 6-81})$$

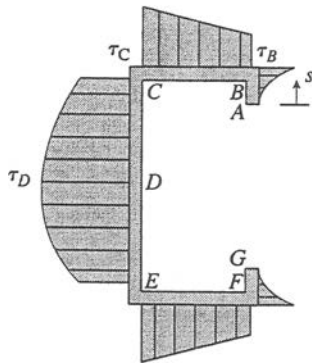
Problem 6.9-10 Derive the following formula for the distance e from the centerline of the wall to the shear center S for the C-section of constant thickness shown in the figure:

$$e = \frac{3bh^2(b + 2a) - 8ba^3}{h^2(h + 6b + 6a) + 4a^2(2a - 3h)}$$

Also, check the formula for the special cases of a channel section ($a = 0$) and a slit rectangular tube ($a = h/2$).



Solution 6.9-10 C-section of constant thickness

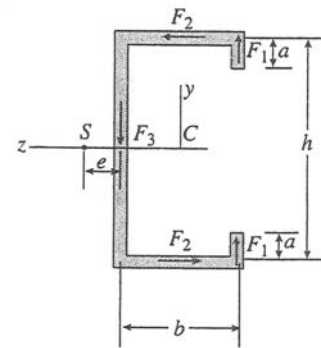


t = thickness

FROM A TO B:

$$Q = st\left(\frac{h}{2} - a + \frac{s}{2}\right) \quad \tau = \frac{VQ}{I_z t} = s\left(\frac{h}{2} - a + \frac{s}{2}\right)\frac{V}{I_z}$$

$$\tau_A = 0 \quad \tau_B = \frac{a}{2}(h - a)\frac{V}{I_z}$$



$$\begin{aligned} F_1 &= \int_0^a \tau t ds = \frac{tV}{I_z} \int_0^a s\left(\frac{h}{2} - a + \frac{s}{2}\right) ds \\ &= \frac{a^2 t(3h - 4a)V}{12I_z} \end{aligned}$$

FROM B TO C:

$$\tau_B = \frac{a}{2}(h-a)\frac{V}{I_z} \quad Q_C = at\left(\frac{h}{2} - \frac{a}{2}\right) + bt\left(\frac{h}{2}\right)$$

$$= \frac{at}{2}(h-a) + \frac{bht}{2}$$

$$\tau_C = \left[\frac{a}{2}(h-a) + \frac{bh}{2}\right]\frac{V}{I_z}$$

$$F_2 = \frac{1}{2}(\tau_B + \tau_C)bt = \frac{bt}{4}[2a(h-a) + bh]\frac{V}{I_z}$$

FROM C TO E:

$$\sum F_{\text{VERT}} = V \quad F_3 - 2F_1 = V$$

$$F_3 = V\left[1 + \frac{a^2t(3h-4a)}{6I_z}\right]$$

Shear force V acts through the shear center S .

$$\therefore \sum M_S = 0 \quad -F_3(e) + F_2h + 2F_1(b+e) = 0$$

Substitute for F_1 , F_2 , and F_3 and solve for e :

$$e = \frac{bt[3h^2(b+2a) - 8a^3]}{12I_z}$$

$$I_z = 2\left(\frac{1}{12}th^3\right) + 2bt\left(\frac{h}{2}\right)^2 - \frac{6}{12}(h-2a)^3$$

$$= \frac{t}{12}[h^2(h+6b+6a) + 4a^2(2a-3h)]$$

$$e = \frac{3bh^2(b+2a) - 8ba^3}{h^2(h+6b+6a) + 4a^2(2a-3h)} \quad \leftarrow$$

CHANNEL SECTION ($a = 0$)

$$e = \frac{3b^2}{h+6b} \quad (\text{agrees with Eq. 6-73 when } t_f = t_w)$$

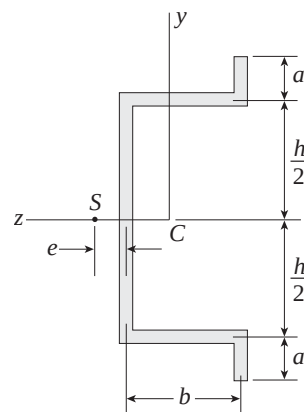
SLIT RECTANGULAR TUBE ($a = h/2$)

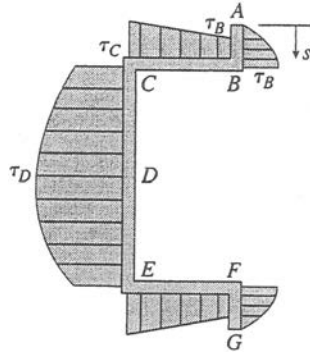
$$e = \frac{b(2h+3b)}{2(h+3b)} \quad (\text{agrees with the result of Prob. 6.9-8})$$

Problem 6.9-11 Derive the following formula for the distance e from the centerline of the wall to the shear center S for the hat section of constant thickness shown in the figure:

$$e = \frac{3bh^2(b+2a) - 8ba^3}{h^2(h+6b+6a) + 4a^2(2a+3h)}$$

Also, check the formula for the special case of a channel section ($a = 0$).



Solution 6.9-11 Hat section of constant thickness

t = thickness

FROM A TO B: $Q = st\left(\frac{h}{2} + a - \frac{s}{2}\right)$

$$\tau = \frac{VQ}{I_z t} = s\left(\frac{h}{2} + a - \frac{s}{2}\right)\frac{V}{I_z}$$

$$\tau_A = 0 \quad \tau_B = \frac{a}{2}(h + a)\frac{V}{I_z}$$

$$F_1 = \int_0^a \tau ds = \frac{tV}{I_z} \int_0^a s\left(\frac{h}{2} + a - \frac{s}{2}\right) ds$$

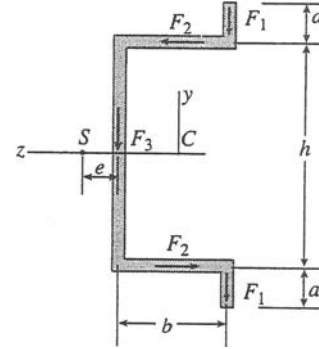
$$= \frac{a^2 t(3h + 4a)V}{12I_z}$$

FROM B TO C: $\tau_B = \frac{a}{2}(h + a)\frac{V}{I_z}$

$$Q_C = at\left(\frac{h}{2} + \frac{a}{2}\right) + bt\left(\frac{h}{2}\right) = \frac{at}{2}(h + a) + \frac{bht}{2}$$

$$\tau_C = \left[\frac{a}{2}(h + a) + \frac{bh}{2}\right]\frac{V}{I_z}$$

$$F_2 = \frac{1}{2}(\tau_B + \tau_C)bt = \frac{bht}{4}[2a(h + a) + bh]\frac{V}{I_z}$$



FROM C TO E:

$$\sum F_{\text{VERT}} = V \quad F_3 + 2F_1 = V$$

$$F_3 = V\left[1 - \frac{a^2 t(3h + 4a)}{6I_z}\right]$$

Shear force V acts through the shear center S .

$$\therefore \sum M_S = 0 \quad -F_3 e + F_2 h - 2F_1(b + e) = 0$$

Substitute for F_1 , F_2 , and F_3 and solve for e :

$$e = \frac{bt[3h^2(b + 2a) - 8a^3]}{12I_z}$$

$$I_z = \frac{1}{12}th^3 + 2bt\left(\frac{h}{2}\right)^2 + \frac{t}{12}(h + 2a)^3 - \frac{1}{12}th^3$$

$$= \frac{t}{12}[h^2(h + 6b + 6a) + 4a^2(2a + 3h)]$$

$$e = \frac{3bh^2(b + 2a) - 8ba^3}{h^2(h + 6b + 6a) + 4a^2(2a + 3h)} \quad \leftarrow$$

CHANNEL SECTION ($a = 0$)

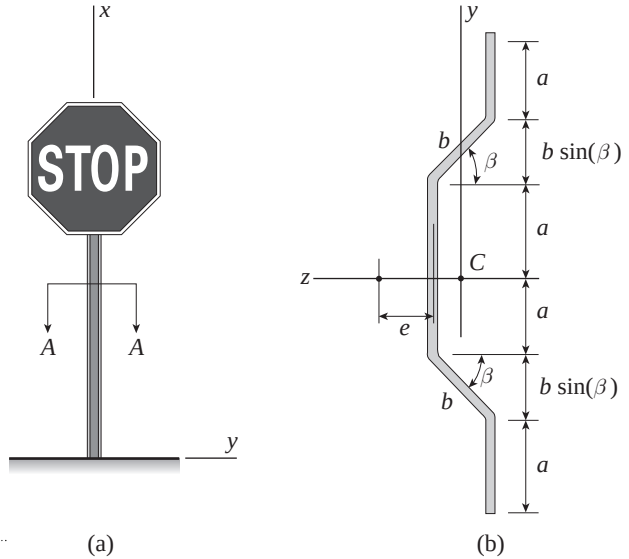
$$e = \frac{3b^2}{h + 6b} \quad (\text{agrees with Eq. 6-73 when } t_f = t_w)$$

Problem 6.9-12 The cross section of a sign post of constant thickness is shown in the figure. Derive the formula below for the distance e from the centerline of the wall of the post to the shear center S :

$$e = \frac{1}{3} t b a (4a^2 + 3ab \sin(\beta) + 3ab + 2 \sin(\beta) b^2) \frac{\cos(\beta)}{I_z}$$

where $I_z = 0$ moment of inertia about the z axis.

Also, compare this formula with that given in Prob. 6.9-11 for the special case of $\beta = 0$ here and $a = h/2$ in both formulas.



Solution 6.9-12

FROM A TO B:

$$Q_{AB} = st \left(2a + b \sin(\beta) - \frac{s}{2} \right)$$

$$\tau_{AB} = \frac{VQ_{AB}}{I_z t}$$

$$F_{AB} = \int \tau dA = \int_0^a \frac{V \left[st \left(2a + b \sin(\beta) - \frac{s}{2} \right) \right]}{I_z t} t ds$$

$$F_{AB} = \frac{V t a^2 (5a + 3b \sin(\beta))}{6 I_z}$$

FROM B TO C:

$$Q_{BC} = at \left(2a + b \sin(\beta) - \frac{a}{2} \right) + ts \left(a + b \sin(\beta) - \frac{s}{2} \sin(\beta) \right)$$

$$Q_{BC} = \frac{3}{2} a^2 t + atb \sin(\beta) + sta + stb \sin(\beta) - \frac{1}{2} s^2 t \sin(\beta)$$

$$\tau_{BC} = \frac{VQ_{BC}}{I_z t}$$

$$F_{BC} = \int \tau dA = \int_0^b \frac{V \left(\frac{3}{2} a^2 t + atb \sin(\beta) + sta + stb \sin(\beta) - \frac{1}{2} s^2 t \sin(\beta) \right)}{I_z t} t ds$$

$$F_{BC} = \frac{V t b \left(9a^2 + 6ab \sin(\beta) + 3ba + 2b^2 \sin(\beta) \right)}{6 I_z}$$

SHEAR FORCE V ACTS THROUGH THE SHEAR CENTER S .

$$\sum M_E = 0 \quad Ve + 2F_{AB}b \cos(\beta) - F_{BC} \cos(\beta)(2a) = 0$$

$$e = 2 \cos(\beta) \frac{(F_{BC}a - F_{AB}b)}{V}$$

$$e = \frac{tba \cos(\beta)}{3I_z} \left(4a^2 + 3ab \sin(\beta) + 3ab + 2b^2 \sin(\beta) \right) \quad \leftarrow$$

Now, compare this formula with that given in Prob. 6.9-11 for the special case of $\beta = 0$ here and $a = h/2$ in both formulas.

FIRST MODIFY ABOVE FORMULA FOR $\beta = 0$ and $a = h/2$:

$$e = \frac{tb \frac{h}{2} \cos(0)}{3I_z} \left[4 \left(\frac{h}{2} \right)^2 + 3 \frac{h}{2} b \sin(0) + 3 \frac{h}{2} b + 2b^2 \sin(0) \right]$$

$$e = \frac{bht \left(h^2 + \frac{3bh}{2} \right)}{6I_z}$$

where I_z for the hat section of Prob. 6.9-11 is as follows:

$$I_z = \frac{th^3}{12} + 2bt \left(\frac{h}{2} \right)^2 + 2 \frac{t \left(\frac{h}{2} \right)^3}{12} + 2t \frac{h}{2} \left(\frac{h}{2} + \frac{h}{4} \right)^2$$

$$I_z = \frac{h^2 t (3b + 4h)}{6}$$

Substituting expression for I_z and simplifying gives:

$$e = \frac{bht \left(h^2 + \frac{3bh}{2} \right)}{\frac{h^2 t (3b + 4h)}{6}}$$

$$e = \frac{b(3b+2h)}{6b+8h} \quad \text{for } \beta = 0 \quad \text{and} \quad a = \frac{h}{2} \quad \leftarrow$$

NOW MODIFY FORMULA FOR e FROM PROB. 6.9-11 AND COMPARE TO ABOVE

$$e = \frac{3bh^2(b+2a) - 8ba^3}{h^2(h+6b+6a) + 4a^2(2a+3h)}$$

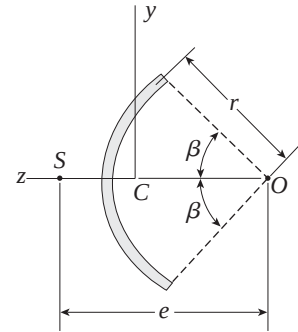
$$e = \frac{3bh^2 \left(b + 2 \frac{h}{2} \right) - 8b \left(\frac{h}{2} \right)^3}{h^2 \left(h + 6b + 6 \frac{h}{2} \right) + 4 \left(\frac{h}{2} \right)^2 \left(2 \frac{h}{2} + 3h \right)}$$

$$e = \frac{b(3b+2h)}{6b+8h} \quad \text{same expressions as that above from sign-post solution} \quad \leftarrow$$

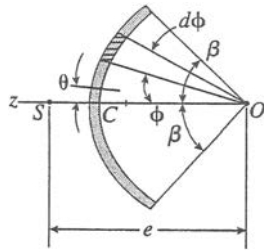
Problem 6.9-13 A cross section in the shape of a circular arc of constant thickness is shown in the figure. Derive the following formula for the distance e from the center of the arc to the shear center S :

$$e = \frac{2r(\sin \beta - \beta \cos \beta)}{\beta - \sin \beta \cos \beta}$$

in which β is in radians. Also, plot a graph showing how the distance e varies as β varies from 0 to π .



Solution 6.9-13 Circular arc



t = thickness r = radius

At angle θ :

$$Q = \int y dA = \int_{\theta}^{\beta} (r \sin \phi) rt d\phi$$

$$= r^2 t (\cos \theta - \cos \beta)$$

$$\tau = \frac{VQ}{I_z t} = \frac{Vr^2 (\cos \theta - \cos \beta)}{I_z}$$

$$I_z = \int y^2 dA = \int_{-\beta}^{\beta} (r \sin \phi)^2 rt d\phi$$

$$= r^3 t (\beta - \sin \beta \cos \beta)$$

$$\tau = \frac{V(\cos \theta - \cos \beta)}{rt(\beta - \sin \beta \cos \beta)}$$

T_0 = moment of shear stresses

At angle θ , $dA = rt d\theta$

$$T_0 = \int \tau r dA = \int_{-\beta}^{\beta} \frac{V(\cos \theta - \cos \beta)}{rt(\beta - \sin \beta \cos \beta)} rt d\theta$$

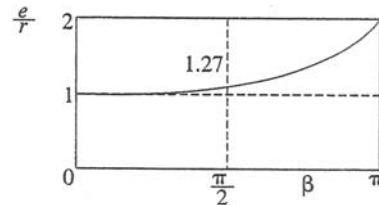
$$= \frac{2Vr(\sin \beta - \beta \cos \beta)}{\beta - \sin \beta \cos \beta}$$

Shear force V acts through the shear center S . Moment of the shear force V about any point must be equal to the moment of the shear stresses about that same point.

$$\therefore \sum M_0 = Ve = T_0 \quad e = T_0/V$$

$$e = \frac{2r(\sin \beta - \beta \cos \beta)}{\beta - \sin \beta \cos \beta} \quad \leftarrow$$

GRAPH



$$\frac{e}{r} = \frac{2(\sin \beta - \beta \cos \beta)}{\beta - \sin \beta \cos \beta}$$

SEMICIRCULAR ARC ($\beta = \pi/2$):

$$\frac{e}{r} = \frac{4}{\pi} \quad (\text{Eq. 6-81})$$

SLIT CIRCULAR ARC ($\beta = \pi$):

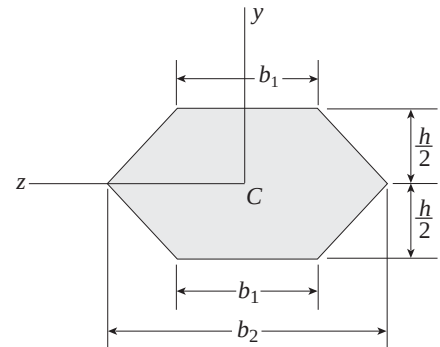
$$\frac{e}{r} = 2 \quad (\text{Prob. 6.9-6})$$

Elastoplastic Bending

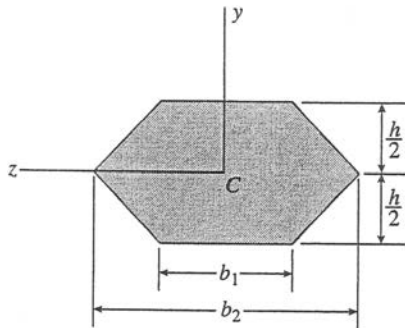
The problems for Section 6.10 are to be solved using the assumption that the material is elastoplastic with yield stress σ_Y .

Problem 6.10-1 Determine the shape factor f for a cross section in the shape of a double trapezoid having the dimensions shown in the figure.

Also, check your result for the special cases of a rhombus ($b_1 = 0$) and a rectangle ($b_1 = b_2$).



Solution 6.10-1 Double trapezoid



Neutral axis passes through the centroid C .

Use Case 8, Appendix E.

SECTION MODULUS S

$$I_z = 2 \left(\frac{h}{2} \right)^3 (3b_1 + b_2) / 12$$

$$= \frac{h^3}{48} (3b_1 + b_2)$$

$$c = h/2 \quad S = \frac{I}{c} = \frac{h^2}{24} (3b_1 + b_2)$$

PLASTIC MODULUS Z (EQ. 6-89)

$$A = 2 \left(\frac{h}{2} \right) (b_1 + b_2) / 2 = \frac{h}{2} (b_1 + b_2)$$

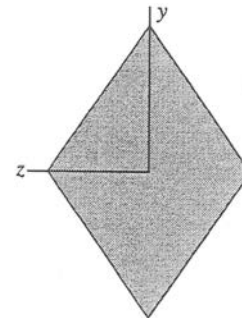
$$\bar{y}_1 = \bar{y}_2 = \frac{1}{3} \left(\frac{h}{2} \right) \left(\frac{2b_1 + b_2}{b_1 + b_2} \right)$$

$$z = \frac{A}{2} (\bar{y}_1 + \bar{y}_2) = \frac{h^2}{12} (2b_1 + b_2)$$

SHAPE FACTOR f (EQ. 6-90)

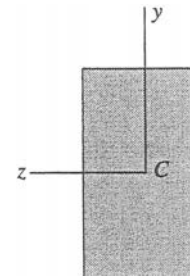
$$f = \frac{Z}{S} = \frac{2(2b_1 + b_2)}{3b_1 + b_2} \leftarrow$$

SPECIAL CASE – RHOMBUS



$$b_1 = 0 \quad f = 2$$

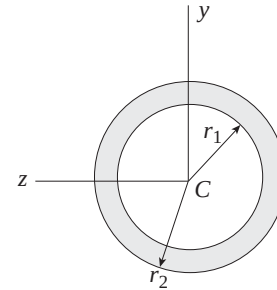
SPECIAL CASE – RECTANGLE



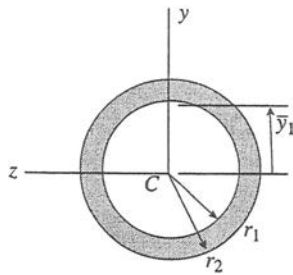
$$b_1 = b_2 \quad f = \frac{3}{2}$$

Problem 6.10-2 (a) Determine the shape factor f for a hollow circular cross section having inner radius r_1 and outer radius r_2 (see figure).

(b) If the section is very thin, what is the shape factor?



Solution 6.10-2 Hollow circular cross sections



Neutral axis passes through the centroid C .

Use Cases 9 and 10, Appendix E.

SECTION MODULUS S

$$I_z = \frac{\pi}{4}(r_2^4 - r_1^4) \quad c = r_2$$

$$S = \frac{I_z}{c} = \frac{\pi}{4r_2}(r_2^4 - r_1^4)$$

PLASTIC MODULUS Z (Eq. 6-89)

$$A = \pi(r_2^2 - r_1^2) \quad \text{For a semicircle, } \bar{y} = \frac{4r}{3\pi}$$

$$\begin{aligned} \bar{y}_1 &= \frac{\sum \bar{y}_i A_i}{\sum A_i} = \frac{\left(\frac{4r_2}{3\pi}\right)\left(\frac{\pi r_2^2}{2}\right) - \left(\frac{4r_1}{3\pi}\right)\left(\frac{\pi r_1^2}{2}\right)}{\pi/2(r_2^2 - r_1^2)} \\ &= \frac{4}{3\pi} \left(\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} \right) \end{aligned}$$

$$\bar{y}_1 = \bar{y}_2 \quad Z = \frac{A}{2} (\bar{y}_1 + \bar{y}_2) = \frac{4}{3} (r_2^3 - r_1^3)$$

(a) SHAPE FACTOR f (Eq. 6-90)

$$f = \frac{Z}{S} = \frac{16r_2(r_2^3 - r_1^3)}{3\pi(r_2^4 - r_1^4)} \quad \leftarrow$$

(b) THIN SECTION ($r_1 \rightarrow r_2$)

Rewrite the expression for the shape factor f .

$$(r_2^3 - r_1^3) = (r_2 - r_1)(r_2^2 + r_1 r_2 + r_1^2)$$

$$(r_2^4 - r_1^4) = (r_2 - r_1)(r_2 + r_1)(r_2^2 + r_1^2)$$

$$\begin{aligned} f &= \frac{16r_2 \left[\frac{r_2^2 + r_1 r_2 + r_1^2}{(r_2 + r_1)(r_2^2 + r_1^2)} \right]}{3\pi} \\ &= \frac{16}{3\pi} \left[\frac{1 + r_1/r_2 + (r_1/r_2)^2}{(1 + r_1/r_2)(1 + r_1^2/r_2^2)} \right] \end{aligned}$$

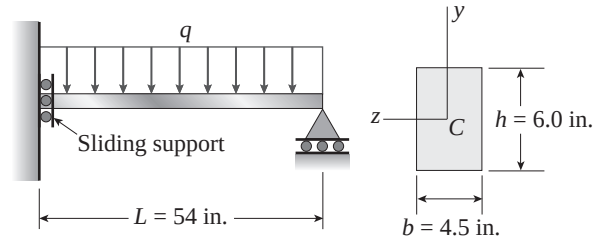
$$\text{Let } r_1/r_2 \rightarrow 1 \quad f = \frac{16}{3\pi} \left(\frac{3}{4} \right) = \frac{4}{\pi} \approx 1.27 \quad \leftarrow$$

SPECIAL CASE OF A SOLID CIRCULAR CROSS SECTION

$$\text{Let } r_1 = 0 \quad f = \frac{16}{3\pi} \left(\frac{1}{1} \right) = \frac{16}{3\pi} \quad (\text{Eq. 6-105})$$

Problem 6.10-3 A propped cantilever beam of length $L = 54$ in. with a sliding support supports a uniform load of intensity q (see figure). The beam is made of steel ($\sigma_Y = 36$ ksi) and has a rectangular cross section of width $b = 4.5$ in. and height $h = 6.0$ in.

What load intensity q will produce a fully plastic condition in the beam?



Solution 6.10-3

$$L = 54 \text{ in.} \quad \sigma_Y = 36 \text{ ksi} \quad b = 4.5 \text{ in.} \quad h = 6.0 \text{ in.}$$

$$\text{MAXIMUM BENDING MOMENT: } M_{\max} = \frac{qL^2}{2}$$

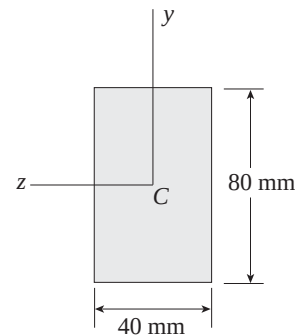
$$\text{PLASTIC MOMENT: } M_P = \frac{\sigma_Y b h^2}{4}$$

$$\text{Let } M_{\max} = M_P \quad \text{gives } q = \frac{\sigma_Y b h^2}{2L^2}$$

$$\text{Therefore, } q = 1000 \text{ lb/in.} \quad \leftarrow$$

Problem 6.10-4 A steel beam of rectangular cross section is 40 mm wide and 80 mm high (see figure). The yield stress of the steel is 210 MPa.

- What percent of the cross-sectional area is occupied by the elastic core if the beam is subjected to a bending moment of 12.0 kN·m acting about the z axis?
- What is the magnitude of the bending moment that will cause 50% of the cross section to yield?



Solution 6.10-4

$$\sigma_Y = 210 \text{ MPa} \quad b = 40 \text{ mm} \quad h = 80 \text{ mm}$$

(a) ELASTIC CORE

$$M = 12.0 \text{ kN} \cdot \text{m} \quad M_Y = \frac{\sigma_Y b h^2}{6}$$

$$M_Y = 9.0 \text{ kN} \cdot \text{m}$$

$$M_P = \frac{\sigma_Y b h^2}{4} \quad M_P = 13.4 \text{ kN} \cdot \text{m}$$

$$M \text{ is between } M_Y \text{ and } M_P$$

$$e = h \sqrt{\frac{1}{2} \left(\frac{3}{2} - \frac{M}{M_Y} \right)} \quad e = 22.678 \text{ mm}$$

Percent of cross-sectional area is

$$\frac{2e}{h} = 56.7\% \quad \leftarrow$$

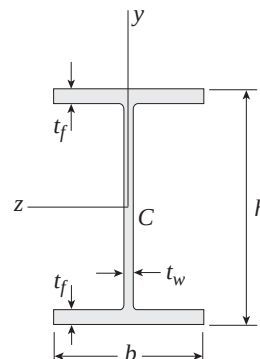
(b) ELASTIC CORE

$$e = \frac{h}{4} \quad e = 20 \text{ mm}$$

$$M = M_Y \left(\frac{3}{2} - \frac{2e^2}{h^2} \right)$$

$$M = 12.3 \text{ kN} \cdot \text{m} \quad \leftarrow$$

Problem 6.10-5 Calculate the shape factor f for the wide-flange beam shown in the figure if $h = 12.2$ in., $b = 8.08$ in., $t_f = 0.64$ in., and $t_w = 0.37$ in.



Probs. 6.10-5 and 6.10-6

Solution 6.10-5

$$h = 12.2 \text{ in.} \quad b = 8.08 \text{ in.}$$

$$t_f = 0.64 \text{ in.} \quad t_w = 0.37 \text{ in.}$$

SECTION MODULUS

$$I = \frac{1}{12}bh^3 - \frac{1}{12}(b - t_w)(h - 2t_f)^3$$

$$I = 386.0 \text{ in.}^4$$

$$c = \frac{h}{2} \quad c = 6.1 \text{ in.} \quad S = \frac{I}{c} \quad S = 63.3 \text{ in.}^3$$

PLASTIC MODULUS

$$Z = \frac{1}{4}[bh^2 - (b - t_w)(h - 2t_f)^2]$$

$$Z = 70.8 \text{ in.}^3$$

SHAPE FACTOR

$$f = \frac{Z}{S} \quad f = 1.12 \quad \leftarrow$$

Problem 6.10-6 Solve the preceding problem for a wide-flange beam with $h = 404$ mm, $b = 140$ mm, $t_f = 11.2$ mm, and $t_w = 6.99$ mm.

Solution 6.10-6

$$h = 404 \text{ mm} \quad b = 140 \text{ mm}$$

$$t_f = 11.2 \text{ mm} \quad t_w = 6.99 \text{ mm}$$

SECTION MODULUS

$$I = \frac{1}{12}bh^3 - \frac{1}{12}(b - t_w)(h - 2t_f)^3$$

$$I = 153.4 \times 10^6 \text{ mm}^4$$

$$c = \frac{h}{2} \quad c = 202.0 \text{ mm} \quad S = \frac{I}{c}$$

$$S = 759.2 \times 10^3 \text{ mm}^3$$

PLASTIC MODULUS

$$Z = \frac{1}{4}[bh^2 - (b - t_w)(h - 2t_f)^2]$$

$$Z = 870.4 \times 10^3 \text{ mm}^3$$

SHAPE FACTOR

$$f = \frac{Z}{S} \quad f = 1.15 \quad \leftarrow$$

Problem 6.10-7 Determine the plastic modulus Z and shape factor f for a W 12 $\times$ 14 wide-flange beam. (Note: Obtain the cross-sectional dimensions and section modulus of the beam from Table F-1a of Appendix F.)

Solution 6.10-7

W 12 $\times$ 14

$$\begin{aligned} h &= 11.9 \text{ in.} & b &= 3.97 \text{ in.} & t_f &= 0.225 \text{ in.} \\ t_w &= 0.200 \text{ in.} & S &= 14.9 \text{ in.}^3 \end{aligned}$$

SHAPE FACTOR

$$f = \frac{Z}{S} \quad f = 1.14 \quad \leftarrow$$

PLASTIC MODULUS

$$Z = \frac{1}{4}[bh^2 - (b - t_w)(h - 2t_f)^2]$$

$$Z = 16.98 \text{ in.}^3 \quad \leftarrow$$

Problem 6.10-8 Solve the preceding problem for a W 250 $\times$ 89 wide-flange beam. (Note: Obtain the cross-sectional dimensions and section modulus of the beam from Table F-1b of Appendix F.)

Solution 6.10-8

W 250 $\times$ 89

$$\begin{aligned} h &= 259 \text{ mm} & b &= 257 \text{ mm} & t_f &= 17.3 \text{ mm} \\ t_w &= 10.7 \text{ mm} & S &= 1090 \times 10^3 \text{ mm}^3 \end{aligned}$$

SHAPE FACTOR

$$f = \frac{Z}{S} \quad f = 1.11 \quad \leftarrow$$

PLASTIC MODULUS

$$Z = \frac{1}{4}[bh^2 - (b - t_w)(h - 2t_f)^2]$$

$$Z = 1.209 \times 10^6 \text{ mm}^3 \quad \leftarrow$$

Problem 6.10-9 Determine the yield moment M_Y , plastic moment M_P , and shape factor f for a W 16 $\times$ 100 wide-flange beam if $\sigma_Y = 36$ ksi. (Note: Obtain the cross-sectional dimensions and section modulus of the beam from Table F-1a of Appendix F.)

Solution 6.10-9

W 16 $\times$ 100

$$\begin{aligned} h &= 17.0 \text{ in.} & b &= 10.4 \text{ in.} & t_f &= 0.985 \text{ in.} \\ t_w &= 0.585 \text{ in.} & S &= 175 \text{ in.}^3 & \sigma_Y &= 36 \text{ ksi} \end{aligned}$$

YIELD MOMENT

$$M_Y = \sigma_Y S \quad M_Y = 525 \text{ k-ft} \quad \leftarrow$$

PLASTIC MODULUS

$$Z = \frac{1}{4}[bh^2 - (b - t_w)(h - 2t_f)^2] \quad Z = 197.1 \text{ in.}^3$$

SHAPE FACTOR

$$f = \frac{Z}{S} \quad f = 1.13 \quad \leftarrow$$

PLASTIC MOMENT

$$M_P = \sigma_Y Z \quad M_P = 591 \text{ k-ft} \quad \leftarrow$$

Problem 6.10-10 Solve the preceding problem for a W 410 $\times$ 85 wide-flange beam. Assume that $\sigma_Y = 250$ MPa. (Note: Obtain the cross-sectional dimensions and section modulus of the beam from Table F-1b of Appendix F.)

Solution 6.10-10W 410 $\times$ 85

$$h = 417 \text{ mm} \quad b = 181 \text{ mm} \quad t_f = 18.2 \text{ mm}$$

$$t_w = 10.9 \text{ mm} \quad S = 1510 \cdot 10^3 \text{ mm}^3$$

$$\sigma_Y = 250 \text{ MPa}$$

YIELD MOMENT

$$M_Y = \sigma_Y S \quad M_Y = 378 \text{ kN} \cdot \text{m} \quad \leftarrow$$

PLASTIC MODULUS

$$Z = \frac{1}{4} \left[bh^2 - (b - t_w)(h - 2t_f)^2 \right]$$

$$Z = 1.708 \times 10^6 \text{ mm}^3$$

PLASTIC MOMENT

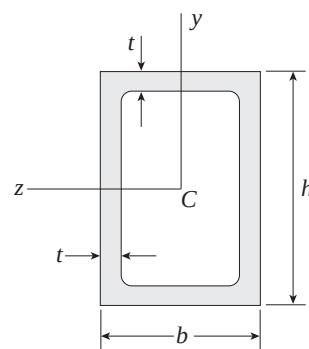
$$M_P = \sigma_Y Z \quad M_P = 427 \text{ kN} \cdot \text{m} \quad \leftarrow$$

SHAPE FACTOR

$$f = \frac{Z}{S} \quad f = 1.13 \quad \leftarrow$$

Problem 6.10-11 A hollow box beam with height $h = 16$ in., width $b = 8$ in., and constant wall thickness $t = 0.75$ in. is shown in the figure. The beam is constructed of steel with yield stress $\sigma_Y = 32$ ksi.

Determine the yield moment M_Y , plastic moment M_P , and shape factor f .



Probs. 6.10-11 and 6.10-12

Solution 6.10-11 Hollow box beam

$$h = 16 \text{ in.} \quad b = 8 \text{ in.}$$

$$t = 0.75 \text{ in.} \quad \sigma_Y = 32 \text{ ksi}$$

SECTION MODULUS ($S = I/c$)

$$I = \frac{1}{12}bh^3 - \frac{1}{12}(b-2t)(h-2t)^3$$

$$= 1079 \text{ in.}^4$$

$$c = \frac{h}{2} = 8.0 \text{ in.} \quad S = \frac{I}{c} = 134.9 \text{ in.}^3$$

YIELD MOMENT (Eq. 6-82)

$$M_Y = \sigma_Y S = 4320 \text{ k-in.} \quad \leftarrow$$

PLASTIC MODULUS

Use (Eq. 6-86) with $t_w = 2t$ and $t_f = t$:

$$Z = \frac{1}{4}[bh^2 - (b-2t)(h-2t)^2]$$

$$= 170.3 \text{ in.}^3$$

PLASTIC MOMENT (Eq. 6-88)

$$M_P = \sigma_Y Z = 5450 \text{ k-in.} \quad \leftarrow$$

SHAPE FACTOR (Eq. 6-90)

$$f = \frac{M_P}{M_Y} = \frac{Z}{S} = 1.26 \quad \leftarrow$$

Problem 6.10-12 Solve the preceding problem for a box beam with dimensions $h = 0.5 \text{ m}$, $b = 0.18 \text{ m}$, and $t = 22 \text{ mm}$. The yield stress of the steel is 210 MPa .

Solution 6.10-12

$$h = 0.5 \text{ m} \quad b = 0.18 \text{ m} \quad t = 22 \text{ mm}$$

$$\sigma_Y = 210 \text{ MPa}$$

SECTION MODULUS

$$I = \frac{1}{12}bh^3 - \frac{1}{12}(b-2t)(h-2t)^3$$

$$I = 800.4 \times 10^6 \text{ mm}^4$$

$$c = \frac{h}{2} \quad c = 250 \text{ mm}$$

$$S = \frac{I}{c} \quad S = 3.202 \times 10^6 \text{ mm}^3$$

YIELD MOMENT

$$M_Y = \sigma_Y S \quad M_Y = 672 \text{ kN} \cdot \text{m} \quad \leftarrow$$

PLASTIC MODULUS

$$Z = \frac{1}{4}[bh^2 - (b-2t)(h-2t)^2]$$

$$Z = 4.180 \times 10^6 \text{ mm}^3$$

PLASTIC MOMENT

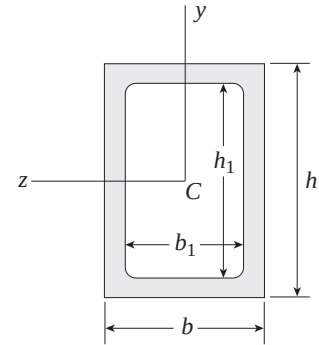
$$M_P = \sigma_Y Z \quad M_P = 878 \text{ kN} \cdot \text{m} \quad \leftarrow$$

SHAPE FACTOR

$$f = \frac{Z}{S} \quad f = 1.31 \quad \leftarrow$$

Problem 6.10-13 A hollow box beam with height $h = 9.5$ in., inside height $h_1 = 8.0$ in., width $b = 5.25$ in., and inside width $b_1 = 4.5$ in. is shown in the figure.

Assuming that the beam is constructed of steel with yield stress $\sigma_Y = 42$ ksi, calculate the yield moment M_Y , plastic moment M_P , and shape factor f .



Probs. 6.10-13 through 6.10-16

Solution 6.10-13

$$h = 9.5 \text{ in.} \quad b = 5.25 \text{ in.} \quad h_1 = 8.0 \text{ in.}$$

$$b_1 = 4.5 \text{ in.} \quad \sigma_Y = 42 \text{ ksi}$$

SECTION MODULUS

$$I = \frac{1}{12} (bh^3 - b_1h_1^3) \quad I = 183.10 \text{ in.}^4$$

$$c = \frac{h}{2} \quad c = 4.75 \text{ in.} \quad S = \frac{I}{c} \quad S = 38.55 \text{ in.}^3$$

YIELD MOMENT

$$M_Y = \sigma_Y S \quad M_Y = 1619 \text{ k-in.} \quad \leftarrow$$

PLASTIC MODULUS

$$Z = \frac{1}{4} (bh^2 - b_1h_1^2) \quad Z = 46.5 \text{ in.}^3$$

PLASTIC MOMENT

$$M_P = \sigma_Y Z \quad M_P = 1951 \text{ k-in.} \quad \leftarrow$$

SHAPE FACTOR

$$f = \frac{Z}{S} \quad f = 1.21 \quad \leftarrow$$

Problem 6.10-14 Solve the preceding problem for a box beam with dimensions $h = 200$ mm, $h_1 = 160$ mm, $b = 150$ mm, and $b_1 = 130$ mm. Assume that the beam is constructed of steel with yield stress $\sigma_Y = 220$ MPa.

Solution 6.10-14 Hollow box beam

$$h = 200 \text{ mm} \quad b = 150 \text{ mm}$$

$$h_1 = 160 \text{ mm} \quad b_1 = 130 \text{ mm} \quad \sigma_Y = 220 \text{ MPa}$$

SECTION MODULUS ($S = I/c$)

$$I = \frac{1}{12} (bh^3 - b_1h_1^3) = 55.63 \times 10^6 \text{ mm}^4$$

$$c = \frac{h}{2} = 100 \text{ mm} \quad S = \frac{I}{c} = 556.3 \times 10^3 \text{ mm}^3$$

YIELD MOMENT (Eq. 6-82)

$$M_Y = \sigma_Y S = 122 \text{ kN} \cdot \text{m} \quad \leftarrow$$

PLASTIC MODULUS

Use (Eq. 6-101) with $b - t_w = b_1$ and $h - 2t_f = h_1$

$$Z = \frac{1}{4} (bh^2 - b_1h_1^2) = 668.0 \times 10^3 \text{ mm}^3$$

PLASTIC MOMENT (Eq. 6-88)

$$M_P = \sigma_Y Z = 147 \text{ kN} \cdot \text{m} \quad \leftarrow$$

SHAPE FACTOR (Eq. 6-90)

$$f = \frac{M_P}{M_Y} = \frac{Z}{S} = 1.20 \quad \leftarrow$$

Problem 6.10-15 The hollow box beam shown in the figure is subjected to a bending moment M of such magnitude that the flanges yield but the webs remain linearly elastic.

- (a) Calculate the magnitude of the moment M if the dimensions of the cross section are $h = 15$ in., $h_1 = 12.75$ in., $b = 9$ in., and $b_1 = 7.5$ in. Also, the yield stress is $\sigma_Y = 33$ ksi.
 (b) What percent of the moment M is produced by the elastic core?

Solution 6.10-15

$$h = 15 \text{ in.} \quad b = 9 \text{ in.} \quad h_1 = 12.75 \text{ in.} \\ b_1 = 7.5 \text{ in.} \quad \sigma_Y = 33 \text{ ksi}$$

ELASTIC CORE

$$S_1 = \frac{1}{6}(b - b_1)h_1^2 \quad S_1 = 40.64 \text{ in.}^3$$

$$M_1 = \sigma_Y S_1 \quad M_1 = 1341 \text{ k-in.}$$

PLASTIC FLANGES

F = force in one flange

$$F = \sigma_Y b \left(\frac{1}{2} \right) (h - h_1) \quad F = 334.1 \text{ k}$$

$$M_2 = F \left(\frac{h + h_1}{2} \right) \quad M_2 = 4636 \text{ k-in.}$$

(a) BENDING MOMENT

$$M = M_1 + M_2 \quad M = 5977 \text{ k-in.} \quad \leftarrow$$

(b) PERCENT DUE TO ELASTIC CORE

$$\frac{M_1}{M} = 22.4\% \quad \leftarrow$$

Problem 6.10-16 Solve the preceding problem for a box beam with dimensions $h = 400$ mm, $h_1 = 360$ mm, $b = 200$ mm, and $b_1 = 160$ mm, and with yield stress $\sigma_Y = 220$ MPa.

Solution 6.10-16 Hollow box beam

$$h = 400 \text{ mm} \quad b = 200 \text{ mm} \\ h_1 = 360 \text{ mm} \quad b_1 = 160 \text{ mm} \quad \sigma_Y = 220 \text{ MPa} \\ \text{(see Fig. 6-51, Example 6-11)}$$

ELASTIC CORE

$$S_1 = \frac{1}{6}(b - b_1)h_1^2 = 864 \times 10^3 \text{ mm}^3$$

$$M_1 = \sigma_Y S_1 = 190.1 \text{ kN} \cdot \text{m}$$

PLASTIC FLANGES

F = force in one flange

$$F = \sigma_Y b \left(\frac{1}{2} \right) (h - h_1) = 880.0 \text{ kN}$$

$$M_2 = F \left(\frac{h + h_1}{2} \right) = 334.4 \text{ kN} \cdot \text{m}$$

(a) BENDING MOMENT

$$M = M_1 + M_2 = 524 \text{ kN} \cdot \text{m} \quad \leftarrow$$

(b) PERCENT DUE TO ELASTIC CORE

$$\text{Percent} = \frac{M_1}{M} (100) = 36\% \quad \leftarrow$$

Problem 6.10-17 A W 10 × 60 wide-flange beam is subjected to a bending moment M of such magnitude that the flanges yield but the web remains linearly elastic.

- (a) Calculate the magnitude of the moment M if the yield stress is $\sigma_Y = 36$ ksi.
 (b) What percent of the moment M is produced by the elastic core?

Solution 6.10-17

W 10 × 60

$$h = 10.2 \text{ in.} \quad b = 10.1 \text{ in.} \quad t_f = 0.680 \text{ in.}$$

$$t_w = 0.420 \text{ in.} \quad \sigma_Y = 36 \text{ ksi}$$

ELASTIC CORE

$$S_1 = \frac{1}{6}(h - 2t_f)^2 t_w \quad S_1 = 5.47 \text{ in.}^3$$

$$M_1 = \sigma_Y S_1 \quad M_1 = 196.9 \text{ k-in.}$$

PLASTIC FLANGES

F = force in one flange

$$F = \sigma_Y b t_f \quad F = 247.2 \text{ k}$$

$$M_2 = F(h - t_f) \quad M_2 = 2354 \text{ k-in.}$$

(a) BENDING MOMENT

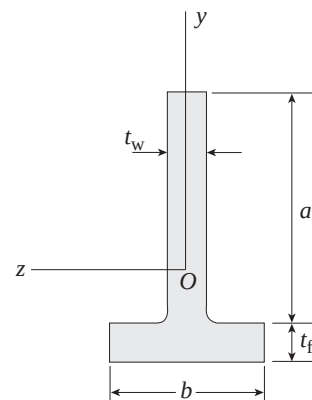
$$M = M_1 + M_2 \quad M = 2551 \text{ k-in.} \quad \leftarrow$$

(b) PERCENT DUE TO ELASTIC CORE

$$\frac{M_1}{M} = 7.7\% \quad \leftarrow$$

Problem 6.10-18 A singly symmetric beam of T-section (see figure) has cross-sectional dimensions $b = 140$ mm, $a = 190.8$ mm, $t_w = 6.99$ mm, and $t_f = 11.2$ mm.

Calculate the plastic modulus Z and the shape factor f .



Solution 6.10-18

$$b = 140 \text{ mm} \quad a = 190.8 \text{ mm} \quad t_w = 6.99 \text{ mm}$$

$$t_f = 11.2 \text{ mm}$$

ELASTIC BENDING

$$c_2 = \frac{\left(\frac{t_f}{2}\right) b t_f + \left(\frac{a}{2} + t_f\right) a t_w}{b t_f + a t_w} \quad c_2 = 52.02 \text{ mm}$$

$$c_1 = a + t_f - c_2 \quad c_1 = 149.98 \text{ mm}$$

$$I_z = \frac{1}{3} t_w c_1^3 + \frac{1}{3} b c_2^3 - \frac{1}{3} (b - t_w) (c_2 - t_f)^3$$

$$I_z = 11.41 \times 10^6 \text{ mm}^4$$

$$S = \frac{I_z}{c_1} \quad S = 76.1 \times 10^3 \text{ mm}^3$$

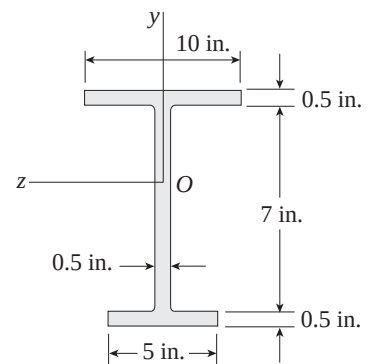
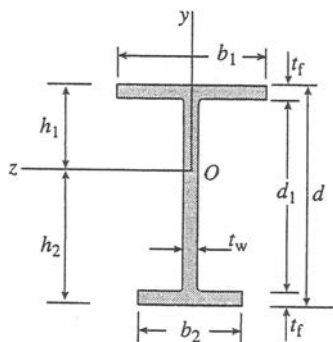
PLASTIC BENDING

$$\begin{aligned}
 A &= bt_f + at_w & A &= 2902 \text{ mm}^2 \\
 h_2 &= \frac{A}{2b} & h_2 &= 10.4 \text{ mm} \\
 h_1 &= a + t_f - h_2 & h_1 &= 191.6 \text{ mm} \\
 y_{2\_bar} &= h_2/2 & y_{2\_bar} &= 5.18 \text{ mm}
 \end{aligned}$$

$$\begin{aligned}
 y_{1\_bar} &= \frac{\frac{1}{2}(b - t_w)(t_f - h_2)^2 + \frac{1}{2}t_w h_1^2}{A/2} \\
 y_{1\_bar} &= 88.50 \text{ mm} \\
 Z &= \frac{A}{2}(y_{1\_bar} + y_{2\_bar}) \\
 Z &= 136 \times 10^3 \text{ mm}^3 \quad \leftarrow \\
 f &= \frac{Z}{S} \quad f = 1.79 \quad \leftarrow
 \end{aligned}$$

Problem 6.10-19 A wide-flange beam of unbalanced cross section has the dimensions shown in the figure.

Determine the plastic moment M_P if $\sigma_Y = 36$ ksi.

**Solution 6.10-19 Unbalanced wide-flange beam**

$$\begin{aligned}
 \sigma_Y &= 36 \text{ ksi} & b_1 &= 10 \text{ in.} & b_2 &= 5 \text{ in.} \\
 t_w &= 0.5 \text{ in.} & d &= 8 \text{ in.} & d_1 &= 7 \text{ in.} \\
 t_f &= 0.5 \text{ in.} & A &= b_1 t_f + b_2 t_f + d_1 t_w = 11.0 \text{ in.}^2
 \end{aligned}$$

NEUTRAL AXIS UNDER FULLY PLASTIC CONDITIONS

$$\frac{A}{2} = h_1 t_w + (b_1 - t_w) t_f$$

from which we get $h_1 = 1.50$ in.

$$h_2 = d - h_1 = 8.50 \text{ in.}$$

PLASTIC MODULUS

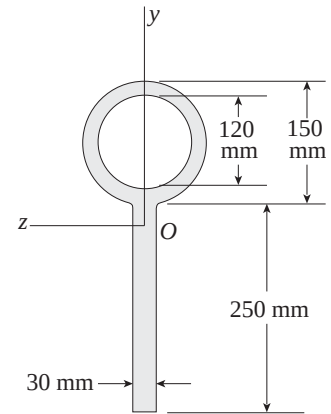
$$\begin{aligned}
 \bar{y}_1 &= \frac{\sum y_i A_i}{A/2} \\
 &= \frac{(h_1/2)(t_w)(h_1) + (h_1 - t_f/2)(b_1 - t_w)(t_f)}{A/2} \\
 &= 1.182 \text{ in.}
 \end{aligned}$$

$$\begin{aligned}
 \bar{y}_2 &= \frac{\sum y_i A_i}{A/2} \\
 &= \frac{(h_2/2)(t_w)(h_2) + (h_2 - t_f/2)(b_2 - t_w)(t_f)}{A/2} \\
 &= 4.477 \text{ in.} \\
 Z &= \frac{A}{2}(\bar{y}_1 + \bar{y}_2) = 31.12 \text{ in.}^3
 \end{aligned}$$

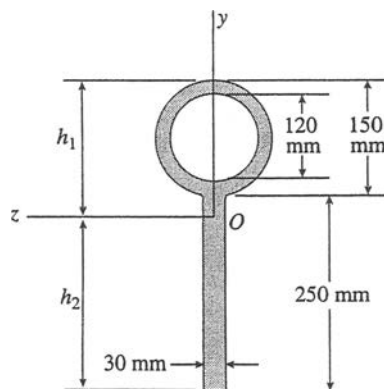
PLASTIC MOMENT

$$M_P = \sigma_y Z = 1120 \text{ k-in.} \quad \leftarrow$$

Problem 6.10-20 Determine the plastic moment M_P for a beam having the cross section shown in the figure if $\sigma_Y = 210 \text{ MPa}$.



Solution 6.10-20 Cross section of beam



$$\sigma_Y = 210 \text{ MPa} \quad d_2 = 150 \text{ mm} \quad d_1 = 120 \text{ mm}$$

NEUTRAL AXIS FOR FULLY PLASTIC CONDITIONS

Cross section is divided into two equal areas.

$$\begin{aligned}
 A &= \frac{\pi}{4} [(150 \text{ mm})^2 - (120 \text{ mm})^2] \\
 &\quad + (250 \text{ mm})(30 \text{ mm}) = 13,862 \text{ mm}^2
 \end{aligned}$$

$$\frac{A}{2} = 6931 \text{ mm}^2$$

$$(h_2)(30 \text{ mm}) = \frac{A}{2} = 6931 \text{ mm}^2$$

$$h_2 = 231.0 \text{ mm}$$

$$h_1 = 150 \text{ mm} + 250 \text{ mm} - h_2 = 169.0 \text{ mm}$$

PLASTIC MODULUS

$$\bar{y}_1 = \frac{\sum y_i A_i}{A/2} \text{ for upper half of cross section}$$

$$\bar{y}_2 = \frac{\sum y_i A_i}{A/2} \text{ for lower half of cross section}$$

$$Z = \frac{A}{2} (\bar{y}_1 + \bar{y}_2) = (\sum y_i A_i)_{\text{upper}} + (\sum y_i A_i)_{\text{lower}}$$

(Dimensions are in millimeters.)

$$\begin{aligned} Z &= (h_1 - 75) \left(\frac{\pi}{4} \right) (d_2^2 - d_1^2) \\ &\quad + \left[\left(\frac{h_1 - 150}{2} \right) (30)(h_1 - 150) \right] \\ &\quad + \left(\frac{h_2}{2} \right) (30)(h_2) \\ &= 598,000 + 5,400 + 800,400 \\ &= 1404 \times 10^3 \text{ mm}^3 \end{aligned}$$

PLASTIC MOMENT

$$\begin{aligned} M_P &= \sigma_P Z = (210 \text{ MPa})(1404 \times 10^3 \text{ mm}^3) \\ &= 295 \text{ kN} \cdot \text{m} \quad \leftarrow \end{aligned}$$

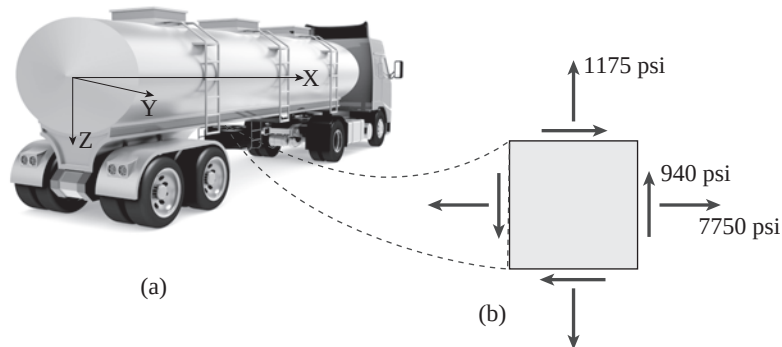
7

Analysis of Stress and Strain

Plane Stress

Problem 7.2-1 The stresses on the bottom surface of a fuel tanker (figure part a) are known to be $\sigma_x = 7750$ psi, $\sigma_y = 1175$ psi, and $\tau_{xy} = 940$ psi (figure part b).

Determine the stresses acting on an element oriented at an angle $\theta = 55^\circ$ from the x axis, where the angle θ is positive when counterclockwise. Show these stresses on a sketch of an element oriented at the angle θ .



Solution 7.2-1

$$\sigma_x = 7750 \text{ psi} \quad \sigma_y = 1175 \text{ psi} \quad \tau_{xy} = 940 \text{ psi}$$

$$\theta = 55^\circ$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\boxed{\sigma_{x1} = 4221 \text{ psi}}$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

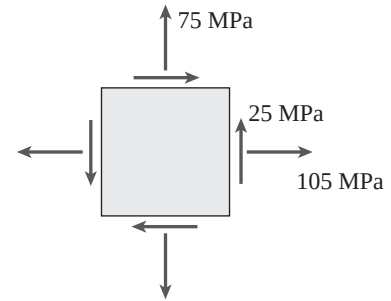
$$\boxed{\tau_{x1y1} = -3411 \text{ psi}}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

$$\boxed{\sigma_{y1} = 4704 \text{ psi}}$$

Problem 7.2-2 Solve the preceding problem for an element in *plane stress* on the bottom surface of a fuel tanker (figure part a); stresses are $\sigma_x = 105$ MPa, $\sigma_y = 75$ MPa, and $\tau_{xy} = 25$ MPa.

Determine the stresses acting on an element oriented at an angle $\theta = 40^\circ$ from the x axis, where the angle θ is positive when counterclockwise. Show these stresses on a sketch of an element oriented at the angle θ .



Solution 7.2-2

$$\sigma_x = 105 \text{ MPa} \quad \sigma_y = 75 \text{ MPa} \quad \tau_{xy} = 25 \text{ MPa}$$

$$\theta = 40^\circ$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\sigma_{x1} = 117.2 \text{ MPa}$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

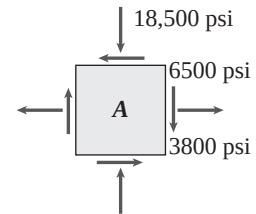
$$\tau_{x1y1} = -10.43 \text{ MPa}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

$$\sigma_{y1} = 62.8 \text{ MPa}$$

Problem 7.2-3 The stresses acting on element **A** on the web of a train rail are found to be 6500 psi tension in the horizontal direction and 18,500 psi compression in the vertical direction (see figure). Also, shear stresses of magnitude 3800 psi act in the directions shown.

Determine the stresses acting on an element oriented at a counterclockwise angle of 30° from the horizontal. Show these stresses on a sketch of an element oriented at this angle.



Solution 7.2-3

$$\sigma_x = 6500 \text{ psi} \quad \sigma_y = -18500 \text{ psi} \quad \tau_{xy} = -3800 \text{ psi}$$

$$\theta = 30^\circ$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\sigma_{x1} = -3041 \text{ psi}$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

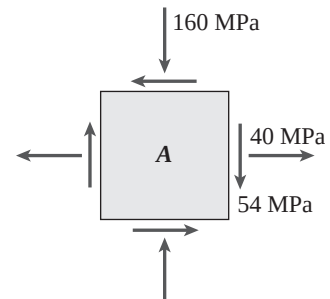
$$\tau_{x1y1} = -12,725 \text{ psi}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

$$\sigma_{y1} = -8959 \text{ psi}$$

Problem 7.2-4 Solve the preceding problem if the stresses acting on element **A** on the web of a train rail are found to be 40 MPa in tension in the horizontal direction and 160 MPa in compression in the vertical direction. Also, shear stresses of magnitude 54 MPa act in the directions shown.

Determine the stresses acting on an element oriented at a counterclockwise angle of 52° from the horizontal. Show these stresses on a sketch of an element oriented at this angle.



Solution 7.2-4

$$\sigma_x = 40 \text{ MPa} \quad \sigma_y = -160 \text{ MPa} \quad \tau_{xy} = -54 \text{ MPa}$$

$$\theta = 52^\circ$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\sigma_{x1} = -136.6 \text{ MPa}$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

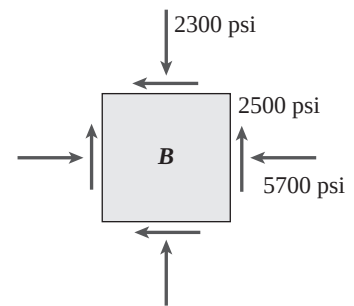
$$\tau_{x1y1} = -84 \text{ MPa}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

$$\sigma_{y1} = 16.6 \text{ MPa}$$

Problem 7.2-5 The stresses acting on element *B* on the web of a train rail are found to be 5700 psi in compression in the horizontal direction and 2300 psi in compression in the vertical direction (see figure). Also, shear stresses of magnitude 2500 psi act in the directions shown.

Determine the stresses acting on an element oriented at a counterclockwise angle of 50° from the horizontal. Show these stresses on a sketch of an element oriented at this angle.

**Solution 7.2-5**

$$\sigma_x = -5700 \text{ psi} \quad \sigma_y = -2300 \text{ psi} \quad \tau_{xy} = 2500 \text{ psi}$$

$$\theta = 50^\circ$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\sigma_{x1} = -1243 \text{ psi}$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

$$\tau_{x1y1} = 1240 \text{ psi}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

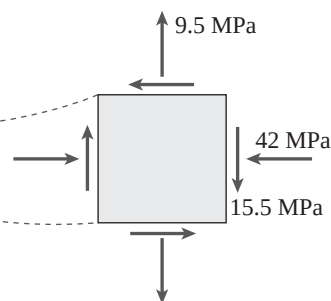
$$\sigma_{y1} = -6757 \text{ psi}$$

Problem 7.2-6 An element in *plane stress* on the fuselage of an airplane (figure part a) is subjected to compressive stresses of magnitude 42 MPa in the horizontal direction and tensile stresses of magnitude 9.5 MPa in the vertical direction (see figure part b). Also, shear stresses of magnitude 15.5 MPa act in the directions shown.

Determine the stresses acting on an element oriented at a clockwise angle of 40° from the horizontal. Show these stresses on a sketch of an element oriented at this angle.



(a)



(b)

Solution 7.2-6

$$\sigma_x = -42 \text{ MPa} \quad \sigma_y = 9.5 \text{ MPa} \quad \tau_{xy} = -15.5 \text{ MPa}$$

$$\theta = -40^\circ$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\boxed{\sigma_{x1} = -5.5 \text{ MPa}}$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

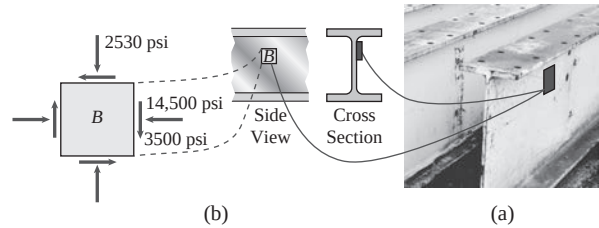
$$\boxed{\tau_{x1y1} = -28.1 \text{ MPa}}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

$$\boxed{\sigma_{y1} = -27 \text{ MPa}}$$

Problem 7.2-7 The stresses acting on element *B* (see figure part a) on the web of a wide-flange beam are found to be 14,500 psi in compression in the horizontal direction and 2530 psi in compression in the vertical direction (see figure part b). Also, shear stresses of magnitude 3500 psi act in the directions shown.

Determine the stresses acting on an element oriented at a counterclockwise angle of 38° from the horizontal. Show these stresses on a sketch of an element oriented at this angle.

**Solution 7.2-7**

$$\sigma_x = -14,500 \text{ psi} \quad \sigma_y = -2530 \text{ psi} \quad \tau_{xy} = -3500 \text{ psi}$$

$$\theta = 38^\circ$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\boxed{\sigma_{x1} = -13,359 \text{ psi}}$$

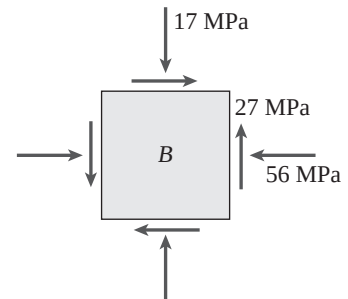
$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

$$\boxed{\tau_{x1y1} = 4960 \text{ psi}}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

$$\boxed{\sigma_{y1} = -3671 \text{ psi}}$$

Problem 7.2-8 Solve the preceding problem if the normal and shear stresses acting on element *B* are 56 MPa, 17 MPa, and 27 MPa (in the directions shown in the figure) and the angle is 40° (clockwise).

**Solution 7.2-8**

$$\sigma_x = -56 \text{ MPa} \quad \sigma_y = -17 \text{ MPa} \quad \tau_{xy} = 27 \text{ MPa}$$

$$\theta = -40^\circ$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\boxed{\sigma_{x1} = -66.5 \text{ MPa}}$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

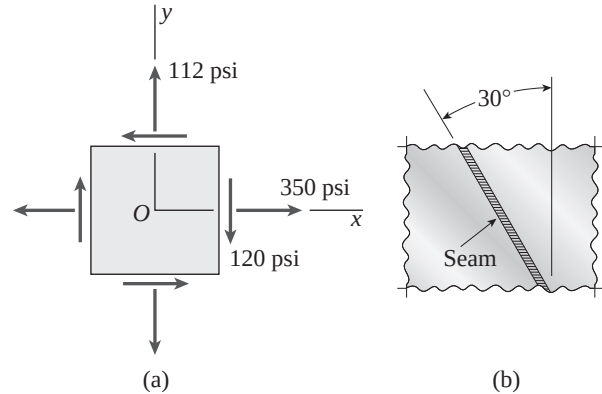
$$\boxed{\tau_{x1y1} = -14.52 \text{ MPa}}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

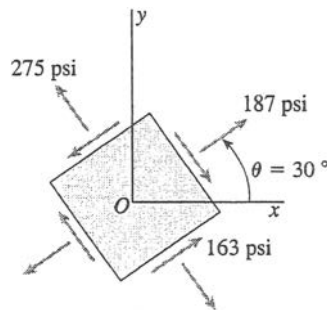
$$\boxed{\sigma_{y1} = -6.52 \text{ MPa}}$$

Problem 7.2-9 The polyethylene liner of a settling pond is subjected to stresses $\sigma_x = 350$ psi, $\sigma_y = 112$ psi, and $\tau_{xy} = -120$ psi, as shown by the plane-stress element in the first part of the figure part a.

Determine the normal and shear stresses acting on a seam oriented at an angle of 30° to the element, as shown in the figure part b. Show these stresses on a sketch of an element having its sides parallel and perpendicular to the seam.



Solution 7.2-9 Plane stress (angle θ)



$$\sigma_x = 350 \text{ psi} \quad \sigma_y = 112 \text{ psi} \quad \tau_{xy} = -120 \text{ psi}$$

$$\theta = 30^\circ$$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$= 187 \text{ psi} \quad \leftarrow$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

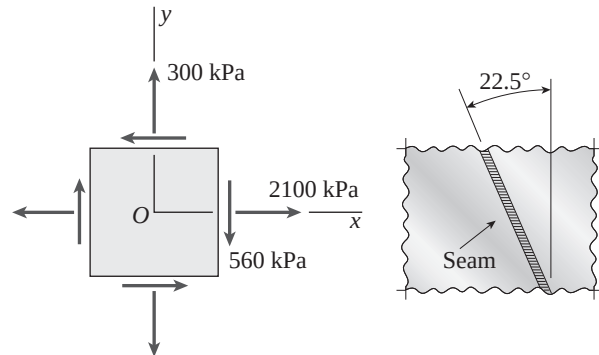
$$= -163 \text{ psi} \quad \leftarrow$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 275 \text{ psi} \quad \leftarrow$$

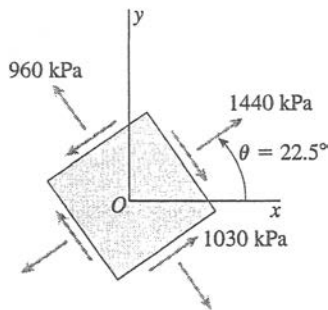
The normal stress on the seam equals 187 psi tension. $\leftarrow$

The shear stress on the seam equals 163 psi, acting clockwise against the seam. $\leftarrow$

Problem 7.2-10 Solve the preceding problem if the normal and shear stresses acting on the element are $\sigma_x = 2100$ kPa, $\sigma_y = 300$ kPa, and $\tau_{xy} = -560$ kPa, and the seam is oriented at an angle of 22.5° to the element.



Solution 7.2-10 Plane stress (angle θ)



$$\sigma_x = 2100 \text{ kPa} \quad \sigma_y = 300 \text{ kPa} \quad \tau_{xy} = -560 \text{ kPa}$$

$$\theta = 22.5^\circ$$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$= 1440 \text{ kPa} \quad \leftarrow$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$= -1030 \text{ kPa} \quad \leftarrow$$

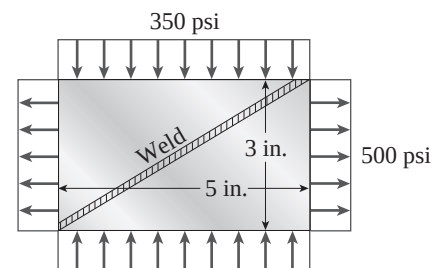
$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 960 \text{ kPa} \quad \leftarrow$$

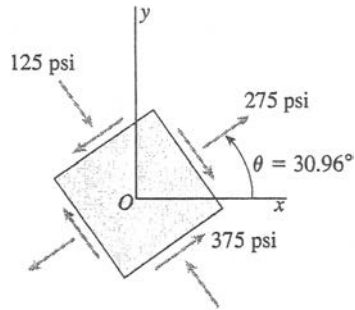
The normal stress on the seam equals 1440 kPa tension. $\leftarrow$

The shear stress on the seam equals 1030 kPa, acting clockwise against the seam. $\leftarrow$

Problem 7.2-11 A rectangular plate of dimensions 3.0 in. $\times$ 5.0 in. is formed by welding two triangular plates (see figure). The plate is subjected to a tensile stress of 500 psi in the long direction and a compressive stress of 350 psi in the short direction.

Determine the normal stress σ_w acting perpendicular to the line of the weld and the shear τ_w acting parallel to the weld. (Assume that the normal stress σ_w is positive when it acts in tension against the weld and the shear stress τ_w is positive when it acts counterclockwise against the weld.)



Solution 7.2-11 Biaxial stress (welded joint)

$$\sigma_x = 500 \text{ psi} \quad \sigma_y = -350 \text{ psi} \quad \tau_{xy} = 0$$

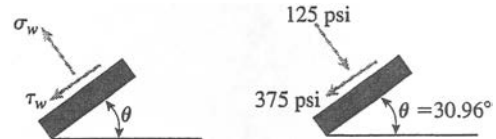
$$\theta = \arctan \frac{3 \text{ in.}}{5 \text{ in.}} = \arctan 0.6 = 30.96^\circ$$

$$\begin{aligned} \sigma_{x_1} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\ &= 275 \text{ psi} \end{aligned}$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta = -375 \text{ psi}$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = -125 \text{ psi}$$

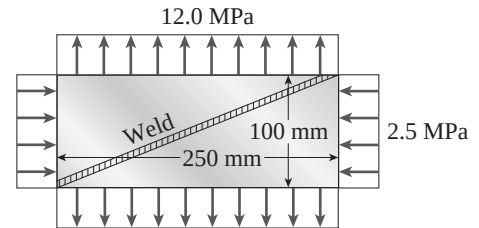
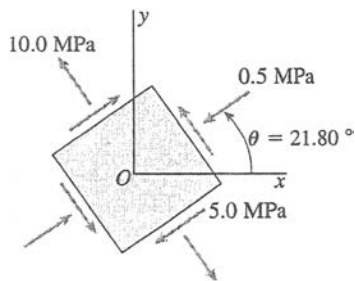
STRESSES ACTING ON THE WELD



$$\sigma_w = -125 \text{ psi} \quad \leftarrow$$

$$\tau_w = 375 \text{ psi} \quad \leftarrow$$

Problem 7.2-12 Solve the preceding problem for a plate of dimensions $100 \text{ mm} \times 250 \text{ mm}$ subjected to a compressive stress of 2.5 MPa in the long direction and a tensile stress of 12.0 MPa in the short direction (see figure).

**Solution 7.2-12 Biaxial stress (welded joint)**

$$\sigma_x = -2.5 \text{ MPa} \quad \sigma_y = 12.0 \text{ MPa} \quad \tau_{xy} = 0$$

$$\theta = \arctan \frac{100 \text{ mm}}{250 \text{ mm}} = \arctan 0.4 = 21.80^\circ$$

$$\begin{aligned} \sigma_{x_1} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\ &= -0.5 \text{ MPa} \end{aligned}$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta = 5.0 \text{ MPa}$$

$$\sigma_{y_1} = \sigma_x + \sigma_y - \sigma_{x_1} = 10.0 \text{ MPa}$$

STRESSES ACTING ON THE WELD

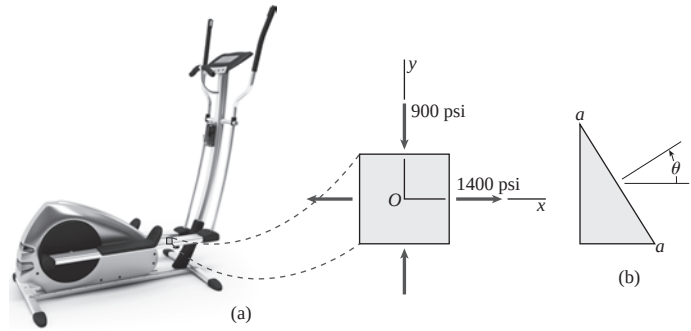


$$\sigma_w = 10.0 \text{ MPa} \quad \leftarrow$$

$$\tau_w = -5.0 \text{ MPa} \quad \leftarrow$$

Problem 7.2-13 At a point on the surface of an elliptical exercise machine the material is in *biaxial stress* with $\sigma_x = 1400$ psi and $\sigma_y = -900$ psi, as shown in figure part a. The figure part b shows an inclined plane *aa* cut through the same point in the material but oriented at an angle θ .

Determine the value of the angle θ between zero and 90° such that no normal stress acts on plane *aa*. Sketch a stress element having plane *aa* as one of its sides and show all stresses acting on the element.



Solution 7.2-13

$$\sigma_x = 1400 \text{ psi} \quad \sigma_y = -900 \text{ psi} \quad \tau_{xy} = 0 \text{ psi} \quad \sigma_{x1} = 0 \text{ psi}$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

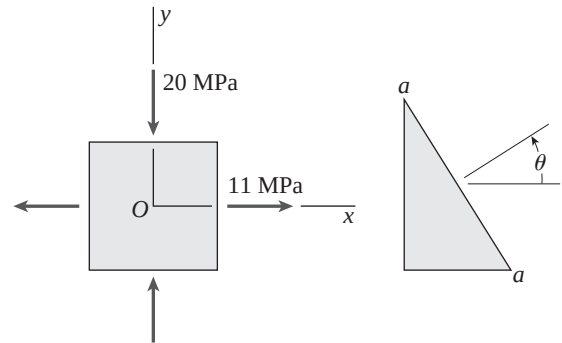
Solving for θ gives

$$\theta = 0.89497 \text{ rad} \quad \boxed{\theta = 51.3^\circ}$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta) = -1122 \text{ psi} \quad \boxed{\tau_{x1y1} = -1122 \text{ psi}}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1} \quad \boxed{\sigma_{y1} = 500 \text{ psi}}$$

Problem 7.2-14 Solve the preceding problem for $\sigma_x = 11$ MPa and $\sigma_y = -20$ MPa (see figure).



Solution 7.2-14

$$\sigma_x = 11 \text{ MPa} \quad \sigma_y = -20 \text{ MPa} \quad \tau_{xy} = 0 \text{ MPa} \quad \sigma_{x1} = 0 \text{ MPa}$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

Solving for θ gives

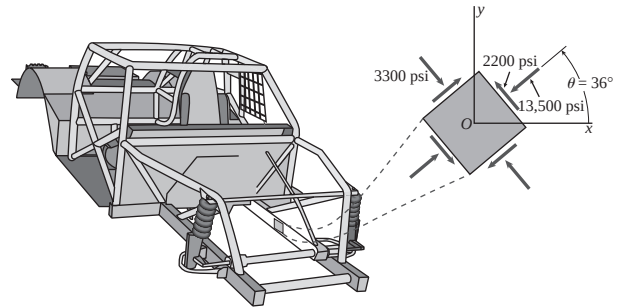
$$\theta = 0.63812 \text{ rad} \quad \boxed{\theta = 36.6^\circ}$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta) = -15 \text{ MPa} \quad \boxed{\tau_{x1y1} = -14.83 \text{ MPa}}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1} \quad \boxed{\sigma_{y1} = -9 \text{ MPa}}$$

Problem 7.2-15 An element in *plane stress* from the frame of a racing car is oriented at a known angle θ (see figure). On this inclined element, the normal and shear stresses have the magnitudes and directions shown in the figure.

Determine the normal and shear stresses acting on an element whose sides are parallel to the xy axes, that is, determine σ_x , σ_y , and τ_{xy} . Show the results on a sketch of an element oriented at $\theta = 0^\circ$.



Solution 7.2-15

TRANSFORM FROM $\theta = 36^\circ$ to $\theta = 0^\circ$

$$\sigma_x = -13,500 \text{ psi} \quad \sigma_y = -3300 \text{ psi} \quad \tau_{xy} = 2200 \text{ psi}$$

$$\theta = -36^\circ \quad (\text{rotate clockwise})$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\sigma_{x1} = -12,068 \text{ psi}$$

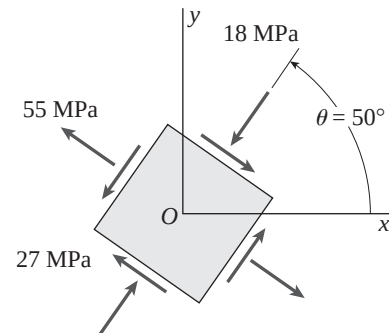
$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

$$\tau_{x1y1} = -4171 \text{ psi}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

$$\sigma_{y1} = -4732 \text{ psi}$$

Problem 7.2-16 Solve the preceding problem for the element shown in the figure.



Solution 7.2-16

TRANSFORM FROM $\theta = 50^\circ$ to $\theta = 0^\circ$

$$\sigma_x = -18 \text{ MPa} \quad \sigma_y = 55 \text{ MPa} \quad \tau_{xy} = -27 \text{ MPa}$$

$$\theta = -50^\circ \quad (\text{rotate clockwise})$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\sigma_{x1} = 51.4 \text{ MPa}$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

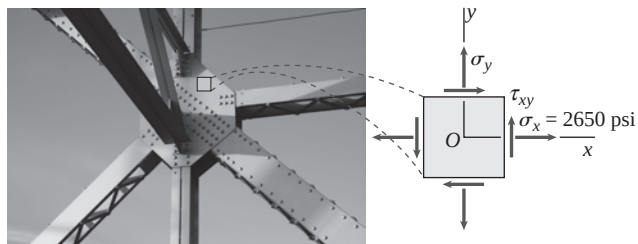
$$\tau_{x1y1} = -31.3 \text{ MPa}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1}$$

$$\sigma_{y1} = -14.4 \text{ MPa}$$

Problem 7.2-17 A gusset plate on a truss bridge is in *plane stress*, with normal stresses σ_x and σ_y and *shear stress* τ_{xy} , as shown in the figure. At counterclockwise angles $\theta = 32^\circ$ and $\theta = 78^\circ$ from the x axis, the normal stress is 4200 psi tension.

If the stress σ_x equals 2650 psi tension, what are the stresses σ_y and τ_{xy} ?



Solution 7.2-17

$\sigma_x = 2650$ psi σ_y unknown τ_{xy} unknown

At $\theta = 32^\circ$ and $\theta = 78^\circ$, $\sigma_{x1} = 4200$ psi

Find σ_y and τ_{xy} :

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

For $\theta = 32^\circ$:

$$\begin{aligned} \sigma_{x1} &= 4200 \text{ psi} \\ 4200 \text{ psi} &= \frac{2650 \text{ psi} + \sigma_y}{2} + \frac{2650 \text{ psi} - \sigma_y}{2} \cos(64^\circ) + \tau_{xy} \sin(64^\circ) \quad (1) \end{aligned}$$

For $\theta = 78^\circ$:

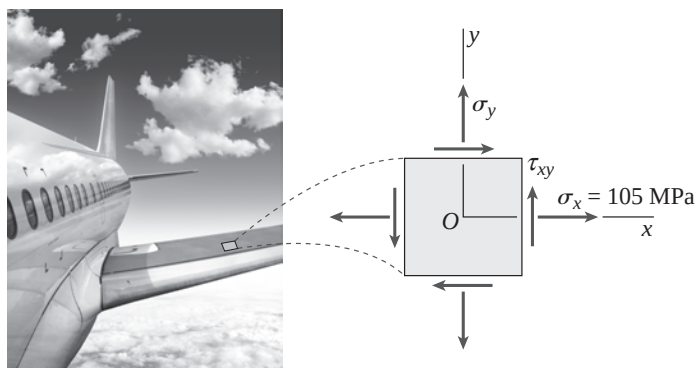
$$\begin{aligned} \sigma_{x1} &= 4200 \text{ psi} \\ 4200 \text{ psi} &= \frac{2650 \text{ psi} + \sigma_y}{2} + \frac{2650 \text{ psi} - \sigma_y}{2} \cos(156^\circ) + \tau_{xy} \sin(156^\circ) \quad (2) \end{aligned}$$

Solve Eqs. (1) and (2):

$$\boxed{\sigma_y = 3673 \text{ psi}} \quad \boxed{\tau_{xy} = 1405 \text{ psi}}$$

Problem 7.2-18 The surface of an airplane wing is subjected to *plane stress* with normal stresses σ_x and σ_y and shear stress τ_{xy} , as shown in the figure. At a counterclockwise angle $\theta = 32^\circ$ from the x axis, the normal stress is 29 MPa in tension, and at an angle $\theta = 46^\circ$, it is 17 MPa in compression.

If the stress σ_x equals 105 MPa in tension, what are the stresses σ_y and τ_{xy} ?



Solution 7.2-18

$\sigma_x = 105 \text{ MPa}$ σ_y unknown τ_{xy} unknown

At $\theta = 32^\circ$, $\sigma_{x1} = 29 \text{ MPa}$, and $\theta = 46^\circ$, $\sigma_{x1} = -17 \text{ MPa}$

Find σ_y and τ_{xy} :

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

For $\theta = 32^\circ$:

$$\sigma_{x1} = 29 \text{ MPa}$$

$$29 \text{ MPa} = \frac{105 \text{ MPa} + \sigma_y}{2} + \frac{105 \text{ MPa} - \sigma_y}{2} \cos(64^\circ) + \tau_{xy} \sin(64^\circ) \quad (1)$$

For $\theta = 46^\circ$:

$$\sigma_{x1} = -17 \text{ MPa}$$

$$-17 \text{ MPa} = \frac{105 \text{ MPa} + \sigma_y}{2} + \frac{105 \text{ MPa} - \sigma_y}{2} \cos(92^\circ) + \tau_{xy} \sin(92^\circ) \quad (2)$$

Solve Eqs. (1) and (2):

$$\boxed{\sigma_y = -77.7 \text{ MPa}} \quad \boxed{\tau_{xy} = -27.5 \text{ MPa}}$$

Problem 7.2-19 At a point on the web of a girder on an overhead bridge crane in a manufacturing facility, the stresses are known to be $\sigma_x = -4300 \text{ psi}$, $\sigma_y = 1700 \text{ psi}$, and $\tau_{xy} = 3100 \text{ psi}$ (the sign convention for these stresses is shown in Figure 7-1). A stress element located at the same point in the structure (but oriented at a counterclockwise angle θ_1 with respect to the x axis) is subjected to the stresses shown in the figure (σ_b , τ_b , and 2100 psi).

Assuming that the angle θ_1 is between zero and 90° , calculate the normal stress σ_b , the shear stress τ_b , and the angle θ_1 .

**Solution 7.2-19**

$\sigma_x = -4300 \text{ psi}$ $\sigma_y = 1700 \text{ psi}$ $\tau_{xy} = 3100 \text{ psi}$

For $\theta = \theta_1$:

$$\sigma_{x1} = 2100 \text{ psi} \quad \sigma_{y1} = \sigma_b \quad \tau_{x1y1} = \tau_b$$

Find σ_b , τ_b , and θ_1 :

Stress σ_b :

$$\sigma_b = \sigma_x + \sigma_y - 2100 \text{ psi} \quad \boxed{\sigma_b = -4700 \text{ psi}}$$

Angle θ_1 :

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

SOLVE NUMERICALLY:

$$\boxed{\theta_1 = 48.04^\circ}$$

Shear stress τ_b :

$$\tau_b = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta_1) + \tau_{xy} \cos(2\theta_1)$$

$$\boxed{\tau_b = 2655 \text{ psi}}$$

Principal Stresses and Maximum Shear Stresses

When solving the problems for Section 7.3, consider only the in-plane stresses (the stresses in the xy plane).

Problem 7.3-1 An element in *plane stress* is subjected to stresses $\sigma_x = 5750$ psi, $\sigma_y = 1100$ psi, and $\tau_{xy} = 750$ psi (see the figure for Prob. 7.2-1).

Determine the principal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-1

$$\sigma_x = 5750 \text{ psi} \quad \sigma_y = 1100 \text{ psi} \quad \tau_{xy} = 750 \text{ psi}$$

PRINCIPAL STRESSES

$$\theta_{p1} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p1} = 8.94^\circ$$

$$\theta_{p2} = \theta_{p1} + 90^\circ \quad \theta_{p2} = 98.94^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

$$\sigma_1 = 5868 \text{ psi}$$

$$\sigma_2 = 982 \text{ psi}$$

Problem 7.3-2 An element in *plane stress* is subjected to stresses $\sigma_x = 105$ MPa, $\sigma_y = 75$ MPa, and $\tau_{xy} = 25$ MPa (see the figure for Prob. 7.2-2).

Determine the principal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-2

$$\sigma_x = 105 \text{ MPa} \quad \sigma_y = 75 \text{ MPa} \quad \tau_{xy} = 25 \text{ MPa}$$

PRINCIPAL STRESSES

$$\theta_{p1} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p1} = 29.52^\circ$$

$$\theta_{p2} = \theta_{p1} + 90^\circ \quad \theta_{p2} = 119.52^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

$$\sigma_1 = 119.2 \text{ MPa}$$

$$\sigma_2 = 60.8 \text{ MPa}$$

Problem 7.3-3 An element in *plane stress* is subjected to stresses $\sigma_x = -5500$ psi, $\sigma_y = -2000$ psi, and $\tau_{xy} = 1900$ psi (see the figure for Prob. 7.2-3).

Determine the principal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-3

$$\sigma_x = -5500 \text{ psi} \quad \sigma_y = -2000 \text{ psi} \quad \tau_{xy} = 1900 \text{ psi}$$

PRINCIPAL STRESSES

$$\theta_{p1} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p1} = -23.68^\circ$$

$$\theta_{p2} = \theta_{p1} + 90^\circ \quad \theta_{p2} = 66.32^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

$$\sigma_1 = -6333 \text{ psi}$$

$$\sigma_2 = -1167 \text{ psi}$$

Problem 7.3-4 The stresses acting on element *A* in the web of a train rail are found to be 40 MPa tension in the horizontal direction and 160 MPa compression in the vertical direction (see figure). Also, shear stresses of magnitude 54 MPa act in the directions shown (see the figure for Prob. 7.2-4).

Determine the principal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-4

$$\sigma_x = 40 \text{ MPa} \quad \sigma_y = -160 \text{ MPa} \quad \tau_{xy} = -54 \text{ MPa}$$

PRINCIPAL STRESSES

$$\theta_{p1} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p1} = -14.2^\circ$$

$$\theta_{p2} = \theta_{p1} + 90^\circ \quad \theta_{p2} = 75.8^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

$$\sigma_1 = 53.6 \text{ MPa} \quad \leftarrow$$

$$\sigma_2 = -173.6 \text{ MPa} \quad \leftarrow$$

Problem 7.3-5 The normal and shear stresses acting on element *A* are 6500 psi, 17,300 psi, and 2900 psi (see the figure for Prob. 7.2-4).

Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-5

$$\sigma_x = 6500 \text{ psi} \quad \sigma_y = 17,300 \text{ psi} \quad \tau_{xy} = 2900 \text{ psi}$$

PRINCIPAL STRESSES

$$\theta_{p1} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p1} = -14.12^\circ$$

$$\theta_{p2} = \theta_{p1} + 90^\circ \quad \theta_{p2} = 75.88^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

$$\sigma_1 = 5771 \text{ psi}$$

$$\sigma_2 = 18,029 \text{ psi}$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = -6129 \text{ psi}$$

Problem 7.3-6 An element in *plane stress* from the fuselage of an airplane is subjected to compressive stresses of magnitude 35 MPa in the horizontal direction and tensile stresses of magnitude 6.5 MPa in the vertical direction. Also, shear stresses of magnitude 12.5 MPa act in the directions shown (see the figure for Prob. 7.2-6).

Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-6

$$\sigma_x = -35 \text{ MPa} \quad \sigma_y = 6.5 \text{ MPa} \quad \tau_{xy} = -12.5 \text{ MPa}$$

PRINCIPAL ANGLES

$$\theta_{p2} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p2} = 15.53^\circ$$

$$\theta_{p1} = \theta_{p2} + 90^\circ \quad \theta_{p1} = 105.53^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

$$\sigma_1 = 10.0 \text{ MPa}$$

$$\sigma_2 = -38.5 \text{ MPa}$$

MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \tau_{\max} = 24.2 \text{ MPa}$$

$$\theta_{s1} = \theta_{p1} - 45^\circ \quad \theta_{s1} = 60.53^\circ$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = -14.3 \text{ MPa}$$

$$\tau_{\max} = 24.2 \text{ MPa}$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{s1}) + \tau_{xy} \sin(2\theta_{s1}) = -14.25 \text{ MPa}$$

$$\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{s1}) - \tau_{xy} \sin(2\theta_{s1}) = -14.25 \text{ MPa}$$

$$\sigma_{y1} = \sigma_1 + \sigma_2 - \sigma_{x1} = -14.25 \text{ MPa}$$

$$\tau_{\max} = 24.2 \text{ MPa} \quad \sigma_{x1} = -14.25 \text{ MPa} \quad \sigma_{y1} = -14.25 \text{ MPa}$$

$$\theta_{s1} = 60.53^\circ$$

Problem 7.3-7 The stresses acting on element *B* in the web of a wide-flange beam are found to be 14,000 psi compression in the horizontal direction and 2600 psi compression in the vertical direction. Also, shear stresses of magnitude 3800 psi act in the directions shown (see the figure for Prob. 7.2-7).

Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-7

$$\sigma_x = -14,000 \text{ psi} \quad \sigma_y = -2600 \text{ psi}$$

$$\tau_{xy} = -3800 \text{ psi}$$

PRINCIPAL ANGLES

$$\theta_{p2} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p2} = 16.85^\circ$$

$$\theta_{p1} = \theta_{p2} + 90^\circ \quad \theta_{p1} = 106.85^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

$$\sigma_1 = -1449 \text{ psi}$$

$$\sigma_2 = -15,151 \text{ psi}$$

MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \tau_{\max} = 6851 \text{ psi} \quad \leftarrow$$

$$\theta_{s1} = \theta_{p1} - 45^\circ \quad \theta_{s1} = 61.8^\circ \quad \leftarrow$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = -8300 \text{ psi} \quad \leftarrow$$

Problem 7.3-8 The normal and shear stresses acting on element *B* are $\sigma_x = -46 \text{ MPa}$, $\sigma_y = -13 \text{ MPa}$, and $\tau_{xy} = 21 \text{ MPa}$ (see figure for Prob. 7.2-8).

Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-8

$$\sigma_x = -46 \text{ MPa} \quad \sigma_y = -13 \text{ MPa} \quad \tau_{xy} = 21 \text{ MPa}$$

PRINCIPAL ANGLES

$$\theta_{p2} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p2} = -25.92^\circ$$

$$\theta_{p1} = \theta_{p2} + 90^\circ \quad \theta_{p1} = 64.08^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

$$\sigma_1 = -2.8 \text{ MPa}$$

$$\sigma_2 = -56.2 \text{ MPa}$$

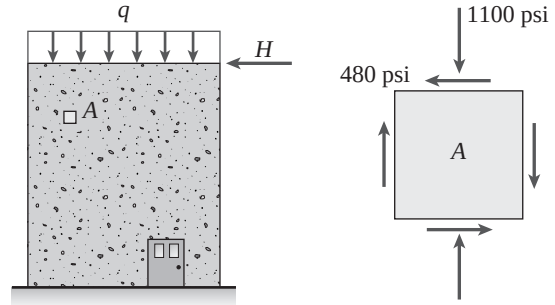
MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \tau_{\max} = 26.7 \text{ MPa} \quad \leftarrow$$

$$\theta_{s1} = \theta_{p1} - 45^\circ \quad \theta_{s1} = 19.08^\circ \quad \leftarrow$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = -29.5 \text{ MPa} \quad \leftarrow$$

Problem 7.3-9 A shear wall in a reinforced concrete building is subjected to a vertical uniform load of intensity q and a horizontal force H , as shown in the first part of the figure. (The force H represents the effects of wind and earthquake loads.) As a consequence of these loads, the stresses at point A on the surface of the wall have the values shown in the second part of the figure (compressive stress equal to 1100 psi and shear stress equal to 480 psi).



- Determine the principal stresses and show them on a sketch of a properly oriented element.
- Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-9 Shear wall

$$\sigma_x = 0 \quad \sigma_y = -1100 \text{ psi} \quad \tau_{xy} = -480 \text{ psi}$$

(a) PRINCIPAL STRESSES

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -0.87273$$

$$2\theta_p = -41.11^\circ \text{ and } \theta_p = -20.56^\circ$$

$$2\theta_p = 138.89^\circ \text{ and } \theta_p = 69.44^\circ$$

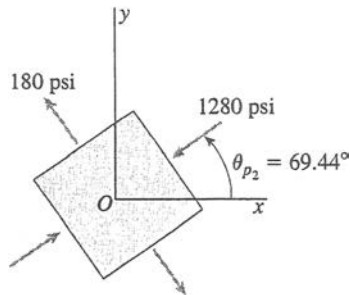
$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\text{For } 2\theta_p = -41.11^\circ: \quad \sigma_{x_1} = 180 \text{ psi}$$

$$\text{For } 2\theta_p = 138.89^\circ: \quad \sigma_{x_1} = -1280 \text{ psi}$$

$$\text{Therefore, } \sigma_1 = 180 \text{ psi and } \theta_{p_1} = -20.56^\circ \left. \vphantom{\begin{matrix} \sigma_1 = 180 \text{ psi} \\ \theta_{p_1} = -20.56^\circ \end{matrix}} \right\} \leftarrow$$

$$\sigma_2 = -1280 \text{ psi and } \theta_{p_2} = 69.44^\circ \left. \vphantom{\begin{matrix} \sigma_2 = -1280 \text{ psi} \\ \theta_{p_2} = 69.44^\circ \end{matrix}} \right\}$$



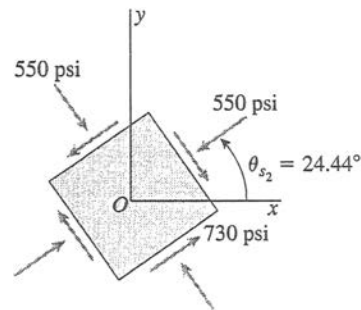
(b) MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 730 \text{ psi}$$

$$\theta_{s_1} = \theta_{p_1} - 45^\circ = -65.56^\circ \text{ and } \tau = 730 \text{ psi}$$

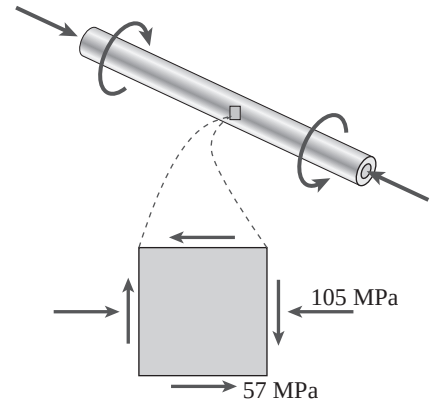
$$\theta_{s_2} = \theta_{p_1} + 45^\circ = 24.44^\circ \text{ and } \tau = -730 \text{ psi} \left. \vphantom{\begin{matrix} \theta_{s_1} = \theta_{p_1} - 45^\circ \\ \theta_{s_2} = \theta_{p_1} + 45^\circ \end{matrix}} \right\} \leftarrow$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} = -550 \text{ psi} \leftarrow$$



Problem 7.3-10 A propeller shaft subjected to combined torsion and axial thrust is designed to resist a shear stress of 57 MPa and a compressive stress of 105 MPa (see figure).

- Determine the principal stresses and show them on a sketch of a properly oriented element.
- Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.



Solution 7.3-10

PLANE STRESS TRANSFORMATIONS

PLANE STRESS VALUES: $\sigma_x = -105$ $\sigma_y = 0$ $\tau_{xy} = -57$ units = MPa

AVERAGE NORMAL STRESS: $\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = -52.5$ $\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 77.494$

PRINCIPAL STRESSES (σ_1, σ_2)

$\sigma_1 = \sigma_{ave} + \tau_{max}$ $\sigma_2 = \sigma_{ave} - \tau_{max}$ $\tau_{max} = \tau_{max}$

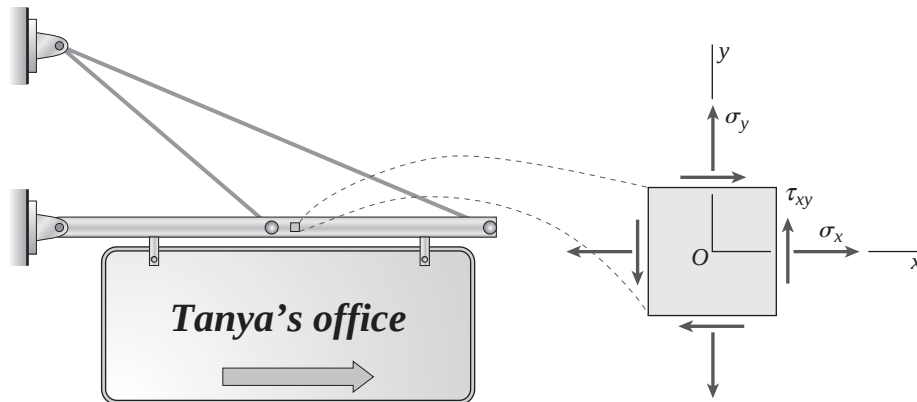
$\sigma_1 = 25$ MPa $\sigma_2 = -130$ MPa $\tau_{max} = 77.5$ MPa

PRINCIPAL ANGLES

$\theta_{p2} = \frac{1}{2} \arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$ $\theta_{p2} = 23.7^\circ$ $\theta_{s2} = \theta_{p2} - 45^\circ = -21.3^\circ$ $\theta_{s1} = \theta_{p2} + 45^\circ = 68.677^\circ$

Problem 7.3-11 The stresses at a point along a beam supporting a sign (see fig.) are $\sigma_x = 2250$ psi, $\sigma_y = 1175$ psi, $\tau_{xy} = -820$ psi.

- Find the principal stresses and show them on a sketch of a properly oriented element.
- Find the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.



Solution 7.3-11

PLANE STRESS TRANSFORMATIONS

PLANE STRESS VALUES: $\sigma_x = 2250$ $\sigma_y = 1175$ $\tau_{xy} = -820$ units = psi

$$\theta_{P1} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2} \quad \boxed{\theta_{P1} = -28.4^\circ} \quad \theta_{P2} = \theta_{P1} + 90^\circ \quad \theta_{P2} = 61.6^\circ$$

$$\theta_{s1} = \theta_{P1} - 45^\circ = -73.4^\circ$$

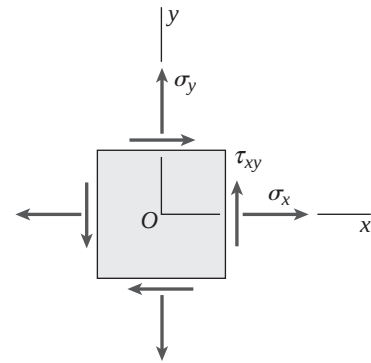
$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{P1}) + \tau_{xy} \sin(2\theta_{P1}) = 2693 \text{ psi}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{P2}) + \tau_{xy} \sin(2\theta_{P2}) = 732 \text{ psi}$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = 980 \quad \text{or} \quad \tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 980 \text{ psi}$$

Problems 7.3-12 An element in *plane stress* (see figure) is subjected to stresses $\sigma_x = 2150$ kPa, $\sigma_y = 375$ kPa, $\tau_{xy} = -460$ kPa

- Determine the principal stresses and show them on a sketch of a properly oriented element.
- Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

**Probs. 7.3-12 through 7.3-16****Solution 7.3-12** $\sigma_x = 2150$ kPa $\sigma_y = 375$ kPa $\tau_{xy} = -460$ kPa

(a) PRINCIPAL STRESSES

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \theta_{p1} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p1} = -13.70^\circ$$

$$\theta_{p2} = 90^\circ + \theta_{p1} \quad \theta_{p2} = 76.30^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

Therefore,

$$\text{For } \theta_{p1} = -13.70^\circ: \quad \sigma_1 = 2262 \text{ kPa} \quad \leftarrow$$

$$\text{For } \theta_{p2} = 76.3^\circ: \quad \sigma_2 = 263 \text{ kPa} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = 145 \text{ psi}$$

$$\theta_{s1} = \theta_{p1} - 45^\circ \quad \theta_{s1} = -58.7^\circ \quad \leftarrow$$

and $\tau_1 = 1000$ kPa

$$\theta_{s2} = \theta_{p1} + 45^\circ \quad \theta_{s2} = 31.3^\circ \quad \leftarrow$$

and $\tau_2 = -1000$ kPa

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = 1263 \text{ kPa}$$

Problems 7.3-13 $\sigma_x = 14,500$ psi, $\sigma_y = 1070$ psi, $\tau_{xy} = 1900$ psi

- (a) Determine the principal stresses and show them on a sketch of a properly oriented element.
 (b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-13

$$\sigma_x = 14,500 \text{ psi} \quad \sigma_y = 1070 \text{ psi} \quad \tau_{xy} = 1900 \text{ psi}$$

(a) PRINCIPAL STRESSES

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \theta_{p1} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p1} = 7.90^\circ$$

$$\theta_{p2} = 90^\circ + \theta_{p1} \quad \theta_{p2} = 97.90^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

Therefore,

$$\text{For } \theta_{p1} = 7.90^\circ: \quad \sigma_1 = 14764 \text{ psi} \quad \leftarrow$$

$$\text{For } \theta_{p2} = 97.9^\circ: \quad \sigma_2 = 806 \text{ psi} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = 6979 \text{ psi}$$

$$\theta_{s1} = \theta_{p1} - 45^\circ \quad \theta_{s1} = -37.1^\circ \quad \leftarrow$$

$$\text{and } \tau_1 = 6979 \text{ psi}$$

$$\theta_{s2} = \theta_{p1} + 45^\circ \quad \theta_{s2} = 52.9^\circ \quad \leftarrow$$

$$\text{and } \tau_2 = -6979 \text{ psi}$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = 7785 \text{ psi}$$

Problems 7.3-14 $\sigma_x = 16.5$ MPa, $\sigma_y = -91$ MPa, $\tau_{xy} = -39$ MPa

- (a) Determine the principal stresses and show them on a sketch of a properly oriented element.
 (b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-14

$$\sigma_x = 16.5 \text{ MPa} \quad \sigma_y = -91 \text{ MPa} \quad \tau_{xy} = -39 \text{ MPa}$$

(a) PRINCIPAL STRESSES

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \theta_{p1} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p1} = -17.98^\circ$$

$$\theta_{p2} = 90^\circ + \theta_{p1} \quad \theta_{p2} = 72.02^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

Therefore,

$$\text{For } \theta_{p1} = -17.98^\circ: \quad \sigma_1 = 29.2 \text{ MPa}$$

$$\text{For } \theta_{p2} = 72.0^\circ: \quad \sigma_2 = -103.7 \text{ MPa}$$

(b) MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = 9631.7 \text{ psi}$$

$$\theta_{s1} = \theta_{p1} - 45^\circ \quad \theta_{s1} = -63.0^\circ \quad \leftarrow$$

$$\text{and } \tau_1 = 66.4 \text{ MPa}$$

$$\theta_{s2} = \theta_{p1} + 45^\circ \quad \theta_{s2} = 27.0^\circ \quad \leftarrow$$

$$\text{and } \tau_2 = -66.4 \text{ MPa}$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = -37.3 \text{ MPa}$$

Problems 7.3-15 $\sigma_x = -3300 \text{ psi}$, $\sigma_y = -11,000 \text{ psi}$, $\tau_{xy} = 4500 \text{ psi}$

- (a) Determine the principal stresses and show them on a sketch of a properly oriented element.
 (b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-15

$$\sigma = -3300 \text{ psi} \quad \sigma_y = -11,000 \text{ psi} \quad \tau_{xy} = 4500 \text{ psi}$$

(a) PRINCIPAL STRESSES

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \theta_{p1} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p1} = 24.73^\circ$$

$$\theta_{p2} = 90^\circ + \theta_{p1} \quad \theta_{p2} = 114.73^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

Therefore,

$$\text{For } \theta_{p1} = 24.7^\circ: \quad \sigma_1 = -1228 \text{ psi}$$

$$\text{For } \theta_{p2} = 114.7^\circ: \quad \sigma_2 = -13,072 \text{ psi}$$

(b) MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = 5922 \text{ psi}$$

$$\theta_{s1} = \theta_{p1} - 45^\circ \quad \theta_{s1} = -20.3^\circ \quad \leftarrow$$

$$\text{and } \tau_1 = 5922 \text{ psi}$$

$$\theta_{s2} = \theta_{p1} + 45^\circ \quad \theta_{s2} = 69.7^\circ \quad \leftarrow$$

$$\text{and } \tau_2 = -5922 \text{ psi}$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = -7150 \text{ psi}$$

Problems 7.3-16 $\sigma_x = -108 \text{ MPa}$, $\sigma_y = 58 \text{ MPa}$, $\tau_{xy} = -58 \text{ MPa}$

- (a) Determine the principal stresses and show them on a sketch of a properly oriented element.
 (b) Determine the maximum shear stresses and associated normal stresses and show them on a sketch of a properly oriented element.

Solution 7.3-16

$$\sigma_x = -108 \text{ MPa} \quad \sigma_y = 58 \text{ MPa} \quad \tau_{xy} = -58 \text{ MPa}$$

(a) PRINCIPAL STRESSES

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \theta_{p2} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2}$$

$$\theta_{p2} = 17.47^\circ$$

$$\theta_{p1} = 90^\circ + \theta_{p2} \quad \theta_{p1} = 107.47^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

Therefore,

$$\text{For } \theta_{p1} = 107.47^\circ: \quad \sigma_1 = 76.3 \text{ MPa} \quad \leftarrow$$

$$\text{For } \theta_{p2} = 17.47^\circ: \quad \sigma_2 = -126.3 \text{ MPa} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESSES

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = 14,686.1 \text{ psi}$$

$$\theta_{s1} = \theta_{p1} - 45^\circ \quad \theta_{s1} = 62.47^\circ \quad \leftarrow$$

and $\tau_1 = 101.3 \text{ MPa}$

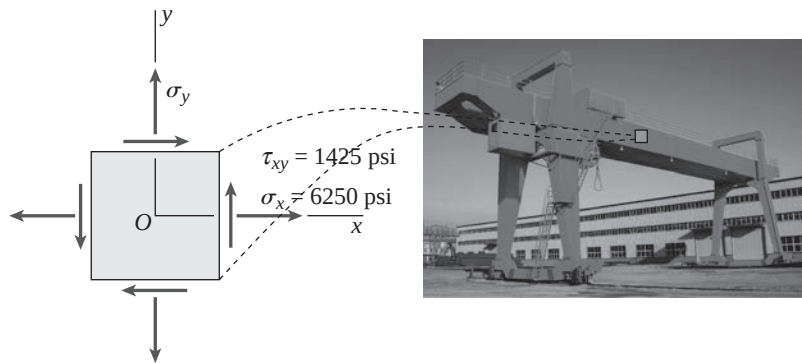
$$\theta_{s2} = \theta_{p1} + 45^\circ \quad \theta_{s2} = 152.47^\circ \quad \leftarrow$$

and $\tau_2 = -101.3 \text{ MPa}$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = -25.0 \text{ MPa}$$

Problem 7.3-17 At a point on the web of a girder on a gantry crane, the stresses acting on the x face of a stress element are $\sigma_x = 6250 \text{ psi}$ and $\tau_{xy} = 1425 \text{ psi}$ (see figure).

What is the allowable range of values for the stress σ_y if the maximum shear stress is limited to $\tau_0 = 2150 \text{ psi}$?



Solution 7.3-17

$$\sigma_x = 6250 \text{ psi} \quad \sigma_y \text{ unknown} \quad \tau_{xy} = 1425 \text{ psi}$$

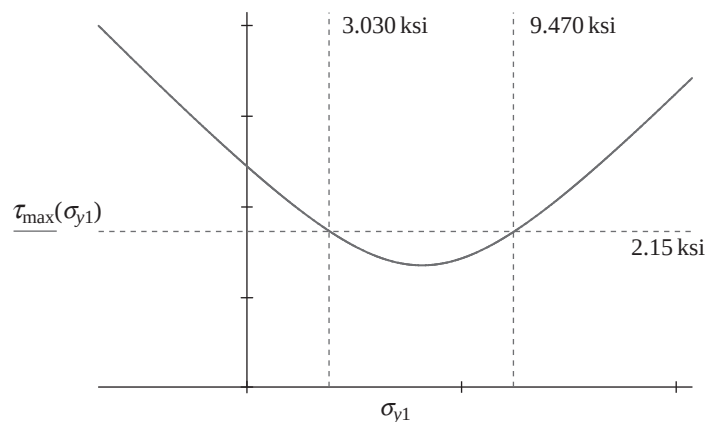
MAXIMUM ALLOWABLE SHEAR STRESSES IS $\tau_{\max} = 2150 \text{ psi}$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (1) \text{ Find the allowable range of values for } \sigma_y.$$

Solve for σ_y :

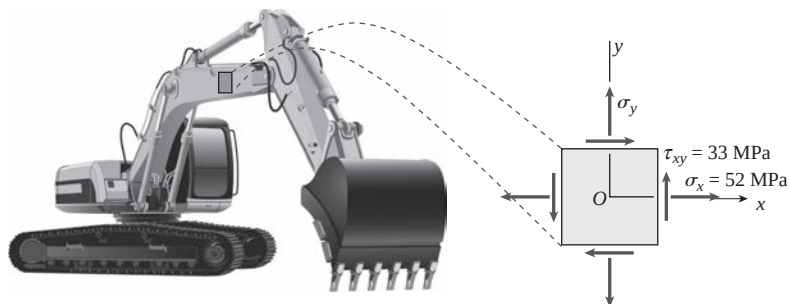
$$\sigma_y = \sigma_x + \left[\begin{array}{l} 2\sqrt{\tau_{\max}^2 - \tau_{xy}^2} \\ -(2\sqrt{\tau_{\max}^2 - \tau_{xy}^2}) \end{array} \right] \quad \sigma_y = \begin{pmatrix} 9470 \\ 3030 \end{pmatrix} \text{ psi} \quad \text{Therefore, } \boxed{3030 \text{ psi} \leq \sigma_y \leq 9470 \text{ psi}}$$

$$\text{From Eq. (1): } \tau_{\max}(\sigma_{y1}) = \sqrt{\left(\frac{\sigma_x - \sigma_{y1}}{2}\right)^2 + \tau_{xy}^2}$$



Problem 7.3-18 The stresses acting on a stress element on the arm of a power excavator (see figure) are $\sigma_x = 52 \text{ MPa}$ and $\tau_{xy} = 33 \text{ MPa}$ (see figure).

What is the allowable range of values for the stress σ_y if the maximum shear stress is limited to $\tau_0 = 37 \text{ MPa}$?



Solution 7.3-18

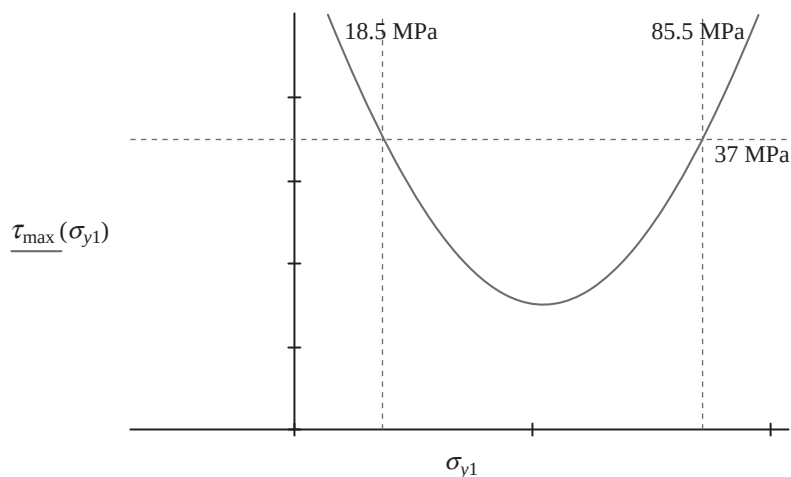
$$\sigma_x = 52 \text{ MPa} \quad \sigma_y \text{ unknown} \quad \tau_{xy} = 33 \text{ MPa}$$

$$\text{MAXIMUM ALLOWABLE SHEAR STRESS IS } \tau_{\max} = 37 \text{ MPa}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (1) \text{ Find the allowable range of values for } \sigma_y.$$

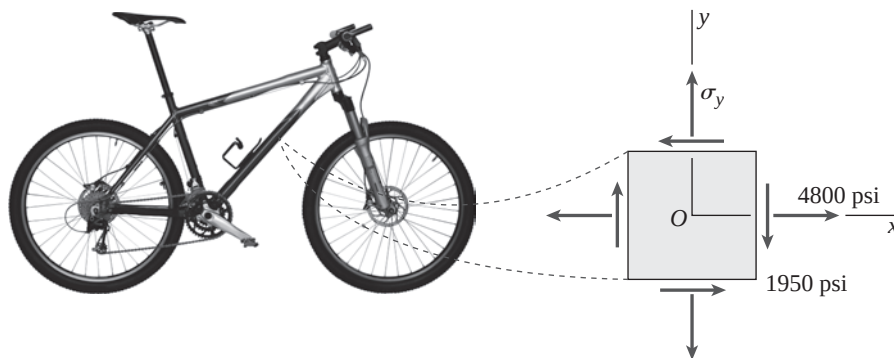
$$\text{Solve for } \sigma_y: \quad \sigma_y = \sigma_x + \left[\begin{array}{c} 2\sqrt{\tau_{\max}^2 - \tau_{xy}^2} \\ -(2\sqrt{\tau_{\max}^2 - \tau_{xy}^2}) \end{array} \right] \quad \sigma_y = \begin{pmatrix} 85.5 \\ 18.5 \end{pmatrix} \text{ MPa} \quad \text{Therefore, } \boxed{18.5 \text{ MPa} \leq \sigma_y \leq 85.5 \text{ MPa}}$$

$$\text{From Eq. (1): } \tau_{\max}(\sigma_{y1}) = \sqrt{\left(\frac{\sigma_x - \sigma_{y1}}{2}\right)^2 + \tau_{xy}^2}$$



Problem 7.3-19 The stresses at a point on the down tube of a bicycle frame are $\sigma_x = 4800 \text{ psi}$ and $\tau_{xy} = -1950 \text{ psi}$ (see figure). It is known that one of the principal stresses equals 6375 psi in tension.

- Determine the stress σ_y .
- Determine the other principal stress and the orientation of the principal planes, then show the principal stresses on a sketch of a properly oriented element.



Solution 7.3-19

$$\sigma_x = 4800 \text{ psi} \quad \sigma_y \text{ unknown} \quad \tau_{xy} = -1950 \text{ psi}$$

(a) STRESS σ_y Because σ_y is smaller than a given principal stress, the given stress is the larger principal stress.

$$\sigma_1 = 6375 \text{ psi} \quad \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \text{ solve, } \sigma_y \rightarrow \frac{27,725 \text{ psi}}{7} \quad \sigma_y = 3961 \text{ psi}$$

(b) PRINCIPAL STRESSES

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \theta_{p1} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2} \quad \theta_{p1} = -38.93^\circ$$

$$\theta_{p2} = 90^\circ + \theta_{p1} \quad \theta_{p2} = 51.07^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

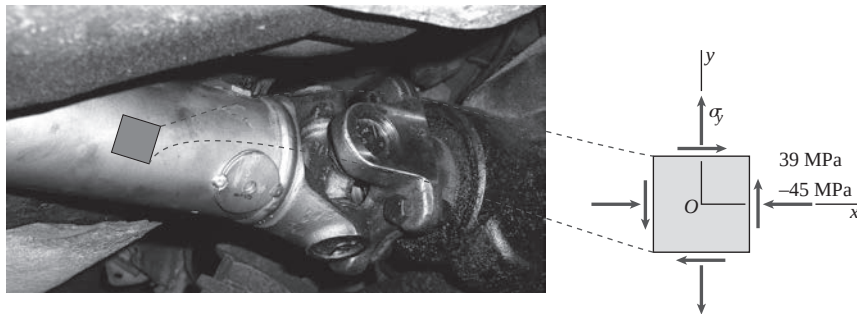
$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

Therefore,

$$\text{For } \boxed{\theta_{p1} = -38.93^\circ}: \quad \boxed{\sigma_1 = 6375 \text{ psi}} \quad \text{For } \boxed{\theta_{p2} = 51.07^\circ}: \quad \boxed{\sigma_2 = 2386 \text{ psi}}$$

Problem 7.3-20 An element in *plane stress* on the surface of an automobile drive shaft (see figure) is subjected to stresses $\sigma_x = -45 \text{ MPa}$ and $\tau_{xy} = 39 \text{ MPa}$ (see figure). It is known that one of the principal stresses equals 41 MPa in tension.

- (a) Determine the stress σ_y .
 (b) Determine the other principal stress and the orientation of the principal planes, then show the principal stresses on a sketch of a properly oriented element.



Solution 7.3-20

$$\sigma_x = -45 \text{ MPa} \quad \sigma_y \text{ unknown} \quad \tau_{xy} = 39 \text{ MPa}$$

(a) STRESS σ_y Because σ_y is smaller than the given principal stress, the given stress is the larger principal stress.

$$\sigma_1 = 41 \text{ MPa} \quad \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{solving gives } \sigma_y = 23.3 \text{ MPa}$$

(b) PRINCIPAL STRESSES

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \theta_{p2} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2} \quad \theta_{p2} = -24.40^\circ$$

$$\theta_{p1} = -90^\circ + \theta_{p2} \quad \theta_{p2} = -24.40^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1})$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2})$$

Therefore,

$$\text{For } \boxed{\theta_{p1} = 65.6^\circ}: \quad \boxed{\sigma_1 = 41 \text{ MPa}} \quad \text{For } \boxed{\theta_{p2} = -24.4^\circ}: \quad \boxed{\sigma_2 = -62.7 \text{ MPa}}$$

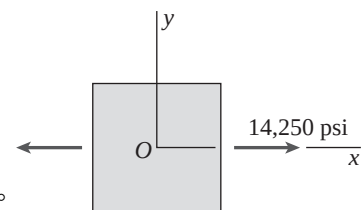
Mohr's Circle

The problems for Section 7.4 are to be solved using Mohr's circle. Consider only the in-plane stresses (the stresses in the xy plane).

Problem 7.4-1 An element in *uniaxial stress* is subjected to tensile stresses $\sigma_x = 14,250$ psi, as shown in the figure. Using Mohr's circle, determine the following.

- The stresses acting on an element oriented at a counterclockwise angle $\theta = 29^\circ$ from the x axis.
- The maximum shear stresses and associated normal stresses.

Show all results on sketches of properly oriented elements.



Solution 7.4-1

MOHR'S CIRCLE - PLANE STRESS TRANSFORMATIONS

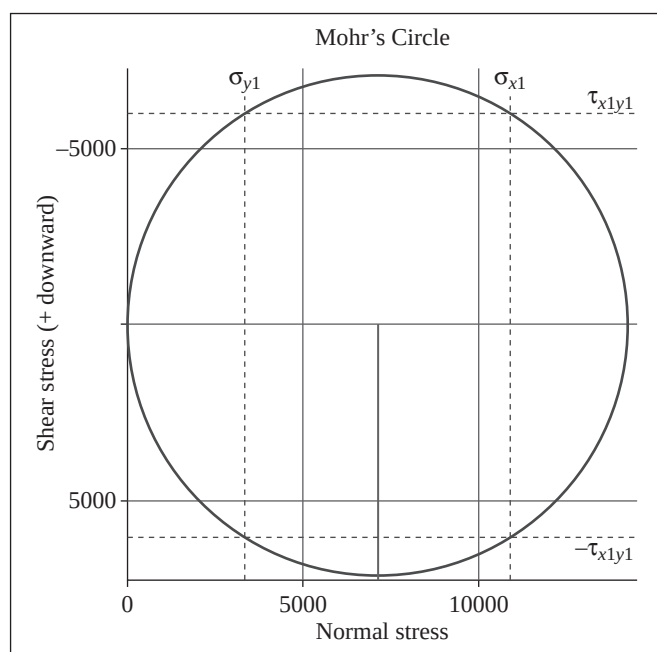
Plane stress values: $\sigma_x = 14,250$ $\sigma_y = 0$ $\tau_{xy} = 0$ units = psiAverage normal stress: $\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = 7125$ radius: $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 7125$ (a) STRESS TRANSFORMATION FOR SOME SPECIFIED ANGLE theta (θ) $\theta = 29^\circ$

$$\sigma_{x1} = \sigma_{ave} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta) \quad \boxed{\sigma_{x1} = 10,901} \text{ psi}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1} \quad \boxed{\sigma_{y1} = 3349} \text{ psi}$$

$$\tau_{x1y1} = -\left(\frac{\sigma_x - \sigma_y}{2}\right) \sin(2\theta) + \tau_{xy} \cos(2\theta) \quad \boxed{\tau_{x1y1} = -6042} \text{ psi}$$

MOHR'S CIRCLE

(b) MAXIMUM SHEAR STRESS AND ASSOCIATED NORMAL STRESSES τ_{max} at $\theta_{s1} = -45^\circ$

$$\theta_{s1} = -45^\circ \quad \theta = \theta_{s1}$$

$$\sigma_{x1} = \sigma_{ave} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta) = 7125 \quad \boxed{\sigma_{x1} = 7125} \text{ psi}$$

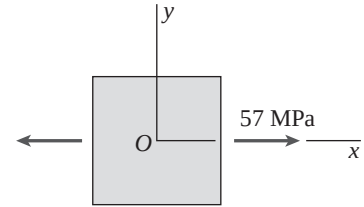
$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1} = 7125 \quad \boxed{\sigma_{y1} = 7125} \text{ psi}$$

$$\tau_{max} = -\left(\frac{\sigma_x - \sigma_y}{2}\right) \sin(2\theta) + \tau_{xy} \cos(2\theta) = 7125 \quad \boxed{\tau_{max} = 7125} \text{ psi}$$

Problem 7.4-2 An element in *uniaxial stress* is subjected to tensile stresses $\sigma_x = 57$ MPa, as shown in the figure. Using Mohr's circle, determine the following.

- The stresses acting on an element oriented at an angle $\theta = -33^\circ$ from the x axis (minus means clockwise).
- The maximum shear stresses and associated normal stresses.

Show all results on sketches of properly oriented elements.



Solution 7.4-2

MOHR'S CIRCLE - PLANE STRESS TRANSFORMATIONS

Plane stress values: $\sigma_x = 57$ $\sigma_y = 0$ $\tau_{xy} = 0$ units = MPa

Average normal stress: $\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = 28.5$ radius: $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 28.5$

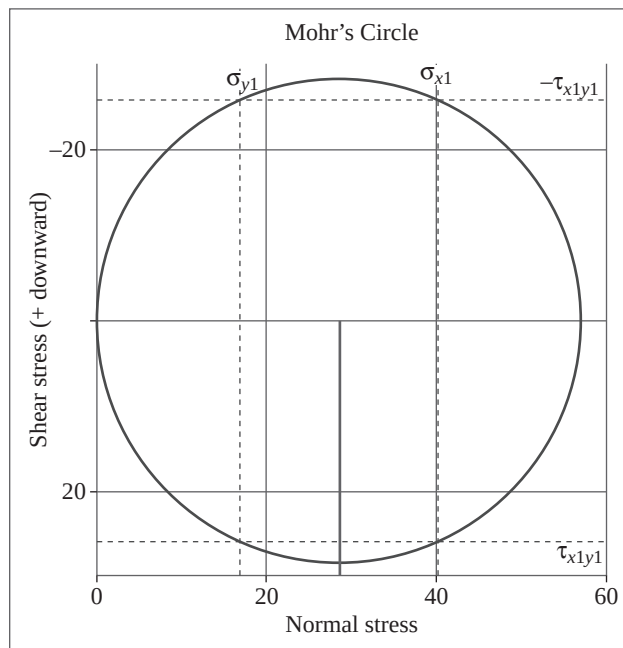
- (a) STRESS TRANSFORMATION FOR SOME SPECIFIED ANGLE theta (θ) $\theta = -33^\circ$

$$\sigma_{x1} = \sigma_{ave} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta) \quad \boxed{\sigma_{x1} = 40.1} \text{ MPa}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1} \quad \boxed{\sigma_{y1} = 16.91} \text{ MPa}$$

$$\tau_{x1y1} = -\left(\frac{\sigma_x - \sigma_y}{2}\right) \sin(2\theta) + \tau_{xy} \cos(2\theta) \quad \boxed{\tau_{x1y1} = 26} \text{ MPa}$$

MOHR'S CIRCLE



(b) MAXIMUM SHEAR STRESS AND ASSOCIATED NORMAL STRESSES $\tau_{\max}$ at $\theta_{s1} = -45^\circ$

$$\sigma_{x1} = \sigma_{ave} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta) \quad \boxed{\sigma_{x1} = 28.5 \text{ MPa}}$$

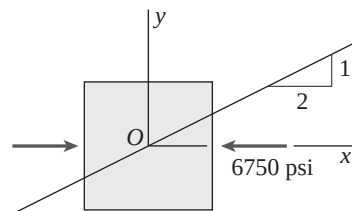
$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1} \quad \boxed{\sigma_{y1} = 28.5 \text{ MPa}}$$

$$\tau_{\max} = -\left(\frac{\sigma_x - \sigma_y}{2}\right) \sin(2\theta) + \tau_{xy} \cos(2\theta) \quad \boxed{\tau_{\max} = 28.5 \text{ MPa}}$$

Problem 7.4-3 An element on the gusset plate in Prob. 7.2-17 in *uniaxial stress* is subjected to compressive stresses of magnitude 6750 psi, as shown in the figure. Using Mohr's circle, determine the following.

- (a) The stresses acting on an element oriented at a slope of 1 on 2 (see figure).
 (b) The maximum shear stresses and associated normal stresses.

Show all results on sketches of properly oriented elements.



Solution 7.4-3

$$\sigma_x = -6750 \text{ psi} \quad \sigma_y = 0 \text{ psi} \quad \tau_{xy} = 0 \text{ psi}$$

(a) ELEMENT AT A SLOPE OF 1 ON 2

$$\theta = \arctan\left(\frac{1}{2}\right) \quad \theta = 26.565^\circ$$

$$2\theta = 53.130^\circ \quad R = \frac{\sigma_x}{2} \quad R = -3375 \text{ psi}$$

$$\text{Point C: } \sigma_c = R \quad \sigma_c = -3375 \text{ psi}$$

$$\text{Point D: } \sigma_{x1} = R + R \cos(2\theta) \quad \boxed{\sigma_{x1} = -5400 \text{ psi}}$$

$$\tau_{x1y1} = -R \sin(2\theta) \quad \boxed{\tau_{x1y1} = 2700 \text{ psi}}$$

$$\text{Point D': } \sigma_{y1} = R - R \cos(2\theta) \quad \boxed{\sigma_{y1} = -1350 \text{ psi}}$$

(b) MAXIMUM SHEAR STRESSES

$$\text{Point S1: } \theta_{s1} = \frac{90^\circ}{2} \quad \boxed{\theta_{s1} = 45^\circ}$$

$$\tau_{\max} = -R \quad \boxed{\tau_{\max} = 3375 \text{ psi}}$$

$$\text{Point S2: } \theta_{s2} = \frac{-90^\circ}{2} \quad \boxed{\theta_{s2} = -45^\circ}$$

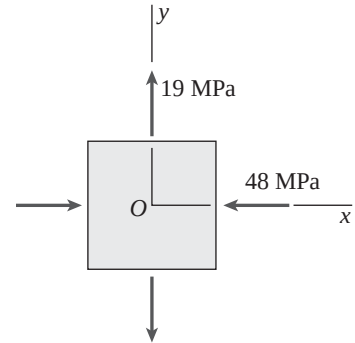
$$\tau_{\max} = R \quad \boxed{\tau_{\max} = -3375 \text{ psi}}$$

$$\sigma_{aver} = R \quad \boxed{\sigma_{aver} = -3375 \text{ psi}}$$

Problem 7.4-4 An element on the top surface of the fuel tanker in Prob. 7.2-1 is in *biaxial stress* and is subjected to stresses $\sigma_x = -48$ MPa and $\sigma_y = 19$ MPa, as shown in the figure. Using Mohr's circle, determine the following.

- The stresses acting on an element oriented at a counterclockwise angle $\theta = 25^\circ$ from the x axis.
- The maximum shear stresses and associated normal stresses.

Show all results on sketches of properly oriented elements.



Solution 7.4-4

$$\sigma_x = -48 \text{ MPa} \quad \sigma_y = 19 \text{ MPa} \quad \tau_{xy} = 0 \text{ MPa}$$

(a) ELEMENT AT $\theta = 25^\circ$ ←

$$2\theta = 50.0^\circ \quad R = \frac{|\sigma_x| + |\sigma_y|}{2} \quad R = 33.5 \text{ MPa}$$

$$\text{Point C: } \sigma_x = \sigma_x + R \quad \sigma_c = -14.5 \text{ MPa}$$

$$\text{Point D: } \sigma_{x1} = \sigma_c - R \cos(2\theta)$$

$$\sigma_{x1} = -36.0 \text{ MPa} \quad \leftarrow$$

$$\tau_{x1y1} = -R \sin(2\theta)$$

$$\tau_{x1y1} = 25.7 \text{ MPa} \quad \leftarrow$$

$$\text{Point D': } \sigma_{y1} = \sigma_c + R \cos(2\theta)$$

$$\sigma_{y1} = 7.0 \text{ MPa}$$

(b) MAXIMUM SHEAR STRESSES

$$\text{Point S1: } \theta_{s1} = \frac{90^\circ}{2}$$

$$\theta_{s1} = 45.0^\circ \quad \leftarrow$$

$$\tau_{\max} = R \quad \tau_{\max} = 33.5 \text{ MPa} \quad \leftarrow$$

$$\text{Point S2: } \theta_{s2} = \frac{-90^\circ}{2}$$

$$\theta_{s2} = -45.0^\circ \quad \leftarrow$$

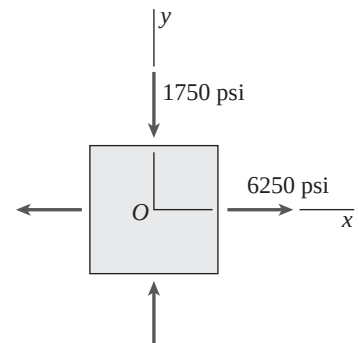
$$\tau_{\max} = -R \quad \tau_{\max} = -33.5 \text{ MPa} \quad \leftarrow$$

$$\sigma_{\text{aver}} = \sigma_c \quad \sigma_{\text{aver}} = -14.5 \text{ MPa} \quad \leftarrow$$

Problem 7.4-5 An element on the top surface of the fuel tanker in Prob. 7.2-1 is in *biaxial stress* and is subjected to stresses $\sigma_x = 6250$ psi and $\sigma_y = -1750$ psi, as shown in the figure. Using Mohr's circle, determine the following.

- The stresses acting on an element oriented at a counterclockwise angle $\theta = 55^\circ$ from the x axis.
- The maximum shear stresses and associated normal stresses.

Show all results on sketches of properly oriented elements.



Solution 7.4-5

$$\sigma_x = 6250 \text{ psi} \quad \sigma_y = -1750 \text{ psi} \quad \tau_{xy} = 0 \text{ psi}$$

(a) ELEMENT AT $\theta = 55^\circ$

$$2\theta = 120^\circ \quad R = \frac{|\sigma_x| + |\sigma_y|}{2} \quad R = 4000 \text{ psi}$$

$$\text{Point C: } \sigma_c = \sigma_x - R \quad \sigma_c = 2250 \text{ psi}$$

$$\text{Point D: } \sigma_{x1} = \sigma_c + R \cos(2\theta)$$

$$\sigma_{x1} = 882 \text{ psi} \quad \leftarrow$$

$$\tau_{x1y1} = -R \sin(2\theta)$$

$$\tau_{x1y1} = -3759 \text{ psi} \quad \leftarrow$$

$$\text{Point D': } \sigma_{y1} = \sigma_c - R \cos(2\theta) \quad \sigma_{y1} = 4250 \text{ psi}$$

(b) MAXIMUM SHEAR STRESSES

$$\text{Point S1: } \theta_{s1} = \frac{-90^\circ}{2}$$

$$\theta_{s1} = -45^\circ \quad \leftarrow$$

$$\tau_{\max} = R \quad \tau_{\max} = 4000 \text{ psi} \quad \leftarrow$$

$$\text{Point S2: } \theta_{s2} = \frac{90^\circ}{2} \quad \theta_{s2} = 45^\circ \quad \leftarrow$$

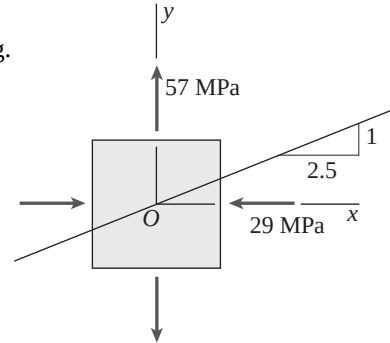
$$\tau_{\max} = R \quad \tau_{\max} = 4000 \text{ psi} \quad \leftarrow$$

$$\sigma_{\text{aver}} = \sigma_c \quad \sigma_{\text{aver}} = 2250 \text{ psi} \quad \leftarrow$$

Problem 7.4-6 An element in *biaxial stress* is subjected to stresses $\sigma_x = -29 \text{ MPa}$ and $\sigma_y = 57 \text{ MPa}$, as shown in the figure. Using Mohr's circle, determine the following.

- (a) The stresses acting on an element oriented at a slope of 1 on 2.5 (see figure).
 (b) The maximum shear stresses and associated normal stresses.

Show all results on sketches of properly oriented elements.

**Solution 7.4-6**

$$\sigma_x = -29 \text{ MPa} \quad \sigma_y = 57 \text{ MPa} \quad \tau_{xy} = 0 \text{ MPa}$$

(a) ELEMENT AT A SLOPE OF 1 ON 2.5

$$\theta = \arctan\left(\frac{1}{2.5}\right) \quad \theta = 21.801^\circ \quad \leftarrow$$

$$2\theta = 43.603^\circ \quad R = \frac{|\sigma_x| + |\sigma_y|}{2} \quad R = 43.0 \text{ MPa}$$

$$\text{Point C: } \sigma_c = \sigma_x + R \quad \sigma_c = 14.0 \text{ MPa}$$

$$\text{Point D: } \sigma_{x1} = \sigma_c - R \cos(2\theta)$$

$$\sigma_{x1} = -17.1 \text{ MPa} \quad \leftarrow$$

$$\tau_{x1y1} = R \sin(2\theta) \quad \tau_{x1y1} = 29.7 \text{ MPa} \quad \leftarrow$$

$$\text{Point D': } \sigma_{y1} = \sigma_c + R \cos(2\theta)$$

$$\sigma_{y1} = 45.1 \text{ MPa}$$

(b) MAXIMUM SHEAR STRESSES

$$\text{Point S1: } \theta_{s1} = \frac{90^\circ}{2} \quad \theta_{s1} = 45.0^\circ \quad \leftarrow$$

$$\tau_{\max} = R \quad \tau_{\max} = 43.0 \text{ MPa} \quad \leftarrow$$

$$\text{Point S2: } \theta_{s2} = \frac{-90^\circ}{2}$$

$$\theta_{s2} = -45.0^\circ \quad \leftarrow$$

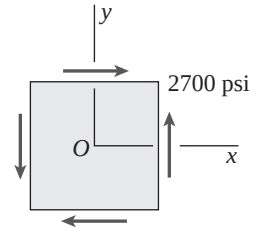
$$\tau_{\max} = -R \quad \tau_{\max} = -43.0 \text{ MPa} \quad \leftarrow$$

$$\sigma_{\text{aver}} = \sigma_c \quad \sigma_{\text{aver}} = 14.0 \text{ MPa} \quad \leftarrow$$

Problem 7.4-7 An element on the surface of a drive shaft is in *pure shear* and is subjected to stresses $\tau_{xy} = 2700$ psi, as shown in the figure. Using Mohr's circle, determine the following.

- The stresses acting on an element oriented at a counterclockwise angle $\theta = 52^\circ$ from the x axis.
- The principal stresses.

Show all results on sketches of properly oriented elements.



Solution 7.4-7

$$\sigma_x = 0 \text{ psi} \quad \sigma_y = 0 \text{ psi} \quad \tau_{xy} = 2700 \text{ psi}$$

(a) ELEMENT AT $\theta = 52^\circ$

$$2\theta = 104.0^\circ \quad R = \tau_{xy} \quad R = 2700 \text{ psi}$$

$$\text{Point D: } \sigma_{x1} = R \cos(2\theta - 90^\circ)$$

$$\sigma_{x1} = 2620 \text{ psi} \quad \leftarrow$$

$$\tau_{x1y1} = -R \sin(2\theta - 90^\circ)$$

$$\tau_{x1y1} = -653 \text{ psi} \quad \leftarrow$$

$$\text{Point D': } \sigma_{y1} = -R \cos(2\theta - 90^\circ)$$

$$\sigma_{y1} = -2620 \text{ psi} \quad \leftarrow$$

(b) PRINCIPAL STRESSES

$$\text{Point P1: } \theta_{p1} = \frac{90^\circ}{2} \quad \theta_{p1} = 45^\circ \quad \leftarrow$$

$$\sigma_1 = R \quad \sigma_1 = 2700 \text{ psi} \quad \leftarrow$$

$$\text{Point P2: } \theta_{p2} = \frac{-90^\circ}{2}$$

$$\theta_{p2} = -45^\circ \quad \leftarrow$$

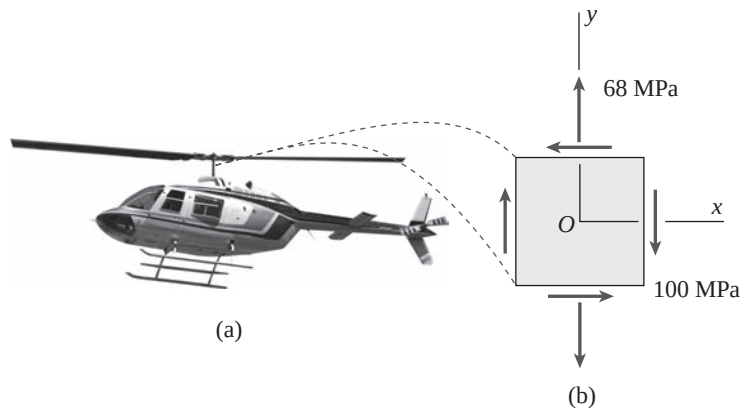
$$\sigma_2 = -R \quad \sigma_2 = -2700 \text{ psi} \quad \leftarrow$$

Problem 7.4-8 The rotor shaft of a helicopter (see figure part a) drives the rotor blades that provide the lifting force and is subjected to a combination of torsion and axial loading (see figure part b).

It is known that normal stress $\sigma_y = 68$ MPa and shear stress $\tau_{xy} = -100$ MPa. Using Mohr's circle, determine the following.

- The stresses acting on an element oriented at a counterclockwise angle $\theta = 22.5^\circ$ from the x axis.
- Find the maximum tensile stress, maximum compressive stress, and maximum shear stress in the shaft.

Show all results on sketches of properly oriented elements.



Solution 7.4-8

MOHR'S CIRCLE - PLANE STRESS TRANSFORMATIONS units = MPa

$$\text{PLANE STRESS VALUES: } \sigma_x = 0 \quad \sigma_y = 68 \quad \tau_{xy} = -100$$

$$\text{AVERAGE NORMAL STRESS: } \sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} = 34 \quad \text{radius: } R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 105.622$$

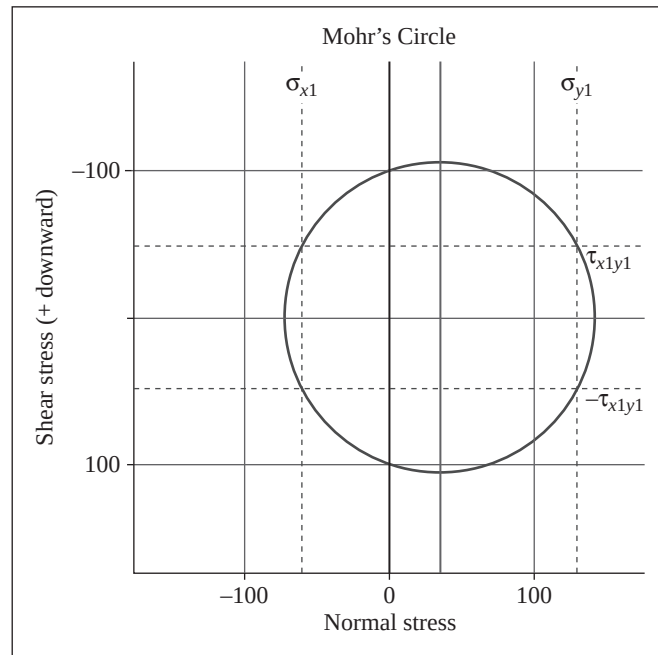
(a) STRESS TRANSFORMATION FOR SOME SPECIFIED ANGLE theta (θ) $\theta = 22.5^\circ$

$$\sigma_{x1} = \sigma_{\text{ave}} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta) \quad \boxed{\sigma_{x1} = -60.8 \text{ MPa}}$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1} \quad \boxed{\sigma_{y1} = 128.75 \text{ MPa}}$$

$$\tau_{x1y1} = -\left(\frac{\sigma_x - \sigma_y}{2}\right) \sin(2\theta) + \tau_{xy} \cos(2\theta) \quad \boxed{\tau_{x1y1} = -46.7 \text{ MPa}}$$

MOHR'S CIRCLE



(b) PRINCIPAL NORMAL STRESSES & ALSO MAXIMUM SHEAR STRESS

$$\theta_{p2} = \frac{\arctan\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)}{2} \quad \theta_{p2} = 35.6^\circ \quad \theta_{p1} = \theta_{p2} + 90^\circ \quad \theta_{p1} = 125.6^\circ$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p1}) + \tau_{xy} \sin(2\theta_{p1}) = 139.6 \quad \boxed{\sigma_1 = 139.6 \text{ MPa}}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_{p2}) + \tau_{xy} \sin(2\theta_{p2}) = -71.6 \quad \boxed{\sigma_2 = -71.6 \text{ MPa}}$$

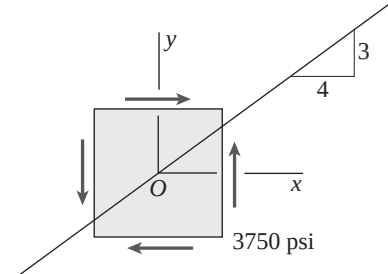
$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = 105.6 \quad \text{or} \quad \tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 105.6 \text{ MPa}$$

Because the principal stresses σ_1 and σ_2 have opposite signs, the maximum in-plane shear stresses are larger than the maximum out-of-plane shear stresses (see Eqs. 7-28a, b, and c and the accompanying discussion). Therefore, the maximum shear stress in the shaft is 105.6 MPa.

Problem 7.4-9 An element in *pure shear* is subjected to stresses $\tau_{xy} = 3750$ psi, as shown in the figure. Using Mohr's circle, determine the following.

- The stresses acting on an element oriented at a slope of 3 on 4 (see figure).
- The principal stresses.

Show all results on sketches of properly oriented elements.



Solution 7.4-9

$$\sigma_x = 0 \text{ psi} \quad \sigma_y = 0 \text{ psi} \quad \tau_{xy} = 3750 \text{ psi}$$

(a) ELEMENT AT A SLOPE OF 3 ON 4

$$\theta = \arctan\left(\frac{3}{4}\right) \quad \theta = 36.870^\circ$$

$$2\theta = 73.740^\circ \quad R = \tau_{xy} \quad R = 3750 \text{ psi}$$

$$\text{Point D: } \sigma_{x1} = R \cos(2\theta - 90^\circ)$$

$$\sigma_{x1} = 3600 \text{ psi} \quad \leftarrow$$

$$\tau_{x1y1} = -R \sin(2\theta - 90^\circ)$$

$$\tau_{x1y1} = 1050 \text{ psi} \quad \leftarrow$$

$$\text{Point D': } \sigma_{y1} = -R \cos(2\theta - 90^\circ)$$

$$\sigma_{y1} = -3600 \text{ psi} \quad \leftarrow$$

(b) PRINCIPAL STRESSES

$$\text{Point P1: } \theta_{p1} = \frac{90^\circ}{2} \quad \theta_{p1} = 45^\circ \quad \leftarrow$$

$$\sigma_1 = R \quad \sigma_1 = 3750 \text{ psi} \quad \leftarrow$$

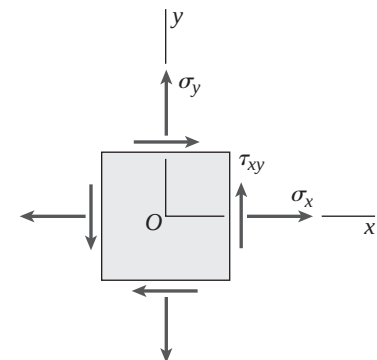
$$\text{Point P2: } \theta_{p2} = \frac{-90^\circ}{2}$$

$$\theta_{p2} = -45^\circ \quad \leftarrow$$

$$\sigma_2 = -R \quad \sigma_2 = -3750 \text{ psi} \quad \leftarrow$$

Problem 7.4-10 An element in *plane stress* is subjected to stresses $\sigma_x = 27$ MPa, $\sigma_y = 14$ MPa, $\tau_{xy} = 6$ MPa, $\theta = 40^\circ$

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ . (Note: The angle θ is positive when counterclockwise and negative when clockwise.)



Probs. 7.4-10 through 7.4-15

Solution 7.4-10

$$\sigma_x = 27 \text{ MPa} \quad \sigma_y = 14 \text{ MPa} \quad \tau_{xy} = 6 \text{ MPa}$$

$$\theta = 40^\circ$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = 20.50 \text{ MPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 8.8459 \text{ MPa}$$

$$\alpha = \arctan\left(\frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}}\right) \quad \alpha = 42.71^\circ$$

$$\beta = 2\theta - \alpha \quad \beta = 37.29^\circ$$

$$\text{Point } D: \sigma_{x1} = \sigma_{\text{aver}} + R \cos(\beta)$$

$$\sigma_{x1} = 27.5 \text{ MPa} \quad \leftarrow$$

$$\tau_{x1y1} = -R \sin(\beta)$$

$$\tau_{x1y1} = -5.36 \text{ MPa} \quad \leftarrow$$

$$\text{Point } D': \sigma_{y1} = \sigma_{\text{aver}} - R \cos(\beta)$$

$$\sigma_{y1} = 13.46 \text{ MPa} \quad \leftarrow$$

Problem 7.4-11 $\sigma_x = 3500 \text{ psi}$, $\sigma_y = 12,200 \text{ psi}$, $\tau_{xy} = -3300 \text{ psi}$, $\theta = -51^\circ$

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ . (Note: The angle θ is positive when counterclockwise and negative when clockwise.)

Solution 7.4-11

$$\sigma_x = 3500 \text{ psi} \quad \sigma_y = 12,200 \text{ psi} \quad \tau_{xy} = -3300 \text{ psi}$$

$$\theta = -51^\circ \quad \leftarrow$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = 7850 \text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 5460 \text{ psi}$$

$$\alpha = \arctan\left(\frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}}\right) \quad \alpha = 37.18^\circ$$

$$\beta = 180^\circ + 2\theta - \alpha \quad \beta = 40.82^\circ$$

$$\text{Point } D: \sigma_{x1} = \sigma_{\text{aver}} + R \cos(\beta)$$

$$\sigma_{x1} = 11,982 \text{ psi} \quad \leftarrow$$

$$\tau_{x1y1} = -R \sin(\beta)$$

$$\tau_{x1y1} = -3569 \text{ psi} \quad \leftarrow$$

$$\text{Point } D': \sigma_{y1} = \sigma_{\text{aver}} - R \cos(\beta)$$

$$\sigma_{y1} = 3718 \text{ psi} \quad \leftarrow$$

Problem 7.4-12 $\sigma_x = -47 \text{ MPa}$, $\sigma_y = -186 \text{ MPa}$, $\tau_{xy} = -29 \text{ MPa}$, $\theta = -33^\circ$

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ . (Note: The angle θ is positive when counterclockwise and negative when clockwise.)

Solution 7.4-12

$$\sigma_x = -47 \text{ MPa} \quad \sigma_y = -186 \text{ MPa} \quad \tau_{xy} = -29 \text{ MPa}$$

$$\theta = -33^\circ$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = -116.50 \text{ MPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 75.3077 \text{ MPa}$$

$$\alpha = \arctan\left(\left|\frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}}\right|\right) \quad \alpha = 22.65^\circ$$

$$\beta = -2\theta - \alpha \quad \beta = 43.35^\circ$$

$$\text{Point } D: \sigma_{x1} = \sigma_{\text{aver}} + R \cos(\beta)$$

$$\sigma_{x1} = -61.7 \text{ MPa} \quad \leftarrow$$

$$\tau_{x1y1} = -R \sin(\beta)$$

$$\tau_{x1y1} = -51.7 \text{ MPa} \quad \leftarrow$$

$$\text{Point } D': \sigma_{y1} = \sigma_{\text{aver}} - R \cos(\beta)$$

$$\sigma_{y1} = -171.3 \text{ MPa} \quad \leftarrow$$

Problem 7.4-13 $\sigma_x = -1720$ psi, $\sigma_y = -680$ psi, $\tau_{xy} = 320$ psi, $\theta = 14^\circ$

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ . (Note: The angle θ is positive when counterclockwise and negative when clockwise.)

Solution 7.4-13

$$\sigma_x = -1720 \text{ psi} \quad \sigma_y = -680 \text{ psi} \quad \tau_{xy} = 320 \text{ psi}$$

$$\theta = 14^\circ$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = -1200 \text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 611 \text{ psi}$$

$$\alpha = \arctan\left(\frac{\tau_{xy}}{|\sigma_x - \sigma_{\text{aver}}|}\right) \quad \alpha = 31.61^\circ$$

$$\beta = 2\theta + \alpha \quad \beta = 59.61^\circ$$

$$\text{Point } D: \sigma_{x1} = \sigma_{\text{aver}} - R \cos(\beta)$$

$$\sigma_{x1} = -1509 \text{ psi} \quad \leftarrow$$

$$\tau_{x1y1} = R \sin(\beta) \quad \tau_{x1y1} = 527 \text{ psi} \quad \leftarrow$$

$$\text{Point } D': \sigma_{y1} = \sigma_{\text{aver}} + R \cos(\beta)$$

$$\sigma_{y1} = -891 \text{ psi} \quad \leftarrow$$

Problem 7.4-14 $\sigma_x = 33$ MPa, $\sigma_y = -9$ MPa, $\tau_{xy} = 29$ MPa, $\theta = 35^\circ$

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ . (Note: The angle θ is positive when counterclockwise and negative when clockwise.)

Solution 7.4-14

$$\sigma_x = 33 \text{ MPa} \quad \sigma_y = -9 \text{ MPa} \quad \tau_{xy} = 29 \text{ MPa}$$

$$\theta = 35^\circ$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = 12.00 \text{ MPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 35.8050 \text{ MPa}$$

$$\alpha = \arctan\left(\left|\frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}}\right|\right) \quad \alpha = 54.09^\circ$$

$$\beta = 2\theta - \alpha \quad \beta = 15.91^\circ$$

$$\text{Point } D: \sigma_{x1} = \sigma_{\text{aver}} + R \cos(\beta)$$

$$\sigma_{x1} = 46.4 \text{ MPa} \quad \leftarrow$$

$$\tau_{x1y1} = -R \sin(\beta)$$

$$\tau_{x1y1} = -9.81 \text{ MPa} \quad \leftarrow$$

$$\text{Point } D': \sigma_{y1} = \sigma_{\text{aver}} - R \cos(\beta)$$

$$\sigma_{y1} = -22.4 \text{ MPa} \quad \leftarrow$$

Problem 7.4-15 $\sigma_x = -5700$ psi, $\sigma_y = 950$ psi, $\tau_{xy} = -2100$ psi, $\theta = 65^\circ$

Using Mohr's circle, determine the stresses acting on an element oriented at an angle θ from the x axis. Show these stresses on a sketch of an element oriented at the angle θ . (Note: The angle θ is positive when counterclockwise and negative when clockwise.)

Solution 7.4-15

$$\sigma_x = -5700 \text{ psi} \quad \sigma_y = 950 \text{ psi} \quad \tau_{xy} = -2100 \text{ psi}$$

$$\theta = 65^\circ$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = -2375 \text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 3933 \text{ psi}$$

$$\alpha = \arctan \left(\frac{|\tau_{xy}|}{|\sigma_x - \sigma_{\text{aver}}|} \right) \quad \alpha = 32.28^\circ$$

$$\beta = 180^\circ - 2\theta + \alpha \quad \beta = 82.28^\circ$$

$$\text{Point } D: \sigma_{x1} = \sigma_{\text{aver}} + R \cos(\beta)$$

$$\sigma_{x1} = -1846 \text{ psi} \quad \leftarrow$$

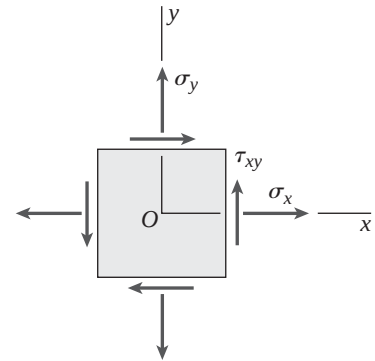
$$\tau_{x1y1} = R \sin(\beta) \quad \tau_{x1y1} = 3897 \text{ psi} \quad \leftarrow$$

$$\text{Point } D': \sigma_{y1} = \sigma_{\text{aver}} - R \cos(\beta)$$

$$\sigma_{y1} = -2904 \text{ psi} \quad \leftarrow$$

Problems 7.4-16 An element in *plane stress* is subjected to stresses $\sigma_x = 2900 \text{ kPa}$, $\sigma_y = 9100 \text{ kPa}$, $\tau_{xy} = -3750 \text{ kPa}$

Using Mohr's circle, determine (a) the principal stresses, and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.



Probs. 7.4-16 through 7.4-23

Solution 7.4-16

$$\sigma_x = 2900 \text{ kPa} \quad \sigma_y = 9100 \text{ kPa} \quad \tau_{xy} = -3750 \text{ kPa}$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = 6000 \text{ kPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 4865.4393 \text{ kPa}$$

$$\alpha = \arctan \left(\left| \frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}} \right| \right) \quad \alpha = 50.42^\circ$$

(a) PRINCIPAL STRESSES

$$\theta_{p1} = \frac{\alpha + 180^\circ}{2} \quad \theta_{p1} = 115.2^\circ \quad \leftarrow$$

$$\theta_{p2} = \frac{\alpha}{2} \quad \theta_{p2} = 25.2^\circ \quad \leftarrow$$

$$\text{Point } P1: \sigma_1 = R + \sigma_{\text{aver}}$$

$$\sigma_1 = 10,865 \text{ kPa} \quad \leftarrow$$

$$\text{Point } P2: \sigma_2 = -R + \sigma_{\text{aver}}$$

$$\sigma_2 = 1135 \text{ kPa} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESSES

$$\theta_{s1} = \frac{90^\circ + \alpha}{2} \quad \theta_{s1} = 70.2^\circ \quad \leftarrow$$

$$\theta_{s2} = 90^\circ + \theta_{s1} \quad \theta_{s2} = 160.2^\circ \quad \leftarrow$$

$$\text{Point } S1: \sigma_{\text{aver}} = 6000 \text{ kPa} \quad \leftarrow$$

$$\tau_{\text{max}} = R \quad \tau_{\text{max}} = 4865 \text{ kPa} \quad \leftarrow$$

Problems 7.4-17 $\sigma_x = 800$ psi, $\sigma_y = -2200$ psi, $\tau_{xy} = 2900$ psi

Using Mohr's circle, determine (a) the principal stresses, and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-17

$$\sigma_x = 800 \text{ psi} \quad \sigma_y = -2200 \text{ psi} \quad \tau_{xy} = 2900 \text{ psi}$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = -700 \text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 3265 \text{ psi}$$

$$\alpha = \arctan \left(\left| \frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}} \right| \right) \quad \alpha = 62.65^\circ$$

(a) PRINCIPAL STRESSES

$$\theta_{p1} = \frac{\alpha}{2} \quad \theta_{p1} = 31.3^\circ \quad \leftarrow$$

$$\theta_{p2} = \frac{180^\circ + \alpha}{2} \quad \theta_{p2} = 121.3^\circ \quad \leftarrow$$

$$\text{Point P1: } \sigma_1 = R + \sigma_{\text{aver}}$$

$$\sigma_1 = 2565 \text{ psi} \quad \leftarrow$$

$$\text{Point P2: } \sigma_2 = -R + \sigma_{\text{aver}}$$

$$\sigma_2 = -3965 \text{ psi} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESSES

$$\theta_{s1} = \frac{-90^\circ + \alpha}{2} \quad \theta_{s1} = -13.7^\circ \quad \leftarrow$$

$$\theta_{s2} = 90^\circ + \theta_{s1} \quad \theta_{s2} = 76.3^\circ \quad \leftarrow$$

$$\text{Point S1: } \sigma_{\text{aver}} = -700 \text{ psi} \quad \leftarrow$$

$$\tau_{\text{max}} = R \quad \tau_{\text{max}} = 3265 \text{ psi}$$

Problems 7.4-18 $\sigma_x = -3.3$ MPa, $\sigma_y = 8.9$ MPa, $\tau_{xy} = -14.1$ MPa

Using Mohr's circle, determine (a) the principal stresses, and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-18

$$\sigma_x = -3.3 \text{ MPa} \quad \sigma_y = 8.9 \text{ MPa}$$

$$\tau_{xy} = -14.1 \text{ MPa}$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = 2.8 \text{ MPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 15.4 \text{ MPa}$$

$$\alpha = \arctan \left(\left| \frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}} \right| \right) \quad \alpha = 66.6^\circ$$

(a) PRINCIPAL STRESSES

$$\theta_{p1} = \frac{\alpha + 180^\circ}{2} \quad \theta_{p1} = 123.3^\circ \quad \leftarrow$$

$$\theta_{p2} = \frac{\alpha}{2} \quad \theta_{p2} = 33.3^\circ$$

$$\text{Point P1: } \sigma_1 = R + \sigma_{\text{aver}}$$

$$\sigma_1 = 18.2 \text{ MPa} \quad \leftarrow$$

$$\text{Point P2: } \sigma_2 = -R + \sigma_{\text{aver}}$$

$$\sigma_2 = -12.6 \text{ MPa} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESSES

$$\theta_{s1} = \frac{90^\circ + \alpha}{2} \quad \theta_{s1} = 78.3^\circ$$

$$\theta_{s2} = 90^\circ + \theta_{s1} \quad \theta_{s2} = 168.3^\circ$$

$$\text{Point S1: } \sigma_{\text{aver}} = 2.8 \text{ MPa} \quad \leftarrow$$

$$\tau_{\text{max}} = R \quad \tau_{\text{max}} = 15.4 \text{ MPa} \quad \leftarrow$$

Problems 7.4-19 $\sigma_x = -11,500$ psi, $\sigma_y = -18,250$ psi, $\tau_{xy} = -7200$ psi

Using Mohr's circle, determine (a) the principal stresses, and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-19

$$\sigma_x = -11,500 \text{ psi} \quad \sigma_y = -18,250 \text{ psi}$$

$$\tau_{xy} = -7200 \text{ psi}$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = -14,875 \text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 7952 \text{ psi}$$

$$\alpha = \arctan \left(\left| \frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}} \right| \right) \quad \alpha = 64.89^\circ$$

(a) PRINCIPAL STRESSES

$$\theta_{p1} = \frac{-\alpha}{2} \quad \theta_{p1} = -32.4^\circ \quad \leftarrow$$

$$\theta_{p2} = \frac{180^\circ - \alpha}{2} \quad \theta_{p2} = 57.6^\circ \quad \leftarrow$$

$$\text{Point } P1: \sigma_1 = R + \sigma_{\text{aver}}$$

$$\sigma_1 = -6923 \text{ psi} \quad \leftarrow$$

$$\text{Point } P2: \sigma_2 = -R + \sigma_{\text{aver}}$$

$$\sigma_2 = -22,827 \text{ psi} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESSES

$$\theta_{s1} = \frac{270^\circ - \alpha}{2} \quad \theta_{s1} = 102.6^\circ \quad \leftarrow$$

$$\theta_{s2} = 90^\circ + \theta_{s1} \quad \theta_{s2} = 192.6^\circ$$

$$\text{Point } S1: \sigma_{\text{aver}} = -14,875 \text{ psi} \quad \leftarrow$$

$$\tau_{\text{max}} = R \quad \tau_{\text{max}} = 7952 \text{ psi} \quad \leftarrow$$

Problems 7.4-20 $\sigma_x = -29.5$ MPa, $\sigma_y = 29.5$ MPa, $\tau_{xy} = 27$ MPa

Using Mohr's circle, determine (a) the principal stresses, and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-20

$$\sigma_x = -29.5 \text{ MPa} \quad \sigma_y = 29.5 \text{ MPa} \quad \tau_{xy} = 27 \text{ MPa}$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = 0 \text{ MPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 39.9906 \text{ MPa}$$

$$\alpha = \arctan \left(\left| \frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}} \right| \right) \quad \alpha = 42.47^\circ$$

(a) PRINCIPAL STRESSES

$$\theta_{p1} = \frac{180^\circ - \alpha}{2} \quad \theta_{p1} = 68.8^\circ \quad \leftarrow$$

$$\theta_{p2} = \theta_{p1} - 90^\circ \quad \theta_{p2} = -21.2^\circ \quad \leftarrow$$

$$\text{Point P1: } \sigma_1 = R \quad \sigma_1 = 40.0 \text{ MPa} \quad \leftarrow$$

$$\text{Point P2: } \sigma_2 = -R \quad \sigma_2 = -40.0 \text{ MPa} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESSES

$$\theta_{s1} = \frac{90^\circ - \alpha}{2} \quad \theta_{s1} = 23.8^\circ \quad \leftarrow$$

$$\theta_{s2} = 90^\circ + \theta_{s1} \quad \theta_{s2} = 113.8^\circ \quad \leftarrow$$

$$\text{Point S1: } \sigma_{\text{aver}} = 0 \text{ MPa} \quad \leftarrow$$

$$\tau_{\text{max}} = R \quad \tau_{\text{max}} = 40.0 \text{ MPa} \quad \leftarrow$$

Problems 7.4-21 $\sigma_x = 2050 \text{ psi}$, $\sigma_y = 6100 \text{ psi}$, $\tau_{xy} = 2750 \text{ psi}$

Using Mohr's circle, determine (a) the principal stresses, and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-21

$$\sigma_x = 2050 \text{ psi} \quad \sigma_y = 6100 \text{ psi} \quad \tau_{xy} = 2750 \text{ psi}$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = 4075 \text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 3415 \text{ psi}$$

$$\alpha = \arctan \left(\left| \frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}} \right| \right) \quad \alpha = 53.63^\circ$$

(a) PRINCIPAL STRESSES

$$\theta_{p1} = \frac{180^\circ - \alpha}{2} \quad \theta_{p1} = 63.2^\circ \quad \leftarrow$$

$$\theta_{p2} = \frac{-\alpha}{2} \quad \theta_{p2} = -26.8^\circ \quad \leftarrow$$

$$\text{Point P1: } \sigma_1 = R + \sigma_{\text{aver}}$$

$$\sigma_1 = 7490 \text{ psi} \quad \leftarrow$$

$$\text{Point P2: } \sigma_2 = -R + \sigma_{\text{aver}}$$

$$\sigma_2 = 660 \text{ psi} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESSES

$$\theta_{s1} = \frac{-90^\circ + \alpha}{2} \quad \theta_{s1} = -18.2^\circ \quad \leftarrow$$

$$\theta_{s2} = 90^\circ + \theta_{s1} \quad \theta_{s2} = 71.8^\circ$$

$$\text{Point S1: } \sigma_{\text{aver}} = 4075 \text{ psi} \quad \leftarrow$$

$$\tau_{\text{max}} = R \quad \tau_{\text{max}} = 3415 \text{ psi} \quad \leftarrow$$

Problems 7.4-22 $\sigma_x = 0$ MPa, $\sigma_y = -23.4$ MPa, $\tau_{xy} = -9.6$ MPa

Using Mohr's circle, determine (a) the principal stresses, and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-22

$$\sigma_x = 0 \text{ MPa} \quad \sigma_y = -23.4 \text{ MPa} \quad \tau_{xy} = -9.6 \text{ MPa}$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = -11.70 \text{ MPa}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 15.1344 \text{ MPa}$$

$$\alpha = \arctan \left(\left| \frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}} \right| \right) \quad \alpha = 39.37^\circ$$

(a) PRINCIPAL STRESSES

$$\theta_{p1} = \frac{-\alpha}{2} \quad \theta_{p1} = -19.68^\circ \quad \leftarrow$$

$$\theta_{p2} = \theta_{p1} + 90^\circ \quad \theta_{p2} = 70.32^\circ \quad \leftarrow$$

$$\text{Point P1: } \sigma_1 = R + \sigma_{\text{aver}}$$

$$\sigma_1 = 3.43 \text{ MPa} \quad \leftarrow$$

$$\text{Point P2: } \sigma_2 = -R + \sigma_{\text{aver}}$$

$$\sigma_2 = -26.8 \text{ MPa} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESSES

$$\theta_{s1} = \frac{-90^\circ - \alpha}{2} \quad \theta_{s1} = -64.7^\circ \quad \leftarrow$$

$$\theta_{s2} = 90^\circ + \theta_{s1} \quad \theta_{s2} = 25.3^\circ$$

$$\text{Point S1: } \sigma_{\text{aver}} = -11.70 \text{ MPa}$$

$$\tau_{\text{max}} = R \quad \tau_{\text{max}} = 15.13 \text{ MPa} \quad \leftarrow$$

Problems 7.4-23 $\sigma_x = 7300$ psi, $\sigma_y = 0$ psi, $\tau_{xy} = 1300$ psi

Using Mohr's circle, determine (a) the principal stresses, and (b) the maximum shear stresses and associated normal stresses. Show all results on sketches of properly oriented elements.

Solution 7.4-23

$$\sigma_x = 7300 \text{ psi} \quad \sigma_y = 0 \text{ psi} \quad \tau_{xy} = 1300 \text{ psi}$$

$$\sigma_{\text{aver}} = \frac{\sigma_x + \sigma_y}{2} \quad \sigma_{\text{aver}} = 3650 \text{ psi}$$

$$R = \sqrt{(\sigma_x - \sigma_{\text{aver}})^2 + \tau_{xy}^2} \quad R = 3875 \text{ psi}$$

$$\alpha = \arctan \left(\left| \frac{\tau_{xy}}{\sigma_x - \sigma_{\text{aver}}} \right| \right) \quad \alpha = 19.60^\circ$$

(a) PRINCIPAL STRESSES

$$\theta_{p1} = \frac{\alpha}{2} \quad \theta_{p1} = 9.80^\circ \quad \leftarrow$$

$$\theta_{p2} = \frac{\alpha + 180^\circ}{2} \quad \theta_{p2} = 99.8^\circ$$

$$\text{Point P1: } \sigma_1 = R + \sigma_{\text{aver}}$$

$$\sigma_1 = 7525 \text{ psi} \quad \leftarrow$$

$$\text{Point P2: } \sigma_2 = -R + \sigma_{\text{aver}}$$

$$\sigma_2 = -225 \text{ psi}$$

(b) MAXIMUM SHEAR STRESSES

$$\theta_{s1} = \frac{-90^\circ + \alpha}{2} \quad \theta_{s1} = -35.2^\circ$$

$$\theta_{s2} = 90^\circ + \theta_{s1} \quad \theta_{s2} = 54.8^\circ$$

$$\text{Point S1: } \sigma_{\text{aver}} = 3650 \text{ psi} \quad \leftarrow$$

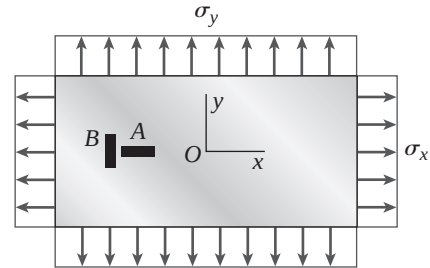
$$\tau_{\text{max}} = R \quad \tau_{\text{max}} = 3875 \text{ psi} \quad \leftarrow$$

Hooke's Law for Plane Stress

When solving the problems for Section 7.5, assume that the material is linearly elastic with modulus of elasticity E and Poisson's ratio ν .

Problem 7.5-1 A rectangular steel plate with thickness $t = 5/8$ in. is subjected to uniform normal stresses σ_x and σ_y , as shown in the figure. Strain gages A and B, oriented in the x and y directions, respectively, are attached to the plate. The gage readings give normal strains $\epsilon_x = 0.00065$ (elongation) and $\epsilon_y = 0.00040$ (elongation).

Knowing that $E = 30 \times 10^6$ psi and $\nu = 0.3$, determine the stresses σ_x and σ_y and the change Δt in the thickness of the plate.



Probs. 7.5-1 and 7.5-2

Solution 7.5-1

$$E = 30(10^6) \text{ psi} \quad \nu = 0.3$$

$$t = \frac{5}{8} \text{ in.} \quad \epsilon_x = 0.00065 \quad \epsilon_y = 0.00040$$

NORMAL STRESSES (Eqs 7-41a, b)

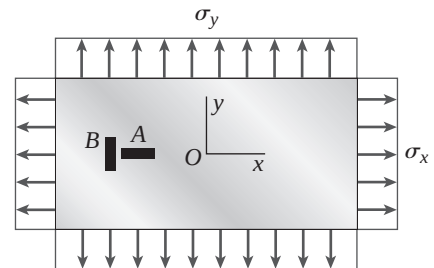
$$\sigma_x = \frac{E}{(1 - \nu^2)} (\epsilon_x + \nu \epsilon_y) \quad \boxed{\sigma_x = 25,385 \text{ psi}}$$

$$\sigma_y = \frac{E}{(1 - \nu^2)} (\epsilon_y + \nu \epsilon_x) \quad \boxed{\sigma_y = 19,615 \text{ psi}}$$

$$\text{CHANGE IN PLATE THICKNESS} \quad \epsilon_z = \frac{-\nu}{E} (\sigma_x + \sigma_y) \quad \epsilon_z = -4.5 \times 10^{-4}$$

$$\Delta t = \epsilon_z t = -2.813 \times 10^{-4} \text{ in.}$$

Problem 7.5-2 Solve the preceding problem if the thickness of the steel plate is $t = 12$ mm, the gage readings are $\epsilon_x = 530 \times 10^{-6}$ (elongation) and $\epsilon_y = -210 \times 10^{-6}$ (shortening), the modulus is $E = 200$ GPa, and Poisson's ratio is $\nu = 0.30$.



Solution 7.5-2

$$E = 200 \text{ GPa} \quad \nu = 0.3$$

$$t = 12 \text{ mm} \quad \epsilon_x = 530 (10^{-6})$$

$$\epsilon_y = -210 (10^{-6})$$

NORMAL STRESSES (Eqs 7-41a, b)

$$\sigma_x = \frac{E}{(1 - \nu^2)} (\epsilon_x + \nu \epsilon_y) \quad \boxed{\sigma_x = 102.6 \text{ MPa}}$$

$$\sigma_y = \frac{E}{(1 - \nu^2)} (\epsilon_y + \nu \epsilon_x) \quad \boxed{\sigma_y = -11.21 \text{ MPa}}$$

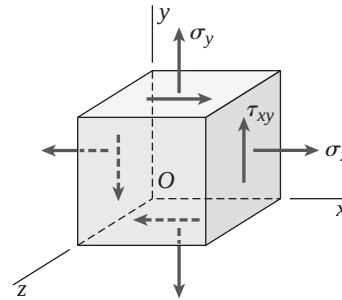
CHANGE IN PLATE THICKNESS

$$\epsilon_z = \frac{-\nu}{E} (\sigma_x + \sigma_y) \quad \epsilon_z = -1.371 \times 10^{-4}$$

$$\Delta t = \epsilon_z t = -1.646 \times 10^{-3} \text{ mm}$$

Problem 7.5-3 Assume that the normal strains ϵ_x and ϵ_y for an element in *plane stress* (see figure) are measured with strain gages.

- Obtain a formula for the normal strain ϵ_z in the z direction in terms of ϵ_x , ϵ_y , and Poisson's ratio ν .
- Obtain a formula for the dilatation e in terms of ϵ_x , ϵ_y , and Poisson's ratio ν .

**Solution 7.5-3 Plane stress**

Given: ϵ_x , ϵ_y , ν

(a) NORMAL STRAIN ϵ_z

$$\text{Eq. (7-40c): } \epsilon_z = -\frac{\nu}{E} (\sigma_x + \sigma_y)$$

$$\text{Eq. (7-41a): } \sigma_x = \frac{E}{(1 - \nu^2)} (\epsilon_x + \nu \epsilon_y)$$

$$\text{Eq. (7-41b): } \sigma_y = \frac{E}{(1 - \nu^2)} (\epsilon_y + \nu \epsilon_x)$$

Substitute σ_x and σ_y into the first equation and simplify:

$$\epsilon_z = -\frac{\nu}{1 - \nu} (\epsilon_x + \epsilon_y) \quad \leftarrow$$

(b) DILATATION

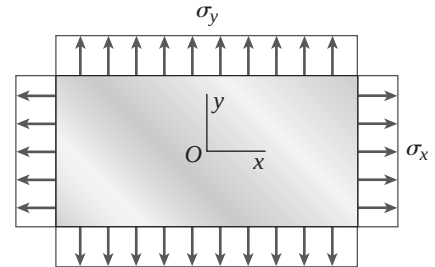
$$\text{Eq. (7-49): } e = \frac{1 - 2\nu}{E} (\sigma_x + \sigma_y)$$

Substitute σ_x and σ_y from above and simplify:

$$e = \frac{1 - 2\nu}{1 - \nu} (\epsilon_x + \epsilon_y) \quad \leftarrow$$

Problem 7.5-4 A cast iron plate in *biaxial stress* is subjected to tensile stresses $\sigma_x = 31$ MPa and $\sigma_y = 17$ MPa (see figure). The corresponding strains in the plate are $\epsilon_x = 240 \times 10^{-6}$ and $\epsilon_y = 85 \times 10^{-6}$.

Determine Poisson's ratio ν and the modulus of elasticity E for the material.



Probs. 7.5-4 through 7.5-7

Solution 7.5-4

$$\sigma_x = 31 \text{ MPa} \quad \sigma_y = 17 \text{ MPa} \quad \epsilon_x = 240(10^{-6}) \quad \epsilon_y = 85(10^{-6})$$

Solve Eqs. 7-40a and b for E and equate the two expressions;
solve for ν ; then substitute ν to find E .

$$\frac{\sigma_x - \nu\sigma_y}{\epsilon_x} - \left(\frac{\sigma_y - \nu\sigma_x}{\epsilon_y} \right) = 0$$

$$\text{so } \boxed{E = \frac{\sigma_x - \nu\sigma_y}{\epsilon_x} = 112 \text{ GPa}}$$

where

$$\text{Eq. 7-40a: } \epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad \text{Eq. 7-40b: } \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

Problem 7.5-5 Solve the preceding problem for a steel plate with $\sigma_x = 11,600$ psi (tension), $\sigma_y = 5,700$ psi (compression), $\epsilon_x = 450 \times 10^{-6}$ (elongation), and $\epsilon_y = -310 \times 10^{-6}$ (shortening).

Solution 7.5-5

$$\sigma_x = 11,600 \text{ psi} \quad \sigma_y = -5,700 \text{ psi} \quad \epsilon_x = 450(10^{-6}) \quad \epsilon_y = -310(10^{-6})$$

Solve Eqs. 7-40a and b for E and equate the two expressions; solve for ν ; then substitute ν to find E .

$$\frac{\sigma_x - \nu\sigma_y}{\epsilon_x} - \left(\frac{\sigma_y - \nu\sigma_x}{\epsilon_y} \right) = 0$$

$$\text{so } \boxed{E = \frac{\sigma_x - \nu\sigma_y}{\epsilon_x} = 29,560 \text{ ksi}}$$

where

$$\text{Eq. 7-40a: } \epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad \text{Eq. 7-40b: } \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

Problem 7.5-6 A rectangular plate in *biaxial stress* (see figure) is subjected to normal stresses $\sigma_x = 67$ MPa (tension) and $\sigma_y = -23$ MPa (compression). The plate has dimensions $400 \times 550 \times 20$ mm and is made of steel with $E = 200$ GPa and $\nu = 0.30$.

- Determine the maximum in-plane shear strain $\gamma_{\max}$ in the plate.
- Determine the change Δt in the thickness of the plate.
- Determine the change ΔV in the volume of the plate.

Solution 7.5-6

(a) CHANGE IN VOLUME $E = 200$ GPa $\nu = 0.30$ $\sigma_x = 67$ MPa $\sigma_y = -23$ MPa

$$G = \frac{E}{2(1 + \nu)} = 76.923 \text{ GPa} \quad b = 400 \text{ mm} \quad h = 550 \text{ mm} \quad t = 20 \text{ mm}$$

PRINCIPAL STRESSES: $\sigma_1 = \sigma_x$ $\sigma_2 = \sigma_y$ $\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = 45 \text{ MPa}$

$$\gamma_{\max} = \frac{\tau_{\max}}{G} = 5.85 \times 10^{-4}$$

(b) CHANGE IN THICKNESS $\epsilon_z = \frac{-\nu}{E}(\sigma_x + \sigma_y) = -6.6 \times 10^{-5}$

$$\Delta t = \epsilon_z t = -1.32 \times 10^{-3} \text{ mm} \quad \text{decrease}$$

(c) CHANGE IN VOLUME $V_0 = bht = 4.4 \times 10^6 \text{ mm}^3$

Eq. 7-49:
$$\Delta V = V_0 \left(\frac{1 - 2\nu}{E} \right) (\sigma_x + \sigma_y) = 387 \text{ mm}^3$$

Problem 7.5-7 Solve the preceding problem for an aluminum plate with $\sigma_x = 12,000$ psi (tension), $\sigma_y = -3000$ psi (compression), dimensions $20 \times 30 \times 0.5$ in., $E = 10.5 \times 10^6$ psi, and $\nu = 0.33$.

Solution 7.5-7 Biaxial stress

$$\sigma_x = 12,000 \text{ psi} \quad \sigma_y = -3000 \text{ psi}$$

$$E = 10.5 \times 10^6 \text{ psi} \quad \nu = 0.33$$

Dimensions of plate: 20 in. $\times$ 30 in. $\times$ 0.5 in.

Shear modulus (Eq. 7-39):

$$G = \frac{E}{2(1 + \nu)} = 3.9474 \times 10^6 \text{ psi}$$

(a) MAXIMUM IN-PLANE SHEAR STRAIN

Principal stresses: $\sigma_1 = 12,000 \text{ psi}$

$$\sigma_2 = -3000 \text{ psi}$$

$$\text{Eq. (7-26): } \tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = 7500 \text{ psi}$$

$$\text{Eq. (7-36): } \gamma_{\max} = \frac{\tau_{\max}}{G} = 1900 \times 10^{-6} \quad \leftarrow$$

(b) CHANGE IN THICKNESS

$$\begin{aligned} \text{Eq. (7-40c): } \varepsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y) \\ &= -282.9 \times 10^{-6} \end{aligned}$$

$$\Delta t = \varepsilon_z t = -141 \times 10^{-6} \text{ in.} \quad \leftarrow$$

(Decrease in thickness)

(c) CHANGE IN VOLUME

$$\text{From Eq. (7-49): } \Delta V = V_0 \left(\frac{1 - 2\nu}{E} \right) (\sigma_x + \sigma_y)$$

$$V_0 = (20)(30)(0.5) = 300 \text{ in.}^3$$

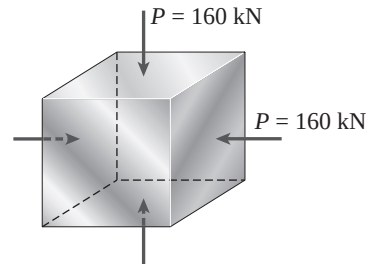
$$\text{Also, } \left(\frac{1 - 2\nu}{E} \right) (\sigma_x + \sigma_y) = 291.4 \times 10^{-6}$$

$$\begin{aligned} \therefore \Delta V &= (300 \text{ in.}^3)(291.4 \times 10^{-6}) \\ &= 0.0874 \text{ in.}^3 \quad \leftarrow \end{aligned}$$

(Increase in volume)

Problem 7.5-8 A brass cube 48 mm on each edge is compressed in two perpendicular directions by forces $P = 160 \text{ kN}$ (see figure).

- (a) Calculate the change ΔV in the volume of the cube and the strain energy U stored in the cube, assuming $E = 100 \text{ GPa}$ and $\nu = 0.34$.
 (b) Repeat part (a) if the cube is made of an aluminum alloy with $E = 73 \text{ GPa}$ and $\nu = 0.33$.

**Solution 7.5-8**

$$b = 48 \text{ mm} \quad P = 160 \text{ kN}$$

$$\text{(a) BRASS CUBE} \quad E_b = 100 \text{ GPa} \quad \nu_b = 0.34$$

$$\sigma_x = \frac{-P}{b^2} = -69.444 \text{ MPa} \quad \sigma_y = \sigma_x$$

$$\text{CHANGE IN VOLUME} \quad \text{Eq. 7-49: } e = \frac{1 - 2\nu_b}{E_b} (\sigma_x + \sigma_y) = -4.444 \times 10^{-4}$$

$$V_0 = b^3 = 1.106 \times 10^5 \text{ mm}^3$$

$$\Delta V_b = e V_0 = -49.2 \text{ mm}^3$$

$$\text{STRAIN ENERGY} \quad \text{Eq. 7-53: } u_b = \frac{1}{2E_b} (\sigma_x^2 + \sigma_y^2 - 2\nu_b \sigma_x \sigma_y) = 0.03183 \text{ MPa}$$

$$U_b = u_b V_0 = 3.52 \text{ J}$$

(b) ALUMINUM ALLOY CUBE $E_a = 73 \text{ GPa}$ $\nu_a = 0.33$

CHANGE IN VOLUME Eq. 7-49: $e = \frac{1 - 2\nu_a}{E_a}(\sigma_x + \sigma_y) = -6.469 \times 10^{-4}$

$$V_0 = b^3 = 1.106 \times 10^5 \text{ mm}^3$$

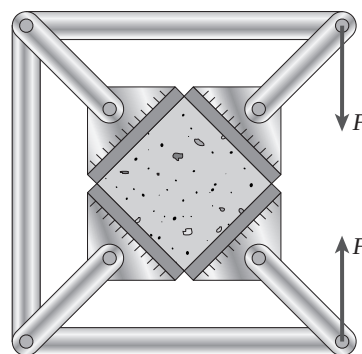
$$\Delta V_a = eV_0 = -71.5 \text{ mm}^3$$

STRAIN ENERGY Eq. 7-53: $u_a = \frac{1}{2E_a}(\sigma_x^2 + \sigma_y^2 - 2\nu_b\sigma_x\sigma_y) = 0.0436 \text{ MPa}$

$$U_a = u_aV_0 = 4.82 \text{ J}$$

Problem 7.5-9 A 4.0-inch cube of concrete ($E = 4.5 \times 10^6 \text{ psi}$, $\nu = 0.2$) is compressed in *biaxial stress* by means of a framework that is loaded as shown in the figure.

Assuming that each load F equals 25 k, determine the change ΔV in the volume of the cube and the strain energy U stored in the cube.



Solution 7.5-9 Biaxial stress – concrete cube

$$b = 4 \text{ in.} \quad F = 25 \text{ kip}$$

PROPERTIES AND NORMAL (BIAXIAL) STRESS $E = 4.5(10^6) \text{ psi}$ $\nu = 0.2$

STATICS $P = \sqrt{2}F = 35.355 \text{ kip}$ $\sigma_x = \frac{-P}{b^2} = -2210 \text{ psi}$ $\sigma_y = \sigma_x$

CHANGE IN VOLUME Eq. 7-49: $e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y) = -5.893 \times 10^{-4}$

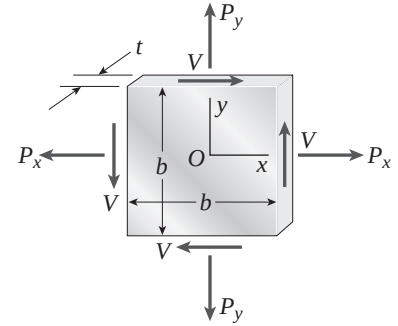
$$V_0 = b^3 = 64 \text{ in.}^3$$

$$\Delta V = eV_0 = -0.0377 \text{ in.}^3$$

STRAIN ENERGY Eq. 7-53: $u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) = 0.868 \text{ psi}$

$$U = uV_0 = 55.6 \text{ in.-lb}$$

Problem 7.5-10 A square plate of width b and thickness t is loaded by normal forces P_x and P_y and by shear forces V , as shown in the figure. These forces produce uniformly distributed stresses acting on the side faces of the plate.



- Calculate the change ΔV in the volume of the plate and the strain energy U stored in the plate if the dimensions are $b = 600$ mm and $t = 40$ mm; the plate is made of magnesium with $E = 41$ GPa and $\nu = 0.35$; and the forces are $P_x = 420$ kN, $P_y = 210$ kN, and $V = 96$ kN.
- Find the maximum permissible thickness of the plate when the strain energy U must be at least 62 J. (Assume that all other numerical values in part (a) are unchanged).
- Find the minimum width b of the square plate of thickness $t = 40$ mm when the change in volume of the plate cannot exceed 0.018% of the original volume.

Solution 7.5-10

(a) CHANGE IN VOLUME AND STRAIN ENERGY STORED

PROPERTIES AND PLANE STRESS STATE $E = 41$ GPa $\nu = 0.35$ $b = 600$ mm $t = 40$ mm

$$P_x = 420 \text{ kN} \quad P_y = 210 \text{ kN} \quad V = 96 \text{ kN} \quad G = \frac{E}{2(1 + \nu)} = 15.185 \text{ GPa}$$

$$\sigma_x = \frac{P_x}{bt} = 17.5 \text{ MPa} \quad \sigma_y = \frac{P_y}{bt} = 8.75 \text{ MPa} \quad \tau_{xy} = \frac{V}{bt} = 4 \text{ MPa}$$

$$\text{CHANGE IN VOLUME} \quad \text{Eq. 7-49: } e = \frac{1 - 2\nu}{E} (\sigma_x + \sigma_y) = 1.921 \times 10^{-4}$$

$$V_0 = b^2 t = 0.0144 \text{ m}^3$$

$$\Delta V = e V_0 = 2766 \text{ mm}^3 \quad \text{increase} \quad \frac{\Delta V}{V_0} = 0.01921\%$$

$$\text{STRAIN ENERGY} \quad \text{Eq. 7-53: } u = \frac{1}{2E} (\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \frac{\tau_{xy}^2}{2G} = 3888 \text{ Pa}$$

$$U = u V_0 = 56 \text{ N}\cdot\text{m}$$

(b) MAXIMUM PLATE THICKNESS t BASED ON REQUIRED STRAIN ENERGY STORED $b = 600$ mm $U_{\text{reqd}} = 62 \text{ N}\cdot\text{m}$

$$U_{\text{reqd}} = (b^2 t) \left[\frac{1}{2E} \left[\left(\frac{P_x}{bt} \right)^2 + \left(\frac{P_y}{bt} \right)^2 - 2\nu \left(\frac{P_x}{bt} \right) \left(\frac{P_y}{bt} \right) \right] + \frac{\left(\frac{V}{bt} \right)^2}{2G} \right] \quad \text{solve numerically} > \boxed{t = 36.1 \text{ mm}}$$

(c) MINIMUM WIDTH b OF SQUARE PLATE OF THICKNESS $t = 40$ mm IF VOLUME CHANGE CANNOT EXCEED 0.018%

$$e_{\text{reqd}} = 0.00018 \quad e_{\text{reqd}} = \frac{1 - 2\nu}{E} \left(\frac{P_x}{bt} + \frac{P_y}{bt} \right) \quad \text{solve numerically} > \boxed{b = 640 \text{ mm}}$$

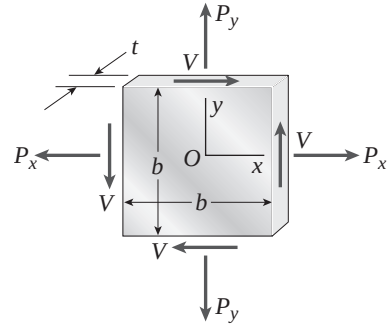
Check:

$$t = 40 \text{ mm} \quad b = 640 \text{ mm} \quad V_0 = b^2 t = 0.01638 \text{ m}^3$$

$$\Delta V = (b^2 t) \left[\frac{1 - 2\nu}{E} \left(\frac{P_x}{bt} + \frac{P_y}{bt} \right) \right] = 2950.244 \text{ mm}^3 \quad \frac{\Delta V}{V_0} = 0.018\%$$

Problem 7.5-11 Solve the preceding problem for an aluminum plate with $b = 10$ in., $t = 0.75$ in., $E = 10,600$ ksi, $\nu = 0.33$, $P_x = 96$ k, $P_y = 24$ k, and $V = 18$ k.

For part (b), assume that the required strain energy stored is 640 in.-lb. In part (c), the change in volume cannot exceed 0.05%.



Solution 7.5-11

(a) CHANGE IN VOLUME AND STRAIN ENERGY STORED

PROPERTIES AND PLANE STRESS STATE $E = 10,600$ ksi $\nu = 0.33$ $b = 10$ in. $t = 0.75$ in.

$$P_x = 96 \text{ kip} \quad P_y = 24 \text{ kip} \quad V = 18 \text{ kip} \quad G = \frac{E}{2(1 + \nu)} = 3984.962 \text{ ksi}$$

$$\sigma_x = \frac{P_x}{bt} = 12.8 \text{ ksi} \quad \sigma_y = \frac{P_y}{bt} = 3.2 \text{ ksi} \quad \tau_{xy} = \frac{V}{bt} = 2.4 \text{ ksi}$$

CHANGE IN VOLUME

$$\text{Eq. 7-49: } e = \frac{1 - 2\nu}{E} (\sigma_x + \sigma_y) = 5.132 \times 10^{-4}$$

$$V_0 = b^2 t = 75 \text{ in.}^3$$

$$\Delta V = e V_0 = 0.0385 \text{ in.}^3 \quad \text{increase} \quad \frac{\Delta V}{V_0} = 0.05132\%$$

STRAIN ENERGY

$$\text{Eq. 7-53: } u = \frac{1}{2E} (\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \frac{\tau_{xy}^2}{2G} = 7.659 \text{ psi}$$

$$U = u V_0 = 574 \text{ in.-lb}$$

(b) MAXIMUM PLATE THICKNESS t BASED ON REQUIRED STRAIN ENERGY STORED $b = 10$ in. $U_{\text{reqd}} = 640 \text{ in.-lb}$

$$U_{\text{reqd}} = (b^2 t) \left[\frac{1}{2E} \left[\left(\frac{P_x}{bt} \right)^2 + \left(\frac{P_y}{bt} \right)^2 - 2\nu \left(\frac{P_x}{bt} \right) \left(\frac{P_y}{bt} \right) \right] + \frac{\left(\frac{V}{bt} \right)^2}{2G} \right] \quad \text{solve numerically} > \boxed{t = 0.673 \text{ in.}}$$

(c) MINIMUM WIDTH b OF SQUARE PLATE OF THICKNESS $t = 0.75$ IN. IF VOLUME CHANGE CANNOT EXCEED 0.05%

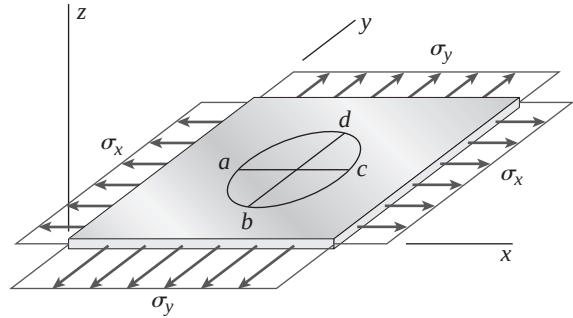
$$e_{\text{reqd}} = 0.0005 \quad e_{\text{reqd}} = \frac{1 - 2\nu}{E} \left(\frac{P_x}{bt} + \frac{P_y}{bt} \right) \quad \text{solve numerically} > \boxed{b = 10.26 \text{ in.}}$$

Check:

$$t = 0.75 \text{ in.} \quad b = 10.26 \text{ in.} \quad V_0 = b^2 t = 78.9507 \text{ in.}^3$$

$$\Delta V = (b^2 t) \left[\frac{1 - 2\nu}{E} \left(\frac{P_x}{bt} + \frac{P_y}{bt} \right) \right] = 0.039 \text{ in.}^3 \quad \frac{\Delta V}{V_0} = 0.05\%$$

Problem 7.5-12 A circle of diameter $d = 200$ mm is etched on a brass plate (see figure). The plate has dimensions $400 \times 400 \times 20$ mm. Forces are applied to the plate, producing uniformly distributed normal stresses $\sigma_x = 59$ MPa and $\sigma_y = -17$ MPa. Calculate the following quantities: (a) the change in length Δac of diameter ac ; (b) the change in length Δbd of diameter bd ; (c) the change Δt in the thickness of the plate; (d) the change ΔV in the volume of the plate, and (e) the strain energy U stored in the plate; (f) maximum permissible thickness of the plate when strain energy U must be at least 78.4 J; and (g) maximum permissible value of normal stress σ_x when the change in volume of the plate cannot exceed 0.015% of the original volume. (Assume $E = 100$ GPa and $\nu = 0.34$.)



Solution 7.5-12

PROPERTIES AND BIAxIAL STRESS STATE $E = 100$ GPa $\nu = 0.34$ $b = 400$ mm $t = 20$ mm $d = 200$ mm

$$\sigma_x = 59 \text{ MPa} \quad \sigma_y = -17 \text{ MPa} \quad G = \frac{E}{2(1 + \nu)} = 37.313 \text{ GPa}$$

(a) CHANGE IN LENGTH OF ac

$$\text{Eq. 7-40a: } \varepsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y) = 6.478 \times 10^{-4} \quad \boxed{\Delta ac = \varepsilon_x d = 0.1296 \text{ mm}} \quad \text{increase}$$

(b) CHANGE IN LENGTH OF bd

$$\text{Eq. 7-40b: } \varepsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x) = -3.706 \times 10^{-4} \quad \boxed{\Delta bd = \varepsilon_y d = -0.074 \text{ mm}} \quad \text{decrease}$$

(c) CHANGE IN THICKNESS OF PLATE

$$\text{Eq. 7-40c: } \varepsilon_z = \frac{-\nu}{E} (\sigma_x + \sigma_y) = -1.428 \times 10^{-4} \quad \boxed{\Delta t = \varepsilon_z t = -2.86 \times 10^{-3} \text{ mm}} \quad \text{decrease}$$

(d) CHANGE IN VOLUME

$$\text{Eq. 7-49: } e = \frac{1 - 2\nu}{E} (\sigma_x + \sigma_y) = 1.344 \times 10^{-4} \quad V_0 = b^2 t = 0.0032 \text{ m}^3 \quad \boxed{\Delta V = e V_0 = 430 \text{ mm}^3} \quad \text{increase}$$

$$\frac{\Delta V}{V_0} = 0.01344\%$$

(e) STRAIN ENERGY $\tau_{xy} = 0$

$$\text{Eq. 7-53: } u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \frac{\tau_{xy}^2}{2G} = 22,260 \text{ Pa} \quad \boxed{U = uV_0 = 71.2 \text{ N}\cdot\text{m}}$$

(f) MAXIMUM PLATE THICKNESS t BASED ON REQUIRED STRAIN ENERGY STORED $b = 400 \text{ mm}$ $U_{\text{reqd}} = 78.4 \text{ N}\cdot\text{m}$

$$U_{\text{reqd}} = (b^2 t) \left[\frac{1}{2E} [(\sigma_x)^2 + (\sigma_y)^2 - 2\nu(\sigma_x)(\sigma_y)] + \frac{(\tau_{xy})^2}{2G} \right] \quad \text{solve numerically} > \boxed{t = 22 \text{ mm}}$$

(g) MAXIMUM STRESS σ_x IF VOLUME CHANGE CANNOT EXCEED 0.015%

$$e_{\text{reqd}} = 0.00015 \quad e_{\text{reqd}} = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y) \quad \boxed{\sigma_{x\text{max}} = \frac{Ee_{\text{reqd}}}{1 - 2\nu} - \sigma_y = 63.9 \text{ MPa}}$$

Check:

$$t = 20 \text{ mm} \quad b = 400 \text{ mm} \quad V_0 = b^2 t = 3.2 \times 10^{-3} \text{ m}^3$$

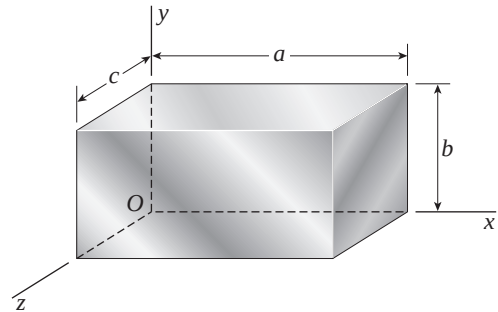
$$\Delta V = (b^2 t) \left[\frac{1 - 2\nu}{E} (\sigma_x + \sigma_y) \right] = 480 \text{ mm}^3 \quad \frac{\Delta V}{V_0} = 0.015\%$$

Triaxial Stress

When solving the problems for Section 7.6, assume that the material is linearly elastic with modulus of elasticity E and Poisson's ratio ν .

Problem 7.6-1 An element of aluminum in the form of a rectangular parallelepiped (see figure) of dimensions $a = 5.5 \text{ in.}$, $b = 4.5 \text{ in.}$ and $c = 3.5 \text{ in.}$ is subjected to *triaxial stresses* $\sigma_x = 12,500 \text{ psi}$, $\sigma_y = -5000 \text{ psi}$, and $\sigma_z = -1400 \text{ psi}$ acting on the x , y , and z faces, respectively.

Determine the following quantities: (a) the maximum shear stress τ_{max} in the material; (b) the changes Δa , Δb , and Δc in the dimensions of the element; (c) the change ΔV in the volume; (d) the strain energy U stored in the element; (e) the maximum value of σ_x when the change in volume must be limited to 0.021%; and (f) the required value of σ_x when the strain energy must be 900 in.-lb. (Assume $E = 10,400 \text{ ksi}$ and $\nu = 0.33$.)



Solution 7.6-1

PROPERTIES AND BIAXIAL STRESS STATE $E = 10,400 \text{ ksi}$ $\nu = 0.33$ $a = 5.5 \text{ in.}$ $b = 4.5 \text{ in.}$ $c = 3.5 \text{ in.}$

$$\sigma_x = 12,500 \text{ psi} \quad \sigma_y = -5000 \text{ psi} \quad \sigma_z = -1400 \text{ psi} \quad G = \frac{E}{2(1 + \nu)} = 3909.774 \text{ ksi}$$

(a) MAXIMUM SHEAR STRESS $\sigma_1 = \sigma_x$ $\sigma_2 = \sigma_z$ $\sigma_3 = \sigma_y$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 8750 \text{ psi}$$

(b) CHANGES IN LENGTHS OF DIMENSIONS a , b , & c

$$\text{Eq. 7-59a, b and c} \quad \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) = 1.405 \times 10^{-3}$$

$$\Delta a = a\varepsilon_x = 7.73 \times 10^{-3} \text{ in.}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_x + \sigma_z) = -8.33 \times 10^{-4}$$

$$\Delta b = \varepsilon_y b = -3.75 \times 10^{-3} \text{ in.}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) = -3.726 \times 10^{-4}$$

$$\Delta c = \varepsilon_z c = -1.3 \times 10^{-3} \text{ in.}$$

(c) CHANGE IN VOLUME

$$\text{Eq. 7-62: } e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) = 1.994 \times 10^{-4} \quad V_0 = abc = 86.625 \text{ in.}^3 \quad \Delta V = eV_0 = 0.0173 \text{ in.}^3$$

increase

$$\frac{\Delta V}{V_0} = 0.01994\%$$

(d) STRAIN ENERGY

$$\text{Eq. 7-63a: } u = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z) = 11.12 \text{ psi}$$

$$U = uV_0 = 964 \text{ in.-lb}$$

(e) FIND MAXIMUM VALUE OF σ_x IF CHANGE IN VOLUME MUST BE LIMITED TO 0.021%

$$e_{\max} = 0.021\% \quad e_{\max} = \frac{1 - 2\nu}{E}(\sigma_{x\max} + \sigma_y + \sigma_z) \quad \sigma_{x\max} = \frac{Ee_{\max}}{1 - 2\nu} - \sigma_y - \sigma_z = 12,824 \text{ psi}$$

$$\text{Check } \sigma_x + \sigma_{x\max} \quad \text{so } e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) = 0.021\%$$

(f) FIND REQUIRED VALUE OF σ_x IF THE STRAIN ENERGY MUST BE 900 IN.-LB

$$U_{\text{reqd}} = 900 \text{ in.-lb}$$

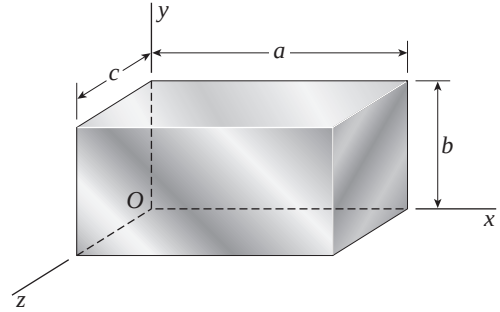
Substitute strain expressions from above then solve for $\sigma_{x,\max}$:

$$U_{\text{reqd}} = V_0 \left[\frac{1}{2} \left[\sigma_{x\max} \left[\frac{\sigma_{x\max}}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \right] + \sigma_y \left[\frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_{x\max} + \sigma_z) \right] + \sigma_z \left[\frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_{x\max} + \sigma_y) \right] \right] \right]$$

$$\text{Solve numerically to get } \sigma_{x\max} = 11,967 \text{ psi}$$

Problem 7.6-2 Solve the preceding problem if the element is steel ($E = 200 \text{ GPa}$, $\nu = 0.30$) with dimensions $a = 300 \text{ mm}$, $b = 150 \text{ mm}$, and $c = 150 \text{ mm}$, and the stresses are $\sigma_x = -62 \text{ MPa}$, $\sigma_y = -45 \text{ MPa}$, and $\sigma_z = -45 \text{ MPa}$.

For part (e), find the maximum value of σ_x if the change in volume must be limited to -0.028% . For part (f), find the required value of σ_x if the strain energy must be 60 J .



Solution 7.6-2

PROPERTIES AND BIAxIAL STRESS STATE $E = 200 \text{ GPa}$ $\nu = 0.30$ $a = 300 \text{ mm}$ $b = 150 \text{ mm}$ $c = 150 \text{ mm}$

$$\sigma_x = -62 \text{ MPa} \quad \sigma_y = -45 \text{ MPa} \quad \sigma_z = -45 \text{ MPa} \quad G = \frac{E}{2(1 + \nu)} = 76923.077 \text{ MPa}$$

(a) MAXIMUM SHEAR STRESS $\sigma_1 = \sigma_y$ $\sigma_2 = \sigma_z$ $\sigma_3 = \sigma_x$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 8.5 \text{ MPa}$$

(b) CHANGES IN LENGTHS OF DIMENSIONS a , b , & c

$$\text{Eq. 7-59a, b and c: } \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E} (\sigma_y + \sigma_z) = -1.75 \times 10^{-4}$$

$$\Delta a = a\varepsilon_x = -0.0525 \text{ mm}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E} (\sigma_x + \sigma_z) = -6.45 \times 10^{-5}$$

$$\Delta b = \varepsilon_y b = -9.67 \times 10^{-3} \text{ mm}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E} (\sigma_x + \sigma_y) = -6.45 \times 10^{-5}$$

$$\Delta c = \varepsilon_z c = -9.67 \times 10^{-3} \text{ mm}$$

(c) CHANGE IN VOLUME

$$\text{Eq. 7-62: } e = \frac{1 - 2\nu}{E} (\sigma_x + \sigma_y + \sigma_z) = -3.04 \times 10^{-4}$$

$$V_0 = abc = 0.00675 \text{ m}^3$$

$$\Delta V = eV_0 = -2.052 \times 10^{-3} \text{ mm}^3$$

decrease

$$\frac{\Delta V}{V_0} = -0.0304\%$$

(d) STRAIN ENERGY

$$\text{Eq. 7-63a: } u = \frac{1}{2} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z) = 8327.5 \text{ Pa}$$

$$U = uV_0 = 56.2 \text{ N}\cdot\text{m}$$

- (e) FIND MAXIMUM VALUE OF
- σ_x
- if change in volume must be limited to
- -0.028%

$$e_{\max} = -0.028\% \quad e_{\max} = \frac{1-2\nu}{E}(\sigma_{x\max} + \sigma_y + \sigma_z) \quad \sigma_{x\max} = \frac{Ee_{\max}}{1-2\nu} - \sigma_y - \sigma_z = -50 \text{ MPa}$$

$$\text{Check } \sigma_x = \sigma_{x\max} \quad \text{so } e = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) = -0.028\%$$

- (f) FIND REQUIRED VALUE OF
- σ_x
- if the strain energy must be 60 J

$$U_{\text{reqd}} = 60 \text{ N}\cdot\text{m}$$

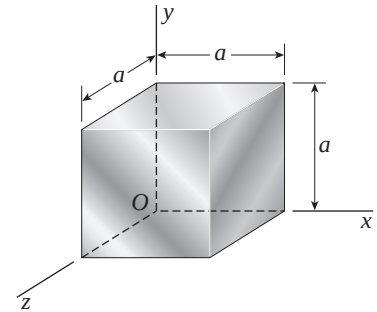
Substitute strain expressions from above, then solve for $\sigma_{x\max}$:

$$U_{\text{reqd}} = V_0 \left[\frac{1}{2} \left[\sigma_{x\max} \left[\frac{\sigma_{x\max}}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \right] + \sigma_y \left[\frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_{x\max} + \sigma_z) \right] + \sigma_z \left[\frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_{x\max} + \sigma_y) \right] \right] \right]$$

$$\text{Solve numerically to get } \boxed{\sigma_{x\max} = -65.1 \text{ MPa}}$$

Problem 7.6-3 A cube of cast iron with sides of length $a = 4.0$ in. (see figure) is tested in a laboratory under *triaxial stress*. Gages mounted on the testing machine show that the compressive strains in the material are $\epsilon_x = -225 \times 10^{-6}$ and $\epsilon_y = \epsilon_z = -37.5 \times 10^{-6}$.

Determine the following quantities: (a) the normal stresses σ_x , σ_y , and σ_z acting on the x , y , and z faces of the cube; (b) the maximum shear stress $\tau_{\max}$ in the material; (c) the change ΔV in the volume of the cube; (d) the strain energy U stored in the cube; (e) the maximum value of σ_x when the change in volume must be limited to 0.028% ; and (f) the required value of ϵ_x when the strain energy must be 38 in.-lb. (Assume $E = 14,000$ ksi and $\nu = 0.25$.)



Solution 7.6-3

PROPERTIES AND TRIAXIAL STRAINS $E = 14,000$ ksi $\nu = 0.25$ $a = 4$ in.

$$\epsilon_x = -225(10^{-6}) \quad \epsilon_y = -37.5(10^{-6}) \quad \epsilon_z = \epsilon_y$$

- (a) NORMAL STRESSES

$$\text{Eq. 7-60a: } \sigma_x = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\epsilon_x + \nu(\epsilon_y + \epsilon_z)] = -4200 \text{ psi}$$

$$\text{Eq. 7-60b: } \sigma_y = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\epsilon_y + \nu(\epsilon_x + \epsilon_z)] = -2100 \text{ psi}$$

$$\text{Eq. 7-60c: } \sigma_z = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\epsilon_z + \nu(\epsilon_y + \epsilon_x)] = -2100 \text{ psi}$$

- (b) MAXIMUM SHEAR STRESS
- $\sigma_1 = \sigma_y$
- $\sigma_2 = \sigma_z$
- $\sigma_3 = \sigma_x$

$$\boxed{\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 1050 \text{ psi}}$$

(c) CHANGE IN VOLUME

$$\text{Eq. 7-61: } e = \varepsilon_x + \varepsilon_y + \varepsilon_z = -3 \times 10^{-4} \quad V_0 = a^3 = 64 \text{ in.}^3 \quad \boxed{\Delta V = eV_0 = -0.0192 \text{ in.}^3}$$

$$\text{decrease } \frac{\Delta V}{V_0} = -0.03\%$$

(d) STRAIN ENERGY

$$\text{Eq. 7-63a: } u = \frac{1}{2} (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z) = 0.5513 \text{ psi} \quad \boxed{U = uV_0 = 35.3 \text{ in.-lb}}$$

(e) MAXIMUM VALUE OF σ_x IF CHANGE IN VOLUME MUST BE LIMITED TO 0.028%

$$e_{\max} = -0.028\% \quad \varepsilon_{x\max} = e_{\max} - \varepsilon_y - \varepsilon_z = -2.05 \times 10^{-4}$$

Check:

$$\varepsilon_x = \varepsilon_{x\max}$$

$$\boxed{\sigma_x = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)] = -3864 \text{ psi}}$$

$$\sigma_y = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z)] = -1988 \text{ psi}$$

$$\sigma_z = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_z + \nu(\varepsilon_y + \varepsilon_x)] = -1988 \text{ psi}$$

$$\text{Eq. 7-59a, b, and c } \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E} (\sigma_y + \sigma_z) = -2.05 \times 10^{-4}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E} (\sigma_x + \sigma_z) = -3.75 \times 10^{-5}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E} (\sigma_x + \sigma_y) = -3.75 \times 10^{-5}$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = -2.8 \times 10^{-4} \quad \boxed{\Delta V = eV_0 = -0.0179 \text{ in.}^3} \quad \text{decrease so } \frac{\Delta V}{V_0} = -0.028\%$$

(f) REQUIRED VALUE OF ε_x IF THE STRAIN ENERGY MUST BE 38 IN.-LB

$$U_{\text{reqd}} = 38 \text{ in.-lb}$$

Substitute stress expressions from above into eq. for u then solve for ε_x :

$$U_x = \frac{E\varepsilon_x}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)]$$

$$U_y = \frac{E\varepsilon_y}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z)]$$

$$U_z = \frac{E\varepsilon_z}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_z + \nu(\varepsilon_y + \varepsilon_x)]$$

$$U_{\text{reqd}} = V_0 \left[\frac{1}{2} (U_x + U_y + U_z) \right]$$

Solve numerically to get $\varepsilon_{x\max} = -235(10^{-6})$

CHECK $\varepsilon_x = \varepsilon_{x\max}$

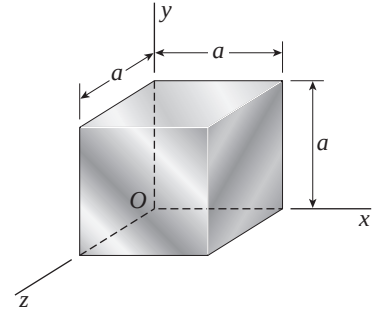
$$U_x = \frac{E\varepsilon_x}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)]$$

$$U_y = \frac{E\varepsilon_y}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z)]$$

$$U_z = \frac{E\varepsilon_z}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_z + \nu(\varepsilon_y + \varepsilon_x)]$$

$$U = V_0 \left[\frac{1}{2} (U_x + U_y + U_z) \right] = 38.022 \text{ in.-lb}$$

Problem 7.6-4 Solve the preceding problem if the cube is granite ($E = 80 \text{ GPa}$, $\nu = 0.25$) with dimensions $a = 89 \text{ mm}$ and compressive strains $\varepsilon_x = -690 \times 10^{-6}$ and $\varepsilon_y = \varepsilon_z = 255 \times 10^{-6}$. For part (e), find the maximum value of σ_x when the change in volume must be limited to 0.11%. For part (f), find the required value of ε_x when the strain energy must be 33 J.



Solution 7.6-4

PROPERTIES AND TRIAXIAL STRAINS $E = 80 \text{ GPa}$ $\nu = 0.25$ $a = 89 \text{ mm}$

$$\varepsilon_x = -690(10^{-6}) \quad \varepsilon_y = -255(10^{-6}) \quad \varepsilon_z = \varepsilon_y$$

(a) NORMAL STRESSES

$$\text{Eq. 7-60a: } \sigma_x = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)] = -82.6 \text{ MPa}$$

$$\text{Eq. 7-60b: } \sigma_y = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z)] = -54.7 \text{ MPa}$$

$$\text{Eq. 7-60c: } \sigma_z = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_z + \nu(\varepsilon_y + \varepsilon_x)] = -54.7 \text{ MPa}$$

(b) MAXIMUM SHEAR STRESS $\sigma_1 = \sigma_y$ $\sigma_2 = \sigma_z$ $\sigma_3 = \sigma_x$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = 13.92 \text{ MPa}$$

(c) CHANGE IN VOLUME

$$\text{Eq. 7-61: } e = \varepsilon_x + \varepsilon_y + \varepsilon_z = -1.2 \times 10^{-3} \quad V_0 = a^3 = 70,4969 \text{ mm}^3 \quad \boxed{\Delta V = eV_0 = -846 \text{ mm}^3}$$

$$\text{decrease } \frac{\Delta V}{V_0} = -0.12\%$$

(d) STRAIN ENERGY

$$\text{Eq. 7-63a: } u = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z) = 42.4368 \text{ kPa} \quad \boxed{U = uV_0 = 29.9 \text{ N}\cdot\text{m}}$$

(e) MAXIMUM VALUE OF σ_x IF CHANGE IN VOLUME MUST BE LIMITED TO 0.11%

$$e_{\max} = -0.11\% \quad \varepsilon_{x\max} = e_{\max} - \varepsilon_y - \varepsilon_z = -5.9 \times 10^{-4}$$

Check:

$$\varepsilon_x = \varepsilon_{x\max}$$

$$\boxed{\sigma_x = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)] = -73 \text{ MPa}}$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z)] = -51.5 \text{ MPa}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_z + \nu(\varepsilon_y + \varepsilon_x)] = -51.5 \text{ MPa}$$

$$\text{Eq. 7-59a, b, and c: } \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) = -5.9 \times 10^{-4}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_x + \sigma_z) = -2.55 \times 10^{-4}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) = -2.55 \times 10^{-4}$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = -1.1 \times 10^{-3} \quad \boxed{\Delta V = eV_0 = -775 \text{ mm}^3} \quad \text{decrease so } \frac{\Delta V}{V_0} = -0.11\%$$

(f) REQUIRED VALUE OF ε_x if the strain energy must be 33 J

$$U_{\text{reqd}} = 33 \text{ N}\cdot\text{m}$$

Substitute stress expressions from above into eq. for u , then solve for ε_x :

$$U_x = \frac{E\varepsilon_x}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)]$$

$$U_y = \frac{E\varepsilon_y}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z)]$$

$$U_z = \frac{E\varepsilon_z}{(1+\nu)(1-2\nu)} [(1-\nu)\varepsilon_z + \nu(\varepsilon_y + \varepsilon_x)]$$

$$U_{\text{reqd}} = V_0 \left[\frac{1}{2}(U_x + U_y + U_z) \right]$$

Solve numerically to get $\boxed{\varepsilon_{x\max} = -741(10^{-6})}$

CHECK $\varepsilon_x = \varepsilon_{x\max}$

$$U_x = \frac{E\varepsilon_x}{(1+\nu)(1-2\nu)} \left[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z) \right]$$

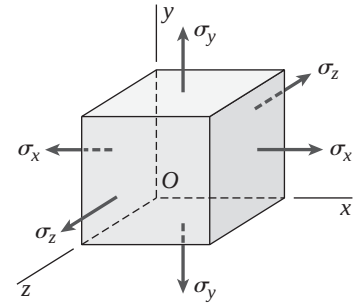
$$U_y = \frac{E\varepsilon_y}{(1+\nu)(1-2\nu)} \left[(1-\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z) \right]$$

$$U_z = \frac{E\varepsilon_z}{(1+\nu)(1-2\nu)} \left[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y) \right]$$

$$U = V_0 \left[\frac{1}{2}(U_x + U_y + U_z) \right] = 32.973 \text{ N}\cdot\text{m}$$

Problem 7.6-5 An element of aluminum is subjected to *triaxial stress* (see figure).

- (a) Find the bulk modulus K for the aluminum if the following stress and strain data is known: normal stresses are $\sigma_x = 5200$ psi (tension), $\sigma_y = -4750$ psi (compression), and $\sigma_z = -3090$ psi (compression) and normal strains in the x and y directions are $\varepsilon_x = 713.8 \times 10^{-6}$ (elongation) and $\varepsilon_y = -502.3 \times 10^{-6}$ (shortening).
- (b) If the element is replaced by one of magnesium, find modulus of elasticity E and Poisson's ratio ν if the following data is given: bulk modulus $K = 6.8 \times 10^6$ psi, normal stresses are $\sigma_x = 4550$ psi (tension), $\sigma_y = -1700$ psi (compression), and $\sigma_z = -1090$ psi (compression) and normal strain in the x direction is $\varepsilon_x = 900 \times 10^{-6}$ (elongation).



Solution 7.6-5

NUMERICAL DATA

$$\varepsilon_x = 713.8(10^{-6}) \quad \varepsilon_y = -502.3(10^{-6}) \quad \sigma_x = 5200 \quad \sigma_y = -4750 \quad \sigma_z = -3090 \quad \text{stresses are in psi}$$

(a) NORMAL STRAINS - ALUMINUM ALLOY

$$\text{Eq. 7-59a, b: } \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \quad \varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_x + \sigma_z)$$

Solve two equations simultaneously for E and ν to get $E = 10,800 \text{ ksi} \quad \nu = 0.32$

$$\text{Eq. 7-67: } \boxed{K_{Al} = \frac{E}{3(1-2\nu)} = 1 \times 10^7 \text{ psi}}$$

(b) MAGNESIUM ALLOY $K_m = 6.8(10^6)$ psi

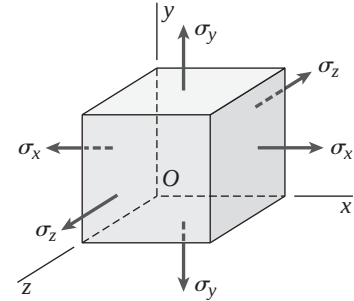
$$\text{Stresses in psi: } \sigma_x = 4550 \quad \sigma_y = -1700 \quad \sigma_z = -1090 \quad \varepsilon_x = 900(10^{-6})$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \quad K_m = \frac{E}{3(1-2\nu)}$$

Solve two equations simultaneously for E and ν to get $\boxed{E = 6139 \text{ ksi}} \quad \boxed{\nu = 0.35}$

Problem 7.6-6 Solve the preceding problem if the material is nylon.

- (a) Find the bulk modulus K for the nylon if the following stress and strain data is known: normal stresses are $\sigma_x = -3.9$ MPa, $\sigma_y = -3.2$ MPa, and $\sigma_z = -1.8$ MPa; normal strains in the x and y directions are $\varepsilon_x = -640 \times 10^{-6}$ (shortening) and $\varepsilon_y = -310 \times 10^{-6}$ (shortening).
- (b) If the element is replaced by one of polyethylene, find modulus of elasticity E and Poisson's ratio ν if the following data is given: bulk modulus $K = 2162$ MPa; normal stresses are $\sigma_x = -3.6$ MPa (compression) $\sigma_y = -2.1$ MPa (compression), and $\sigma_z = -2.1$ MPa (compression) and normal strain in the x direction is $\varepsilon_x = -1480 \times 10^{-6}$ (shortening).



Solution 7.6-6

NUMERICAL DATA

$$\varepsilon_x = -640(10^{-6}) \quad \varepsilon_y = -310(10^{-6}) \quad \sigma_x = -3.9 \quad \sigma_y = -3.2 \quad \sigma_z = -1.8 \quad \text{Stresses are in MPa.}$$

(a) NORMAL STRAINS - NYLON

$$\text{Eq. 7-59a, b: } \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \quad \varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_x + \sigma_z)$$

Solve two equations simultaneously for E and ν to get $E = 2.969$ GPa $\nu = 0.40$

$$\text{Eq. 7-67: } K = \frac{E}{3(1 - 2\nu)} = 4.95 \text{ GPa}$$

(b) MAGNESIUM ALLOY $K_m = 2162$ MPa

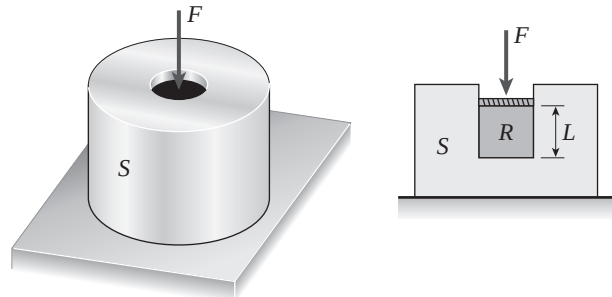
$$\text{Stresses in MPa} \quad \sigma_x = -3.6 \quad \sigma_y = -2.1 \quad \sigma_z = -2.1 \quad \varepsilon_x = -1480(10^{-6})$$

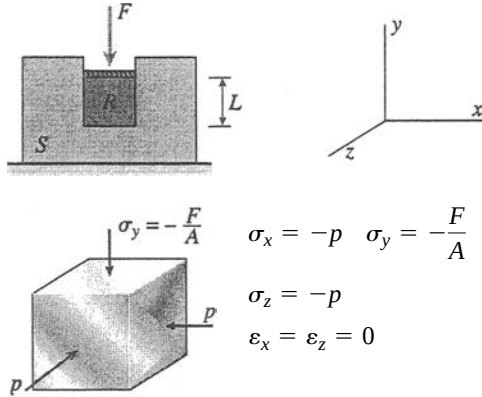
$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \quad K_m = \frac{E}{3(1 - 2\nu)}$$

Solve two equations simultaneously for E and ν to get $E = 1.297$ GPa $\nu = 0.40$

Problem 7.6-7 A rubber cylinder R of length L and cross-sectional area A is compressed inside a steel cylinder S by a force F that applies a uniformly distributed pressure to the rubber (see figure).

- (a) Derive a formula for the lateral pressure p between the rubber and the steel. (Disregard friction between the rubber and the steel, and assume that the steel cylinder is rigid when compared to the rubber.)
- (b) Derive a formula for the shortening δ of the rubber cylinder.



Solution 7.6-7 Rubber cylinder**(a) LATERAL PRESSURE**

$$\text{Eq. (7-59a): } \varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

$$\text{or } 0 = -p - \nu\left(-\frac{F}{A} - p\right)$$

$$\text{Solve for } p: p = \frac{\nu}{1 - \nu} \left(\frac{F}{A} \right) \quad \leftarrow$$

(b) SHORTENING

$$\text{Eq. (7-59b): } \varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x)$$

$$= -\frac{F}{EA} - \frac{\nu}{E}(-2p)$$

Substitute for p and simplify:

$$\varepsilon_y = \frac{F}{EA} \frac{(1 + \nu)(-1 + 2\nu)}{1 - \nu}$$

(Positive ε_y represents an increase in strain, that is, elongation.)

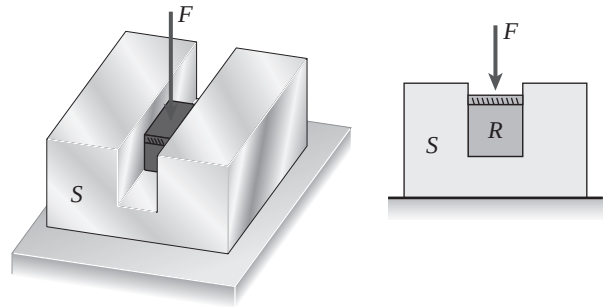
$$\delta = -\varepsilon_y L$$

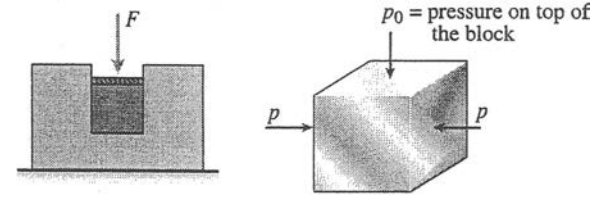
$$\delta = \frac{(1 + \nu)(1 - 2\nu)}{(1 - \nu)} \left(\frac{FL}{EA} \right) \quad \leftarrow$$

(Positive δ represents a shortening of the rubber cylinder.)

Problem 7.6-8 A block R of rubber is confined between plane parallel walls of a steel block S (see figure). A uniformly distributed pressure p_0 is applied to the top of the rubber block by a force F .

- Derive a formula for the lateral pressure p between the rubber and the steel. (Disregard friction between the rubber and the steel, and assume that the steel block is rigid when compared to the rubber.)
- Derive a formula for the dilatation e of the rubber.
- Derive a formula for the strain-energy density u of the rubber.



Solution 7.6-8 Block of rubber

$$\begin{aligned} \sigma_x &= -p \\ \sigma_y &= -p_0 \quad \sigma_z = 0 \\ \varepsilon_x &= 0 \quad \varepsilon_y \neq 0 \quad \varepsilon_z \neq 0 \end{aligned}$$

(a) LATERAL PRESSURE

$$\begin{aligned} \text{Eq. (7-59a): } \varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \\ \text{OR } 0 &= -p - \nu(-p_0) \quad \therefore p = \nu p_0 \quad \leftarrow \end{aligned}$$

(b) DILATATION

$$\begin{aligned} \text{Eq. (7-62): } e &= \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \\ &= \frac{1 - 2\nu}{E}(-p - p_0) \end{aligned}$$

Substitute for p :

$$e = -\frac{(1 + \nu)(1 - 2\nu)p_0}{E} \quad \leftarrow$$

(c) STRAIN ENERGY DENSITY

Eq. (7-63b):

$$u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E}(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z)$$

Substitute for σ_x , σ_y , σ_z , and p :

$$u = \frac{(1 - \nu^2)p_0^2}{2E} \quad \leftarrow$$

Problem 7.6-9 A solid spherical ball of magnesium alloy ($E = 6.5 \times 10^6$ psi, $\nu = 0.35$) is lowered into the ocean to a depth of 8000 ft. The diameter of the ball is 9.0 in.

- (a) Determine the decrease Δd in diameter, the decrease ΔV in volume, and the strain energy U of the ball.
 (b) At what depth will the volume change be equal to 0.0324% of the original volume?

Solution 7.6-9

NUMERICAL DATA

$$E = 6.5(10^6) \text{ psi} \quad \nu = 0.35 \quad h = 8000 \text{ ft} \quad d = 9 \text{ in.} \quad \gamma_s = 63.8 \frac{\text{lb}}{\text{ft}^3} \quad \sigma_0 = \gamma_s h$$

(a) DECREASE IN DIAMETER, DECREASE IN VOLUME, STRAIN ENERGY

$$\text{Eq. 7-65: } \varepsilon_0 = \frac{\sigma_0}{E}(1 - 2\nu) = 1.636 \times 10^{-4} \quad \boxed{\Delta d = \varepsilon_0 d = 1.472 \times 10^{-3} \text{ in.}}$$

$$\text{Eq. 7-66: } e = 3\varepsilon_0 = 4.908 \times 10^{-4} \quad V_0 = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = 381.704 \text{ in.}^3$$

$$\boxed{\Delta V = eV_0 = 0.187 \text{ in.}^3} \quad \frac{\Delta V}{V_0} = 0.049\%$$

$$\text{Eq. 7-63: with } x, y \text{ and } z \text{ normal stresses} = \sigma_0 \quad u = \frac{3(1 - 2\nu)}{2E}\sigma_0^2 = 0.87 \text{ psi}$$

$$\boxed{U = uV_0 = 332 \text{ in.-lb}}$$

(b) DEPTH AT WHICH VOLUME CHANGE IS 0.0324% OF ORIGINAL VOLUME

$$e = 3\frac{(\gamma_s h)}{E}(1 - 2\nu) \quad e = 0.0324\% \quad \boxed{h = \frac{eE}{3\gamma_s(1 - 2\nu)} = 5282 \text{ ft}}$$

Problem 7.6-10 A solid steel sphere ($E = 210$ GPa, $\nu = 0.3$) is subjected to hydrostatic pressure p such that its volume is reduced by 0.4%.

- Calculate the pressure p .
- Calculate the volume modulus of elasticity K for the steel.
- Calculate the strain energy U stored in the sphere if its diameter is $d = 150$ mm.

Solution 7.6-10 Steel sphere

$$E = 210 \text{ GPa} \quad \nu = 0.3$$

Hydrostatic pressure: $V_0 = \text{Initial volume}$

$$\Delta V = 0.004V_0$$

$$\text{Dilatation: } e = \frac{\Delta V}{V_0} = 0.004$$

(a) PRESSURE

$$\text{Eq. (7-66): } e = \frac{3\sigma_0(1 - 2\nu)}{E}$$

$$\text{or } \sigma_0 = \frac{Ee}{3(1 - 2\nu)} = 700 \text{ MPa}$$

$$\text{Pressure } p = \sigma_0 = 700 \text{ MPa} \quad \leftarrow$$

(b) VOLUME MODULUS OF ELASTICITY

$$\text{Eq. (7-69): } K = \frac{\sigma_0}{e} = \frac{700 \text{ MPa}}{0.004} = 175 \text{ GPa} \quad \leftarrow$$

(c) STRAIN ENERGY ($d = \text{diameter}$)

$$d = 150 \text{ mm} \quad r = 75 \text{ mm}$$

From Eq. (7-63b) with $\sigma_x = \sigma_y = \sigma_z = \sigma_0$:

$$u = \frac{3(1 - 2\nu)\sigma_0^2}{2E} = 1.40 \text{ MPa}$$

$$V_0 = \frac{4\pi r^3}{3} = 1767 \times 10^{-6} \text{ m}^3$$

$$U = uV_0 = 2470 \text{ N} \cdot \text{m} = 2470 \text{ J} \quad \leftarrow$$

Problem 7.6-11 A solid bronze sphere (volume modulus of elasticity $K = 14.5 \times 10^6$ psi) is suddenly heated around its outer surface. The tendency of the heated part of the sphere to expand produces uniform tension in all directions at the center of the sphere.

If the stress at the center is 12,000 psi, what is the strain? Also, calculate the unit volume change e and the strain-energy density u at the center.

Solution 7.6-11 Bronze sphere (heated)

$$K = 14.5 \times 10^6 \text{ psi}$$

$$\sigma_0 = 12,000 \text{ psi (tension at the center)}$$

STRAIN AT THE CENTER OF THE SPHERE

$$\text{Eq. (7-65): } \varepsilon_0 = \frac{\sigma_0}{E}(1 - 2\nu)$$

$$\text{Eq. (7-67): } K = \frac{E}{3(1 - 2\nu)}$$

Combine the two equations:

$$\varepsilon_0 = \frac{\sigma_0}{3K} = 276 \times 10^{-6} \quad \leftarrow$$

UNIT VOLUME CHANGE AT THE CENTER

$$\text{Eq. (7-68): } e = \frac{\sigma_0}{K} = 828 \times 10^{-6} \quad \leftarrow$$

STRAIN ENERGY DENSITY AT THE CENTER

$$\text{Eq. (7-63b) with } \sigma_x = \sigma_y = \sigma_z = \sigma_0:$$

$$u = \frac{3(1 - 2\nu)\sigma_0^2}{2E} = \frac{\sigma_0^2}{2K}$$

$$u = 4.97 \text{ psi} \quad \leftarrow$$

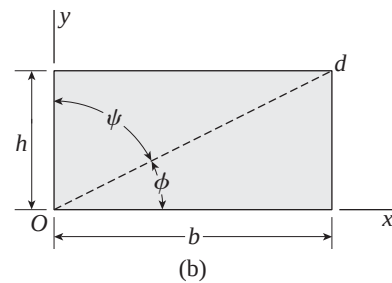
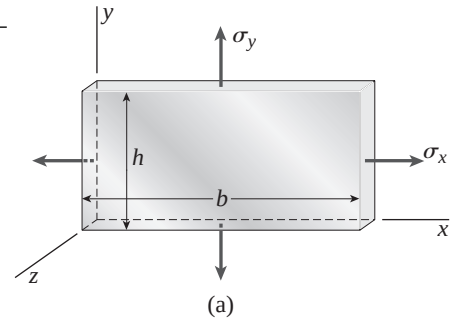
Plane Strain

When solving the problems for Section 7.7, consider only the in-plane strains (the strains in the xy plane) unless stated otherwise. Use the transformation equations of plane strain except when Mohr's circle is specified (Prob. 7.7-23 through 7.7-28).

Problem 7.7-1 A thin rectangular plate in *biaxial stress* is subjected to stresses σ_x and σ_y , as shown in part a of the figure. The width and height of the plate are $b = 7.5$ in. and $h = 2.5$ in., respectively. Measurements show that the normal strains in the x and y directions are $\epsilon_x = 285 \times 10^{-6}$ and $\epsilon_y = -190 \times 10^{-6}$, respectively.

With reference to part b of the figure, which shows a two-dimensional view of the plate, determine the following quantities.

- The increase Δd in the length of diagonal.
- The change $\Delta\phi$ in the angle ϕ between diagonal Od and the x axis.
- The change $\Delta\psi$ in the angle ψ between diagonal Od and the y axis.



Probs. 7.7-1 and 7.7-2

Solution 7.7-1

NUMERICAL DATA

$$b = 7.5 \text{ in.} \quad h = 2.5 \text{ in.} \quad \epsilon_x = 285(10^{-6}) \quad \epsilon_y = -190(10^{-6}) \quad \gamma_{xy} = 0$$

$$\phi = \arctan\left(\frac{h}{b}\right) = 18.435^\circ \quad L_d = \sqrt{b^2 + h^2} = 7.906 \text{ in.}$$

- (a) INCREASE IN LENGTH OF DIAGONAL $\theta = \phi$

$$\epsilon_{x1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta) = 2.375 \times 10^{-4} \quad \Delta d = \epsilon_{x1} L_d = 1.878 \times 10^{-3} \text{ in.}$$

- (b) CHANGE IN ANGLE ϕ

$$\text{Eq. 7-79: } \alpha = -(\epsilon_x - \epsilon_y) \sin(\theta) \cos(\theta) - \gamma_{xy} \sin^2(\theta) = -1.425 \times 10^{-4} \quad \Delta\phi = -\alpha = 1.425 \times 10^{-4}$$

(decrease, radians)

- (c) CHANGE IN ANGLE ψ

$$\text{Angle } \psi \text{ increases by the same amount that angle } \phi \text{ decreases so } \Delta\psi = -\alpha = 1.425 \times 10^{-4}$$

(increase, radians)

Problem 7.7-2 Solve the preceding problem if $b = 180$ mm and $h = 70$ mm, respectively. Measurements show that the normal strains in the x and y directions are $\epsilon_x = 390 \times 10^{-6}$, and $\epsilon_y = -240 \times 10^{-6}$, respectively.

Solution 7.7-2

NUMERICAL DATA

$$b = 180 \text{ mm} \quad h = 60 \text{ mm} \quad \epsilon_x = 390(10^{-6}) \quad \epsilon_y = -240(10^{-6}) \quad \gamma_{xy} = 0$$

$$\phi = \arctan\left(\frac{h}{b}\right) = 18.435^\circ \quad L_d = \sqrt{b^2 + h^2} = 189.737 \text{ mm}$$

(a) INCREASE IN LENGTH OF DIAGONAL $\theta = \phi$

$$\epsilon_{x1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta) = 3.27 \times 10^{-4} \quad \Delta d = \epsilon_{x1} L_d = 0.062 \text{ mm}$$

(b) CHANGE IN ANGLE ϕ

$$\text{Eq. 7-79: } \alpha = -(\epsilon_x - \epsilon_y) \sin(\theta) \cos(\theta) - \gamma_{xy} \sin(\theta)^2 = -1.89 \times 10^{-4} \quad \Delta\phi = -\alpha = 1.89 \times 10^{-4}$$

(decrease, radians)

(c) CHANGE IN ANGLE ψ

Angle ψ increases by the same amount that angle ϕ decreases so

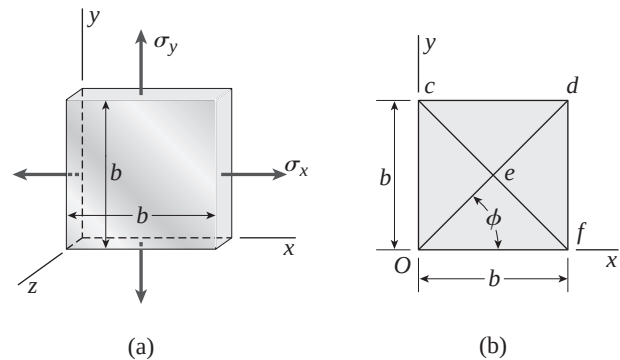
$$\Delta\psi = -\alpha = 1.89 \times 10^{-4}$$

(increase, radians)

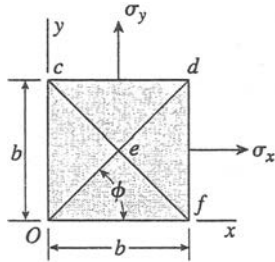
Problem 7.7-3 A thin square plate in *biaxial stress* is subjected to stresses σ_x and σ_y , as shown in part a of the figure. The width of the plate is $b = 12.0$ in. Measurements show that the normal strains in the x and y directions are $\epsilon_x = 427 \times 10^{-6}$ and $\epsilon_y = 113 \times 10^{-6}$, respectively.

With reference to part b of the figure, which shows a two-dimensional view of the plate, determine the following quantities.

- The increase Δd in the length of diagonal Od .
- The change $\Delta\phi$ in the angle ϕ between diagonal Od and the x axis.
- The shear strain γ associated with diagonals Od and cf (that is, find the decrease in angle ced).



PROBS. 7.7-3 and 7.7-4

Solution 7.7-3 Square plate in biaxial stress

$$b = 12.0 \text{ in.} \quad \epsilon_x = 427 \times 10^{-6}$$

$$\epsilon_y = 113 \times 10^{-6}$$

$$\phi = 45^\circ \quad \gamma_{xy} = 0$$

$$L_d = b\sqrt{2} = 16.97 \text{ in.}$$

(a) INCREASE IN LENGTH OF DIAGONAL

$$\epsilon_{x_1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\text{For } \theta = \phi = 45^\circ: \epsilon_{x_1} = 270 \times 10^{-6}$$

$$\Delta d = \epsilon_{x_1} L_d = 0.00458 \text{ in.} \quad \leftarrow$$

(b) CHANGE IN ANGLE ϕ

$$\text{Eq. (7-79): } \alpha = -(\epsilon_x - \epsilon_y) \sin \theta \cos \theta - \gamma_{xy} \sin^2 \theta$$

$$\text{For } \theta = \phi = 45^\circ: \alpha = -157 \times 10^{-6} \text{ rad}$$

Minus sign means line Od rotates clockwise (angle ϕ decreases.)

$$\Delta \phi = 157 \times 10^{-6} \text{ rad (decrease)} \quad \leftarrow$$

(c) SHEAR STRAIN BETWEEN DIAGONALS

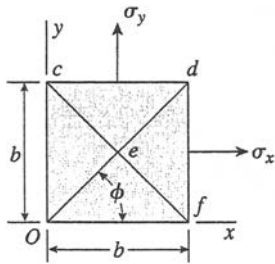
$$\text{Eq. (7-83b): } \frac{\gamma_{x_1 y_1}}{2} = -\frac{\epsilon_x - \epsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta$$

$$\text{For } \theta = \phi = 45^\circ: \gamma_{x_1 y_1} = -314 \times 10^{-6} \text{ rad}$$

(Negative strain means angle ced increases.)

$$\gamma = -314 \times 10^{-6} \text{ rad} \quad \leftarrow$$

Problem 7.7-4 Solve the preceding problem if $b = 225 \text{ mm}$, $\epsilon_x = 845 \times 10^{-6}$, and $\epsilon_y = 211 \times 10^{-6}$.

Solution 7.7-4 Square plate in biaxial stress

$$b = 225 \text{ mm} \quad \epsilon_x = 845 \times 10^{-6}$$

$$\epsilon_y = 211 \times 10^{-6} \quad \phi = 45^\circ \quad \gamma_{xy} = 0$$

$$L_d = b\sqrt{2} = 318.2 \text{ mm}$$

(a) INCREASE IN LENGTH OF DIAGONAL

$$\epsilon_{x_1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\text{For } \theta = \phi = 45^\circ: \epsilon_{x_1} = 528 \times 10^{-6}$$

$$\Delta d = \epsilon_{x_1} L_d = 0.168 \text{ mm} \quad \leftarrow$$

(b) CHANGE IN ANGLE ϕ

$$\text{Eq. (7-79): } \alpha = -(\epsilon_x - \epsilon_y) \sin \theta \cos \theta - \gamma_{xy} \sin^2 \theta$$

$$\text{For } \theta = \phi = 45^\circ: \alpha = -317 \times 10^{-6} \text{ rad}$$

Minus sign means line Od rotates clockwise (angle ϕ decreases).

$$\Delta \phi = 317 \times 10^{-6} \text{ rad (decrease)} \quad \leftarrow$$

(c) SHEAR STRAIN BETWEEN DIAGONALS

$$\text{Eq. (7-83b): } \frac{\gamma_{x_1 y_1}}{2} = -\frac{\epsilon_x - \epsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta$$

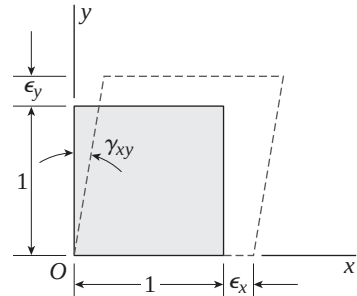
$$\text{For } \theta = \phi = 45^\circ: \gamma_{x_1 y_1} = -634 \times 10^{-6} \text{ rad}$$

(Negative strain means angle ced increases.)

$$\gamma = -634 \times 10^{-6} \text{ rad}$$

Problem 7.7-5 An element of material subjected to *plane strain* (see figure) has strains of $\epsilon_x = 280 \times 10^{-6}$ and $\epsilon_y = 420 \times 10^{-6}$, and $\gamma_{xy} = 150 \times 10^{-6}$.

Calculate the strains for an element oriented at an angle $\theta = 35^\circ$ and show these strains on a sketch of a properly oriented element.



Probs. 7.7-5 through 7.7-10

Solution 7.7-5

NUMERICAL DATA

$$\epsilon_x = 280(10^{-6}) \quad \epsilon_y = 420(10^{-6}) \quad \gamma_{xy} = 150(10^{-6}) \quad \theta = 35^\circ$$

STRAINS AT ANGLE θ

$$\epsilon_{x1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta) = 3.97 \times 10^{-4}$$

$$\gamma_{x1y1} = 2 \left[-\left(\frac{\epsilon_x - \epsilon_y}{2} \right) \sin(2\theta) + \frac{\gamma_{xy}}{2} \cos(2\theta) \right] = 1.829 \times 10^{-4}$$

$$\epsilon_{y1} = \epsilon_x + \epsilon_y - \epsilon_{x1} \quad \boxed{\epsilon_{y1} = 3.03 \times 10^{-4}}$$

Problem 7.7-6 Solve the preceding problem for the following data: $\epsilon_x = 190 \times 10^{-6}$ and $\epsilon_y = -230 \times 10^{-6}$, and $\gamma_{xy} = 160 \times 10^{-6}$, $\theta = 40^\circ$.

Solution 7.7-6

NUMERICAL DATA

$$\epsilon_x = 190(10^{-6}) \quad \epsilon_y = -230(10^{-6}) \quad \gamma_{xy} = 160(10^{-6}) \quad \theta = 40^\circ$$

STRAINS AT ANGLE θ

$$\epsilon_{x1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta) = 9.53 \times 10^{-5}$$

$$\gamma_{x1y1} = 2 \left[-\left(\frac{\epsilon_x - \epsilon_y}{2} \right) \sin(2\theta) + \frac{\gamma_{xy}}{2} \cos(2\theta) \right] = -3.86 \times 10^{-4}$$

$$\epsilon_{y1} = \epsilon_x + \epsilon_y - \epsilon_{x1} \quad \boxed{\epsilon_{y1} = -1.35 \times 10^{-4}}$$

Problem 7.7-7 The strains for an element of material in *plane strain* (see figure) are as follows: $\epsilon_x = 480 \times 10^{-6}$, $\epsilon_y = 140 \times 10^{-6}$, and $\gamma_{xy} = -350 \times 10^{-6}$.

Determine the principal strains and maximum shear strains, and show these strains on sketches of properly oriented elements.

Solution 7.7-7 Element in plane strain

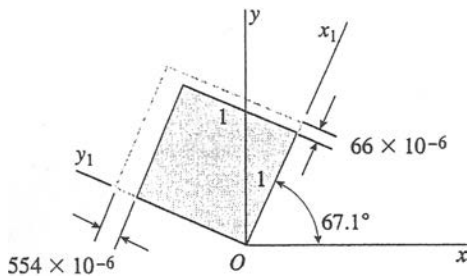
$$\begin{aligned}\epsilon_x &= 480 \times 10^{-6} & \epsilon_y &= 140 \times 10^{-6} \\ \gamma_{xy} &= -350 \times 10^{-6}\end{aligned}$$

PRINCIPAL STRAINS

$$\begin{aligned}\epsilon_{1,2} &= \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= 310 \times 10^{-6} \pm 244 \times 10^{-6} \\ \epsilon_1 &= 554 \times 10^{-6} & \epsilon_2 &= 66 \times 10^{-6} \\ \tan 2\theta_p &= \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} = -1.0294 \\ 2\theta_p &= -45.8^\circ \text{ and } 134.2^\circ \\ \theta_p &= -22.9^\circ \text{ and } 67.1^\circ\end{aligned}$$

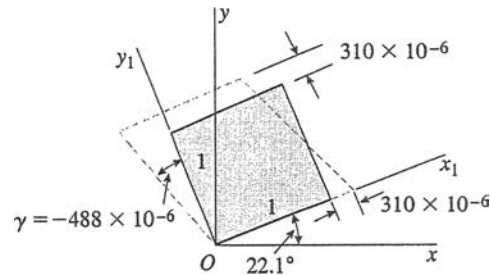
For $\theta_p = -22.9^\circ$:

$$\begin{aligned}\epsilon_{x_1} &= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \\ &= 554 \times 10^{-6} \\ \therefore \theta_{p_1} &= -22.9^\circ & \epsilon_1 &= 554 \times 10^{-6} \quad \leftarrow \\ \theta_{p_2} &= 67.1^\circ & \epsilon_2 &= 66 \times 10^{-6} \quad \leftarrow\end{aligned}$$



MAXIMUM SHEAR STRAINS

$$\begin{aligned}\frac{\gamma_{\max}}{2} &= \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= 244 \times 10^{-6} \\ \gamma_{\max} &= 488 \times 10^{-6} \\ \theta_{s_1} &= \theta_{p_1} - 45^\circ = -67.9^\circ \text{ or } 112.1^\circ \\ \gamma_{\max} &= 488 \times 10^{-6} \quad \leftarrow \\ \theta_{s_2} &= \theta_{s_1} + 90^\circ = 22.1^\circ \\ \gamma_{\min} &= -488 \times 10^{-6} \quad \leftarrow \\ \epsilon_{\text{aver}} &= \frac{\epsilon_x + \epsilon_y}{2} = 310 \times 10^{-6}\end{aligned}$$



Problem 7.7-8 Solve the preceding problem for the following strains: $\epsilon_x = 120 \times 10^{-6}$, $\epsilon_y = -450 \times 10^{-6}$, and $\gamma_{xy} = -360 \times 10^{-6}$.

Solution 7.7-8 Element in plane strain

$$\epsilon_x = 120 \times 10^{-6} \quad \epsilon_y = -450 \times 10^{-6}$$

$$\gamma_{xy} = -360 \times 10^{-6}$$

PRINCIPAL STRAINS

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= -165 \times 10^{-6} \pm 377 \times 10^{-6}$$

$$\epsilon_1 = 172 \times 10^{-6} \quad \epsilon_2 = -502 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} = -0.6316$$

$$2\theta_p = 327.7^\circ \quad \text{and} \quad 147.7^\circ$$

$$\theta_p = 163.9^\circ \quad \text{and} \quad 73.9^\circ$$

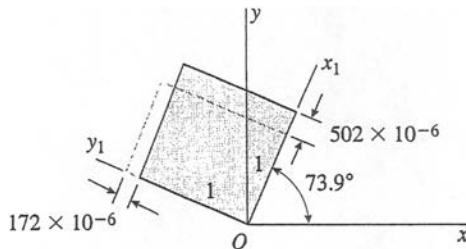
For $\theta_p = 163.9^\circ$:

$$\epsilon_{x_1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$= 172 \times 10^{-6}$$

$$\therefore \theta_{p_1} = 163.9^\circ \quad \epsilon_1 = 172 \times 10^{-6} \quad \leftarrow$$

$$\theta_{p_2} = 73.9^\circ \quad \epsilon_2 = -502 \times 10^{-6} \quad \leftarrow$$



MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{\max}}{2} = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 337 \times 10^{-6}$$

$$\gamma_{\max} = 674 \times 10^{-6}$$

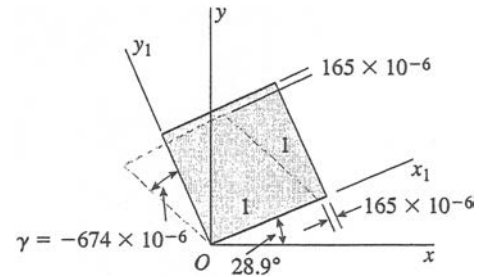
$$\theta_{s_1} = \theta_{p_1} - 45^\circ = 118.9^\circ$$

$$\gamma_{\max} = 674 \times 10^{-6} \quad \leftarrow$$

$$\theta_{s_2} = \theta_{s_1} - 90^\circ = 28.9^\circ$$

$$\gamma_{\min} = -674 \times 10^{-6} \quad \leftarrow$$

$$\epsilon_{\text{aver}} = \frac{\epsilon_x + \epsilon_y}{2} = -165 \times 10^{-6}$$



Problem 7.7-9 A element of material in *plane strain* (see figure) is subjected to strains $\epsilon_x = 480 \times 10^{-6}$, $\epsilon_y = 70 \times 10^{-6}$, and $\gamma_{xy} = 420 \times 10^{-6}$.

Determine the following quantities: (a) the strains for an element oriented at an angle $\theta = 75^\circ$, (b) the principal strains, and (c) the maximum shear strains. Show the results on sketches of properly oriented elements.

Solution 7.7-9 Element in plane strain

$$\epsilon_x = 480 \times 10^{-6} \quad \epsilon_y = 70 \times 10^{-6}$$

$$\gamma_{xy} = 420 \times 10^{-6}$$

$$\epsilon_{x_1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

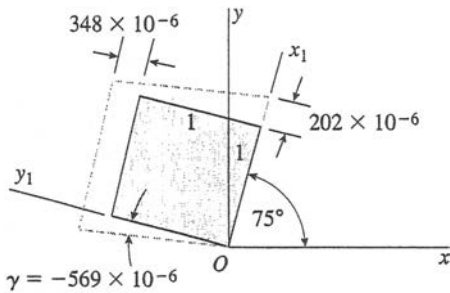
$$\frac{\gamma_{x_1 y_1}}{2} = -\frac{\epsilon_x - \epsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta$$

$$\epsilon_{y_1} = \epsilon_x + \epsilon_y - \epsilon_{x_1}$$

(a) For $\theta = 75^\circ$:

$$\epsilon_{x_1} = 202 \times 10^{-6} \quad \gamma_{x_1 y_1} = -569 \times 10^{-6}$$

$$\epsilon_{y_1} = 348 \times 10^{-6}$$



(b) PRINCIPAL STRAINS

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 275 \times 10^{-6} \pm 293 \times 10^{-6}$$

$$\epsilon_1 = 568 \times 10^{-6} \quad \epsilon_2 = -18 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} = 1.0244$$

$$2\theta_p = 45.69^\circ \quad \text{and} \quad 225.69^\circ$$

$$\theta_p = 22.85^\circ \quad \text{and} \quad 112.85^\circ$$

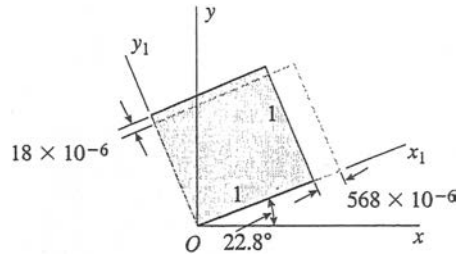
For $\theta_p = 22.85^\circ$:

$$\epsilon_{x_1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$= 568 \times 10^{-6}$$

$$\therefore \theta_{p_1} = 22.8^\circ \quad \epsilon_1 = 568 \times 10^{-6} \quad \leftarrow$$

$$\theta_{p_2} = 112.8^\circ \quad \epsilon_2 = -18 \times 10^{-6} \quad \leftarrow$$



(c) MAXIMUM SHEAR STRAINS

$$\frac{\gamma_{\max}}{2} = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 293 \times 10^{-6}$$

$$\gamma_{\max} = 587 \times 10^{-6}$$

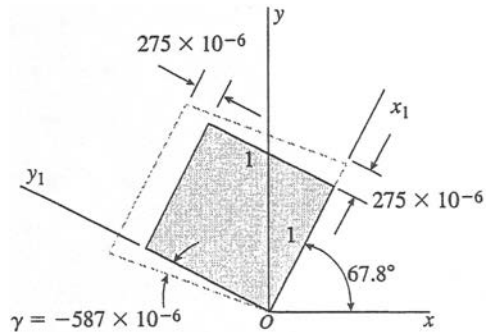
$$\theta_{s_1} = \theta_{p_1} - 45^\circ = -22.2^\circ \quad \text{or} \quad 157.8^\circ$$

$$\gamma_{\max} = 587 \times 10^{-6} \quad \leftarrow$$

$$\theta_{s_2} = \theta_{s_1} + 90^\circ = 67.8^\circ$$

$$\gamma_{\min} = -587 \times 10^{-6} \quad \leftarrow$$

$$\epsilon_{\text{aver}} = \frac{\epsilon_x + \epsilon_y}{2} = 275 \times 10^{-6}$$



Problem 7.7-10 Solve the preceding problem for the following data: $\epsilon_x = -1120 \times 10^{-6}$, $\epsilon_y = -430 \times 10^{-6}$, $\gamma_{xy} = 780 \times 10^{-6}$, and $\theta = 45^\circ$.

Solution 7.7-10 Element in plane strain

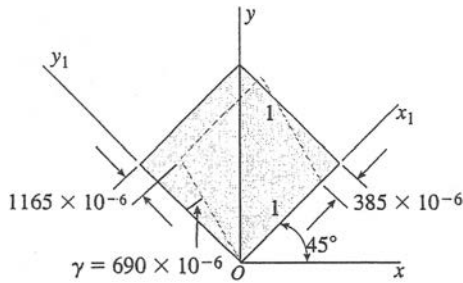
$$\begin{aligned}\epsilon_x &= -1120 \times 10^{-6} & \epsilon_y &= -430 \times 10^{-6} \\ \gamma_{xy} &= 780 \times 10^{-6} \\ \epsilon_{x_1} &= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta\end{aligned}$$

$$\frac{\gamma_{x_1 y_1}}{2} = -\frac{\epsilon_x - \epsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta$$

$$\epsilon_{y_1} = \epsilon_x + \epsilon_y - \epsilon_{x_1}$$

For $\theta = 45^\circ$:

$$\begin{aligned}\epsilon_{x_1} &= -385 \times 10^{-6} & \gamma_{x_1 y_1} &= 690 \times 10^{-6} \\ \epsilon_{y_1} &= -1165 \times 10^{-6}\end{aligned}$$



PRINCIPAL STRAINS

$$\begin{aligned}\epsilon_{1,2} &= \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= -775 \times 10^{-6} \pm 521 \times 10^{-6} \\ \epsilon_1 &= -254 \times 10^{-6} & \epsilon_2 &= -1296 \times 10^{-6}\end{aligned}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} = -1.1304$$

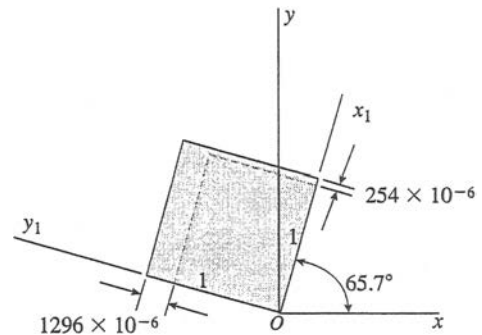
$$2\theta_p = 131.5^\circ \text{ and } 311.5^\circ$$

$$\theta_p = 65.7^\circ \text{ and } 155.7^\circ$$

For $\theta_p = 65.7^\circ$:

$$\begin{aligned}\epsilon_{x_1} &= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \\ &= -254 \times 10^{-6}\end{aligned}$$

$$\begin{aligned}\therefore \theta_{p_1} &= 65.7^\circ & \epsilon_1 &= -254 \times 10^{-6} \quad \leftarrow \\ \theta_{p_2} &= 155.7^\circ & \epsilon_2 &= -1296 \times 10^{-6} \quad \leftarrow\end{aligned}$$



MAXIMUM SHEAR STRAINS

$$\begin{aligned}\frac{\gamma_{\max}}{2} &= \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= 521 \times 10^{-6}\end{aligned}$$

$$\gamma_{\max} = 1041 \times 10^{-6}$$

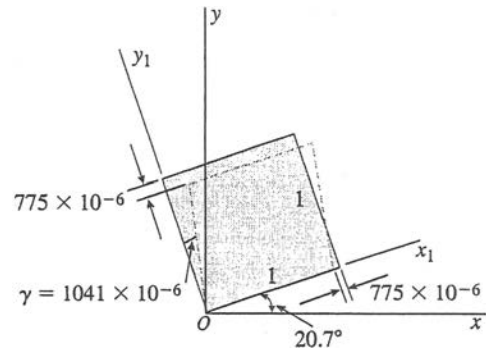
$$\theta_{s_1} = \theta_{p_1} - 45^\circ = 20.7^\circ$$

$$\gamma_{\max} = 1041 \times 10^{-6} \quad \leftarrow$$

$$\theta_{s_2} = \theta_{s_1} + 90^\circ = 110.7^\circ$$

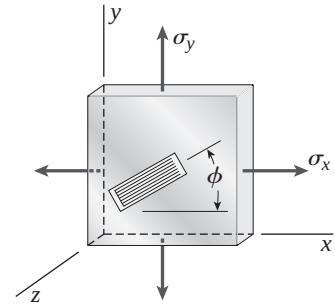
$$\gamma_{\min} = -1041 \times 10^{-6} \quad \leftarrow$$

$$\epsilon_{\text{aver}} = \frac{\epsilon_x + \epsilon_y}{2} = -775 \times 10^{-6}$$



Problem 7.7-11 A brass plate with modulus of elasticity $E = 16 \times 10^6$ psi and Poisson's ratio $\nu = 0.34$ is loaded in *biaxial stress* by normal stresses σ_x and σ_y (see figure). A strain gage is bonded to the plate at an angle $\phi = 35^\circ$.

If the stress σ_x is 10,700 psi and the strain measured by the gage is $\varepsilon = 390 \times 10^{-6}$, what is the maximum in-plane shear stress $(\tau_{\max})_{xy}$ and shear strain $(\gamma_{\max})_{xy}$? What is the maximum shear strain $(\gamma_{\max})_{xz}$ in the xz plane? What is the maximum shear strain $(\gamma_{\max})_{yz}$ in the yz plane?



Probs. 7.7-11 and 7.7-12

Solution 7.7-11

NUMERICAL DATA

$$E = 16(10^6) \text{ psi} \quad \nu = 0.34 \quad \phi = 35^\circ \quad \sigma_x = 10,700 \text{ psi} \quad \varepsilon = 390(10^{-6}) \quad \gamma_{xy} = 0$$

x AND y -DIR. STRAINS IN BIAxIAL STRESS - Eq. 7-40

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \quad \varepsilon_z = \frac{-\nu}{E}(\sigma_x + \sigma_y)$$

STRAIN TRANSFORMATION FOR ANGLE θ (Eq. 7-83a) $\varepsilon_{x1} = \varepsilon \quad \theta = \phi$

$$\varepsilon_{x1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta)$$

Substitute ε_x and ε_y expressions into above equation, then solve for σ_y : $\sigma_y = 2549$ psi

MAXIMUM IN-PLANE SHEAR STRESS

$$\tau_{\max xy} = \frac{\sigma_x - \sigma_y}{2} = 4076 \text{ psi}$$

USE NORMAL STRESSES IN EXPRESSIONS ABOVE TO FIND NORMAL STRAINS:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = 6.15 \times 10^{-4} \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = -6.81 \times 10^{-5}$$

$$\varepsilon_z = \frac{-\nu}{E}(\sigma_x + \sigma_y) = -2.82 \times 10^{-4}$$

MAXIMUM SHEAR STRAINS - USE EQ. 7-87 $\gamma_{xy} = 0 \quad \gamma_{xz} = 0 \quad \gamma_{yz} = 0$

$$\gamma_{xy\max} = 2 \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 6.83 \times 10^{-4}$$

$$\gamma_{xz\max} = 2 \sqrt{\left(\frac{\varepsilon_x - \varepsilon_z}{2}\right)^2 + \gamma_{xz}^2} = 8.96 \times 10^{-4}$$

$$\gamma_{yz\max} = 2 \sqrt{\left(\frac{\varepsilon_y - \varepsilon_z}{2}\right)^2 + \gamma_{yz}^2} = 2.13 \times 10^{-4}$$

Problem 7.7-12 Solve the preceding problem if the plate is made of aluminum with $E = 72$ GPa and Poisson's ratio $\nu = 0.33$; the plate is loaded in *biaxial stress* with normal stress $\sigma_x = 79$ MPa, angle $\phi = 18^\circ$, and the strain measured by the gage is $\varepsilon = 925 \times 10^{-6}$.

Solution 7.7-12

NUMERICAL DATA

$$E = 72(10^3) \text{ MPa} \quad \nu = 0.33 \quad \phi = 18^\circ \quad \sigma_x = 79 \text{ MPa} \quad \varepsilon = 925(10^{-6}) \quad \gamma_{xy} = 0$$

x AND y -DIR. STRAINS IN BIAXIAL STRESS - EQS. 7-40

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \quad \varepsilon_z = \frac{-\nu}{E}(\sigma_x + \sigma_y)$$

STRAIN TRANSFORMATION FOR ANGLE θ (EQ. 7-83a) $\varepsilon_{x1} = \varepsilon \quad \theta = \phi$

$$\varepsilon_{x1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta)$$

Substitute ε_x and ε_y expressions into above equation, then solve for σ_y : $\sigma_y = 11.66$ MPa

MAXIMUM IN-PLANE SHEAR STRESS

$$\tau_{\max xy} = \frac{\sigma_x - \sigma_y}{2} = 33.7 \text{ MPa}$$

USE NORMAL STRESSES IN EXPRESSIONS ABOVE TO FIND NORMAL STRAINS:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = 1.04 \times 10^{-3} \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = -2 \times 10^{-4}$$

$$\varepsilon_z = \frac{-\nu}{E}(\sigma_x + \sigma_y) = -4.16 \times 10^{-4}$$

MAXIMUM SHEAR STRAINS - USE EQ. 7-87 $\gamma_{xy} = 0 \quad \gamma_{xz} = 0 \quad \gamma_{yz} = 0$

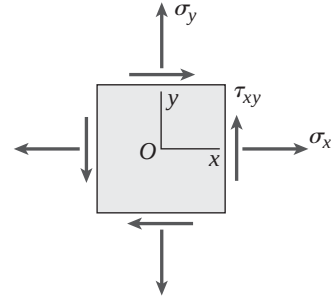
$$\gamma_{xy\max} = 2 \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 1.244 \times 10^{-3}$$

$$\gamma_{xz\max} = 2 \sqrt{\left(\frac{\varepsilon_x - \varepsilon_z}{2}\right)^2 + \gamma_{xz}^2} = 1.459 \times 10^{-3}$$

$$\gamma_{yz\max} = 2 \sqrt{\left(\frac{\varepsilon_y - \varepsilon_z}{2}\right)^2 + \gamma_{yz}^2} = 2.15 \times 10^{-4}$$

Problem 7.7-13 An element in *plane stress* is subjected to stresses $\sigma_x = -8400$ psi, $\sigma_y = 1100$ psi, and $\tau_{xy} = -1700$ psi (see figure). The material is aluminum with modulus of elasticity $E = 10,000$ ksi and Poisson's ratio $\nu = 0.33$.

Determine the following quantities: (a) the strains for an element oriented at an angle $\theta = 30^\circ$, (b) the principal strains, and (c) the maximum shear strains. Show the results on sketches of properly oriented elements.



Probs. 7.7-13 and 7.7-14

Solution 7.7-13 Element in plane strain

$$\begin{aligned}\sigma_x &= -8400 \text{ psi} & \sigma_y &= 1100 \text{ psi} \\ \tau_{xy} &= -1700 \text{ psi} & E &= 10,000 \text{ ksi} & \nu &= 0.33\end{aligned}$$

HOOKÉ'S LAW (EQS. 7-35a AND b)

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = -876.3 \times 10^{-6}$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = 387.2 \times 10^{-6}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2\tau_{xy}(1+\nu)}{E} = -452.2 \times 10^{-6}$$

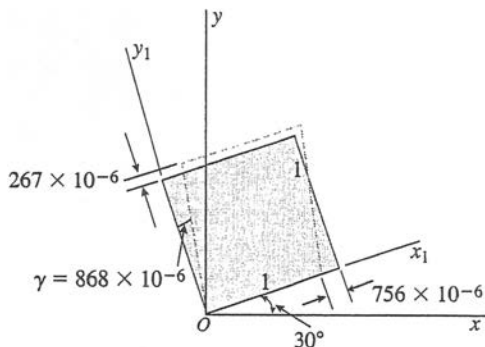
FOR $\theta = 30^\circ$:

$$\begin{aligned}\epsilon_{x_1} &= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \\ &= -756 \times 10^{-6}\end{aligned}$$

$$\begin{aligned}\frac{\gamma_{x_1y_1}}{2} &= -\frac{\epsilon_x - \epsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta \\ &= 434 \times 10^{-6}\end{aligned}$$

$$\gamma_{x_1y_1} = 868 \times 10^{-6}$$

$$\epsilon_{y_1} = \epsilon_x + \epsilon_y - \epsilon_{x_1} = 267 \times 10^{-6}$$



PRINCIPAL STRAINS

$$\begin{aligned}\epsilon_{1,2} &= \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= -245 \times 10^{-6} \pm 671 \times 10^{-6} \\ \epsilon_1 &= 426 \times 10^{-6} & \epsilon_2 &= -961 \times 10^{-6}\end{aligned}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} = 0.3579$$

$$2\theta_p = 19.7^\circ \text{ and } 199.7^\circ$$

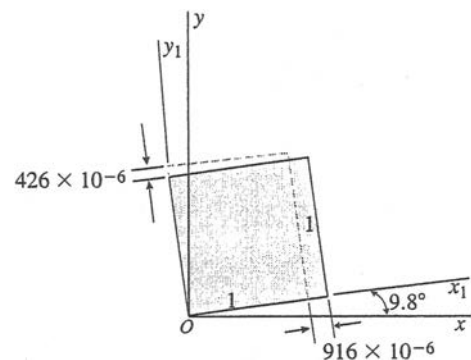
$$\theta_p = 9.8^\circ \text{ and } 99.8^\circ$$

FOR $\theta_p = 9.8^\circ$:

$$\begin{aligned}\epsilon_{x_1} &= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \\ &= -916 \times 10^{-6}\end{aligned}$$

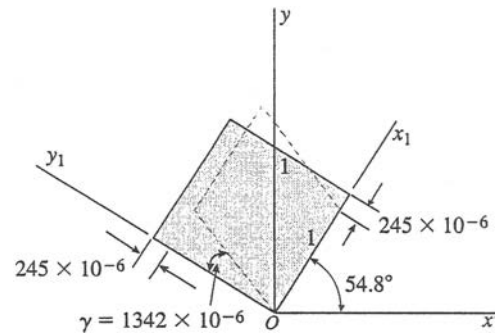
$$\therefore \theta_{p_1} = 99.8^\circ \quad \epsilon_1 = 426 \times 10^{-6} \quad \leftarrow$$

$$\theta_{p_2} = 9.8^\circ \quad \epsilon_2 = -916 \times 10^{-6} \quad \leftarrow$$



MAXIMUM SHEAR STRAINS

$$\begin{aligned}\frac{\gamma_{\max}}{2} &= \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= 671 \times 10^{-6} \\ \gamma_{\max} &= 1342 \times 10^{-6} \\ \theta_{s_1} &= \theta_{p_1} - 45^\circ = 54.8^\circ \\ \gamma_{\max} &= 1342 \times 10^{-6} \quad \leftarrow \\ \theta_{s_2} &= \theta_{s_1} + 90^\circ = 144.8^\circ \\ \gamma_{\min} &= -1342 \times 10^{-6} \quad \leftarrow \\ \varepsilon_{\text{aver}} &= \frac{\varepsilon_x + \varepsilon_y}{2} = -245 \times 10^{-6}\end{aligned}$$



Problem 7.7-14 Solve the preceding problem for the following data: $\sigma_x = -150$ MPa, $\sigma_y = -210$ MPa, $\tau_{xy} = -16$ MPa, and $\theta = 50^\circ$. The material is brass with $E = 100$ GPa and $\nu = 0.34$.

Solution 7.7-14 Element in plane strain

$$\begin{aligned}\sigma_x &= -150 \text{ MPa} & \sigma_y &= -210 \text{ MPa} \\ \tau_{xy} &= -16 \text{ MPa} & E &= 100 \text{ GPa} & \nu &= 0.34\end{aligned}$$

HOOKE'S LAW (EQS. 7-35a AND b)

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = -786 \times 10^{-6}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = -1590 \times 10^{-6}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2\tau_{xy}(1 + \nu)}{E} = -429 \times 10^{-6}$$

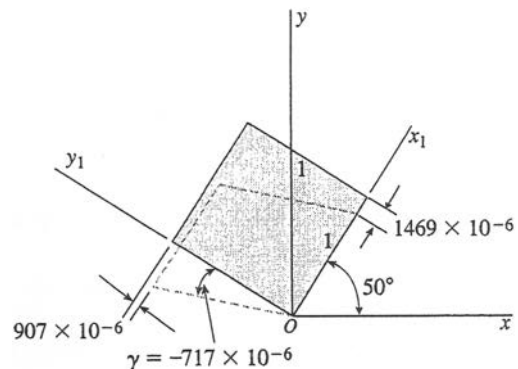
FOR $\theta = 50^\circ$:

$$\begin{aligned}\varepsilon_{x_1} &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \\ &= -1469 \times 10^{-6}\end{aligned}$$

$$\begin{aligned}\frac{\gamma_{x_1y_1}}{2} &= -\frac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta \\ &= -358.5 \times 10^{-6}\end{aligned}$$

$$\gamma_{x_1y_1} = -717 \times 10^{-6}$$

$$\varepsilon_{y_1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x_1} = -907 \times 10^{-6}$$



PRINCIPAL STRAINS

$$\begin{aligned}\varepsilon_{1,2} &= \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= -1188 \times 10^{-6} \pm 456 \times 10^{-6} \\ \varepsilon_1 &= -732 \times 10^{-6} & \varepsilon_2 &= -1644 \times 10^{-6}\end{aligned}$$

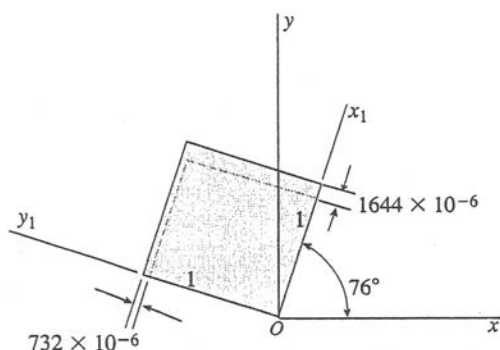
$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = -0.5333$$

$$2\theta_p = 151.9^\circ \text{ and } 331.9^\circ$$

$$\theta_p = 76.0^\circ \text{ and } 166.0^\circ$$

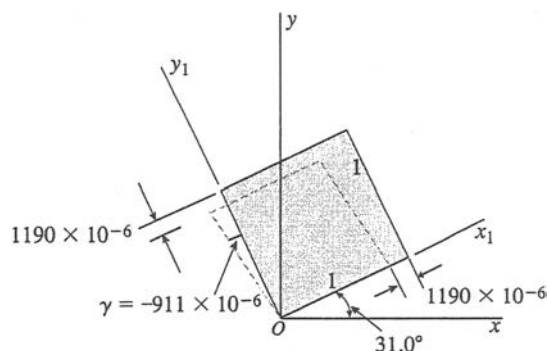
For $\theta_p = 76.0^\circ$:

$$\begin{aligned}\epsilon_{x_1} &= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \\ &= -1644 \times 10^{-6} \\ \therefore \theta_{p_1} &= 166.0^\circ \quad \epsilon_1 = -732 \times 10^{-6} \quad \leftarrow \\ \theta_{p_2} &= 76.0^\circ \quad \epsilon_2 = -1644 \times 10^{-6} \quad \leftarrow\end{aligned}$$



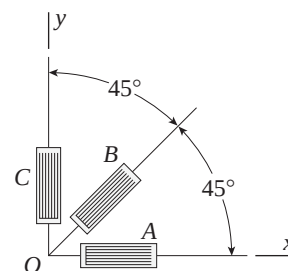
MAXIMUM SHEAR STRAINS

$$\begin{aligned}\frac{\gamma_{\max}}{2} &= \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= 456 \times 10^{-6} \\ \gamma_{\max} &= 911 \times 10^{-6} \\ \theta_{s_1} &= \theta_{p_1} - 45^\circ = 121.0^\circ \\ \gamma_{\max} &= 911 \times 10^{-6} \quad \leftarrow \\ \theta_{s_2} &= \theta_{s_1} - 90^\circ = 31.0^\circ \\ \gamma_{\min} &= -911 \times 10^{-6} \quad \leftarrow \\ \epsilon_{\text{aver}} &= \frac{\epsilon_x + \epsilon_y}{2} = -1190 \times 10^{-6}\end{aligned}$$



Problem 7.7-15 During a test of an airplane wing, the strain gage readings from a 45° rosette (see figure) are as follows: gage A, 520×10^{-6} ; gage B, 360×10^{-6} ; and gage C, -80×10^{-6} .

Determine the principal strains and maximum shear strains, and show them on sketches of properly oriented elements.



Probs. 7.7-15 and 7.7-16

Solution 7.7-15 45° strain rosette

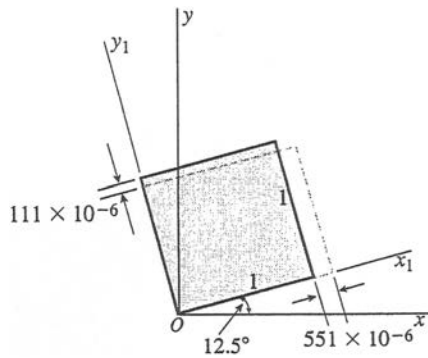
$$\begin{aligned}\varepsilon_A &= 520 \times 10^{-6} & \varepsilon_B &= 360 \times 10^{-6} \\ \varepsilon_C &= -80 \times 10^{-6}\end{aligned}$$

FROM EQS. (7-89a AND b) OF EXAMPLE 7-9:

$$\begin{aligned}\varepsilon_x &= \varepsilon_A = 520 \times 10^{-6} & \varepsilon_y &= \varepsilon_C = -80 \times 10^{-6} \\ \gamma_{xy} &= 2\varepsilon_B - \varepsilon_A - \varepsilon_C = 280 \times 10^{-6}\end{aligned}$$

PRINCIPAL STRAINS

$$\begin{aligned}\varepsilon_{1,2} &= \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= 220 \times 10^{-6} \pm 331 \times 10^{-6} \\ \varepsilon_1 &= 551 \times 10^{-6} & \varepsilon_2 &= -111 \times 10^{-6}\end{aligned}$$



$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = 0.4667$$

$$2\theta_p = 25.0^\circ \text{ and } 205.0^\circ$$

$$\theta_p = 12.5^\circ \text{ and } 102.5^\circ$$

For $\theta_p = 12.5^\circ$:

$$\begin{aligned}\varepsilon_{x_1} &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \\ &= 551 \times 10^{-6}\end{aligned}$$

$$\therefore \theta_{p_1} = 12.5^\circ \quad \varepsilon_1 = 551 \times 10^{-6} \quad \leftarrow$$

$$\theta_{p_2} = 102.5^\circ \quad \varepsilon_2 = -111 \times 10^{-6} \quad \leftarrow$$

MAXIMUM SHEAR STRAINS

$$\begin{aligned}\frac{\gamma_{\max}}{2} &= \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= 331 \times 10^{-6}\end{aligned}$$

$$\gamma_{\max} = 662 \times 10^{-6}$$

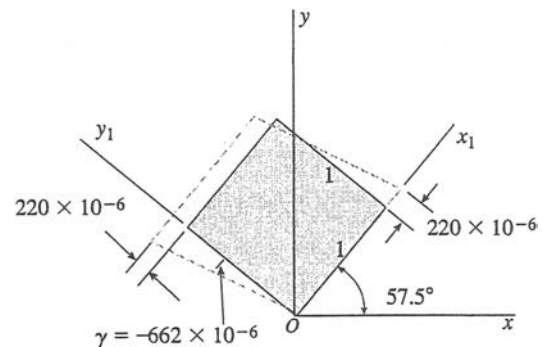
$$\theta_{s_1} = \theta_{p_1} - 45^\circ = -32.5^\circ \text{ or } 147.5^\circ$$

$$\gamma_{\max} = 662 \times 10^{-6} \quad \leftarrow$$

$$\theta_{s_2} = \theta_{s_1} + 90^\circ = 57.5^\circ$$

$$\gamma_{\min} = -662 \times 10^{-6} \quad \leftarrow$$

$$\varepsilon_{\text{aver}} = \frac{\varepsilon_x + \varepsilon_y}{2} = 220 \times 10^{-6}$$



Problem 7.7-16 A 45° strain rosette (see figure) mounted on the surface of an automobile frame gives the following readings: gage A, 310×10^{-6} ; gage B, 180×10^{-6} ; and gage C, -160×10^{-6} .

Determine the principal strains and maximum shear strains, and show them on sketches of properly oriented elements.

Solution 7.7-16 45° strain rosette

$$\varepsilon_A = 310 \times 10^{-6} \quad \varepsilon_B = 180 \times 10^{-6}$$

$$\varepsilon_C = -160 \times 10^{-6}$$

FROM EQS. (7-89a AND b) OF EXAMPLE 7-9:

$$\varepsilon_x = \varepsilon_A = 310 \times 10^{-6} \quad \varepsilon_y = \varepsilon_C = -160 \times 10^{-6}$$

$$\gamma_{xy} = 2\varepsilon_B - \varepsilon_A - \varepsilon_C = 210 \times 10^{-6}$$

PRINCIPAL STRAINS

$$\begin{aligned} \varepsilon_{1,2} &= \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= 75 \times 10^{-6} \pm 257 \times 10^{-6} \\ \varepsilon_1 &= 332 \times 10^{-6} \quad \varepsilon_2 = -182 \times 10^{-6} \end{aligned}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = 0.4468$$

$$2\theta_p = 24.1^\circ \quad \text{and} \quad 204.1^\circ$$

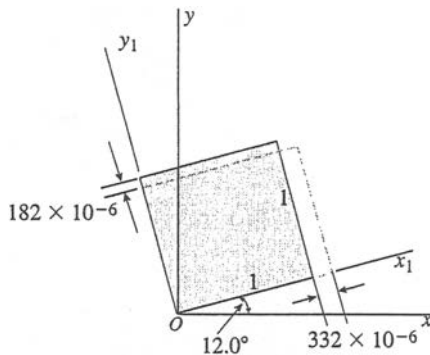
$$\theta_p = 12.0^\circ \quad \text{and} \quad 102.0^\circ$$

FOR $\theta_p = 12.0^\circ$:

$$\begin{aligned} \varepsilon_{x_1} &= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \\ &= 332 \times 10^{-6} \end{aligned}$$

$$\therefore \theta_{p1} = 12.0^\circ \quad \varepsilon_1 = 332 \times 10^{-6} \quad \leftarrow$$

$$\theta_{p2} = 102.0^\circ \quad \varepsilon_2 = -182 \times 10^{-6} \quad \leftarrow$$



MAXIMUM SHEAR STRAINS

$$\begin{aligned} \frac{\gamma_{\max}}{2} &= \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\ &= 257 \times 10^{-6} \end{aligned}$$

$$\gamma_{\max} = 515 \times 10^{-6}$$

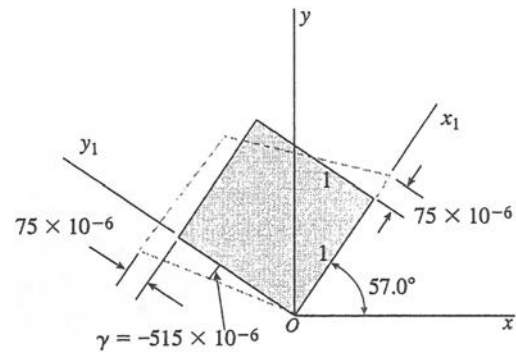
$$\theta_{s1} = \theta_{p1} - 45^\circ = -33.0^\circ \quad \text{or} \quad 147.0^\circ$$

$$\gamma_{\max} = 515 \times 10^{-6} \quad \leftarrow$$

$$\theta_{s2} = \theta_{s1} + 90^\circ = 57.0^\circ$$

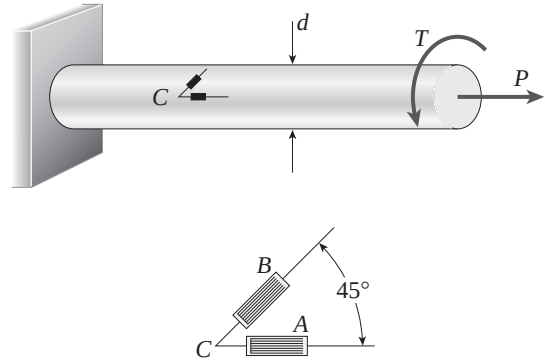
$$\gamma_{\min} = -515 \times 10^{-6} \quad \leftarrow$$

$$\varepsilon_{\text{aver}} = \frac{\varepsilon_x + \varepsilon_y}{2} = 75 \times 10^{-6}$$



Problem 7.7-17 A solid circular bar of diameter $d = 1.25$ in. is subjected to an axial force P and a torque T (see figure). Strain gages A and B mounted on the surface of the bar give readings $\varepsilon_A = 140 \times 10^{-6}$ and $\varepsilon_B = -60 \times 10^{-6}$. The bar is made of steel having $E = 30 \times 10^6$ psi and $\nu = 0.29$.

- Determine the axial force P and the torque T .
- Determine the maximum shear strain $\gamma_{\max}$ and the maximum shear stress $\tau_{\max}$ in the bar.



Solution 7.7-17

NUMERICAL DATA

$$E = 30(10^6) \text{ psi} \quad \nu = 0.29 \quad \varepsilon_A = 140(10^{-6}) \quad \varepsilon_B = -60(10^{-6}) \quad d = 1.25 \text{ in.}$$

$$\text{at } \theta_A = 0 \quad \text{at } \theta_B = 45^\circ$$

- AXIAL FORCE AND TORQUE

Plane stress:

$$\sigma_x = \frac{P}{\left(\frac{\pi d^2}{4}\right)} = \frac{4P}{\pi d^2} \quad \sigma_y = 0 \quad \tau_{xy} = \frac{T\left(\frac{d}{2}\right)}{\left(\frac{\pi d^4}{32}\right)} = \frac{16T}{\pi d^3}$$

FIND P

$$\varepsilon_x = \frac{\sigma_x}{E} \quad \varepsilon_x = \varepsilon_A \quad \text{so} \quad \boxed{P = \frac{\pi d^2}{4} (E \varepsilon_A) = 5154 \text{ lb}}$$

$$\text{Eq. 7-35b:} \quad \varepsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x) \quad \text{so} \quad \varepsilon_y = -\nu \varepsilon_x = -4.06 \times 10^{-5} \quad < \text{ where } \varepsilon_x = \varepsilon_A$$

FIND T

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \quad \tau_{xy} = \frac{E}{2(1 + \nu)} \gamma_{xy} \quad \text{Use strain transformation Eq. 7-83a to find } \gamma_{xy}:$$

$$\varepsilon_{x1} = \varepsilon_B \quad \theta = 45^\circ \quad \varepsilon_B = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta)$$

$$\text{Solving for } \gamma_{xy}: \quad \gamma_{xy} = -2.194(10^{-4})$$

$$\boxed{T = \frac{\pi d^3}{16} \left[\frac{E}{2(1 + \nu)} \gamma_{xy} \right] = -978 \text{ in.-lb}}$$

(b) MAXIMUM SHEAR STRAIN AND MAXIMUM SHEAR STRESS

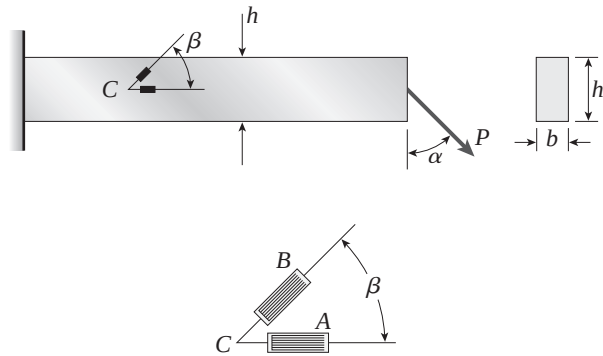
$$\gamma_{xy} = -2.194 \times 10^{-4} \quad \varepsilon_x = 1.4 \times 10^{-4} \quad \varepsilon_y = -4.06 \times 10^{-5}$$

$$\gamma_{\max} = 2 \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 2.84 \times 10^{-4}$$

$$\tau_{\max} = \frac{E}{2(1 + \nu)} \gamma_{\max} = 3304 \text{ psi}$$

Problem 7.7-18 A cantilever beam of rectangular cross section (width $b = 20$ mm, height $h = 175$ mm) is loaded by a force P that acts at the mid-height of the beam and is inclined at an angle α to the vertical (see figure). Two strain gages are placed at point C, which also is at the mid-height of the beam. Gage A measures the strain in the horizontal direction and gage B measures the strain at an angle $\beta = 60^\circ$ to the horizontal. The measured strains are $\varepsilon_A = 145 \times 10^{-6}$ and $\varepsilon_B = 165 \times 10^{-6}$.

Determine the force P and the angle α , assuming the material is steel with $E = 200$ GPa and $\nu = 1/3$.



Probs. 7.7-18

Solution 7.7-18

NUMERICAL DATA

$$E = 200(10^9) \text{ Pa} \quad \varepsilon_A = 145(10^{-6}) \quad \varepsilon_B = -165(10^{-6}) \quad b = 20 \text{ mm} \quad h = 175 \text{ mm}$$

$$\nu = \frac{1}{3} \quad \text{at } \theta_A = 0 \quad \text{at } \theta_B = 60^\circ \quad \text{or} \quad \beta = 60^\circ \quad \theta_B = \beta$$

COMBINED STRESS - AXIAL FORCE AND SHEAR

Plane stress: axial & transverse shear stresses

$$\sigma_x = \frac{P \sin(\alpha)}{bh} = \frac{P \sin(\alpha)}{bh} \quad \sigma_y = 0 \quad \tau_{xy} = \frac{-3}{2} \left(\frac{V}{A} \right) \quad \tau_{xy} = \frac{-3}{2} \frac{(P \cos(\alpha))}{bh} = -\frac{3P \cos(\alpha)}{2bh}$$

STRAIN AT $\theta = 0$

$$\varepsilon_x = \frac{\sigma_x}{E} \quad \varepsilon_x = \varepsilon_A \quad \text{so} \quad \varepsilon_A = \frac{P \sin(\alpha)}{Ebh} \quad < \text{Equation (1)}$$

$$\text{Eq. 7-35b: } \varepsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x) \quad \text{so} \quad \varepsilon_y = -\nu \varepsilon_x = -4.833 \times 10^{-5} \quad < \text{where } \varepsilon_x = \varepsilon_A$$

STRAIN AT ANGLE β

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \quad \tau_{xy} = \frac{E}{2(1 + \nu)} \gamma_{xy} \quad \gamma_{xy} = \left(-\frac{3P \cos(\alpha)}{2bh} \right) \left[\frac{2(1 + \nu)}{E} \right]$$

Use strain transformation Eq. 7-83a; substitute ε_A and γ_{xy} from above, then solve for P and angle α (or solve Equations (1) & (2) simultaneously)

$$\varepsilon_{x1} = \varepsilon_B \quad \theta = \theta_B \quad \varepsilon_B = \left(\frac{\varepsilon_A + \varepsilon_y}{2} + \frac{\varepsilon_A - \varepsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta) \right) < \text{Equation (2)}$$

OR

$$\varepsilon_A = \frac{P \sin(\alpha)}{Ebh} \quad (1)$$

$$\varepsilon_B = \frac{\varepsilon_A + \varepsilon_y}{2} + \frac{\varepsilon_A - \varepsilon_y}{2} \cos(2\theta) + \frac{\left(-\frac{3P \cos(\alpha)}{2bh} \right) \left[\frac{2(1 + \nu)}{E} \right]}{2} \sin(2\theta) \quad (2)$$

Solving Eqs (1) & (2) for P and angle α : $P = 121.4$ kN $\alpha = 56.7$ degrees

Problem 7.7-19 Solve the preceding problem if the cross-sectional dimensions are $b = 1.5$ in. and $h = 5.0$ in., the gage angle is $\beta = 75^\circ$, the measured strains are $\varepsilon_A = 209 \times 10^{-6}$ and $\varepsilon_B = 110 \times 10^{-6}$, and the material is a magnesium alloy with modulus $E = 6.0 \times 10^6$ psi and Poisson's ratio $\nu = 0.35$.

Solution 7.7-19

NUMERICAL DATA

$$E = 6(10^6) \text{ psi} \quad \varepsilon_A = 209(10^{-6}) \quad \varepsilon_B = -110(10^{-6}) \quad b = 1.5 \text{ in.} \quad h = 5 \text{ in.}$$

$$\nu = 0.35 \quad \text{at } \theta_A = 0 \quad \text{at } \theta_B = 75^\circ \quad \text{or } \beta = 75^\circ \quad \theta_B = \beta$$

COMBINED STRESS - AXIAL FORCE AND SHEAR

Plane stress: axial & transverse shear stresses

$$\sigma_x = \frac{P \sin(\alpha)}{(bh)} = \frac{P \sin(\alpha)}{bh} \quad \sigma_y = 0 \quad \tau_{xy} = \frac{-3}{2} \left(\frac{V}{A} \right) \quad \tau_{xy} = \frac{-3}{2} \frac{(P \cos(\alpha))}{bh} = -\frac{3P \cos(\alpha)}{2bh}$$

STRAIN AT $\theta = 0$

$$\varepsilon_x = \frac{\sigma_x}{E} \quad \varepsilon_x = \varepsilon_A \quad \text{so} \quad \varepsilon_A = \frac{P \sin(\alpha)}{Ebh} < \text{Equation (1)}$$

$$\text{Eq. 7-35b: } \varepsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x) \quad \text{so} \quad \varepsilon_y = -\nu \varepsilon_x = -7.315 \times 10^{-5} < \text{where } \varepsilon_x = \varepsilon_A$$

STRAIN AT ANGLE β

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \quad \tau_{xy} = \frac{E}{2(1 + \nu)} \gamma_{xy} \quad \gamma_{xy} = \left(-\frac{3P \cos(\alpha)}{2bh} \right) \left[\frac{2(1 + \nu)}{E} \right]$$

Use strain transformation Eq. 7-83a; substitute ϵ_A and γ_{xy} from above, then solve for P and angle α (or solve equations (1) & (2) simultaneously)

$$\epsilon_{x1} = \epsilon_B \quad \theta = \theta_B \quad \epsilon_B = \left(\frac{\epsilon_A + \epsilon_y}{2} + \frac{\epsilon_A - \epsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta) \right) < \text{Equation (2)}$$

OR

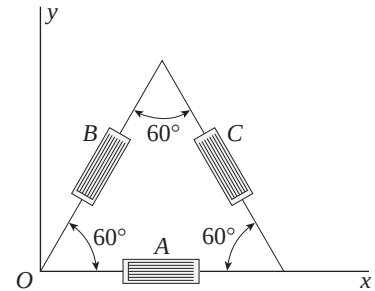
$$\epsilon_A = \frac{P \sin(\alpha)}{Ebh} \quad (1)$$

$$\epsilon_B = \frac{\epsilon_A + \epsilon_y}{2} + \frac{\epsilon_A - \epsilon_y}{2} \cos(2\theta) + \frac{\left(-\frac{3P \cos(\alpha)}{2bh} \right) \left[\frac{2(1 + \nu)}{E} \right]}{2} \sin(2\theta) \quad (2)$$

Solving Eqs. (1) & (2) for P and angle α : $P = 9726$ lb $\alpha = 75.2$ degrees

Problem 7.7-20 A 60° strain rosette, or *delta rosette*, consists of three electrical-resistance strain gages arranged as shown in the figure. Gage A measures the normal strain ϵ_a in the direction of the x axis. Gages B and C measure the strains ϵ_b and ϵ_c in the inclined directions shown.

Obtain the equations for the strains ϵ_x , ϵ_y , and γ_{xy} associated with the xy axis.



Solution 7.7-20 Delta rosette (60° strain rosette)

STRAIN GAGES

Gage A at $\theta = 0^\circ$ Strain = ϵ_A

Gage B at $\theta = 60^\circ$ Strain = ϵ_B

Gage C at $\theta = 120^\circ$ Strain = ϵ_C

FOR $\theta = 0^\circ$: $\epsilon_x = \epsilon_A \leftarrow$

FOR $\theta = 60^\circ$:

$$\epsilon_{x1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$\epsilon_B = \frac{\epsilon_A + \epsilon_y}{2} + \frac{\epsilon_A - \epsilon_y}{2} (\cos 120^\circ) + \frac{\gamma_{xy}}{2} (\sin 120^\circ)$$

$$\epsilon_B = \frac{\epsilon_A}{4} + \frac{3\epsilon_y}{4} + \frac{\gamma_{xy}\sqrt{3}}{4} \quad (1)$$

FOR $\theta = 120^\circ$:

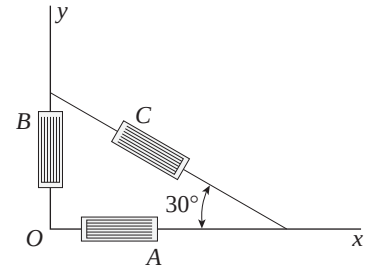
$$\begin{aligned}\epsilon_{x_1} &= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta \\ \epsilon_C &= \frac{\epsilon_A + \epsilon_y}{2} + \frac{\epsilon_A - \epsilon_y}{2} (\cos 240^\circ) + \frac{\gamma_{xy}}{2} (\sin 240^\circ) \\ \epsilon_C &= \frac{\epsilon_A}{4} + \frac{3\epsilon_y}{4} - \frac{\gamma_{xy}\sqrt{3}}{4}\end{aligned}\quad (2)$$

SOLVE EQS. (1) AND (2):

$$\begin{aligned}\epsilon_y &= \frac{1}{3}(2\epsilon_B + 2\epsilon_C - \epsilon_A) \quad \leftarrow \\ \gamma_{xy} &= \frac{2}{\sqrt{3}}(\epsilon_B - \epsilon_C) \quad \leftarrow\end{aligned}$$

Problem 7.7-21 On the surface of a structural component in a space vehicle, the strains are monitored by means of three strain gages arranged as shown in the figure. During a certain maneuver, the following strains were recorded: $\epsilon_a = 1100 \times 10^{-6}$, $\epsilon_b = 200 \times 10^{-6}$, and $\epsilon_c = 200 \times 10^{-6}$.

Determine the principal strains and principal stresses in the material, which is a magnesium alloy for which $E = 6000$ ksi and $\nu = 0.35$. (Show the principal strains and principal stresses on sketches of properly oriented elements.)



Solution 7.7-21 30-60-90° strain rosette

Magnesium alloy: $E = 6000$ ksi $\nu = 0.35$

STRAIN GAGES

Gage A at $\theta = 0^\circ$ $\epsilon_A = 1100 \times 10^{-6}$

Gage B at $\theta = 90^\circ$ $\epsilon_B = 200 \times 10^{-6}$

Gage C at $\theta = 150^\circ$ $\epsilon_C = 200 \times 10^{-6}$

FOR $\theta = 0^\circ$: $\epsilon_x = \epsilon_A = 1100 \times 10^{-6}$

FOR $\theta = 90^\circ$: $\epsilon_y = \epsilon_B = 200 \times 10^{-6}$

FOR $\theta = 150^\circ$:

$$\epsilon_{x_1} = \epsilon_C = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$200 \times 10^{-6} = 650 \times 10^{-6} + 225 \times 10^{-6}$$

$$-0.43301\gamma_{xy}$$

Solve for γ_{xy} : $\gamma_{xy} = 1558.9 \times 10^{-6}$

PRINCIPAL STRAINS

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= 650 \times 10^{-6} \pm 900 \times 10^{-6}$$

$$\epsilon_1 = 1550 \times 10^{-6} \quad \epsilon_2 = -250 \times 10^{-6}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} = \sqrt{3} = 1.7321$$

$$2\theta_p = 60^\circ \quad \theta_p = 30^\circ$$

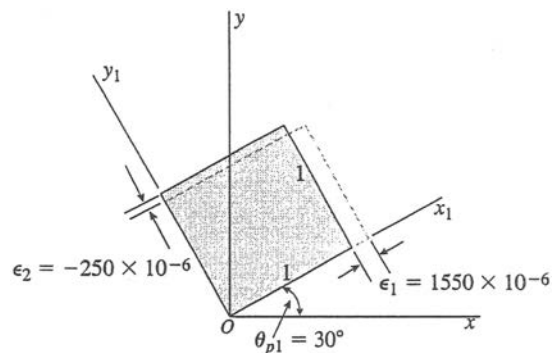
FOR $\theta_p = 30^\circ$:

$$\epsilon_{x_1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

$$= 1550 \times 10^{-6}$$

$$\therefore \theta_{p1} = 30^\circ \quad \epsilon_1 = 1550 \times 10^{-6} \quad \leftarrow$$

$$\theta_{p2} = 120^\circ \quad \epsilon_2 = -250 \times 10^{-6} \quad \leftarrow$$

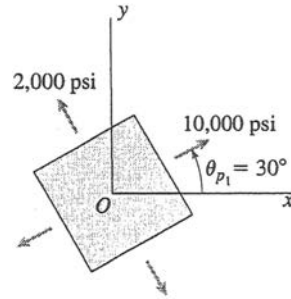


PRINCIPAL STRESSES (see Eqs. 7-41)

$$\sigma_1 = \frac{E}{1 - \nu^2} (\epsilon_1 + \nu \epsilon_2) \quad \sigma_2 = \frac{E}{1 - \nu^2} (\epsilon_2 + \nu \epsilon_1)$$

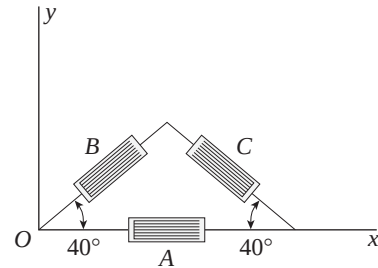
Substitute numerical values:

$$\sigma_1 = 10,000 \text{ psi} \quad \sigma_2 = 2000 \text{ psi} \quad \leftarrow$$



Problem 7.7-22 The strains on the surface of an experimental device made of pure aluminum ($E = 70 \text{ GPa}$, $\nu = 0.33$) and tested in a space shuttle were measured by means of strain gages. The gages were oriented as shown in the figure, and the measured strains were $\epsilon_a = 1100 \times 10^{-6}$, $\epsilon_b = 1496 \times 10^{-6}$, and $\epsilon_c = -39.44 \times 10^{-6}$.

What is the stress σ_x in the x direction?



Solution 7.7-22 40-40-100° strain rosette

Pure aluminum: $E = 70 \text{ GPa}$ $\nu = 0.33$

STRAIN GAGES

Gage A at $\theta = 0^\circ$ $\epsilon_A = 1100 \times 10^{-6}$

Gage B at $\theta = 40^\circ$ $\epsilon_B = 1496 \times 10^{-6}$

Gage C at $\theta = 140^\circ$ $\epsilon_C = -39.44 \times 10^{-6}$

FOR $\theta = 0^\circ$: $\epsilon_x = \epsilon_A = 1100 \times 10^{-6}$

FOR $\theta = 40^\circ$:

$$\epsilon_{x_1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

Substitute $\epsilon_{x_1} = \epsilon_B = 1496 \times 10^{-6}$ and

$\epsilon_x = 1100 \times 10^{-6}$; then simplify and rearrange:

$$0.41318\epsilon_y + 0.49240\gamma_{xy} = 850.49 \times 10^{-6} \quad (1)$$

FOR $\theta = 140^\circ$:

$$\epsilon_{x_1} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

Substitute $\epsilon_{x_1} = \epsilon_C = -39.44 \times 10^{-6}$ and

$\epsilon_x = 1100 \times 10^{-6}$; then simplify and rearrange:

$$0.41318\epsilon_y - 0.49240\gamma_{xy} = -684.95 \times 10^{-6} \quad (2)$$

SOLVE EQS. (1) AND (2):

$$\epsilon_y = 200.3 \times 10^{-6} \quad \gamma_{xy} = 1559.2 \times 10^{-6}$$

HOOKE'S LAW

$$\sigma_x = \frac{E}{1 - \nu^2} (\epsilon_x + \nu \epsilon_y) = 91.6 \text{ MPa} \quad \leftarrow$$

Problem 7.7-23 Solve Prob. 7.7-5 by using Mohr's circle for plane strain.

Solution 7.7-23

NUMERICAL DATA

$$\varepsilon_x = 280(10^{-6})$$

$$\varepsilon_y = 420(10^{-6})$$

$$\gamma_{xy} = 150(10^{-6}) \quad \theta = 35^\circ$$

MOHR'S CIRCLE

$$R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 1.026 \times 10^{-4}$$

$$\frac{\varepsilon_x - \varepsilon_y}{2} = -7 \times 10^{-5} \quad \frac{\gamma_{xy}}{2} = 7.5 \times 10^{-5}$$

$$\alpha = \arctan\left(\frac{\frac{\gamma_{xy}}{2}}{\frac{\varepsilon_x - \varepsilon_y}{2}}\right) = -46.975^\circ \quad \varepsilon_{\text{ave}} = \frac{\varepsilon_x + \varepsilon_y}{2} = 3.5 \times 10^{-4} < \text{center of circle}$$

$$\varepsilon_A = \varepsilon_x \quad \gamma_A = \gamma_{xy} \quad \varepsilon_B = \varepsilon_y \quad \gamma_B = -\gamma_{xy} \quad \beta = 180^\circ - |\alpha| - 2\theta = 63.025^\circ$$

$$\varepsilon_D = \varepsilon_{\text{ave}} + R \cos(\beta) = 3.97 \times 10^{-4} < \text{at rotation } \theta = 35^\circ \text{ from point A (same as } \varepsilon_{x1} \text{ below)}$$

$$\gamma_D = 2R \sin(\beta) = 1.829 \times 10^{-4} < \text{at rotation } \theta = 35^\circ \text{ from point A (same as } \gamma_{x1y1} \text{ below)}$$

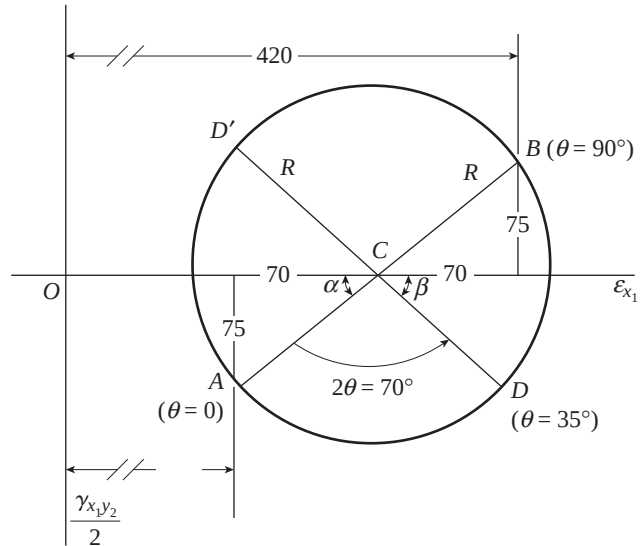
$$\varepsilon_{D'} = \varepsilon_{\text{ave}} - R \cos(\beta) = 3.03 \times 10^{-4} < \text{at rotation } \theta = 90 + 35^\circ \text{ from point A (same as } \varepsilon_{y1} \text{ below)}$$

OR STRAINS AT ANGLE θ USING STRAIN TRANSFORMATION FORMULAS

$$\varepsilon_{x1} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos(2\theta) + \frac{\gamma_{xy}}{2} \sin(2\theta) = 3.97 \times 10^{-4}$$

$$\gamma_{x1y1} = 2 \left[-\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right) \sin(2\theta) + \frac{\gamma_{xy}}{2} \cos(2\theta) \right] = 1.829 \times 10^{-4}$$

$$\varepsilon_{y1} = \varepsilon_x + \varepsilon_y - \varepsilon_{x1} \quad \varepsilon_{y1} = 3.03 \times 10^{-4}$$

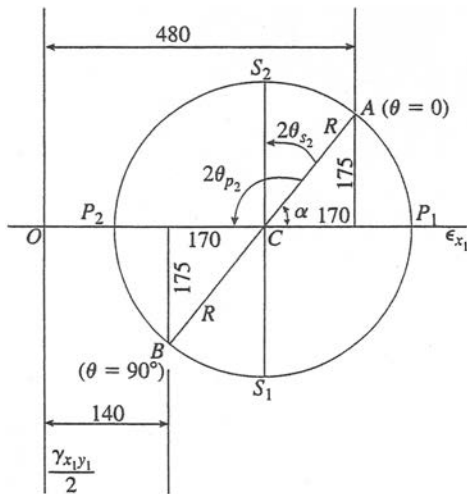


Problem 7.7-25 Solve Prob. 7.7-7 by using Mohr's circle for plane strain.

Solution 7.7-25 Element in plane strain

$$\varepsilon_x = 480 \times 10^{-6} \quad \varepsilon_y = 140 \times 10^{-6}$$

$$\gamma_{xy} = -350 \times 10^{-6} \quad \frac{\gamma_{xy}}{2} = -175 \times 10^{-6}$$



$$R = \sqrt{(175 \times 10^{-6})^2 + (170 \times 10^{-6})^2} = 243.98 \times 10^{-6}$$

$$\alpha = \arctan \frac{175}{170} = 45.83^\circ$$

$$\text{POINT C: } \varepsilon_{x_1} = 310 \times 10^{-6}$$

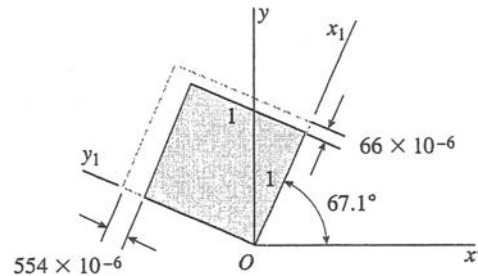
PRINCIPAL STRAINS

$$2\theta_{p_2} = 180^\circ - \alpha = 134.2^\circ \quad \theta_{p_2} = 67.1^\circ$$

$$2\theta_{p_1} = 2\theta_{p_2} + 180^\circ = 314.2^\circ \quad \theta_{p_1} = 157.1^\circ$$

$$\text{Point } P_1: \varepsilon_1 = 310 \times 10^{-6} + R = 554 \times 10^{-6}$$

$$\text{Point } P_2: \varepsilon_2 = 310 \times 10^{-6} - R = 66 \times 10^{-6}$$



MAXIMUM SHEAR STRAINS

$$2\theta_{s_2} = 90^\circ - \alpha = 44.17^\circ \quad \theta_{s_2} = 22.1^\circ$$

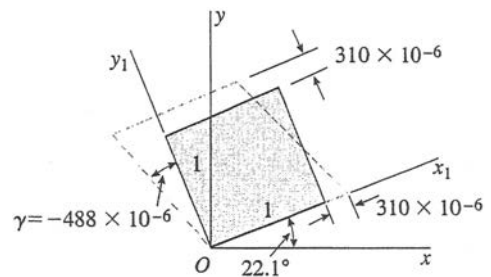
$$2\theta_{s_1} = 2\theta_{s_2} + 180^\circ = 224.17^\circ \quad \theta_{s_1} = 112.1^\circ$$

$$\text{Point } S_1: \varepsilon_{\text{aver}} = 310 \times 10^{-6}$$

$$\gamma_{\text{max}} = 2R = 488 \times 10^{-6}$$

$$\text{Point } S_2: \varepsilon_{\text{aver}} = 310 \times 10^{-6}$$

$$\gamma_{\text{min}} = -488 \times 10^{-6}$$

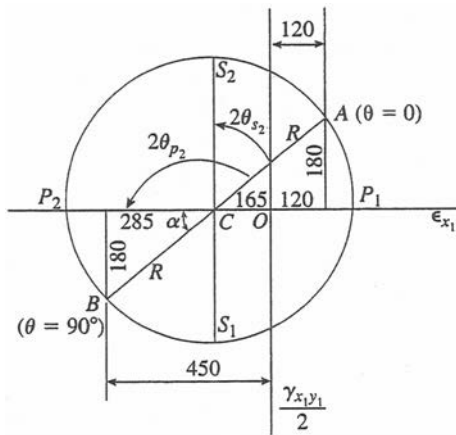


Problem 7.7-26 Solve Prob. 7.7-8 by using Mohr's circle for plane strain.

Solution 7.7-26 Element in plane strain

$$\varepsilon_x = 120 \times 10^{-6} \quad \varepsilon_y = -450 \times 10^{-6}$$

$$\gamma_{xy} = -360 \times 10^{-6} \quad \frac{\gamma_{xy}}{2} = -180 \times 10^{-6}$$



$$R = \sqrt{(285 \times 10^{-6})^2 + (180 \times 10^{-6})^2} = 337.08 \times 10^{-6}$$

$$\alpha = \arctan \frac{180}{285} = 32.28^\circ$$

$$\text{Point C: } \varepsilon_{x_1} = -165 \times 10^{-6}$$

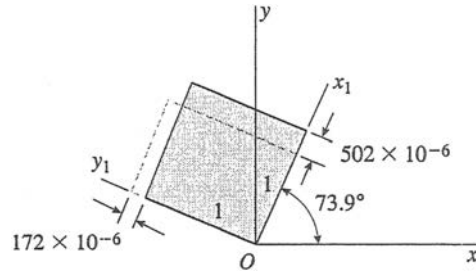
PRINCIPAL STRAINS

$$2\theta_{p_2} = 180^\circ - \alpha = 147.72^\circ \quad \theta_{p_2} = 73.9^\circ$$

$$2\theta_{p_1} = 2\theta_{p_2} + 180^\circ = 327.72^\circ \quad \theta_{p_1} = 163.9^\circ$$

$$\text{Point } P_1: \varepsilon_1 = R - 165 \times 10^{-6} = 172 \times 10^{-6}$$

$$\text{Point } P_2: \varepsilon_2 = -165 \times 10^{-6} - R = -502 \times 10^{-6}$$



MAXIMUM SHEAR STRAINS

$$2\theta_{s_2} = 90^\circ - \alpha = 57.72^\circ \quad \theta_{s_2} = 28.9^\circ$$

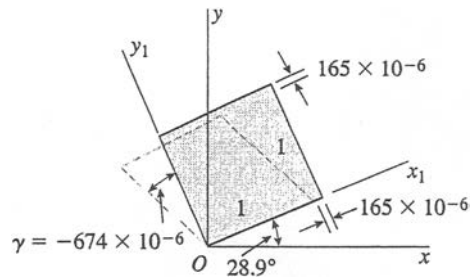
$$2\theta_{s_1} = 2\theta_{s_2} + 180^\circ = 237.72^\circ \quad \theta_{s_1} = 118.9^\circ$$

$$\text{Point } S_1: \varepsilon_{\text{aver}} = -165 \times 10^{-6}$$

$$\gamma_{\text{max}} = 2R = 674 \times 10^{-6}$$

$$\text{Point } S_2: \varepsilon_{\text{aver}} = -165 \times 10^{-6}$$

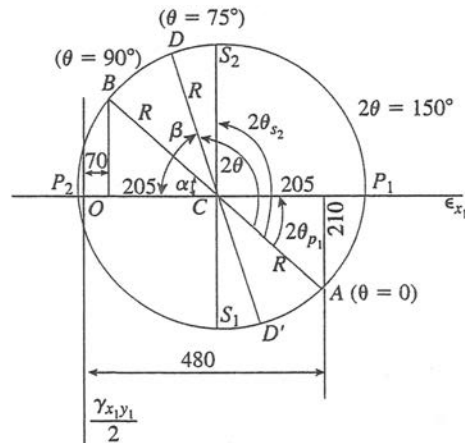
$$\gamma_{\text{min}} = -674 \times 10^{-6}$$



Problem 7.7-27 Solve Prob. 7.7-9 by using Mohr's circle for plane strain.

Solution 7.7-27 Element in plane strain

$$\begin{aligned}\epsilon_x &= 480 \times 10^{-6} & \epsilon_y &= 70 \times 10^{-6} \\ \gamma_{xy} &= 420 \times 10^{-6} & \frac{\gamma_{xy}}{2} &= 210 \times 10^{-6} \quad \theta = 75^\circ\end{aligned}$$



$$\begin{aligned}R &= \sqrt{(205 \times 10^{-6})^2 + (210 \times 10^{-6})^2} \\ &= 293.47 \times 10^{-6}\end{aligned}$$

$$\alpha = \arctan \frac{210}{205} = 45.69^\circ$$

$$\beta = \alpha + 180^\circ - 2\theta = 75.69^\circ$$

$$\text{Point C: } \epsilon_{x_1} = 275 \times 10^{-6}$$

Point D ($\theta = 75^\circ$):

$$\epsilon_{x_1} = 275 \times 10^{-6} - R \cos \beta = 202 \times 10^{-6}$$

$$\frac{\gamma_{x_1 y_1}}{2} = -R \sin \beta = -284.36 \times 10^{-6}$$

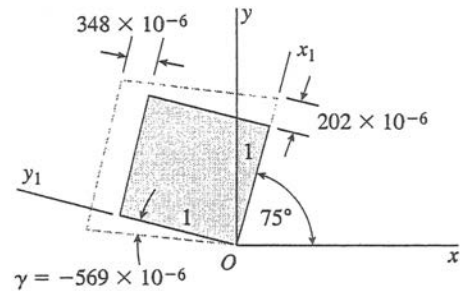
$$\gamma_{x_1 y_1} = -569 \times 10^{-6}$$

Point D' ($\theta = 165^\circ$):

$$\epsilon_{x_1} = 275 \times 10^{-6} + R \cos \beta = 348 \times 10^{-6}$$

$$\frac{\gamma_{x_1 y_1}}{2} = R \sin \beta = 284.36 \times 10^{-6}$$

$$\gamma_{x_1 y_1} = 569 \times 10^{-6}$$



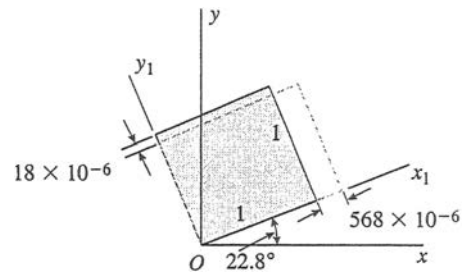
PRINCIPAL STRAINS

$$2\theta_{p_1} = \alpha = 45.69^\circ \quad \theta_{p_1} = 22.8^\circ$$

$$2\theta_{p_2} = 2\theta_{p_1} + 180^\circ = 225.69^\circ \quad \theta_{p_2} = 112.8^\circ$$

$$\text{Point } P_1: \epsilon_1 = 275 \times 10^{-6} + R = 568 \times 10^{-6}$$

$$\text{Point } P_2: \epsilon_2 = 275 \times 10^{-6} - R = -18 \times 10^{-6}$$



MAXIMUM SHEAR STRAINS

$$2\theta_{s_2} = 90^\circ + \alpha = 135.69^\circ \quad \theta_{s_2} = 67.8^\circ$$

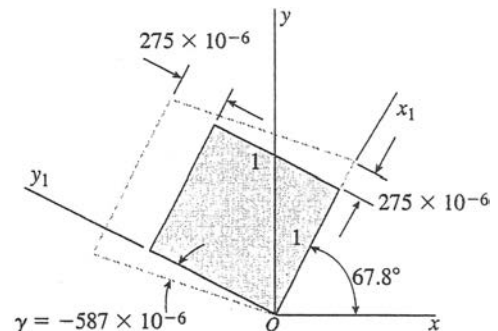
$$2\theta_{s_1} = 2\theta_{s_2} + 180^\circ = 315.69^\circ \quad \theta_{s_1} = 157.8^\circ$$

$$\text{Point } S_1: \epsilon_{\text{aver}} = 275 \times 10^{-6}$$

$$\gamma_{\text{max}} = 2R = 587 \times 10^{-6}$$

$$\text{Point } S_2: \epsilon_{\text{aver}} = 275 \times 10^{-6}$$

$$\gamma_{\text{min}} = -587 \times 10^{-6}$$



8

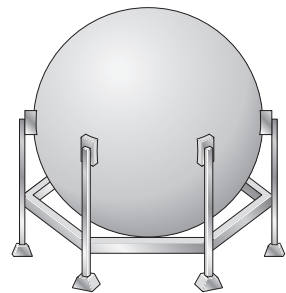
Applications of Plane Stress (Pressure Vessels, Beams, and Combined Loadings)

Spherical Pressure Vessels

When solving the problems for Section 8.2, assume that the given radius or diameter is an inside dimension and that all internal pressures are gage pressures.

Problem 8.2-1 A large spherical tank (see figure) contains gas at a pressure of 420 psi. The tank is 45 ft in diameter and is constructed of high-strength steel having a yield stress in tension of 80 ksi.

- (a) Determine the required thickness (to the nearest 1/4 inch) of the wall of the tank if a factor of safety of 3.5 with respect to yielding is required.
- (b) If the tank wall thickness is 2.25 in., what is the maximum permissible internal pressure?



Probs. 8.2-1 and 8.2-2

Solution 8.2-1

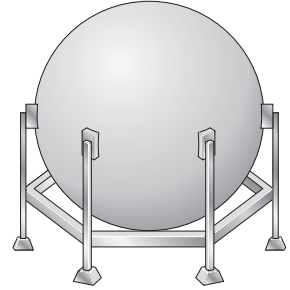
$$(a) \quad p = 420 \text{ psi} \quad d = 45 \text{ ft} \quad \sigma_Y = 80 \text{ ksi} \quad FS_Y = 3.5 \quad \sigma_a = \frac{\sigma_Y}{FS_Y} = 22.857 \text{ ksi}$$

$$t = \frac{p \frac{d}{2}}{2\sigma_a} = 2.481 \text{ in.} \quad \boxed{\text{use } t = 2.5 \text{ in.}}$$

$$(b) \quad t = 2.25 \text{ in.} \quad \boxed{p_{\max} = \frac{4\sigma_a t}{d} = 381 \text{ psi}}$$

Problem 8.2-2 Solve the preceding problem if the internal pressure is 3.85 MPa, the diameter is 20 m, the yield stress is 590 MPa, and the factor of safety is 3.0.

- Determine the required thickness to the nearest millimeter.
- If the tank wall thickness is 85 mm, what is the maximum permissible internal pressure?



Solution 8.2-2

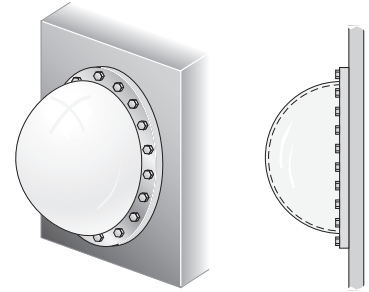
$$(a) \quad p = 3.85 \text{ MPa} \quad d = 20 \text{ m} \quad \sigma_Y = 590 \text{ MPa} \quad FS_Y = 3.0 \quad \sigma_a = \frac{\sigma_Y}{FS_Y} = 196.667 \text{ MPa}$$

$$t = \frac{p \frac{d}{2}}{2\sigma_a} = 97.881 \text{ mm} \quad \boxed{\text{use } t = 98 \text{ mm}}$$

$$(b) \quad t = 85 \text{ mm} \quad \boxed{p_{\max} = \frac{4\sigma_a t}{d} = 3.34 \text{ MPa}}$$

Problem 8.2-3 A hemispherical window (or *viewport*) in a decompression chamber (see figure) is subjected to an internal air pressure of 85 psi. The port is attached to the wall of the chamber by 14 bolts.

- Find the tensile force F in each bolt and the tensile stress σ in the viewport if the radius of the hemisphere is 7.5 in. and its thickness is 1.25 in.
- If the yield stress for each of the 14 bolts is 50 ksi and the factor of safety is 3.0, find the required bolt diameter.
- If the stress in the viewport is limited to 250 psi, find the required radius of the hemisphere.



Solution 8.2-3 Hemispherical viewport

$$(a) \quad r = 7.5 \text{ in.} \quad p = 85 \text{ psi} \quad n = 14 \quad t = 1.25 \text{ in.}$$

$$T = pA \quad F = \frac{T}{n} \quad \boxed{F = \frac{p(\pi r^2)}{n} = 1073 \text{ lb}} \quad \boxed{\sigma = \frac{pr}{2t} = 2.55 \text{ psi}}$$

$$F = 1073 \text{ lb} \quad \sigma = 255 \text{ psi}$$

$$(b) \quad \sigma_Y = 50 \text{ ksi} \quad FS = 3 \quad \sigma_a = \frac{\sigma_Y}{FS} = 16.667 \text{ ksi}$$

$$F = \sigma_a \left(\frac{\pi}{4} d_b^2 \right) \quad \boxed{d_b = \sqrt{\frac{4F}{\pi \sigma_a}} = 0.286 \text{ in.}}$$

$$(c) \quad \sigma = 250 \text{ psi} \quad \boxed{r = \frac{2\sigma t}{p} = 7.35 \text{ in.}}$$

Problem 8.2-4 A rubber ball (see figure) is inflated to a pressure of 65 kPa. At that pressure the diameter of the ball is 240 mm and the wall thickness is 1.25 mm. The rubber has modulus of elasticity $E = 3.7$ MPa and Poisson's ratio $\nu = 0.48$.

- Determine the maximum stress and strain in the ball.
- If the strain must be limited to 0.425, find the minimum required wall thickness of the ball.

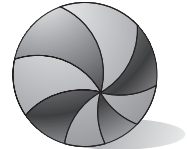
Solution 8.2-4 Rubber ball

$$(a) \quad p = 65 \text{ kPa} \quad t = 1.25 \text{ mm} \quad E = 3.7 \text{ MPa} \quad \nu = 0.48 \quad d = 240 \text{ mm} \quad r = \frac{d}{2}$$

$$\sigma_{\max} = \frac{pr}{2t} = 3.12 \text{ MPa}$$

$$\epsilon_{\max} = \frac{pr}{2tE}(1 - \nu) = 0.438$$

$$(b) \quad \epsilon_{\max} = 0.425 \quad t_{\text{reqd}} = \frac{pr}{2\epsilon_{\max}E}(1 - \nu) = 1.29 \text{ mm}$$



Problem 8.2-5 A rubber ball (see figure) is inflated to a pressure of 65 kPa. At that pressure the diameter of the ball is 240 mm and the wall thickness is 1.25 mm. The rubber has modulus of elasticity $E = 3.7$ MPa and Poisson's ratio $\nu = 0.48$.

- Solve the preceding problem if the pressure is 8.5 psi, the diameter is 10 in., the wall thickness is 0.05 in., the modulus of elasticity is 200 psi, and Poisson's ratio is 0.48.
- If the strain must be limited to 1.01, find the maximum acceptable inflation pressure.

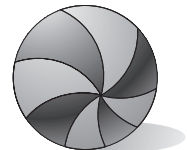
Solution 8.2-5 Rubber ball

$$(a) \quad p = 8.5 \text{ psi} \quad t = 0.05 \text{ in.} \quad E = 200 \text{ psi} \quad \nu = 0.48 \quad d = 10 \text{ in.} \quad r = \frac{d}{2}$$

$$\sigma_{\max} = \frac{pr}{2t} = 425 \text{ psi}$$

$$\epsilon_{\max} = \frac{pr}{2tE}(1 - \nu) = 1.105$$

$$(b) \quad \epsilon_{\max} = 1.01 \quad p_{\max} = \frac{2\epsilon_{\max}tE}{r(1 - \nu)} = 7.77 \text{ psi}$$



Problem 8.2-6 A spherical steel pressure vessel (diameter 500 mm, thickness 10 mm) is coated with brittle lacquer that cracks when the strain reaches 150×10^{-6} (see figure).

- What internal pressure p will cause the lacquer to develop cracks? (Assume $E = 205$ GPa and $\nu = 0.30$.)
- If the strain is measured at 125×10^{-6} , what is the internal pressure at that point?

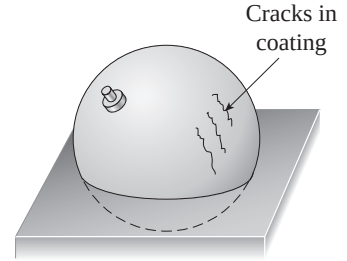
Solution 8.2-6 Spherical vessel with brittle coating

$$(a) \quad t = 10 \text{ mm} \quad E = 205 \text{ GPa} \quad \nu = 0.30 \quad d = 500 \text{ mm} \quad r = \frac{d}{2} \quad \varepsilon_{\max} = 150(10^{-6})$$

$$p_{\max} = \frac{2\varepsilon_{\max}tE}{r(1-\nu)} = 3.51 \text{ MPa}$$

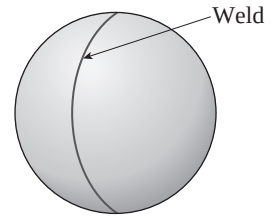
$$(b) \quad \varepsilon = 125(10^{-6})$$

$$p_{\max} = \frac{2\varepsilon tE}{r(1-\nu)} = 2.93 \text{ MPa}$$



Problem 8.2-7 A spherical tank of diameter 48 in. and wall thickness 1.75 in. contains compressed air at a pressure of 2200 psi. The tank is constructed of two hemispheres joined by a welded seam (see figure).

- What is the tensile load f (lb per in. of length of weld) carried by the weld?
- What is the maximum shear stress $\tau_{\max}$ in the wall of the tank?
- What is the maximum normal strain ε in the wall? (For steel, assume $E = 30 \times 10^6$ psi and $\nu = 0.29$.)

**Probs. 8.2-7 and 8.2-8****Solution 8.2-7**

$$r = 24 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$$

$$t = 1.75 \text{ in.} \quad \nu = 0.29 \text{ (steel)}$$

(a) TENSILE LOAD CARRIED BY WELD

$$T = \text{Total load} \quad f = \text{load per inch}$$

$$T = pA = p\pi r^2 \quad c = \text{circumference of tank} = 2\pi r$$

$$f = \frac{T}{c} = \frac{p(\pi r^2)}{2\pi r} = \frac{pr}{2} = \frac{(2200 \text{ psi})(24 \text{ in.})}{2}$$

$$= 26.4 \text{ k/in.} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESS IN WALL (Eq. 8-7)

$$\tau_{\max} = \frac{pr}{4t} = \frac{(2200 \text{ psi})(24 \text{ in.})}{4(1.75 \text{ in.})} = 7543 \text{ psi}$$

(c) MAXIMUM NORMAL STRAIN IN WALL (Eq. 8-10)

$$\varepsilon_{\max} = \frac{pr(1-\nu)}{2tE} = \frac{(2200 \text{ psi})(24 \text{ in.})(0.71)}{2(1.75 \text{ in.})(30 \times 10^6 \text{ psi})}$$

$$= 3.57 \times 10^{-4} \quad \leftarrow$$

Problem 8.2-8 Solve the preceding problem for the following data: diameter 1.0 m, thickness 48 mm, pressure 22 MPa, modulus 210 GPa, and Poisson's ratio 0.29.

Solution 8.2-8

$$r = 0.5 \text{ m} \quad E = 210 \text{ GPa}$$

$$t = 48 \text{ mm} \quad \nu = 0.29 \text{ (steel)}$$

(a) TENSILE LOAD CARRIED BY WELD

$$T = \text{Total load} \quad f = \text{load per inch}$$

$$T = pA = p\pi r^2 \quad c = \text{circumference of tank} = 2\pi r$$

$$f = \frac{T}{c} = \frac{p(\pi r^2)}{2\pi r} = \frac{pr}{2} = \frac{(22 \text{ MPa})(0.5 \text{ m})}{2}$$

$$= 5.5 \text{ MN/m} \quad \leftarrow$$

(b) MAXIMUM SHEAR STRESS IN WALL (EQ. 8-7)

$$\tau_{\max} = \frac{pr}{4t} = \frac{(22 \text{ MPa})(0.5 \text{ m})}{4(48 \text{ mm})}$$

$$= 57.3 \text{ MPa} \quad \leftarrow$$

(c) MAXIMUM NORMAL STRAIN IN WALL (EQ. 8-10)

$$\epsilon_{\max} = \frac{pr(1 - \nu)}{2tE} = \frac{(22 \text{ MPa})(0.5 \text{ m})0.71}{2(48 \text{ mm})(210 \text{ GPa})}$$

$$= 3.87 \times 10^{-4} \quad \leftarrow$$

Problem 8.2-9 A spherical stainless-steel tank having a diameter of 26 in. is used to store propane gas at a pressure of 2075 psi. The properties of the steel are as follows: yield stress in tension, 140,000 psi; yield stress in shear, 65,000 psi; modulus of elasticity, 30×10^6 psi; and Poisson's ratio, 0.28. The desired factor of safety with respect to yielding is 2.8. Also, the normal strain must not exceed 1250×10^{-6} .

(a) Determine the minimum permissible thickness $t_{\min}$ of the tank.

(b) If the tank thickness is 0.30 in. and normal strain is measured at 990×10^{-6} , what is the internal pressure in the tank at that point?

Solution 8.2-9

$$(a) \quad E = 30(10^6) \text{ psi} \quad \nu = 0.28 \quad d = 26 \text{ in.} \quad r = \frac{d}{2} \quad \epsilon_{\max} = 1250(10^{-6}) \quad FS = 2.8$$

$$p = 2075 \text{ psi} \quad \sigma_Y = 140,000 \text{ psi} \quad \tau_Y = 65,000 \text{ psi}$$

$$\text{TENSION} \quad t_1 = \frac{pr}{2\left(\frac{\sigma_Y}{FS}\right)} = 0.27 \text{ in.} \quad \text{SHEAR} \quad t_2 = \frac{pr}{4\left(\frac{\tau_Y}{FS}\right)} = 0.291 \text{ in.} \quad < \text{controls}$$

$$\text{STRAIN} \quad t_3 = \frac{pr}{2\epsilon_{\max}E}(1 - \nu) = 0.259 \text{ in.}$$

$$(b) \quad \epsilon = 990(10^{-6}) \quad t = 0.3 \text{ in.}$$

$$p = \frac{2\epsilon t E}{r(1 - \nu)} = 1904 \text{ psi}$$

Problem 8.2-10 Solve the preceding problem if the diameter is 480 mm, the pressure is 20 MPa, the yield stress in tension is 975 MPa, the yield stress in shear is 460 MPa, the factor of safety is 2.75, the modulus of elasticity is 210 GPa, Poisson's ratio is 0.28, and the normal strain must not exceed 1190×10^{-6} . For part (b) assume that the tank thickness is 8 mm and the measured normal strain is 990×10^{-6} .

Solution 8.2-10

$$(a) \quad E = 210 \text{ GPa} \quad \nu = 0.28 \quad d = 480 \text{ mm} \quad r = \frac{d}{2} \quad \varepsilon_{\max} = 1190(10^{-6}) \quad FS = 2.75$$

$$p = 20 \text{ MPa} \quad \sigma_Y = 975 \text{ MPa} \quad \tau_Y = 460 \text{ MPa}$$

$$\text{TENSION} \quad t_1 = \frac{pr}{2\left(\frac{\sigma_Y}{FS}\right)} = 6.77 \text{ mm} \quad \text{SHEAR} \quad t_2 = \frac{pr}{4\left(\frac{\tau_Y}{FS}\right)} = 7.17 \text{ mm} \quad < \text{controls}$$

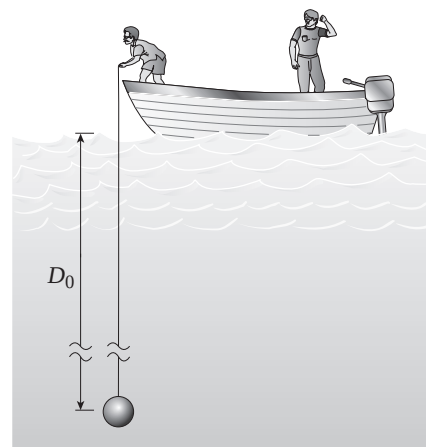
$$\text{STRAIN} \quad t_3 = \frac{pr}{2\varepsilon_{\max}E}(1 - \nu) = 6.915 \text{ mm}$$

$$(b) \quad \varepsilon = 990(10^{-6}) \quad t = 8 \text{ mm}$$

$$p = \frac{2\varepsilon t E}{r(1 - \nu)} = 19.25 \text{ MPa}$$

Problem 8.2-11 A hollow pressurized sphere having radius $r = 4.8$ in. and wall thickness $t = 0.4$ in. is lowered into a lake (see figure). The compressed air in the tank is at a pressure of 24 psi (gage pressure when the tank is out of the water).

At what depth D_0 will the wall of the tank be subjected to a compressive stress of 90 psi?

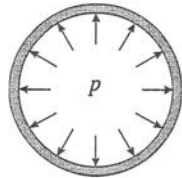


Solution 8.2-11 Pressurized sphere under water

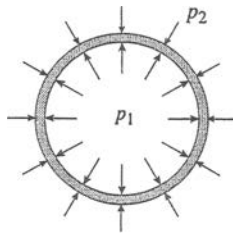
CROSS SECTION

$$r = 4.8 \text{ in.} \quad p_1 = 24 \text{ psi}$$

$$t = 0.4 \text{ in.} \quad \gamma = \text{density of water} = 62.4 \text{ lb/ft}^3$$

(1) IN AIR: $p_1 = 24 \text{ psi}$ 

(1) IN AIR

(2) UNDER WATER: $p_1 = 24 \text{ psi}$ 

(2) UNDER WATER

 $D_0 = \text{depth of water (in.)}$

$$p_2 = \gamma D_0 = \left(\frac{62.4 \text{ lb/ft}^3}{1728 \text{ in.}^3/\text{ft}^3} \right) D_0 = 0.036111 D_0 \text{ (psi)}$$

Compressive stress in tank wall equals 90 psi.
(Note: σ is positive in tension.)

$$\sigma = \frac{pr}{2t} = \frac{(p_1 - p_2)r}{2t} \quad \sigma = -90 \text{ psi}$$

$$\begin{aligned} -90 \text{ psi} &= \frac{(24 \text{ psi} - 0.036111 D_0)(4.8 \text{ in.})}{2(0.4 \text{ in.})} \\ &= 144 - 0.21667 D_0 \end{aligned}$$

$$\text{Solve for } D_0: D_0 = \frac{234}{0.21667}$$

$$= 1080 \text{ in.} = 90 \text{ ft} \quad \leftarrow$$

Cylindrical Pressure Vessels

When solving the problems for Section 8.3, assume that the given radius or diameter is an inside dimension and that all internal pressures are gage pressures.

Problem 8.3-1 A scuba tank (see figure) is being designed for an internal pressure of 2640 psi with a factor of safety of 2.0 with respect to yielding. The yield stress of the steel is 65,000 psi in tension and 32,000 psi in shear.

- If the diameter of the tank is 7.0 in., what is the minimum required wall thickness?
- If the wall thickness is 0.25 in., what is the maximum acceptable internal pressure?

Solution 8.3-1 Scuba tank

$$(a) \quad p = 2640 \text{ psi} \quad \sigma_Y = 65,000 \text{ psi} \quad \tau_Y = 32,000 \text{ psi} \quad FS = 2 \quad d = 7 \text{ in.} \quad r = \frac{d}{2}$$

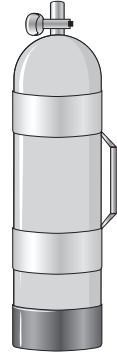
$$\text{TENSION} \quad t_1 = \frac{pr}{\frac{\sigma_Y}{FS}} = 0.284 \text{ in.}$$

$$\text{SHEAR} \quad t_2 = \frac{pr}{2 \frac{\tau_Y}{FS}} = 0.289 \text{ in.} \quad < \text{controls}$$

$$(b) \quad t = 0.25 \text{ in.}$$

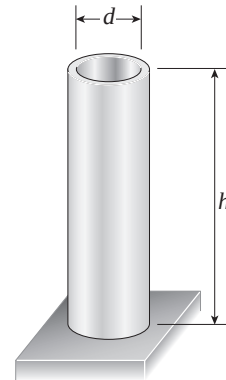
$$p_{\max} = \frac{t}{r} \left(\frac{\sigma_Y}{FS} \right) = 2321 \text{ psi}$$

$$\text{or} \quad p_{\max} = 2 \frac{t}{r} \left(\frac{\tau_Y}{FS} \right) = 2286 \text{ psi} \quad < \text{controls}$$



Problem 8.3-2 A tall standpipe with an open top (see figure) has diameter $d = 2.2 \text{ m}$ and wall thickness $t = 20 \text{ mm}$.

- What height h of water will produce a circumferential stress of 12 MPa in the wall of the standpipe?
- What is the axial stress in the wall of the tank due to the water pressure?

**Solution 8.3-2**

$$d = 2.2 \text{ m} \quad r = 1.1 \text{ m} \quad t = 20 \text{ mm}$$

$$\text{weight density of water } \gamma = 9.81 \text{ kN/m}^3$$

$$\text{height of water} \quad h$$

$$\text{water pressure} \quad p = \gamma h$$

(a) HEIGHT OF WATER

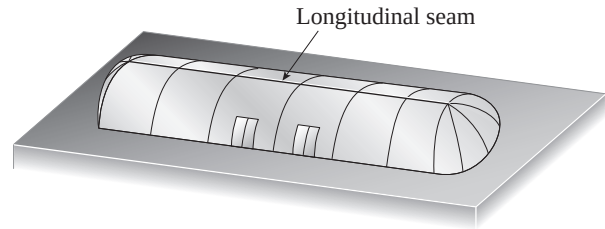
$$\sigma_1 = \frac{pr}{t} = 12 \text{ MPa} = \frac{0.00981h(1.1 \text{ m})}{20 \text{ mm}}$$

$$h = \frac{12(20)}{0.00981(1.1)} = 22.2 \text{ m}$$

(b) AXIAL STRESS IN THE WALL DUE TO WATER PRESSURE ALONE

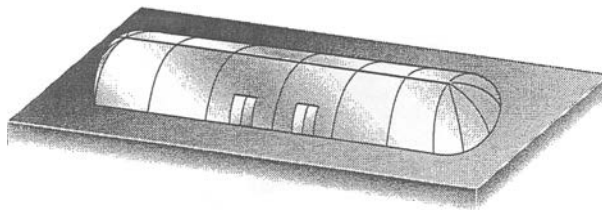
Because the top of the tank is open, the internal pressure of the water produces no axial (longitudinal) stresses in the wall of the tank. Axial stress equals zero. ←

Problem 8.3-3 An inflatable structure used by a traveling circus has the shape of a half-circular cylinder with closed ends (see figure). The fabric and plastic structure is inflated by a small blower and has a radius of 40 ft when fully inflated. A longitudinal seam runs the entire length of the “ridge” of the structure.



If the longitudinal seam along the ridge tears open when it is subjected to a tensile load of 540 pounds per inch of seam, what is the factor of safety n against tearing when the internal pressure is 0.5 psi and the structure is fully inflated?

Solution 8.3-3 Inflatable structure



Half-circular cylinder

$r = 40 \text{ ft} = 480 \text{ in.}$

Internal pressure $p = 0.5 \text{ psi}$

T = tensile force per unit length of longitudinal seam

Seam tears when $T = T_{\max} = 540 \text{ lb/in.}$

Find factor of safety against tearing.

CIRCUMFERENTIAL STRESS (EQ. 8-11)

$\sigma_1 = \frac{pr}{t}$ where t = thickness of fabric

Actual value of T due to internal pressure $= \sigma_1 t$

$\therefore T = \sigma_1 t = pr = (0.5 \text{ psi})(480 \text{ in.}) = 240 \text{ lb/in.}$

FACTOR OF SAFETY

$$n = \frac{T_{\max}}{T} = \frac{540 \text{ lb/in.}}{240 \text{ lb/in.}} = 2.25 \quad \leftarrow$$

Problem 8.3-4 A thin-walled cylindrical pressure vessel of radius r is subjected simultaneously to internal gas pressure p and a compressive force F acting at the ends (see figure).



- What should be the magnitude of compressive force F in order to produce pure shear in the wall of the cylinder?
- If force $F = 190 \text{ kN}$, internal pressure $p = 12 \text{ MPa}$, inner diameter $= 200 \text{ mm}$, and allowable normal and shear stresses are 110 MPa and 60 MPa , respectively, what is the required thickness of the vessel?

Solution 8.3-4 Cylindrical pressure vessel

$$(a) \left(\frac{pr}{t} + \frac{pr}{2t} \right) - \frac{F}{2\pi r t} = 0 \text{ solve, } F = 3\pi p r^2$$

$$3\pi(12 \text{ MPa})(100 \text{ mm})^2 = 1131 \text{ kN}$$

$$\boxed{F = 3\pi p r^2}$$

$$(b) \begin{array}{lll} F = 190 \text{ kN} & p = 12 \text{ MPa} & r = 100 \text{ mm} \\ \sigma_a = 110 \text{ MPa} & \tau_a = 60 \text{ MPa} & A = 2\pi r t \end{array}$$

CIRCUMFERENTIAL NORMAL STRESS:

$$\sigma_1 = \frac{pr}{t} \quad \boxed{t_1 = \frac{pr}{\sigma_a} = 10.91 \text{ mm}} < \text{controls}$$

LONGITUDINAL NORMAL STRESS:

$$\sigma_2 = \frac{pr}{2t} - \frac{F}{A} \quad \sigma_a = \frac{pr}{2t} - \frac{F}{2\pi r t}$$

$$t_2 = \frac{\frac{pr}{2} - \frac{F}{2\pi r}}{\sigma_a} = 2.706 \text{ mm}$$

$$\text{IN-PLANE SHEAR STRESS: } \tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\tau_a = \frac{\frac{pr}{t} - \left(\frac{pr}{2t} - \frac{F}{2\pi r t} \right)}{2}$$

$$\tau_a = \frac{pr}{4t} + \frac{F}{4\pi r t}$$

$$t_3 = \frac{\frac{pr}{4} + \frac{F}{4\pi r}}{\tau_a} = 7.52 \text{ mm}$$

OUT-OF-PLANE SHEAR STRESSES:

$$\tau_{\max} = \frac{\sigma_1}{2} \quad t_4 = \frac{pr}{2\tau_2} = 10 \text{ mm}$$

OR

$$\tau_{\max} = \frac{\sigma_2}{2} \quad t_5 = \frac{\frac{pr}{2} - \frac{F}{2\pi r}}{2\tau_a} = 2.48 \text{ mm}$$

Problem 8.3-5 A strain gage is installed in the longitudinal direction on the surface of an aluminum beverage can (see figure). The radius-to-thickness ratio of the can is 200. When the lid of the can is popped open, the strain changes by $\varepsilon_0 = 187 \times 10^{-6}$.

- (a) What was the internal pressure p in the can? (Assume $E = 10 \times 10^6$ psi and $\nu = 0.33$.)
 (b) What is the change in strain in the radial direction when the lid is opened?

Solution 8.3-5 Aluminum can

$$(a) \text{ ratio} = 200 \quad \varepsilon_0 = 187(10^{-6}) \quad E = 10(10^6) \text{ psi} \quad \nu = 0.33$$

$$\text{From Eq. 8-19a: } \varepsilon_L = \frac{p \times \text{ratio}}{2E} (1 - 2\nu)$$

$$p = \frac{\varepsilon_0(2E)}{\text{ratio}(1 - 2\nu)} = 55 \text{ psi} \quad \boxed{p = 55 \text{ psi}}$$

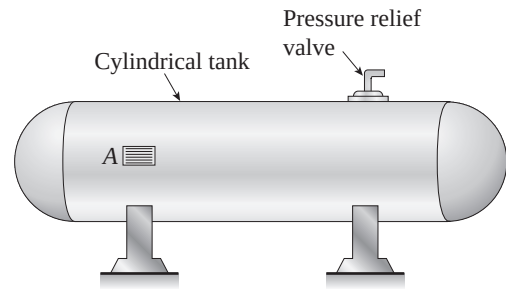
$$(b) \text{ From Eq. 8-19b: } \boxed{\varepsilon_r = \frac{p \times \text{ratio}}{2E} (2 - \nu) = 9.18 \times 10^{-4}}$$

$$\frac{\varepsilon_r}{\varepsilon_0} = 4.912 \quad \frac{2 - \nu}{1 - 2\nu} = 4.912$$



Problem 8.3-6 A circular cylindrical steel tank (see figure) contains a volatile fuel under pressure. A strain gage at point A records the longitudinal strain in the tank and transmits this information to a control room. The ultimate shear stress in the wall of the tank is 98 MPa, and a factor of safety of 2.8 is required.

- (a) At what value of the strain should the operators take action to reduce the pressure in the tank? (Data for the steel are as follows: modulus of elasticity $E = 210$ GPa and Poisson's ratio $\nu = 0.30$.)
 (b) What is the associated strain in the radial direction?



Solution 8.3-6

$$(a) \tau_U = 98 \text{ MPa} \quad FS = 2.8 \quad \tau_{\max} = \frac{\tau_U}{FS} \quad E = 210 \text{ GPa} \quad \nu = 0.30$$

$$\text{From Eq 8-19a: } \epsilon_L = \frac{pr}{2tE}(1 - 2\nu) \quad \text{or} \quad \epsilon_L = \frac{\tau_{\max}}{E}(1 - 2\nu) = 6.67 \times 10^{-5}$$

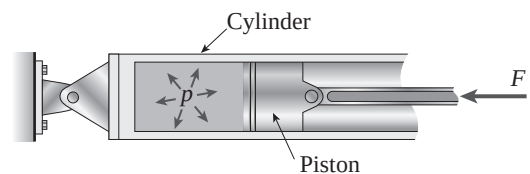
$$\epsilon_{\max} = \epsilon_L \quad \boxed{\epsilon_{\max} = 6.67 \times 10^{-5}}$$

$$(b) \text{ From Eq 8-19b: } \epsilon_r = \frac{\tau_{\max}}{E}(2 - \nu) = 2.83 \times 10^{-4} \quad \frac{\epsilon_r}{\epsilon_L} = 4.25$$

$$\frac{2 - \nu}{1 - 2\nu} = 4.25$$

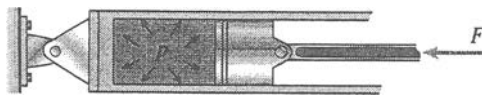
Problem 8.3-7 A cylinder filled with oil is under pressure from a piston, as shown in the figure. The diameter d of the piston is 1.80 in. and the compressive force F is 3500 lb. The maximum allowable shear stress τ_{allow} in the wall of the cylinder is 5500 psi.

What is the minimum permissible thickness $t_{\min}$ of the cylinder wall? (See figure)



Probs. 8.3-7 and 8.3-8

Solution 8.3-7 Cylinder with internal pressure



$$d = 1.80 \text{ in.} \quad r = 0.90 \text{ in.}$$

$$F = 3500 \text{ lb} \quad \tau_{\text{allow}} = 5500 \text{ psi}$$

Find minimum thickness $t_{\min}$.

$$\text{Pressure in cylinder: } p = \frac{F}{A} = \frac{F}{\pi r^2}$$

Maximum shear stress (Eq. 8-16):

$$\tau_{\max} = \frac{pr}{2t} = \frac{F}{2\pi rt}$$

Minimum thickness:

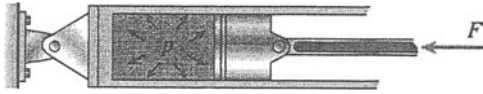
$$t_{\min} = \frac{F}{2\pi r \tau_{\text{allow}}}$$

Substitute numerical values:

$$t_{\min} = \frac{3500 \text{ lb}}{2\pi(0.90 \text{ in.})(5500 \text{ psi})} = 0.113 \text{ in.} \quad \leftarrow$$

Problem 8.3-8 Solve the preceding problem if $d = 90$ mm, $F = 42$ kN, and $\tau_{\text{allow}} = 40$ MPa.

Solution 8.3-8 Cylinder with internal pressure



$$d = 90 \text{ mm} \quad r = 45 \text{ mm}$$

$$F = 42.0 \text{ kN} \quad \tau_{\text{allow}} = 40 \text{ MPa}$$

Find minimum thickness t_{min} .

$$\text{Pressure in cylinder: } p = \frac{F}{A} = \frac{F}{\pi r^2}$$

Maximum shear stress (Eq. 8-16):

$$\tau_{\text{max}} = \frac{pr}{2t} = \frac{F}{2\pi r t}$$

Minimum thickness:

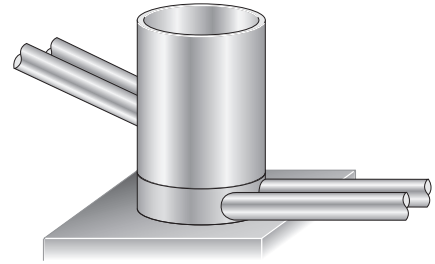
$$t_{\text{min}} = \frac{F}{2\pi r \tau_{\text{allow}}}$$

Substitute numerical values:

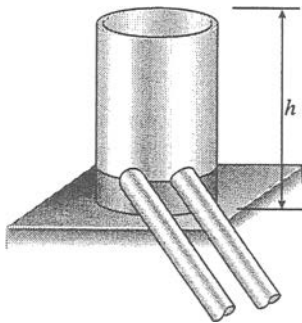
$$t_{\text{min}} = \frac{42.0 \text{ kN}}{2\pi(45 \text{ mm})(40 \text{ MPa})} = 3.71 \text{ mm} \quad \leftarrow$$

Problem 8.3-9 A standpipe in a water-supply system (see figure) is 12 ft in diameter and 6 inches thick. Two horizontal pipes carry water out of the standpipe; each is 2 ft in diameter and 1 inch thick. When the system is shut down and water fills the pipes but is not moving, the hoop stress at the bottom of the standpipe is 130 psi.

- What is the height h of the water in the standpipe?
- If the bottoms of the pipes are at the same elevation as the bottom of the standpipe, what is the hoop stress in the pipes?



Solution 8.3-9 Vertical standpipe



$$d = 12 \text{ ft} = 144 \text{ in.} \quad r = 72 \text{ in.} \quad t = 6 \text{ in.}$$

$$\gamma = 62.4 \text{ lb/ft}^3 = \frac{62.4}{1728} \text{ lb/in.}^3$$

$$\sigma_1 = \text{hoop stress at bottom of standpipe} = 130 \text{ psi}$$

(a) FIND HEIGHT h OF WATER IN THE STANDPIPE

$$p = \text{pressure at bottom of standpipe} = \gamma h$$

$$\text{From Eq. (8-11): } \sigma_1 = \frac{pr}{t} = \frac{\gamma hr}{t} \quad \text{or} \quad h = \frac{\sigma_1 t}{\gamma r}$$

Substitute numerical values:

$$h = \frac{(130 \text{ psi})(6 \text{ in.})}{\left(\frac{62.4}{1728} \text{ lb/in.}^3\right)(72 \text{ in.})} = 300 \text{ in.}$$

$$= 25 \text{ ft} \quad \leftarrow$$

HORIZONTAL PIPES

$$d_1 = 2 \text{ ft} = 24 \text{ in.} \quad r_1 = 12 \text{ in.} \quad t_1 = 1.0 \text{ in.}$$

(b) FIND HOOP STRESS σ_1 IN THE PIPES

Since the pipes are 2 ft in diameter, the depth of water to the center of the pipes is about 24 ft.

$$h_1 \approx 24 \text{ ft} = 288 \text{ in.} \quad p_1 = \gamma h_1$$

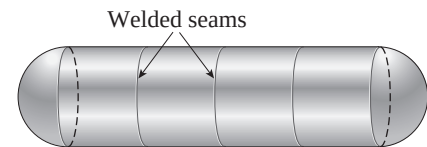
$$\begin{aligned} \sigma_1 &= \frac{p_1 r_1}{t_1} = \frac{\gamma h_1 r_1}{t_1} \\ &= \frac{\left(\frac{62.4}{1728} \text{ lb/in.}^3\right)(288 \text{ in.})(12 \text{ in.})}{1.0 \text{ in.}} \\ &= 125 \text{ psi} \end{aligned}$$

Based on the average pressure in the pipes:

$$\sigma_1 \approx 125 \text{ psi} \quad \leftarrow$$

Problem 8.3-10 A cylindrical tank with hemispherical heads is constructed of steel sections that are welded circumferentially (see figure). The tank diameter is 1.25 m, the wall thickness is 22 mm, and the internal pressure is 1750 kPa.

- Determine the maximum tensile stress σ_h in the heads of the tank.
- Determine the maximum tensile stress σ_c in the cylindrical part of the tank.
- Determine the tensile stress σ_w acting perpendicular to the welded joints.
- Determine the maximum shear stress τ_h in the heads of the tank.
- Determine the maximum shear stress τ_c in the cylindrical part of the tank.



Probs. 8.3-10 and 8.3-11

Solution 8.3-10

$$d = 1.25 \text{ m} \quad r = \frac{d}{2} \quad t = 22 \text{ mm} \quad p = 1750 \text{ kPa}$$

(a) MAXIMUM TENSILE STRESS IN HEMISPHERES (EQ. 8-5)

$$\sigma_h = \frac{pr}{2t} \quad \sigma_h = 24.9 \text{ MPa} \quad \leftarrow$$

(b) MAXIMUM STRESS IN CYLINDER (EQ. 8-11)

$$\sigma_c = \frac{pr}{t} \quad \sigma_c = 49.7 \text{ MPa} \quad \leftarrow$$

(c) TENSILE STRESS IN WELDS (EQ. 8-12)

$$\sigma_w = \frac{pr}{2t} \quad \sigma_w = 24.9 \text{ MPa} \quad \leftarrow$$

(d) MAXIMUM SHEAR STRESS IN HEMISPHERES (EQ. 8-7)

$$\tau_h = \frac{pr}{4t} \quad \tau_h = 12.43 \text{ MPa} \quad \leftarrow$$

(e) MAXIMUM SHEAR STRESS IN CYLINDER (EQ. 8-16)

$$\tau_c = \frac{pr}{2t} \quad \tau_c = 24.9 \text{ MPa} \quad \leftarrow$$

Problem 8.3-11 A cylindrical tank with diameter $d = 18 \text{ in.}$ is subjected to internal gas pressure $p = 450 \text{ psi}$. The tank is constructed of steel sections that are welded circumferentially (see figure). The heads of the tank are hemispherical. The allowable tensile and shear stresses are 8200 psi and 3000 psi, respectively. Also, the allowable tensile stress perpendicular to a weld is 6250 psi.

Determine the minimum required thickness $t_{\min}$ of (a) the cylindrical part of the tank, and (b) the hemispherical heads.

Solution 8.3-11

$$d = 18 \text{ in.} \quad r = \frac{d}{2} \quad p = 450 \text{ psi}$$

$$\sigma_{\text{allow}} = 8200 \text{ psi} \quad (\text{tension})$$

$$\tau_{\text{allow}} = 3000 \text{ psi} \quad (\text{shear})$$

$$\text{WELD} \quad \sigma_a = 6250 \text{ psi} \quad (\text{tension})$$

(a) FIND MINIMUM THICKNESS OF CYLINDER

$$\text{TENSION} \quad \sigma_{\text{max}} = \frac{pr}{t}$$

$$t_{\text{min}} = \frac{pr}{\sigma_{\text{allow}}} \quad t_{\text{min}} = 0.494 \text{ in.}$$

$$\text{SHEAR} \quad \tau_{\text{max}} = \frac{pr}{2t}$$

$$t_{\text{min}} = \frac{pr}{2\tau_{\text{allow}}} \quad t_{\text{min}} = 0.675 \text{ in.}$$

$$\text{WELD} \quad \sigma = \frac{pr}{2t}$$

$$t_{\text{min}} = \frac{pr}{2\sigma_a} \quad t_{\text{min}} = 0.324 \text{ in.}$$

$$t_{\text{min}} = 0.675 \text{ in.} \quad \leftarrow$$

(b) FIND MINIMUM THICKNESS OF HEMISPHERES

$$\text{TENSION} \quad \sigma_{\text{max}} = \frac{pr}{2t}$$

$$t_{\text{min}} = \frac{pr}{2\sigma_{\text{allow}}} \quad t_{\text{min}} = 0.247 \text{ in.}$$

$$\text{SHEAR} \quad \tau_{\text{max}} = \frac{pr}{4t}$$

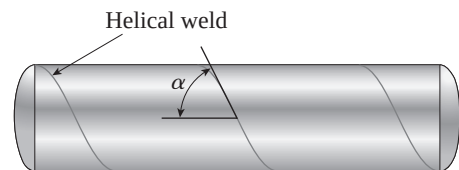
$$t_{\text{min}} = \frac{pr}{4\tau_{\text{allow}}} \quad t_{\text{min}} = 0.338 \text{ in.}$$

$$t_{\text{min}} = 0.338 \text{ in.} \quad \leftarrow$$

Problem 8.3-12 A pressurized steel tank is constructed with a helical weld that makes an angle $\alpha = 55^\circ$ with the longitudinal axis (see figure). The tank has radius $r = 0.6 \text{ m}$, wall thickness $t = 18 \text{ mm}$, and internal pressure $p = 2.8 \text{ MPa}$. Also, the steel has modulus of elasticity $E = 200 \text{ GPa}$ and Poisson's ratio $\nu = 0.30$.

Determine the following quantities for the cylindrical part of the tank.

- The circumferential and longitudinal stresses.
- The maximum in-plane and out-of-plane shear stresses.
- The circumferential and longitudinal strains.
- The normal and shear stresses acting on planes parallel and perpendicular to the weld (show these stresses on a properly oriented stress element).

**Probs. 8.3-12 and 8.3-13****Solution 8.3-12**

$$\alpha = 55^\circ \quad r = 0.6 \text{ m} \quad t = 18 \text{ mm}$$

$$p = 2.8 \text{ MPa} \quad E = 200 \text{ GPa} \quad \nu = 0.3$$

(a) CIRCUMFERENTIAL STRESS

$$\sigma_1 = \frac{pr}{t} \quad \sigma_1 = 93.3 \text{ MPa} \quad \leftarrow$$

LONGITUDINAL STRESS

$$\sigma_2 = \frac{pr}{2t} \quad \sigma_2 = 46.7 \text{ MPa} \quad \leftarrow$$

(b) IN-PLANE SHEAR STRESS

$$\tau_1 = \frac{\sigma_1 - \sigma_2}{2} \quad \tau_1 = 23.3 \text{ MPa} \quad \leftarrow$$

OUT-OF-PLANE SHEAR STRESS

$$\tau_2 = \frac{\sigma_1}{2} \quad \tau_2 = 46.7 \text{ MPa} \quad \leftarrow$$

(c) CIRCUMFERENTIAL STRAIN

$$\varepsilon_1 = \frac{\sigma_1}{2E}(2 - \nu) \quad \varepsilon_1 = 3.97 \times 10^{-4} \quad \leftarrow$$

LONGITUDINAL STRAIN

$$\varepsilon_2 = \frac{\sigma_2}{E}(1 - 2\nu) \quad \varepsilon_2 = 9.33 \times 10^{-5} \quad \leftarrow$$

(d) STRESS ON WELD

$$\theta = 90^\circ - \alpha \quad \theta = 35^\circ$$

$$\sigma_x = \sigma_2 \quad \sigma_x = 46.667 \text{ MPa} \quad \sigma_y = \sigma_1$$

$$\sigma_y = 93.333 \text{ MPa} \quad \tau_{xy} = 0$$

For $\theta = 35^\circ$:

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\sigma_{x1} = 62.0 \text{ MPa} \quad \leftarrow$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

$$\tau_{x1y1} = 21.9 \text{ MPa} \quad \leftarrow$$

$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1} \quad \sigma_{y1} = 78.0 \text{ MPa} \quad \leftarrow$$

Problem 8.3-13 Solve the preceding problem for a welded tank with $\alpha = 62^\circ$, $r = 19$ in., $t = 0.65$ in., $p = 240$ psi, $E = 30 \times 10^6$ psi, and $\nu = 0.30$.

Solution 8.3-13

$$\alpha = 62^\circ \quad r = 19 \text{ in.} \quad t = 0.65 \text{ in.}$$

$$p = 240 \text{ psi} \quad E = 30 \times 10^6 \text{ psi} \quad \nu = 0.3$$

(a) CIRCUMFERENTIAL STRESS

$$\sigma_1 = \frac{pr}{t} \quad \sigma_1 = 7015 \text{ psi} \quad \leftarrow$$

LONGITUDINAL STRESS

$$\sigma_2 = \frac{pr}{2t} \quad \sigma_2 = 3508 \text{ psi} \quad \leftarrow$$

(b) IN-PLANE SHEAR STRESS

$$\tau_1 = \frac{\sigma_1 - \sigma_2}{2} \quad \tau_1 = 1754 \text{ psi} \quad \leftarrow$$

OUT-OF-PLANE SHEAR STRESS

$$\tau_2 = \frac{\sigma_1}{2} \quad \tau_2 = 3508 \text{ psi} \quad \leftarrow$$

(c) CIRCUMFERENTIAL STRAIN

$$\varepsilon_1 = \frac{\sigma_1}{2E}(2 - \nu) \quad \varepsilon_1 = 1.988 \times 10^{-4} \quad \leftarrow$$

LONGITUDINAL STRAIN

$$\varepsilon_2 = \frac{\sigma_2}{E}(1 - 2\nu) \quad \varepsilon_2 = 4.68 \times 10^{-5} \quad \leftarrow$$

(d) STRESS ON WELD

$$\theta = 90^\circ - \alpha \quad \theta = 28^\circ$$

$$\sigma_x = \sigma_2 \quad \sigma_x = 3.508 \times 10^3 \text{ psi} \quad \sigma_y = \sigma_1$$

$$\sigma_y = 7.015 \times 10^3 \text{ psi} \quad \tau_{xy} = 0$$

For $\theta = 28^\circ$:

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta) + \tau_{xy} \sin(2\theta)$$

$$\sigma_{x1} = 4281 \text{ psi} \quad \leftarrow$$

$$\tau_{x1y1} = -\frac{\sigma_x - \sigma_y}{2} \sin(2\theta) + \tau_{xy} \cos(2\theta)$$

$$\tau_{x1y1} = 1454 \text{ psi} \quad \leftarrow$$

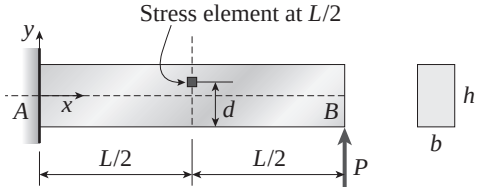
$$\sigma_{y1} = \sigma_x + \sigma_y - \sigma_{x1} \quad \sigma_{y1} = 6242 \text{ psi} \quad \leftarrow$$

Maximum Stresses in Beams

When solving the problems for Section 8.4, consider only the in-plane stresses and disregard the weights of the beams

Problem 8.4-1 A cantilever beam ($L = 6$ ft) with rectangular cross section ($b = 3.5$ in., $h = 12$ in.) supports upward load $P = 35$ kips at its free end.

- (a) Find the state of stress (σ_x , σ_y , τ_{xy} in MPa) on a plane-stress element at $L/2$ which is $d = 8$ in. up from the bottom of the beam. Find the principal normal stresses and maximum shear stress. Show these stresses on sketches of properly oriented elements.
- (b) Repeat part (a) if axial compressive centroidal load $N = 40$ kips is added at B .



Solution 8.4-1

(a) $b = 3.5$ in. $h = 12$ in. $d = 8$ in. $P = 35$ kip $N = 40$ kip $L = 6$ ft

$$A = bh = 42 \text{ in.}^2 \quad I = \frac{bh^3}{12} = 504 \text{ in.}^4 \quad M = \frac{PL}{2} = 105 \text{ kip-ft}$$

$$Q = (h - d)b \left[\frac{h}{2} - \left(\frac{h - d}{2} \right) \right] = 56 \text{ in.}^3 \quad V = P = 35 \text{ kip}$$

$$\sigma_y = 0 \quad \sigma_x = \frac{-M \left(d - \frac{h}{2} \right)}{I} = -5 \text{ ksi} \quad \tau_{xy} = \frac{VQ}{Ib} = 1.111 \text{ ksi}$$

PRINCIPAL STRESSES $\theta_p = \frac{1}{2} \arctan \text{ or } \tan^{-1} \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right) \quad \theta_p = -11.981^\circ$

Let $\theta_1 = \theta_p$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_1) + \tau_{xy} \sin(2\theta_1) \quad \sigma_{x1} = -5.236 \text{ ksi}$$

Let $\theta_2 = \theta_p + \frac{\pi}{2} = 78.019^\circ$

$$\sigma_{x2} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_2) + \tau_{xy} \sin(2\theta_2) \quad \sigma_{x2} = 0.236 \text{ ksi}$$

Therefore,

$$\begin{array}{llll} \sigma_1 = \sigma_{x1} & \theta_{p1} = \theta_1 & \sigma_2 = \sigma_{x2} & \theta_{p2} = \theta_2 \\ \boxed{\sigma_1 = -5.24 \text{ ksi}} & \boxed{\theta_{p1} = -11.98^\circ} & \boxed{\sigma_2 = 0.236 \text{ ksi}} & \boxed{\theta_{p2} = 78.02^\circ} \end{array}$$

MAXIMUM SHEAR STRESS

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \quad \boxed{\tau_{\max} = 2.74 \text{ ksi}}$$

(b)

$$\boxed{\sigma_y = 0} \quad \boxed{\sigma_x = \frac{-N}{A} - \frac{M\left(d - \frac{h}{2}\right)}{I} = -5.95 \text{ ksi}} \quad \boxed{\tau_{xy} = \frac{VQ}{Ib} = 1.11 \text{ ksi}}$$

PRINCIPAL STRESSES $\theta_p = \frac{1}{2} \arctan \text{ or } \tan^{-1} \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right) \quad \theta_p = -10.236^\circ$

Let $\theta_1 = \theta_p$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_1) + \tau_{xy} \sin(2\theta_1) \quad \sigma_{x1} = -6.153 \text{ ksi}$$

Let $\theta_2 = \theta_p + \frac{\pi}{2} = 79.764^\circ$

$$\sigma_{x2} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_2) + \tau_{xy} \sin(2\theta_2) \quad \sigma_{x2} = 0.201 \text{ ksi}$$

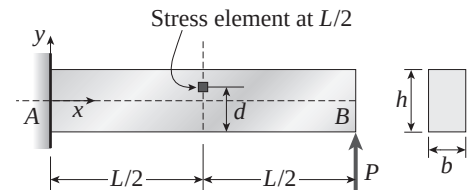
Therefore,

$$\begin{array}{llll} \sigma_1 = \sigma_{x1} & \theta_{p1} = \theta_1 & \sigma_2 = \sigma_{x2} & \theta_{p2} = \theta_2 \\ \boxed{\sigma_1 = -6.15 \text{ ksi}} & \boxed{\theta_{p1} = -10.24^\circ} & \boxed{\sigma_2 = 0.201 \text{ ksi}} & \boxed{\theta_{p2} = 79.76^\circ} \end{array}$$

MAXIMUM SHEAR STRESS

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \boxed{\tau_{\max} = 3.18 \text{ ksi}}$$

Problem 8.4-2 Solve the preceding problem for the following data: $P = 160 \text{ kN}$, $N = 200 \text{ kN}$, $L = 2 \text{ m}$, $b = 95 \text{ mm}$, $h = 300 \text{ mm}$, and $d = 200 \text{ mm}$.



Solution 8.4-2

(a) $b = 95 \text{ mm}$ $h = 300 \text{ mm}$ $d = 200 \text{ mm}$ $P = 160 \text{ kN}$ $N = 200 \text{ kN}$ $L = 2 \text{ m}$

$$A = bh = 2.85 \times 10^4 \text{ mm}^2 \quad I = \frac{bh^3}{12} = 2.138 \times 10^8 \text{ mm}^4 \quad M = \frac{PL}{2} = 160 \text{ kN}\cdot\text{m}$$

$$Q = (h - d)b \left[\frac{h}{2} - \left(\frac{h - d}{2} \right) \right] = 9.5 \times 10^5 \text{ mm}^3 \quad V = P = 1.6 \times 10^5 \text{ N}$$

$$\boxed{\sigma_y = 0} \quad \boxed{\sigma_x = \frac{-M\left(d - \frac{h}{2}\right)}{I} = -37.4 \text{ MPa}} \quad \boxed{\tau_{xy} = \frac{VQ}{Ib} = 7.49 \text{ MPa}}$$

PRINCIPAL STRESSES $\theta_p = \frac{1}{2} \arctan \text{ or } \tan^{-1} \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right) \quad \theta_p = -10.901^\circ$

Let $\theta_1 = \theta_p$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_1) + \tau_{xy} \sin(2\theta_1) \quad \sigma_{x1} = -38.868 \text{ MPa}$$

Let $\theta_2 = \theta_p + \frac{\pi}{2} = 79.099^\circ$

$$\sigma_{x2} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_2) + \tau_{xy} \sin(2\theta_2) \quad \sigma_{x2} = 1.442 \text{ MPa}$$

Therefore,

$\sigma_1 = \sigma_{x1}$	$\theta_{p1} = \theta_1$	$\sigma_2 = \sigma_{x2}$	$\theta_{p2} = \theta_2$
$\sigma_1 = -38.9 \text{ MPa}$	$\theta_{p1} = -10.9^\circ$	$\sigma_2 = 1.442 \text{ MPa}$	$\theta_{p2} = 79.1^\circ$

MAXIMUM SHEAR STRESS

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \quad \tau_{\max} = 20.2 \text{ MPa}$$

(b)

$\sigma_y = 0$	$\sigma_x = \frac{-N}{A} - \frac{M \left(d - \frac{h}{2} \right)}{I} = -44.4 \text{ MPa}$	$\tau_{xy} = \frac{VQ}{Ib} = 7.49 \text{ MPa}$

PRINCIPAL STRESSES $\theta_p = \frac{1}{2} \arctan \text{ or } \tan^{-1} \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right) \quad \theta_p = -9.308^\circ$

Let $\theta_1 = \theta_p$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_1) + \tau_{xy} \sin(2\theta_1) \quad \sigma_{x1} = -45.671 \text{ MPa}$$

Let $\theta_2 = \theta_p + \frac{\pi}{2} = 80.692^\circ$

$$\sigma_{x2} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos(2\theta_2) + \tau_{xy} \sin(2\theta_2) \quad \sigma_{x2} = 1.227 \text{ MPa}$$

Therefore,

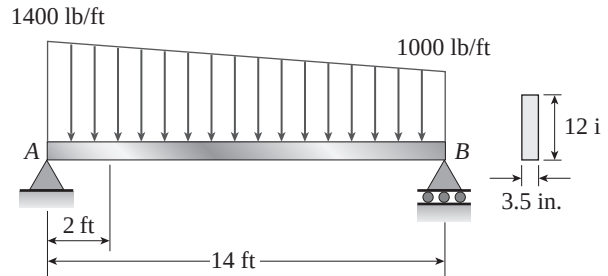
$\sigma_1 = \sigma_{x1}$	$\theta_{p1} = \theta_1$	$\sigma_2 = \sigma_{x2}$	$\theta_{p2} = \theta_2$
$\sigma_1 = -45.7 \text{ MPa}$	$\theta_{p1} = -9.3^\circ$	$\sigma_2 = 1.227 \text{ MPa}$	$\theta_{p2} = 80.7^\circ$

MAXIMUM SHEAR STRESS

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \quad \tau_{\max} = 23.4 \text{ MPa}$$

Problem 8.4-3 A simple beam of rectangular cross section (width, 3.5 in.; height, 12 in.) carries a trapezoidally distributed load of 1400 lb/ft at A and 1000 lb/ft at B on a span of 14 ft (see figure).

Find the principal stresses σ_1 and σ_2 and the maximum shear stress $\tau_{\max}$ at a cross section 2 ft from the left-hand support at each of the following locations: (a) the neutral axis, (b) 2 in. above the neutral axis, and (c) the top of the beam. (Disregard the direct compressive stresses produced by the uniform load bearing against the top of the beam.)



Solution 8.4-3

NUMERICAL DATA

$$b = 3.5 \text{ in.} \quad h = 12 \text{ in.} \quad A = bh \quad I = \frac{bh^3}{12} \quad I = 504 \text{ in.}^4 \quad c = 2 \text{ ft} \quad L = 14 \text{ ft}$$

$$q_A = 1400 \frac{\text{lb}}{\text{ft}} \quad q_B = 1000 \frac{\text{lb}}{\text{ft}}$$

STATICS

$$R_A = \frac{1}{L} \left[\frac{1}{2} (q_B) (L) \frac{L}{3} + \frac{1}{2} (q_A) L \frac{2L}{3} \right] = 8866.667 \text{ lb} \quad q_c = q_B + \frac{L-c}{L} (q_A - q_B) = 1342.857 \frac{\text{lb}}{\text{ft}}$$

$$M = R_A c - \left[\frac{1}{2} (q_c) (c) \frac{c}{3} + \frac{1}{2} (q_A) c \frac{2c}{3} \right] = 14.971 \text{ kip-ft}$$

$$V = R_A - \frac{1}{2} (q_A + q_c) c = 6.124 \text{ kip}$$

(a) NEUTRAL AXIS

$$\sigma_x = 0 \quad \sigma_y = 0 \quad \tau_{xy} = -\frac{3V}{2A} \quad \tau_{xy} = -218.707 \text{ psi}$$

$$\text{Pure shear:} \quad \sigma_1 = -\tau_{xy} \quad \sigma_2 = -\sigma_1 \quad \tau_{\max} = \sigma_1$$

$$\boxed{\sigma_1 = 219 \text{ psi}} \quad \boxed{\sigma_2 = -219 \text{ psi}} \quad \boxed{\tau_{\max} = 219 \text{ psi}}$$

(b) 2 IN. ABOVE THE NEUTRAL AXIS $y = 2 \text{ in.}$ $d = 4 \text{ in.}$

$$\sigma_x = -\frac{My}{I} \quad \sigma_x = -712.925 \text{ psi} \quad \sigma_y = 0$$

$$Q = bd \left(\frac{h}{2} - \frac{d}{2} \right) \quad Q = 56 \text{ in.}^3$$

$$\tau_{xy} = -\frac{VQ}{Ib} \quad \tau_{xy} = -194.407 \text{ psi}$$

$$\sigma_1 = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2} \right)^2 + \tau_{xy}^2} \quad \boxed{\sigma_1 = 49.6 \text{ psi}}$$

$$\sigma_2 = \frac{\sigma_x}{2} - \sqrt{\left(\frac{\sigma_x}{2} \right)^2 + \tau_{xy}^2} \quad \boxed{\sigma_2 = -762 \text{ psi}}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x}{2} \right)^2 + \tau_{xy}^2} \quad \boxed{\tau_{\max} = 406 \text{ psi}}$$

(c) TOP OF THE BEAM

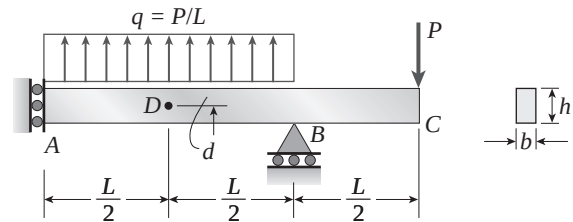
$$\sigma_x = -\frac{M\left(\frac{h}{2}\right)}{I} \quad \sigma_x = -2.139 \times 10^3 \text{ psi} \quad \sigma_y = 0 \quad \tau_{xy} = 0$$

Uniaxial stress:

$$\begin{aligned} \sigma_1 &= \sigma_y & \sigma_1 &= 0 \text{ psi} \\ \sigma_2 &= \sigma_x & \sigma_2 &= -2139 \text{ psi} \\ \tau_{\max} &= \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2} & \tau_{\max} &= 1069 \text{ psi} \end{aligned}$$

Problem 8.4-4 An overhanging beam ABC has a guided support at A, a rectangular cross section, and supports an upward uniform load $q = P/L$ over AB and a downward concentrated load P at the free end C (see figure). The span length from A to B is L , and the length of the overhang is $L/2$. The cross section has width b and height h . Point D is located midway between the supports at a distance d from the top face of the beam.

Knowing that the maximum tensile stress (principal stress) at point D is $\sigma_1 = 38 \text{ MPa}$, determine the magnitude of the load P . Data for the beam are as follows: $L = 1.75 \text{ m}$, $b = 50 \text{ mm}$, $h = 220 \text{ mm}$, and $d = 55 \text{ mm}$.

**Solution 8.4-4**

$$L = 1.75 \text{ m} \quad b = 50 \text{ mm} \quad h = 220 \text{ mm} \quad d = 55 \text{ mm} \quad \sigma_1 = 38 \text{ MPa} \quad q = \frac{P}{L}$$

STATICS

$$R_B = -qL + P \quad V_D = q\frac{L}{2} \quad V_D = \frac{P}{2} \quad M_D = R_B\frac{L}{2} + q\frac{L}{2}\frac{L}{4} - PL$$

$$M_D = (-qL + P)\frac{L}{2} + q\frac{L}{2}\frac{L}{4} - PL = -\frac{7LP}{8}$$

BEAM CROSS SECTION PROPERTIES

$$I = \frac{bh^3}{12} = 4.437 \times 10^7 \text{ mm}^4 \quad Q = bd\left(\frac{h}{2} - \frac{d}{2}\right) = 2.269 \times 10^5 \text{ mm}^3 \quad y = \frac{h}{2} - d$$

STRESSES AT D

$$\sigma_y = 0 \quad \sigma_x = \frac{-M_D y}{I} \quad \sigma_x = \frac{-\left(\frac{-7}{8} PL\right) y}{I} \quad \tau_{xy} = \frac{-V_D Q}{Ib} \quad \tau_{xy} = \frac{\left(\frac{-P}{2}\right) Q}{Ib}$$

PRINCIPAL STRESSES

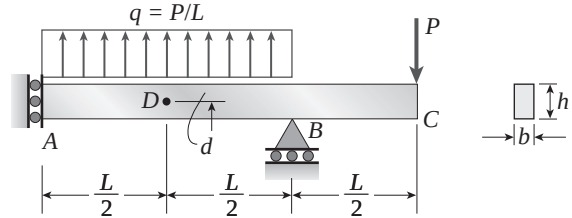
$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{or} \quad \sigma_1 = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$$y = 55 \text{ mm} \quad Q = 2.269 \times 10^5 \text{ mm}^3 \quad I = 4.437 \times 10^7 \text{ mm}^4 \quad L = 1.75 \text{ m} \quad b = 50 \text{ mm} \quad \sigma_1 = 38 \text{ MPa}$$

Substitute expressions and solve for P :

$$\frac{-\left(\frac{-7}{8}PL\right)y}{\frac{I}{2}} + \sqrt{\left[\frac{-\left(\frac{-7}{8}PL\right)y}{\frac{I}{2}}\right]^2 + \left[\left(\frac{-P}{2}\right)Q\right]^2} = \sigma_1 \quad \text{solving} \quad \boxed{P = 20 \text{ kN}}$$

Problem 8.4-5 Solve the preceding problem if the stress and dimensions are as follows: $\sigma_1 = 2450$ psi, $L = 80$ in., $b = 2.5$ in., $h = 10$ in., and $d = 2.5$ in.



Solution 8.4-5

$$L = 80 \text{ in.} \quad b = 2.5 \text{ in.} \quad h = 10 \text{ in.} \quad d = 2.5 \text{ in.} \quad \sigma_1 = 2450 \text{ psi} \quad q = \frac{P}{L}$$

STATICS

$$R_B = -qL + P \quad V_D = q\frac{L}{2} \quad V_D = \frac{P}{2} \quad M_D = R_B\frac{L}{2} + q\frac{L}{2}\frac{L}{4} - PL$$

$$M_D = (-qL + P)\frac{L}{2} + q\frac{L}{2}\frac{L}{4} - PL = -\frac{7LP}{8}$$

BEAM CROSS SECTION PROPERTIES

$$I = \frac{bh^3}{12} = 208.333 \text{ in.}^4 \quad Q = bd\left(\frac{h}{2} - \frac{d}{2}\right) = 23.438 \text{ in.}^3 \quad y = \frac{h}{2} - d$$

STRESSES AT D

$$\sigma_y = 0 \quad \sigma_x = \frac{-M_D y}{I} \quad \sigma_x = \frac{-\left(\frac{-7}{8}PL\right)y}{I} \quad \tau_{xy} = \frac{-V_D Q}{Ib} \quad \tau_{xy} = \frac{\left(\frac{-P}{2}\right)Q}{Ib}$$

PRINCIPAL STRESSES

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{or} \quad \sigma_1 = \frac{\sigma_x}{2} + \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

$$y = 2.5 \text{ in.} \quad Q = 23.438 \text{ in.}^3 \quad I = 208.333 \text{ in.}^4 \quad L = 80 \text{ in.} \quad b = 2.5 \text{ in.}$$

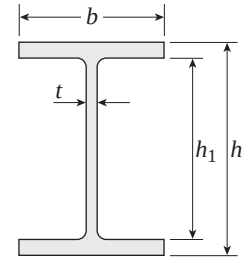
$$\sigma_1 = 2450 \text{ psi}$$

Substitute expressions and solve for P :

$$\frac{-\left(\frac{-7}{8}PL\right)y}{\frac{I}{2}} + \sqrt{\left[\frac{-\left(\frac{-7}{8}PL\right)y}{\frac{I}{2}}\right]^2 + \left[\left(\frac{-P}{2}\right)Q\right]^2} = \sigma_1 \quad \text{solving} \quad \boxed{P = 2.91 \text{ kip}}$$

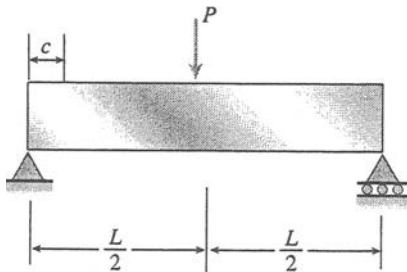
Problem 8.4-6 A beam of wide-flange cross section (see figure) has the following dimensions: $b = 120$ mm, $t = 10$ mm, $h = 300$ mm, and $h_1 = 260$ mm. The beam is simply supported with span length $L = 3.0$ m. A concentrated load $P = 120$ kN acts at the midpoint of the span.

At a cross section located 1.0 m from the left-hand support, determine the principal stresses σ_1 and σ_2 and the maximum shear stress $\tau_{\max}$ at each of the following locations: (a) the top of the beam, (b) the top of the web, and (c) the neutral axis.



Probs. 8.4-6 and 8.4-7

Solution 8.4-6 Simply supported beam



$$P = 120 \text{ kN} \quad L = 3.0 \text{ m} \quad c = 1.0 \text{ m}$$

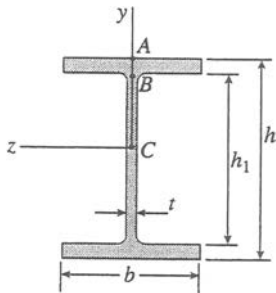
$$M = \frac{Pc}{2} = 60 \text{ kN} \cdot \text{m} \quad V = \frac{P}{2} = 60 \text{ kN}$$

$$b = 120 \text{ mm} \quad t = 10 \text{ mm}$$

$$h = 300 \text{ mm} \quad h_1 = 260 \text{ mm}$$

$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = 108.89 \times 10^6 \text{ mm}^4$$

(a) TOP OF THE BEAM (POINT A)



$$\begin{aligned} \sigma_x &= \frac{Mc}{I} = -\frac{(60 \text{ kN} \cdot \text{m})(150 \text{ mm})}{108.89 \times 10^6 \text{ mm}^4} \\ &= -82.652 \text{ MPa} \end{aligned}$$

$$\sigma_y = 0 \quad \tau_{xy} = 0$$

$$\text{Uniaxial stress: } \sigma_1 = 0$$

$$\sigma_2 = -82.7 \text{ MPa}$$

$$\tau_{\max} = 41.3 \text{ MPa}$$

(b) TOP OF THE WEB (POINT B)

$$\begin{aligned} \sigma_x &= -\frac{My}{I} = -\frac{(60 \text{ kN} \cdot \text{m})(130 \text{ mm})}{108.89 \times 10^6 \text{ mm}^4} \\ &= -71.63 \text{ MPa} \end{aligned}$$

$$\sigma_y = 0$$

$$Q = (b)\left(\frac{h-h_1}{2}\right)\left(\frac{h+h_1}{4}\right) = 336 \times 10^3 \text{ mm}^3$$

$$\begin{aligned} \tau_{xy} &= -\frac{VQ}{It} = -\frac{(60 \text{ kN})(336 \times 10^3 \text{ mm}^3)}{(108.89 \times 10^6 \text{ mm}^4)(10 \text{ mm})} \\ &= -18.51 \text{ MPa} \end{aligned}$$

$$\begin{aligned} \sigma_{1,2} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ &= -35.82 \pm 40.32 \text{ MPa} \end{aligned}$$

$$\sigma_1 = 4.5 \text{ MPa}, \sigma_2 = -76.1 \text{ MPa}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 40.3 \text{ MPa}$$

(c) NEUTRAL AXIS (POINT C)

$$\sigma_x = 0 \quad \sigma_y = 0$$

$$\begin{aligned} Q &= b\left(\frac{h}{2}\right)\left(\frac{h}{4}\right) - (b-t)\left(\frac{h_1}{2}\right)\left(\frac{h_1}{4}\right) \\ &= 420.5 \times 10^3 \text{ mm}^3 \end{aligned}$$

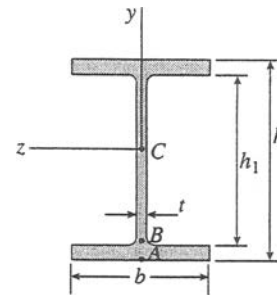
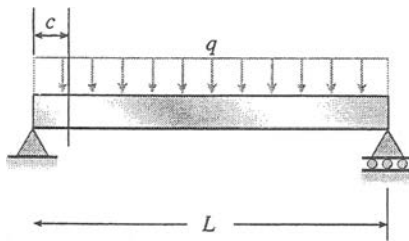
$$\begin{aligned}\tau_{xy} &= -\frac{VQ}{It} = -\frac{(60 \text{ kN})(420.5 \times 10^3 \text{ mm}^3)}{(108.89 \times 10^6 \text{ mm}^4)(10 \text{ mm})} \\ &= -23.17 \text{ MPa}\end{aligned}$$

$$\left. \begin{aligned}\text{Pure shear: } \sigma_1 &= 23.2 \text{ MPa,} \\ \sigma_2 &= -23.2 \text{ MPa} \\ \tau_{\max} &= 23.2 \text{ MPa}\end{aligned} \right\} \leftarrow$$

Problem 8.4-7 A beam of wide-flange cross section (see figure) has the following dimensions: $b = 5 \text{ in.}$, $t = 0.5 \text{ in.}$, $h = 12 \text{ in.}$, and $h_1 = 10.5 \text{ in.}$ The beam is simply supported with span length $L = 10 \text{ ft}$ and supports a uniform load $q = 6 \text{ k/ft}$.

Calculate the principal stresses σ_1 and σ_2 and the maximum shear stress $\tau_{\max}$ at a cross section located 3 ft from the left-hand support at each of the following locations: (a) the bottom of the beam, (b) the bottom of the web, and (c) the neutral axis.

Solution 8.4-7 Simply supported beam



$$q = 6.0 \text{ k/ft} \quad L = 10 \text{ ft} = 120 \text{ in.}$$

$$c = 3 \text{ ft} = 36 \text{ in.} \quad M = \frac{qLc}{2} - \frac{qc^2}{2} = 756,000 \text{ lb-in.}$$

$$V = \frac{qL}{2} - qc = 12,000 \text{ lb}$$

$$b = 5.0 \text{ in.} \quad t = 0.5 \text{ in.} \quad h = 12 \text{ in.} \quad h_1 = 10.5 \text{ in.}$$

$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = 285.89 \text{ in.}^4$$

(a) BOTTOM OF THE BEAM (POINT A)

$$\sigma_x = -\frac{Mc}{I} = \frac{(756,000 \text{ lb-in.})(-6.0 \text{ in.})}{285.89 \text{ in.}^4}$$

$$= 15,866 \text{ psi}$$

$$\sigma_y = 0 \quad \tau_{xy} = 0$$

$$\left. \begin{aligned}\text{Uniaxial stress: } \sigma_1 &= 15,870 \text{ psi,} \\ \sigma_2 &= 0 \quad \tau_{\max} = 7930 \text{ psi}\end{aligned} \right\} \leftarrow$$

(b) BOTTOM OF THE WEB (POINT B)

$$\sigma_x = -\frac{My}{I} = -\frac{(756,000 \text{ lb-in.})(-5.25 \text{ in.})}{285.89 \text{ in.}^4}$$

$$= 13,883 \text{ psi}$$

$$\sigma_y = 0 \quad Q = b\left(\frac{h-h_1}{2}\right)\left(\frac{h+h_1}{4}\right) = 21.094 \text{ in.}^3$$

$$\tau_{xy} = -\frac{VQ}{It} = -\frac{(12,000 \text{ lb})(21.094 \text{ in.}^3)}{(285.89 \text{ in.}^4)(0.5 \text{ in.})}$$

$$= -1771 \text{ psi}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 6941.5 \pm 7163.9 \text{ psi}$$

$$\sigma_1 = 14,100 \text{ psi}, \sigma_2 = -220 \text{ psi} \quad \leftarrow$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 7160 \text{ psi} \quad \leftarrow$$

(c) NEUTRAL AXIS (POINT C)

$$\sigma_x = 0 \quad \sigma_y = 0$$

$$Q = b\left(\frac{h}{2}\right)\left(\frac{h}{4}\right) - (b - t)\left(\frac{h_1}{2}\right)\left(\frac{h_1}{4}\right)$$

$$= 27.984 \text{ in.}^3$$

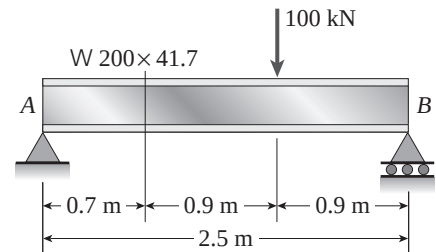
$$\tau_{xy} = -\frac{VQ}{It} = -\frac{(12,000 \text{ lb})(27.984 \text{ in.}^3)}{(285.89 \text{ in.}^4)(0.5 \text{ in.})}$$

$$= -2349 \text{ psi}$$

$$\left. \begin{array}{l} \text{Pure shear: } \sigma_1 = 2350 \text{ psi,} \\ \sigma_2 = -2350 \text{ psi,} \\ \tau_{\max} = 2350 \text{ psi} \end{array} \right\} \quad \leftarrow$$

Problem 8.4-8 A W 200 × 41.7 wide-flange beam (see Table F-1(b), Appendix F) is simply supported with a span length of 2.5 m (see figure). The beam supports a concentrated load of 100 kN at 0.9 m from support B.

At a cross section located 0.7 m from the left-hand support, determine the principal stresses σ_1 and σ_2 and the maximum shear stress $\tau_{\max}$ at each of the following locations: (a) the top of the beam, (b) the top of the web, and (c) the neutral axis.

**Solution 8.4-8**

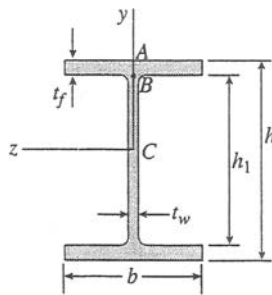
$$R_B = 100 \text{ kN} \left(\frac{0.7 + 0.9}{2.5} \right) \quad R_B = 64 \text{ kN} \quad (\text{upward})$$

$$R_A = 100 \text{ kN} - R_B \quad R_A = 36 \text{ kN} \quad (\text{upward})$$

At the point D,

$$M = R_A(0.7 \text{ m}) \quad M = 25.2 \text{ kN} \cdot \text{m}$$

$$V = R_A \quad V = 36 \text{ kN}$$

**W200 × 41.7**

$$I = 40.8 \times 10^6 \text{ mm}^4$$

$$b = 166 \text{ mm}$$

$$t_w = 7.24 \text{ mm}$$

$$t_f = 11.8 \text{ mm}$$

$$h = 205 \text{ mm}$$

$$h_1 = h - 2t_f$$

$$h_1 = 181.4 \text{ mm}$$

(a) TOP OF THE BEAM (POINT A)

$$\sigma_x = -\frac{M\left(\frac{h}{2}\right)}{I} \quad \sigma_x = -63.309 \text{ MPa}$$

$$\sigma_y = 0 \quad \tau_{xy} = 0$$

$$\text{Uniaxial stress: } \sigma_1 = \sigma_y \quad \sigma_2 = \sigma_x \quad \tau_{\max} = \left| \frac{\sigma_x}{2} \right|$$

$$\sigma_1 = 0 \quad \leftarrow \quad \sigma_2 = -63.3 \text{ MPa} \quad \leftarrow$$

$$\tau_{\max} = 31.7 \text{ MPa} \quad \leftarrow$$

(b) TOP OF THE WEB (POINT B)

$$\sigma_x = -\frac{M\left(\frac{h_1}{2}\right)}{I} \quad \sigma_x = -56.021 \text{ MPa}$$

$$\sigma_y = 0$$

$$Q = b\left(\frac{h - h_1}{2}\right)\left(\frac{h + h_1}{4}\right)$$

$$Q = 1.892 \times 10^5 \text{ mm}^3$$

$$\tau_{xy} = -\frac{VQ}{It_w} \quad \tau_{xy} = -23.061 \text{ MPa}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = 8.27 \text{ MPa} \quad \leftarrow$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = -64.3 \text{ MPa} \quad \leftarrow$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = 36.3 \text{ MPa} \quad \leftarrow$$

(c) NEUTRAL AXIS (POINT C)

$$\sigma_x = 0 \quad \sigma_y = 0$$

$$Q = b\left(\frac{h}{2}\right)\left(\frac{h}{4}\right) - (b - t_w)\left(\frac{h_1}{2}\right)\left(\frac{h_1}{4}\right)$$

$$Q = 2.19 \times 10^5 \text{ mm}^3$$

$$\tau_{xy} = -\frac{VQ}{It_w} \quad \tau_{xy} = -26.69 \text{ MPa}$$

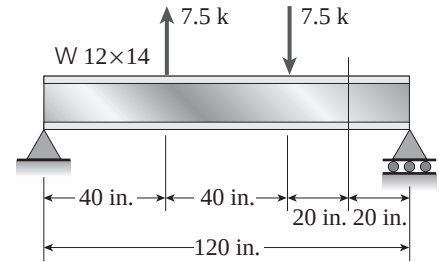
$$\text{Pure shear: } \sigma_1 = |\tau_{xy}| \quad \sigma_1 = 26.7 \text{ MPa} \quad \leftarrow$$

$$\sigma_2 = -\sigma_1 \quad \sigma_2 = -26.7 \text{ MPa} \quad \leftarrow$$

$$\tau_{\max} = \tau_{xy} \quad \tau_{\max} = -26.7 \text{ MPa} \quad \leftarrow$$

Problem 8.4-9 A W 12 × 14 wide-flange beam (see Table F-1(a), Appendix F) is simply supported with a span length of 120 in. (see figure). The beam supports two anti-symmetrically placed concentrated loads of 7.5 k each.

At a cross section located 20 in. from the right-hand support, determine the principal stresses σ_1 and σ_2 and the maximum shear stress $\tau_{\max}$ at each of the following locations: (a) the top of the beam, (b) the top of the web, and (c) the neutral axis.

**Solution 8.4-9**

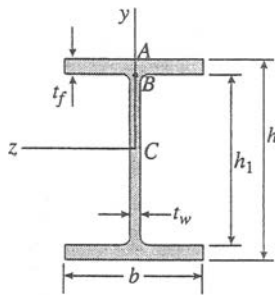
$$R_B = \frac{7.5 \cdot 80 - 7.5 \cdot 40}{120} \text{ k} \quad R_B = 2.5 \text{ k} \quad (\text{upward})$$

$$R_A = -R_B \quad R_A = -2.5 \text{ k} \quad (\text{downward})$$

At Section C-C

$$M = R_B \cdot 20 \text{ in.} \quad M = 50 \text{ k} \cdot \text{in.}$$

$$V = -R_B \quad V = -2.5 \text{ k}$$

**W12 × 14**

$$I = 88.6 \text{ in.}^4$$

$$b = 3.970 \text{ in.}$$

$$t_w = 0.200 \text{ in.}$$

$$t_f = 0.225 \text{ in.}$$

$$h = 11.91 \text{ in.}$$

$$h_1 = h - 2t_f$$

$$h_1 = 11.46 \text{ in.}$$

(a) TOP OF THE BEAM (POINT A)

$$\sigma_x = -\frac{M\left(\frac{h}{2}\right)}{I} \quad \sigma_x = -3.361 \times 10^3 \text{ psi}$$

$$\sigma_y = 0 \quad \tau_{xy} = 0$$

$$\text{Uniaxial stress: } \sigma_1 = \sigma_y \quad \sigma_2 = \sigma_x \quad \tau_{\max} = \left|\frac{\sigma_x}{2}\right|$$

$$\sigma_1 = 0 \quad \leftarrow \quad \sigma_2 = -3361 \text{ psi} \quad \leftarrow$$

$$\tau_{\max} = 1680 \text{ psi} \quad \leftarrow$$

(b) TOP OF THE WEB (POINT B)

$$\sigma_x = -\frac{M\left(\frac{h_1}{2}\right)}{I} \quad \sigma_x = -3.234 \times 10^3 \text{ psi}$$

$$\sigma_y = 0$$

$$Q = b \left(\frac{h - h_1}{2} \right) \left(\frac{h + h_1}{4} \right) \quad Q = 5.219 \text{ in.}^3$$

$$\tau_{xy} = -\frac{VQ}{It_w} \quad \tau_{xy} = 736.289 \text{ psi}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = 159.8 \text{ psi} \quad \leftarrow$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = -3393 \text{ psi} \quad \leftarrow$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = 1777 \text{ psi} \quad \leftarrow$$

(c) NEUTRAL AXIS (POINT C)

$$\sigma_x = 0 \quad \sigma_y = 0$$

$$Q = b \left(\frac{h}{2} \right) \left(\frac{h}{4} \right) - (b - t_w) \left(\frac{h_1}{2} \right) \left(\frac{h_1}{4} \right)$$

$$Q = 8.502 \text{ in.}^3$$

$$\tau_{xy} = -\frac{VQ}{It_w} \quad \tau_{xy} = 1.2 \times 10^3 \text{ psi}$$

Pure shear:

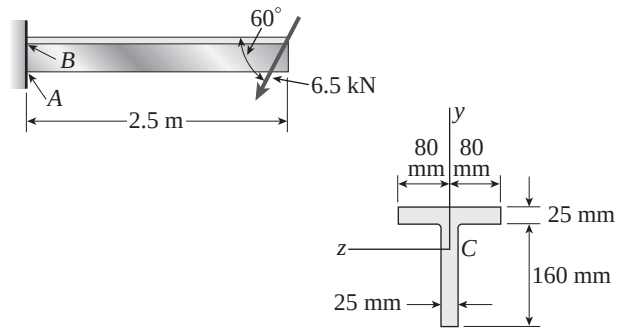
$$\sigma_1 = |\tau_{xy}| \quad \sigma_1 = 1200 \text{ psi} \quad \leftarrow$$

$$\sigma_2 = -\sigma_1 \quad \sigma_2 = -1200 \text{ psi} \quad \leftarrow$$

$$\tau_{\max} = \tau_{xy} \quad \tau_{\max} = 1200 \text{ psi} \quad \leftarrow$$

Problem 8.4-10 A cantilever beam of T-section is loaded by an inclined force of magnitude 6.5 kN (see figure). The line of action of the force is inclined at an angle of 60° to the horizontal and intersects the top of the beam at the end cross section. The beam is 2.5 m long and the cross section has the dimensions shown.

Determine the principal stresses σ_1 and σ_2 and the maximum shear stress $\tau_{\max}$ at points A and B in the web of the beam near the support.

**Solution 8.4-10**

$$P = 6.5 \text{ kN} \quad L = 2.5 \text{ m} \quad A = 2(160 \text{ mm})(25 \text{ mm})$$

$$A = 8 \times 10^3 \text{ mm}^2 \quad b = 160 \text{ mm} \quad t = 25 \text{ mm}$$

LOCATION OF CENTROID C From Eq. (12-7b) in Chapter 12:

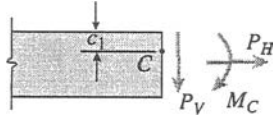
$$c_2 = \frac{\sum (y_i A_i)}{A} \quad c_2 = \frac{(160 \text{ mm})(25 \text{ mm}) \left(160 + \frac{25}{2} \right) \text{ mm} + (160 \text{ mm})(25 \text{ mm})(80 \text{ mm})}{A}$$

$$c_2 = 126.25 \text{ mm} \quad c_1 = 185 \text{ mm} - c_2 \quad c_1 = 58.75 \text{ mm}$$

MOMENT OF INERTIA

$$I_z = \frac{1}{3} t c_2^3 + \frac{1}{3} b c_1^3 - \frac{1}{3} (b - t) (c_1 - t)^3 \quad I_z = 2.585 \times 10^7 \text{ mm}^4$$

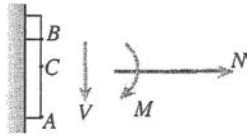
EQUIVALENT LOADS AT FREE END OF BEAM



$$P_H = -P \cos(60^\circ) \quad P_H = -3.25 \text{ kN} \quad P_v = P \sin(60^\circ)$$

$$P_v = 5.629 \text{ kN} \quad M_C = P_H c_1 \quad M_C = -190.938 \text{ N} \cdot \text{m}$$

STRESS RESULTANTS AT FIXED END OF BEAM



$$N_0 = P_H \quad N_0 = -3.25 \text{ kN}$$

$$V = P_v \quad V = 5.629 \text{ kN}$$

$$M = -M_C - P_v L \quad M = -1.388 \times 10^4 \text{ N} \cdot \text{m}$$

Stress at point A (bottom of web):

$$\sigma_x = \frac{N_0}{A} + \frac{M c_2}{I_z} \quad \sigma_x = -68.194 \text{ MPa} \quad \sigma_y = 0 \quad \tau_{xy} = 0$$

Uniaxial stress: $\sigma_1 = \sigma_y \quad \sigma_2 = \sigma_x \quad \tau_{\max} = \left| \frac{\sigma_x}{2} \right|$

$\sigma_1 = 0$	$\sigma_2 = -68.2 \text{ MPa}$	$\tau_{\max} = 34.1 \text{ MPa}$

Stress at point B (top of web):

$$\sigma_x = \frac{N_0}{A} - \frac{M(c_1 - t)}{I_z} \quad \sigma_x = 17.715 \text{ MPa} \quad \sigma_y = 0$$

$$Q = bt \left(c_1 - \frac{t}{2} \right) \quad Q = 1.85 \times 10^5 \text{ mm}^3$$

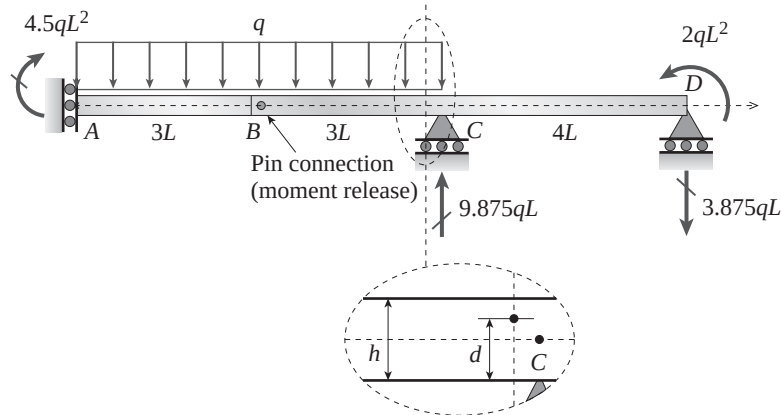
$$\tau_{xy} = -\frac{VQ}{I_z t} \quad \tau_{xy} = -1.611 \text{ MPa}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \quad \sigma_1 = 17.86 \text{ MPa} \quad \leftarrow$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \quad \sigma_2 = -0.145 \text{ MPa} \quad \leftarrow$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \quad \tau_{\max} = 9.00 \text{ MPa} \quad \leftarrow$$

Problem 8.4-11 Beam $ABCD$ has a sliding support at A , roller supports at C and D , and a pin connection at B (see figure). Assume that the beam has a rectangular cross section ($b = 4$ in., $h = 12$ in.). Uniform load q acts on ABC and a concentrated moment is applied at D . Let load variable $q = 1750$ lb/ft, and assume that dimension variable $L = 4$ ft. First, use statics to confirm the reaction moment at A and the reaction forces at C and D , as given in the figure. Then find the ratio of the magnitudes of the principal stresses (σ_1/σ_2) just left of support C at distance $d = 8$ in. up from the bottom.

**Solution 8.4-11**

$$b = 4 \text{ in.} \quad h = 12 \text{ in.} \quad d = 8 \text{ in.} \quad L = 4 \text{ ft} \quad q = 1750 \frac{\text{lb}}{\text{ft}} \quad I_z = \frac{bh^3}{12} = 576 \text{ in.}^4$$

SHEAR & MOMENT AT SECTION JUST LEFT OF C

$$V_C = -6qL = -42 \text{ kip} \quad M_C = -13.5qL^2 = -378 \text{ kip-ft}$$

STRESSES ON ELEMENT AT d ABOVE BOTTOM OF BEAM NEAR C

$$\sigma_y = 0 \quad \sigma_x = \frac{-M_C \left(d - \frac{h}{2} \right)}{I_z} = 15.75 \text{ ksi}$$

$$\tau_{xy} = \frac{V_C Q}{I_z b} \quad Q = (h - d)(b) \left[\frac{h}{2} - \left(\frac{h - d}{2} \right) \right] = 64 \text{ in.}^3 \quad < \text{Note that } V_C \text{ is neg., which means positive } \tau_{xy} \text{ on } +x \text{ face of element.}$$

$$\tau_{xy} = \frac{(-V_C)Q}{I_z b} = 1.167 \text{ ksi}$$

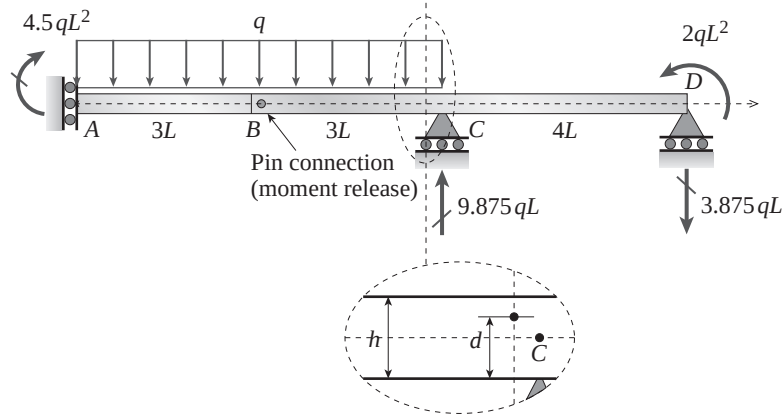
PRINCIPAL STRESSES AND RATIO & MAXIMUM SHEAR STRESS

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 15.836 \text{ ksi} \quad \tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 7.961 \text{ ksi}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = -0.086 \text{ ksi}$$

$$\frac{\sigma_1}{\sigma_2} = -184$$

Problem 8.4-12 Solve the preceding problem using the following numerical data: $b = 90$ mm, $h = 280$ mm, $d = 210$ mm, $q = 14$ kN/m and $L = 1.2$ m.



Solution 8.4-12

$$b = 90 \text{ mm} \quad h = 280 \text{ mm} \quad d = 210 \text{ mm} \quad L = 1.2 \text{ m} \quad q = 14 \frac{\text{kN}}{\text{m}} \quad I_z = \frac{bh^3}{12} = 1.646 \times 10^8 \text{ mm}^4$$

SHEAR & MOMENT AT SECTION JUST LEFT OF C

$$V_C = -6qL = -1.008 \times 10^5 \text{ N} \quad M_C = -13.5qL^2 = -2.722 \times 10^5 \text{ N}\cdot\text{m}$$

STRESSES ON ELEMENT AT d ABOVE BOTTOM OF BEAM NEAR C

$$\sigma_y = 0 \quad \sigma_x = \frac{-M_C \left(d - \frac{h}{2} \right)}{I_z} = 115.714 \text{ MPa}$$

$$\tau_{xy} = \frac{V_C Q}{I_z b} \quad Q = (h - d)(b) \left[\frac{h}{2} - \left(\frac{h - d}{2} \right) \right] = 6.615 \times 10^5 \text{ mm}^3 \quad \text{< Note that } V_C \text{ is neg, which means positive } \tau_{xy} \text{ on } +x \text{ face of element.}$$

$$\tau_{xy} = \frac{(-V_C) Q}{I_z b} = 4.5 \text{ MPa}$$

PRINCIPAL STRESSES AND RATIO & MAXIMUM SHEAR STRESS

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 115.889 \text{ MPa} \quad \tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 58.032 \text{ MPa}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = -0.175 \text{ MPa}$$

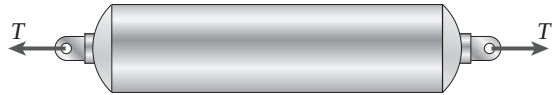
$$\frac{\sigma_1}{\sigma_2} = -663$$

Combined Loadings

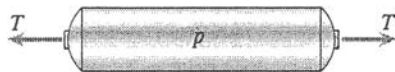
The problems for Section 8.5 are to be solved assuming that the structures behave linearly elastically and that the stresses caused by two or more loads may be superimposed to obtain the resultant stresses acting at a point. Consider both in-plane and out-of-plane shear stresses unless otherwise specified.

Problem 8.5-1 A cylindrical tank having diameter $d = 2.5$ in. is subjected to internal gas pressure $p = 600$ psi and an external tensile load $T = 1000$ lb (see figure).

Determine the minimum thickness t of the wall of the tank based upon an allowable shear stress of 3000 psi.



Solution 8.5-1 Cylindrical tank with tensile load



$$T = 1000 \text{ lb} \quad t = \text{thickness}$$

$$p = 600 \text{ psi}$$

$$d = 2.5 \text{ in.} \quad \tau_{\text{allow}} = 3000 \text{ psi}$$

CIRCUMFERENTIAL STRESS (TENSION)

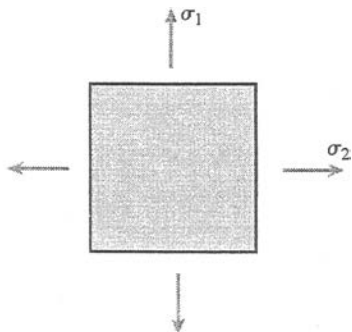
$$\sigma_1 = \frac{pr}{t} = \frac{(600 \text{ psi})(1.25 \text{ in.})}{t} = \frac{750}{t}$$

$$\text{Units: } \sigma_1 = \text{psi} \quad t = \text{inches} \quad \sigma_2 = \text{psi}$$

LONGITUDINAL STRESS (TENSION)

$$\begin{aligned} \sigma_2 &= \frac{pr}{2t} + \frac{T}{A} = \frac{pr}{2t} + \frac{T}{2\pi r t} \\ &= \frac{375}{t} + \frac{1000 \text{ lb}}{2\pi(1.25 \text{ in.})t} = \frac{375}{t} + \frac{127.32}{t} + \frac{502.32}{t} \end{aligned}$$

BIAXIAL STRESS



IN-PLANE SHEAR STRESS (CASE 1)

$$\tau_{\text{max}} = \frac{\sigma_1 - \sigma_2}{2} = \frac{247.68}{2t} = \frac{123.84}{t}$$

$$3000 \text{ psi} = \frac{123.84}{t}$$

$$\text{Solving, } t_1 = 0.0413 \text{ in.}$$

OUT-OF-PLANE SHEAR STRESSES

$$\text{Case 2: } \tau_{\text{max}} = \frac{\sigma_1}{2} = \frac{375}{t}; 3000 = \frac{375}{t}$$

$$\text{Solving, } t_2 = 0.125 \text{ in.}$$

$$\text{Case 3: } \tau_{\text{max}} = \frac{\sigma_2}{2} = \frac{251.16}{t}; 3000 = \frac{251.16}{t}$$

$$\text{Solving, } t_3 = 0.0837 \text{ in.}$$

CASE 2, OUT-OF-PLANE SHEAR STRESS GOVERNS

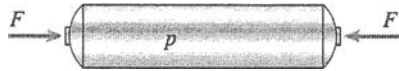
$$t_{\text{min}} = 0.125 \text{ in.} \quad \leftarrow$$

Problem 8.5-2 A cylindrical tank subjected to internal pressure p is simultaneously compressed by an axial force $F = 72$ kN (see figure). The cylinder has diameter $d = 100$ mm and wall thickness $t = 4$ mm.

Calculate the maximum allowable internal pressure $p_{\max}$ based upon an allowable shear stress in the wall of the tank of 60 MPa.



Solution 8.5-2 Cylindrical tank with compressive force



$$F = 72 \text{ kN}$$

p = internal pressure

$$d = 100 \text{ mm} \quad t = 4 \text{ mm} \quad \tau_{\text{allow}} = 60 \text{ MPa}$$

CIRCUMFERENTIAL STRESS (TENSION)

$$\sigma_1 = \frac{pr}{t} = \frac{p(50 \text{ mm})}{4 \text{ mm}} = 12.5 p$$

$$\text{Units: } \sigma_1 = \text{MPa} \quad p = \text{MPa}$$

LONGITUDINAL STRESS (TENSION)

$$\begin{aligned} \sigma_2 &= \frac{pr}{2t} - \frac{F}{A} = \frac{pr}{2t} - \frac{F}{2\pi r t} \\ &= 6.25p - \frac{72,000 \text{ N}}{2\pi(50 \text{ mm})(4 \text{ mm})} \\ &= 6.25p - 57.296 \text{ MPa} \end{aligned}$$

$$\text{Units: } \sigma_2 = \text{MPa} \quad p = \text{MPa}$$

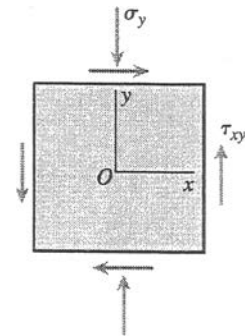
BIAXIAL STRESS

IN-PLANE SHEAR STRESS (CASE 1)

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = 3.125 p + 28.648 \text{ MPa}$$

$$60 \text{ MPa} = 3.125 p + 28.648 \text{ MPa}$$

$$\text{Solving, } p_1 = 10.03 \text{ MPa}$$



OUT-OF-PLANE SHEAR STRESSES

$$\text{Case 2: } \tau_{\max} = \frac{\sigma_1}{2} = 6.25p; \quad 60 \text{ MPa} = 6.25p$$

$$\text{Solving, } p_2 = 9.60 \text{ MPa}$$

$$\text{Case 3: } \tau_{\max} = \frac{\sigma_2}{2} = 3.125p - 28.648 \text{ MPa}$$

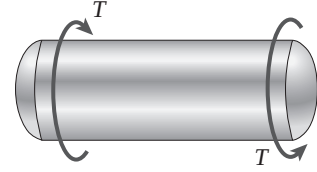
$$60 \text{ MPa} = 3.125p - 28.648 \text{ MPa}$$

$$\text{Solving, } p_3 = 28.37 \text{ MPa}$$

CASE 2, OUT-OF-PLANE SHEAR STRESS GOVERNS

$$p_{\max} = 9.60 \text{ MPa} \quad \leftarrow$$

Problem 8.5-3 A cylindrical pressure vessel having radius $r = 14$ in. and wall thickness $t = 0.5$ in. is subjected to internal pressure $p = 375$ psi. In addition, a torque $T = 90$ k-ft acts at each end of the cylinder (see figure).



- Determine the maximum tensile stress $\sigma_{\max}$ and the maximum in-plane shear stress $\tau_{\max}$ in the wall of the cylinder.
- If the allowable in-plane shear stress is 4.5 ksi, what is the maximum allowable torque T ?
- If $T = 150$ k-ft and allowable in-plane shear and allowable normal stresses are 4.5 ksi and 11.5 ksi, respectively, what is the minimum required wall thickness?

Solution 8.5-3 Cylindrical pressure vessel

$$(a) \quad r = 14 \text{ in.} \quad t = 0.5 \text{ in.} \quad p = 375 \text{ psi} \quad T = 90 \text{ kip-ft} \quad I_p = \frac{\pi[r^4 - (r - t)^4]}{2} = 0.394 \text{ ft}^4$$

$$\sigma_x = \frac{pr}{2t} = 5250 \text{ psi} \quad \sigma_y = 2\sigma_x = 10,500 \text{ psi} \quad \tau_{xy} = \frac{-Tr}{I_p} = -1851 \text{ psi}$$

PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESS

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 11.09 \text{ ksi}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 3.21 \text{ ksi}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 4.66 \text{ ksi}$$

$$(b) \quad \tau_a = 4.5 \text{ ksi} \quad p = 375 \text{ psi} \quad t = 0.5 \text{ in.}$$

$$\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \tau_a \quad \text{or} \quad \sqrt{\left(\frac{\frac{pr}{2t} - \frac{pr}{t}}{2}\right)^2 + \left(\frac{T_{\max} r}{I_p}\right)^2} = \tau_a$$

$$\text{Solve numerically for } T_{\max}: \quad T_{\max} = 178 \text{ kip-ft}$$

$$(c) \quad T = 150 \text{ kip-ft} \quad \sigma_a = 11.5 \text{ ksi} \quad \tau_a = 4.5 \text{ ksi} \quad t = 0.5 \text{ in.} \quad p = 375 \text{ psi}$$

$$\frac{pr}{2t} + \frac{pr}{t} + \sqrt{\left(\frac{\frac{pr}{2t} - \frac{pr}{t}}{2}\right)^2 + \left[\frac{-Tr}{\frac{\pi[r^4 - (r - t)^4]}{2}}\right]^2} = \sigma_a \quad \text{Solve numerically for } t_{\min}: \quad t_{\min} = 0.519 \text{ in.}$$

^controls

$$\text{OR} \quad \sqrt{\left(\frac{\frac{pr}{2t} - \frac{pr}{t}}{2}\right)^2 + \left[\frac{-Tr}{\frac{\pi[r^4 - (r - t)^4]}{2}}\right]^2} = \tau_a \quad \text{Solve numerically for } t_{\min}: \quad t_{\min} = 0.449 \text{ in.}$$

Problem 8.5-4 A pressurized cylindrical tank with flat ends is loaded by torques T and tensile forces P (see figure). The tank has inner radius of $r = 125$ mm and wall thickness $t = 6.5$ mm. The internal pressure $p = 7.25$ MPa and the torque $T = 850$ N·m.



- What is the maximum permissible value of the forces P if the allowable tensile stress in the wall of the cylinder is 80 MPa?
- If forces $P = 114$ kN, what is the maximum acceptable internal pressure in the tank?

Solution 8.5-4 Cylindrical tank

- (a) PLANE STRESS STATE ON SURFACE OF TANK $A = 2\pi rt$ $I_p = 2\pi r^3 t$

$$\sigma_x = \frac{pr}{2t} + \frac{P}{A} \rightarrow \frac{pr}{2t} + \frac{P}{2\pi rt} \quad \sigma_y = \frac{pr}{t} \quad \tau_{xy} = \frac{-Tr}{I_p} = \frac{T}{2\pi r^2 t}$$

MAXIMUM PRINCIPAL STRESS; EQUATE TO ALLOWABLE STRESS AND SOLVE FOR $P_{\max}$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{pr}{4t} - \frac{P}{4\pi rt}\right)^2 + \frac{T^2}{4\pi^2 r^4 t^2}} + \frac{3pr}{4t} + \frac{P}{4\pi rt}$$

$$\sqrt{\left(\frac{pr}{4t} - \frac{P}{4\pi rt}\right)^2 + \frac{T^2}{4\pi^2 r^4 t^2}} + \frac{3pr}{4t} + \frac{P}{4\pi rt} = \sigma_a \quad \text{solve, } P = \frac{T^2 - 2\pi^2 p^2 r^6 + 6\pi^2 pr^5 t \sigma_a - 4\pi^2 r^4 t^2 \sigma_a^2}{2\pi pr^4 - 2\pi r^3 t \sigma_a}$$

EVALUATE USING GIVEN NUMERICAL DATA $r = 125$ mm $t = 6.5$ mm $p = 7.25$ MPa $T = 850$ Nm $\sigma_a = 80$ MPa

$$P_{\max} = \frac{T^2 - 2\pi^2 p^2 r^6 + 6\pi^2 pr^5 t \sigma_a - 4\pi^2 r^4 t^2 \sigma_a^2}{2\pi pr^4 - 2\pi r^3 t \sigma_a} = 52.7 \text{ kN}$$

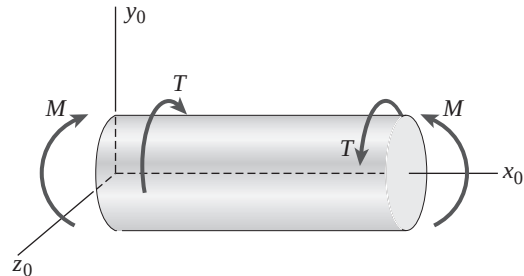
- (b) $P = 114$ kN $t = 6.5$ mm $r = 125$ mm $\sigma_a = 80$ MPa $T = 850$ N·m

$$\sqrt{\left(\frac{pr}{4t} - \frac{P}{4\pi rt}\right)^2 + \frac{T^2}{4\pi^2 r^4 t^2}} + \frac{3pr}{4t} + \frac{P}{4\pi rt} = \sigma_a \quad \text{solve numerically for max pressure:}$$

$$P_{\max} = 6.00 \text{ MPa}$$

Problem 8.5-5 A cylindrical pressure vessel with flat ends is subjected to a torque T and a bending moment M (see figure). The outer radius is 12.0 in. and the wall thickness is 1.0 in. The loads are as follows: $T = 800$ k-in., $M = 1000$ k-in., and the internal pressure $p = 900$ psi.

Determine the maximum tensile stress σ_t , maximum compressive stress σ_c , and maximum shear stress $\tau_{\max}$ in the wall of the cylinder.



Solution 8.5-5 Cylindrical pressure vessel

Internal pressure: $p = 900$ psi
 Bending moment: $M = 1000$ k-in.
 Torque: $T = 800$ k-in.
 Outer radius: $r_2 = 12$ in.
 Wall thickness: $t = 1.0$ in.
 Mean radius: $r = r_2 - t/2 = 11.5$ in.
 Outer diameter: $d_2 = 24$ in.
 Inner diameter: $d_1 = 22$ in.

MOMENT OF INERTIA

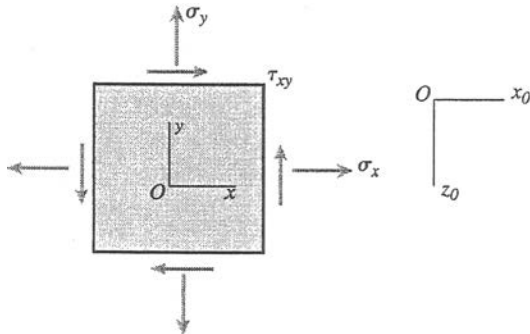
$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 4787.0 \text{ in.}^4$$

$$I_p = 2I = 9574.0 \text{ in.}^4$$

NOTE: Since the stresses due to T and p are the same everywhere in the cylinder, the maximum stresses occur at the top and bottom of the cylinder where the bending stresses are the largest.

PART (a). TOP OF THE CYLINDER

Stress element on the top of the cylinder as seen from above.



$$\begin{aligned}\sigma_x &= \frac{pr}{2t} - \frac{Mr_2}{I} = 5175.0 \text{ psi} - 2506.8 \text{ psi} \\ &= 2668.2 \text{ psi}\end{aligned}$$

$$\sigma_y = \frac{pr}{t} = 10,350 \text{ psi}$$

$$\tau_{xy} = \frac{Tr_2}{I_p} = -1002.7 \text{ psi}$$

PRINCIPAL STRESSES

$$\begin{aligned}\sigma_{1,2} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ &= 6509.1 \text{ psi} \pm 3969.6 \text{ psi} \\ \sigma_1 &= 10,479 \text{ psi} \quad \sigma_2 = 2540 \text{ psi}\end{aligned}$$

MAXIMUM SHEAR STRESSES

In-plane: $\tau = 3970$ psi

Out-of-plane:

$$\begin{aligned}\tau &= \frac{\sigma_1}{2} \quad \text{or} \quad \frac{\sigma_2}{2} \quad \tau = \frac{\sigma_1}{2} = 5240 \text{ psi} \\ \therefore \tau_{\max} &= 5240 \text{ psi}\end{aligned}$$

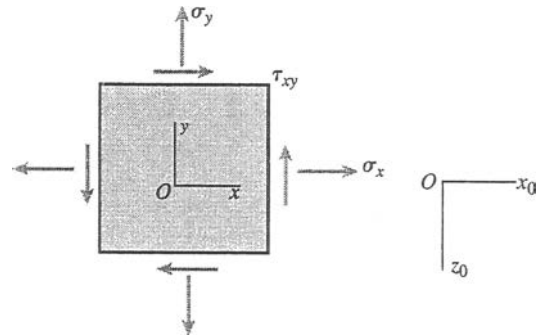
MAXIMUM STRESSES FOR THE TOP OF THE CYLINDER

$\sigma_t = 10,480$ psi $\sigma_c = 0$ (No compressive stresses)

$\tau_{\max} = 5240$ psi

PART (b). BOTTOM OF THE CYLINDER

Stress element on the bottom of the cylinder as seen from below.



$$\begin{aligned}\sigma_x &= \frac{pr}{2t} + \frac{Mr_2}{I} = 5175.0 \text{ psi} + 2506.8 \text{ psi} \\ &= 7681.8 \text{ psi}\end{aligned}$$

$$\sigma_y = \frac{pr}{t} = 10,350 \text{ psi}$$

$$\tau_{xy} = -\frac{Tr_2}{I_p} = -1002.7 \text{ psi}$$

PRINCIPAL STRESSES

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 9015.9 \text{ psi} \pm 1668.9 \text{ psi}$$

$$\sigma_1 = 10,685 \text{ psi} \quad \sigma_2 = 7347 \text{ psi}$$

MAXIMUM SHEAR STRESSES

In-plane: $\tau = 1669 \text{ psi}$

Out-of-plane:

$$\tau = \frac{\sigma_1}{2} \quad \text{or} \quad \frac{\sigma_2}{2} \quad \tau = \frac{\sigma_1}{2} = 5340 \text{ psi}$$

$$\therefore \tau_{\max} = 5340 \text{ psi}$$

MAXIMUM STRESSES FOR THE BOTTOM OF THE CYLINDER

$$\sigma_t = 10,680 \text{ psi} \quad \sigma_c = 0 \text{ (No compressive stresses)}$$

$$\tau_{\max} = 5340 \text{ psi}$$

PART (c). ENTIRE CYLINDER

The largest stresses are at the bottom of the cylinder.

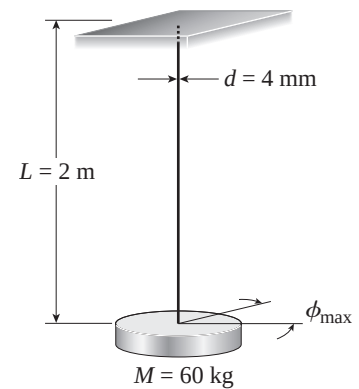
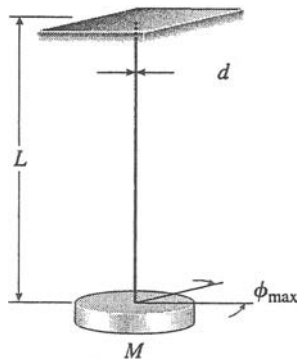
$$\sigma_t = 10,680 \text{ psi} \quad \leftarrow$$

$$\sigma_c = 0 \text{ (No compressive stresses)} \quad \leftarrow$$

$$\tau_{\max} = 5340 \text{ psi} \quad \leftarrow$$

Problem 8.5-6 The torsional pendulum shown in the figure consists of a horizontal circular disk of mass $M = 60 \text{ kg}$ suspended by a vertical steel wire ($G = 80 \text{ GPa}$) of length $L = 2 \text{ m}$ and diameter $d = 4 \text{ mm}$.

Calculate the maximum permissible angle of rotation $\phi_{\max}$ of the disk (that is, the maximum amplitude of torsional vibrations) so that the stresses in the wire do not exceed 100 MPa in tension or 50 MPa in shear.

**Solution 8.5-6 Torsional pendulum**

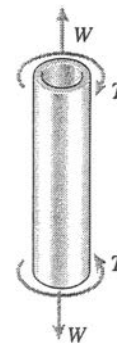
$$L = 2.0 \text{ m} \quad d = 4.0 \text{ mm}$$

$$M = 60 \text{ kg} \quad G = 80 \text{ GPa}$$

$$\sigma_{\text{allow}} = 100 \text{ MPa} \quad \tau_{\text{allow}} = 50 \text{ MPa}$$

$$A = \frac{\pi d^2}{4} = 12.5664 \text{ mm}^2$$

$$W = Mg = (60 \text{ kg})(9.81 \text{ m/s}^2) = 588.6 \text{ N}$$



$$\text{TORQUE: } T = \frac{GI_p \phi_{\max}}{L} \quad (\text{EQ. 3-17})$$

$$\text{SHEAR STRESS: } \tau = \frac{Tr}{I_p} \quad (\text{EQ. 3-13})$$

$$\tau = \left(\frac{GI_p \phi_{\max}}{L} \right) \left(\frac{r}{I_p} \right) = \frac{Gr \phi_{\max}}{L} = (80 \times 10^6 \text{ Pa}) \phi_{\max}$$

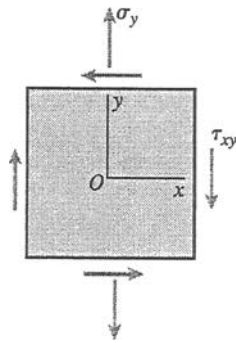
$$\tau = 80 \phi_{\max} \quad \text{Units: } \tau = \text{MPa} \quad \phi_{\max} = \text{radians}$$

$$\text{TENSILE STRESS} \quad \sigma_x = \frac{W}{A} = 46.839 \text{ MPa}$$

$$\sigma_x = 0$$

$$\sigma_y = \sigma_t = 46.839 \text{ MPa}$$

$$\tau_{xy} = -80 \phi_{\max} \text{ (MPa)}$$



PRINCIPAL STRESSES

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

$$\sigma_{1,2} = 23.420 \pm \sqrt{(23.420)^2 + 6400 \phi_{\max}^2} \quad (\text{MPa})$$

Note that σ_1 is positive and σ_2 is negative. Therefore, the maximum in-plane shear stress is greater than the maximum out-of-plane shear stress.

MAXIMUM ANGLE OF ROTATION BASED ON TENSILE STRESS

$$\sigma_1 = \text{maximum tensile stress} \quad \sigma_{\text{allow}} = 100 \text{ MPa}$$

$$\therefore 100 \text{ MPa} = 23.420 \pm \sqrt{(23.420)^2 + 6400 \phi_{\max}^2}$$

$$(100 - 23.420)^2 = (23.420)^2 + 6400 \phi_{\max}^2$$

$$5316 = 6400 \phi_{\max}^2 \quad \phi_{\max} = 0.9114 \text{ rad} = 52.2^\circ$$

MAXIMUM ANGLE OF ROTATION BASED ON IN-PLANE SHEAR STRESS

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = \sqrt{(23.420)^2 + 6400 \phi_{\max}^2}$$

$$\tau_{\text{allow}} = 50 \text{ MPa} \quad 50 = \sqrt{(23.420)^2 + 6400 \phi_{\max}^2}$$

$$(50)^2 = (23.420)^2 + 6400 \phi_{\max}^2$$

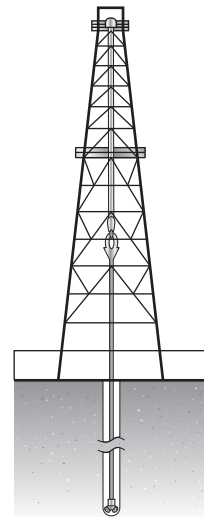
$$\text{Solving, } \phi_{\max} = 0.5522 \text{ rad} = 31.6^\circ$$

SHEAR STRESS GOVERNS

$$\phi_{\max} = 0.552 \text{ rad} = 31.6^\circ \quad \leftarrow$$

Problem 8.5-7 The hollow drill pipe for an oil well (see figure) is 6.2 in. in outer diameter and 0.75 in. in thickness. Just above the bit, the compressive force in the pipe (due to the weight of the pipe) is 62 k and the torque (due to drilling) is 185 k-in.

Determine the maximum tensile, compressive, and shear stresses in the drill pipe.

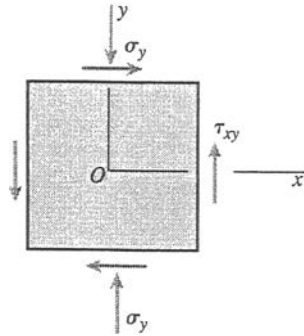


Solution 8.5-7 P = compressive force T = torque d_2 = outer diameter d_1 = inner diameter $P = 62 \text{ k}$ $T = 185 \text{ k} \cdot \text{in.}$ $d_2 = 6.2 \text{ in.}$ $t = 0.75 \text{ in.}$ $d_1 = d_2 - 2t$ $d_1 = 4.7 \text{ in.}$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) \quad A = 12.841 \text{ in.}^2$$

$$I_p = \frac{\pi}{32}(d_2^4 - d_1^4) \quad I_p = 97.16 \text{ in.}^4$$

STRESSES AT THE OUTER SURFACE



$$\sigma_y = -\frac{P}{A} \quad \sigma_y = -4828 \text{ psi}$$

$$\sigma_x = 0$$

$$\tau_{xy} = \frac{T\left(\frac{d_2}{2}\right)}{I_p} \quad \tau_{xy} = 5903 \text{ psi}$$

PRINCIPAL STRESSES

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = 3963 \text{ psi}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = -8791 \text{ psi}$$

MAXIMUM TENSILE STRESS $\sigma_t = \sigma_1$

$$\sigma_t = 3963 \text{ psi} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS $\sigma_c = \sigma_2$

$$\sigma_c = -8791 \text{ psi} \quad \leftarrow$$

MAXIMUM IN-PLANE SHEAR STRESS

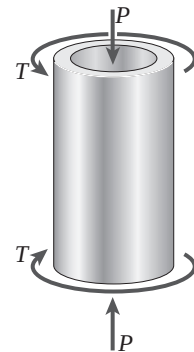
$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = 6377 \text{ psi} \quad \leftarrow$$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-8 A segment of a generator shaft is subjected to a torque T and an axial force P , as shown in the figure. The shaft is hollow (outer diameter $d_2 = 300 \text{ mm}$ and inner diameter $d_1 = 250 \text{ mm}$) and delivers 1800 kW at 4.0 Hz.

If the compressive force $P = 540 \text{ kN}$, what are the maximum tensile, compressive, and shear stresses in the shaft?

**Probs. 8.5-8 and 8.5-9**

Solution 8.5-8 P = compressive force P_0 = power f = frequency

$$T = \text{torque} = \frac{P_0}{2\pi f}$$

 d_2 = outer diameter d_1 = inner diameter f = 4.0 Hz P = 540 kN P_0 = 1800 kW

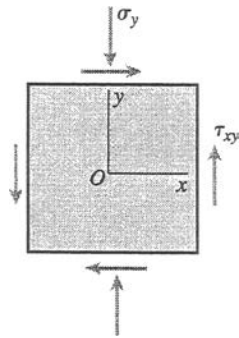
$$T = \frac{P_0}{2\pi f} \quad T = 7.162 \times 10^4 \text{ N} \cdot \text{m}$$

 d_2 = 300 mm d_1 = 250 mm

$$A = \frac{\pi}{4} (d_2^2 - d_1^2) \quad I_p = \frac{\pi}{32} (d_2^4 - d_1^4)$$

$$A = 2.16 \times 10^4 \text{ mm}^2 \quad I_p = 4.117 \times 10^8 \text{ mm}^4$$

STRESSES AT THE OUTER SURFACE



$$\sigma_y = -\frac{P}{A} \quad \sigma_y = -25.002 \text{ MPa}$$

$$\sigma_x = 0$$

$$\tau_{xy} = \frac{T \left(\frac{d_2}{2} \right)}{I_p} \quad \tau_{xy} = 26.093 \text{ MPa}$$

PRINCIPAL STRESSES

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = 16.432 \text{ MPa}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = -41.434 \text{ MPa}$$

MAXIMUM TENSILE STRESS $\sigma_t = \sigma_1$

$$\sigma_t = 16.43 \text{ MPa} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS $\sigma_c = \sigma_2$

$$\sigma_c = -41.4 \text{ MPa} \quad \leftarrow$$

MAXIMUM IN-PLANE SHEAR STRESS

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = 28.9 \text{ MPa} \quad \leftarrow$$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-9 A segment of a generator shaft of hollow circular cross section is subjected to a torque $T = 240$ k-in. (see figure). The outer and inner diameters of the shaft are 8.0 in. and 6.25 in., respectively.

What is the maximum permissible compressive load P that can be applied to the shaft if the allowable in-plane shear stress is $\tau_{\text{allow}} = 6250$ psi?

Solution 8.5-9

P = compressive force T = torque

d_2 = outer diameter d_1 = inner diameter

$T = 240 \text{ k} \cdot \text{in.}$

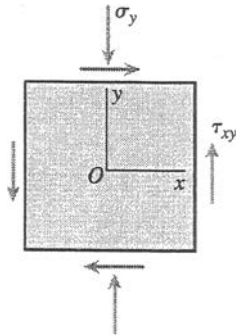
$d_2 = 8 \text{ in.}$ $d_1 = 6.25 \text{ in.}$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) \quad A = 19.586 \text{ in.}^2$$

$$\tau_{\text{allow}} = 6250 \text{ psi} \quad I_p = \frac{\pi}{32}(d_2^4 - d_1^4)$$

$$I_p = 252.321 \text{ in.}^4$$

STRESSES AT THE OUTER SURFACE



$$\sigma_y = -\frac{P}{A}$$

$$\sigma_x = 0$$

$$\tau_{xy} = \frac{T \left(\frac{d_2}{2} \right)}{I_p} \quad \tau_{xy} = 3805 \text{ psi}$$

MAXIMUM IN-PLANE SHEAR STRESS $\tau_{\text{max}} = \tau_{\text{allow}}$

$$\tau_{\text{max}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

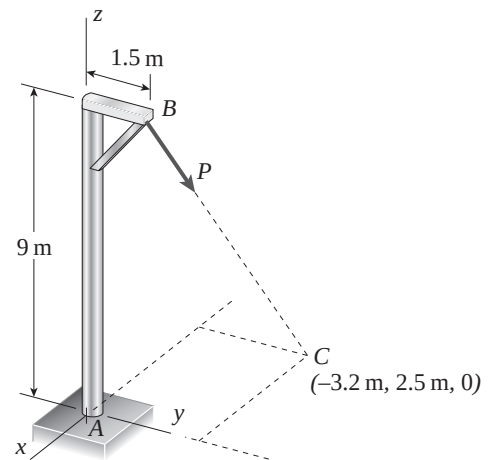
$$\sigma_y = \sqrt{(\tau_{\text{max}}^2 - \tau_{xy}^2) \cdot 4} \quad \sigma_y = 9917 \text{ psi}$$

$$P = \sigma_y A \quad P = 194.2 \text{ k} \quad \leftarrow$$

NOTE: The maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-10 A post having a hollow circular cross section supports a load $P = 3.2 \text{ kN}$ acting at the end of an arm that is $b = 1.5 \text{ m}$ long (see figure). The height of the post is $L = 9 \text{ m}$, and its section modulus is $S = 2.65 \times 10^5 \text{ mm}^3$. Assume that outer radius of the post, $r_2 = 123 \text{ mm}$, and inner radius $r_1 = 117 \text{ mm}$.

- Calculate the maximum tensile stress σ_{max} and maximum in-plane shear stress τ_{max} at point A on the outer surface of the post along the x axis due to the load P . Load P acts at B along line BC.
- If the maximum tensile stress and maximum inplane shear stress at point A are limited to 90 MPa and 38 MPa , respectively, what is the largest permissible value of the load P ?



Solution 8.5-10

$$(a) \ r_{BC} = \begin{pmatrix} -3.2 \text{ m} \\ 2.5 \text{ m} - 1.5 \text{ m} \\ -9 \text{ m} \end{pmatrix} \quad e_{BC} = \frac{r_{BC}}{|r_{BC}|} = \begin{pmatrix} -0.333 \\ 0.104 \\ -0.937 \end{pmatrix}$$

$$P = 3.2 \text{ kN} \quad P_B = P e_{BC} = \begin{pmatrix} -1.066 \\ 0.333 \\ -2.999 \end{pmatrix} \text{ kN}$$

$$r_{AC} = \begin{pmatrix} -3.2 \text{ m} \\ 2.5 \text{ m} \\ 0 \end{pmatrix}$$

RESULTANT FORCES AND MOMENT AT BASE OF POLE

$$R_A = p_B = \begin{pmatrix} -1.0662 \\ 0.33319 \\ -2.9987 \end{pmatrix} \text{ kN}$$

$$M_A = r_{AC} \times p_B = \begin{pmatrix} -7.49675 \\ -9.59584 \\ 1.59931 \end{pmatrix} \text{ kN}\cdot\text{m}$$

SECTION PROPERTIES FOR POLE

$$r_2 = 123 \text{ mm} \quad r_1 = 117 \text{ mm} \quad A = \pi(r_2^2 - r_1^2) = 4.524 \times 10^3 \text{ mm}^2 \quad \alpha = \frac{r_2^2 + r_2 r_1 + r_1^2}{r_2^2 + r_1^2} = 1.499$$

$$I = \frac{\pi}{4}(r_2^4 - r_1^4) = 3.259 \times 10^7 \text{ mm}^4 \quad S = \frac{I}{r_2} = 2.65 \times 10^5 \text{ mm}^3$$

PLANE STRESS STATE AT A

$$\sigma_x = 0 \quad \sigma_y = \frac{R_{A3}}{A} - \frac{M_{A2} r_2}{I} = 35.551 \text{ MPa} \quad \tau_{xy} = \frac{M_{A3} r_2}{2I} + \frac{R_{A2}}{A} \left(\frac{4}{3} \alpha \right) = 3.165 \text{ MPa}$$

PRINCIPAL NORMAL STRESSES AND MAXIMUM IN-PLANE SHEAR STRESS AT A

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 35.83 \text{ MPa} \quad \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = -0.28 \text{ MPa}$$

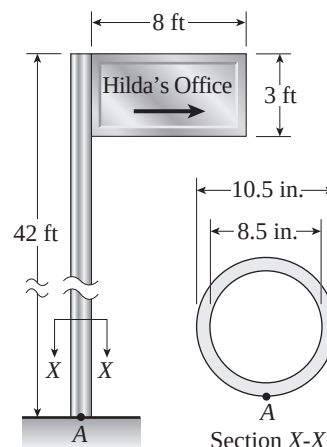
$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 18.05 \text{ MPa}$$

$$(b) \text{ STRESSES AT A ARE PROPORTIONAL TO LOAD } P \quad \sigma_a = 90 \text{ MPa} \quad \tau_a = 38 \text{ MPa}$$

$$P_{\max 1} = \frac{\sigma_a}{\sigma_1} P = 8.04 \text{ kN} \quad P_{\max 2} = \frac{\tau_a}{\tau_{\max}} P = 6.73 \text{ kN} \quad \text{Lower value controls.}$$

Problem 8.5-11 A sign is supported by a pole of hollow circular cross section, as shown in the figure. The outer and inner diameters of the pole are 10.5 in. and 8.5 in., respectively. The pole is 42 ft high and weighs 4.0 k. The sign has dimensions 8 ft $\times$ 3 ft and weighs 500 lb. Note that its center of gravity is 53.25 in. from the axis of the pole. The wind pressure against the sign is 35 lb/ft².

- Determine the stresses acting on a stress element at point A, which is on the outer surface of the pole at the “front” of the pole, that is, the part of the pole nearest to the viewer.
- Determine the maximum tensile, compressive, and shear stresses at point A.



Solution 8.5-11

PIPE: $d_2 = 10.5 \text{ in.}$ $d_1 = 8.5 \text{ in.}$ $c = \frac{d_2}{2}$

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) \quad A = 29.845 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) \quad I = 340.421 \text{ in.}^4$$

$$I_p = 2I \quad I_p = 680.842 \text{ in.}^4$$

$$Q = \frac{1}{12}(d_2^3 - d_1^3) \quad Q = 45.292 \text{ in.}^3$$

$$W_1 = 4000 \text{ lb}$$

SIGN: $A_s = (8)(3) \text{ ft}^2$ Area $A_s = 24 \text{ ft}^2$

Height from the base to the center of gravity of the sign

$$h = \left(42 - \frac{3}{2}\right) \text{ ft}$$

Horizontal distance from the center of gravity of the sign to the axis of the pipe

$$b = \left((4)12 + \frac{10.5}{2}\right) \text{ in.}$$

$$b = 53.25 \text{ in.}$$

$$W_2 = 500 \text{ lb}$$

WIND PRESSURE $p_w = 35 \text{ lb/ft}^2$

$$P = p_w A_s \quad P = 840 \text{ lb}$$

STRESS RESULTANTS AT THE BASE

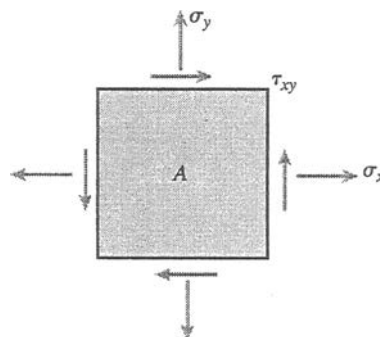
$$M = Ph \quad M = 4.082 \times 10^5 \text{ lb} \cdot \text{in.}$$

$$T = Pb \quad T = 4.473 \times 10^4 \text{ lb} \cdot \text{in.}$$

$$V = P \quad V = 840 \text{ lb}$$

$$N_z = W_1 + W_2 \quad N_z = 4.5 \times 10^3 \text{ lb}$$

(a) STRESSES AT POINT A



$$\sigma_x = 0 \quad \leftarrow$$

$$\sigma_y = -\frac{N_z}{A} + \frac{Md_2}{2I} \quad \sigma_y = 6145 \text{ psi} \quad \leftarrow$$

$$\sigma_{xy} = \frac{Td_2}{2I_p} \quad \tau_{xy} = 345 \text{ psi} \quad \leftarrow$$

(b) MAXIMUM STRESSES AT POINT A

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\text{Max. tensile stress: } \sigma_1 = 6164 \text{ psi} \quad \leftarrow$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\text{Max. compressive stress: } \sigma_2 = -19.30 \text{ psi} \quad \leftarrow$$

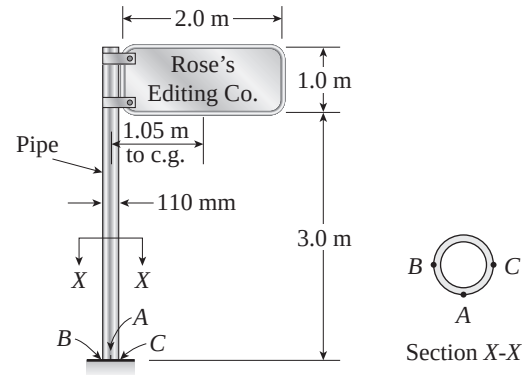
$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\text{Max. shear stress: } \tau_{\max} = 3092 \text{ psi} \quad \leftarrow$$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-12 A sign is supported by a pipe (see figure) having outer diameter 110 mm and inner diameter 90 mm. The dimensions of the sign are $2.0 \text{ m} \times 1.0 \text{ m}$, and its lower edge is 3.0 m above the base. Note that the center of gravity of the sign is 1.05 m from the axis of the pipe. The wind pressure against the sign is 1.5 kPa.

Determine the maximum in-plane shear stresses due to the wind pressure on the sign at points A, B, and C, located on the outer surface at the base of the pipe.

**Solution 8.5-12**

PIPE: $d_2 = 110 \text{ mm}$ $d_1 = 90 \text{ mm}$ $t = 10 \text{ mm}$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) \quad I = 3.966 \times 10^6 \text{ mm}^4$$

$$I_p = 2I \quad I_p = 7.933 \times 10^6 \text{ mm}^4$$

$$Q = \frac{1}{12} (d_2^3 - d_1^3)$$

$$Q = 5.017 \times 10^4 \text{ mm}^3$$

SIGN: $A = 2 \text{ m}^2$ Area

$$h = \left(3 + \frac{1}{2} \right) \text{ m} \quad \text{Height from the base to the center of gravity of the sign}$$

$$b = 1.05 \text{ m} \quad \text{horizontal distance from the center of gravity of the sign to the axis of the pipe}$$

WIND PRESSURE $p_w = 1.5 \text{ kPa}$

$$P = p_w A \quad P = 3 \text{ kN}$$

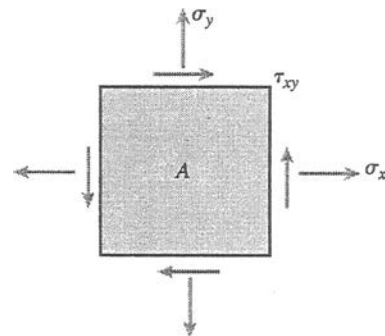
STRESS RESULTANTS AT THE BASE

$$M = Ph \quad M = 10.5 \text{ kN} \cdot \text{m}$$

$$T = Pb \quad T = 3.15 \text{ kN} \cdot \text{m}$$

$$V = P \quad V = 3 \text{ kN}$$

MAXIMUM SHEAR STRESS AT POINT A



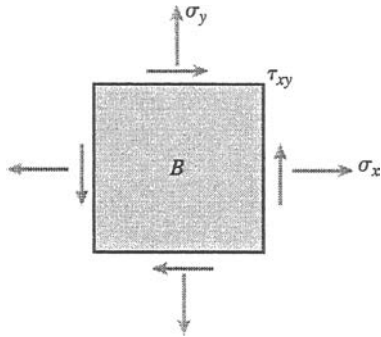
$$\sigma_x = 0$$

$$\sigma_y = \frac{Md_2}{2I} \quad \sigma_y = 145.603 \text{ MPa}$$

$$\tau_{xy} = \frac{Td_2}{2I_p} \quad \tau_{xy} = 21.84 \text{ MPa}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = 76.0 \text{ MPa} \quad \leftarrow$$

MAXIMUM SHEAR STRESS AT POINT *B*

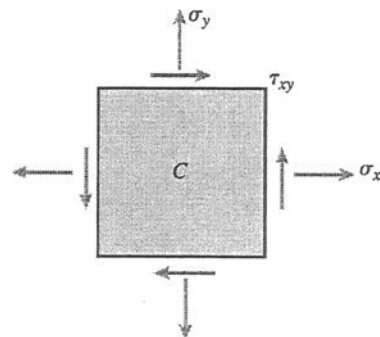
$$\sigma_x = 0$$

$$\sigma_y = 0$$

$$\tau_{xy} = \frac{Td_2}{2I_p} - \frac{VQ}{I(2t)} \quad \tau_{xy} = 19.943 \text{ MPa}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\text{Pure shear:} \quad \tau_{\max} = 19.94 \text{ MPa} \quad \leftarrow$$

MAXIMUM SHEAR STRESS AT POINT *C*

$$\sigma_x = 0$$

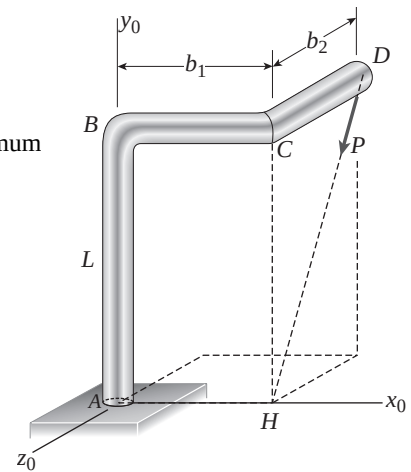
$$\sigma_y = 0$$

$$\tau_{xy} = \frac{Td_2}{2I_p} + \frac{VQ}{I(2t)} \quad \tau_{xy} = 23.738 \text{ MPa}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\text{Pure shear:} \quad \tau_{\max} = 23.7 \text{ MPa} \quad \leftarrow$$

Problem 8.5-13 A bracket *ABCD* having a hollow circular cross section consists of a vertical arm *AB* ($L = 6$ ft), a horizontal arm *BC* parallel to the x_0 axis, and a horizontal arm *CD* parallel to the z_0 axis (see figure). The arms *BC* and *CD* have lengths $b_1 = 3.6$ ft and $b_2 = 2.2$ ft, respectively. The outer and inner diameters of the bracket are $d_2 = 7.5$ in. and $d_1 = 6.8$ in. An inclined load $P = 2200$ lb acts at point *D* along line *DH*. Determine the maximum tensile, compressive, and shear stresses in the vertical arm.



Solution 8.5-13FORCE COMPONENTS IN y AND z DIRECTIONS

$$P_y = -P \frac{L}{\sqrt{b_2^2 + L^2}} \quad P_z = P \frac{b_2}{\sqrt{b_2^2 + L^2}}$$

$$P_D = \begin{pmatrix} 0 \\ P_y \\ P_z \end{pmatrix} \quad r_{AD} = \begin{pmatrix} b_1 \\ L \\ -b_2 \end{pmatrix}$$

FORCE AND MOMENT STRESS RESULTANTS (NOT REACTIONS) AT A

$$R_A = P_D \rightarrow \begin{pmatrix} 0 \\ \frac{LP}{\sqrt{L^2 + b_2^2}} \\ \frac{Pb_2}{\sqrt{L^2 + b_2^2}} \end{pmatrix} \quad M_A = r_{AD} \times P_D = \begin{pmatrix} 0 \\ \frac{Pb_1b_2}{\sqrt{L^2 + b_2^2}} \\ \frac{LPb_1}{\sqrt{L^2 + b_2^2}} \end{pmatrix}$$

STRESSES ON ELEMENT 1 AT COORD. $(0, 0, d_2/2)$ - see Example 8-6 for additional guidance.

$$\sigma_x = 0 \quad \sigma_y = \frac{R_{A_2}}{A} + \frac{M_{A_1} \left(\frac{d_2}{2} \right)}{I} = \frac{LP}{A\sqrt{L^2 + b_2^2}} \quad \tau_{xy} = \frac{M_{A_2} \left(\frac{d_2}{2} \right)}{2I} = \frac{Pb_1b_2d_2}{4I\sqrt{L^2 + b_2^2}}$$

STRESSES ON ELEMENT 2 AT COORD. $(d_2/2, 0, 0)$

$$\sigma_x = 0 \quad \sigma_y = \frac{R_{A_2}}{A} + \frac{M_{A_3} \left(\frac{d_2}{2} \right)}{I} = \frac{LP}{A\sqrt{L^2 + b_2^2}} - \frac{LPb_1d_2}{2I\sqrt{L^2 + b_2^2}}$$

$$\tau_{xy} = \frac{M_{A_2} \left(\frac{d_2}{2} \right)}{2I} + \frac{4}{3} \left(\frac{-R_{A_3}}{A} \right) \left(\frac{d_1^2 + d_1d_2 + d_2^2}{d_1^2 + d_2^2} \right)$$

NUMERICAL VALUES AND CROSS-SECTIONAL PROPERTIES

$$b_1 = 3.6 \text{ ft} \quad b_2 = 2.2 \text{ ft} \quad P = 2200 \text{ lb} \quad d_2 = 7.5 \text{ in.} \quad d_1 = 6.8 \text{ in.} \quad L = 6 \text{ ft}$$

$$A = \frac{\pi}{4} (d_2^2 - d_1^2) \quad A = 7.862 \text{ in.}^2 \quad I = \frac{\pi}{64} (d_2^4 - d_1^4) \quad I = 50.36 \text{ in.}^4$$

STRESSES ON ELEMENT 1 AT COORD. $(0, 0, d_2/2)$ - see Example 8-6 for additional guidance.

$$\sigma_x = 0 \quad \sigma_y = \frac{R_{A_2}}{A} + \frac{M_{A_1} \left(\frac{d_2}{2} \right)}{I} = -263 \text{ psi} \quad \tau_{xy} = \frac{M_{A_2} \left(\frac{d_2}{2} \right)}{2I} = -1218 \text{ psi}$$

Element 1 - principal normal stresses & maximum in-plane shear stress:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 1094 \text{ psi} \quad \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = -1357 \text{ psi}$$

^ max. tensile stress at base of pole

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 1225 \text{ psi}$$

Because the principal stresses σ_1 and σ_2 have opposite signs, the maximum in-plane shear stresses are larger than the maximum out-of-plane shear stresses (see Eqs. 7-28a, b, and c and the accompanying discussion).

STRESSES ON ELEMENT 2 AT COORD. $(d_2/2, 0, 0)$

$$\sigma_x = 0 \quad \sigma_y = \frac{R_{A2}}{A} + \frac{M_{A3}\left(\frac{d_2}{2}\right)}{I} = -6907 \text{ psi}$$

$$\tau_{xy} = \frac{M_{A2}\left(\frac{d_2}{2}\right)}{2I} + \frac{4}{3}\left(\frac{-R_{A3}}{A}\right)\left(\frac{d_1^2 + d_1d_2 + d_2^2}{d_1^2 + d_2^2}\right) = -1411 \text{ psi}$$

Element 2 - principal normal stresses & maximum in-plane shear stress:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 277 \text{ psi} \quad \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = -7184 \text{ psi}$$

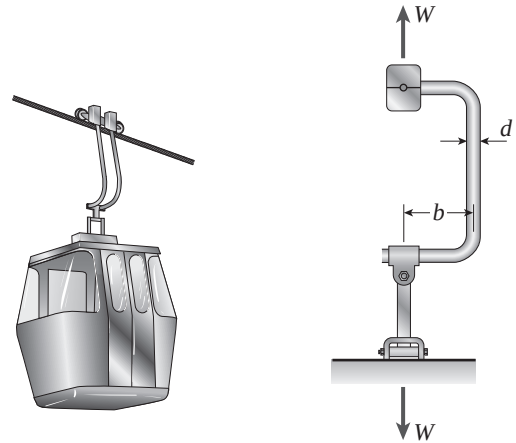
^ max. compressive stress at base of pole

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 3731 \text{ psi}$$

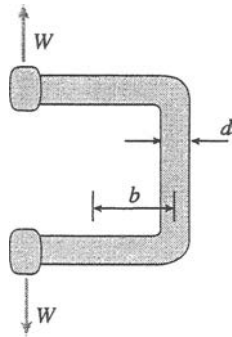
^ max. shear stress at base of pole

Problem 8.5-14 A gondola on a ski lift is supported by two bent arms, as shown in the figure. Each arm is offset by the distance $b = 180$ mm from the line of action of the weight force W . The allowable stresses in the arms are 100 MPa in tension and 50 MPa in shear.

If the loaded gondola weighs 12 kN, what is the minimum diameter d of the arms?



Solution 8.5-14 Gondola on a ski lift



$$b = 180 \text{ mm} \quad W = \frac{12 \text{ kN}}{2} = 6 \text{ kN}$$

$$\sigma_{\text{allow}} = 100 \text{ MPa (tension)} \quad \tau_{\text{allow}} = 50 \text{ MPa}$$

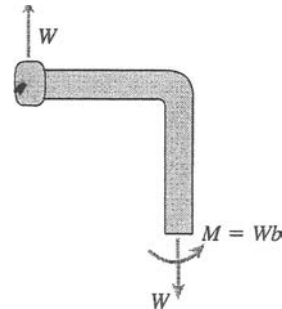
Find $d_{\min}$:

$$A = \frac{\pi d^2}{4} \quad S = \frac{\pi d^3}{32}$$

MAXIMUM TENSILE STRESS

$$\sigma_t = \frac{W}{A} + \frac{M}{S} = \frac{4W}{\pi d^2} + \frac{32Wb}{\pi d^3}$$

$$\text{or} \quad \left(\frac{\pi \sigma_t}{4W} \right) d^3 - d - 8b = 0$$



SUBSTITUTE NUMERICAL VALUES:

$$\frac{\pi \sigma_t}{4W} = \frac{\pi \sigma_{\text{allow}}}{4W} = \frac{\pi(100 \text{ MPa})}{4(6 \text{ kN})} = 13,089.97 \frac{1}{\text{m}^2}$$

$$8b = 1.44 \text{ m}$$

$$13,090d^3 - d - 1.44 = 0 \quad (d = \text{meters})$$

Solve numerically: $d = 0.04845 \text{ m}$

$$\therefore d_{\min} = 48.4 \text{ mm} \quad \leftarrow$$

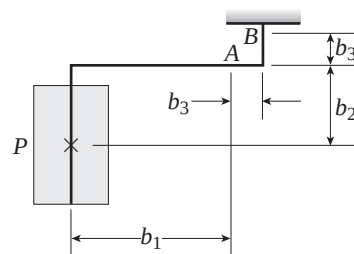
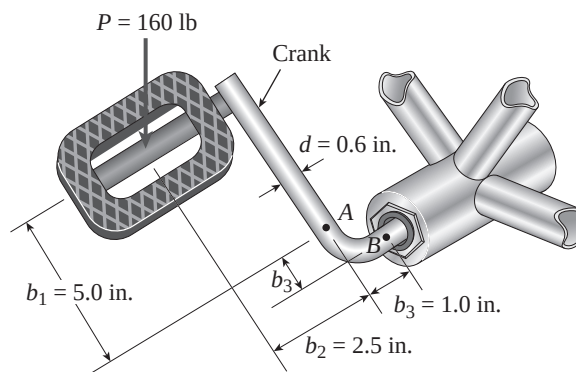
MAXIMUM SHEAR STRESS

$$\tau_{\max} = \frac{\sigma_t}{2} \text{ (uniaxial stress)}$$

Since τ_{allow} is one-half of σ_{allow} , the minimum diameter for shear is the same as for tension.

Problem 8.5-15 Determine the maximum tensile, compressive, and shear stresses at points A and B on the bicycle pedal crank shown in the figure.

The pedal and crank are in a horizontal plane and points A and B are located on the top of the crank. The load $P = 160$ lb acts in the vertical direction and the distances (in the horizontal plane) between the line of action of the load and points A and B are $b_1 = 5.0$ in., $b_2 = 2.5$ in. and $b_3 = 1.0$ in. Assume that the crank has a solid circular cross section with diameter $d = 0.6$ in.



Top view

Solution 8.5-15

$$P = 160 \text{ lb} \quad d = 0.6 \text{ in.}$$

$$b_1 = 5.0 \text{ in.} \quad b_2 = 2.5 \text{ in.}$$

$$b_3 = 1.0 \text{ in.} \quad S = \frac{\pi d^3}{32}$$

STRESS RESULTANTS ON CROSS SECTION AT **point A**:

$$\text{Torque:} \quad T_A = Pb_2$$

$$\text{Moment:} \quad M_A = Pb_1$$

$$\text{Shear Force:} \quad V_A = P$$

STRESS RESULTANTS at **point B**:

$$T_B = P(b_1 + b_3)$$

$$M_B = P(b_2 + b_3)$$

$$V_B = P$$

STRESS AT POINT A:

$$\tau = \frac{16T_A}{\pi d^3} \quad \tau = 9.431 \times 10^3 \text{ psi}$$

$$\sigma = \frac{M_A}{S} \quad \sigma = 3.773 \times 10^4 \text{ psi}$$

(The shear force V produces no shear stresses at point A.)

PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESS

$$\sigma_x = 0 \quad \sigma_y = \sigma \quad \tau_{xy} = -\tau$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = 3.995 \times 10^4 \text{ psi}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = -2.226 \times 10^3 \text{ psi}$$

MAXIMUM TENSILE STRESS

$$\sigma_t = \sigma_1 \quad \sigma_t = 39,950 \text{ psi} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS

$$\sigma_c = \sigma_2 \quad \sigma_c = -2226 \text{ psi} \quad \leftarrow$$

MAXIMUM IN-PLANE SHEAR STRESS

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = 21,090 \text{ psi} \quad \leftarrow$$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

STRESS AT POINT B

$$\tau = \frac{16T_B}{\pi d^3} \quad \tau = 2.264 \times 10^4 \text{ psi}$$

$$\sigma = \frac{M_B}{S} \quad \sigma = 2.641 \times 10^4 \text{ psi}$$

(The shear force V produces no shear stresses at point A.)

PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESS

$$\sigma_x = 0 \quad \sigma_y = \sigma \quad \tau_{xy} = -\tau$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = 3.941 \times 10^4 \text{ psi}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = -1.3 \times 10^4$$

MAXIMUM TENSILE STRESS

$$\sigma_t = \sigma_1 \quad \sigma_t = 39,410 \text{ psi} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS

$$\sigma_c = \sigma_2 \quad \sigma_c = -13,000 \text{ psi} \quad \leftarrow$$

MAXIMUM IN-PLANE SHEAR STRESS

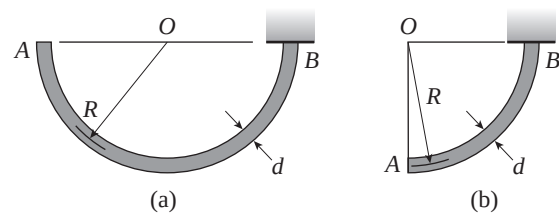
$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = 26,210 \text{ psi} \quad \leftarrow$$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-16 A semicircular bar AB lying in a horizontal plane is supported at B (see figure part a). The bar has centerline radius R and weight q per unit of length (total weight of the bar equals πqR). The cross section of the bar is circular with diameter d .

- Obtain formulas for the maximum tensile stress σ_t , maximum compressive stress σ_c , and maximum in-plane shear stress $\tau_{\max}$ at the top of the bar at the support due to the weight of the bar.
- Repeat part (a) if the bar is a quarter-circular segment (see figure part b) but has the same total weight as the semicircular bar.



Solution 8.5-16 Semicircular bar

$$(a) \quad c = \frac{2R}{\pi} \quad W = \pi q R \quad M_B = Wc = 2R^2 q \quad T_B = WR = \pi R^2 q$$

$$\sigma_B = \frac{M_B \left(\frac{d}{2} \right)}{\left(\frac{\pi d^4}{64} \right)} = \frac{64 R^2 q}{\pi d^3} \quad \tau_B = \frac{T_B \left(\frac{d}{2} \right)}{\frac{\pi d^4}{32}} = \frac{16 R^2 q}{d^3}$$

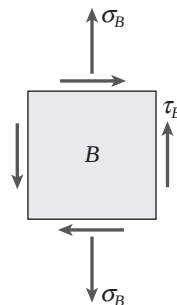
PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESS

$$\sigma_x = 0 \quad \sigma_y = \sigma_B \quad \tau_{xy} = \tau_B$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = \frac{16 \sqrt{\pi^2 + 4} \sqrt{\frac{R^4 q^2}{d^6}}}{\pi} + \frac{32 R^2 q}{\pi d^3} \quad \text{OR} \quad \sigma_1 = 29.15 \frac{q R^2}{d^3}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = \frac{32 R^2 q}{\pi d^3} - \frac{16 \sqrt{\pi^2 + 4} \sqrt{\frac{R^4 q^2}{d^6}}}{\pi} \quad \text{OR} \quad \sigma_2 = -8.78 \frac{q R^2}{d^3}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 16 \sqrt{\frac{R^4 q^2}{d^6} + \frac{4 R^4 q^2}{\pi^2 d^6}} \quad \text{OR} \quad \tau_{\max} = 18.97 \frac{q R^2}{d^3}$$

(b) QUARTER-CIRCULAR BAR OF SAME TOTAL WEIGHT W AS SEMICIRCULAR BAR $W = \pi q R$

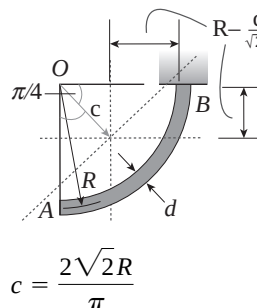
$$\text{Appendix E, Case 23:} \quad \beta = \frac{\pi}{4} \quad y_{\text{bar}} = \frac{R \sin(\beta)}{\beta} = \frac{2 \sqrt{2} R}{\pi} \quad \text{so} \quad c = \frac{2 \sqrt{2} R}{\pi}$$

$$M_B = W \left(R - \frac{c}{\sqrt{2}} \right) = \pi R q \left(R - \frac{2R}{\pi} \right) \quad R - \frac{c}{\sqrt{2}} = R - \frac{2R}{\pi}$$

$$T_B = M_B$$

$$\sigma_B = \frac{M_B \left(\frac{d}{2} \right)}{\left(\frac{\pi d^4}{64} \right)} = \frac{32 R q \left(R - \frac{2R}{\pi} \right)}{d^3}$$

$$\tau_B = \frac{T_B \left(\frac{d}{2} \right)}{\frac{\pi d^4}{32}} = \frac{16 R q \left(R - \frac{2R}{\pi} \right)}{d^3}$$



PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESS

$$\sigma_x = 0 \quad \sigma_y = \sigma_B \quad \tau_{xy} = \tau_B$$

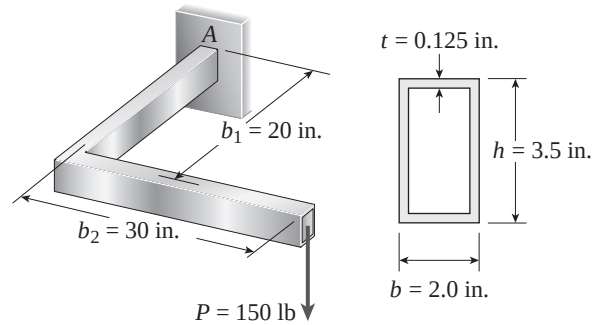
$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{16\left(R^2q + d^3\sqrt{\frac{2R^4q^2}{d^6}}\right)(\pi - 2)}{\pi d^3} \quad \text{OR} \quad \sigma_t = 14.04 \frac{qR^2}{d^3}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{16\left(R^2q - d^3\sqrt{\frac{2R^4q^2}{d^6}}\right)(\pi - 2)}{\pi d^3} \quad \text{OR} \quad \sigma_c = -2.41 \frac{qR^2}{d^3}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 16\sqrt{2}\sqrt{\frac{R^2q^2\left(R - \frac{2R}{\pi}\right)^2}{d^6}} \quad \text{OR} \quad \tau_{\max} = 8.22 \frac{qR^2}{d^3}$$

Problem 8.5-17 An L-shaped bracket lying in a horizontal plane supports a load $P = 150$ lb (see figure). The bracket has a hollow rectangular cross section with thickness $t = 0.125$ in. and outer dimensions $b = 2.0$ in. and $h = 3.5$ in. The centerline lengths of the arms are $b_1 = 20$ in. and $b_2 = 30$ in.

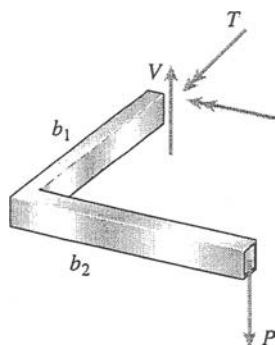
Considering only the load P , calculate the maximum tensile stress σ_t , maximum compressive stress σ_c , and maximum shear stress $\tau_{\max}$ at point A, which is located on the top of the bracket at the support.



Solution 8.5-17 L-shaped bracket

$$P = 150 \text{ lb} \quad b_1 = 20 \text{ in.} \quad b_2 = 30 \text{ in.} \\ t = 0.125 \text{ in.} \quad h = 3.5 \text{ in.} \quad b = 2.0 \text{ in.}$$

FREE-BODY DIAGRAM OF BRACKET



STRESS RESULTANTS AT THE SUPPORT

$$\text{Torque: } T = Pb_2 = (150 \text{ lb})(30 \text{ in.}) = 4500 \text{ lb-in.}$$

$$\text{Moment: } M = Pb_1 = (150 \text{ lb})(20 \text{ in.}) = 3000 \text{ lb-in.}$$

$$\text{Shear force: } V = P = 150 \text{ lb}$$

PROPERTIES OF THE CROSS SECTION

For torsion:

$$A_m = (b - t)(h - t) = (1.875 \text{ in.})(3.375 \text{ in.}) = 6.3281 \text{ in.}^2$$

$$\text{For bending: } c = \frac{h}{2} = 1.75 \text{ in.}$$

$$I = \frac{1}{12}(bh^3) - \frac{1}{12}(b - 2t)(h - 2t)^3 \\ = \frac{1}{12}(2.0 \text{ in.})(3.5 \text{ in.})^3 - \frac{1}{12}(1.75 \text{ in.})(3.25 \text{ in.})^3 \\ = 2.1396 \text{ in.}^4$$

STRESSES AT POINT A ON THE TOP OF THE BRACKET

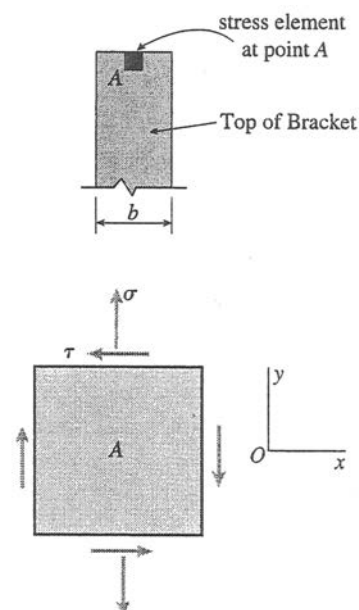
$$\tau = \frac{T}{2tA_m} = \frac{4500 \text{ lb-in.}}{2(0.125 \text{ in.})(6.3281 \text{ in.}^2)} = 2844 \text{ psi}$$

$$\sigma = \frac{Mc}{I} = \frac{(3000 \text{ lb-in.})(1.75 \text{ in.})}{2.1396 \text{ in.}^4} = 2454 \text{ psi}$$

(The shear force V produces no stresses at point A.)

STRESS ELEMENT AT POINT A

(This view is looking downward at the top of the bracket.)



$$\sigma_x = 0 \quad \sigma_y = \sigma = 2454 \text{ psi}$$

$$\tau_{xy} = -\tau = -2844 \text{ psi}$$

PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESS

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ = 1227 \text{ psi} \pm 3097 \text{ psi}$$

$$\sigma_1 = 4324 \text{ psi} \quad \sigma_2 = -1870 \text{ psi}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 3097 \text{ psi}$$

MAXIMUM TENSILE STRESS:

$$\sigma_t = 4320 \text{ psi} \quad \leftarrow$$

MAXIMUM COMPRESSIVE STRESS:

$$\sigma_c = -1870 \text{ psi} \quad \leftarrow$$

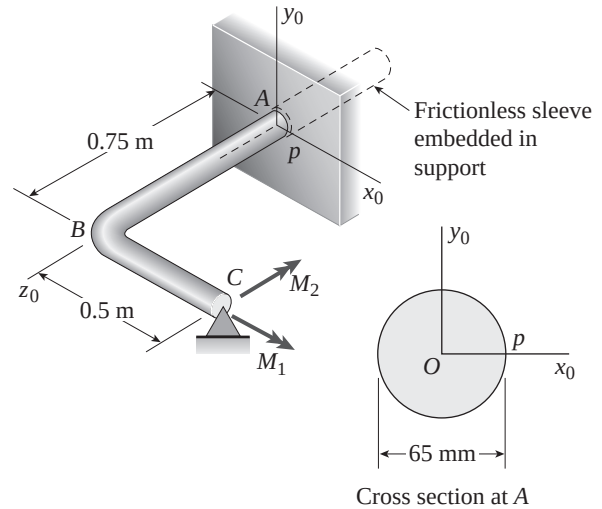
MAXIMUM SHEAR STRESS:

$$\tau_{\max} = 3100 \text{ psi} \quad \leftarrow$$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear stress is larger than the maximum out-of-plane shear stress.

Problem 8.5-18 A horizontal bracket ABC consists of two perpendicular arms AB of length 0.75 m, and BC of length of 0.5 m. The bracket has a solid circular cross section with diameter equal to 65 mm. The bracket is inserted in a frictionless sleeve at A (which is slightly larger in diameter) so is free to rotate about the z_0 axis at A , and is supported by a pin at C . Moments are applied at point C as follows: $M_1 = 1.5 \text{ kN} \cdot \text{m}$ in the x direction and $M_2 = 1.0 \text{ kN} \cdot \text{m}$ acts in the $(-z)$ direction.

Considering only the moments M_1 and M_2 , calculate the maximum tensile stress σ_t , the maximum compressive stress σ_c , and the maximum in-plane shear stress $\tau_{\max}$ at point p , which is located at support A on the side of the bracket at midheight.



Solution 8.5-18

$$d = 65 \text{ mm}$$

$$b_1 = 0.5 \text{ m} \quad \text{length of arm } BC$$

$$b_2 = 0.75 \text{ m} \quad \text{length of arm } AB$$

$$M_1 = 1.5 \text{ kN} \cdot \text{m}$$

$$M_2 = 1.0 \text{ kN} \cdot \text{m}$$

PROPERTIES OF THE CROSS SECTION

$$d = 65 \text{ mm} \quad r = \frac{d}{2}$$

$$A = \frac{\pi}{4} d^2 \quad A = 3.318 \times 10^3 \text{ mm}^2$$

$$I = \frac{\pi}{64} d^4 \quad I = 8.762 \times 10^5 \text{ mm}^4$$

$$I_p = 2I \quad I_p = 1.752 \times 10^6 \text{ mm}^4$$

$$Q = \frac{2}{3} r^3 \quad Q = 2.289 \times 10^4 \text{ mm}^3$$

STRESS RESULTANTS AT SUPPORT A

$$N_z = 0 \quad \text{Axial force}$$

$$M_y = 0$$

$$M_x = -M_2 \frac{b_2}{b_1} + M_1 \quad M_x = 0$$

$$T = 0 \quad \text{Torsional frictionless sleeve at support A } (T = M_z)$$

$$V_y = -\frac{M_2}{b_1}$$

$$\sigma_x = 0$$

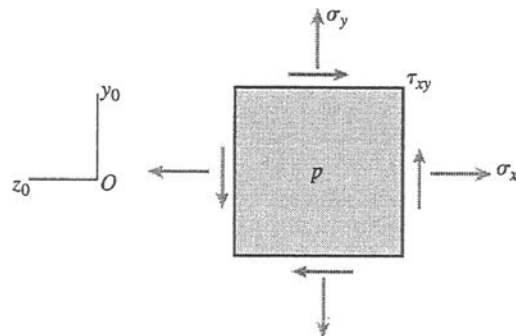
$$\sigma_y = 0$$

$$\tau_{xy} = \frac{V_y Q}{I d} \quad \tau_{xy} = -0.804 \text{ MPa}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

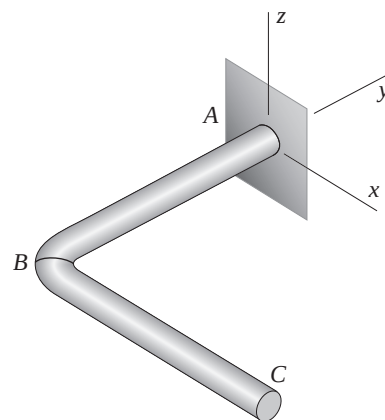
$$\text{Pure shear:} \quad \tau_{\max} = 0.804 \text{ MPa} \quad \leftarrow$$

STRESSES AT POINT p ON THE SIDE OF THE BRACKET



Problem 8.5-19 An arm ABC lying in a horizontal plane and supported at A (see figure) is made of two identical solid steel bars AB and BC welded together at a right angle. Each bar is 22 in. long.

- (a) Knowing that the maximum tensile stress (principal stress) at the top of the bar at support A due solely to the weights of the bars is 1025 psi, determine the diameter d of the bars.
- (b) If the allowable tensile stress is 1475 psi and each bar has diameter $d = 2.0$ in., what is the maximum downward load P that can be applied at C (in addition to self-weight)?



Solution 8.5-19

- (a) RESULTANT BENDING MOMENT AND TORQUE AT A due to uniformly distributed self-weight, w :

$$w = \gamma_s \left(\frac{\pi d^2}{4} \right) \quad M_A = wLL + wL \frac{L}{2} = \frac{3\pi L^2 d^2 \gamma_s}{8} \quad T_A = wL \frac{L}{2} = \frac{\pi L^2 d^2 \gamma_s}{8}$$

$$\text{NORMAL TENSILE STRESS ON TOP OF BAR AT } A \quad \sigma_A = \frac{M_A \left(\frac{d}{2} \right)}{\left(\frac{\pi d^4}{64} \right)} = \frac{12L^2 \gamma_s}{d}$$

$$\text{SHEAR STRESS ON TOP OF BAR AT } A \text{ DUE TO TORQUE } T_A \quad \tau_A = \frac{T_A \left(\frac{d}{2} \right)}{\left(\frac{\pi d^4}{32} \right)} = \frac{2L^2 \gamma_s}{d}$$

PLANE STRESS STATE AT TOP OF BAR

$$\sigma_x = 0 \quad \sigma_y = \sigma_A = \frac{12L^2 \gamma_s}{d} \quad \tau_{xy} = -\tau_A = -\frac{2L^2 \gamma_s}{d}$$

MAXIMUM PRINCIPAL STRESS AT TOP OF BAR AT A

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 2\sqrt{10} \sqrt{\frac{L^4 \gamma_s^2}{d^2} + \frac{6L^2 \gamma_s}{d}}$$

EQUATE σ_1 TO KNOWN STRESS VALUE AND SOLVE FOR REQUIRED DIAMETER d

$$2\sqrt{10} \sqrt{\frac{L^4 \gamma_s^2}{d^2} + \frac{6L^2 \gamma_s}{d}} = \sigma_t \quad \text{solving for required diameter } d: \quad d_{\min} = \frac{2L^2 \gamma_s (\sqrt{10} + 3)}{\sigma_t}$$

SUBSTITUTE NUMERICAL DATA

$$\sigma_t = 1025 \text{ psi} \quad L = 22 \text{ in.} \quad \gamma_s = 490 \frac{\text{lb}}{\text{ft}^3} \quad \text{so} \quad d_{\min} = \frac{2L^2 \gamma_s (\sqrt{10} + 3)}{\sigma_t} = 1.65 \text{ in.}$$

(b) RESULTANT BENDING MOMENT AND TORQUE AT A due to uniformly distributed self-weight, w :

$$w = \gamma_s \left(\frac{\pi d^2}{4} \right) \quad M_A = wL + wL \frac{L}{2} + PL \quad T_A = wL \frac{L}{2} + PL = \frac{\pi \gamma_s L^2 d^2}{8} + PL$$

$$\text{NORMAL TENSILE STRESS ON TOP OF BAR AT A} \quad \sigma_A = \frac{M_A \left(\frac{d}{2} \right)}{\left(\frac{\pi d^4}{64} \right)} = \frac{32 \left(\frac{3\pi \gamma_s L^2 d^2}{8} + PL \right)}{\pi d^3}$$

$$\text{SHEAR STRESS ON TOP OF BAR AT A DUE TO TORQUE } T_A \quad \tau_A = \frac{T_A \left(\frac{d}{2} \right)}{\left(\frac{\pi d^4}{32} \right)} = \frac{16 \left(\frac{\pi \gamma_s L^2 d^2}{8} + PL \right)}{\pi d^3}$$

PLANE STRESS STATE AT TOP OF BAR

$$\sigma_x = 0 \quad \sigma_y = \sigma_A = \frac{12\pi \gamma_s L^2 d^2 + 32PL}{\pi d^3} \quad \tau_{xy} = -\tau_A \rightarrow -\frac{2\pi \gamma_s L^2 d^2 + 16PL}{\pi d^3}$$

MAXIMUM PRINCIPAL STRESS AT TOP OF BAR AT A

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = \sqrt{\frac{(2\pi \gamma_s L^2 d^2 + 16PL)^2}{\pi^2 d^6} + \frac{(12\pi \gamma_s L^2 d^2 + 32PL)^2}{4\pi^2 d^6}} + \frac{12\pi \gamma_s L^2 d^2 + 32PL}{2\pi d^3}$$

EQUATE σ_1 TO ALLOWABLE STRESS VALUE AND SOLVE FOR PERMISSIBLE LOAD P

$$\sqrt{\frac{(2\pi \gamma_s L^2 d^2 + 16PL)^2}{\pi^2 d^6} + \frac{(12\pi \gamma_s L^2 d^2 + 32PL)^2}{4\pi^2 d^6}} + \frac{12\pi \gamma_s L^2 d^2 + 32PL}{2\pi d^3} = \sigma_a$$

SUBSTITUTE NUMERICAL DATA

$$\sigma_a = 1475 \text{ psi} \quad L = 22 \text{ in.} \quad \gamma_s = 490 \frac{\text{lb}}{\text{ft}^3} \quad d = 2 \text{ in.}$$

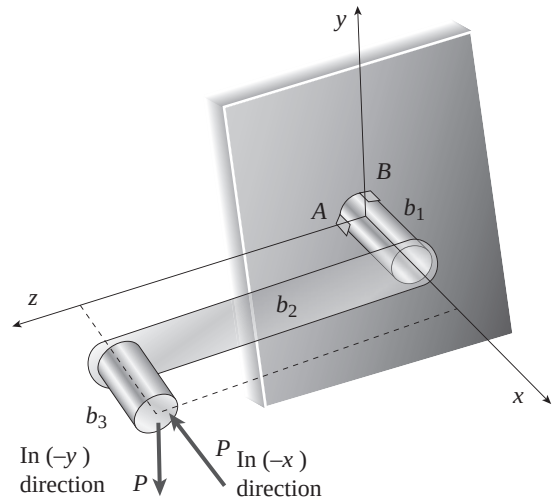
Solving for permissible load P :

$$P_{\max} = -\frac{\pi d^3 \sigma_a + 2\pi L^2 d^2 \gamma_s - \pi \sqrt{2} d^6 \sqrt{\frac{d \sigma_a^2 - 4L^2 \sigma_a \gamma_s}{d^7}}}{16L} = 19.25 \text{ lb}$$

Note that each bar weighs 19.6 lb.

Problem 8.5-20 A crank arm consists of a solid segment of length b_1 and diameter d , a segment of length b_2 , and a segment of length b_3 as shown in the figure. Two loads P act as shown, one parallel to $-x$ and another parallel to $-y$. Each load P equals 1.2 kN. The crankshaft dimensions are $b_1 = 75 \text{ mm}$, $b_2 = 125 \text{ mm}$, and $b_3 = 35 \text{ mm}$. The diameter of the upper shaft is $d = 22 \text{ mm}$.

- Determine the maximum tensile, compressive, and shear stresses at point A, which is located on the surface of the shaft at the z axis.
- Determine the maximum tensile, compressive, and shear stresses at point B, which is located on the surface of the shaft at the y axis.



Solution 8.5-20

NUMERICAL DATA

$$P = 1.2 \text{ kN} \quad b_1 = 75 \text{ mm} \quad b_2 = 125 \text{ mm} \quad b_3 = 35 \text{ mm} \quad d = 22 \text{ mm}$$

Properties of the cross section:

$$r = \frac{d}{2} \quad A = \frac{\pi}{4} d^2 \quad A = 380.133 \text{ mm}^2$$

$$I = \frac{\pi}{64} d^4 \quad I = 1.15 \times 10^4 \text{ mm}^4 \quad I_p = 2I \quad I_p = 2.3 \times 10^4 \text{ mm}^4$$

Reactions at the support:

$$V_x = P \quad (\text{Axial force in } x \text{ dir.})$$

$$V_y = P \quad (\text{Shear force in } y \text{ dir.})$$

$$V_z = 0 \quad (\text{Shear force in } z \text{ dir.})$$

$$M_x = -Pb_2 \quad (\text{Torsional moment}) \quad M_x = -150 \text{ kN}\cdot\text{mm}$$

$$M_y = Pb_2 \quad (\text{Bending moment}) \quad M_y = 150 \text{ kN}\cdot\text{mm}$$

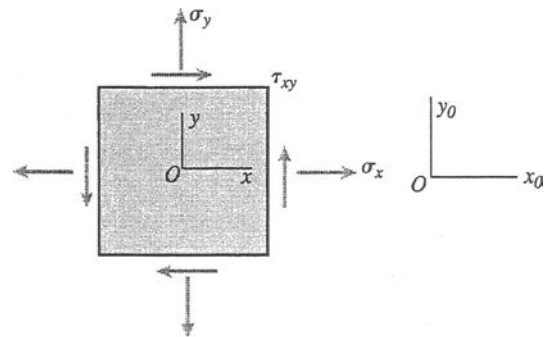
$$M_z = P(b_1 + b_3) \quad (\text{Bending moment}) \quad M_z = 132 \text{ kN}\cdot\text{mm}$$

(a) STRESSES AT POINT A

$$\sigma_x = -\frac{V_x}{A} - \frac{M_y}{I} r \quad \sigma_x = -146.647 \text{ MPa} \quad (\text{compressive})$$

$$\sigma_y = 0$$

$$\tau_{xy} = \frac{M_x r}{I_p} \quad \tau_{xy} = -71.745 \text{ MPa}$$



PRINCIPAL NORMAL STRESSES AND MAXIMUM SHEAR STRESS AT A

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{Max. tensile stress:} \quad \boxed{\sigma_1 = 29.3 \text{ MPa}}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{Max. compressive stress:} \quad \boxed{\sigma_2 = -175.9 \text{ MPa}}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{Max. shear stress:} \quad \boxed{\tau_{\max} = 102.6 \text{ MPa}}$$

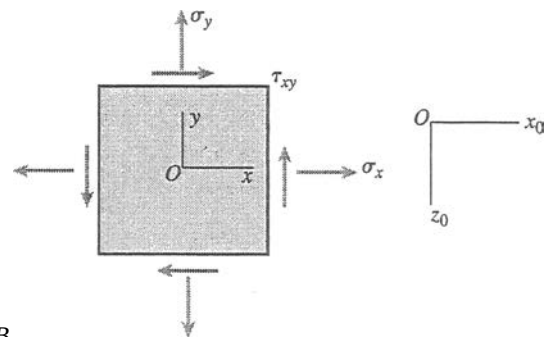
(b) STRESSES AT POINT B

$$\sigma_x = -\frac{V_x}{A} - \frac{M_z}{I} r \quad \sigma_x = 123.115 \text{ MPa} \quad (\text{tensile})$$

$$\sigma_y = 0$$

$$\tau_{xy} = \frac{M_x d}{2I_p} - \frac{4}{3} \frac{V_z}{A} \quad \tau_{xy} = -71.745 \text{ MPa}$$

$$\frac{M_x d}{2I_p} = -71.745 \text{ MPa}$$



PRINCIPAL NORMAL STRESSES AND MAXIMUM SHEAR STRESS AT B

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{Max. tensile stress:} \quad \boxed{\sigma_1 = 156.1 \text{ MPa}}$$

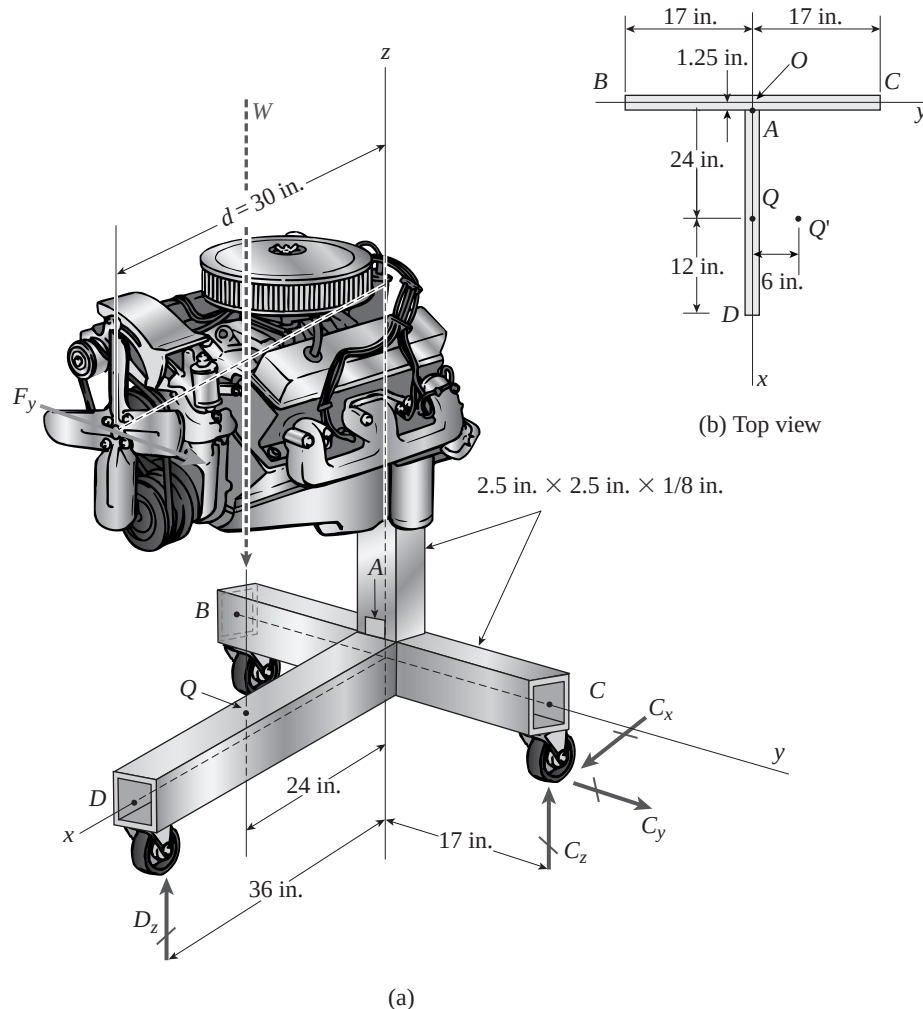
$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{Max. compressive stress: } \boxed{\sigma_2 = -33 \text{ MPa}}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{Max. shear stress: } \boxed{\tau_{\max} = 94.5 \text{ MPa}}$$

Note: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-21 A moveable steel stand supports an automobile engine weighing $W = 750 \text{ lb}$ as shown in the figure part a. The stand is constructed of $2.5 \text{ in.} \times 2.5 \text{ in.} \times 1/8 \text{ in.}$ -thick steel tubing. Once in position the stand is restrained by pin supports at B and C. Of interest are stresses at point A at the base of the vertical post; point A has coordinates $(x = 1.25, y = 0, z = 1.25)$ in inches. Neglect the weight of the stand.

- Initially, the engine weight acts in the $-z$ direction through point Q, which has coordinates $(24, 0, 1.25)$. Find the maximum tensile, compressive, and shear stresses at point A.
- Repeat part (a) assuming now that, during repair, the engine is rotated about its own longitudinal axis (which is parallel to the x axis) so that W acts through Q' (with coordinates $(24, 6, 1.25)$) and force $F_y = 200 \text{ lb}$ is applied parallel to the y axis at distance $d = 30 \text{ in.}$



Solution 8.5-21

$$W = 750 \text{ lb} \quad F_y = 200 \text{ lb}$$

$$b = 2.5 \text{ in.} \quad t = \frac{1}{8} \text{ in.} \quad d = 30 \text{ in.} \quad x_1 = 24 \text{ in.}$$

$$A = b^2 - (b - 2t)^2 \quad A = 1 \text{ in.}^2 \quad A_m = (b - t)^2$$

$$A_m = 6 \text{ in.}^2 \quad y_1 = 6 \text{ in.} \quad c = \frac{b}{2}$$

$$I = \frac{1}{12} [b^4 - (b - 2t)^4] \quad I = 1 \text{ in.}^4$$

(a) ENGINE WEIGHT ACTS THROUGH X AXIS (POINT Q)

$$M_x = 0 \quad M_y = Wx_1$$

SHEAR & NORMAL STRESSES AT A

$$\sigma = \frac{-W}{A} - \frac{M_y c}{I} \quad \sigma = -20,730 \text{ psi} \quad \tau = 0$$

PRINCIPAL STRESSES & MAX. SHEAR STRESS

$$\sigma_x = 0 \quad \sigma_y = \sigma \quad \tau_{xy} = \tau$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = 0 \quad \leftarrow$$

$$\sigma_2 = -20,730 \text{ psi} \quad \leftarrow$$

$$\tau_{\max} = 10,365 \text{ psi} \quad \leftarrow$$

(b) ENGINE WEIGHT ACTS THROUGH POINT Q' & FORCE F_y ACTS IN y DIR

$$M_y = 18,000 \text{ in.-lb} \quad M_x = Wy_1$$

SHEAR & NORMAL STRESSES AT A

$$\sigma = \frac{-W}{A} - \frac{M_y c}{I} \quad \sigma = -20,730 \text{ psi}$$

same since element A lies on NA for bending about x axis

SHEAR STRESS DEPENDS ON TRANSVERSE SHEAR DUE TO F_y & TORSION DUE TO M_z (MAX. TRANSVERSE SHEAR IN WEB - SEE PROB. 5.10-11)

$$b_1 = b - 2t \quad b_1 = 2 \text{ in.}$$

$$Q = \frac{1}{8} (b^3 - b_1^3) \quad Q = 1 \text{ in.}^3$$

$$\tau_t = \frac{F_y Q}{I(2t)} \quad \tau_t = 378 \text{ psi}$$

SHEAR DUE TO TORSIONAL MOMENT M_z USING APPROX. THEORY (EQ. 3-81)

$$M_z = F_y d \quad M_z = 6000 \text{ in.-lb} \quad \tau_T = \frac{M_z}{2tA_m}$$

$$\tau_T = 4255 \text{ psi}$$

$$\tau = \tau_t + \tau_T \quad \tau = 4633 \text{ psi}$$

PRINCIPAL STRESSES & MAX. SHEAR STRESS

$$\sigma_x = 0 \quad \sigma_y = \sigma \quad \tau_{xy} = \tau$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

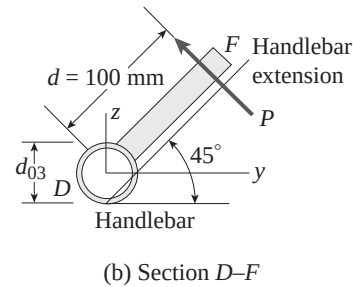
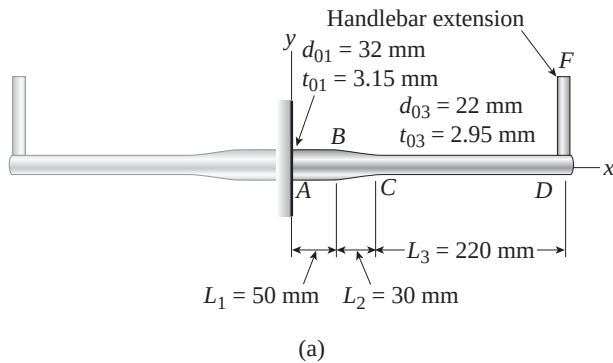
$$\sigma_1 = 988 \text{ psi} \quad \leftarrow$$

$$\sigma_2 = -21,719 \text{ psi} \quad \leftarrow$$

$$\tau_{\max} = 11,354 \text{ psi} \quad \leftarrow$$

Problem 8.5-22 A mountain bike rider going uphill applies force $P = 65 \text{ N}$ to each end of the handlebars $ABCD$, made of aluminum alloy 7075-T6, by pulling on the handlebar extenders (DF on right handlebar segment). Consider the right half of the handlebar assembly only (assume the bars are fixed at the fork at A). Segments AB and CD are prismatic with lengths L_1 and L_3 and with outer diameters and thicknesses d_{01} , t_{01} and d_{03} , t_{03} , respectively, as shown. Segment BC of length L_2 , however, is tapered, and outer diameter and thickness vary linearly between dimensions at B and C . Consider shear, torsion, and bending effects only for segment AD ; assume DF is rigid.

Find maximum tensile, compressive, and shear stresses adjacent to support A . Show where each maximum stress value occurs.



Solution 8.5-22

$$P = 65 \text{ N}$$

$$L_1 = 50 \text{ mm}$$

$$L_2 = 30 \text{ mm}$$

$$L_3 = 220 \text{ mm}$$

$$d_{01} = 32 \text{ mm}$$

$$d_{03} = 22 \text{ mm}$$

$$d = 100 \text{ mm}$$

PROPERTIES OF THE CROSS SECTION AT POINT A

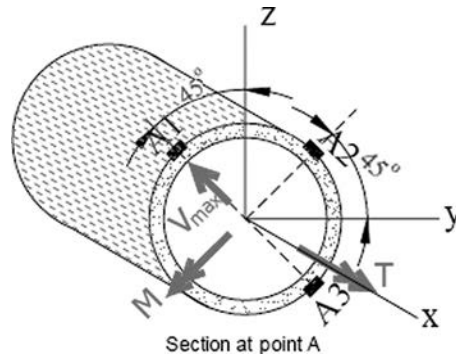
$$r = \frac{d_{01}}{2} \quad t_{01} = 3.15 \text{ mm}$$

$$A = \frac{\pi}{4} [d_{01}^2 - (d_{01} - t_{01})^2] \quad A = 150.543 \text{ mm}^2$$

$$I = \frac{\pi}{64} [d_{01}^4 - (d_{01} - t_{01})^4] \quad I = 1.747 \times 10^4 \text{ mm}^4$$

$$I_p = 2I \quad I_p = 3.493 \times 10^4 \text{ mm}^4$$

$$Q = \frac{1}{12} [d_{01}^3 - (d_{01} - t_{01})^3] \quad Q = 729.625 \text{ mm}^3$$



STRESS RESULTANTS AT POINT A1

$$N_x = 0 \quad (\text{Axial force in } x \text{ dir.})$$

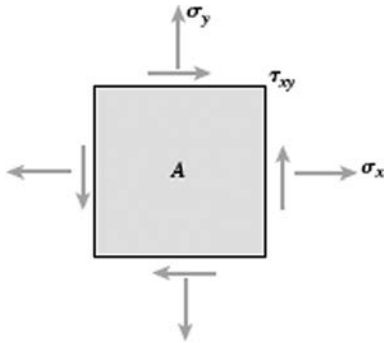
$$V_{\max.} = P \quad (\text{Max. shear force})$$

$$V_{\max.} = 0.065 \text{ kN}$$

$$T = Pd \quad (\text{Torsional moment}) \quad T = 6.5 \text{ kN} \cdot \text{mm}$$

$$M = P(L_1 + L_2 + L_3) \quad (\text{Bending moment})$$

$$M = 19.5 \text{ kN} \cdot \text{mm}$$



$$\sigma_x = -\frac{M}{I}r$$

$$\sigma_x = -17.863 \text{ MPa (compressive stress)}$$

$$\sigma_y = 0$$

$$\tau_{xy} = \frac{Td_{01}}{2I_p} \quad \tau_{xy} = 2.977 \text{ MPa}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\text{MAX. TENSILE STRESS} \quad \sigma_1 = 0.483 \text{ MPa} \quad \leftarrow$$

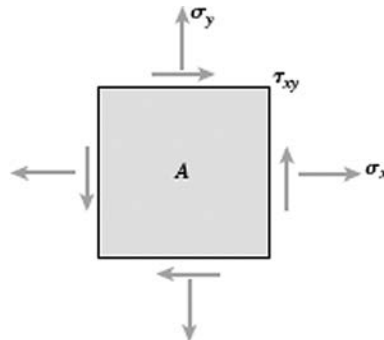
$$\text{MAX. COMPRESSIVE STRESS}$$

$$\sigma_2 = -18.35 \text{ MPa} \quad \leftarrow$$

$$\text{MAX. SHEAR STRESS} \quad \tau_{\max} = 9.42 \text{ MPa} \quad \leftarrow$$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

STRESS RESULTANTS AT POINT A2



$$\sigma_x = 0$$

$$\sigma_y = 0$$

$$\tau_{xy} = \frac{Td_{01}}{2I_p} + \frac{V_{\max}Q}{I2t_{01}} \quad \tau_{xy} = 3.408 \text{ MPa}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\text{MAX. TENSILE STRESS} \quad \sigma_1 = 3.41 \text{ MPa} \quad \leftarrow$$

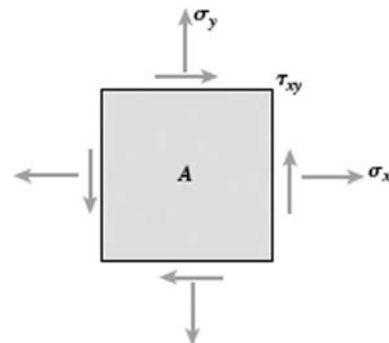
$$\text{MAX. COMPRESSIVE STRESS}$$

$$\sigma_2 = -3.41 \text{ MPa} \quad \leftarrow$$

$$\text{MAX. SHEAR STRESS} \quad \tau_{\max} = 3.41 \text{ MPa} \quad \leftarrow$$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

STRESS RESULTANTS AT POINT A3



$$\sigma_x = \frac{M}{I}r \quad \sigma_x = 17.863 \text{ MPa (tensile stress)}$$

$$\sigma_y = 0$$

$$\tau_{xy} = \frac{Td_{01}}{2I_p} \quad \tau_{xy} = 2.977 \text{ MPa}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

MAX. TENSILE STRESS $\sigma_1 = 18.35 \text{ MPa}$ ←

MAX. COMPRESSIVE STRESS

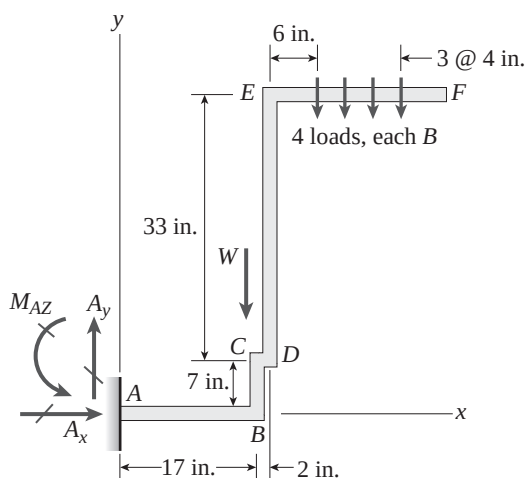
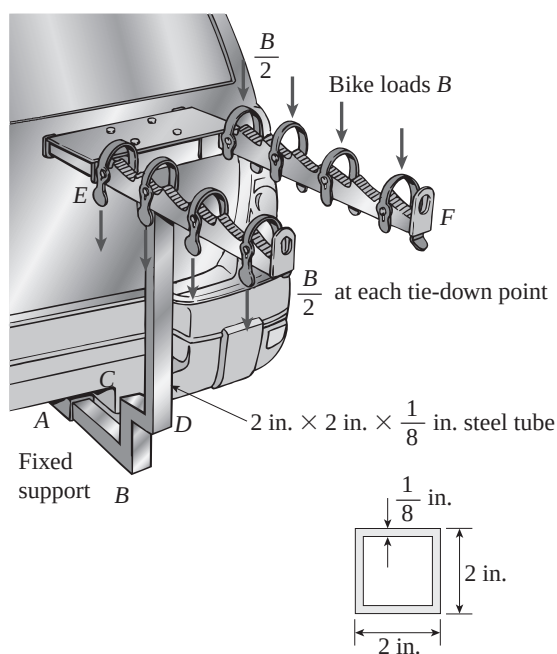
$\sigma_2 = -0.483 \text{ MPa}$ ←

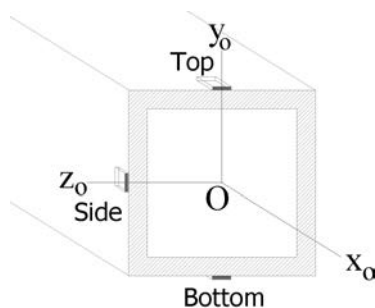
MAX. SHEAR STRESS $\tau_{\max} = 9.42 \text{ MPa}$ ←

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-23 Determine the maximum tensile, compressive, and shear stresses acting on the cross section of the tube at point A of the hitch bicycle rack shown in the figure.

The rack is made up of 2 in. $\times$ 2 in. steel tubing which is $1/8$ in. thick. Assume that the weight of each of four bicycles is distributed evenly between the two support arms so that the rack can be represented as a cantilever beam (*ABCDEF*) in the *xy* plane. The overall weight of the rack alone is $W = 60 \text{ lb}$ directed through *C*, and the weight of each bicycle is $B = 30 \text{ lb}$.



Solution 8.5-23

Cross section at the point A

Cross section: $t = 0.125$ in. Thickness

$d_2 = 2$ in. Outer width

$d_1 = d_2 - 2t$ Inner width

$A = d_2^2 - d_1^2$ $A = 0.938$ in.²

$I = \frac{d_2^4}{12} - \frac{d_1^4}{12}$ $I = 0.552$ in.⁴

$Q = (d_2 - 2t)t \left(\frac{d_2}{2} - \frac{t}{2} \right) + 2 \frac{d_2}{2} t \frac{d_2}{4}$

$Q = 0.33$ in.³

The distance between point A and the center of load W:

$b_1 = 17$ in.

The distance between the point A and the center of a bike load B:

$b_2 = \left(17 + 2 + 6 + \frac{3.4}{2} \right)$ in.

$W = 60$ lb $B = 30$ lb

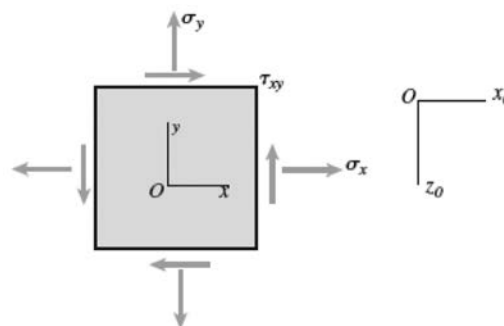
REACTIONS AT SUPPORT A

$M_{Az} = Wb_1 + 4Bb_2$ $M_{Az} = 4.74 \times 10^3$ lb · in.

$A_y = W + 4B$ $A_y = 180$ lb

$A_x = 0$

TOP OF THE CROSS SECTION (AT POINT A)



$$\sigma_x = \frac{M_{Az} d_2}{2I}$$

$$\sigma_x = 8.591 \times 10^3 \text{ psi (tensile stress)}$$

$$\sigma_y = 0$$

$$\tau_{xy} = 0$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

MAX. TENSILE STRESS

$$\sigma_1 = 8591 \text{ psi} \quad \leftarrow$$

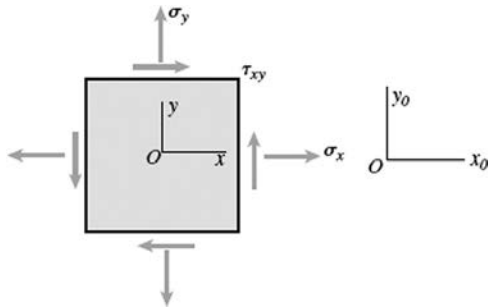
$$\text{MAX. COMPRESSIVE STRESS} \quad \sigma_2 = 0 \quad \leftarrow$$

MAX. SHEAR STRESS

$$\tau_{\max} = 4295 \text{ psi} \quad \leftarrow$$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

SIDE OF THE CROSS SECTION (AT POINT A)



$$\sigma_x = 0$$

$$\sigma_y = 0$$

$$\tau_{xy} = \frac{A_y Q}{I 2t} \quad \tau_{xy} = 430.726 \text{ psi}$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\text{MAX. TENSILE STRESS} \quad \sigma_1 = 431 \text{ psi} \quad \leftarrow$$

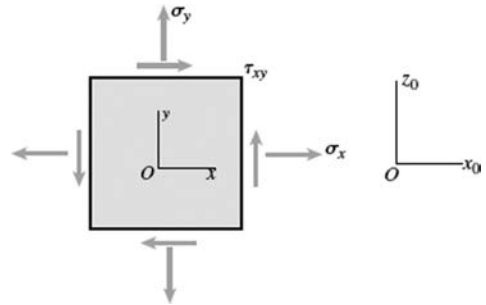
$$\text{MAX. COMPRESSIVE STRESS}$$

$$\sigma_2 = -431 \text{ psi} \quad \leftarrow$$

$$\text{MAX. SHEAR STRESS} \quad \tau_{\max} = 431 \text{ psi} \quad \leftarrow$$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

BOTTOM OF THE CROSS SECTION (AT POINT A)



$$\sigma_x = -\frac{M_{Az} d_2}{2I}$$

$$\sigma_x = -8.591 \times 10^3 \text{ psi (compressive stress)}$$

$$\sigma_y = 0$$

$$\tau_{xy} = 0$$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\text{MAX. TENSILE STRESS} \quad \sigma_1 = 0 \quad \leftarrow$$

$$\text{MAX. COMPRESSIVE STRESS}$$

$$\sigma_2 = -8591 \text{ psi} \quad \leftarrow$$

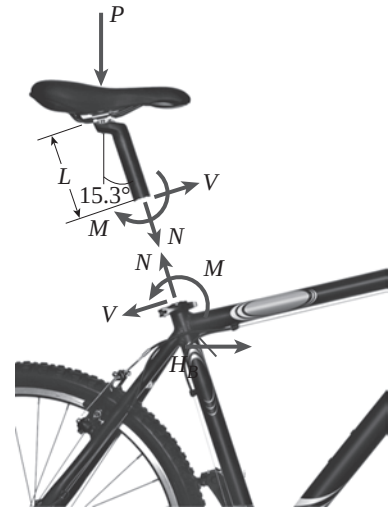
$$\text{MAX. SHEAR STRESS}$$

$$\tau_{\max} = 4295 \text{ psi} \quad \leftarrow$$

NOTE: Since the principal stresses have opposite signs, the maximum in-plane shear is larger than the maximum out-of-plane shear stress.

Problem 8.5-24 The mountain bike from Prob. 1.2-26 is shown in the figure below. To account for impact, crashes, and other loading uncertainties, a *design load* $P = 5000$ N is used to design the seat post. The length of the seat post is $L = 254$ mm.

- Find the required diameter of the seat post if it is to be constructed using an *aluminum alloy* with the ultimate stress $\sigma_U = 550$ MPa and a factor of safety of 2.8. Consider only axial and flexural normal stresses in your design.
- Repeat part (a) if a *titanium alloy* is used instead. Assume the ultimate stress $\sigma_U = 900$ MPa and a factor of safety of 2.5.



Solution 8.5-24

$$\begin{aligned}
 Wt &= 5000 \text{ N} & N_w &= Wt \cos(15.3^\circ) = 4822.787 \text{ N} \\
 & & V_w &= Wt \sin(15.3^\circ) = 1319.365 \text{ N} \\
 & & M_w &= V_w(254 \text{ mm}) = 3.351 \times 10^5 \text{ N}\cdot\text{mm} < \text{Seat post height is } 254 \text{ mm.}
 \end{aligned}$$

$$(a) \quad \sigma_{aAl} = \frac{550 \text{ MPa}}{2.8} = 196.429 \text{ MPa} \quad \text{OR} \quad \sigma_{aTi} = \frac{900 \text{ MPa}}{2.5} = 360 \text{ MPa}$$

$$\sigma_a = \sigma_{aAl}$$

$$d^3 - \left(\frac{4N_w}{\pi\sigma_a} \right) d - \frac{32M_w}{\pi\sigma_a} = 0 \quad \text{Solve numerically for } d:$$

$$d_{Al} = 26.3 \text{ mm}$$

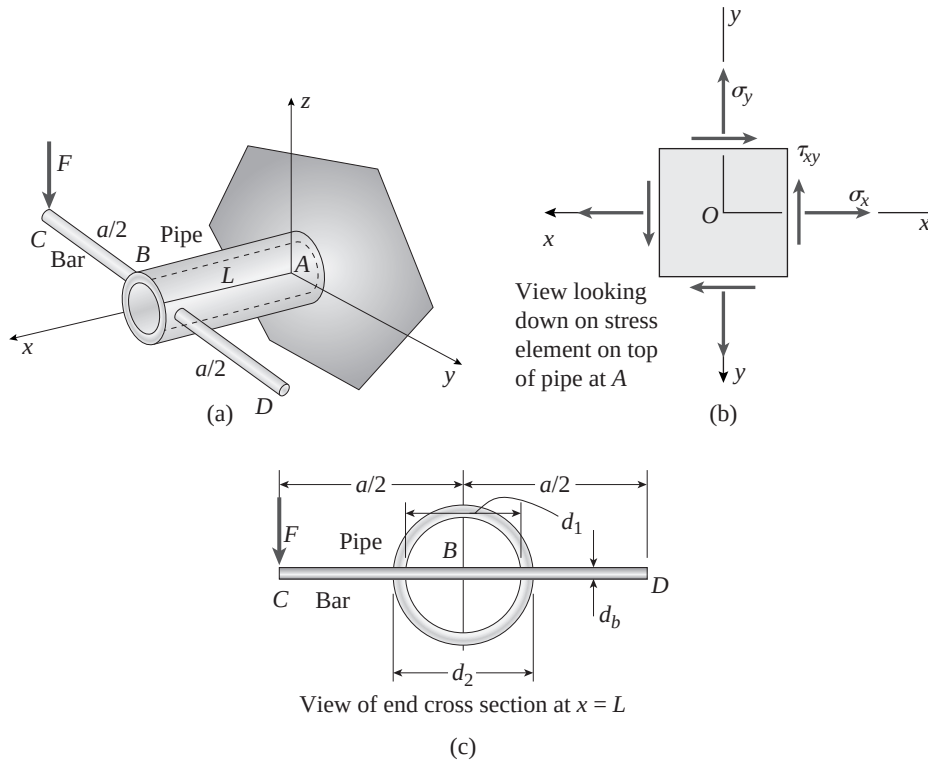
$$(b) \quad \sigma_a = \sigma_{aTi}$$

$$d^3 - \left(\frac{4N_w}{\pi\sigma_a} \right) d - \frac{32M_w}{\pi\sigma_a} = 0 \quad \text{Solve numerically for } d:$$

$$d_{Ti} = 21.4 \text{ mm}$$

Problem 8.5-25 A plumber's valve wrench is used to replace valves in plumbing fixtures. A simplified model of the wrench (see figure part a) consists of pipe AB (length L , outer diameter d_2 , inner diameter d_1) which is fixed at A and has holes of diameter d_b on either side of the pipe at B . A solid cylindrical bar CBD (length a , diameter d_b) is inserted into the holes at B and *only one* force $F = 240$ N is applied in the $-z$ direction at C to loosen the fixture valve at A (see figure part c). Let $G = 11,800$ ksi, $\nu = 0.30$, $L = 4$ in., $a = 4.5$ in., $d_2 = 1.25$ in., $d_1 = 1$ in., $d_b = 0.25$ in.

Find the state of plane stress on the top of the pipe near A (at coordinates $x = 0$, $y = 0$, $z = d_2/2$) and show all stresses on a plane stress element (see figure part b). **Compute** the principal stresses and maximum shear stress, and **show** on properly rotated stress elements.

**Solution 8.5-25 Plumbers wrench**

$$a = 4.5 \text{ in.} \quad L = 4 \text{ in.} \quad G = 11,800 \text{ ksi} \quad d_2 = 1.25 \text{ in.} \quad d_1 = 1 \text{ in.} \quad F = 55 \text{ lb}$$

$$\text{STATE OF PLANE STRESS AT ELEMENT AT A} \quad I_p = \frac{\pi}{32}(d_2^4 - d_1^4) = 0.14151 \text{ in.}^4 \quad T = F \frac{a}{2} = 123.75 \text{ lb-in.}$$

$$\sigma_y = 0 \quad \sigma_x = \frac{(FL) \frac{d_2}{2}}{\frac{I_2}{2}} = 1943 \text{ psi} \quad \tau_{xy} = \frac{-T \frac{d_2}{2}}{I_p} = -547 \text{ psi}$$

PRINCIPAL STRESSES AND $\tau_{\max}$

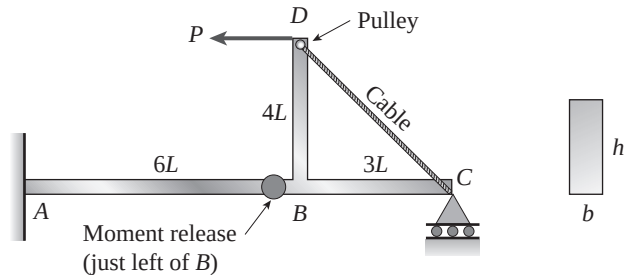
$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 2087 \text{ psi}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = -143.2 \text{ psi}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 1115 \text{ psi}$$

Problem 8.5-26 A compound beam $ABCD$ has a cable with force P anchored at C . The cable passes over a pulley at D and force P acts in the $-x$ direction. There is a moment release just left of B . Neglect the self-weight of the beam and cable. Cable force $P = 450$ N and dimension variable $L = 0.25$ m. The beam is of rectangular cross section, $b = 20$ mm, $h = 50$ mm.

- Calculate the maximum normal stresses and maximum in-plane shear stress on the bottom surface of the beam at support A .
- Repeat part (a) for a plane stress element located at midheight of the beam at A .
- If the maximum tensile stress and maximum in-plane shear stress at point A are limited to 90 MPa and 42 MPa, respectively, what is the largest permissible value of the cable force P ?



Solution 8.5-26

REACTIONS AT SUPPORT A

$$H_A = P \quad V_A = \frac{4}{3}P \quad M_A = 8PL$$

BEAM CROSS-SECTIONAL PROPERTIES

$$A = bh \quad I = \frac{bh^3}{12}$$

- (a) MAXIMUM STRESSES ON BOTTOM SURFACE OF BEAM AT A

$$\sigma_x = \frac{-H_A}{A} - \frac{M_A \left(\frac{h}{2} \right)}{I} = -108.45 \text{ MPa} \quad \sigma_y = 0 \quad \tau_{xy} = 0$$

$$\boxed{\sigma_1 = 0} \quad \boxed{\sigma_2 = \sigma_x = -108.4 \text{ MPa}} \quad \boxed{\tau_{\max} = \frac{\sigma_x}{2} = -54.2 \text{ MPa}}$$

- (b) MAXIMUM STRESSES AT MIDHEIGHT OF BEAM AT A

$$\sigma_x = \frac{-H_A}{A} = -0.45 \text{ MPa} \quad \sigma_y = 0 \quad \tau_{xy} = \frac{-3}{2} \frac{V_A}{A} = -0.9 \text{ MPa}$$

$$\boxed{\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 0.703 \text{ MPa}}$$

$$\boxed{\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = -1.153 \text{ MPa}}$$

$$\boxed{\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 0.928 \text{ MPa}}$$

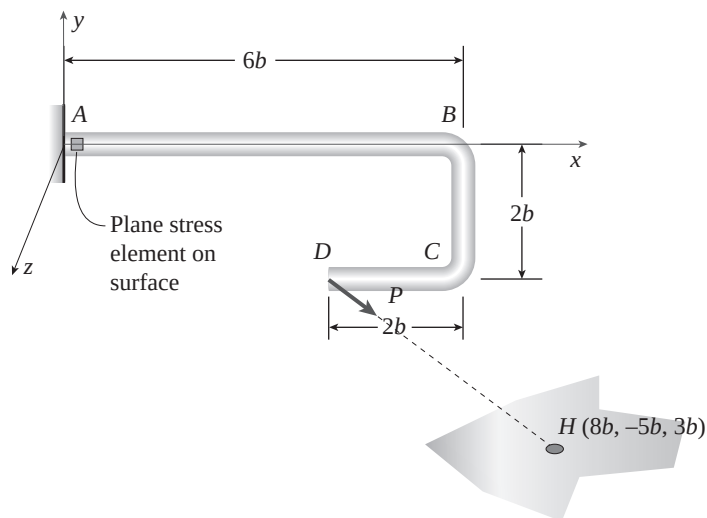
(c) MAXIMUM PERMISSIBLE VALUE OF CABLE FORCE P $\sigma_a = 90 \text{ MPa}$ $\tau_a = 42 \text{ MPa}$

$P_{\max}$ is controlled by $\sigma_x = \sigma_a$ and $\tau_{\max}$ at bottom of beam at A.

$$\sigma_x = \frac{-P}{bh} - \frac{(8PL)\left(\frac{h}{2}\right)}{\left(\frac{bh^3}{12}\right)} = -\frac{P}{bh} - \frac{48LP}{bh^2} \quad \text{so} \quad P_{\max 1} = \frac{\sigma_a}{\left(\frac{1}{bh} + \frac{48L}{bh^2}\right)} = 373 \text{ N}$$

$$\tau_{\max} = \frac{\sigma_x}{2} = \frac{P}{2bh} - \frac{24LP}{bh^2} \quad \text{so} \quad P_{\max 2} = \frac{\tau_a}{\left(\frac{1}{2bh} + \frac{24L}{bh^2}\right)} = 349 \text{ N}$$

Problem 8.5-27 A steel hanger bracket $ABCD$ has solid circular cross section with diameter $d = 2$ in. Dimension variable $b = 6$ in. (see figure). Load $P = 1200$ lb is applied at D along a line DH ; the coordinates of point H are $(8b, -5b, 3b)$. Find normal and shear stresses on a plane stress element on the surface of the bracket at A . Then find the principal stresses and maximum shear stress. Show each stress state on properly rotated elements.



Solution 8.5-27

NUMERICAL DATA

$$d = 2 \text{ in.} \quad b = 6 \text{ in.} \quad P = 1200 \text{ lb} \quad A = \frac{\pi d^2}{4} = 3.142 \text{ in.}^2 \quad I = \frac{\pi d^4}{64} = 0.785 \text{ in.}^4 \quad I_p = 2I = 1.571 \text{ in.}^4$$

POSITION, UNIT AND LOAD VECTORS

$$r_{DH} = \begin{bmatrix} 8b - 4b \\ -5b - (-2b) \\ 3b \end{bmatrix} = \begin{pmatrix} 24 \text{ in.} \\ -18 \text{ in.} \\ 18 \text{ in.} \end{pmatrix} \quad e_{DH} = \frac{r_{DH}}{|r_{DH}|} = \begin{pmatrix} 0.686 \\ -0.514 \\ 0.514 \end{pmatrix} \quad P_D = P e_{DH} = \begin{pmatrix} 823.193 \\ -617.395 \\ 617.395 \end{pmatrix} \text{ lb}$$

$$r_{AD} = \begin{pmatrix} 4b \\ -2b \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \text{ ft}$$

RESULTANT LOAD AND MOMENT VECTORS AT A

$$R_A = P_D = \begin{pmatrix} 823.193 \\ -617.395 \\ 617.395 \end{pmatrix} \text{ lb} \quad M_A = r_{AD} \times P_D = \begin{pmatrix} -617.395 \\ -1234.79 \\ -411.597 \end{pmatrix} \text{ lb-ft}$$

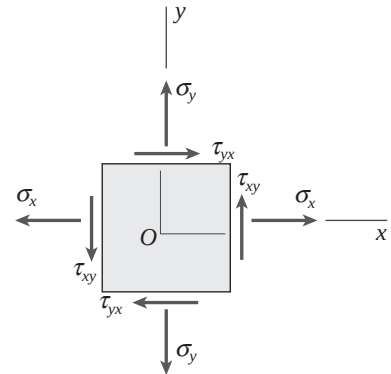
PLANE STRESS STATE AT A

$$\sigma_x = \frac{R_{A1}}{A} + \frac{M_{A2} \left(\frac{d}{2} \right)}{I} = -18.604 \text{ ksi} \quad \boxed{\sigma_x = -18.6 \text{ ksi}}$$

$$\boxed{\sigma_y = 0}$$

$$\tau_{xy} = \frac{4 R_{A2}}{3 A} + \frac{(-M_{A1}) \left(\frac{d}{2} \right)}{I_p} = 4.45 \text{ ksi}$$

$$\boxed{\tau_{xy} = 4.45 \text{ ksi}}$$



PRINCIPAL STRESSES AND MAXIMUM SHEAR STRESS

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 1.012 \text{ ksi}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = -19.62 \text{ ksi}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = 10.31 \text{ ksi}$$

9

Deflections of Beams

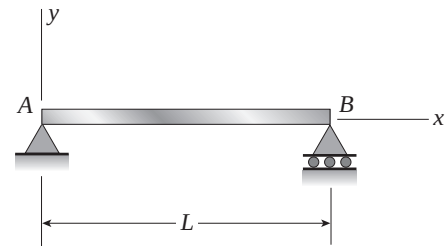
Differential Equations of the Deflection Curve

The beams described in the problems for Section 9.2 have constant flexural rigidity EI .

Problem 9.2-1 The deflection curve for a simple beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 x}{360EI} (7L^4 - 10L^2 x^2 + 3x^4)$$

Describe the load acting on the beam.



Probs. 9.2-1 and 9.2-2

Solution 9.2-1 Simple beam

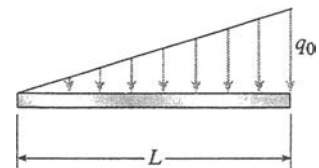
$$v = -\frac{q_0 x}{360EI} (7L^4 - 10L^2 x^2 + 3x^4)$$

Take four consecutive derivatives and obtain:

$$v'''' = -\frac{q_0}{EI}$$

$$\text{From Eq. (9-14c): } q = -EIv'''' = \frac{q_0 x}{L} \quad \leftarrow$$

The load is a downward triangular load of maximum intensity q_0 . $\leftarrow$



Problem 9.2-2 The deflection curve for a simple beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin \frac{\pi x}{L}$$

- (a) Describe the load acting on the beam.
 (b) Determine the reactions R_A and R_B at the supports.
 (c) Determine the maximum bending moment $M_{\max}$.

Solution 9.2-2 Simple beam

$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin \frac{\pi x}{L}$$

$$v' = -\frac{q_0 L^3}{\pi^3 EI} \cos \frac{\pi x}{L}$$

$$v'' = \frac{q_0 L^2}{\pi^2 EI} \sin \frac{\pi x}{L}$$

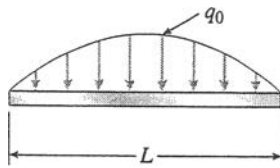
$$v''' = \frac{q_0 L}{\pi EI} \cos \frac{\pi x}{L}$$

$$v'''' = -\frac{q_0}{EI} \sin \frac{\pi x}{L}$$

(a) LOAD (EQ. 9-14c)

$$q = -EIv'''' = q_0 \sin \frac{\pi x}{L} \quad \leftarrow$$

The load has the shape of a sine curve, acts downward, and has maximum intensity q_0 . $\leftarrow$



(b) REACTIONS (EQ. 9-14b)

$$V = EIv''' = \frac{q_0 L}{\pi} \cos \frac{\pi x}{L}$$

$$\text{At } x = 0: V = R_A = \frac{q_0 L}{\pi} \quad \leftarrow$$

$$\text{At } x = L: V = -R_B = -\frac{q_0 L}{\pi};$$

$$R_B = \frac{q_0 L}{\pi} \quad \leftarrow$$

(c) MAXIMUM BENDING MOMENT (EQ. 9-14a)

$$M = EIv'' = \frac{q_0 L^2}{\pi^2} \sin \frac{\pi x}{L}$$

$$\text{For maximum moment, } x = \frac{L}{2};$$

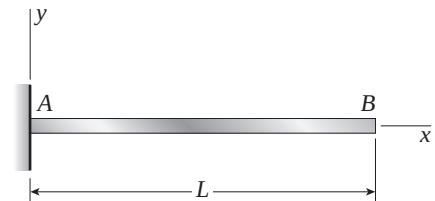
$$M_{\max} = \frac{q_0 L^2}{\pi^2} \quad \leftarrow$$

Problem 9.2-3 The deflection curve for a cantilever beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 x^2}{120LEI} (10L^3 - 10L^2x + 5Lx^2 - x^3)$$

Describe the load acting on the beam.

Probs. 9.2-3 and 9.2-4



Solution 9.2-3 Cantilever beam

$$v = -\frac{q_0 x^2}{120LEI} (10L^3 - 10L^2x + 5Lx^2 - x^3)$$

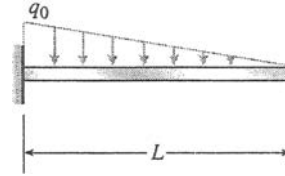
Take four consecutive derivatives and obtain:

$$v'''' = -\frac{q_0}{LEI} (L - x)$$

From Eq. (9-14c):

$$q = -EIv'''' = q_0 \left(1 - \frac{x}{L} \right) \quad \leftarrow$$

The load is a downward triangular load of maximum intensity q_0 . $\leftarrow$



Problem 9.2-4 The deflection curve for a cantilever beam AB (see figure) is given by the following equation:

$$v = -\frac{q_0 x^2}{360L^2EI} (45L^4 - 40L^3x + 15L^2x^2 - x^4)$$

- (a) Describe the load acting on the beam.
 (b) Determine the reactions R_A and M_A at the support.

Solution 9.2-4 Cantilever beam

$$v = -\frac{q_0 x^2}{360L^2EI} (45L^4 - 40L^3x + 15L^2x^2 - x^4)$$

$$v'' = -\frac{q_0}{60L^2EI} (15L^4x - 20L^3x^2 + 10L^2x^3 - x^5)$$

$$v'' = -\frac{q_0}{12L^2EI} (3L^4 - 8L^3x + 6L^2x^2 - x^4)$$

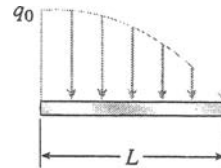
$$v''' = -\frac{q_0}{3L^2EI} (-2L^3 + 3L^2x - x^3)$$

$$v'''' = -\frac{q_0}{L^2EI} (L^2 - x^2)$$

(a) LOAD (EQ. 9-14c)

$$q = -EIv'''' = q_0 \left(1 - \frac{x^2}{L^2} \right) \quad \leftarrow$$

The load is a downward parabolic load of maximum intensity q_0 . $\leftarrow$



(b) REACTIONS R_A AND M_A (EQ. 9-14b AND EQ. 9-14a)

$$V = EIv''' = -\frac{q_0}{3L^2} (-2L^3 + 3L^2x - x^3)$$

$$\text{At } x = 0: V = R_A = \frac{2q_0L}{3} \quad \leftarrow$$

$$M = EIv'' = -\frac{q_0}{12L^2} (3L^4 - 8L^3x + 6L^2x^2 - x^4)$$

$$\text{At } x = 0: M = M_A = -\frac{q_0L^2}{4} \quad \leftarrow$$

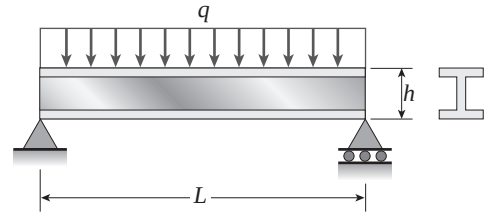
Note: Reaction R_A is positive upward.
 Reaction M_A is positive clockwise (minus means M_A is counterclockwise).

Deflection Formulas

Problems 9.3-1 through 9.3-7 require the calculation of deflections using the formulas derived in Examples 9-1, 9-2, and 9-3. All beams have constant flexural rigidity EI .

Problem 9.3-1 A wide-flange beam (W 12 $\times$ 35) supports a uniform load on a simple span of length $L = 14$ ft (see figure).

Calculate the maximum deflection $\delta_{\max}$ at the midpoint and the angles of rotation θ at the supports if $q = 1.8$ k/ft and $E = 30 \times 10^6$ psi. Use the formulas of Example 9-1.



Probs. 9.3-1 through 9.3-3

Solution 9.3-1 Simple beam (uniform load)

W 12 $\times$ 35 $L = 14$ ft = 168 in.

$q = 1.8$ k/ft = 150 lb/in. $E = 30 \times 10^6$ psi

$I = 285$ in.⁴

MAXIMUM DEFLECTION (EQ. 9-24)

$$\delta_{\max} = \frac{5qL^4}{384EI} = \frac{5(150 \text{ lb/in.})(168 \text{ in.})^4}{384(30 \times 10^6 \text{ psi})(285 \text{ in.}^4)} = 0.182 \text{ in.} \quad \leftarrow$$

ANGLE OF ROTATION AT THE SUPPORTS
(EQS. 9-25 AND 9-26)

$$\begin{aligned} \theta = \theta_A = \theta_B &= \frac{qL^3}{24EI} \\ &= \frac{(150 \text{ lb/in.})(168 \text{ in.})^3}{24(30 \times 10^6 \text{ psi})(285 \text{ in.}^4)} \\ &= 0.003466 \text{ rad} = 0.199^\circ \quad \leftarrow \end{aligned}$$

Problem 9.3-2 A uniformly loaded steel wide-flange beam with simple supports (see figure) has a downward deflection of 10 mm at the midpoint and angles of rotation equal to 0.01 radians at the ends.

Calculate the height h of the beam if the maximum bending stress is 90 MPa and the modulus of elasticity is 200 GPa. (Hint: Use the formulas of Example 9-1.)

Solution 9.3-2 Simple beam (uniform load)

$\delta = \delta_{\max} = 10$ mm $\theta = \theta_A = \theta_B = 0.01$ rad

$\sigma = \sigma_{\max} = 90$ MPa $E = 200$ GPa

Calculate the height h of the beam.

$$\text{Eq. (9-24): } \delta = \delta_{\max} = \frac{5qL^4}{384EI} \text{ or } q = \frac{384EI\delta}{5L^4} \quad (1)$$

$$\text{Eq. (9-25): } \theta = \theta_A = \frac{qL^3}{24EI} \text{ or } q = \frac{24EI\theta}{L^3} \quad (2)$$

$$\text{Equate (1) and (2) and solve for } L: L = \frac{16\delta}{5\theta} \quad (3)$$

$$\text{Flexure formula: } \sigma = \frac{Mc}{I} = \frac{Mh}{2I}$$

Maximum bending moment:

$$M = \frac{qL^2}{8} \quad \therefore \sigma = \frac{qL^2h}{16I} \quad (4)$$

$$\text{Solve Eq. (4) for } h: h = \frac{16I\sigma}{qL^2} \quad (5)$$

Substitute for q from Eq. (2) and for L from Eq. (3):

$$h = \frac{32\sigma\delta}{15E\theta^2} \quad \leftarrow$$

Substitute numerical values:

$$h = \frac{32(90 \text{ MPa})(10 \text{ mm})}{15(200 \text{ GPa})(0.01 \text{ rad})^2} = 96 \text{ mm} \quad \leftarrow$$

Problem 9.3-3 What is the span length L of a uniformly loaded simple beam of wide-flange cross section (see figure) if the maximum bending stress is 12,000 psi, the maximum deflection is 0.1 in., the height of the beam is 12 in., and the modulus of elasticity is 30×10^6 psi? (Use the formulas of Example 9-1.)

Solution 9.3-3 Simple beam (uniform load)

$$\sigma = \sigma_{\max} = 12,000 \text{ psi} \quad \delta = \delta_{\max} = 0.1 \text{ in.}$$

$$h = 12 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$$

Calculate the span length L .

$$\text{Eq. (9-18): } \delta = \delta_{\max} = \frac{5qL^4}{384EI} \text{ or } q = \frac{384EI\delta}{5L^4} \quad (1)$$

$$\text{Flexure formula: } \sigma = \frac{Mc}{I} = \frac{Mh}{2I}$$

Maximum bending moment:

$$M = \frac{qL^2}{8} \quad \therefore \sigma = \frac{qL^2h}{16I} \quad (2)$$

$$\text{Solve Eq. (2) for } q: q = \frac{16I\sigma}{L^2h} \quad (3)$$

Equate Eqs. (1) and (2) and solve for L :

$$L^2 = \frac{24 Eh\delta}{5\sigma} \quad L = \sqrt{\frac{24 Eh\delta}{5\sigma}} \quad \leftarrow$$

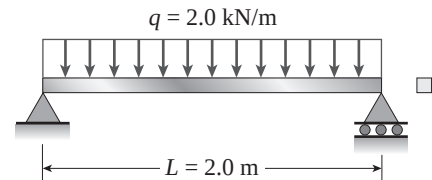
Substitute numerical values:

$$L^2 = \frac{24(30 \times 10^6 \text{ psi})(12 \text{ in.})(0.1 \text{ in.})}{5(12,000 \text{ psi})} = 14,400 \text{ in.}^2$$

$$L = 120 \text{ in.} = 10 \text{ ft} \quad \leftarrow$$

Problem 9.3-4 Calculate the maximum deflection $\delta_{\max}$ of a uniformly loaded simple beam (see figure) if the span length $L = 2.0$ m, the intensity of the uniform load $q = 2.0$ kN/m, and the maximum bending stress $\sigma = 60$ MPa.

The cross section of the beam is square, and the material is aluminum having modulus of elasticity $E = 70$ GPa. (Use the formulas of Example 9-1.)



Solution 9.3-4 Simple beam (uniform load)

$$L = 2.0 \text{ m} \quad q = 2.0 \text{ kN/m}$$

$$\sigma = \sigma_{\max} = 60 \text{ MPa} \quad E = 70 \text{ GPa}$$

CROSS SECTION (square; b = width)

$$I = \frac{b^4}{12} \quad S = \frac{b^3}{6}$$

$$\text{Maximum deflection (Eq. 9-24): } \delta = \frac{5qL^4}{384 EI} \quad (1)$$

$$\text{Substitute for } I: \delta = \frac{5qL^4}{32 Eb^4} \quad (2)$$

$$\text{Flexure formula with } M = \frac{qL^2}{8}: \sigma = \frac{M}{S} = \frac{qL^2}{8S}$$

$$\text{Substitute for } S: \sigma = \frac{3qL^2}{4b^3} \quad (3)$$

$$\text{Solve for } b^3: b^3 = \frac{3qL^2}{4\sigma} \quad (4)$$

$$\text{Substitute } b \text{ into Eq. (2): } \delta_{\max} = \frac{5L\sigma}{24E} \left(\frac{4L\sigma}{3q} \right)^{1/3} \quad \leftarrow$$

(The term in parentheses is nondimensional.)

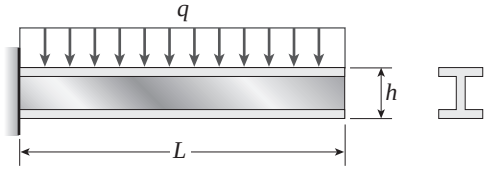
Substitute numerical values:

$$\frac{5L\sigma}{24E} = \frac{5(2.0 \text{ m})(60 \text{ MPa})}{24(70 \text{ GPa})} = \frac{1}{2800} \text{ m} = \frac{1}{2.8} \text{ mm}$$

$$\left(\frac{4L\sigma}{3q} \right)^{1/3} = \left[\frac{4(2.0 \text{ m})(60 \text{ MPa})}{3(2000 \text{ N/m})} \right]^{1/3} = 10(80)^{1/3}$$

$$\delta_{\max} = \frac{10(80)^{1/3}}{2.8} \text{ mm} = 15.4 \text{ mm} \quad \leftarrow$$

Problem 9.3-5 A cantilever beam with a uniform load (see figure) has a height h equal to $1/8$ of the length L . The beam is a steel wideflange section with $E = 28 \times 10^6$ psi and an allowable bending stress of 17,500 psi in both tension and compression. Calculate the ratio δ/L of the deflection at the free end to the length, assuming that the beam carries the maximum allowable load. (Use the formulas of Example 9-2.)



Solution 9.3-5 Cantilever beam (uniform load)

$$\frac{h}{L} = \frac{1}{8} \quad E = 28 \times 10^6 \text{ psi} \quad \sigma = 17,500 \text{ psi}$$

Calculate the ratio δ/L .

$$\text{Maximum deflection (Eq. 9-32): } \delta_{\max} = \frac{qL^4}{8EI} \quad (1)$$

$$\therefore \frac{\delta}{L} = \frac{qL^3}{8EI} \quad (2)$$

$$\text{Flexure formula with } M = \frac{qL^2}{2}:$$

$$\sigma = \frac{Mc}{I} = \left(\frac{qL^2}{2} \right) \left(\frac{h}{2I} \right) = \frac{qL^2 h}{4I}$$

Solve for q :

$$q = \frac{4I\sigma}{L^2 h} \quad (3)$$

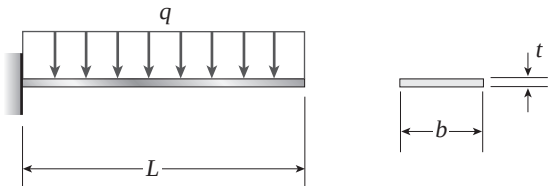
Substitute q from Eq. (3) into Eq. (2):

$$\frac{\delta}{L} = \frac{\sigma}{2E} \left(\frac{L}{h} \right) \quad \leftarrow$$

Substitute numerical values:

$$\frac{\delta}{L} = \frac{17,500 \text{ psi}}{2(28 \times 10^6 \text{ psi})} (8) = \frac{1}{400} \quad \leftarrow$$

Problem 9.3-6 A gold-alloy microbeam attached to a silicon wafer behaves like a cantilever beam subjected to a uniform load (see figure). The beam has length $L = 27.5 \mu\text{m}$ and rectangular cross section of width $b = 4.0 \mu\text{m}$ and thickness $t = 0.88 \mu\text{m}$. The total load on the beam is $17.2 \mu\text{N}$. If the deflection at the end of the beam is $2.46 \mu\text{m}$, what is the modulus of elasticity E_g of the gold alloy? (Use the formulas of Example 9-2.)



Solution 9.3-6 Gold-alloy microbeam

Cantilever beam with a uniform load.

$$L = 27.5 \mu\text{m} \quad b = 4.0 \mu\text{m} \quad t = 0.88 \mu\text{m}$$

$$qL = 17.2 \mu\text{N} \quad \delta_{\max} = 2.46 \mu\text{m}$$

Determine E_g .

$$\text{Eq. (9-32): } \delta = \frac{qL^4}{8E_g I} \text{ or } E_g = \frac{qL^4}{8I\delta_{\max}}$$

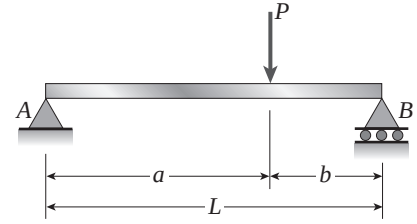
$$I = \frac{bt^3}{12} \quad E_g = \frac{3qL^4}{2bt^3\delta_{\max}} \quad \leftarrow$$

Substitute numerical values:

$$\begin{aligned} E_g &= \frac{3(17.2 \mu\text{N})(27.5 \mu\text{m})^3}{2(4.0 \mu\text{m})(0.88 \mu\text{m})^3(2.46 \mu\text{m})} \\ &= 80.02 \times 10^9 \text{ N/m}^2 \quad \text{or} \quad E_g = 80.0 \text{ GPa} \quad \leftarrow \end{aligned}$$

Problem 9.3-7 Obtain a formula for the ratio $\delta_c/\delta_{\max}$ of the deflection at the midpoint to the maximum deflection for a simple beam supporting a concentrated load P (see figure).

From the formula, plot a graph of $\delta_c/\delta_{\max}$ versus the ratio a/L that defines the position of the load ($0.5 < a/L < 1$). What conclusion do you draw from the graph? (Use the formulas of Example 9-3.)



Solution 9.3-7 Simple beam (concentrated load)

$$\text{Eq. (9-41): } \delta_c = \frac{Pb(3L^2 - 4b^2)}{48EI} \quad (a \geq b)$$

$$\text{Eq. (9-40): } \delta_{\max} = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}LEI} \quad (a \geq b)$$

$$\frac{\delta_c}{\delta_{\max}} = \frac{(3\sqrt{3}L)(3L^2 - 4b^2)}{16(L^2 - b^2)^{3/2}} \quad (a \geq b)$$

Replace the distance b by the distance a by substituting $L - a$ for b :

$$\frac{\delta_c}{\delta_{\max}} = \frac{(3\sqrt{3}L)(-L^2 + 8aL - 4a^2)}{16(2aL - a^2)^{3/2}}$$

Divide numerator and denominator by L^2 :

$$\frac{\delta_c}{\delta_{\max}} = \frac{(3\sqrt{3}L)\left(-1 + 8\frac{a}{L} - 4\frac{a^2}{L^2}\right)}{16L\left(2\frac{a}{L} - \frac{a^2}{L^2}\right)^{3/2}}$$

$$\frac{\delta_c}{\delta_{\max}} = \frac{(3\sqrt{3}L)\left(-1 + 8\frac{a}{L} - 4\frac{a^2}{L^2}\right)}{16\left(2\frac{a}{L} - \frac{a^2}{L^2}\right)^{3/2}} \quad \leftarrow$$

ALTERNATIVE FORM OF THE RATIO

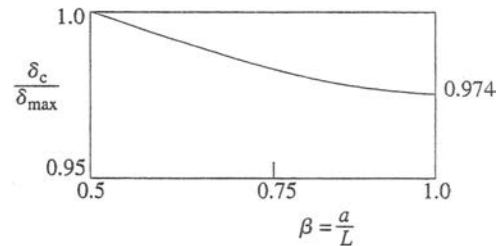
$$\text{Let } \beta = \frac{a}{L}$$

$$\frac{\delta_c}{\delta_{\max}} = \frac{(3\sqrt{3})(-1 + 8\beta - 4\beta^2)}{16(2\beta - \beta^2)^{3/2}} \quad \leftarrow$$

GRAPH OF $\delta_c/\delta_{\max}$ VERSUS $\beta = a/L$

Because $a \geq b$, the ratio β varies from 0.5 to 1.0.

β	$\frac{\delta_c}{\delta_{\max}}$
0.5	1.0
0.6	0.996
0.7	0.988
0.8	0.981
0.9	0.976
1.0	0.974

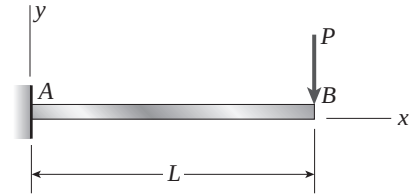


Note: The deflection δ_c at the midpoint of the beam is almost as large as the maximum deflection $\delta_{\max}$. The greatest difference is only 2.6% and occurs when the load reaches the end of the beam ($\beta = 1$).

Deflections by Integration of the Bending-Moment Equation

Problems 9.3-8 through 9.3-17 are to be solved by integrating the second-order differential equation of the deflection curve (the bending-moment equation). The origin of coordinates is at the left-hand end of each beam, and all beams have constant flexural rigidity EI .

Problem 9.3-8 Derive the equation of the deflection curve for a cantilever beam AB supporting a load P at the free end (see figure). Also, determine the deflection δ_B and angle of rotation θ_B at the free end. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-8 Cantilever beam (concentrated load)

BENDING-MOMENT EQUATION (Eq. 9-16a)

$$EIv'' = M = -P(L - x)$$

$$EIv' = -PLx + \frac{Px^2}{2} + C_1$$

$$\text{B.C. } v'(0) = 0 \quad \therefore C_1 = 0$$

$$EIv = -\frac{PLx^2}{2} + \frac{Px^3}{6} + C_2$$

$$\text{B.C. } v(0) = 0 \quad \therefore C_2 = 0$$

$$v = -\frac{Px^2}{6EI}(3L - x) \quad \leftarrow$$

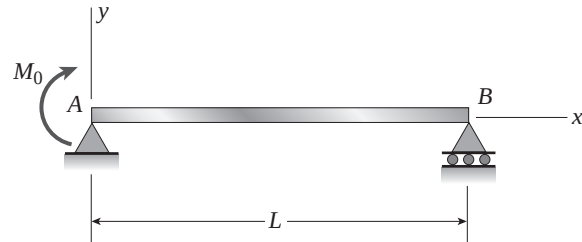
$$v' = -\frac{Px}{2EI}(2L - x)$$

$$\delta_B = -v(L) = \frac{PL^3}{3EI} \quad \leftarrow$$

$$\theta_B = -v'(L) = \frac{PL^2}{2EI} \quad \leftarrow$$

(These results agree with Case 4, Table H-1.)

Problem 9.3-9 Derive the equation of the deflection curve for a simple beam AB loaded by a couple M_0 at the left-hand support (see figure). Also, determine the maximum deflection $\delta_{\max}$. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-9 Simple beam (couple M_0)

BENDING-MOMENT EQUATION (Eq. 9-16a)

$$EIv'' = M = M_0\left(1 - \frac{x}{L}\right)$$

$$EIv' = M_0\left(x - \frac{x^2}{2L}\right) + C_1$$

$$EIv = M_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + C_1x + C_2$$

$$\text{B.C. } v(0) = 0 \quad \therefore C_2 = 0$$

$$\text{B.C. } v(L) = 0 \quad \therefore C_1 = -\frac{M_0L}{3}$$

$$v = -\frac{M_0x}{6EI}(2L^2 - 3Lx + x^2) \quad \leftarrow$$

MAXIMUM DEFLECTION

$$v' = -\frac{M_0}{6LEI}(2L^2 - 6Lx + 3x^2)$$

Set $v' = 0$ and solve for x :

$$x_1 = L\left(1 - \frac{\sqrt{3}}{3}\right) \quad \leftarrow$$

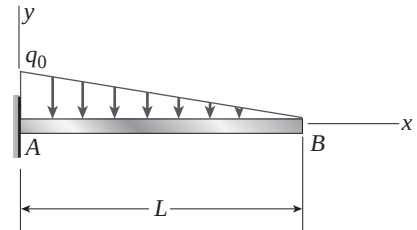
Substitute x_1 into the equation for v :

$$\begin{aligned}\delta_{\max} &= -(v)_{x=x_1} \\ &= \frac{M_0 L^2}{9\sqrt{3}EI} \quad \leftarrow\end{aligned}$$

(These results agree with Case 7, Table H-2.)

Problem 9.3-10 A cantilever beam AB supporting a triangularly distributed load of maximum intensity q_0 is shown in the figure.

Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (Note: Use the second-order differential equation of the deflection curve.)

**Solution 9.3-10 Cantilever beam (triangular load)**

BENDING-MOMENT EQUATION (EQ. 9-16a)

$$EIv'' = M = -\frac{q_0}{6L}(L-x)^3$$

$$EIv' = \frac{q_0}{24L}(L-x)^4 + C_1$$

$$\text{B.C. } v'(0) = 0 \quad \therefore C_1 = -\frac{q_0 L^3}{24}$$

$$EIv = -\frac{q_0}{120L}(L-x)^5 - \frac{q_0 L^3 x}{24} + C_2$$

$$\text{B.C. } v(0) = 0 \quad \therefore C_2 = \frac{q_0 L^4}{120}$$

$$v = -\frac{q_0 x^2}{120LEI}(10L^3 - 10L^2x + 5Lx^2 - x^3) \quad \leftarrow$$

$$v' = -\frac{q_0 x}{24LEI}(4L^3 - 6L^2x + 4Lx^2 - x^3)$$

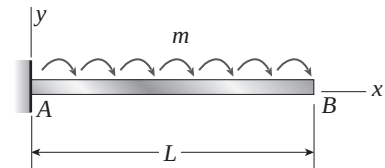
$$\delta_B = -v(L) = \frac{q_0 L^4}{30EI} \quad \leftarrow$$

$$\theta_B = -v'(L) = \frac{q_0 L^3}{24EI} \quad \leftarrow$$

(These results agree with Case 8, Table H-1.)

Problem 9.3-11 A cantilever beam AB is acted upon by a uniformly distributed moment (bending moment, not torque) of intensity m per unit distance along the axis of the beam (see figure).

Derive the equation of the deflection curve and then obtain formulas for the deflection δ_B and angle of rotation θ_B at the free end. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-11 Cantilever beam (distributed moment)

BENDING-MOMENT EQUATION (EQ. 9-16a)

$$EIv'' = M = -m(L - x)$$

$$EIv' = -m\left(Lx - \frac{x^2}{2}\right) + C_1$$

$$\text{B.C. } v'(0) = 0 \quad \therefore C_1 = 0$$

$$EIv = -m\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2$$

$$\text{B.C. } v(0) = 0 \quad \therefore C_2 = 0$$

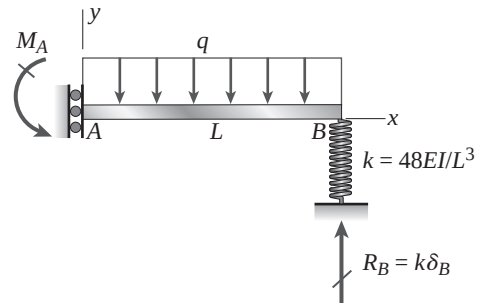
$$v = -\frac{mx^2}{6EI}(3L - x) \quad \leftarrow$$

$$v' = -\frac{mx}{2EI}(2L - x)$$

$$\delta_B = -v(L) = \frac{mL^3}{3EI} \quad \leftarrow$$

$$\theta_B = -v'(L) = \frac{mL^2}{2EI} \quad \leftarrow$$

Problem 9.3-12 The beam shown in the figure has a guided support at A and a spring support at B. The guided support permits vertical movement but no rotation. Derive the equation of the deflection curve and determine the deflection δ_B at end B due to the uniform load of intensity q . (Note: Use the second-order differential equation of the deflection curve.)

**Solution 9.3-12**

BENDING-MOMENT EQUATION

$$EIv'' = M(x) = \frac{qL^2}{2} - \frac{qx^2}{2}$$

$$EIv' = \frac{qL^2x}{2} - \frac{qx^3}{24} + C_1$$

$$EIv = \frac{qL^2x^2}{2} - \frac{qx^4}{24} + C_1x + C_2$$

$$\text{B.C. } v'(0) = 0 \quad C_1 = 0$$

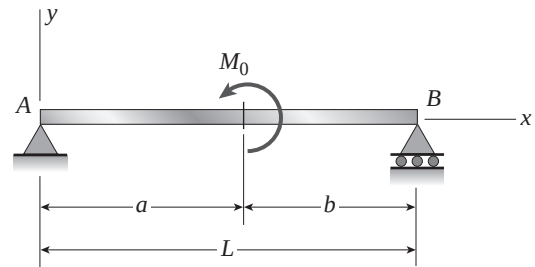
$$\text{B.C. } v(L) = \frac{qL}{k} = -\frac{qL^4}{48EI} \quad C_2 = -\frac{11qL^4}{48}$$

$$v(x) = -\frac{q}{48EI}(2x^4 - 12x^2L^2 + 11L^4) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{qL^4}{48EI} \quad \leftarrow$$

Note that $R_B = k\delta_B = qL$ which agrees with $\sum F_{\text{vert}} = 0$

Problem 9.3-13 Derive the equations of the deflection curve for a simple beam AB loaded by a couple M_0 acting at distance a from the left-hand support (see figure). Also, determine the deflection δ_0 at the point where the load is applied. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-13 Simple beam (couple M_0)

BENDING-MOMENT EQUATION (EQ. 9-16a)

$$EIv'' = M = \frac{M_0 x}{L} \quad (0 \leq x \leq a)$$

$$EIv' = \frac{M_0 x^2}{2L} + C_1 \quad (0 \leq x \leq a)$$

$$EIv'' = M = -\frac{M_0}{L}(L - x) \quad (a \leq x \leq L)$$

$$EIv' = -\frac{M_0}{L}\left(Lx - \frac{x^2}{2}\right) + C_2 \quad (a \leq x \leq L)$$

B.C. 1 $(v')_{\text{Left}} = (v')_{\text{Right}}$ at $x = a$

$$\therefore C_2 = C_1 + M_0 a$$

$$EIv = \frac{M_0 x^3}{6L} + C_1 x + C_3 \quad (0 \leq x \leq a)$$

B.C. 2 $v(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -\frac{M_0 x^2}{2} + \frac{M_0 x^3}{6L} + C_1 x + M_0 a x + C_4 \quad (a \leq x \leq L)$$

B.C. 3 $v(L) = 0 \quad \therefore C_4 = -M_0 L\left(a - \frac{L}{3}\right) - C_1 L$

B.C. 4 $(v)_{\text{Left}} = (v)_{\text{Right}}$ at $x = a$

$$\therefore C_4 = -\frac{M_0 a^2}{2}$$

$$C_1 = \frac{M_0}{6L}(2L^2 - 6aL + 3a^2)$$

$$v = -\frac{M_0 x}{6LEI}(6aL - 3a^2 - 2L^2 - x^2) \quad (0 \leq x \leq a) \quad \leftarrow$$

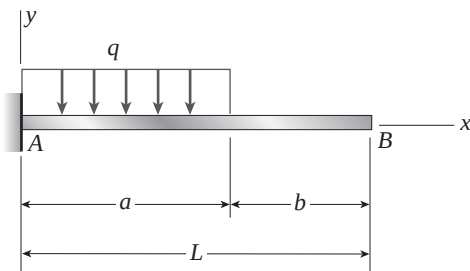
$$v = -\frac{M_0}{6LEI}(3a^2L - 3a^2x - 2L^2x + 3Lx^2 - x^3) \quad (a \leq x \leq L) \quad \leftarrow$$

$$\delta_0 = -v(a) = \frac{M_0 a(L - a)(2a - L)}{3LEI}$$

$$= \frac{M_0 ab(2a - L)}{3LEI} \quad \leftarrow$$

Note: δ_0 is positive downward. The preceding results agree with Case 9, Table H-2.

Problem 9.3-14 Derive the equations of the deflection curve for a cantilever beam AB carrying a uniform load of intensity q over part of the span (see figure). Also, determine the deflection δ_B at the end of the beam. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-14 Cantilever beam (partial uniform load)

BENDING-MOMENT EQUATION (EQ. 9-16a)

$$EIv'' = M = -\frac{q}{2}(a-x)^2 = -\frac{q}{2}(a^2 - 2ax + x^2) \quad (0 \leq x \leq a)$$

$$EIv' = -\frac{q}{2}\left(a^2x - ax^2 + \frac{x^3}{3}\right) + C_1 \quad (0 \leq x \leq a)$$

$$\text{B.C. 1 } v'(0) = 0 \quad \therefore C_1 = 0$$

$$EIv'' = M = 0 \quad (a \leq x \leq L)$$

$$EIv' = C_2 \quad (a \leq x \leq L)$$

$$\text{B.C. 2 } (v')_{\text{Left}} = (v')_{\text{Right}} \text{ at } x = a \quad \therefore C_2 = -\frac{qa^3}{6}$$

$$EIv = -\frac{q}{2}\left(\frac{a^2x^2}{2} - \frac{ax^3}{3} + \frac{x^4}{12}\right) + C_3 \quad (0 \leq x \leq a)$$

$$\text{B.C. 3 } v(0) = 0 \quad \therefore C_3 = 0$$

$$EIv = C_2x + C_4 = -\frac{qa^3x}{6} + C_4 \quad (a \leq x \leq L)$$

$$\text{B.C. 4 } (v)_{\text{Left}} = (v)_{\text{Right}} \text{ at } x = a \quad \therefore C_4 = \frac{qa^4}{24}$$

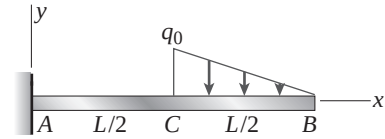
$$v = -\frac{qx^2}{24EI}(6a^2 - 4ax + x^2) \quad (0 \leq x \leq a) \quad \leftarrow$$

$$v = -\frac{qa^3}{24EI}(4x - a) \quad (a \leq x \leq L) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{qa^3}{24EI}(4L - a) \quad \leftarrow$$

(These results agree with Case 2, Table H-1.)

Problem 9.3-15 Derive the equations of the deflection curve for a cantilever beam AB supporting a distributed load of peak intensity q_0 acting over one-half of the length (see figure). Also, obtain formulas for the deflections δ_B and δ_C at points B and C, respectively. (Note: Use the second-order differential equation of the deflection curve.)

**Solution 9.3-15**

BENDING-MOMENT EQUATION

$$\text{For } 0 \leq x \leq \frac{L}{2},$$

$$EIv'' = M(x) = \frac{q_0Lx}{4} - \frac{q_0L^2}{6}$$

$$EIv' = \frac{q_0Lx^2}{8} - \frac{q_0L^2x}{6} + C_1$$

$$EIv = \frac{q_0Lx^3}{24} - \frac{q_0L^2x^2}{12} + C_1x + C_2$$

$$\text{B.C. } v'(0) = 0 \quad C_1 = 0$$

$$\text{B.C. } v(0) = 0 \quad C_2 = 0$$

$$v'\left(\frac{L}{2}\right) = -\frac{5q_0L^3}{96EI}$$

$$v(x) = \frac{q_0L}{24EI}(x^3 - 2Lx^2) \quad \leftarrow$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{q_0L^4}{64EI} \quad \leftarrow$$

$$\text{For } \frac{L}{2} \leq x \leq L,$$

$$EIv'' = M(x) = \frac{q_0Lx}{4} - \frac{q_0L^2}{6} - \frac{q_0}{L}(L-x)$$

$$\left(x - \frac{L}{2}\right)^2 - \frac{1}{2}\left[q_0 - \frac{2q_0}{L}\right]$$

$$(L-x)\left]\left(x - \frac{L}{2}\right)^2 \frac{2}{3}\right]$$

$$EIv'' = M(x) = \frac{-q_0}{3L}(-3L^2x + L^3 + 3Lx^2 - x^3)$$

$$EIv' = -\frac{q_0}{3L} \left(\frac{-3}{2}L^2x^2 + L^3x + Lx^3 - \frac{x^4}{4} \right) + C_3$$

$$EIv = -\frac{q_0}{3L} \left(\frac{-1}{2}L^2x^3 + \frac{1}{2}L^3x^2 + \frac{1}{4}Lx^4 - \frac{1}{20}x^5 \right) + C_3x + C_4$$

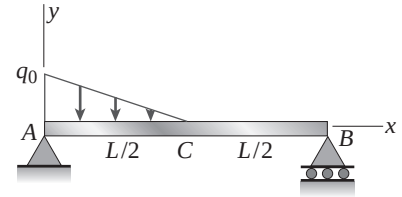
$$\text{B.C. } v'\left(\frac{L}{2}\right) = -\frac{5q_0L^3}{96EI} \quad C_3 = \frac{5}{192}q_0L^3$$

$$\text{B.C. } v\left(\frac{L}{2}\right) = -\frac{q_0L^4}{64EI} \quad C_4 = \frac{-1}{320}q_0L^4$$

$$v(x) = \frac{-q_0}{960LEI}(-160L^2x^3 + 160L^3x^2 + 80Lx^4 - 16x^5 - 25L^4x + 3L^5) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{7q_0L^4}{160EI} \quad \leftarrow$$

Problem 9.3-16 Derive the equations of the deflection curve for a simple beam AB with a distributed load of peak intensity q_0 acting over the left-hand half of the span (see figure). Also, determine the deflection δ_C at the midpoint of the beam. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-16

BENDING-MOMENT EQUATION

For $0 \leq x \leq \frac{L}{2}$,

$$EIv'' = M(x) = \frac{5q_0Lx}{24} - \frac{2q_0}{L} \left(\frac{L}{2} - x \right) \left(\frac{x^2}{2} \right) - \frac{1}{2} \left[q_0 - \frac{2q_0}{L} \left(\frac{L}{2} - x \right) \right] x \frac{2}{3}x$$

$$EIv'' = \frac{q_0}{24L} (5L^2x - 12x^2L + 8x^3)$$

$$EIv' = \frac{q_0}{24L} \left(\frac{5L^2x^2}{2} - 4x^3L + 2x^4 \right) + C_1 \quad (1)$$

$$EIv = \frac{q_0}{24L} \left(\frac{5L^2x^3}{6} - x^4L + \frac{2x^5}{5} \right) + C_1x + C_2$$

$$\text{B.C. } v(0) = 0 \quad C_2 = 0$$

$$EIv = \frac{q_0}{24L} \left(\frac{5L^2x^3}{6} - x^4L + \frac{2x^5}{5} \right) + C_1x \quad (2)$$

For $\frac{L}{2} \leq x \leq L$,

$$EIv'' = M(x) = \frac{5q_0Lx}{24} - \frac{1}{2}q_0 \frac{L}{2} \left(x - \frac{L}{6} \right)$$

$$EIv'' = \frac{Lq_0}{24} (-x + L)$$

$$EIv' = \frac{Lq_0}{24} \left(\frac{-x^2}{2} + Lx \right) + C_3 \quad (3)$$

$$EIv = \frac{Lq_0}{24} \left(\frac{-x^3}{6} + \frac{Lx^2}{2} \right) + C_3x + C_4 \quad (4)$$

$$\text{B.C. } v(L) = 0 \quad 0 = \frac{q_0L^4}{72} + C_3L + C_4 \quad (5)$$

$$\text{B.C. } v'_L\left(\frac{L}{2}\right) = v'_R\left(\frac{L}{2}\right)$$

$$\frac{1}{96}q_0L^3 + C_1 = \frac{1}{64}q_0L^3 + C_3 \quad (6)$$

$$\text{B.C. } v_L\left(\frac{L}{2}\right) = v_R\left(\frac{L}{2}\right)$$

$$\frac{13}{5760}q_0L^4 + \frac{1}{2}C_1L = \frac{5}{1152}q_0L^4 + \frac{1}{2}C_3L + C_4 \quad (7)$$

From Eqs. (5) through (7),

$$C_1 = \frac{-53}{5760}q_0L^3 \quad C_3 = \frac{-83}{5760}q_0L^3$$

$$C_4 = \frac{1}{1920}q_0L^4$$

$$\text{For } 0 \leq x \leq \frac{L}{2},$$

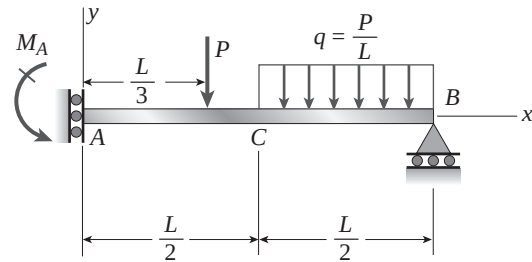
$$v(x) = \frac{q_0x}{5760EI} (200x^2L^2 - 240x^3L + 96x^4 - 53L^4) \quad \leftarrow$$

$$\text{For } \frac{L}{2} \leq x \leq L,$$

$$v(x) = \frac{-Lq_0}{5760EI} (40x^3 - 120Lx^2 + 83L^2x - 3L^3) \quad \leftarrow$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{3q_0L^4}{1280EI} \quad \leftarrow$$

Problem 9.3-17 The beam shown in the figure has a guided support at A and a roller support at B. The guided support permits vertical movement but no rotation. Derive the equation of the deflection curve and determine the deflection δ_A at end A and also δ_C at point C due to the uniform load of intensity $q = P/L$ applied over segment CB and load P at $x = L/3$. (Note: Use the second-order differential equation of the deflection curve.)



Solution 9.3-17

BENDING-MOMENT EQUATION

$$\text{For } 0 \leq x \leq \frac{L}{3}, \quad EIv'' = M(x) = \frac{19}{24}PL$$

$$EIv' = \frac{19}{24}PLx + C_1$$

$$EIv = \frac{19}{48}PLx^2 + C_1x + C_2$$

$$\text{B.C. } v'(0) = 0 \quad C_1 = 0 \quad EIv' = \frac{19}{24}PLx \quad EIv = \frac{19}{48}PLx^2 + C_2$$

$$\text{For } \frac{L}{3} \leq x \leq \frac{L}{2}, \quad EIv'' = M(x) = \frac{19}{24}PL - P\left(x - \frac{L}{3}\right)$$

$$EIv'' = M(x) = \frac{19}{24}PL - Px + \frac{PL}{3}$$

$$EIv' = \frac{19}{24}PLx - \frac{Px^2}{2} + \frac{PLx}{3} + C_3$$

$$EIv = \frac{19}{48}PLx^2 - \frac{Px^3}{6} + \frac{PLx^2}{6} + C_3x + C_4$$

$$\text{For } \frac{L}{2} \leq x \leq L, \quad EIv'' = M(x) = \frac{19}{24}PL - Px + \frac{PL}{3} - \frac{P}{L}\left(x - \frac{L}{2}\right)^2 \frac{1}{2}$$

$$EIv'' = M(x) = \frac{19}{24}PL - Px + \frac{PL}{3} - \frac{Px^2}{2L} + \frac{Px}{2} - \frac{PL}{8}$$

$$EIv' = \frac{19}{24}PLx - \frac{Px^2}{2} + \frac{PLx}{3} - \frac{Px^3}{6L} + \frac{Px^2}{4} - \frac{PLx}{8} + C_5$$

$$EIv = \frac{19}{48}PLx^2 - \frac{Px^3}{6} + \frac{PLx^2}{6} - \frac{Px^4}{24L} + \frac{Px^3}{12} - \frac{PLx^2}{16} + C_5x + C_6$$

$$\text{B.C. } v(L) = 0 \quad 0 = \frac{19}{48}PLL^2 - \frac{PL^3}{6} + \frac{PLL^2}{6} - \frac{PL^4}{24L} + \frac{PL^3}{12} - \frac{PLL^2}{16} + C_5L + C_6 \quad (1)$$

$$\text{B.C. } v'_L\left(\frac{L}{3}\right) = v'_R\left(\frac{L}{3}\right) \quad 0 = -\frac{P\left(\frac{L}{3}\right)^2}{2} + \frac{PL\left(\frac{L}{3}\right)}{3} + C_3 \quad (2)$$

$$\text{B.C. } v_L\left(\frac{L}{3}\right) = v_R\left(\frac{L}{3}\right) \quad C_2 = -\frac{P\left(\frac{L}{3}\right)^3}{6} + \frac{PL\left(\frac{L}{3}\right)^2}{6} + C_3\left(\frac{L}{3}\right) + C_4 \quad (3)$$

$$\text{B.C. } v'_L\left(\frac{L}{2}\right) = v'_R\left(\frac{L}{2}\right) \quad C_3 = -\frac{P\left(\frac{L}{2}\right)^3}{6L} + \frac{P\left(\frac{L}{2}\right)^2}{4} - \frac{PL\left(\frac{L}{2}\right)}{8} + C_5 \quad (4)$$

$$\text{B.C. } v_L(a) = v_R(a) \quad C_3\frac{L}{2} + C_4 = -\frac{P\left(\frac{L}{2}\right)^4}{24L} + \frac{P\left(\frac{L}{2}\right)^3}{12} - \frac{PL\left(\frac{L}{2}\right)^2}{16} + C_5\left(\frac{L}{2}\right) + C_6 \quad (5)$$

From Eqs. (1) through (5),

$$C_2 = \frac{-3565}{10368}PL^3 \quad C_3 = \frac{-1}{18}PL^2 \quad C_4 = \frac{-389}{1152}PL^3 \quad C_5 = \frac{-5}{144}PL^2 \quad C_6 = \frac{-49}{144}PL^3$$

$$\text{For } 0 \leq x \leq \frac{L}{3}, \quad v(x) = \frac{-PL}{10368EI}(-4104x^2 + 3565L^2) \quad \leftarrow$$

$$\text{For } \frac{L}{3} \leq x \leq \frac{L}{2}, \quad v(x) = \frac{-P}{1152EI}(-648Lx^2 + 192x^3 + 64L^2x + 389L^3) \quad \leftarrow$$

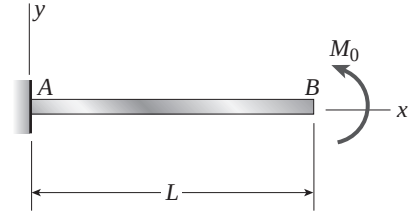
$$\text{For } \frac{L}{2} \leq x \leq L, \quad v(x) = \frac{-P}{144EI}(-72L^2x^2 + 12x^3L + 6x^4 + 5L^3x + 49L^4) \quad \leftarrow$$

$$\delta_A = -v(0) = \frac{3565PL^3}{10368EI} \quad \leftarrow \quad \delta_C = -v\left(\frac{L}{3}\right) = \frac{3109PL^3}{10368EI} \quad \leftarrow$$

Deflections by Integration of the Shear-Force and Load Equations

The beams described in the problems for Section 9.4 have constant flexural rigidity EI . Also, the origin of coordinates is at the left-hand end of each beam.

Problem 9.4-1 Derive the equation of the deflection curve for a cantilever beam AB when a couple M_0 acts counterclockwise at the free end (see figure). Also, determine the deflection δ_B and slope θ_B at the free end. Use the third-order differential equation of the deflection curve (the shear-force equation).



Solution 9.4-1 Cantilever beam (couple M_0)

SHEAR-FORCE EQUATION (EQ. 9-16b).

$$EIv''' = V = 0$$

$$EIv'' = C_1$$

B.C. 1 $M = M_0$ $EIv'' = M = M_0 = C_1$

$$EIv' = C_1x + C_2 = M_0x + C_2$$

B.C. 2 $v'(0) = 0$ $\therefore C_2 = 0$

$$EIv = \frac{M_0x^2}{2} + C_3$$

B.C. 3 $v(0) = 0$ $\therefore C_3 = 0$

$$v = \frac{M_0x^2}{2EI} \quad \leftarrow$$

$$v' = \frac{M_0x}{EI}$$

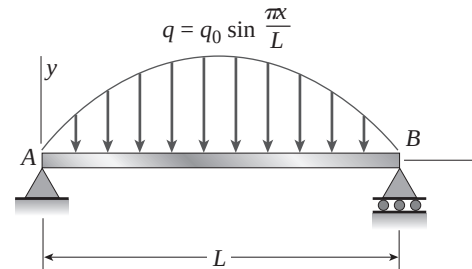
$$\delta_B = v(L) = \frac{M_0L^2}{2EI} \text{ (upward)} \quad \leftarrow$$

$$\theta_B = v'(L) = \frac{M_0L}{EI} \text{ (counterclockwise)} \quad \leftarrow$$

(These results agree with Case 6, Table H-1.)

Problem 9.4-2 A simple beam AB is subjected to a distributed load of intensity $q = q_0 \sin \pi x/L$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection $\delta_{\max}$ at the midpoint of the beam. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-2 Simple beam (sine load)

LOAD EQUATION (EQ. 9-16c).

$$EIv'''' = -q = -q_0 \sin \frac{\pi x}{L}$$

$$EIv''' = q_0 \left(\frac{L}{\pi} \right) \cos \frac{\pi x}{L} + C_1$$

$$EIv'' = q_0 \left(\frac{L}{\pi} \right)^2 \sin \frac{\pi x}{L} + C_1x + C_2$$

B.C. 1 $EIv'' = M$ $EIv''(0) = 0$ $\therefore C_2 = 0$

B.C. 2 $EIv''(L) = 0$ $\therefore C_1 = 0$

$$EIv' = -q_0 \left(\frac{L}{\pi} \right)^3 \cos \frac{\pi x}{L} + C_3$$

$$EIv = -q_0 \left(\frac{L}{\pi} \right)^4 \sin \frac{\pi x}{L} + C_3x + C_4$$

$$\text{B.C. 3 } v(0) = 0 \quad \therefore C_4 = 0$$

$$\text{B.C. 4 } v(L) = 0 \quad \therefore C_3 = 0$$

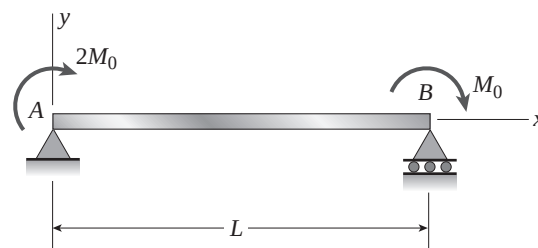
$$v = -\frac{q_0 L^4}{\pi^4 EI} \sin \frac{\pi x}{L} \quad \leftarrow$$

$$\delta_{\max} = -v\left(\frac{L}{2}\right) = \frac{q_0 L^4}{\pi^4 EI} \quad \leftarrow$$

(These results agree with Case 13, Table H-2.)

Problem 9.4-3 The simple beam AB shown in the figure has moments $2M_0$ and M_0 acting at the ends.

Derive the equation of the deflection curve, and then determine the maximum deflection $\delta_{\max}$. Use the third-order differential equation of the deflection curve (the shear-force equation).



Solution 9.4-3 Simple beam with two couples

$$\text{Reaction at support A: } R_A = \frac{3M_0}{L} \quad (\text{downward})$$

$$\text{Shear force in beam: } V = -R_A = -\frac{3M_0}{L}$$

SHEAR-FORCE EQUATION (EQ. 9-16b)

$$EIv''' = V = -\frac{3M_0}{L}$$

$$EIv'' = -\frac{3M_0 x}{L} + C_1$$

$$\text{B.C. 1 } EIv'' = M \quad EIv''(0) = 2M_0 \quad \therefore C_1 = 2M_0$$

$$EIv' = -\frac{3M_0 x^2}{2L} + 2M_0 x + C_2$$

$$EIv = -\frac{M_0 x^3}{2L} + M_0 x^2 + C_2 x + C_3$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_3 = 0$$

$$\text{B.C. 3 } v(L) = 0 \quad \therefore C_2 = -\frac{M_0 L}{2}$$

$$v = -\frac{M_0 x}{2LEI} (L^2 - 2Lx + x^2)$$

$$= -\frac{M_0 x}{2LEI} (L - x)^2 \quad \leftarrow$$

$$v' = -\frac{M_0}{2LEI} (L - x)(L - 3x)$$

MAXIMUM DEFLECTION

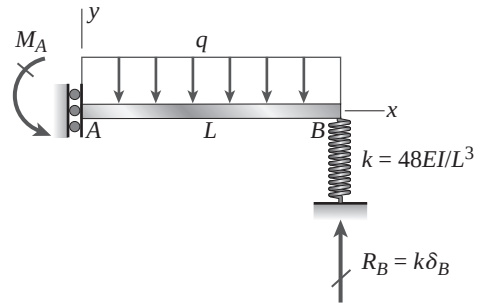
Set $v' = 0$ and solve for x :

$$x_1 = L \text{ and } x_2 = \frac{L}{3}$$

Maximum deflection occurs at $x_2 = \frac{L}{3}$.

$$\delta_{\max} = -v\left(\frac{L}{3}\right) = \frac{2M_0 L^2}{27EI} \quad (\text{downward}) \quad \leftarrow$$

Problem 9.4-4 A beam with a uniform load has a guided support at one end and spring support at the other. The spring has stiffness $k = 48EI/L^3$. Derive the equation of the deflection curve by starting with the third-order differential equation (the shear-force equation). Also, determine the angle of rotation θ_B at support B.



Solution 9.4-4

SHEAR-FORCE EQUATION

$$EIv''' = V = -qx$$

$$EIv'' = -\frac{qx^2}{2} + C_1$$

$$\text{B.C. } v''(L) = M(L) = 0 \quad C_1 = \frac{qL^2}{2}$$

$$EIv'' = \frac{qL^2}{2} - \frac{qx^2}{2}$$

$$EIv' = \frac{qL^2x}{2} - \frac{qx^3}{6} + C_2$$

$$EIv = \frac{qL^2x^2}{4} - \frac{qx^4}{24} + C_2x + C_3$$

$$\text{B.C. } v'(0) = 0 \quad C_2 = 0$$

$$\text{B.C. } v(L) = \frac{qL}{k} = -\frac{qL^4}{48EI}$$

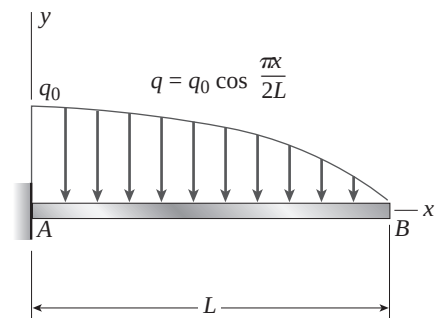
$$C_3 = -\frac{11qL^4}{48}$$

$$v(x) = -\frac{q}{48EI} (2x^4 - 12x^2L^2 + 11L^4) \quad \leftarrow$$

$$\theta_B = -v'(L) = -\frac{qL^3}{3EI} \quad (\text{counterclockwise}) \quad \leftarrow$$

Problem 9.4-5 The distributed load acting on a cantilever beam AB has an intensity q given by the expression $q_0 \cos \pi x/2L$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_B at the free end. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-5 Cantilever beam (cosine load)

LOAD EQUATION (EQ. 9-16c)

$$EIv'''' = -q = -q_0 \cos \frac{\pi x}{2L}$$

$$EIv''' = -q_0 \left(\frac{2L}{\pi} \right) \sin \frac{\pi x}{2L} + C_1$$

$$\text{B.C. 1 } EIv''' = V \quad EIv'''(L) = 0 \quad \therefore C_1 = \frac{2q_0 L}{\pi}$$

$$EIv'' = q_0 \left(\frac{2L}{\pi} \right)^2 \cos \frac{\pi x}{2L} + \frac{2q_0 Lx}{\pi} + C_2$$

$$\text{B.C. 2 } EIv'' = M \quad EIv''(L) = 0 \quad \therefore C_2 = -\frac{2q_0 L^2}{\pi}$$

$$EIv' = q_0 \left(\frac{2L}{\pi} \right)^3 \sin \frac{\pi x}{2L} + \frac{q_0 Lx^2}{\pi} - \frac{2q_0 L^2 x}{\pi} + C_3$$

$$\text{B.C. 3 } v'(0) = 0 \quad \therefore C_3 = 0$$

$$EIv = -q_0 \left(\frac{2L}{\pi} \right)^4 \cos \frac{\pi x}{2L} + \frac{q_0 Lx^3}{3\pi} - \frac{q_0 L^2 x^2}{\pi} + C_4$$

$$\text{B.C. 4 } v(0) = 0 \quad \therefore C_4 = \frac{16q_0 L^4}{\pi^4}$$

$$v = -\frac{q_0 L}{3\pi^4 EI}$$

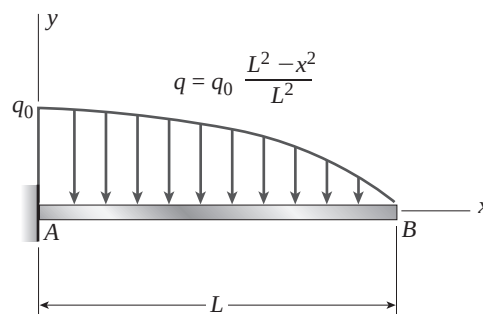
$$\left(48L^3 \cos \frac{\pi x}{2L} - 48L^3 + 3\pi^3 Lx^2 - \pi^3 x^3 \right) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{2q_0 L^4}{3\pi^4 EI} (\pi^3 - 24) \quad \leftarrow$$

(These results agree with Case 10, Table H-1.)

Problem 9.4-6 A cantilever beam AB is subjected to a parabolically varying load of intensity $q = q_0(L^2 - x^2)/L^2$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_B and angle of rotation θ_B at the free end. Use the fourth-order differential equation of the deflection curve (the load equation).

**Solution 9.4-6 Cantilever beam (parabolic load)**

LOAD EQUATION (EQ. 9-16c)

$$EIv'''' = -q = -\frac{q_0}{L^2}(L^2 - x^2)$$

$$EIv''' = -\frac{q_0}{L^2} \left(L^2 x - \frac{x^3}{3} \right) + C_1$$

$$\text{B.C. 1 } EIv''' = V \quad EIv'''(L) = 0 \quad \therefore C_1 = \frac{2q_0 L}{3}$$

$$EIv'' = -\frac{q_0}{L^2} \left(\frac{L^2 x^2}{2} - \frac{x^4}{12} \right) + \frac{2q_0 L}{3} x + C_2$$

$$\text{B.C. 2 } EIv'' = M \quad EIv''(L) = 0 \quad \therefore C_2 = -\frac{q_0 L^2}{4}$$

$$EIv' = -\frac{q_0}{L^2} \left(\frac{L^2 x^3}{6} - \frac{x^5}{60} \right) + \frac{q_0 Lx^2}{3} - \frac{q_0 L^2 x}{4} + C_3$$

$$\text{B.C. 3 } v'(0) = 0 \quad \therefore C_3 = 0$$

$$EIv = -\frac{q_0}{L^2} \left(\frac{L^2 x^4}{24} - \frac{x^6}{360} \right) + \frac{q_0 Lx^3}{9} - \frac{q_0 L^2 x^2}{8} + C_4$$

$$\text{B.C. 4 } v(0) = 0 \quad \therefore C_4 = 0$$

$$v = -\frac{q_0 x^2}{360 L^2 EI} (45L^4 - 40L^3 x + 15L^2 x^2 - x^4) \quad \leftarrow$$

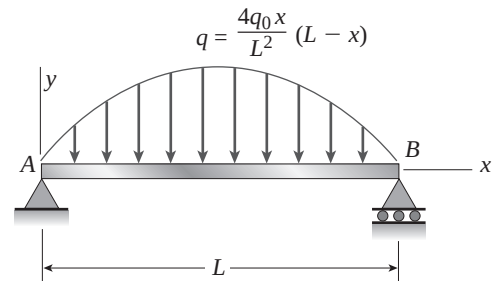
$$\delta_B = -v(L) = \frac{19q_0 L^4}{360 EI} \quad \leftarrow$$

$$v' = -\frac{q_0 x}{60 L^2 EI} (15L^4 - 20L^3 x + 10L^2 x^2 - x^4)$$

$$\theta_B = -v'(L) = \frac{q_0 L^3}{15 EI} \quad \leftarrow$$

Problem 9.4-7 A beam on simple supports is subjected to a parabolically distributed load of intensity $q = 4q_0 x(L - x)/L^2$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the maximum deflection $\delta_{\max}$. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-7 Simple beam (parabolic load)

LOAD EQUATION (Eq. 9-16c)

$$EIv'''' = -q = -\frac{4q_0 x}{L^2} (L - x) = -\frac{4q_0}{L^2} (Lx - x^2)$$

$$EIv''' = -\frac{2q_0}{3L^2} (3Lx^2 - 2x^3) + C_1$$

$$EIv'' = -\frac{q_0}{3L^2} (2Lx^3 - x^4) + C_1 x + C_2$$

$$\text{B.C. 1 } EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_2 = 0$$

$$\text{B.C. 2 } EIv''(L) = 0 \quad \therefore C_1 = \frac{q_0 L}{3}$$

$$EIv' = -\frac{q_0}{30L^2} (-5L^3 x^2 + 5Lx^4 - 2x^5) + C_3$$

$$\text{B.C. 3 (Symmetry)} \quad v'\left(\frac{L}{2}\right) = 0 \quad \therefore C_3 = -\frac{q_0 L^3}{30}$$

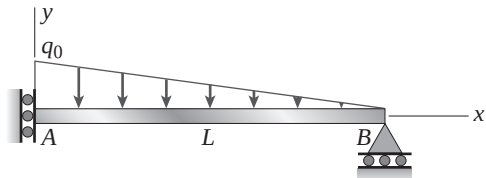
$$EIv = -\frac{q_0}{30L^2} \left(L^5 x - \frac{5L^3 x^3}{3} + Lx^5 - \frac{x^6}{3} \right) + C_4$$

$$\text{B.C. 4 } v(0) = 0 \quad \therefore C_4 = 0$$

$$v = -\frac{q_0 x}{90L^2 EI} (3L^5 - 5L^3 x^2 + 3Lx^4 - x^5) \quad \leftarrow$$

$$\delta_{\max} = -v\left(\frac{L}{2}\right) = \frac{61q_0 L^4}{5760 EI} \quad \leftarrow$$

Problem 9.4-8 Derive the equation of the deflection curve for beam AB, with guided support at A and roller at B, carrying a triangularly distributed load of maximum intensity q_0 (see figure). Also, determine the maximum deflection $\delta_{\max}$ of the beam. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-8

LOAD EQUATION

$$EIv'''' = -q = -q_0 + \frac{q_0x}{L}$$

$$EIv''' = -q_0x + \frac{q_0x^2}{2L} + C_1$$

$$\text{B.C. } v'''(0) = V(0) = 0 \quad C_1 = 0$$

$$EIv''' = -q_0x + \frac{q_0x^2}{2L}$$

$$EIv'' = -\frac{q_0x^2}{2} + \frac{q_0x^3}{6L} + C_2$$

$$\text{B.C. } v''(L) = M(L) = 0 \quad C_2 = \frac{q_0L^2}{3}$$

$$EIv'' = -\frac{q_0x^2}{2} + \frac{q_0x^3}{6L} + \frac{q_0L^2}{3}$$

$$EIv' = -\frac{q_0x^3}{6} + \frac{q_0x^4}{24L} + \frac{q_0L^2x}{3} + C_3$$

$$EIv = -\frac{q_0x^4}{24} + \frac{q_0x^5}{120L} + \frac{q_0L^2x^2}{6} + C_3x + C_4$$

$$\text{B.C. } v'(0) = 0 \quad C_3 = 0$$

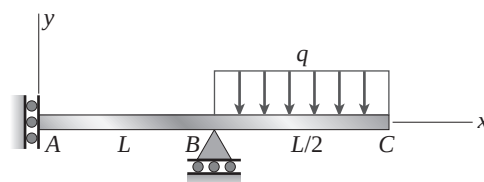
$$\text{B.C. } v(L) = 0 \quad C_4 = -\frac{2q_0L^4}{15}$$

$$v(x) = \frac{q_0}{120EI}(-5x^4L + x^5 + 20L^3x^2 - 16L^5) \quad \leftarrow$$

MAXIMUM DEFLECTION

$$\delta_{\max} = -v(0) = \frac{2q_0L^4}{15EI} \quad \leftarrow$$

Problem 9.4-9 Derive the equations of the deflection curve for beam ABC, with guided support at A and roller support at B, supporting a uniform load of intensity q acting on the overhang portion of the beam (see figure). Also, determine deflection δ_C and angle of rotation θ_C . Use the fourth-order differential equation of the deflection curve (the load equation).

**Solution 9.4-9**

LOAD EQUATION

$$EIv'''' = -q = 0 \quad (0 \leq x \leq L)$$

$$EIv''' = C_1 \quad (0 \leq x \leq L)$$

$$EIv'' = C_1x + C_2 \quad (0 \leq x \leq L)$$

$$\text{B.C. } v'''(0) = V(0) = 0 \quad C_1 = 0$$

$$\text{B.C. } v''(0) = M(0) = -\frac{qL^2}{8} \quad C_2 = -\frac{qL^2}{8}$$

$$EIv'' = -\frac{qL^2}{8}$$

$$EIv' = -\frac{qL^2x}{8} + C_3$$

$$\text{B.C. } v'(0) = 0 \quad C_3 = 0$$

$$EIv = -\frac{qL^2x^2}{16} + C_4$$

$$\text{B.C. } v(L) = 0 \quad C_4 = \frac{qL^4}{16}$$

$$v(x) = -\frac{qL^2}{16EI}(x^2 - L^2) \quad (0 \leq x \leq L) \quad \leftarrow$$

LOAD EQUATION

$$EIv'''' = -q \quad \left(L \leq x \leq \frac{3L}{2} \right)$$

$$EIv''' = -qx + C_5 \quad \left(L \leq x \leq \frac{3L}{2} \right)$$

$$EIv'' = \frac{-qx^2}{2} + C_5x + C_6 \quad \left(L \leq x \leq \frac{3L}{2} \right)$$

$$\text{B.C. } v''' \left(\frac{3L}{2} \right) = V \left(\frac{3L}{2} \right) = 0 \quad C_5 = \frac{3qL}{2}$$

$$\text{B.C. } v'' \left(\frac{3L}{2} \right) = M \left(\frac{3L}{2} \right) = 0 \quad C_6 = \frac{9qL^2}{8}$$

$$EIv'' = \frac{-qx^2}{2} + \frac{3qLx}{2} - \frac{9qL^2}{8}$$

$$EIv' = \frac{-qx^3}{6} + \frac{3qLx^2}{4} - \frac{9qL^2x}{8} + C_7$$

$$\text{B.C. } v'_L(L) = v'_R(L)$$

$$-\frac{qL^3}{8} = \frac{-qL^3}{6} + \frac{3qL^3}{4} - \frac{9qL^3}{8} + C_7$$

$$C_7 = \frac{5}{12}qL^3$$

$$EIv' = \frac{-qx^3}{6} + \frac{3qLx^2}{4} - \frac{9qL^2x}{8}$$

$$+ \frac{5}{12}qL^3$$

$$EIv = \frac{-qx^4}{24} + \frac{3qLx^3}{12} - \frac{9qL^2x^2}{16}$$

$$+ \frac{5}{12}qL^3x + C_8$$

$$\text{B.C. } v(L) = 0 \quad C_8 = \frac{-1}{16}qL^4$$

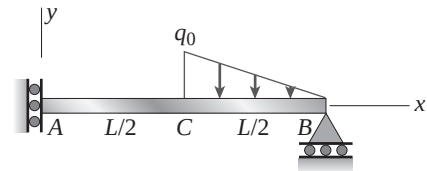
$$v(x) = \frac{-q}{48EI}(-20xL^3 + 27L^2x^2$$

$$-12Lx^3 + 2x^4 + 3L^4) \quad \left(L \leq x \leq \frac{3L}{2} \right) \quad \leftarrow$$

$$\delta_C = -v \left(\frac{3L}{2} \right) = \frac{9qL^4}{128EI} \quad \leftarrow$$

$$\theta_C = -v' \left(\frac{3L}{2} \right) = \frac{7qL^3}{48EI} \quad (\text{clockwise}) \quad \leftarrow$$

Problem 9.4-10 Derive the equations of the deflection curve for beam AB, with guided support at A and roller support at B, supporting a distributed load of maximum intensity q_0 acting on the right-hand half of the beam (see figure). Also, determine deflection δ_A , angle of rotation θ_B , and deflection δ_C at the midpoint. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-10

LOAD EQUATION

$$EIv'''' = -q = 0 \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$EIv'''' = C_1 \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$EIv'' = C_1x + C_2 \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$\text{B.C. } v'''(0) = V(0) = 0 \quad C_1 = 0$$

$$\text{B.C. } v''(0) = M(0) = \frac{q_0L^2}{12} \quad C_2 = \frac{q_0L^2}{12}$$

$$EIv'' = \frac{q_0L^2}{12} \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$EIv' = \frac{q_0L^2x}{12} + C_3 \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$\text{B.C. } v'(0) = 0 \quad C_3 = 0$$

$$EIv = \frac{q_0 L^2 x^2}{24} + C_4 \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

LOAD EQUATION

$$EIv'''' = -q_0 + \frac{2q_0}{L} \left(x - \frac{L}{2}\right) \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$EIv'''' = -2q_0 + \frac{2q_0 x}{L}$$

$$EIv''' = -2q_0 x + \frac{q_0 x^2}{L} + C_5 \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$EIv'' = -q_0 x^2 + \frac{q_0 x^3}{3L} + C_5 x + C_6 \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$\text{B.C. } v''' \left(\frac{L}{2}\right) = v' \left(\frac{L}{2}\right) = 0 \quad C_5 = \frac{3q_0 L}{4}$$

$$\text{B.C. } v'' \left(\frac{L}{2}\right) = M \left(\frac{L}{2}\right) = \frac{q_0 L^2}{12} \quad C_6 = \frac{q_0 L^2}{12}$$

$$EIv'' = -q_0 x^2 + \frac{q_0 x^3}{3L} + \frac{3q_0 L x}{4} - \frac{q_0 L^2}{12}$$

$$EIv' = -\frac{q_0 x^3}{3} + \frac{q_0 x^4}{12L} + \frac{3q_0 L x^2}{8}$$

$$- \frac{q_0 L^2 x}{12} + C_7$$

$$\text{B.C. } v'_L \left(\frac{L}{2}\right) = v'_R \left(\frac{L}{2}\right) \quad C_7 = \frac{5}{192} q_0 L^3$$

$$EIv' = -\frac{q_0 x^3}{3} + \frac{q_0 x^4}{12L} + \frac{3q_0 L x^2}{8}$$

$$+ \frac{q_0 L^2 x}{12} + \frac{5q_0 L^3}{192}$$

$$EIv = -\frac{q_0 x^4}{12} + \frac{q_0 x^5}{60L} + \frac{q_0 L x^3}{8} - \frac{q_0 L^2 x^2}{24} + \frac{5q_0 L^3 x}{192} + C_8$$

$$\text{B.C. } v(L) = 0 \quad C_8 = \frac{-41}{960} q_0 L^4$$

$$EIv = -\frac{q_0 x^4}{12} + \frac{q_0 x^5}{60L} + \frac{q_0 L x^3}{8} - \frac{q_0 L^2 x^2}{24} + \frac{5q_0 L^3 x}{192} - \frac{41}{960} q_0 L^4$$

$$\left(\frac{L}{2} \leq x \leq L\right) \quad \leftarrow$$

$$\text{B.C. } v_L \left(\frac{L}{2}\right) = v_R \left(\frac{L}{2}\right)$$

$$\frac{q_0 L^2 \left(\frac{L}{2}\right)^2}{24} + C_4 = -\frac{q_0 \left(\frac{L}{2}\right)^4}{12} + \frac{q_0 \left(\frac{L}{2}\right)^5}{60L}$$

$$+ \frac{q_0 L \left(\frac{L}{2}\right)^3}{8} - \frac{q_0 L^2 \left(\frac{L}{2}\right)^2}{24} + \frac{5q_0 L^3 \frac{L}{2}}{192} - \frac{41}{960} q_0 L^4$$

$$C_4 = \frac{-19}{480} q_0 L^4$$

$$EIv = \frac{q_0 L^2 x^2}{24} - \frac{19}{480} q_0 L^4 \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$v(x) = -\frac{q_0 L^2}{480 EI} (-20x^2 + 19L^2)$$

$$\left(0 \leq x \leq L\right) \quad \leftarrow$$

$$v(x) = -\frac{q_0}{960 LEI} (80x^4 L - 16x^5 - 120L^2 x^3 + 40L^3 x^2 - 25L^4 x + 41L^5)$$

$$\left(\frac{L}{2} \leq x \leq L\right) \quad \leftarrow$$

$$\delta_A = -v(0) = \frac{19}{480 EI} q_0 L^4 \quad \leftarrow$$

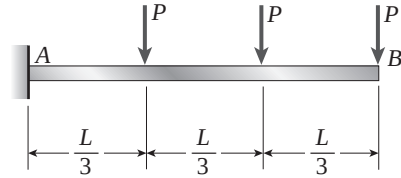
$$\theta_B = -v'(L) = -\frac{13}{192 EI} q_0 L^3 \quad \leftarrow$$

$$\delta_C = -v \left(\frac{L}{2}\right) = \frac{7}{240 EI} q_0 L^4 \quad \leftarrow$$

Method of Superposition

The problems for Section 9.5 are to be solved by the method of superposition. All beams have constant flexural rigidity EI .

Problem 9.5-1 A cantilever beam AB carries three equally spaced concentrated loads, as shown in the figure. Obtain formulas for the angle of rotation θ_B and deflection δ_B at the free end of the beam.



Solution 9.5-1 Cantilever beam with 3 loads

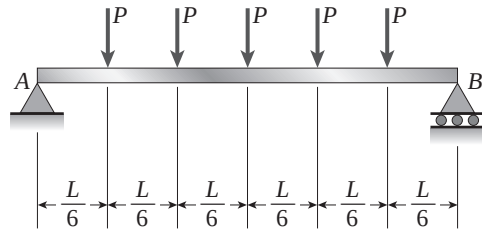
Table H-1, Cases 4 and 5:

$$\theta_B = \frac{P\left(\frac{L}{3}\right)^2}{2EI} + \frac{P\left(\frac{2L}{3}\right)^2}{2EI} + \frac{PL^2}{2EI} = \frac{7PL^2}{9EI} \quad \leftarrow$$

$$\begin{aligned} \delta_B &= \frac{P\left(\frac{L}{3}\right)^2}{6EI} \left(3L - \frac{L}{3}\right) + \frac{P\left(\frac{2L}{3}\right)^2}{6EI} \left(3L - \frac{2L}{3}\right) \\ &\quad + \frac{PL^3}{3EI} = \frac{5PL^3}{9EI} \quad \leftarrow \end{aligned}$$

Problem 9.5-2 A simple beam AB supports five equally spaced loads P (see figure).

- Determine the deflection δ_1 at the midpoint of the beam.
- If the same total load ($5P$) is distributed as a uniform load on the beam, what is the deflection δ_2 at the midpoint?
- Calculate the ratio of δ_1 to δ_2 .



Solution 9.5-2 Simple beam with 5 loads

(a) Table H-2, Cases 4 and 6:

$$\begin{aligned} \delta_1 &= \frac{P\left(\frac{L}{6}\right)}{24EI} \left[3L^2 - 4\left(\frac{L}{6}\right)^2\right] \\ &\quad + \frac{P\left(\frac{L}{3}\right)}{24EI} \left[3L^2 - 4\left(\frac{L}{3}\right)^2\right] + \frac{PL^3}{48EI} \\ &= \frac{11PL^3}{144EI} \quad \leftarrow \end{aligned}$$

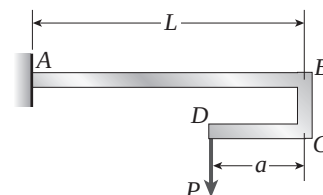
(b) Table H-2, Case 1 with $qL = 5P$:

$$\delta_2 = \frac{5qL^4}{384EI} = \frac{25PL^3}{384EI} \quad \leftarrow$$

$$(c) \frac{\delta_1}{\delta_2} = \frac{11}{144} \left(\frac{384}{25}\right) = \frac{88}{75} = 1.173 \quad \leftarrow$$

Problem 9.5-3 The cantilever beam AB shown in the figure has an extension BCD attached to its free end. A force P acts at the end of the extension.

- Find the ratio a/L so that the vertical deflection of point B will be zero.
- Find the ratio a/L so that the angle of rotation at point B will be zero.



Solution 9.5-3 Cantilever beam with extension

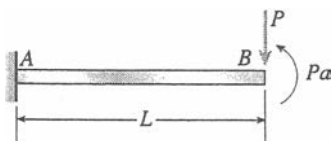
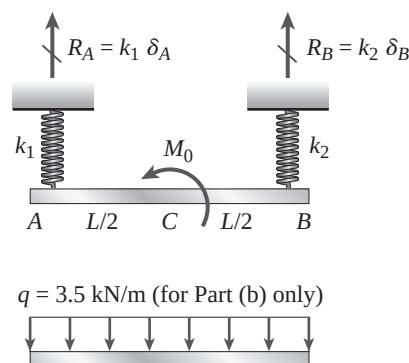


Table H-1, Cases 4 and 6

$$\begin{aligned} \text{(a)} \quad \delta_B &= \frac{PL^3}{3EI} - \frac{PaL^2}{2EI} = 0 & \frac{a}{L} &= \frac{2}{3} \quad \leftarrow \\ \text{(b)} \quad \theta_B &= \frac{PL^2}{2EI} - \frac{PaL}{EI} = 0 & \frac{a}{L} &= \frac{1}{2} \quad \leftarrow \end{aligned}$$

Problem 9.5-4 Beam ACB hangs from two springs, as shown in the figure. The springs have stiffnesses k_1 and k_2 and the beam has flexural rigidity EI .

- What is the downward displacement of point C , which is at the midpoint of the beam, when the moment M_0 is applied? Data for the structure are as follows: $M_0 = 10.0 \text{ kN}\cdot\text{m}$, $L = 1.8 \text{ m}$, $EI = 216 \text{ kN}\cdot\text{m}^2$, $k_1 = 250 \text{ kN/m}$, and $k_2 = 160 \text{ kN/m}$.
- Repeat part (a), but remove M_0 and apply uniform load $q = 3.5 \text{ kN/m}$ to the entire beam.



Solution 9.5-4

$$M_0 = 10.0 \text{ kN}\cdot\text{m} \quad L = 1.8 \text{ m} \quad EI = 216 \text{ kN}\cdot\text{m}^2$$

$$k_1 = 250 \text{ kN/m} \quad k_2 = 160 \text{ kN/m}$$

$$q = 3.5 \text{ kN/m}$$

$$\text{(a)} \quad R_A = \frac{M_0}{L} \quad R_B = -\frac{M_0}{L}$$

$$\delta_A = \frac{R_A}{k_1} \quad \delta_B = \frac{R_B}{k_2}$$

$$\delta_A = 22.22 \text{ mm} \quad (\text{downward})$$

$$\delta_B = -34.72 \text{ mm} \quad (\text{upward})$$

Table H-2, Case 8:

$$\delta_C = 0 + \frac{1}{2}(\delta_A + \delta_B)$$

$$\delta_C = -6.25 \text{ mm} \quad (\text{upward}) \quad \leftarrow$$

$$(b) R_A = \frac{qL}{2} \quad R_B = R_A$$

$$\delta_A = \frac{R_A}{k_1} \quad \delta_B = \frac{R_B}{k_2}$$

$$\delta_A = 12.60 \text{ mm}$$

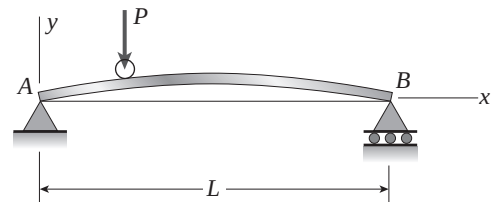
$$\delta_B = 19.69 \text{ mm}$$

Table H-2, Case 1:

$$\delta_C = \frac{5qL^4}{384EI} + \frac{1}{2}(\delta_A + \delta_B)$$

$$\delta_C = 18.36 \text{ mm (downward)} \quad \leftarrow$$

Problem 9.5-5 What must be the equation $y = f(x)$ of the axis of the slightly curved beam AB (see figure) *before* the load is applied in order that the load P , moving along the bar, always stays at the same level?



Solution 9.5-5 Slightly curved beam

Let x = distance to load P

δ = downward deflection at load P

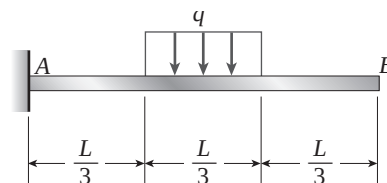
Table H-2, Case 5:

$$\delta = \frac{P(L-x)x}{6LEI} [L^2 - (L-x)^2 - x^2] = \frac{Px^2(L-x)^2}{3LEI}$$

Initial upward displacement of the beam must equal δ .

$$\therefore y = \frac{Px^2(L-x)^2}{3LEI} \quad \leftarrow$$

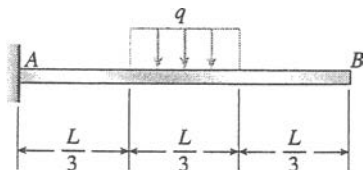
Problem 9.5-6 Determine the angle of rotation θ_B and deflection δ_B at the free end of a cantilever beam AB having a uniform load of intensity q acting over the middle third of its length (see figure).



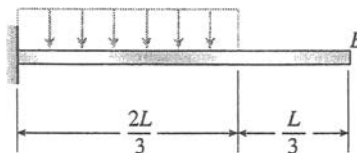
Solution 9.5-6 Cantilever beam (partial uniform load)

q = intensity of uniform load

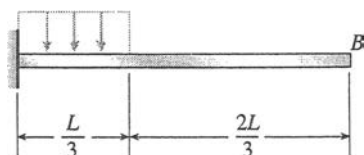
Original load on the beam:



Load No. 1:



Load No. 2:



SUPERPOSITION:

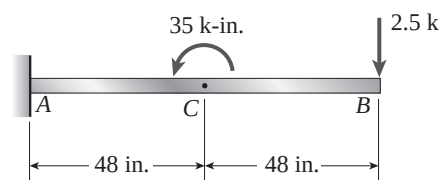
Original load = Load No. 1 minus Load No. 2

Table H-1, Case 2:

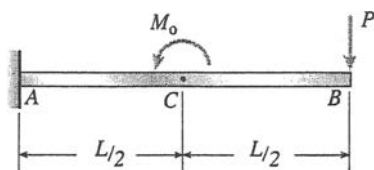
$$\theta_B = \frac{q}{6EI} \left(\frac{2L}{3} \right)^3 - \frac{q}{6EI} \left(\frac{L}{3} \right)^3 = \frac{7qL^3}{162EI} \quad \leftarrow$$

$$\begin{aligned} \delta_B &= \frac{q}{24EI} \left(\frac{2L}{3} \right)^3 \left(4L - \frac{2L}{3} \right) - \frac{q}{24EI} \left(\frac{L}{3} \right)^3 \left(4L - \frac{L}{3} \right) \\ &= \frac{23qL^4}{648EI} \quad \leftarrow \end{aligned}$$

Problem 9.5-7 The cantilever beam ACB shown in the figure has flexural rigidity $EI = 2.1 \times 10^6 \text{ k-in.}^2$. Calculate the downward deflections δ_C and δ_B at points C and B , respectively, due to the simultaneous action of the moment of 35 k-in. applied at point C and the concentrated load of 2.5 k applied at the free end B .



Solution 9.5-7 Cantilever beam (two loads)



$$EI = 2.1 \times 10^6 \text{ k-in.}^2$$

$$M_0 = 35 \text{ k-in.}$$

$$P = 2.5 \text{ k}$$

$$L = 96 \text{ in.}$$

Table H-1, Cases 4, 6, and 7:

$$\begin{aligned} \delta_C &= -\frac{M_0(L/2)^2}{2EI} + \frac{P(L/2)^2}{6EI} \left(3L - \frac{L}{2} \right) \\ &= -\frac{M_0L^2}{8EI} + \frac{5PL^3}{48EI} \quad (+ = \text{downward deflection}) \end{aligned}$$

$$\begin{aligned} \delta_B &= -\frac{M_0(L/2)}{2EI} \left(2L - \frac{L}{2} \right) + \frac{PL^3}{3EI} \\ &= -\frac{3M_0L^2}{8EI} + \frac{PL^3}{3EI} \quad (+ = \text{downward deflection}) \end{aligned}$$

SUBSTITUTE NUMERICAL VALUES:

$$\delta_C = -0.01920 \text{ in.} + 0.10971 \text{ in.}$$

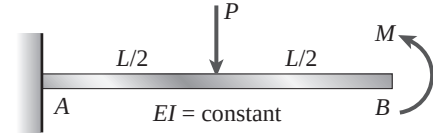
$$= 0.0905 \text{ in.} \quad \leftarrow$$

$$\delta_B = -0.05760 \text{ in.} + 0.35109 \text{ in.}$$

$$= 0.293 \text{ in.} \quad \leftarrow$$

Problem 9.5-8 A cantilever beam is subjected to load P at midspan and Counterclockwise moment M at B (see figure).

- Find an *expression* for moment M in terms of the load P so that the reaction moment at A , M_A , is equal to zero.
- Find an *expression* for moment M in terms of the load P so that the deflection $\delta_B = 0$; *also*, what is rotation θ_B ?
- Find an *expression* for moment M in terms of the load P so that the rotation $\theta_B = 0$; *also*, what is deflection δ_B ?



Solution 9.5-8

- (a) USE STATICS TO FIND MOMENT AT A

$$M_A = P\frac{L}{2} - M \quad M_A = 0 \quad \text{solve, } M = \frac{LP}{2}$$

- (b) USE DEFLECTION δ EQUATIONS FROM TABLE H-1, CASES 5 AND 6

$$\delta_B = \frac{-P\left(\frac{L}{2}\right)^3}{3EI} - \frac{P\left(\frac{L}{2}\right)^2}{2EI}\left(\frac{L}{2}\right) + \frac{ML^2}{2EI} = \frac{L^2M}{2EI} - \frac{5L^3P}{48EI}$$

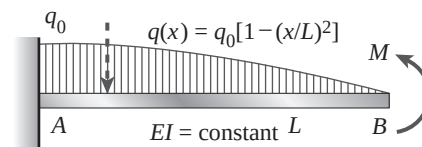
$$\delta_B = 0 \text{ solve, } M = \frac{5LP}{24}$$

$$\theta_B = -\frac{P\left(\frac{L}{2}\right)^2}{2EI} + \frac{ML}{EI} = \frac{LM}{EI} - \frac{L^2P}{8EI} \quad \text{so} \quad \theta_B = \frac{L\left(\frac{5LP}{24}\right)}{EI} - \frac{L^2P}{8EI} = \frac{L^2P}{12EI}$$

- (c) USE ANGLE OF ROTATION θ EQUATIONS FROM TABLE H-1, CASES 5 AND 6

$$\frac{LM}{EI} - \frac{L^2P}{8EI} = 0 \quad \text{solve, } M = \frac{LP}{8} \quad \delta_B = \frac{L^2\left(\frac{PL}{8}\right)}{2EI} - \frac{5L^3P}{48EI} = -\frac{L^3P}{24EI}$$

Problem 9.5-9 A cantilever beam is subjected to a quadratic distributed load $q(x)$ over the length of the beam (see figure). Find an expression for moment M in terms of the peak distributed load intensity q_0 so that the deflection $\delta_B = 0$.



Solution 9.5-9

FOR MOMENT M , FROM TABLE H-1, CASE 6 $\delta_{BM} = \frac{ML^2}{2EI}$ (upward) (1)

FIND AN EXPRESSION FOR DEFLECTION AT B DUE TO $q(x)$ $q(x) = q_0 \left[1 - \left(\frac{x}{L} \right)^2 \right]$

From solution for Problem 9.4-6:

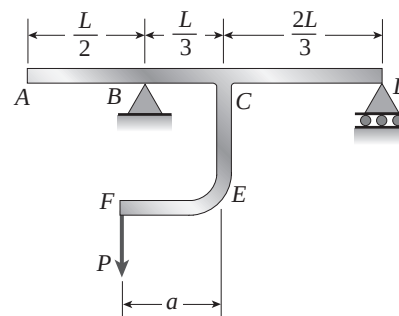
$$\delta_B = -v(L) = \frac{19q_0L^4}{360EI} \quad (\text{downward}) \quad (2)$$

EQUATE EXPRESSIONS IN EQS. (1) AND (2) AND SOLVE FOR M

$$M = \frac{2EI}{L^2} \left(\frac{19}{360} \frac{q_0L^4}{EI} \right) = \frac{19L^2q_0}{180} \quad M = \frac{19L^2q_0}{180} \quad (\text{counterclockwise})$$

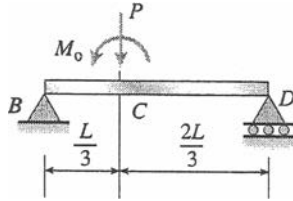
Problem 9.5-10 A beam $ABCD$ consisting of a simple span BD and an overhang AB is loaded by a force P acting at the end of the bracket CEF (see figure).

- Determine the deflection δ_A at the end of the overhang.
- Under what conditions is this deflection upward? Under what conditions is it downward?



Solution 9.5-10 Beam with bracket and overhang

Consider part *BD* of the beam.



$$M_0 = Pa$$

Table H-2, Cases 5 and 9:

$$\begin{aligned}\theta_B &= \frac{P(L/3)(2L/3)(5L/3)}{6EI} \\ &\quad + \frac{Pa}{6EI} \left[6\left(\frac{L^2}{3}\right) - 3\left(\frac{L^2}{9}\right) - 2L^2 \right] \\ &= \frac{PL}{162EI}(10L - 9a) \quad (+ = \text{clockwise angle})\end{aligned}$$

(a) DEFLECTION AT THE END OF THE OVERHANG

$$\delta_A = \theta_B \left(\frac{L}{2} \right) = \frac{PL^2}{324EI}(10L - 9a) \quad \leftarrow$$

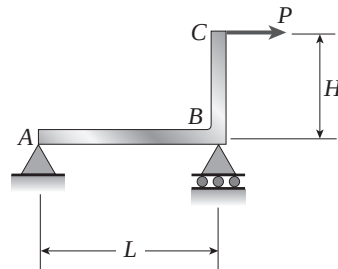
(+ = upward deflection)

(b) Deflection is upward when $\frac{a}{L} < \frac{10}{9}$ and downward when $\frac{a}{L} > \frac{10}{9} \quad \leftarrow$

Problem 9.5-11 A horizontal load P acts at end C of the bracket *ABC* shown in the figure.

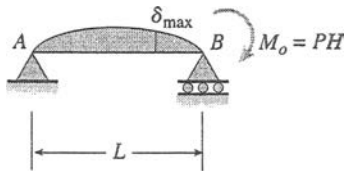
- Determine the deflection δ_C of point C .
- Determine the maximum upward deflection $\delta_{\max}$ of member *AB*.

Note: Assume that the flexural rigidity EI is constant throughout the frame. Also, disregard the effects of axial deformations and consider only the effects of bending due to the load P .

**Solution 9.5-11 Bracket ABC**

BEAM *AB*

$$M_0 = PH$$



$$\text{Table H-2, Case 7: } \theta_B = \frac{M_0 L}{3EI} = \frac{PHL}{3EI}$$

(a) ARM *BC* Table G-1, Case 4

$$\begin{aligned}\delta_C &= \frac{PH^3}{3EI} + \theta_B H = \frac{PH^3}{3EI} + \frac{PH^2 L}{3EI} \\ &= \frac{PH^2}{3EI}(L + H) \quad \leftarrow\end{aligned}$$

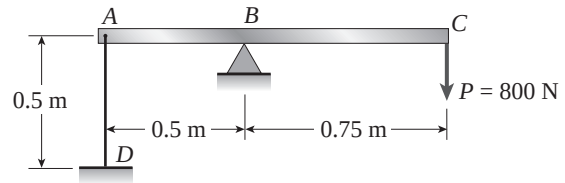
(b) MAXIMUM DEFLECTION OF BEAM *AB*

Table H-2, Case 7:

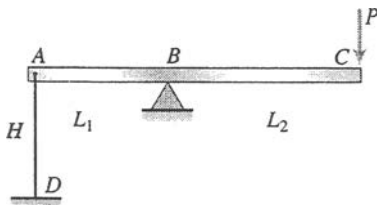
$$\delta_{\max} = \frac{M_0 L^2}{9\sqrt{3}EI} = \frac{PHL^2}{9\sqrt{3}EI} \quad \leftarrow$$

Problem 9.5-12 A beam ABC having flexural rigidity $EI = 75 \text{ kN}\cdot\text{m}^2$ is loaded by a force $P = 800 \text{ N}$ at end C and tied down at end A by a wire having axial rigidity $EA = 900 \text{ kN}$ (see figure).

What is the deflection at point C when the load P is applied?



Solution 9.5-12 Beam tied down by a wire



$$EI = 75 \text{ kN}\cdot\text{m}^2$$

$$P = 800 \text{ N}$$

$$EA = 900 \text{ kN}$$

$$H = 0.5 \text{ m} \quad L_1 = 0.5 \text{ m}$$

$$L_2 = 0.75 \text{ m}$$

CONSIDER BC AS A CANTILEVER BEAM

$$\text{Table H-1, Case 4: } \delta'_C = \frac{PL_2^3}{3EI}$$

CONSIDER AB AS A SIMPLE BEAM

$$M_0 = PL_2$$

$$\text{Table H-2, Case 7: } \theta'_B = \frac{M_0 L_1}{3EI} = \frac{PL_1 L_2}{3EI}$$

CONSIDER THE STRETCHING OF WIRE AD

$$\delta'_A = (\text{Force in AD}) \left(\frac{H}{EA} \right) = \left(\frac{PL_2}{L_1} \right) \left(\frac{H}{EA} \right) = \frac{PL_2 H}{EAL_1}$$

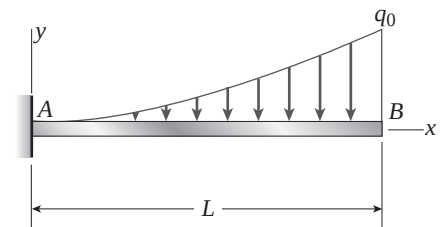
DEFLECTION δ_C OF POINT C

$$\begin{aligned} \delta_C &= \delta'_C + \theta'_B(L_2) + \delta'_A \left(\frac{L_2}{L_1} \right) \\ &= \frac{PL_2^3}{3EI} + \frac{PL_1 L_2^2}{3EI} + \frac{PL_2^2 H}{EAL_1^2} \quad \leftarrow \end{aligned}$$

Substitute numerical values:

$$\delta_C = 1.50 \text{ mm} + 1.00 \text{ mm} + 1.00 \text{ mm} = 3.50 \text{ mm} \quad \leftarrow$$

Problem 9.5-13 Determine the angle of rotation θ_B and deflection δ_B at the free end of a cantilever beam AB supporting a parabolic load defined by the equation $q(x) = q_0 x^2/L^2$ (see figure).



Solution 9.5-13 Cantilever beam (parabolic load)

LOAD: $q = \frac{q_0 x^2}{L^2}$ $qdx = \text{element of load}$

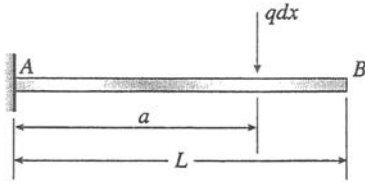


Table H-1, Case 5: (Set a equal to x)

$$\theta_B = \int_0^L \frac{(qdx)(x^2)}{2EI} = \frac{1}{2EI} \int_0^L \left(\frac{q_0 x^2}{L^2} \right) x^2 dx$$

$$= \frac{q_0}{2EI L^2} \int_0^L x^4 dx = \frac{q_0 L^3}{10EI} \quad \leftarrow$$

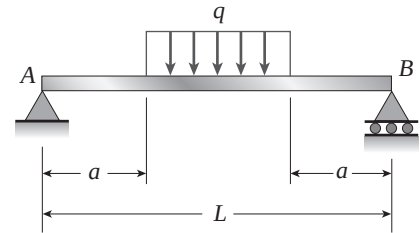
$$\delta_B = \int_0^L \frac{(qdx)(x^2)}{6EI} (3L - x)$$

$$= \frac{1}{6EI} \int_0^L \left(\frac{q_0 x^2}{L^2} \right) (x^2)(3L - x) dx$$

$$= \frac{q_0}{6EI L^2} \int_0^L (x^4)(3L - x) dx = \frac{13q_0 L^4}{180EI} \quad \leftarrow$$

Problem 9.5-14 A simple beam AB supports a uniform load of intensity q acting over the middle region of the span (see figure).

Determine the angle of rotation θ_A at the left-hand support and the deflection $\delta_{\max}$ at the midpoint.

**Solution 9.5-14 Simple beam (partial uniform load)**

LOAD: $qdx = \text{element of load}$

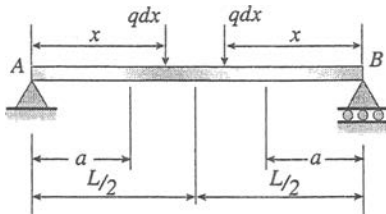


Table H-2, Case 6: $\theta_A = \frac{Pa(L - a)}{2EI}$

Replace P by qdx . Replace a by x .

Integrate x from a to $L/2$.

$$\theta_A = \int_a^{L/2} \frac{qdx}{2EI} (x)(L - x) = \frac{q}{2EI} \int_a^{L/2} (xL - x^2) dx$$

$$= \frac{q}{24EI} (L^3 - 6a^2L + 4a^3) \quad \leftarrow$$

Table H-2, Case 6: $\delta_{\max} = \frac{Pa}{24EI} (3L^2 - 4a^2)$

Replace P by qdx . Replace a by x .

Integrate x from a to $L/2$.

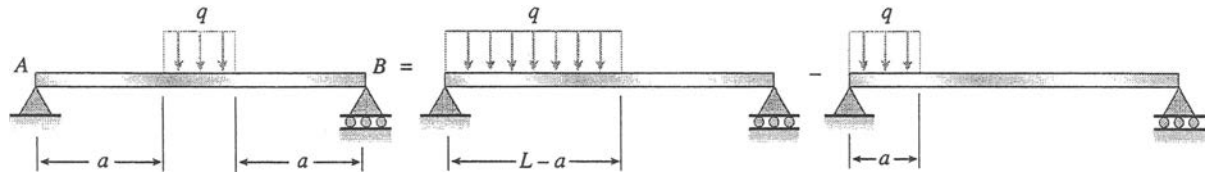
$$\begin{aligned}
 \delta_{\max} &= \int_a^{L/2} \frac{qdx}{24EI} (x)(3L^2 - 4x^2) \\
 &= \frac{q}{24EI} \int_a^{L/2} (3L^2x - 4x^3) dx \\
 &= \frac{q}{384EI} (5L^4 - 24a^2L^2 + 16a^4) \quad \leftarrow
 \end{aligned}$$

ALTERNATE SOLUTION (not recommended; algebra is extremely lengthy)

Table H-2, Case 3:

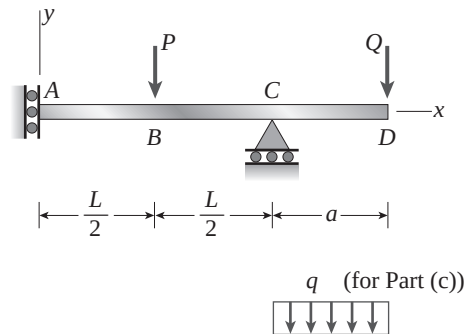
$$\begin{aligned}
 \theta_A &= \frac{q(L-a)^2}{24LEI} [2L - (L-a)]^2 - \frac{qa^2}{24LEI} (2L-a)^2 \\
 &= \frac{q}{24EI} (L^3 - 6La^2 + 4a^3) \quad \leftarrow
 \end{aligned}$$

$$\begin{aligned}
 \delta_{\max} &= \frac{q(L/2)}{24LEI} \left[(L-a)^4 - 4L(L-a)^3 \right. \\
 &\quad + 4L^2(L-a)^2 + 2(L-a)^2 \left(\frac{L}{2} \right)^2 \\
 &\quad \left. - 4L(L-a) \left(\frac{L}{2} \right)^2 + L \left(\frac{L}{2} \right)^3 \right] \\
 &= \frac{qa^2}{24LEI} \left[-La^2 + 4L^2 \left(\frac{L}{2} \right) + a^2 \left(\frac{L}{2} \right) \right. \\
 &\quad \left. - 6L \left(\frac{L}{2} \right)^2 + 2 \left(\frac{L}{2} \right)^3 \right] \\
 \delta_{\max} &= \frac{q}{384EI} (5L^4 - 24L^2a^2 + 16a^4) \quad \leftarrow
 \end{aligned}$$



Problem 9.5-15 The overhanging beam ABCD supports two concentrated loads P and Q (see figure).

- For what ratio P/Q will the deflection at point B be zero?
- For what ratio will the deflection at point D be zero?
- If Q is replaced by uniform load with intensity q (on the overhang), repeat parts (a) and (b) but find ratio $P/(qa)$.



Solution 9.5-15

(a) DEFLECTION AT POINT B

Table H-2, Cases 6 and 10:

$$\begin{aligned}
 \delta_B &= \frac{P \left(\frac{L}{2} \right)}{6EI} \left[3 \left(\frac{L}{2} \right) (2L) - 3 \left(\frac{L}{2} \right)^2 \right. \\
 &\quad \left. - \left(\frac{L}{2} \right)^2 \right] - \frac{Qa \left(\frac{L}{2} \right)}{2EI} \left[(2L) - \left(\frac{L}{2} \right) \right]
 \end{aligned}$$

$$\delta_B = 0 \quad \frac{P}{Q} = \frac{9a}{4L} \quad \leftarrow$$

(b) DEFLECTION AT POINT D

Table H-2 Case 6; Table H-1 Case 4; Table H-2 Case 10:

$$\delta_D = -\frac{P\left(\frac{L}{2}\right)\left[(2L) - \frac{L}{2}\right]}{2EI} (a)$$

$$+ \frac{Qa^3}{3EI} + \frac{Qa(2L)}{2EI} (a)$$

$$\delta_D = 0 \quad \frac{P}{Q} = \frac{8a(3L + a)}{9L^2} \quad \leftarrow$$

(c.1) DEFLECTION AT POINT B

Table H-2 Cases 6 and 10:

$$\delta_B = \frac{P\left(\frac{L}{2}\right)}{6EI} \left[3\left(\frac{L}{2}\right)(2L) - 3\left(\frac{L}{2}\right)^2 \right]$$

$$- \left(\frac{L}{2}\right)^2 \left[-\frac{\left(\frac{qa^2}{2}\right)\left(\frac{L}{2}\right)}{2EI} \right] \left[(2L) - \left(\frac{L}{2}\right) \right]$$

$$\delta_B = 0 \quad \frac{P}{qa} = \frac{9a}{8L} \quad \leftarrow$$

(c.2) DEFLECTION AT POINT D

Table H-2 Case 6; Table H-1 Case 1; Table H-2 Case 10:

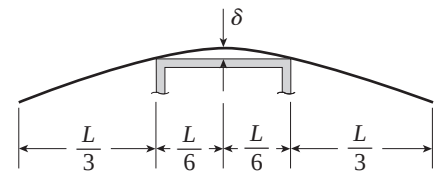
$$\delta_D = -\frac{P\left(\frac{L}{2}\right)\left[(2L) - \frac{L}{2}\right]}{2EI} (a)$$

$$+ \frac{qa^4}{8EI} + \frac{\left(\frac{qa^2}{2}\right)(2L)}{2EI} (a)$$

$$\delta_D = 0 \quad \frac{P}{qa} = \frac{a(4L + a)}{3L^2} \quad \leftarrow$$

Problem 9.5-16 A thin metal strip of total weight W and length L is placed across the top of a flat table of width $L/3$ as shown in the figure.

What is the clearance δ between the strip and the middle of the table? (The strip of metal has flexural rigidity EI .)

**Solution 9.5-16 Thin metal strip**

W = total weight $q = \frac{W}{L}$

EI = flexural rigidity

FREE BODY DIAGRAM (the part of the strip above the table)

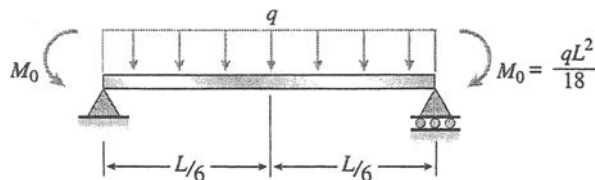


Table H-2, Cases 1 and 10:

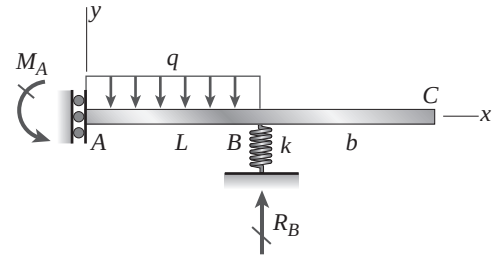
$$\delta = -\frac{5q}{384EI} \left(\frac{L}{3}\right)^4 + \frac{M_0}{8EI} \left(\frac{L}{3}\right)^2$$

$$= -\frac{5qL^4}{31,104EI} + \frac{qL^4}{1296EI}$$

$$= \frac{19qL^4}{31,104EI}$$

But $q = \frac{W}{L}$: $\therefore \delta = \frac{19WL^3}{31,104EI} \quad \leftarrow$

Problem 9.5-17 An overhanging beam ABC with flexural rigidity $EI = 15 \text{ k-in.}^2$ is supported by a guided support at A and by a spring of stiffness k at point B (see figure). Span AB has length $L = 30 \text{ in.}$ and carries a uniform load. The overhang BC has length $b = 15 \text{ in.}$ For what stiffness k of the spring will the uniform load produce no deflection at the free end C ?



Solution 9.5-17

$$EI = 15 \text{ kip}\cdot\text{in.}^2 \quad L = 30 \text{ in.} \quad b = 15 \text{ in.}$$

$$R_B = qL$$

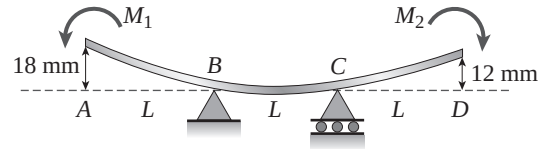
Table H-2, Case 1:

$$\delta_C = \theta_B b - \delta_B = \frac{q(2L)^3}{24EI}(b) - \frac{qL}{k}$$

$$\text{for } \delta_C = 0 \quad k = \frac{3EI}{bL^2}$$

$$\text{Therefore, } k = 3.33 \text{ lb/in.} \quad \leftarrow$$

Problem 9.5-18 A beam $ABCD$ rests on simple supports at B and C (see figure). The beam has a slight initial curvature so that end A is 18 mm above the elevation of the supports and end D is 12 mm above. What moments M_1 and M_2 , acting at points A and D , respectively, will move points A and D downward to the level of the supports? (The flexural rigidity EI of the beam is $2.5 \times 10^6 \text{ N}\cdot\text{m}^2$ and $L = 2.5 \text{ m.}$)



Solution 9.5-18

$$EI = 2.5 \times 10^6 \text{ N}\cdot\text{m}^2 \quad L = 2.5 \text{ m} \quad \delta_A = 18 \text{ mm}$$

$$\delta_D = 12 \text{ mm}$$

Table H-2, Case 7:

$$\theta_B = \frac{M_1 L}{3EI} + \frac{M_2 L}{6EI} \quad \theta_C = \frac{M_1 L}{6EI} + \frac{M_2 L}{3EI}$$

DEFLECTION AT POINT A AND D

Table H-1, Case 6:

$$\delta_A = \frac{M_1 L^2}{2EI} + \theta_B L \quad \delta_D = \frac{M_2 L^2}{2EI} + \theta_C L$$

$$\delta_A = \frac{M_1 L^2}{2EI} + \left(\frac{M_1 L}{3EI} + \frac{M_2 L}{6EI} \right) L$$

$$\delta_D = \frac{M_2 L^2}{2EI} + \left(\frac{M_1 L}{6EI} + \frac{M_2 L}{3EI} \right) L$$

$$5M_1 + M_2 = \frac{6\delta_A EI}{L^2} \quad (1)$$

$$5M_2 + M_1 = \frac{6\delta_D EI}{L^2} \quad (2)$$

SOLVE EQS. (1) AND (2)

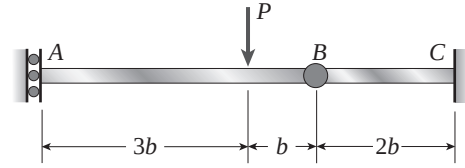
$$M_1 = \frac{EI(5\delta_A - \delta_D)}{4L^2} \quad M_2 = \frac{EI(5\delta_D - \delta_A)}{4L^2}$$

Therefore,

$$M_1 = 7800 \text{ N}\cdot\text{m} \quad \leftarrow$$

$$M_2 = 4200 \text{ N}\cdot\text{m} \quad \leftarrow$$

Problem 9.5-19 The compound beam ABC shown in the figure has a guided support at A and a fixed support at C . The beam consists of two members joined by a pin connection (i.e., moment release) at B . Find the deflection δ under the load P .



Solution 9.5-19

Table H-1, Case 4:

$$\delta_B = \frac{P(2b)^3}{3EI}$$

DEFLECTION UNDER THE LOAD P

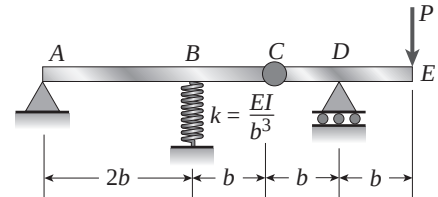
Table H-2, Case 6:

$$\delta = \frac{P(b)}{6EI} \left[3(b)(8b) - 3(b)^2 - (b)^2 \right] + \delta_B$$

$$\delta = \frac{P(b)}{6EI} \left[3(b)(8b) - 3b^2 - b^2 \right] + \frac{P(2b)^3}{3EI}$$

$$\delta = \frac{6Pb^3}{EI} \quad \leftarrow$$

Problem 9.5-20 A compound beam $ABCDE$ (see figure) consists of two parts (ABC and CDE) connected by a hinge (i.e., moment release) at C . The elastic support at B has stiffness $k = EI/b^3$. Determine the deflection δ_E at the free end E due to the load P acting at that point.



Solution 9.5-20

CONSIDER BEAM ABC

$$R_B = \frac{3P}{2} \quad \delta_B = \frac{R_B}{k} = \frac{3P}{2k} \quad (\text{upward})$$

Table H-2, Case 7; Table H-1, Case 4:

$$\begin{aligned} \delta_C &= \frac{Pb(2b)}{3EI}b + \frac{Pb^3}{3EI} + \delta_B \left(\frac{2b+b}{2b} \right) \\ &= \frac{Pb(2b)}{3EI}b + \frac{Pb^3}{3EI} + \frac{3P}{2k} \left(\frac{2b+b}{2b} \right) \end{aligned}$$

$$\delta_C = \frac{P(4b^3k + 9EI)}{4EI k} \quad (\text{upward})$$

CONSIDER BEAM CDE

Table H-2, Case 7; Table H-1, Case 4:

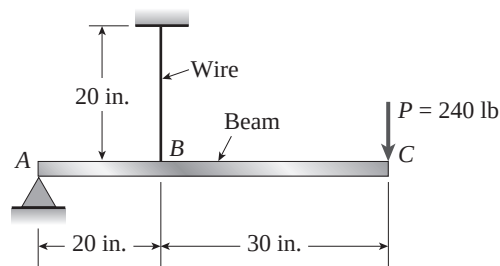
$$\begin{aligned} \delta_E &= \frac{(Pb)(b)}{3EI}b + \frac{Pb^3}{3EI} + \delta_C = \frac{(Pb)(b)}{3EI}b \\ &\quad + \frac{Pb^3}{3EI} + \frac{P(4b^3k + 9EI)}{4EI k} \end{aligned}$$

$$\text{For } k = \frac{EI}{b^3},$$

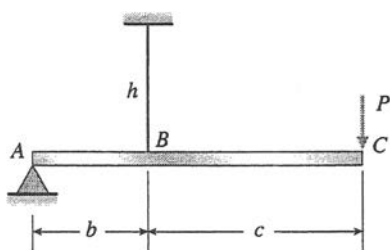
$$\delta_E = \frac{47Pb^3}{12EI} \quad \leftarrow$$

Problem 9.5-21 A steel beam ABC is simply supported at A and held by a high-strength steel wire at B (see figure). A load $P = 240$ lb acts at the free end C . The wire has axial rigidity $EA = 1500 \times 10^3$ lb, and the beam has flexural rigidity $EI = 36 \times 10^6$ lb-in.²

What is the deflection δ_C of point C due to the load P ?



Solution 9.5-21 Beam supported by a wire

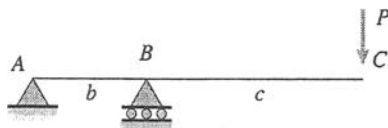


$$P = 240 \text{ lb} \quad b = 20 \text{ in.} \quad c = 30 \text{ in.} \quad h = 20 \text{ in.}$$

$$\text{Beam: } EI = 36 \times 10^6 \text{ lb-in.}^2$$

$$\text{Wire: } EA = 1500 \times 10^3 \text{ lb}$$

(1) ASSUME THAT POINT B IS ON A SIMPLE SUPPORT



$$\begin{aligned} \delta'_C &= \frac{Pc^3}{3EI} + \theta'_B c = \frac{Pc^3}{3EI} + (Pc) \left(\frac{b}{3EI} \right) c \\ &= \frac{Pc^2}{3EI} (b + c) \quad (\text{downward}) \end{aligned}$$

(2) ASSUME THAT THE WIRE STRETCHES

T = tensile force in the wire

$$= \frac{P}{b}(b + c)$$

$$\delta_B = \frac{Th}{EA} = \frac{Ph(b + c)}{EAb}$$

$$\delta''_C = \delta_B \left(\frac{b + c}{b} \right) = \frac{Ph(b + c)^2}{EAb^2} \quad (\text{downward})$$

(3) DEFLECTION AT POINT C

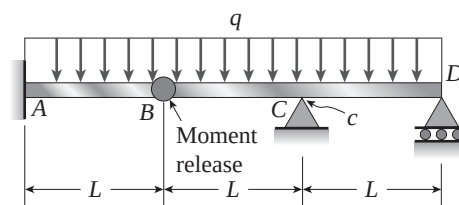
$$\delta_C = \delta'_C + \delta''_C = P(b + c) \left[\frac{c^2}{3EI} + \frac{h(b + c)}{EAb^2} \right] \quad \leftarrow$$

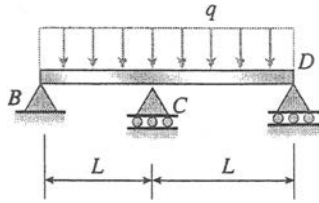
Substitute numerical values:

$$\delta_C = 0.10 \text{ in.} + 0.02 \text{ in.} = 0.12 \text{ in.} \quad \leftarrow$$

Problem 9.5-22 The compound beam shown in the figure consists of a cantilever beam AB (length L) that is pin-connected to a simple beam BD (length $2L$). After the beam is constructed, a clearance c exists between the beam and a support at C , midway between points B and D . Subsequently, a uniform load is placed along the entire length of the beam.

What intensity q of the load is needed to close the gap at C and bring the beam into contact with the support?



Solution 9.5-22 Compound beamBEAM BCD WITH A SUPPORT AT B 

$$\delta'_c = \frac{5q(2L)^4}{384EI}$$

$$= \frac{5qL^4}{24EI}$$

 δ''_c = downward displacement of point C due to δ_B

$$\delta''_c = \frac{1}{2}\delta_B = \frac{11qL^4}{48EI}$$

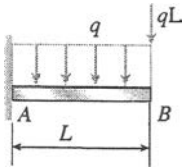
DOWNWARD DISPLACEMENT OF POINT C

$$\delta_c = \delta'_c + \delta''_c = \frac{5qL^4}{24EI} + \frac{11qL^4}{48EI} = \frac{7qL^4}{16EI}$$

$$c = \text{clearance} \quad c = \delta_c = \frac{7qL^4}{16EI}$$

INTENSITY OF LOAD TO CLOSE THE GAP

$$q = \frac{16Eic}{7L^4} \quad \leftarrow$$

CANTILEVER BEAM AB 

$$\delta_B = \frac{qL^4}{8EI} + \frac{(qL)L^3}{3EI}$$

$$= \frac{11qL^4}{24EI} \quad (\text{downward})$$

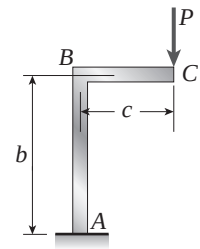
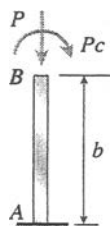
Problem 9.5-23 Find the horizontal deflection δ_h and vertical deflection δ_v at the free end C of the frame ABC shown in the figure. (The flexural rigidity EI is constant throughout the frame.)Note: Disregard the effects of axial deformations and consider only the effects of bending due to the load P .**Solution 9.5-23 Frame ABC** MEMBER AB : δ_h = horizontal deflection of point B

Table H-1, Case 6:

$$\delta_h = \frac{(Pc)b^2}{2EI} = \frac{Pcb^2}{2EI}$$

$$\theta_B = \frac{Pcb}{EI}$$

Since member BC does not change in length, δ_h is also the horizontal displacement of point C .

$$\therefore \delta_h = \frac{Pcb^2}{2EI} \quad \leftarrow$$

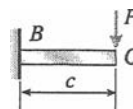
MEMBER BC WITH B FIXED AGAINST ROTATION:

Table H-1, Case 4:

$$\delta'_c = \frac{Pc^3}{3EI}$$

VERTICAL DEFLECTION OF POINT C

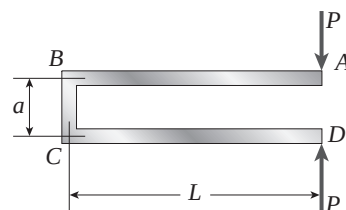
$$\delta_c = \delta_v = \delta'_c + \theta_B c = \frac{Pc^3}{3EI} + \frac{Pcb}{EI}(c)$$

$$= \frac{Pc^2}{3EI}(c + 3b)$$

$$\delta_v = \frac{Pc^2}{3EI}(c + 3b) \quad \leftarrow$$

Problem 9.5-24 The frame $ABCD$ shown in the figure is squeezed by two collinear forces P acting at points A and D . What is the decrease δ in the distance between points A and D when the loads P are applied? (The flexural rigidity EI is constant throughout the frame.)

Note: Disregard the effects of axial deformations and consider only the effects of bending due to the loads P .



Solution 9.5-24 Frame $ABCD$

MEMBER BC :

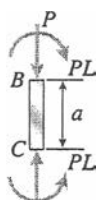


Table H-1, Case 10: $\theta_B = \frac{(PL)a}{2EI} = \frac{PLa}{2EI}$

MEMBER BA :

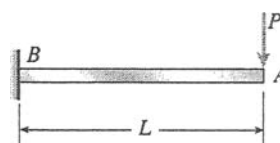


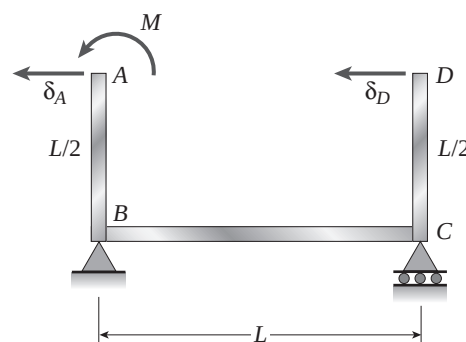
Table H-1, Case 4: $\delta_A = \frac{PL^3}{3EI} + \theta_B L$
 $= \frac{PL^3}{3EI} + \frac{PLa}{2EI}(L)$
 $= \frac{PL^2}{6EI}(2L + 3a)$

DECREASE IN DISTANCE BETWEEN POINTS A AND D

$\delta = 2\delta_A = \frac{PL^2}{3EI}(2L + 3a) \quad \leftarrow$

Problem 9.5-25 A framework $ABCD$ is acted on by counterclockwise moment M at A (see figure). Assume that EI is constant.

- Find expressions for reactions at supports B and C .
- Find expressions for angles of rotation at A , B , C , and D .
- Find expressions for horizontal deflections δ_A and δ_D .
- If length $L_{AB} = L/2$, find length L_{CD} in terms of L so that the absolute value of the ratio $|\delta_A/\delta_D| = 1$.



Solution 9.5-25(a) USE STATICS TO FIND REACTIONS AT B AND C

$$\sum F_H = 0 \quad H_B = 0 \quad \sum M_C = 0 \quad V_B = \frac{M}{L} \quad \sum F_V = 0 \quad V_C = -V_B$$

(b) USE SUPERPOSITION AND TABLES H-1 AND H-2

$$\theta_A = \frac{5ML}{6EI} \quad \text{From Table H-1, Case 6, } \theta_{A1} = \frac{M\left(\frac{L}{2}\right)}{EI} \quad \text{From Table H-2, Case 7, } \theta_A = \frac{5ML}{6EI}$$

$$\theta_A = \theta_{A1} + \theta_{A2} \frac{5LM}{6EI} \quad (\text{counterclockwise})$$

$$\theta_B = \frac{ML}{3EI} \quad \text{From Table H-2, Case 7, } \theta_{\text{left}} = \frac{ML}{3EI} \quad \text{and } \theta_B = \theta_{\text{left}} \quad (\text{counterclockwise})$$

$$\theta_C = \frac{-ML}{6EI} \quad \text{From Table H-2, Case 7, } \theta_{\text{right}} = \frac{ML}{6EI} \quad \text{and } \theta_C = -\theta_{\text{right}} \quad (\text{counterclockwise})$$

$$\theta_D = \theta_C$$

(c) SUPERPOSITION AND TABULATED EXPRESSIONS (APPENDIX H)

$$\text{Table H-1, Case 6 } (\delta_B) \text{ and H-2, Case 7 } (\theta_B): \quad \delta_B + \theta_B \frac{L}{2}$$

$$\delta_A = \frac{M\left(\frac{L}{2}\right)^2}{2EI} + \frac{ML}{3EI} \left(\frac{L}{2}\right) = \frac{7L^2M}{24EI} \quad \frac{7}{24} = 0.29167 < \delta_A \text{ to the left}$$

Table H-2, Case 7: Use θ_B

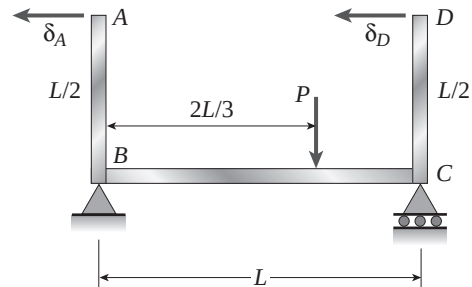
$$\delta_D = \frac{ML}{6EI} \left(\frac{L}{2}\right) = \frac{L^2M}{12EI} \quad \frac{1}{12} = 0.083 < \delta_D \text{ to the right}$$

(d) EQUATE ABOVE EXPRESSIONS FOR δ_A and δ_D and then solve for L_{CD}

$$\frac{L_{CD}^2 M}{12EI} = \frac{7L^2 M}{24EI} \quad \text{Solve, } L_{CD}: \quad L_{CD} = \frac{\sqrt{14}}{3} L = 1.871L$$

Problem 9.5-26 A framework $ABCD$ is acted on by force P at $2L/3$ from B (see figure). Assume that EI is constant.

- Find expressions for reactions at supports B and C .
- Find expressions for angles of rotation at A , B , C , and D .
- Find expressions for horizontal deflections δ_A and δ_D .
- If length $L_{AB} = L/2$, find length L_{CD} in terms of L for the absolute value of the ratio $|\delta_A/\delta_D| = 1$.



Solution 9.5-26(a) USE STATICS TO FIND REACTIONS AT B AND C

$$\sum F_H = 0 \quad H_B = 0 \quad \sum M_C = 0 \quad V_B = \frac{1}{L} \left(P \frac{L}{3} \right) = \frac{P}{3} \quad \sum F_V = 0 \quad V_C = P - V_B = \frac{2P}{3}$$

(b) ANGLES OF ROTATION AT A , B , C , AND D : USE EXPRESSIONS FOR θ_A and θ_B in Table H-2, Case 5, with $a = 2L/3$, $b = L/3$

$$\theta_B = \left(\frac{-4}{81} \right) \frac{ML}{EI} \quad \theta_A = \theta_B \quad \theta_C = \left(\frac{5}{81} \right) \frac{ML}{EI} \quad \theta_D = \theta_C$$

(c) BARS AB AND CD EXPERIENCE RIGID BODY ROTATIONS (NO BENDING); ASSUME SMALL ROTATIONS SO THAT $\delta = \theta \times \text{BAR LENGTH}$

$$\delta_A = -\theta_B \left(\frac{L}{2} \right) = \frac{2L^2 M}{81EI} \quad \frac{2}{81} = 0.02469 \quad < \text{to the right}$$

$$\delta_D = \theta_C \left(\frac{L}{2} \right) = \frac{5L^2 M}{162EI} \quad \frac{5}{162} = 0.03086 \quad < \text{to the left}$$

(d) EQUATE EXPRESSIONS FOR δ_A AND δ_D THEN SOLVE FOR L_{CD}

$$\frac{(L_{CD})^2 (5M)}{162EI} = \frac{2L^2 M}{81EI} \quad \text{Solve, } L_{CD}: \quad L_{CD} = \frac{2\sqrt{5}L}{5} = 0.894L$$

Problem 9.5-27 A beam $ABCDE$ has simple supports at B and D and symmetrical overhangs at each end (see figure). The center span has length L and each overhang has length b . A uniform load of intensity q acts on the beam.

- (a) Determine the ratio b/L so that the deflection δ_C at the midpoint of the beam is equal to the deflections δ_A and δ_E at the ends.
 (b) For this value of b/L , what is the deflection δ_C at the midpoint?

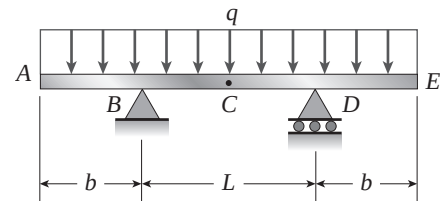
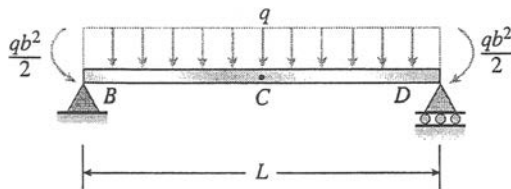
**Solution 9.5-27 Beam with overhangs**BEAM BCD :

Table H-2, Case 1 and Case 10:

$$\begin{aligned} \theta_B &= \frac{qL^3}{24EI} - \frac{qb^2}{2} \left(\frac{L}{2EI} \right) \\ &= \frac{qL}{24EI} (L^2 - 6b^2) \quad (\text{clockwise is positive}) \end{aligned}$$

$$\delta_C = \frac{5qL^4}{384EI} - \frac{qb^2}{2} \left(\frac{L^2}{8EI} \right) = \frac{qL^2}{384EI} (5L^2 - 24b^2)$$

(downward is positive) (1)

BEAM AB:

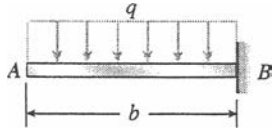


Table H-1, Case 1:

$$\begin{aligned} \delta_A &= \frac{qb^4}{8EI} - \theta_B b = \frac{qb^4}{8EI} - \frac{qL}{24EI} (L^2 - 6b^2)b \\ &= \frac{qb}{24EI} (3b^3 + 6b^2L - L^3) \end{aligned}$$

(downward is positive)

DEFLECTION δ_C EQUALS DEFLECTION δ_A

$$\frac{qL^2}{384EI} (5L^2 - 24b^2) = \frac{qb}{24EI} (3b^3 + 6b^2L - L^3)$$

Rearrange and simplify the equation:

$$48b^4 + 96b^3L + 24b^2L^2 - 16bL^3 - 5L^4 = 0$$

or

$$48\left(\frac{b}{L}\right)^4 + 96\left(\frac{b}{L}\right)^3 + 24\left(\frac{b}{L}\right)^2 - 16\left(\frac{b}{L}\right) - 5 = 0$$

(a) RATIO $\frac{b}{L}$

Solve the preceding equation numerically:

$$\frac{b}{L} = 0.40301 \quad \text{Say,} \quad \frac{b}{L} = 0.4030 \quad \leftarrow$$

(b) DEFLECTION δ_C (EQ. 1)

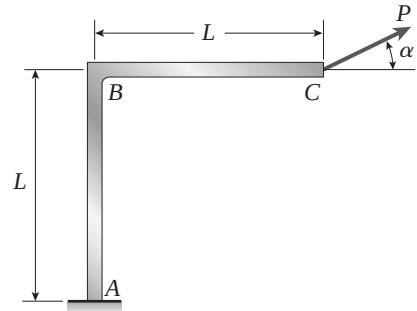
$$\begin{aligned} \delta_C &= \frac{qL^2}{384EI} (5L^2 - 24b^2) \\ &= \frac{qL^2}{384EI} [5L^2 - 24(0.40301L)^2] \\ &= 0.002870 \frac{qL^4}{EI} \end{aligned}$$

(downward deflection) $\leftarrow$

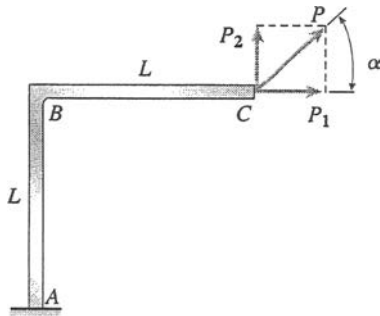
Problem 9.5-28 A frame ABC is loaded at point C by a force P acting at an angle α to the horizontal (see figure). Both members of the frame have the same length and the same flexural rigidity.

Determine the angle α so that the deflection of point C is in the same direction as the load. (Disregard the effects of axial deformations and consider only the effects of bending due to the load P .)

Note: A direction of loading such that the resulting deflection is in the same direction as the load is called a *principal direction*. For a given load on a planar structure, there are two principal directions, perpendicular to each other.



Solution 9.5-28 Principal directions for a frame



P_1 and P_2 are the components of the load P

$$P_1 = P \cos \alpha$$

$$P_2 = P \sin \alpha$$

If P_1 acts alone: $\delta'_H = \frac{P_1 L^3}{3EI}$ (to the right)

$$\delta'_V = \theta_B L = \left(\frac{P_1 L^2}{2EI} \right) L = \frac{P_1 L^3}{2EI}$$

(downward)

If P_2 acts alone: $\delta_H'' = \frac{P_2 L^3}{2EI}$ (to the left)

$$\delta_v'' = \frac{P_2 L^3}{3EI} + \theta_B L = \frac{P_2 L^3}{3EI} + \left(\frac{P_2 L^2}{EI} \right) L = \frac{4P_2 L^3}{3EI}$$

(upward)

DEFLECTIONS DUE TO THE LOAD P

$$\delta_H = \frac{P_1 L^3}{3EI} - \frac{P_2 L^3}{2EI} = \frac{L^3}{6EI} (2P_1 - 3P_2)$$

(to the right)

$$\delta_v = -\frac{P_1 L^3}{2EI} + \frac{4P_2 L^3}{3EI} = \frac{L^3}{6EI} (-3P_1 + 8P_2)$$

(upward)

$$\frac{\delta_v}{\delta_H} = \frac{-3P_1 + 8P_2}{2P_1 - 3P_2}$$

$$= \frac{-3P \cos \alpha + 8P \sin \alpha}{2P \cos \alpha - 3P \sin \alpha} = \frac{-3 + 8 \tan \alpha}{2 - 3 \tan \alpha}$$

PRINCIPAL DIRECTIONS

The deflection of point C is in the same direction as the load P .

$$\therefore \tan \alpha = \frac{P_2}{P_1} = \frac{\delta_v}{\delta_H} \quad \text{or} \quad \tan \alpha = \frac{-3 + 8 \tan \alpha}{2 - 3 \tan \alpha}$$

Rearrange and simplify: $\tan^2 \alpha + 2 \tan \alpha - 1 = 0$ (quadratic equation)

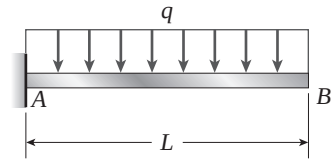
Solving, $\tan \alpha = -1 \pm \sqrt{2}$:

$$\alpha = 22.5^\circ, 112.5^\circ, -67.5^\circ, -157.5^\circ \quad \leftarrow$$

Moment-Area Method

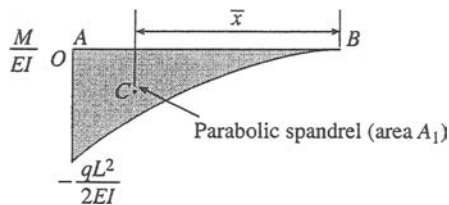
The problems for Section 9.6 are to be solved by the moment-area method. All beams have constant flexural rigidity EI .

Problem 9.6-1 A cantilever beam AB is subjected to a uniform load of intensity q acting throughout its length (see figure). Determine the angle of rotation θ_B and the deflection δ_B at the free end.



Solution 9.6-1 Cantilever beam (uniform load)

$\frac{M}{EI}$ DIAGRAM



ANGLE OF ROTATION

Use absolute values of areas.

Appendix E, Case 18: $A_1 = \frac{1}{3}(L) \left(\frac{qL^2}{2EI} \right) = \frac{qL^3}{6EI}$

$$\bar{x} = \frac{3L}{4}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_1 = \frac{qL^3}{6EI}$$

$$\theta_A = 0 \quad \theta_B = \frac{qL^3}{6EI} \quad (\text{clockwise}) \quad \leftarrow$$

DEFLECTION

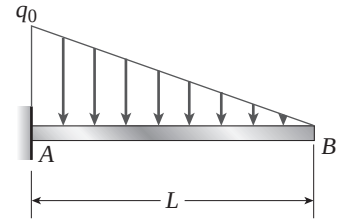
Q_1 = First moment of area A_1 with respect to B

$$Q_1 = A_1 \bar{x} = \left(\frac{qL^3}{6EI} \right) \left(\frac{3L}{4} \right) = \frac{qL^4}{8EI}$$

$$\delta_B = Q_1 = \frac{qL^4}{8EI} \quad (\text{downward}) \quad \leftarrow$$

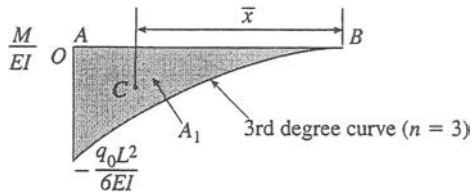
(These results agree with Case 1, Table H-1.)

Problem 9.6-2 The load on a cantilever beam AB has a triangular distribution with maximum intensity q_0 (see figure). Determine the angle of rotation θ_B and the deflection δ_B at the free end.



Solution 9.6-2 Cantilever beam (triangular load)

$\frac{M}{EI}$ DIAGRAM



ANGLE OF ROTATION

Use absolute values of areas.

Appendix E, Case 20:

$$A_1 = \frac{bh}{n+1} = \frac{1}{4}(L)\left(\frac{q_0L^2}{6EI}\right) = \frac{q_0L^3}{24EI}$$

$$\bar{x} = \frac{b(n+1)}{n+2} = \frac{4L}{5}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_1 = \frac{q_0L^3}{24EI}$$

$$\theta_A = 0 \quad \theta_B = \frac{q_0L^3}{24EI} \quad (\text{clockwise}) \quad \leftarrow$$

DEFLECTION

Q_1 = First moment of area A_1 with respect to B

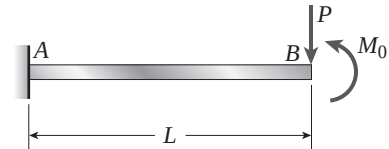
$$Q_1 = A_1\bar{x} = \left(\frac{q_0L^3}{24EI}\right)\left(\frac{4L}{5}\right) = \frac{q_0L^4}{30EI}$$

$$\delta_B = Q_1 = \frac{q_0L^4}{30EI} \quad (\text{downward}) \quad \leftarrow$$

(These results agree with Case 8, Table H-1.)

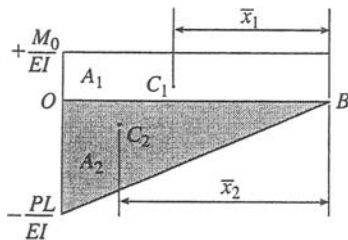
Problem 9.6-3 A cantilever beam AB is subjected to a concentrated load P and a couple M_0 acting at the free end (see figure).

Obtain formulas for the angle of rotation θ_B and the deflection δ_B at end B .



Solution 9.6-3 Cantilever beam (force P and couple M_0)

$\frac{M}{EI}$ DIAGRAM



NOTE: A_1 is the M/EI diagram for M_0 (rectangle). A_2 is the M/EI diagram for P (triangle).

ANGLE OF ROTATION

Use the sign conventions for the moment-area theorems (page 762 of textbook).

$$A_1 = \frac{M_0L}{EI} \quad \bar{x}_1 = \frac{L}{2} \quad A_2 = -\frac{PL^2}{2EI} \quad \bar{x}_2 = \frac{2L}{3}$$

$$A_0 = A_1 + A_2 = \frac{M_0 L}{EI} - \frac{PL^2}{2EI}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_0 \quad \theta_A = 0$$

$$\theta_B = A_0 = \frac{M_0 L}{EI} - \frac{PL^2}{2EI}$$

(θ_B is positive when counterclockwise)

DEFLECTION

Q = first moment of areas A_1 and A_2 with respect to point B

$$Q = A_1 \bar{x}_1 + A_2 \bar{x}_2 = \frac{M_0 L^2}{2EI} - \frac{PL^3}{3EI}$$

$$t_{B/A} = Q = \delta_B \quad \delta_B = \frac{M_0 L^2}{2EI} - \frac{PL^3}{3EI}$$

(δ_B is positive when upward)

FINAL RESULTS

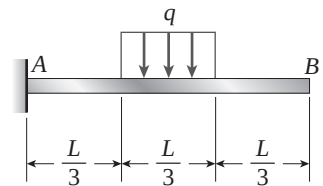
To match the sign conventions for θ_B and δ_B used in Appendix H, change the signs as follows.

$$\theta_B = \frac{PL^2}{2EI} - \frac{M_0 L}{EI} \quad (\text{positive clockwise}) \quad \leftarrow$$

$$\delta_B = \frac{PL^3}{3EI} - \frac{M_0 L^2}{2EI} \quad (\text{positive downward}) \quad \leftarrow$$

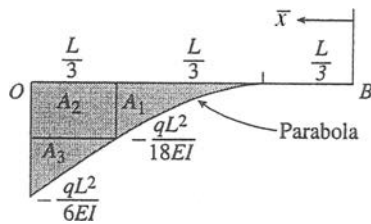
(These results agree with Cases 4 and 6, Table H-1.)

Problem 9.6-4 Determine the angle of rotation θ_B and the deflection δ_B at the free end of a cantilever beam AB with a uniform load of intensity q acting over the middle third of the length (see figure).



Solution 9.6-4 Cantilever beam with partial uniform load

$\frac{M}{EI}$ DIAGRAM



ANGLE OF ROTATION

Use absolute values of areas. Appendix E, Cases 1, 6, and 18:

$$A_1 = \frac{1}{3} \left(\frac{L}{3} \right) \left(\frac{qL^2}{18EI} \right) = \frac{qL^3}{162EI}$$

$$\bar{x}_1 = \frac{L}{3} + \frac{3}{4} \left(\frac{L}{3} \right) = \frac{7L}{12}$$

$$A_2 = \left(\frac{L}{3} \right) \left(\frac{qL^2}{18EI} \right) = \frac{qL^3}{54EI} \quad \bar{x}_2 = \frac{2L}{3} + \frac{L}{6} = \frac{5L}{6}$$

$$A_3 = \frac{1}{2} \left(\frac{L}{3} \right) \left(\frac{qL^2}{9EI} \right) = \frac{qL^3}{54EI}$$

$$\bar{x}_3 = \frac{2L}{3} + \frac{2}{3} \left(\frac{L}{3} \right) = \frac{8L}{9}$$

$$A_0 = A_1 + A_2 + A_3 = \frac{7qL^3}{162EI}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_0$$

$$\theta_A = 0 \quad \theta_B = \frac{7qL^3}{162EI} \quad (\text{clockwise}) \quad \leftarrow$$

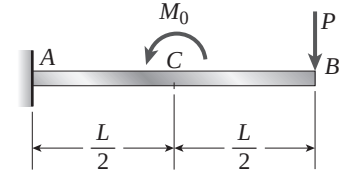
DEFLECTION

Q = first moment of area A_0 with respect to point B

$$Q = A_1 \bar{x}_1 + A_2 \bar{x}_2 + A_3 \bar{x}_3 = \frac{23qL^4}{648EI}$$

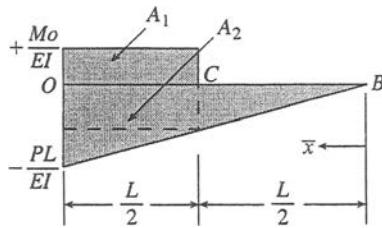
$$\delta_B = Q = \frac{23qL^4}{648EI} \quad (\text{downward}) \quad \leftarrow$$

Problem 9.6-5 Calculate the deflections δ_B and δ_C at points B and C, respectively, of the cantilever beam ACB shown in the figure. Assume $M_0 = 36$ k-in., $P = 3.8$ k, $L = 8$ ft, and $EI = 2.25 \times 10^9$ lb-in.²



Solution 9.6-5 Cantilever beam (force P and couple M_0)

$\frac{M}{EI}$ DIAGRAM



NOTE: A_1 is the M/EI diagram for M_0 (rectangle). A_2 is the M/EI diagram for P (triangle).

Use the sign conventions for the moment-area theorems (page 762 of textbook).

DEFLECTION δ_B

Q_B = first moment of areas A_1 and A_2 with respect to point B

$$\begin{aligned} &= A_1 \bar{x}_1 + A_2 \bar{x}_2 = \left(\frac{M_0}{EI} \right) \left(\frac{L}{2} \right) \left(\frac{3L}{4} \right) \\ &\quad - \frac{1}{2} \left(\frac{PL}{EI} \right) (L) \left(\frac{2L}{3} \right) \\ &= \frac{L^2}{24EI} (9M_0 - 8PL) \end{aligned}$$

$$t_{B/A} = Q_B = \delta_B \quad \delta_B = \frac{L^2}{24EI} (9M_0 - 8PL)$$

(δ_B is positive when upward)

DEFLECTION δ_C

Q_C = first moment of areas A_1 and left-hand part of A_2 with respect to point C

$$\begin{aligned} &= \left(\frac{M_0}{EI} \right) \left(\frac{L}{2} \right) \left(\frac{L}{4} \right) - \left(\frac{PL}{2EI} \right) \left(\frac{L}{2} \right) \left(\frac{L}{4} \right) \\ &\quad - \frac{1}{2} \left(\frac{PL}{2EI} \right) \left(\frac{L}{2} \right) \left(\frac{L}{3} \right) \\ &= \frac{L^2}{48EI} (6M_0 - 5PL) \end{aligned}$$

$$t_{C/A} = Q_C = \delta_C \quad \delta_C = \frac{L^2}{48EI} (6M_0 - 5PL)$$

(δ_C is positive when upward)

ASSUME DOWNWARD DEFLECTIONS ARE POSITIVE
(change the signs of δ_B and δ_C)

$$\delta_B = \frac{L^2}{24EI} (8PL - 9M_0) \quad \leftarrow$$

$$\delta_C = \frac{L^2}{48EI} (5PL - 6M_0) \quad \leftarrow$$

SUBSTITUTE NUMERICAL VALUES:

$$M_0 = 36 \text{ k-in.} \quad P = 3.8 \text{ k}$$

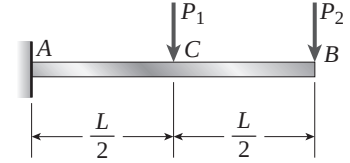
$$L = 8 \text{ ft} = 96 \text{ in.} \quad EI = 2.25 \times 10^6 \text{ k-in.}^2$$

$$\delta_B = 0.4981 \text{ in.} - 0.0553 \text{ in.} = 0.443 \text{ in.} \quad \leftarrow$$

$$\delta_C = 0.1556 \text{ in.} - 0.0184 \text{ in.} = 0.137 \text{ in.} \quad \leftarrow$$

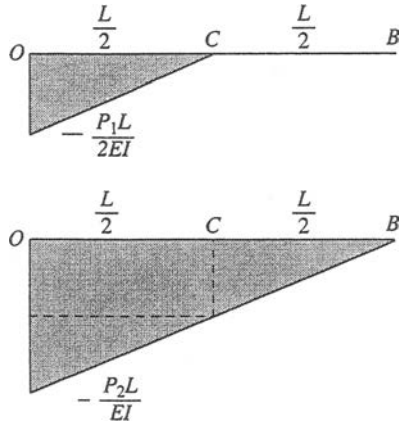
Problem 9.6-6 A cantilever beam ACB supports two concentrated loads P_1 and P_2 as shown in the figure.

Calculate the deflections δ_B and δ_C at points B and C , respectively. Assume $P_1 = 10$ kN, $P_2 = 5$ kN, $L = 2.6$ m, $E = 200$ GPa, and $I = 20.1 \times 10^6$ mm⁴.



Solution 9.6-6 Cantilever beam (forces P_1 and P_2)

$\frac{M}{EI}$ DIAGRAMS



$$P_1 = 10 \text{ kN} \quad P_2 = 5 \text{ kN} \quad L = 2.6 \text{ m}$$

$$E = 200 \text{ GPa} \quad I = 20.1 \times 10^6 \text{ mm}^4$$

Use absolute values of areas.

DEFLECTION δ_B

$\delta_B = t_{B/A} = Q_B =$ first moment of areas with respect to point B

$$\begin{aligned} \delta_B &= \frac{1}{2} \left(\frac{P_1 L}{2EI} \right) \left(\frac{L}{2} \right) \left(\frac{L}{2} + \frac{L}{3} \right) + \frac{1}{2} \left(\frac{P_2 L}{EI} \right) (L) \left(\frac{2L}{3} \right) \\ &= \frac{5P_1 L^3}{48EI} + \frac{P_2 L^3}{3EI} \quad (\text{downward}) \quad \leftarrow \end{aligned}$$

DEFLECTION δ_C

$\delta_C = t_{C/A} = Q_C =$ first moment of areas to the left of point C with respect to point C

$$\begin{aligned} \delta_C &= \frac{1}{2} \left(\frac{P_1 L}{2EI} \right) \left(\frac{L}{2} \right) \left(\frac{L}{3} \right) + \left(\frac{P_2 L}{2EI} \right) \left(\frac{L}{2} \right) \left(\frac{L}{4} \right) \\ &\quad + \frac{1}{2} \left(\frac{P_2 L}{2EI} \right) \left(\frac{L}{2} \right) \left(\frac{L}{3} \right) \\ &= \frac{P_1 L^3}{24EI} + \frac{5P_2 L^3}{48EI} \quad (\text{downward}) \quad \leftarrow \end{aligned}$$

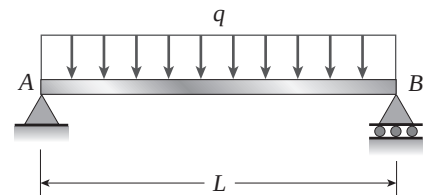
SUBSTITUTE NUMERICAL VALUES:

$$\delta_B = 4.554 \text{ mm} + 7.287 \text{ mm} = 11.84 \text{ mm} \quad \leftarrow$$

$$\delta_C = 1.822 \text{ mm} + 2.277 \text{ mm} = 4.10 \text{ mm} \quad \leftarrow$$

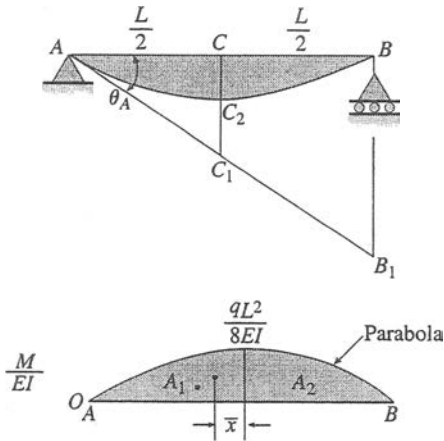
(deflections are downward)

Problem 9.6-7 Obtain formulas for the angle of rotation θ_A at support A and the deflection $\delta_{\max}$ at the midpoint for a simple beam AB with a uniform load of intensity q (see figure).



Solution 9.6-7 Simple beam with a uniform load

DEFLECTION CURVE AND $\frac{M}{EI}$ DIAGRAM


$$\delta_{\max} = \text{maximum deflection (distance } CC_2)$$

Use absolute values of areas.

ANGLE OF ROTATION AT END A

Appendix E, Case 17:

$$A_1 = A_2 = \frac{2}{3} \left(\frac{L}{2} \right) \left(\frac{qL^2}{8EI} \right) = \frac{qL^3}{24EI}$$

$$\bar{x}_1 = \frac{3}{8} \left(\frac{L}{2} \right) = \frac{3L}{16}$$

$t_{B/A} = BB_1$ = first moment of areas A_1 and A_2 with respect to point B

$$= (A_1 + A_2) \left(\frac{L}{2} \right) = \frac{qL^4}{24EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{qL^3}{24EI} \quad (\text{clockwise}) \quad \leftarrow$$

DEFLECTION $\delta_{\max}$ AT THE MIDPOINT C

$$\text{Distance } CC_1 = \frac{1}{2} (BB_1) = \frac{qL^4}{48EI}$$

$t_{C_2/A} = C_2C_1$ = first moment of area A_1 with respect to point C

$$= A_1 \bar{x}_1 = \left(\frac{qL^3}{24EI} \right) \left(\frac{3L}{16} \right) = \frac{qL^4}{128EI}$$

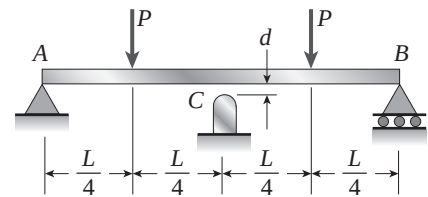
$$\delta_{\max} = CC_2 = CC_1 - C_2C_1 = \frac{qL^4}{48EI} - \frac{qL^4}{128EI}$$

$$= \frac{5qL^4}{384EI} \quad (\text{downward}) \quad \leftarrow$$

(These results agree with Case 1 of Table H-2.)

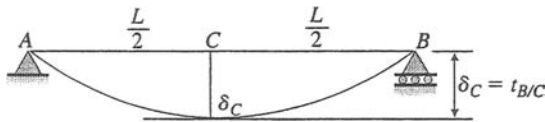
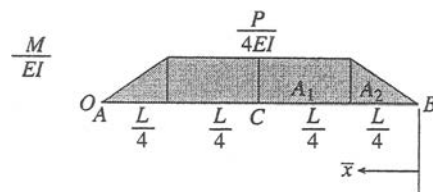
Problem 9.6-8 A simple beam AB supports two concentrated loads P at the positions shown in the figure. A support C at the midpoint of the beam is positioned at distance d below the beam before the loads are applied.

Assuming that $d = 10$ mm, $L = 6$ m, $E = 200$ GPa, and $I = 198 \times 10^6$ mm⁴, calculate the magnitude of the loads P so that the beam just touches the support at C .



Solution 9.6-8 Simple beam with two equal loads

DEFLECTION CURVE AND $\frac{M}{EI}$ DIAGRAM

 δ_C = deflection at the midpoint C

$$A_1 = \frac{PL^2}{16EI} \quad \bar{x}_1 = \frac{3L}{8}$$

$$A_2 = \frac{PL^2}{32EI} \quad \bar{x}_2 = \frac{L}{6}$$

Use absolute values of areas.

DEFLECTION δ_C AT MIDPOINT OF BEAM

At point C, the deflection curve is horizontal.

$\delta_C = t_{B/C}$ = first moment of area between B and C with respect to B

$$= A_1\bar{x}_1 + A_2\bar{x}_2 = \frac{PL^2}{16EI} \left(\frac{3L}{8} \right) + \frac{PL^2}{32EI} \left(\frac{L}{6} \right)$$

$$= \frac{11PL^3}{384EI}$$

d = gap between the beam and the support at C

MAGNITUDE OF LOAD TO CLOSE THE GAP

$$\delta = d = \frac{11PL^3}{384EI} \quad P = \frac{384EId}{11L^3} \quad \leftarrow$$

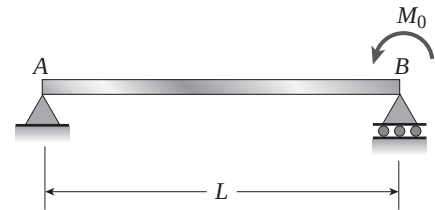
SUBSTITUTE NUMERICAL VALUES

$$d = 10 \text{ mm} \quad L = 6 \text{ m} \quad E = 200 \text{ GPa}$$

$$I = 198 \times 10^6 \text{ mm}^4 \quad P = 64 \text{ kN} \quad \leftarrow$$

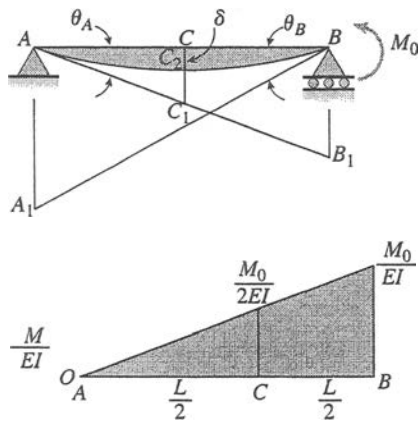
Problem 9.6-9 A simple beam AB is subjected to a load in the form of a couple M_0 acting at end B (see figure).

Determine the angles of rotation θ_A and θ_B at the supports and the deflection δ at the midpoint.



Solution 9.6-9 Simple beam with a couple M_0

DEFLECTION CURVE AND $\frac{M}{EI}$ DIAGRAM



δ = deflection at the midpoint C

δ = distance CC_2

Use absolute values of areas.

ANGLE OF ROTATION θ_A

$t_{B/A} = BB_1$ = first moment of area between A and B with respect to B

$$= \frac{1}{2} \left(\frac{M_0}{EI} \right) (L) \left(\frac{L}{3} \right) = \frac{M_0 L^2}{6EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{M_0 L}{6EI} \quad (\text{clockwise}) \quad \leftarrow$$

ANGLE OF ROTATION θ_B

$t_{A/B} = AA_1$ = first moment of area between A and B with respect to A

$$= \frac{1}{2} \left(\frac{M_0}{EI} \right) (L) \left(\frac{2L}{3} \right) = \frac{M_0 L^2}{3EI}$$

$$\theta_B = \frac{AA_1}{L} = \frac{M_0 L}{3EI} \quad (\text{counterclockwise}) \quad \leftarrow$$

DEFLECTION δ AT THE MIDPOINT C

$$\text{Distance } CC_1 = \frac{1}{2}(BB_1) = \frac{M_0 L^2}{12EI}$$

 $t_{C_2/A} = C_2 C_1 = \text{first moment of area between } A \text{ and } C \text{ with respect to } C$

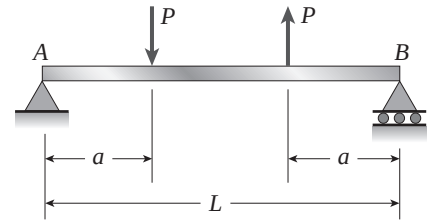
$$= \frac{1}{2} \left(\frac{M_0}{2EI} \right) \left(\frac{L}{2} \right) \left(\frac{L}{6} \right) = \frac{M_0 L^2}{48EI}$$

$$\begin{aligned} \delta &= CC_1 - C_2 C_1 = \frac{M_0 L^2}{12EI} - \frac{M_0 L^2}{48EI} \\ &= \frac{M_0 L^2}{16EI} \quad (\text{downward}) \quad \leftarrow \end{aligned}$$

(These results agree with Case 7 of Table H-2.)

Problem 9.6-10 The simple beam AB shown in the figure supports two equal concentrated loads P , one acting downward and the other upward.

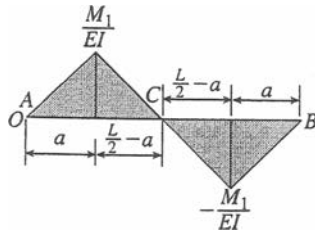
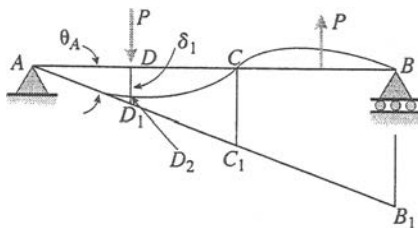
Determine the angle of rotation θ_A at the left-hand end, the deflection δ_1 under the downward load, and the deflection δ_2 at the midpoint of the beam.



Solution 9.6-10 Simple beam with two loads

Because the beam is symmetric and the load is antisymmetric, the deflection at the midpoint is zero.

$$\therefore \delta_2 = 0 \quad \leftarrow$$



$$\frac{M_1}{EI} = \frac{Pa(L-2a)}{LEI}$$

$$A_1 = \frac{1}{2} \left(\frac{M_1}{EI} \right) (a) = \frac{Pa^2(L-2a)}{2LEI}$$

$$A_2 = \frac{1}{2} \left(\frac{M_1}{EI} \right) \left(\frac{L}{2} - a \right) = \frac{Pa(L-2a)^2}{4LEI}$$

ANGLE OF ROTATION θ_A AT END A
 $t_{C/A} = CC_1 = \text{first moment of area between } A \text{ and } C \text{ with respect to } C$

$$\begin{aligned} &= A_1 \left(\frac{L}{2} - a + \frac{a}{3} \right) + A_2 \left(\frac{2}{3} \right) \left(\frac{L}{2} - a \right) \\ &= \frac{Pa(L-a)(L-2a)}{12EI} \end{aligned}$$

$$\theta_A = \frac{CC_1}{L/2} = \frac{Pa(L-a)(L-2a)}{6LEI} \quad (\text{clockwise}) \quad \leftarrow$$

DEFLECTION δ_1 UNDER THE DOWNWARD LOAD

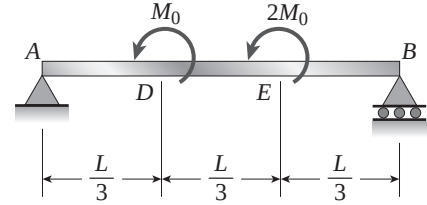
$$\begin{aligned} \text{Distance } DD_1 &= \left(\frac{a}{L/2} \right) (CC_1) \\ &= \frac{Pa^2(L-a)(L-2a)}{6LEI} \end{aligned}$$

 $t_{D_2/A} = D_2 D_1 = \text{first moment of area between } A \text{ and } D \text{ with respect to } D$

$$= A_1 \left(\frac{a}{3} \right) = \frac{Pa^3(L-2a)}{6LEI}$$

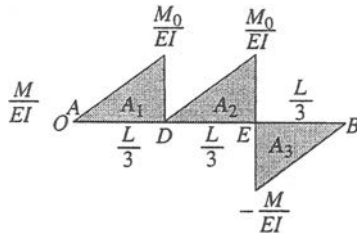
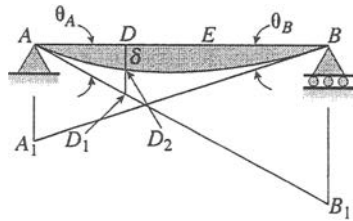
$$\begin{aligned} \delta_1 &= DD_1 - D_2 D_1 \\ &= \frac{Pa^2(L-2a)^2}{6LEI} \quad (\text{downward}) \quad \leftarrow \end{aligned}$$

Problem 9.6-11 A simple beam AB is subjected to couples M_0 and $2M_0$ acting as shown in the figure. Determine the angles of rotation θ_A and θ_B at the ends of the beam and the deflection δ at point D where the load M_0 is applied.



Solution 9.6-11 Simple beam with two couples

DEFLECTION CURVE AND $\frac{M}{EI}$ DIAGRAM



$$A_1 = A_2 = \frac{1}{2} \left(\frac{M_0}{EI} \right) \left(\frac{L}{3} \right) = \frac{M_0 L}{6EI} \quad A_3 = -\frac{M_0 L}{6EI}$$

ANGLE OF ROTATION θ_A AT END A

$t_{B/A} = BB_1$ = first moment of area between A and B with respect to B

$$= A_1 \left(\frac{2L}{3} + \frac{L}{9} \right) + A_2 \left(\frac{L}{3} + \frac{L}{9} \right) + A_3 \left(\frac{2L}{9} \right)$$

$$= \frac{M_0 L^2}{6EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{M_0 L}{6EI} \quad (\text{clockwise}) \quad \leftarrow$$

ANGLE OF ROTATION θ_B AT END B

$t_{A/B} = AA_1$ = first moment of area between A and B with respect to A

$$= A_1 \left(\frac{2L}{9} \right) + A_2 \left(\frac{L}{3} + \frac{2L}{9} \right) + A_3 \left(\frac{2L}{3} + \frac{L}{9} \right) = 0$$

$$\theta_B = \frac{AA_1}{L} = 0 \quad \leftarrow$$

DEFLECTION δ AT POINT D

$$\text{Distance } DD_1 = \frac{1}{3}(BB_1) = \frac{M_0 L^2}{18EI}$$

$t_{D_2/A} = D_2D_1$ = first moment of area between A and D with respect to D

$$= A_1 \left(\frac{L}{9} \right) = \frac{M_0 L^2}{54EI}$$

$$\delta = DD_1 - D_2D_1 = \frac{M_0 L^2}{27EI}$$

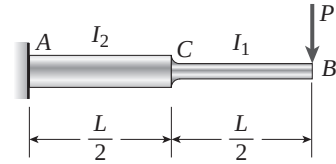
(downward) $\leftarrow$

Note: This deflection is also the maximum deflection.

Nonprismatic Beams

Problem 9.7-1 The cantilever beam ACB shown in the figure has moments of inertia I_2 and I_1 in parts AC and CB , respectively.

- Using the method of superposition, determine the deflection δ_B at the free end due to the load P .
- Determine the ratio r of the deflection δ_B to the deflection δ_1 at the free end of a prismatic cantilever with moment of inertia I_1 carrying the same load.
- Plot a graph of the deflection ratio r versus the ratio I_2/I_1 of the moments of inertia. (Let I_2/I_1 vary from 1 to 5.)

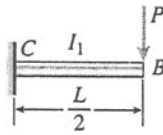


Solution 9.7-1 Cantilever beam (nonprismatic)

Use the method of superposition.

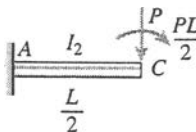
(a) DEFLECTION δ_B AT THE FREE END

(1) Part CB of the beam:



$$(\delta_B)_1 = \frac{P}{3EI_1} \left(\frac{L}{2} \right)^3 = \frac{PL^3}{24EI_1}$$

(2) Part AC of the beam:



$$\delta_C = \frac{P(L/2)^3}{3EI_2} + \frac{(PL/2)(L/2)^2}{2EI_2} = \frac{5PL^3}{48EI_2}$$

$$\theta_C = \frac{P(L/2)^2}{2EI_2} + \frac{(PL/2)(L/2)}{EI_2} = \frac{3PL^2}{8EI_2}$$

$$(\delta_B)_2 = \delta_C + \theta_C \left(\frac{L}{2} \right) = \frac{7PL^3}{24EI_2}$$

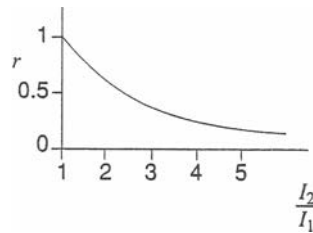
(3) Total deflection at point B

$$\delta_B = (\delta_B)_1 + (\delta_B)_2 = \frac{PL^3}{24EI_1} \left(1 + \frac{7I_1}{I_2} \right) \quad \leftarrow$$

(b) PRISMATIC BEAM $\delta_1 = \frac{PL^3}{3EI_1}$

$$\text{Ratio: } r = \frac{\delta_B}{\delta_1} = \frac{1}{8} \left(1 + \frac{7I_1}{I_2} \right) \quad \leftarrow$$

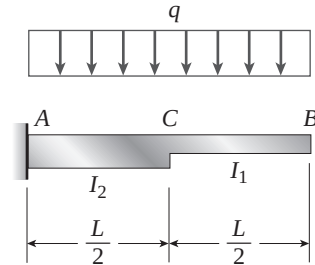
(c) GRAPH OF RATIO



$\frac{I_2}{I_1}$	r
1	1.00
2	0.56
3	0.42
4	0.34
5	0.30

Problem 9.7-2 The cantilever beam ACB shown in the figure supports a uniform load of intensity q throughout its length. The beam has moments of inertia I_2 and I_1 in parts AC and CB , respectively.

- Using the method of superposition, determine the deflection δ_B at the free end due to the uniform load.
- Determine the ratio r of the deflection δ_B to the deflection δ_1 at the free end of a prismatic cantilever with moment of inertia I_1 carrying the same load.
- Plot a graph of the deflection ratio r versus the ratio I_2/I_1 of the moments of inertia. (Let I_2/I_1 vary from 1 to 5.)

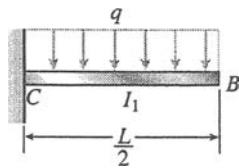


Solution 9.7-2 Cantilever beam (nonprismatic)

Use the method of superposition

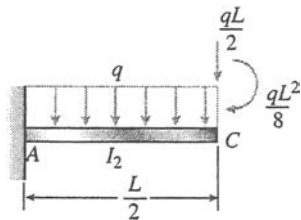
(a) DEFLECTION δ_B AT THE FREE END

(1) Part CB of the beam:



$$(\delta_B)_1 = \frac{q}{8EI_1} \left(\frac{L}{2} \right)^4 = \frac{qL^4}{128EI_1}$$

(2) Part AC of the beam:



$$\begin{aligned} \delta_C &= \frac{q(L/2)^4}{8EI_2} + \frac{\left(\frac{qL}{2} \right) (L/2)^3}{3EI_2} \\ &\quad + \frac{\left(\frac{qL^2}{8} \right) \left(\frac{L}{2} \right)^2}{2EI_2} = \frac{17qL^4}{384EI_2} \\ \theta_C &= \frac{q(L/2)^3}{6EI_2} + \frac{(qL/2)(L/2)^2}{2EI_2} + \frac{(qL^2/8)(L/2)}{EI_2} \\ &= \frac{7qL^3}{48EI_2} \\ (\delta_B)_2 &= \delta_C + \theta_C \left(\frac{L}{2} \right) = \frac{15qL^4}{128EI_2} \end{aligned}$$

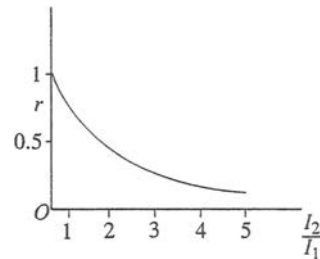
(3) Total deflection at point B

$$\delta_B = (\delta_B)_1 + (\delta_B)_2 = \frac{qL^4}{128EI_1} \left(1 + \frac{15I_1}{I_2} \right) \quad \leftarrow$$

(b) PRISMATIC BEAM $\delta_1 = \frac{qL^4}{8EI_1}$

$$\text{Ratio: } r = \frac{\delta_B}{\delta_1} = \frac{1}{16} \left(1 + \frac{15I_1}{I_2} \right) \quad \leftarrow$$

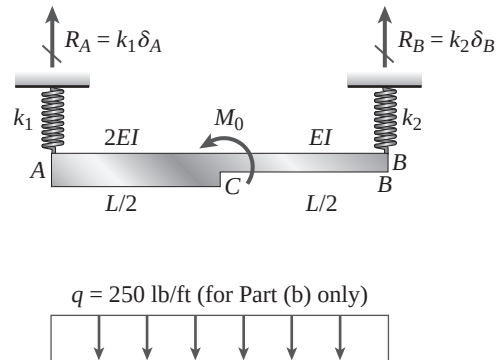
(c) GRAPH OF RATIO



$\frac{I_2}{I_1}$	r
1	1.00
2	0.53
3	0.38
4	0.30
5	0.25

Problem 9.7-3 Beam ACB hangs from two springs, as shown in the figure. The springs have stiffnesses k_1 and k_2 and the beam has flexural rigidity EI .

- (a) What is the downward displacement of point C , which is at the midpoint of the beam, when the moment M_0 is applied? Data for the structure are as follows: $M_0 = 7.5$ k-ft, $L = 6$ ft, $EI = 520$ k-ft², $k_1 = 17$ k/ft, and $k_2 = 11$ k/ft.
- (b) Repeat part (a), but remove M_0 and instead, apply uniform load q over the entire beam.



Solution 9.7-3

$$M_0 = 7.5 \text{ k-ft} \quad L = 6 \text{ ft} \quad EI = 520 \text{ k-ft}^2 \quad k_1 = 17 \text{ k/ft} \quad k_2 = 11 \text{ k/ft} \quad q = 250 \text{ lb/ft}$$

- (a) BENDING-MOMENT EQUATIONS—MOMENT M_0 AT C

$$2EIv'' = M = \frac{M_0 x}{L} \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$2EIv' = \frac{M_0 x^2}{2L} + C_1 \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$2EIv = \frac{M_0 x^3}{6L} + C_1 x + C_2 \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$\text{B.C. } v(0) = 0 \quad C_2 = 0 \quad 2EIv = \frac{M_0 x^3}{6L} + C_1 x \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$EIv'' = -\frac{M_0}{2} + \frac{M_0 \left(x - \frac{L}{2}\right)}{L} = -M_0 + \frac{M_0 x}{L} \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$EIv' = -M_0 x + \frac{M_0 x^2}{2L} + C_3 \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$EIv = -\frac{M_0 x^2}{2} + \frac{M_0 x^3}{6L} + C_3 x + C_4 \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$\text{B.C. } v(L) = 0 \quad -\frac{M_0 L^2}{2} + \frac{M_0 L^3}{6L} + C_3 L + C_4 = 0 \quad (1)$$

$$\text{B.C. } v'_L \left(\frac{L}{2}\right) = v'_R \left(\frac{L}{2}\right) \quad \frac{1}{2} \left[\frac{M_0 \left(\frac{L}{2}\right)^2}{2L} + C_1 \right] = -M_0 + \frac{M_0 \left(\frac{L}{2}\right)^2}{2L} + C_3 \quad (2)$$

$$\text{B.C. } v_L \left(\frac{L}{2}\right) = v_R \left(\frac{L}{2}\right) \quad \frac{1}{2} \left[\frac{M_0 \left(\frac{L}{2}\right)^3}{6L} + C_1 \frac{L}{2} \right] = -\frac{M_0 \left(\frac{L}{2}\right)^2}{2} + \frac{M_0 \left(\frac{L}{2}\right)^3}{6L} + C_3 \frac{L}{2} + C_4 \quad (3)$$

From Eqs. (1), (2), and (3),

$$C_1 = 0 \quad C_3 = \frac{7}{16}M_0L \quad C_4 = \frac{-5}{48}M_0L^2 \quad \leftarrow$$

Therefore,

$$v(x) = \frac{M_0x^3}{12EI L} \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$v(x) = \frac{M_0}{48EI L}(-24x^2L + 8x^3 + 21L^2x - 5L^3) \quad \left(\frac{L}{2} \leq x \leq L\right)$$

DEFLECTION AT A AND B

$$R_A = \frac{M_0}{L} \quad R_B = -\frac{M_0}{L}$$

$$\delta_A = \frac{R_A}{k_1} \quad \delta_B = \frac{R_B}{k_2}$$

$$\delta_A = 0.88 \text{ in. (downward)} \quad \delta_B = -1.36 \text{ in. (upward)}$$

DEFLECTION AT POINT C

$$\delta_C = -v\left(\frac{L}{2}\right) + \frac{1}{2}(\delta_A + \delta_B)$$

$$\delta_C = -\frac{M_0\left(\frac{L}{2}\right)^3}{12EI L} + \frac{1}{2}(\delta_A + \delta_B)$$

$$\delta_C = -0.31 \text{ in. (upward)} \quad \leftarrow$$

(b) BENDING-MOMENT EQUATIONS-UNIFORM LOAD q

$$2EIv'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$2EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1 \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$2EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} + C_1x + C_2 \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$\text{B.C. } v(0) = 0 \quad C_2 = 0 \quad 2EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} + C_1x \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$EIv'' = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_3 \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} + C_3x + C_4 \quad \left(\frac{L}{2} \leq x \leq L\right)$$

$$\text{B.C. } v(L) = 0 \quad \frac{qLL^3}{12} - \frac{qL^4}{24} + C_3L + C_4 = 0 \quad (1)$$

$$\text{B.C. } v'_L\left(\frac{L}{2}\right) = v'_R\left(\frac{L}{2}\right) \quad \frac{1}{2} \left[\frac{qL\left(\frac{L}{2}\right)^2}{4} - \frac{q\left(\frac{L}{2}\right)^3}{6} + C_1 \right] = \frac{qL\left(\frac{L}{2}\right)^2}{4} - \frac{q\left(\frac{L}{2}\right)^3}{6} + C_3 \quad (2)$$

$$\begin{aligned} \text{B.C. } v_L\left(\frac{L}{2}\right) &= v_R\left(\frac{L}{2}\right) \quad \frac{1}{2} \left[\frac{qL\left(\frac{L}{2}\right)^3}{12} - \frac{q\left(\frac{L}{2}\right)^4}{24} + C_1\frac{L}{2} \right] \\ &= \frac{qL\left(\frac{L}{2}\right)^3}{12} - \frac{q\left(\frac{L}{2}\right)^4}{24} + C_3\frac{L}{2} + C_4 \end{aligned} \quad (3)$$

From Eqs. (1), (2), and (3),

$$C_1 = \frac{-7}{128}qL^3 \quad C_3 = \frac{-37}{768}qL^3 \quad C_4 = \frac{5}{768}qL^4$$

Therefore

$$\begin{aligned} v(x) &= -\frac{qx}{768EI}(-32Lx^2 + 16x^3 + 21L^3) \quad \left(0 \leq x \leq \frac{L}{2}\right) \\ v(x) &= \frac{q}{768EI}(64Lx^3 - 32x^4 - 37L^3x + 5L^4) \quad \left(\frac{L}{2} \leq x \leq L\right) \end{aligned}$$

DEFLECTION AT A AND B

$$R_A = \frac{qL}{2} \quad R_B = \frac{qL}{2}$$

$$\delta_A = \frac{R_A}{k_1} \quad \delta_B = \frac{R_B}{k_2}$$

$$\delta_A = 0.53 \text{ in. (downward)} \quad \delta_B = 0.82 \text{ in. (downward)}$$

DEFLECTION AT POINT C

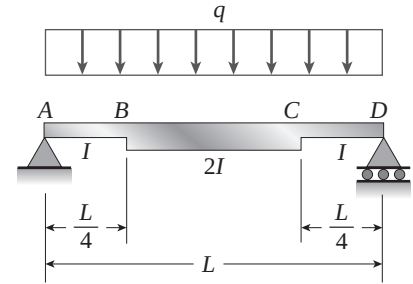
$$\delta_C = -v\left(\frac{L}{2}\right) + \frac{1}{2}(\delta_A + \delta_B)$$

$$\delta_C = \frac{q}{768EI} \left[-32L\left(\frac{L}{2}\right)^2 + 16\left(\frac{L}{2}\right)^3 + 21L^3 \right] + \frac{1}{2}(\delta_A + \delta_B)$$

$$\delta_C = 0.75 \text{ in. (downward)} \quad \leftarrow$$

Problem 9.7-4 A simple beam $ABCD$ has moment of inertia I near the supports and moment of inertia $2I$ in the middle region, as shown in the figure. A uniform load of intensity q acts over the entire length of the beam.

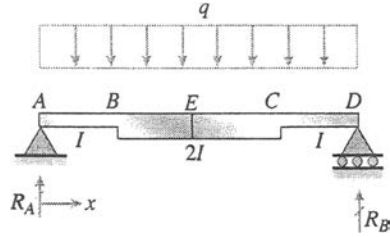
Determine the equations of the deflection curve for the left-hand half of the beam. Also, find the angle of rotation θ_A at the left-hand support and the deflection $\delta_{\max}$ at the midpoint.



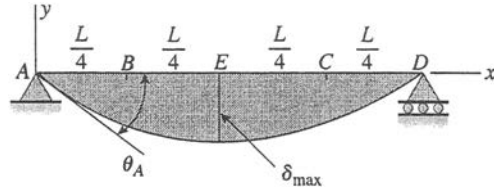
Solution 9.7-4 Simple beam (nonprismatic)

Use the bending-moment equation (Eq. 9-16a).

REACTIONS, BENDING MOMENT, AND DEFLECTION CURVE



$$R_A = R_B = \frac{qL}{2} \quad M = Rx - \frac{qx^2}{2} = \frac{qLx}{2} - \frac{qx^2}{2}$$



BENDING-MOMENT EQUATIONS FOR THE LEFT-HAND HALF OF THE BEAM

$$EIv'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(0 \leq x \leq \frac{L}{4}\right) \quad (1)$$

$$E(2I)v'' = M = \frac{qLx}{2} - \frac{qx^2}{2} \quad \left(\frac{L}{4} \leq x \leq \frac{L}{2}\right) \quad (2)$$

INTEGRATE EACH EQUATION

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1 \quad \left(0 \leq x \leq \frac{L}{4}\right) \quad (3)$$

$$2EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_2 \quad \left(\frac{L}{4} \leq x \leq \frac{L}{2}\right) \quad (4)$$

B.C. 1 Symmetry: $v'\left(\frac{L}{2}\right) = 0$

From Eq. (4): $C_2 = -\frac{qL^3}{24}$

$$2EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{qL^3}{24} \quad \left(\frac{L}{4} \leq x \leq \frac{L}{2}\right) \quad (5)$$

SLOPE AT POINT B (FROM THE RIGHT)

Substitute $x = \frac{L}{4}$ into Eq. (5):

$$EIv'_B = -\frac{11qL^3}{768} \quad (6)$$

B.C. 2 CONTINUITY OF SLOPES AT POINT B

$$(v'_B)_{\text{left}} = (v'_B)_{\text{right}}$$

From Eqs. (3) and (6):

$$\frac{qL}{4} \left(\frac{L}{4}\right)^2 - \frac{q}{6} \left(\frac{L}{4}\right)^3 + C_1 = -\frac{11qL^3}{768} \quad \therefore C_1 = -\frac{7qL^3}{256}$$

SLOPE OF THE BEAM (FROM EQS. 3 AND 5)

$$EIv' = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{7qL^3}{256} \quad \left(0 \leq x \leq \frac{L}{4}\right) \quad (7)$$

$$EIv' = \frac{qLx^2}{8} - \frac{qx^3}{12} - \frac{qL^3}{48} \quad \left(\frac{L}{4} \leq x \leq \frac{L}{2}\right) \quad (8)$$

ANGLE OF ROTATION θ_A (FROM EQ. 7)

$$\theta_A = -v'(0) = \frac{7qL^3}{256EI} \quad (\text{positive clockwise}) \quad \leftarrow$$

INTEGRATE EQS. (7) AND (8)

$$EIv = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{7qL^3x}{256} + C_3 \quad \left(0 \leq x \leq \frac{L}{4}\right) \quad (9)$$

$$EIv = \frac{qLx^3}{24} - \frac{qx^4}{48} - \frac{qL^3x}{48} + C_4 \quad \left(\frac{L}{4} \leq x \leq \frac{L}{2}\right) \quad (10)$$

B.C. 3 DEFLECTION AT SUPPORT A

$$v(0) = 0 \quad \text{From Eq. (9), } C_3 = 0$$

DEFLECTION AT POINT B (FROM THE LEFT)

Substitute $x = \frac{L}{4}$ into Eq. (9) with $C_3 = 0$.

$$EIv_B = -\frac{35qL^4}{6144} \quad (11)$$

B.C. 4 CONTINUITY OF DEFLECTIONS AT POINT B

$$(v_B)_{\text{Right}} = (v_B)_{\text{Left}}$$

From Eqs. (10) and (11),

$$\begin{aligned} \frac{qL}{24} \left(\frac{L}{4}\right)^3 - \frac{q}{48} \left(\frac{L}{4}\right)^4 - \frac{qL^3}{48} \left(\frac{L}{4}\right) + C_4 &= -\frac{35qL^4}{6144} \\ \therefore C_4 &= -\frac{13qL^4}{12,288} \end{aligned}$$

DEFLECTION OF THE BEAM (FROM EQS. 9 AND 10)

$$v = -\frac{qx}{768EI} (21L^3 - 64Lx^2 + 32x^3) \quad \left(0 \leq x \leq \frac{L}{4}\right) \quad \leftarrow$$

$$v = -\frac{q}{12,288EI} (13L^4 + 256L^3x - 512Lx^3 + 256x^4) \quad \left(\frac{L}{4} \leq x \leq \frac{L}{2}\right) \quad \leftarrow$$

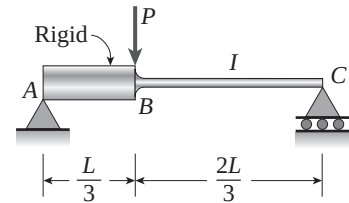
MAXIMUM DEFLECTION (AT THE MIDPOINT E)

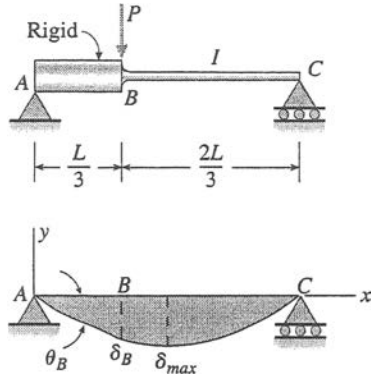
(From the preceding equation for v .)

$$\delta_{\max} = -v\left(\frac{L}{2}\right) = \frac{31qL^4}{4096EI} \quad (\text{positive downward}) \quad \leftarrow$$

Problem 9.7-5 A beam ABC has a rigid segment from A to B and a flexible segment with moment of inertia I from B to C (see figure). A concentrated load P acts at point B.

Determine the angle of rotation θ_A of the rigid segment, the deflection δ_B at point B, and the maximum deflection $\delta_{\max}$.



Solution 9.7-5 Simple beam with a rigid segment

FROM A TO B,

$$v = -\frac{3\delta_B x}{L} \quad \left(0 \leq x \leq \frac{L}{3}\right) \quad (1)$$

$$v' = -\frac{3\delta_B}{L} \quad \left(0 \leq x \leq \frac{L}{3}\right) \quad (2)$$

FROM B TO C,

$$EIv'' = M = \frac{PL}{3} - \frac{Px}{3} \quad (3)$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} + C_1$$

$$\text{B.C. 1 At } x = L/3, \quad v' = -\frac{3\delta_B}{L}$$

$$\therefore C_1 = -\frac{5PL^2}{54} - \frac{3EI\delta_B}{L}$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{6} - \frac{5PL^2}{54} - \frac{3EI\delta_B}{L} \quad \left(\frac{L}{3} \leq x \leq L\right) \quad (4)$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} + C_2 \quad \left(\frac{L}{3} \leq x \leq L\right)$$

$$\text{B.C. 2 } v(L) = 0 \quad \therefore C_2 = -\frac{PL^3}{54} + 3EI\delta_B$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} - \frac{5PL^2x}{54} - \frac{3EI\delta_B x}{L} - \frac{PL^2}{54} + 3EI\delta_B \quad \left(\frac{L}{3} \leq x \leq L\right) \quad (5)$$

$$\text{B.C. 3 At } x = \frac{L}{3}, (\nu_B)_{\text{left}} = (\nu_B)_{\text{right}} \text{ from Eqs. (1) and (5)}$$

$$\therefore \delta_B = \frac{8PL^3}{729EI} \quad \leftarrow$$

$$\theta_A = \frac{\delta_B}{L/3} = \frac{8PL^2}{243EI} \quad \leftarrow$$

Substitute for δ_B in Eq. (5) and simplify:

$$v = \frac{P}{486EI} (7L^3 - 61L^2x + 81Lx^2 - 27x^3) \quad \left(\frac{L}{3} \leq x \leq L\right) \quad (6)$$

Also,

$$v' = \frac{P}{486EI} (-61L^2 + 162Lx - 81x^2) \quad \left(\frac{L}{3} \leq x \leq L\right) \quad (7)$$

MAXIMUM DEFLECTION

$$v' = 0 \text{ gives } x_1 = \frac{L}{9} (9 - 2\sqrt{5}) = 0.5031L$$

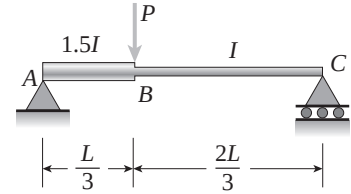
Substitute x_1 in Eq. (6) and simplify:

$$v_{\max} = -\frac{40\sqrt{5}PL^3}{6561EI}$$

$$\delta_{\max} = -v_{\max} = \frac{40\sqrt{5}PL^3}{6561EI} = 0.01363 \frac{PL^3}{EI} \quad \leftarrow$$

Problem 9.7-6 A simple beam ABC has moment of inertia $1.5I$ from A to B and I from B to C (see figure). A concentrated load P acts at point B.

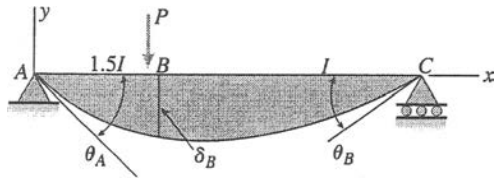
Obtain the equations of the deflection curves for both parts of the beam. From the equations, determine the angles of rotation θ_A and θ_C at the supports and the deflection δ_B at point B.



Solution 9.7-6 Simple beam (nonprismatic)

Use the bending-moment equation (Eq. 9-16a).

DEFLECTION CURVE



BENDING-MOMENT EQUATIONS

$$E\left(\frac{3I}{2}\right)v'' = M = \frac{2Px}{3} \quad \left(0 \leq x \leq \frac{L}{3}\right) \quad (1)$$

$$EIv'' = M = \frac{PL}{3} - \frac{Px}{3} \quad \left(\frac{L}{3} \leq x \leq L\right) \quad (2)$$

INTEGRATE EACH EQUATION

$$EIv' = \frac{4Px^2}{18} + C_1 \quad \left(0 \leq x \leq \frac{L}{3}\right) \quad (3)$$

$$EIv' = \frac{PLx}{3} - \frac{Px^2}{2} + C_2 \quad \left(\frac{L}{3} \leq x \leq L\right) \quad (4)$$

B.C. 1 Continuity of slopes at point B:

$$(v'_B)_{\text{left}} = (v'_B)_{\text{right}}$$

From Eqs. (3) and (4),

$$\frac{4P}{18}\left(\frac{L}{3}\right)^2 + C_1 = \frac{PL}{3}\left(\frac{L}{3}\right) - \frac{P}{6}\left(\frac{L}{3}\right)^2 + C_2$$

$$C_2 = C_1 - \frac{11PL^2}{162} \quad (5)$$

INTEGRATE EQS. (3) AND (4)

$$EIv = \frac{4Px^3}{54} + C_1x + C_3 \quad \left(0 \leq x \leq \frac{L}{3}\right) \quad (6)$$

$$EIv = \frac{PLx^2}{6} - \frac{Px^3}{18} + C_2x + C_4 \quad \left(\frac{L}{3} \leq x \leq L\right) \quad (7)$$

B.C. 2 Deflection at support A:

$$v(0) = 0 \quad \text{From Eq. (6):} \quad C_3 = 0 \quad (8)$$

B.C. 3 Deflection at support C:

$$v(L) = 0 \quad \text{From Eq. (7):} \quad C_4 = -\frac{PL^3}{9} - C_2L \quad (9)$$

B.C. 4 Continuity of deflections at point B:

$$(v_B)_{\text{left}} = (v_B)_{\text{right}}$$

From Eqs. (6), (8), and (9),

$$\frac{4P}{54}\left(\frac{L}{3}\right)^3 + C_1\left(\frac{L}{3}\right) = \frac{PL}{6}\left(\frac{L}{3}\right)^2 - \frac{P}{18}\left(\frac{L}{3}\right)^3 + C_2\left(\frac{L}{3}\right) + C_4 \quad (10)$$

$$C_1L = \frac{10PL^3}{243} + C_2L + 3C_4$$

SOLVE EQS. (5), (8), (9), AND (10)

$$C_1 = -\frac{38PL^2}{729} \quad C_2 = -\frac{175PL^2}{1458} \quad C_3 = 0$$

$$C_4 = \frac{13PL^3}{1458}$$

SLOPES OF THE BEAM (FROM EQS. 3 AND 4)

$$v' = -\frac{2P}{729EI}(19L^2 - 81x^2) \quad \left(0 \leq x \leq \frac{L}{3}\right) \quad (11)$$

$$v' = -\frac{P}{1458EI}(175L^2 - 486Lx + 243x^2) \quad \left(\frac{L}{3} \leq x \leq L\right) \quad (12)$$

ANGLE OF ROTATION θ_A (FROM EQ. 11)

$$\theta_A = -v'(0) = \frac{38PL^2}{729EI} \quad (\text{positive clockwise}) \quad \leftarrow$$

ANGLE OF ROTATION θ_C (FROM EQ. 12)

$$\theta_C = v'(L) = \frac{34PL^2}{729EI} \quad (\text{positive counterclockwise}) \quad \leftarrow$$

DEFLECTIONS OF THE BEAM

Substitute C_1 , C_2 , C_3 , and C_4 into Eqs. (6) and (7):

$$v = -\frac{2Px}{729EI}(19L^2 - 27x^2) \quad \left(0 \leq x \leq \frac{L}{3}\right) \quad \leftarrow$$

$$v = -\frac{P}{1458EI}(-13L^3 + 175L^2x - 243Lx^2 + 81x^3) \quad \left(\frac{L}{3} \leq x \leq L\right) \quad \leftarrow$$

DEFLECTION AT POINT B $\left(x = \frac{L}{3}\right)$

$$\delta_B = -v\left(\frac{L}{3}\right) = \frac{32PL^3}{2187EI} \quad (\text{positive downward}) \quad \leftarrow$$

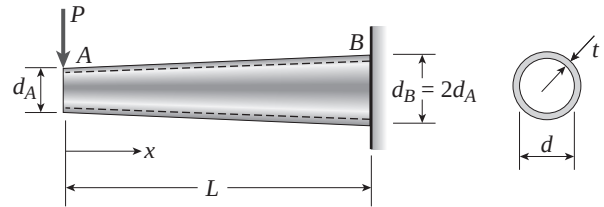
Problem 9.7-7 The tapered cantilever beam AB shown in the figure has thin-walled, hollow circular cross sections of constant thickness t . The diameters at the ends A and B are d_A and $d_B = 2d_A$, respectively. Thus, the diameter d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{L}(L + x)$$

$$I = \frac{\pi t d^3}{8} = \frac{\pi t d_A^3}{8L^3}(L + x)^3 = \frac{I_A}{L^3}(L + x)^3$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P .



Solution 9.7-7 Tapered cantilever beam

$$M = -Px \quad EIv'' = -Px \quad I = \frac{I_A}{L^3}(L + x)^3$$

$$v'' = -\frac{Px}{EI} = -\frac{PL^3}{EI_A} \left[\frac{x}{(L + x)^3} \right] \quad (1)$$

INTEGRATE EQ. (1)

From Appendix D: $\int \frac{xdx}{(L + x)^3} = -\frac{L + 2x}{2(L + x)^2}$

$$v' = \frac{PL^3}{EI_A} \left[\frac{L + 2x}{2(L + x)^2} \right] + C_1$$

B.C. 1 $v'(L) = 0 \quad \therefore C_1 = -\frac{3PL^2}{8EI_A}$

$$v' = \frac{PL^3}{EI_A} \left[\frac{L + 2x}{2(L + x)^2} \right] - \frac{3PL^2}{8EI_A}$$

or

$$v' = \frac{PL^3}{EI_A} \left[\frac{L}{2(L + x)^2} \right] + \frac{PL^3}{EI_A} \left[\frac{x}{(L + x)^2} \right] - \frac{3PL^2}{8EI_A} \quad (2)$$

INTEGRATE EQ. (2)

From Appendix D:

$$\int \frac{dx}{(L + x)^2} = -\frac{1}{L + x}$$

$$\int \frac{xdx}{(L + x)^2} = \frac{L}{L + x} + \ln(L + x)$$

$$\begin{aligned}
 v &= \frac{PL^3}{EI_A} \left(\frac{L}{2} \right) \left(-\frac{1}{L+x} \right) + \frac{PL^3}{EI_A} \left[\frac{L}{L+x} + \ln(L+x) \right] \\
 &\quad - \frac{3PL^2}{8EI_A} x + C_2 \\
 &= \frac{PL^3}{EI_A} \left[\frac{L}{2(L+x)} + \ln(L+x) - \frac{3x}{8L} \right] + C_2 \quad (3) \\
 \text{B.C. 2 } v(L) &= 0 \quad \therefore C_2 = \frac{PL^3}{EI_A} \left[\frac{1}{8} - \ln(2L) \right]
 \end{aligned}$$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (3).

$$v = \frac{PL^3}{EI_A} \left[\frac{L}{2(L+x)} - \frac{3x}{8L} + \frac{1}{8} + \ln\left(\frac{L+x}{2L}\right) \right] \quad \leftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

$$\begin{aligned}
 \delta_A &= -v(0) = \frac{PL^3}{8EI_A} (8\ln 2 - 5) \\
 &= 0.06815 \frac{PL^3}{EI_A} \text{ (positive downward)} \quad \leftarrow
 \end{aligned}$$

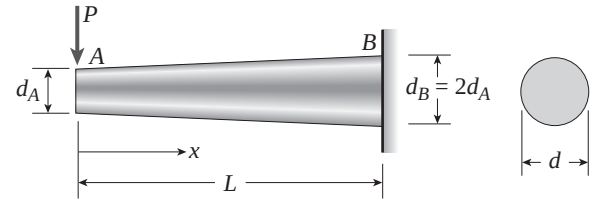
$$\text{Note: } \ln \frac{1}{2} = -\ln 2$$

Problem 9.7-8 The tapered cantilever beam AB shown in the figure has a solid circular cross section. The diameters at the ends A and B are d_A and $d_B = 2d_A$, respectively. Thus, the diameter d and moment of inertia I at distance x from the free end are, respectively,

$$\begin{aligned}
 d &= \frac{d_A}{L} (L+x) \\
 I &= \frac{\pi d^4}{64} = \frac{\pi d_A^4}{64L^4} (L+x)^4 = \frac{I_A}{L^4} (L+x)^4
 \end{aligned}$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P .

**Solution 9.7-8 Tapered cantilever beam**

$$M = -Px \quad EIv'' = -Px \quad I = \frac{I_A}{L^4} (L+x)^4$$

$$v'' = -\frac{Px}{EI} = -\frac{PL^4}{EI_A} \left[\frac{x}{(L+x)^4} \right] \quad (1)$$

INTEGRATE EQ. (1)

$$\text{From Appendix D: } \int \frac{xdx}{(L+x)^4} = -\frac{L+3x}{6(L+x)^3}$$

$$v' = \frac{PL^4}{EI_A} \left[\frac{L+3x}{6(L+x)^3} \right] + C_1$$

$$\text{B.C. 1 } v'(L) = 0 \quad \therefore C_1 = -\frac{PL^2}{12EI_A}$$

$$v' = \frac{PL^4}{EI_A} \left[\frac{L+3x}{6(L+x)^3} \right] - \frac{PL^2}{12EI_A}$$

or

$$\begin{aligned}
 v' &= \frac{PL^4}{EI_A} \left[\frac{L}{6(L+x)^3} \right] + \frac{PL^4}{EI_A} \left[\frac{x}{2(L+x)^3} \right] \\
 &\quad - \frac{PL^2}{12EI_A} \quad (2)
 \end{aligned}$$

INTEGRATE EQ. (2)

From Appendix D: $\int \frac{dx}{(L+x)^3} = -\frac{1}{2(L+x)^2}$

$$\int \frac{xdx}{(L+x)^3} = \frac{-(L+2x)}{2(L+x)^2}$$

$$v = \frac{PL^4}{EI_A} \left(\frac{L}{6} \right) \left(-\frac{1}{2} \right) \left(\frac{1}{L+x} \right)^2 + \frac{PL^4}{EI_A} \left(\frac{1}{2} \right) \left[-\frac{L+2x}{2(L+x)^2} \right]$$

$$- \frac{PL^2}{12EI_A} x + C_2$$

$$= \frac{PL^3}{EI_A} \left[-\frac{L^2}{12(L+x)^2} - \frac{L(L+2x)}{4(L+x)^2} - \frac{x}{12L} \right] + C_2 \quad (3)$$

B.C. 2 $v(L) = 0 \quad \therefore C_2 = \frac{PL^3}{EI_A} \left(\frac{7}{24} \right)$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (3)

$$v = \frac{PL^3}{24EI_A} \left[7 - \frac{4L(2L+3x)}{(L+x)^2} - \frac{2x}{L} \right] \quad \leftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

$$\delta_A = -v(0) = \frac{PL^3}{24EI_A} \quad (\text{positive downward}) \quad \leftarrow$$

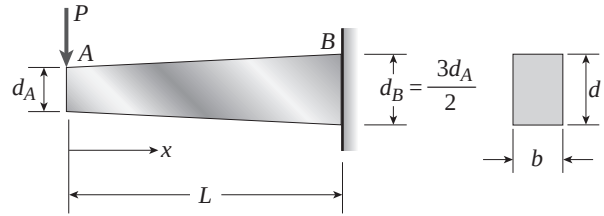
Problem 9.7-9 A tapered cantilever beam AB supports a concentrated load P at the free end (see figure). The cross sections of the beam are rectangular with constant width b , depth d_A at support A , and depth $d_B = 3d_A/2$ at the support. Thus, the depth d and moment of inertia I at distance x from the free end are, respectively,

$$d = \frac{d_A}{2L} (2L+x)$$

$$I = \frac{bd^3}{12} = \frac{bd_A^3}{96L^3} (2L+x)^3 = \frac{I_A}{8L^3} (2L+x)^3$$

in which I_A is the moment of inertia at end A of the beam.

Determine the equation of the deflection curve and the deflection δ_A at the free end of the beam due to the load P .



Solution 9.7-9 Tapered cantilever beam

$$M = -Px \quad EIv'' = -Px \quad I = \frac{I_A}{8L^3} (2L+x)^3$$

$$v'' = -\frac{Px}{EI} = -\frac{8PL^3}{EI_A} \left[\frac{x}{(2L+x)^3} \right] \quad (1)$$

INTEGRATE EQ. (1)

From Appendix D: $\int \frac{xdx}{(2L+x)^3} = -\frac{2L+2x}{2(2L+x)^2}$

$$v' = \frac{8PL^3}{EI_A} \left[\frac{L+x}{(2L+x)^3} \right] + C_1$$

B.C. 1 $v'(L) = 0 \quad \therefore C_1 = -\frac{16PL^2}{9EI_A}$

$$v' = \frac{8PL^3}{EI_A} \left[\frac{L+x}{(2L+x)^2} \right] - \frac{16PL^2}{9EI_A}$$

or

$$v' = \frac{8PL^3}{EI_A} \left[\frac{L}{(2L+x)^2} \right] + \frac{8PL^3}{EI_A} \left[\frac{x}{(2L+x)^2} \right] - \frac{16PL^2}{9EI_A} \quad (2)$$

INTEGRATE EQ. (2)

From Appendix D: $\int \frac{dx}{(2L+x)^2} = -\frac{1}{2L+x}$

$$\int \frac{x dx}{(2L+x)^2} = \frac{2L}{2L+x} + \ln(2L+x)$$

$$v = \frac{8PL^3}{EI_A} \left(-\frac{L}{2L+x} \right) + \frac{8PL^3}{EI_A} \left[\frac{2L}{2L+x} + \ln(2L+x) \right] - \frac{16PL^2}{9EI_A} x + C_2$$

$$= \frac{PL^3}{EI_A} \left[\frac{8L}{2L+x} + 8 \ln(2L+x) - \frac{16x}{9L} \right] + C_2 \quad (3)$$

B.C. 2 $v(L) = 0 \quad \therefore C_2 = -\frac{8PL^3}{EI_A} \left[\frac{1}{9} + \ln(3L) \right]$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (3).

$$v = \frac{8PL^3}{EI_A} \left[\frac{L}{2L+x} - \frac{2x}{9L} - \frac{1}{9} + \ln\left(\frac{2L+x}{3L}\right) \right] \leftarrow$$

DEFLECTION δ_A AT END A OF THE BEAM

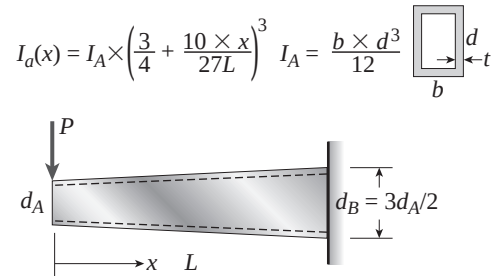
$$\delta_A = -v(0) = \frac{8PL^2}{EI_A} \left[\ln\left(\frac{3}{2}\right) - \frac{7}{18} \right]$$

$$= 0.1326 \frac{PL^3}{EI_A} \quad (\text{positive downward}) \quad \leftarrow$$

Note: $\ln \frac{2}{3} = -\ln \frac{3}{2}$

Problem 9.7-10 A tapered cantilever beam AB supports a concentrated load P at the free end (see figure). The cross sections of the beam are rectangular tubes with constant width b and *outer tube* depth d_A at A, and *outer tube* depth $d_B = 3d_A/2$ at support B. The tube thickness is constant, $t = d_A/20$. I_A is the moment of inertia of the *outer tube* at end A of the beam.

If the moment of inertia of the tube is approximated as $I_a(x)$ as defined, find the *equation* of the deflection curve and the deflection δ_A at the free end of the beam due to the load P .

**Solution 9.7-10**

BENDING-MOMENT EQUATIONS

$$EIv'' = M = -Px$$

$$v'' = \frac{-Px}{EI_a(x)} = \frac{-Px}{EI_A \left(\frac{3}{4} + \frac{10x}{27L} \right)^3} = \frac{-P}{EI_A} \frac{x}{\left(\frac{3}{4} + \frac{10x}{27L} \right)^3}$$

From Appendix D: $\int \frac{x}{(a+bx)^3} dx = -\frac{a+2bx}{2b^2(a+bx)^2}$

$$v' = \frac{-P}{EI_A} \left[\frac{\frac{3}{4} + 2\frac{10}{27L}x}{2\left(\frac{10}{27L}\right)^2 \left(\frac{3}{4} + \frac{10}{27L}x\right)^2} \right] + C_1$$

$$v' = \frac{PL^3}{EI_A} \frac{19,683}{50} \left[\frac{81L}{(81L+40x)^2} + \frac{80x}{(81L+40x)^2} \right] + C_1$$

From Appendix D: $\int \frac{1}{(a+bx)^2} dx = \frac{-1}{b(a+bx)}$

$$\int \frac{x}{(a+bx)^2} dx = \frac{1}{b^2} \left(\frac{a}{a+bx} + \ln(a+bx) \right)$$

$$v = \frac{PL^3}{EI_A} \frac{19,683}{50} \left[\frac{-81L}{40(81L+40x)} + \frac{80}{40^2} \left(\frac{81L}{81L+40x} + \ln(81L+40x) \right) \right] + C_1x + C_2$$

$$v = \frac{19,683PL^3}{2000EI_A} \left(\frac{81L + 162\ln(81L+40x)L + 80\ln(81L+40x)x}{81L+40x} \right) + C_1x + C_2$$

B.C. $v'(L) = 0$ $C_1 = \frac{-3,168,963}{732,050} \frac{PL^2}{EI_A}$

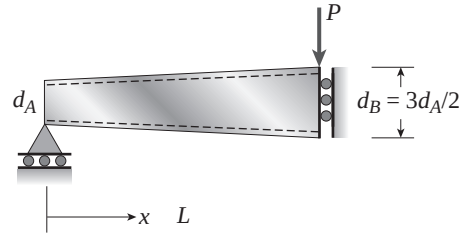
B.C. $v(L) = 0$ $C_2 = \frac{-19,683PL^3}{29,282,000EI_A} (3361 + 29,282\ln(121L))$

$$v(x) = \frac{19,683PL^3}{2000EI_A} \left(\frac{81L}{81L+40x} + 2\ln\left(\frac{81}{121} + \frac{40x}{121L}\right) - \frac{6440x}{14,641L} - \frac{3361}{14,641} \right) \leftarrow$$

$$\delta_A = -v(0) = \frac{19,683PL^3}{7,320,500EI_A} \left(-2820 + 14,641\ln\left(\frac{11}{9}\right) \right) = 0.317 \frac{PL^3}{EI_A} \leftarrow$$

Problem 9.7-11 Repeat Prob. 9.7-10, but now use the tapered propped cantilever tube AB, with guided support at B, shown in the figure which supports a concentrated load P at the guided end.

Find the equation of the deflection curve and the deflection δ_B at the guided end of the beam due to the load P .



Solution 9.7-11

BENDING-MOMENT EQUATIONS

$$EIv'' = M = Px$$

$$v'' = \frac{Px}{EI_d(x)} = \frac{Px}{EI_A \left(\frac{3}{4} + \frac{10x}{27L} \right)^3} = \frac{P}{EI_A} \frac{x}{\left(\frac{3}{4} + \frac{10x}{27L} \right)^3}$$

From Appendix D: $\int \frac{x}{(a+bx)^3} dx = -\frac{a+2bx}{2b^2(a+bx)^2}$

$$v' = \frac{P}{EI_A} \left[\frac{\frac{3}{4} + 2\frac{10}{27L}x}{2\left(\frac{10}{27L}\right)^2\left(\frac{3}{4} + \frac{10}{27L}x\right)^2} \right] + C_1$$

$$v' = -\frac{PL^3}{EI_A} \frac{19,683}{50} \left[\frac{81L}{(81L+40x)^2} + \frac{80x}{(81L+40x)^2} \right] + C_1$$

From Appendix D: $\int \frac{1}{(a+bx)^2} dx = \frac{-1}{b(a+bx)}$

$$\int \frac{x}{(a+bx)^2} dx = \frac{1}{b^2} \left(\frac{a}{a+bx} + \ln(a+bx) \right)$$

$$v = -\frac{PL^3}{EI_A} \frac{19,683}{50} \left[\frac{-81L}{40(81L+40x)} + \frac{80}{40^2} \left(\frac{81L}{81L+40x} + \ln(81L+40x) \right) \right] + C_1x + C_2$$

$$v = -\frac{19,683PL^3}{2000EI_A} \left(\frac{81L + 162\ln(81L+40x)L + 80\ln(81L+40x)x}{81L+40x} \right) + C_1x + C_2$$

B.C. $v'(L) = 0$ $C_1 = \frac{3,168,963}{732,050} \frac{PL^2}{EI_A}$

B.C. $v(L) = 0$ $C_2 = \frac{19,683PL^3}{2000EI_A} (1 + 2\ln(81L))$

$$v(x) = -\frac{19,683PL^3}{2000EI_A} \left(\frac{81L}{81L+40x} + 2\ln\left(1 + \frac{40x}{81L}\right) - \frac{6440x}{14,641L} - 1 \right) \leftarrow$$

$$\delta_B = -v(L) = \frac{19,683PL^3}{7,320,500EI_A} \left(-2820 + 14,641\ln\left(\frac{11}{9}\right) \right) = 0.317 \frac{PL^3}{EI_A} \leftarrow$$

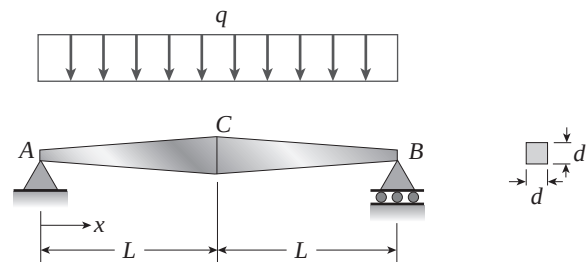
Problem 9.7-12 A simple beam ACB is constructed with square cross sections and a double taper (see figure). The depth of the beam at the supports is d_A and at the midpoint is $d_C = 2d_A$. Each half of the beam has length L . Thus, the depth d and moment of inertia I at distance x from the left-hand end are, respectively,

$$d = \frac{d_A}{L}(L+x)$$

$$I = \frac{d^4}{12} = \frac{d_A^4}{12L^4}(L+x)^4 = \frac{I_A}{L^4}(L+x)^4$$

in which I_A is the moment of inertia at end A of the beam. (These equations are valid for x between 0 and L , that is, for the left-hand half of the beam.)

- Obtain equations for the slope and deflection of the left-hand half of the beam due to the uniform load.
- From those equations obtain formulas for the angle of rotation θ_A at support A and the deflection δ_C at the midpoint.



Solution 9.7-12 Simple beam with a double taper

L = length of one-half of the beam

$$I = \frac{I_A}{L^4} (L+x)^4 \quad (0 \leq x \leq L)$$

(x is measured from the left-hand support A)

Reactions: $R_A = R_B = qL$

$$\text{Bending moment: } M = R_A x - \frac{qx^2}{2} = qLx - \frac{qx^2}{2}$$

From Eq. (9-16a):

$$EIv'' = M = qLx - \frac{qx^2}{2}$$

$$v'' = \frac{qL^5x}{EI_A(L+x)^4} - \frac{qL^4x^2}{2EI_A(L+x)^4} \quad (0 \leq x \leq L) \quad (1)$$

INTEGRATE EQ. (1)

$$\text{From Appendix D: } \int \frac{xdx}{(L+x)^4} = -\frac{L+3x}{6(L+x)^3}$$

$$\int \frac{x^2dx}{(L+x)^4} = -\frac{L^2+3Lx+3x^2}{3(L+x)^3}$$

$$v' = \frac{qL^5}{EI_A} \left[-\frac{L+3x}{6(L+x)^3} \right]$$

$$- \frac{qL^4}{2EI_A} \left[-\frac{L^2+3Lx+3x^2}{3(L+x)^3} \right] + C_1$$

$$= \frac{qL^4x^2}{2EI_A(L+x)^3} + C_1 \quad (0 \leq x \leq L) \quad (2)$$

$$\text{B.C. 1 (symmetry) } v'(L) = 0 \quad \therefore C_1 = -\frac{qL^3}{16EI_A}$$

SLOPE OF THE BEAM

Substitute C_1 into Eq. (2).

$$v' = \frac{qL^4x^2}{2EI_A(L+x)^3} - \frac{qL^3}{16EI_A}$$

$$= -\frac{qL^3}{16EI_A} \left[1 - \frac{8Lx^2}{(L+x)^3} \right] \quad (0 \leq x \leq L) \quad (3)$$

ANGLE OF ROTATION AT SUPPORT A

$$\theta_A = -v'(0) = \frac{qL^3}{16EI_A} \quad (\text{positive clockwise}) \quad \leftarrow$$

INTEGRATE EQ. (3)

$$\text{From Appendix D: } \int \frac{x^2dx}{(L+x)^3} = \frac{L(3L+4x)}{2(L+x)^2} + \ln(L+x)$$

$$v = -\frac{qL^3}{16EI_A} \left[x - \frac{8L^2(3L+4x)}{2(L+x)^2} \right. \\ \left. - 8L \ln(L+x) \right] + C_2 \quad (0 \leq x \leq L) \quad (4)$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_2 = -\frac{qL^4}{2EI_A} \left(\frac{3}{2} + \ln L \right)$$

DEFLECTION OF THE BEAM

Substitute C_2 into Eq. (4) and simplify. (The algebra is lengthy.)

$$v = -\frac{qL^4}{2EI_A} \left[\frac{(9L^2+14Lx+x^2)x}{8L(L+x)^2} - \ln \left(1 + \frac{x}{L} \right) \right] \\ (0 \leq x \leq L) \quad \leftarrow$$

DEFLECTION AT THE MIDPOINT C OF THE BEAM

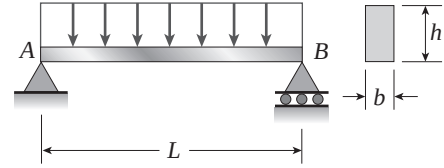
$$\delta_C = -v(L) = \frac{qL^4}{8EI_A} (3 - 4 \ln 2) = 0.02843 \frac{qL^4}{EI_A} \\ (\text{positive downward}) \quad \leftarrow$$

Strain Energy

The beams described in the problems for Section 9.8 have constant flexural rigidity EI .

Problem 9.8-1 A uniformly loaded simple beam AB (see figure) of span length L and rectangular cross section (b = width, h = height) has a maximum bending stress $\sigma_{\max}$ due to the uniform load.

Determine the strain energy U stored in the beam.



Solution 9.8-1 Simple beam with a uniform load

Given: $L, b, h, \sigma_{\max}$ Find: U (strain energy)

$$\text{Bending moment: } M = \frac{qLx}{2} - \frac{qx^2}{2}$$

$$\begin{aligned} \text{Strain energy (Eq. 9-95a): } U &= \int_0^L \frac{M^2 dx}{2EI} \\ &= \frac{q^2 L^5}{240EI} \quad (1) \end{aligned}$$

$$\text{Maximum stress: } \sigma_{\max} = \frac{M_{\max} c}{I} = \frac{M_{\max} h}{2I}$$

$$M_{\max} = \frac{qL^2}{8} \quad \sigma_{\max} = \frac{qL^2 h}{16I}$$

$$\text{Solve for } q: q = \frac{16I\sigma_{\max}}{L^2 h}$$

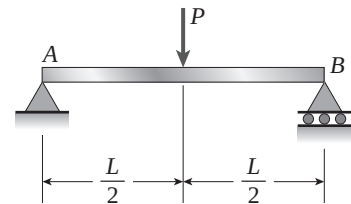
Substitute q into Eq. (1):

$$U = \frac{16I\sigma_{\max}^2 L}{15h^2 E}$$

$$\text{Substitute } I = \frac{bh^3}{12}: U = \frac{4bhL\sigma_{\max}^2}{45E} \quad \leftarrow$$

Problem 9.8-2 A simple beam AB of length L supports a concentrated load P at the midpoint (see figure).

- Evaluate the strain energy of the beam from the bending moment in the beam.
- Evaluate the strain energy of the beam from the equation of the deflection curve.
- From the strain energy, determine the deflection δ under the load P .



Solution 9.8-2 Simple beam with a concentrated load

$$(a) \text{ BENDING MOMENT } M = \frac{Px}{2} \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

Strain energy (Eq. 9-95a):

$$U = 2 \int_0^{L/2} \frac{M^2 dx}{2EI} = \frac{P^2 L^3}{96EI} \quad \leftarrow$$

(b) DEFLECTION CURVE

From Table H-2, Case 4:

$$v = -\frac{Px}{48EI} (3L^2 - 4x^2) \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$\frac{dv}{dx} = -\frac{P}{16EI} (L^2 - 4x^2) \quad \frac{d^2v}{dx^2} = \frac{Px}{2EI}$$

Strain energy (Eq. 9-95b):

$$U = 2 \int_0^{L/2} \frac{EI}{2} \left(\frac{d^2v}{dx^2} \right)^2 dx = EI \int_0^{L/2} \left(\frac{Px}{2EI} \right)^2 dx$$

$$= \frac{P^2 L^3}{96EI} \quad \leftarrow$$

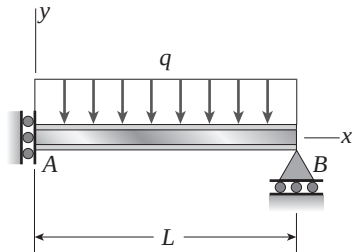
(c) DEFLECTION δ UNDER THE LOAD P

From Eq. (9-97a):

$$\delta = \frac{2U}{P} = \frac{PL^3}{48EI} \quad \leftarrow$$

Problem 9.8-3 A propped cantilever beam AB of length L , and with guided support at A , supports a uniform load of intensity q (see figure).

- Evaluate the strain energy of the beam from the bending moment in the beam.
- Evaluate the strain energy of the beam from the equation of the deflection curve.



Solution 9.8-3

(a) BENDING-MOMENT EQUATIONS

Measure x from end B

$$M = qLx - \frac{qx^2}{2}$$

Strain Energy (Eq. 9-95a):

$$U = \int_0^L \frac{M^2}{2EI} dx$$

$$= \int_0^L \frac{1}{2EI} \left(qLx - \frac{qx^2}{2} \right)^2 dx = \frac{q^2 L^5}{15EI} \quad \leftarrow$$

(b) DEFLECTION CURVE

Measure x from end B :

$$v = -\frac{qx}{24EI} (8L^3 - 4Lx^2 + x^3)$$

$$\frac{d}{dx} v = -\frac{q}{24EI} (8L^3 - 12Lx^2 + 4x^3)$$

$$\frac{d^2}{dx^2} v = -\frac{q}{2EI} (-2Lx + x^2)$$

Strain energy (Eq. 9-95b):

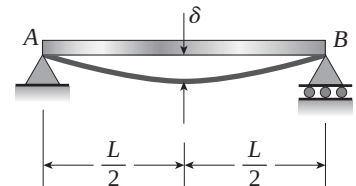
$$U = \int_0^L \frac{EI}{2} \left(\frac{d^2}{dx^2} v \right)^2 dx$$

$$U = \int_0^L \frac{EI}{2} \left[-\frac{q}{2EI} (-2Lx + x^2) \right]^2 dx$$

$$U = \frac{q^2 L^5}{15EI} \quad \leftarrow$$

Problem 9.8-4 A simple beam AB of length L is subjected to loads that produce a symmetric deflection curve with maximum deflection δ at the midpoint of the span (see figure).

How much strain energy U is stored in the beam if the deflection curve is
(a) a parabola, and (b) a half-wave of a sine curve?



Solution 9.8-4 Simple beam (symmetric deflection curve)

GIVEN: L, EI, δ δ = maximum deflection at midpoint

Determine the strain energy U .

Assume the deflection v is positive downward.

(a) DEFLECTION CURVE IS A PARABOLA

$$v = \frac{4\delta x}{L^2}(L-x) \quad \frac{dv}{dx} = \frac{4\delta}{L^2}(L-2x)$$

$$\frac{d^2v}{dx^2} = -\frac{8\delta}{L^2}$$

Strain energy (Eq. 9-95b):

$$\begin{aligned} U &= \int_0^L \frac{EI}{2} \left(\frac{d^2v}{dx^2} \right)^2 dx = \frac{EI}{2} \int_0^L \left(-\frac{8\delta}{L^2} \right)^2 dx \\ &= \frac{32EI\delta^2}{L^3} \quad \leftarrow \end{aligned}$$

(b) DEFLECTION CURVE IS A SINE CURVE

$$v = \delta \sin \frac{\pi x}{L} \quad \frac{dv}{dx} = \frac{\pi\delta}{L} \cos \frac{\pi x}{L}$$

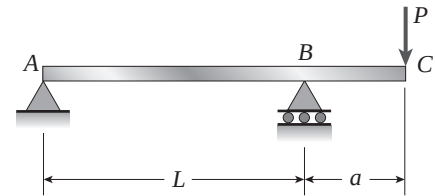
$$\frac{d^2v}{dx^2} = -\frac{\pi^2\delta}{L^2} \sin \frac{\pi x}{L}$$

Strain energy (Eq. 9-95b):

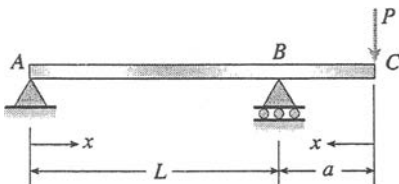
$$\begin{aligned} U &= \int_0^L \frac{EI}{2} \left(\frac{d^2v}{dx^2} \right)^2 dx = \frac{EI}{2} \int_0^L \left(-\frac{\pi^2\delta}{L^2} \right)^2 \sin^2 \frac{\pi x}{L} dx \\ &= \frac{\pi^4 EI \delta^2}{4L^3} \quad \leftarrow \end{aligned}$$

Problem 9.8-5 A beam ABC with simple supports at A and B and an overhang BC supports a concentrated load P at the free end C (see figure).

- Determine the strain energy U stored in the beam due to the load P .
- From the strain energy, find the deflection δ_C under the load P .
- Calculate the numerical values of U and δ_C if the length L is 8 ft, the overhang length a is 3 ft, the beam is a W 10 $\times$ 12 steel wide-flange section, and the load P produces a maximum stress of 12,000 psi in the beam. (Use $E = 29 \times 10^6$ psi.)

**Solution 9.8-5 Simple beam with an overhang**

(a) STRAIN ENERGY (use Eq. 9-95a)



From A to B , $M = -\frac{Pax}{L}$

$$U_{AB} = \int \frac{M^2 dx}{2EI} = \int_0^L \frac{1}{2EI} \left(-\frac{Pax}{L} \right)^2 dx = \frac{P^2 a^2 L}{6EI}$$

From B to C , $M = -Px$

$$U_{BC} = \int_0^a \frac{1}{2EI} (-Px)^2 dx = \frac{P^2 a^3}{6EI}$$

Total strain energy:

$$U = U_{AB} + U_{BC} = \frac{P^2 a^2}{6EI} (L + a) \quad \leftarrow$$

(b) DEFLECTION δ_C UNDER THE LOAD P

From Eq. (9-97a),

$$\delta_C = \frac{2U}{P} = \frac{Pa^2}{3EI} (L + a) \quad \leftarrow$$

(c) CALCULATE U AND δ_c Data: $L = 8 \text{ ft} = 96 \text{ in.}$ $a = 3 \text{ ft} = 36 \text{ in.}$ $W 10 \times 12$ $E = 29 \times 10^6 \text{ psi}$ $\sigma_{\max} = 12,000 \text{ psi}$ $I = 53.8 \text{ in.}^4$ $c = \frac{d}{2} = \frac{9.87}{2} = 4.935 \text{ in.}$ Express load P in terms of maximum stress:

$$\sigma_{\max} = \frac{Mc}{I} = \frac{M_{\max}c}{I} = \frac{Pac}{I}; \quad \therefore P = \frac{\sigma_{\max} I}{ac}$$

$$U = \frac{P^2 a^2 (L+a)}{6EI} = \frac{\sigma_{\max}^2 I (L+a)}{6c^2 E}$$

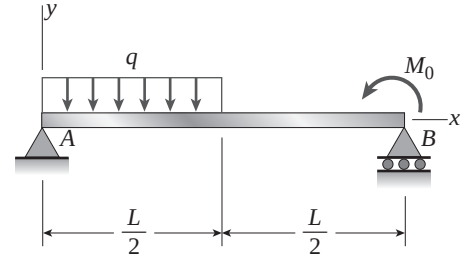
$$= 241 \text{ in.-lb} \quad \leftarrow$$

$$\delta_c = \frac{Pa^2(L+a)}{3EI} = \frac{\sigma_{\max} a (L+a)}{3cE}$$

$$= 0.133 \text{ in.} \quad \leftarrow$$

Problem 9.8-6 A simple beam ACB supporting a uniform load q over the first half of the beam and a couple of moment M_0 at end B is shown in the figure.

Determine the strain energy U stored in the beam due to the load q and the couple M_0 acting simultaneously.

**Solution 9.8-6**

FROM A TO MID-SPAN

Bending-moment equations:

$$M = \left(\frac{3qL}{8} + \frac{M_0}{L} \right) x - \frac{qx^2}{2}$$

STRAIN ENERGY (EQ. 9-95a)

$$\begin{aligned} U_1 &= \int_0^{L/2} \frac{M^2}{2EI} dx \\ &= \int_0^{L/2} \frac{1}{2EI} \left[\left(\frac{3qL}{8} + \frac{M_0}{L} \right) x - \frac{qx^2}{2} \right]^2 dx \\ U_1 &= \frac{L}{3840EI} \left(3L^4 q^2 + 30qL^2 M_0 + 80M_0^2 \right) \end{aligned}$$

FROM MID-SPAN TO B

Bending-moment equations:

$$M = \left(\frac{3qL}{8} + \frac{M_0}{L} \right) x - \frac{qL}{2} \left(x - \frac{L}{4} \right)$$

STRAIN ENERGY (EQ. 9-95a)

$$\begin{aligned} U_2 &= \int_{L/2}^L \frac{M^2}{2EI} dx = \int_{L/2}^L \frac{1}{2EI} \left[\left(\frac{3qL}{8} + \frac{M_0}{L} \right) x - \frac{qL}{2} \left(x - \frac{L}{4} \right) \right]^2 dx \end{aligned}$$

$$U_2 = \frac{L}{3072EI} \left(L^4 q^2 + 32qL^2 M_0 + 448M_0^2 \right)$$

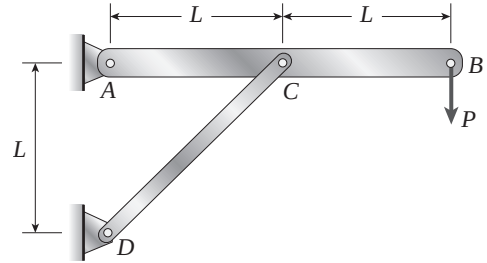
STRAIN ENERGY OF THE ENTIRE BEAM

$$\begin{aligned} U &= U_1 + U_2 = \frac{L}{15,360EI} \left(17L^4 q^2 \right. \\ &\quad \left. + 280qL^2 M_0 + 2560M_0^2 \right) \quad \leftarrow \end{aligned}$$

Problem 9.8-7 The frame shown in the figure consists of a beam ACB supported by a strut CD . The beam has length $2L$ and is continuous through joint C . A concentrated load P acts at the free end B .

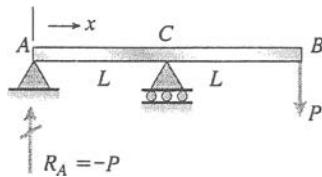
Determine the vertical deflection δ_B at point B due to the load P .

Note: Let EI denote the flexural rigidity of the beam, and let EA denote the axial rigidity of the strut. Disregard axial and shearing effects in the beam, and disregard any bending effects in the strut.



Solution 9.8-7 Frame with beam and strut

BEAM ACB



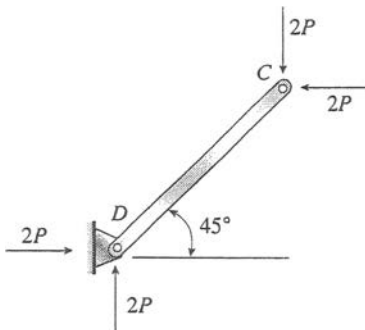
For part AC of the beam, $M = -Px$

$$U_{AC} = \int \frac{M^2 dx}{2EI} = \frac{1}{2EI} \int_0^L (-Px)^2 dx = \frac{P^2 L^3}{6EI}$$

$$\text{For part } CB \text{ of the beam, } U_{CB} = U_{AC} = \frac{P^2 L^3}{6EI}$$

$$\text{Entire beam, } U_{\text{beam}} = U_{AC} + U_{CB} = \frac{P^2 L^3}{3EI}$$

STRUT CD



L_{CD} = length of strut

$$= \sqrt{2}L$$

F = axial force in strut CD

$$= 2\sqrt{2}P$$

$$U_{\text{strut}} = \frac{F^2 L_{CD}}{2EA} \quad (\text{Eq. 2-39a})$$

$$U_{\text{strut}} = \frac{(2\sqrt{2}P)^2 (\sqrt{2}L)}{2EA} = \frac{4\sqrt{2}P^2 L}{EA}$$

FRAME

$$U = U_{\text{beam}} + U_{\text{strut}} = \frac{P^2 L^3}{3EI} + \frac{4\sqrt{2}P^2 L}{EA}$$

DEFLECTION δ_B AT POINT B

From Eq. (9-97a),

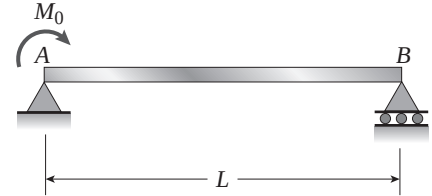
$$\delta_B = \frac{2U}{P} = \frac{2PL^3}{3EI} + \frac{8\sqrt{2}PL}{EA} \quad \leftarrow$$

Castigliano's Theorem

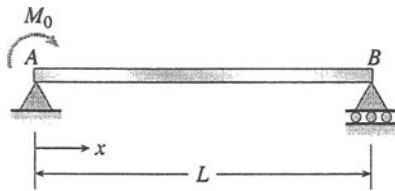
The beams described in the problems for Section 9.9 have constant flexural rigidity EI .

Problem 9.9-1 A simple beam AB of length L is loaded at the left-hand end by a couple of moment M_0 (see figure).

Determine the angle of rotation θ_A at support A . (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-1 Simple beam with couple M_0



$$R_A = \frac{M_0}{L} \quad (\text{downward})$$

$$\begin{aligned} M &= M_0 - R_A x = M_0 - \frac{M_0 x}{L} \\ &= M_0 \left(1 - \frac{x}{L} \right) \end{aligned}$$

STRAIN ENERGY

$$U = \int \frac{M^2 dx}{2EI} = \frac{M_0^2}{2EI} \int_0^L \left(1 - \frac{x}{L} \right)^2 dx = \frac{M_0^2 L}{6EI}$$

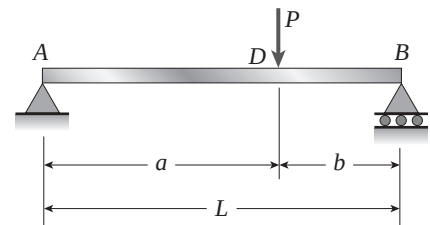
CASTIGLIANO'S THEOREM

$$\theta_A = \frac{dU}{dM_0} = \frac{M_0 L}{3EI} \quad (\text{clockwise}) \quad \leftarrow$$

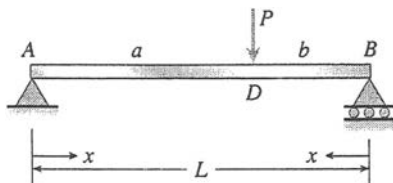
(This result agrees with Case 7, Table H-2)

Problem 9.9-2 The simple beam shown in the figure supports a concentrated load P acting at distance a from the left-hand support and distance b from the right-hand support.

Determine the deflection δ_D at point D where the load is applied. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-2 Simple beam with load P



$$R_A = \frac{Pb}{L} \quad R_B = \frac{Pa}{L}$$

$$M_{AD} = R_A x = \frac{Pbx}{L}$$

$$M_{DB} = R_B x = \frac{Pax}{L}$$

STRAIN ENERGY
$$U = \int \frac{M^2 dx}{2EI}$$

$$U_{AD} = \frac{1}{2EI} \int_0^a \left(\frac{Pbx}{L} \right)^2 dx = \frac{P^2 a^3 b^2}{6EIL^2}$$

$$U_{DB} = \frac{1}{2EI} \int_0^b \left(\frac{Pax}{L} \right)^2 dx = \frac{P^2 a^2 b^3}{6EIL^2}$$

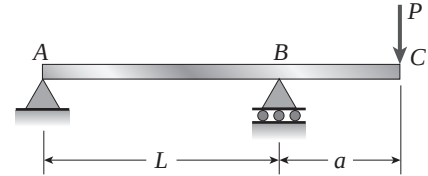
$$U = U_{AD} + U_{DB} = \frac{P^2 a^2 b^2}{6LEI}$$

CASTIGLIANO'S THEOREM

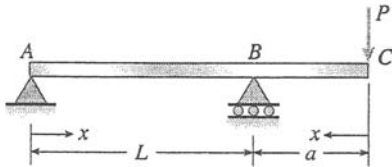
$$\delta_D = \frac{dU}{dP} = \frac{Pa^2 b^2}{3LEI} \quad (\text{downward}) \quad \leftarrow$$

Problem 9.9-3 An overhanging beam ABC supports a concentrated load P at the end of the overhang (see figure). Span AB has length L and the overhang has length a .

Determine the deflection δ_C at the end of the overhang. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-3 Overhanging beam



$$R_A = \frac{Pa}{L} \quad (\text{downward})$$

$$M_{AB} = -R_A x = -\frac{Pax}{L}$$

$$M_{CB} = -Px$$

$$\text{STRAIN ENERGY} \quad U = \int \frac{M^2 dx}{2EI}$$

$$U_{AB} = \frac{1}{2EI} \int_0^L \left(-\frac{Pax}{L} \right)^2 dx = \frac{P^2 a^2 L}{6EI}$$

$$U_{CB} = \frac{1}{2EI} \int_0^a (-Px)^2 dx = \frac{P^2 a^3}{6EI}$$

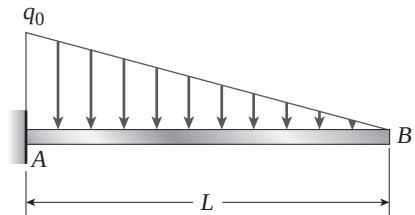
$$U = U_{AB} + U_{CB} = \frac{P^2 a^2}{6EI} (L + a)$$

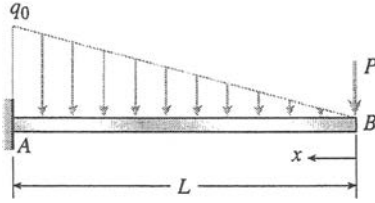
CASTIGLIANO'S THEOREM

$$\delta_C = \frac{dU}{dP} = \frac{Pa^2}{3EI} (L + a) \quad (\text{downward}) \quad \leftarrow$$

Problem 9.9-4 The cantilever beam shown in the figure supports a triangularly distributed load of maximum intensity q_0 .

Determine the deflection δ_B at the free end B. (Obtain the solution by determining the strain energy of the beam and then using Castigliano's theorem.)



Solution 9.9-4 Cantilever beam with triangular load

P = fictitious load corresponding to deflection δ_B

$$M = -Px - \frac{q_0 x^3}{6L}$$

STRAIN ENERGY

$$\begin{aligned} U &= \int \frac{M^2 dx}{2EI} = \frac{1}{2EI} \int_0^L \left(-Px - \frac{q_0 x^3}{6L} \right)^2 dx \\ &= \frac{P^2 L^3}{6EI} + \frac{P q_0 L^4}{30EI} + \frac{q_0^2 L^5}{42EI} \end{aligned}$$

CASTIGLIANO'S THEOREM

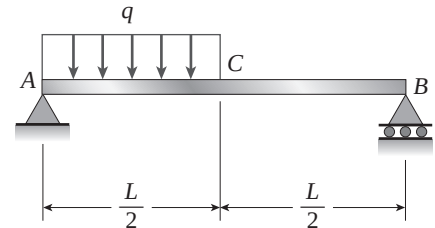
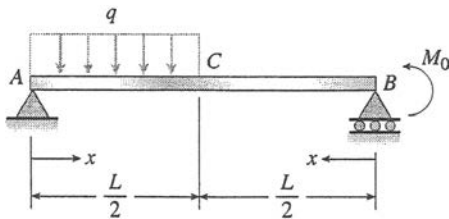
$$\delta_B = \frac{\partial U}{\partial P} = \frac{PL^3}{3EI} + \frac{q_0 L^4}{30EI} \quad (\text{downward})$$

(This result agrees with Cases 1 and 8 of Table H-1.)

$$\text{SET } P = 0: \delta_B = \frac{q_0 L^4}{30EI} \quad \leftarrow$$

Problem 9.9-5 A simple beam ACB supports a uniform load of intensity q on the left-hand half of the span (see figure).

Determine the angle of rotation θ_B at support B. (Obtain the solution by using the modified form of Castigliano's theorem.)

**Solution 9.9-5 Simple beam with partial uniform load**

M_0 = fictitious load corresponding to angle of rotation θ_B

$$R_A = \frac{3qL}{8} + \frac{M_0}{L} \quad R_B = \frac{qL}{8} - \frac{M_0}{L}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT AC

$$M_{AC} = R_A x - \frac{qx^2}{2} = \left(\frac{3qL}{8} + \frac{M_0}{L} \right) x - \frac{qx^2}{2} \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$\frac{\partial M_{AC}}{\partial M_0} = \frac{x}{L}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT CB

$$M_{CB} = R_B x + M_0 = \left(\frac{qL}{8} - \frac{M_0}{L} \right) x + M_0 \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$\frac{\partial M_{CB}}{\partial M_0} = -\frac{x}{L} + 1$$

MODIFIED CASTIGLIANO'S THEOREM (Eq. 9-118)

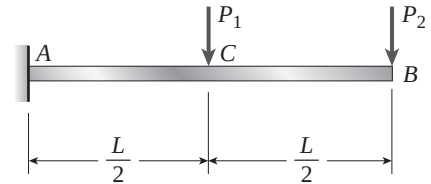
$$\begin{aligned}\theta_B &= \int \left(\frac{M}{EI} \right) \left(\frac{\partial M}{\partial M_0} \right) dx \\ &= \frac{1}{EI} \int_0^{L/2} \left[\left(\frac{3qL}{8} + \frac{M_0}{L} \right) x - \frac{qx^2}{2} \right] \left[\frac{x}{L} \right] dx \\ &\quad + \frac{1}{EI} \int_0^{L/2} \left[\left(\frac{qL}{8} - \frac{M_0}{L} \right) x + M_0 \right] \left[1 - \frac{x}{L} \right] dx\end{aligned}$$

SET FICTITIOUS LOAD M_0 EQUAL TO ZERO

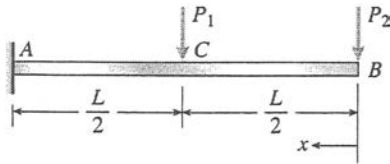
$$\begin{aligned}\theta_B &= \frac{1}{EI} \int_0^{L/2} \left(\frac{3qLx}{8} - \frac{qx^2}{2} \right) \left(\frac{x}{L} \right) dx \\ &\quad + \frac{1}{EI} \int_0^{L/2} \left(\frac{qLx}{8} \right) \left(1 - \frac{x}{L} \right) dx \\ &= \frac{qL^3}{128EI} + \frac{qL^3}{96EI} \\ &= \frac{7qL^3}{384EI} \quad (\text{counterclockwise}) \quad \leftarrow\end{aligned}$$

(This result agrees with Case 2, Table H-2.)

Problem 9.9-6 A cantilever beam ACB supports two concentrated loads P_1 and P_2 , as shown in the figure. Determine the deflections δ_C and δ_B at points C and B , respectively. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-6 Cantilever beam with loads P_1 and P_2



BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT CB

$$M_{CB} = -P_2x \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$\frac{\partial M_{CB}}{\partial P_1} = 0 \quad \frac{\partial M_{CB}}{\partial P_2} = -x$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR SEGMENT AC

$$M_{AC} = -P_1 \left(x - \frac{L}{2} \right) - P_2x \quad \left(\frac{L}{2} \leq x \leq L \right)$$

$$\frac{\partial M_{AC}}{\partial P_1} = \frac{L}{2} - x \quad \frac{\partial M_{AC}}{\partial P_2} = -x$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_C

$$\begin{aligned}\delta_C &= \frac{1}{EI} \int_0^{L/2} (M_{CB}) \left(\frac{\partial M_{CB}}{\partial P_1} \right) dx \\ &\quad + \frac{1}{EI} \int_{L/2}^L (M_{AC}) \left(\frac{\partial M_{AC}}{\partial P_1} \right) dx \\ &= 0 + \frac{1}{EI} \int_{L/2}^L \left[-P_1 \left(x - \frac{L}{2} \right) - P_2x \right] \left(\frac{L}{2} - x \right) dx \\ &= \frac{L^3}{48EI} (2P_1 + 5P_2) \quad \leftarrow\end{aligned}$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_B

$$\begin{aligned}\delta_B &= \frac{1}{EI} \int_0^{L/2} (M_{CB}) \left(\frac{\partial M_{CB}}{\partial P_2} \right) dx \\ &\quad + \frac{1}{EI} \int_{L/2}^L (M_{AC}) \left(\frac{\partial M_{AC}}{\partial P_2} \right) dx\end{aligned}$$

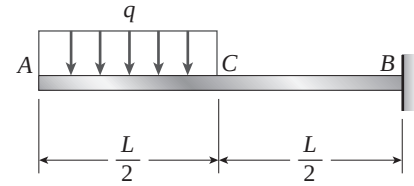
$$= \frac{1}{EI} \int_0^{L/2} (-P_2 x)(-x) dx + \frac{1}{EI} \int_{L/2}^L \left[-P_1 \left(x - \frac{L}{2} \right) - P_2 x \right] (-x) dx$$

$$= \frac{P_2 L^3}{24EI} + \frac{L^3}{48EI} (5P_1 + 14P_2) = \frac{L^3}{48EI} (5P_1 + 16P_2) \quad \leftarrow$$

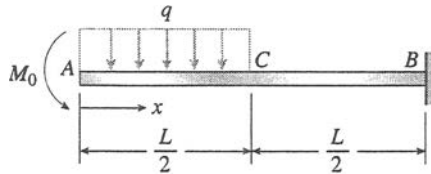
(These results can be verified with the aid of Cases 4 and 5, Table H-1.)

Problem 9.9-7 The cantilever beam ACB shown in the figure is subjected to a uniform load of intensity q acting between points A and C .

Determine the angle of rotation θ_A at the free end A . (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-7 Cantilever beam with partial uniform load



M_0 = fictitious load corresponding to the angle of rotation θ_A

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT AC

$$M_{AC} = -M_0 - \frac{qx^2}{2} \quad \left(0 \leq x \leq \frac{L}{2} \right)$$

$$\frac{\partial M_{AC}}{\partial M_0} = -1$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT CB

$$M_{CB} = -M_0 - \frac{qL}{2} \left(x - \frac{L}{4} \right) \quad \left(\frac{L}{2} \leq x \leq L \right)$$

$$\frac{\partial M_{CB}}{\partial M_0} = -1$$

MODIFIED CASTIGLIANO'S THEOREM (Eq. 9-118)

$$\begin{aligned} \theta_A &= \int \left(\frac{M}{EI} \right) \left(\frac{\partial M}{\partial M_0} \right) dx \\ &= \frac{1}{EI} \int_0^{L/2} \left(-M_0 - \frac{qx^2}{2} \right) (-1) dx \\ &\quad + \frac{1}{EI} \int_{L/2}^L \left[-M_0 - \frac{qL}{2} \left(x - \frac{L}{4} \right) \right] (-1) dx \end{aligned}$$

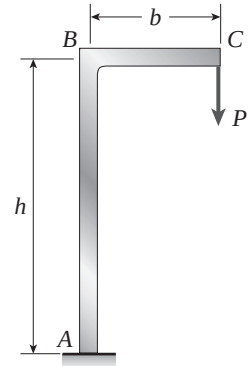
SET FICTITIOUS LOAD M_0 EQUAL TO ZERO

$$\begin{aligned} \theta_A &= \frac{1}{EI} \int_0^{L/2} \frac{qx^2}{2} dx + \frac{1}{EI} \int_{L/2}^L \left(\frac{qL}{2} \right) \left(x - \frac{L}{4} \right) dx \\ &= \frac{qL^3}{48EI} + \frac{qL^3}{8EI} \\ &= \frac{7qL^3}{48EI} \quad (\text{counterclockwise}) \quad \leftarrow \end{aligned}$$

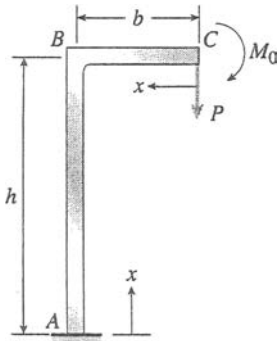
(This result can be verified with the aid of Case 3, Table H-1.)

Problem 9.9-8 The frame ABC supports a concentrated load P at point C (see figure). Members AB and BC have lengths h and b , respectively.

Determine the vertical deflection δ_C and angle of rotation θ_C at end C of the frame. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-8 Frame with concentrated load



P = concentrated load acting at point C
(corresponding to the deflection δ_C)
 M_0 = fictitious moment corresponding to the
angle of rotation θ_C

BENDING MOMENT AND PARTIAL DERIVATIVES FOR
MEMBER AB

$$M_{AB} = Pb + M_0 \quad (0 \leq x \leq h)$$

$$\frac{\partial M_{AB}}{\partial P} = b \quad \frac{\partial M_{AB}}{\partial M_0} = 1$$

BENDING MOMENT AND PARTIAL DERIVATIVES FOR
MEMBER BC

$$M_{BC} = Px + M_0 \quad (0 \leq x \leq b)$$

$$\frac{\partial M_{BC}}{\partial P} = x \quad \frac{\partial M_{BC}}{\partial M_0} = 1$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_C

$$\begin{aligned} \delta_C &= \int \left(\frac{M}{EI} \right) \left(\frac{\partial M}{\partial P} \right) dx \\ &= \frac{1}{EI} \int_0^h (Pb + M_0)(b) dx + \frac{1}{EI} \int_0^b (Px + M_0)(x) dx \end{aligned}$$

Set $M_0 = 0$:

$$\begin{aligned} \delta_C &= \frac{1}{EI} \int_0^h Pb^2 dx + \frac{1}{EI} \int_0^b Px^2 dx \\ &= \frac{Pb^2}{3EI} (3h + b) \quad (\text{downward}) \quad \leftarrow \end{aligned}$$

MODIFIED CASTIGLIANO'S THEOREM FOR ANGLE OF
ROTATION θ_C

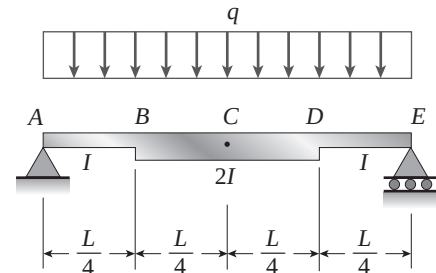
$$\begin{aligned} \theta_C &= \int \left(\frac{M}{EI} \right) \left(\frac{\partial M}{\partial M_0} \right) dx \\ &= \frac{1}{EI} \int_0^h (Pb + M_0)(1) dx + \frac{1}{EI} \int_0^b (Px + M_0)(1) dx \end{aligned}$$

Set $M_0 = 0$:

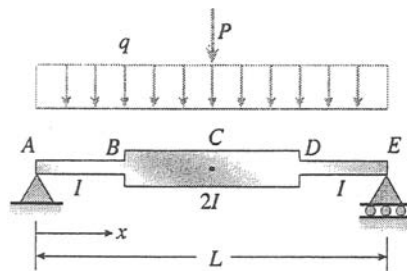
$$\begin{aligned} \theta_C &= \frac{1}{EI} \int_0^h Pb dx + \frac{1}{EI} \int_0^b Pxdx \\ &= \frac{Pb}{2EI} (2h + b) \quad (\text{clockwise}) \quad \leftarrow \end{aligned}$$

Problem 9.9-9 A simple beam $ABCDE$ supports a uniform load of intensity q (see figure). The moment of inertia in the central part of the beam (BCD) is twice the moment of inertia in the end parts (AB and DE).

Find the deflection δ_C at the midpoint C of the beam. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-9 Nonprismatic beam



P = fictitious load corresponding to the deflection δ_C at the midpoint

$$R_A = \frac{qL}{2} + \frac{P}{2}$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR THE LEFT-HAND HALF OF THE BEAM (A TO C)

$$M_{AC} = \frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2} \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$\frac{\partial M_{AC}}{\partial P} = \frac{x}{2} \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

MODIFIED CASTIGLIANO'S THEOREM (Eq. 9-118)

Integrate from A to C and multiply by 2.

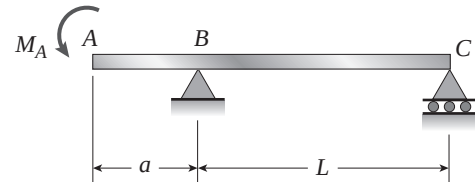
$$\begin{aligned} \delta_C &= 2 \int \left(\frac{M_{AC}}{EI} \right) \left(\frac{\partial M_{AC}}{\partial P} \right) dx \\ &= 2 \left(\frac{1}{EI} \right) \int_0^{L/4} \left(\frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2} \right) \left(\frac{x}{2} \right) dx \\ &\quad + 2 \left(\frac{1}{2EI} \right) \int_{L/4}^{L/2} \left(\frac{qLx}{2} - \frac{qx^2}{2} + \frac{Px}{2} \right) \left(\frac{x}{2} \right) dx \end{aligned}$$

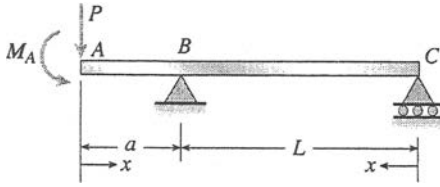
SET FICTITIOUS LOAD P EQUAL TO ZERO

$$\begin{aligned} \delta_C &= \frac{2}{EI} \int_0^{L/4} \left(\frac{qLx}{2} - \frac{qx^2}{2} \right) \left(\frac{x}{2} \right) dx \\ &\quad + \frac{1}{EI} \int_{L/4}^{L/2} \left(\frac{qLx}{2} - \frac{qx^2}{2} \right) \left(\frac{x}{2} \right) dx \\ &= \frac{13qL^4}{6,144EI} + \frac{67qL^4}{12,288EI} \\ \delta_C &= \frac{31qL^4}{4096EI} \quad (\text{downward}) \quad \leftarrow \end{aligned}$$

Problem 9.9-10 An overhanging beam ABC is subjected to a couple M_A at the free end (see figure). The lengths of the overhang and the main span are a and L , respectively.

Determine the angle of rotation θ_A and deflection δ_A at end A . (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-10 Overhanging beam ABC

M_A = couple acting at the free end A (corresponding to the angle of rotation θ_A)

P = fictitious load corresponding to the deflection δ_A

BENDING MOMENT AND PARTIAL DERIVATIVES
FOR SEGMENT AB

$$M_{AB} = -M_A - Px \quad (0 \leq x \leq a)$$

$$\frac{\partial M_{AB}}{\partial M_A} = -1 \quad \frac{\partial M_{AB}}{\partial P} = -x$$

BENDING MOMENT AND PARTIAL DERIVATIVES
FOR SEGMENT BC

$$\text{Reaction at support C: } R_C = \frac{M_A}{L} + \frac{Pa}{L} \quad (\text{downward})$$

$$M_{BC} = -R_C x = -\frac{M_A x}{L} - \frac{Pax}{L} \quad (0 \leq x \leq L)$$

$$\frac{\partial M_{BC}}{\partial M_A} = -\frac{x}{L} \quad \frac{\partial M_{BC}}{\partial P} = -\frac{ax}{L}$$

MODIFIED CASTIGLIANO'S THEOREM FOR ANGLE OF
ROTATION θ_A

$$\begin{aligned} \theta_A &= \int \left(\frac{M}{EI} \right) \left(\frac{\partial M}{\partial M_A} \right) dx \\ &= \frac{1}{EI} \int_0^a (-M_A - Px)(-1) dx \\ &\quad + \frac{1}{EI} \int_0^L \left(-\frac{M_A x}{L} - \frac{Pax}{L} \right) \left(-\frac{x}{L} \right) dx \end{aligned}$$

Set $P = 0$:

$$\begin{aligned} \theta_A &= \frac{1}{EI} \int_0^a M_A dx + \frac{1}{EI} \int_0^L \left(\frac{M_A x}{L} \right) \left(\frac{x}{L} \right) dx \\ &= \frac{M_A}{3EI} (L + 3a) \quad (\text{counterclockwise}) \quad \leftarrow \end{aligned}$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_A

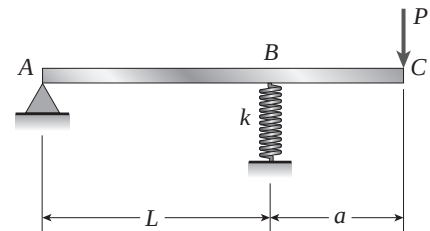
$$\begin{aligned} \delta_A &= \int \left(\frac{M}{EI} \right) \left(\frac{\partial M}{\partial P} \right) dx \\ &= \frac{1}{EI} \int_0^a (-M_A - Px)(-x) dx \\ &\quad + \frac{1}{EI} \int_0^L \left(-\frac{M_A x}{L} - \frac{Pax}{L} \right) \left(-\frac{ax}{L} \right) dx \end{aligned}$$

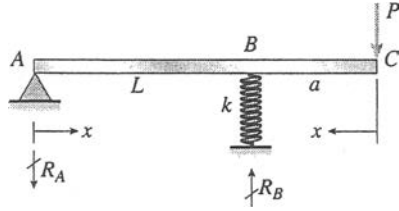
Set $P = 0$:

$$\begin{aligned} \delta_A &= \frac{1}{EI} \int_0^a M_A x dx + \frac{1}{EI} \int_0^L \left(\frac{M_A x}{L} \right) \left(\frac{ax}{L} \right) dx \\ &= \frac{M_A a}{6EI} (2L + 3a) \quad (\text{downward}) \quad \leftarrow \end{aligned}$$

Problem 9.9-11 An overhanging beam ABC rests on a simple support at A and a spring support at B (see figure). A concentrated load P acts at the end of the overhang. Span AB has length L , the overhang has length a , and the spring has stiffness k .

Determine the downward displacement δ_C of the end of the overhang. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-11 Beam with spring support

$$R_A = \frac{Pa}{L} \quad (\text{downward})$$

$$R_B = \frac{P}{L}(L + a) \quad (\text{upward})$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT AB

$$M_{AB} = -R_A x = -\frac{Pax}{L} \quad \frac{dM_{AB}}{dP} = -\frac{ax}{L} \quad (0 \leq x \leq L)$$

BENDING MOMENT AND PARTIAL DERIVATIVE FOR SEGMENT BC

$$M_{BC} = -Px \frac{dM_{BC}}{dP} = -x \quad (0 \leq x \leq a)$$

STRAIN ENERGY OF THE SPRING (EQ. 2-40a)

$$U_S = \frac{R_B^2}{2k} = \frac{P^2(L + a)^2}{2kL^2}$$

STRAIN ENERGY OF THE BEAM (EQ. 9-95a)

$$U_B = \int \frac{M^2 dx}{2EI}$$

TOTAL STRAIN ENERGY U

$$U = U_B + U_S = \int \frac{M^2 dx}{2EI} + \frac{P^2(L + a)^2}{2kL^2}$$

APPLY CASTIGLIANO'S THEOREM (EQ. 9-117)

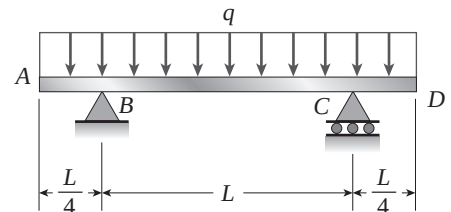
$$\begin{aligned} \delta_C &= \frac{dU}{dP} = \frac{d}{dP} \int \frac{M^2 dx}{2EI} + \frac{d}{dP} \left[\frac{P^2(L + a)^2}{2kL^2} \right] \\ &= \frac{d}{dP} \int \frac{M^2 dx}{2EI} + \frac{P(L + a)^2}{kL^2} \end{aligned}$$

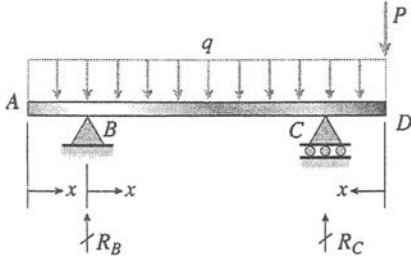
DIFFERENTIATE UNDER THE INTEGRAL SIGN (MODIFIED CASTIGLIANO'S THEOREM)

$$\begin{aligned} \delta_C &= \int \left(\frac{M}{EI} \right) \left(\frac{dM}{dP} \right) dx + \frac{P(L + a)^2}{kL^2} \\ &= \frac{1}{EI} \int_0^L \left(-\frac{Pax}{L} \right) \left(-\frac{ax}{L} \right) dx \\ &\quad + \frac{1}{EI} \int_0^a (-Px)(-x) dx + \frac{P(L + a)^2}{kL^2} \\ &= \frac{Pa^2L}{3EI} + \frac{Pa^3}{3EI} + \frac{P(L + a)^2}{kL^2} \\ \delta_C &= \frac{Pa^2(L + a)}{3EI} + \frac{P(L + a)^2}{kL^2} \quad \leftarrow \end{aligned}$$

Problem 9.9-12 A symmetric beam ABCD with overhangs at both ends supports a uniform load of intensity q (see figure).

Determine the deflection δ_D at the end of the overhang. (Obtain the solution by using the modified form of Castigliano's theorem.)



Solution 9.9-12 Beam with overhangs

q = intensity of uniform load

P = fictitious load corresponding to the deflection δ_D

$\frac{L}{4}$ = length of segments AB and CD

L = length of span BC

$$R_B = \frac{3qL}{4} - \frac{P}{4} \quad R_C = \frac{3qL}{4} + \frac{5P}{4}$$

BENDING MOMENTS AND PARTIAL DERIVATIVES

SEGMENT AB

$$M_{AB} = -\frac{qx^2}{2} \quad \frac{\partial M_{AB}}{\partial P} = 0 \quad \left(0 \leq x \leq \frac{L}{4}\right)$$

SEGMENT BC

$$\begin{aligned} M_{BC} &= -\left[q\left(x + \frac{L}{4}\right)\right]\left[\frac{1}{2}\left(x + \frac{L}{4}\right)\right] + R_B x \\ &= -\frac{q}{2}\left(x + \frac{L}{4}\right)^2 + \left(\frac{3qL}{4} - \frac{P}{4}\right)x \\ &\quad (0 \leq x \leq L) \end{aligned}$$

$$\frac{\partial M_{BC}}{\partial P} = -\frac{x}{4}$$

$$\text{SEGMENT CD} \quad M_{CD} = -\frac{qx^2}{2} - Px \quad \left(0 \leq x \leq \frac{L}{4}\right)$$

$$\frac{\partial M_{CD}}{\partial P} = -x$$

MODIFIED CASTIGLIANO'S THEOREM FOR DEFLECTION δ_D

$$\begin{aligned} \delta_D &= \int \left(\frac{M}{EI}\right) \left(\frac{\partial M}{\partial P}\right) dx \\ &= \frac{1}{EI} \int_0^{L/4} \left(-\frac{qx^2}{2}\right)(0) dx \\ &\quad + \frac{1}{EI} \int_0^L \left[-\frac{q}{2}\left(x + \frac{L}{4}\right)^2 + \left(\frac{3qL}{4} - \frac{P}{4}\right)x\right] \left[-\frac{x}{4}\right] dx \\ &\quad + \frac{1}{EI} \int_0^{L/4} \left(-\frac{qx^2}{2} - Px\right)(-x) dx \end{aligned}$$

SET $P = 0$:

$$\begin{aligned} \delta_D &= \frac{1}{EI} \int_0^L \left[-\frac{q}{2}\left(x + \frac{L}{4}\right)^2 + \frac{3qL}{4}x\right] \left[-\frac{x}{4}\right] dx \\ &\quad + \frac{1}{EI} \int_0^{L/4} \left(-\frac{qx^2}{2}\right)(-x) dx \\ &= -\frac{5qL^4}{768EI} + \frac{qL^4}{2048EI} = -\frac{37qL^4}{6144EI} \end{aligned}$$

(Minus means the deflection is opposite in direction to the fictitious load P .)

$$\therefore \delta_D = \frac{37qL^4}{6144EI} \quad (\text{upward}) \quad \leftarrow$$

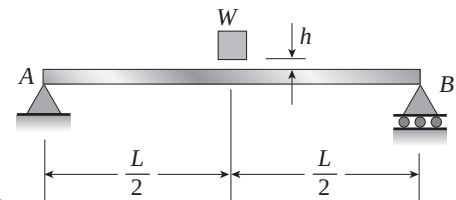
Deflections Produced by Impact

The beams described in the problems for Section 9.10 have constant flexural rigidity EI . Disregard the weights of the beams themselves, and consider only the effects of the given loads.

Problem 9.10-1 A heavy object of weight W is dropped onto the midpoint of a simple beam AB from a height h (see figure).

Obtain a formula for the maximum bending stress $\sigma_{\max}$ due to the falling weight in terms of h , σ_{st} , and δ_{st} , where σ_{st} is the maximum bending stress and δ_{st} is the deflection at the midpoint when the weight W acts on the beam as a statically applied load.

Plot a graph of the ratio $\sigma_{\max}/\sigma_{st}$ (that is, the ratio of the dynamic stress to the static stress) versus the ratio h/δ_{st} . (Let h/δ_{st} vary from 0 to 10.)



Solution 9.10-1 Weight W dropping onto a simple beam

MAXIMUM DEFLECTION (Eq. 9-131)

$$\delta_{\max} = \delta_{\text{st}} + (\delta_{\text{st}}^2 + 2h\delta_{\text{st}})^{1/2}$$

MAXIMUM BENDING STRESS

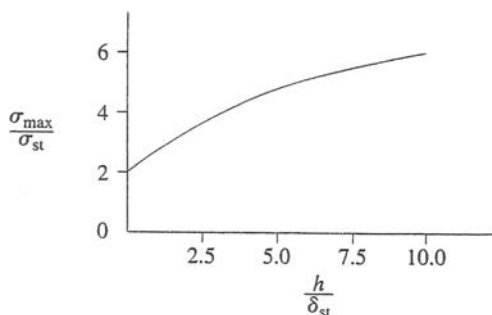
For a linearly elastic beam, the bending stress σ is proportional to the deflection δ

$$\therefore \frac{\sigma_{\max}}{\sigma_{\text{st}}} = \frac{\delta_{\max}}{\delta_{\text{st}}} = 1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2}$$

$$\sigma_{\max} = \sigma_{\text{st}} \left[1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2} \right] \quad \leftarrow$$

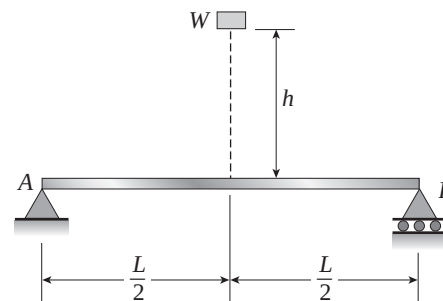
$\frac{h}{\delta_{\text{st}}}$	$\frac{\sigma_{\max}}{\sigma_{\text{st}}}$
0	2.00
2.5	3.45
5.0	4.33
7.5	5.00
10.0	5.58

Note: $\delta_{\text{st}} = \frac{WL^3}{48EI}$ for a simple beam with a load at the midpoint.

GRAPH OF RATIO $\sigma_{\max}/\sigma_{\text{st}}$ 

Problem 9.10-2 An object of weight W is dropped onto the midpoint of a simple beam AB from a height h (see figure). The beam has a rectangular cross section of area A .

Assuming that h is very large compared to the deflection of the beam when the weight W is applied statically, obtain a formula for the maximum bending stress $\sigma_{\max}$ in the beam due to the falling weight.



Solution 9.10-2 Weight W dropping onto a simple beam

Height h is very large.

MAXIMUM DEFLECTION (Eq. 9-132)

$$\delta_{\max} = \sqrt{2h\delta_{\text{st}}}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ .

$$\therefore \frac{\sigma_{\max}}{\sigma_{\text{st}}} = \frac{\delta_{\max}}{\delta_{\text{st}}} = \sqrt{\frac{2h}{\delta_{\text{st}}}}$$

$$\sigma_{\max} = \sqrt{\frac{2h\sigma_{\text{st}}^2}{\delta_{\text{st}}}} \quad (1)$$

$$\begin{aligned} \sigma_{\text{st}} &= \frac{M}{S} = \frac{WL}{4S} & \sigma_{\text{st}}^2 &= \frac{W^2L^2}{16S^2} \\ \delta_{\text{st}} &= \frac{WL^3}{48EI} & \frac{\sigma_{\text{st}}^2}{\delta_{\text{st}}} &= \frac{3WEI}{S^2L} \end{aligned} \quad (2)$$

For a RECTANGULAR BEAM (with b , depth d):

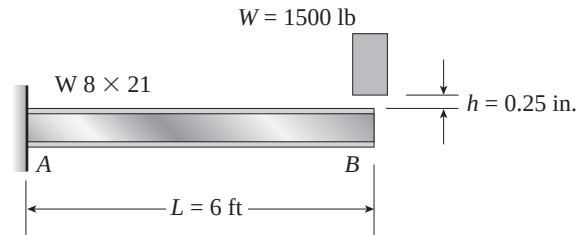
$$I = \frac{bd^3}{12} \quad S = \frac{bd^2}{6} \quad \frac{I}{S^2} = \frac{3}{bd} = \frac{3}{A} \quad (3)$$

Substitute Eqs. (2) and (3) into Eq. (1):

$$\sigma_{\max} = \sqrt{\frac{18WhE}{AL}} \quad \leftarrow$$

Problem 9.10-3 A cantilever beam AB of length $L = 6$ ft is constructed of a $W 8 \times 21$ wide-flange section (see figure). A weight $W = 1500$ lb falls through a height $h = 0.25$ in. onto the end of the beam.

Calculate the maximum deflection $\delta_{\max}$ of the end of the beam and the maximum bending stress $\sigma_{\max}$ due to the falling weight. (Assume $E = 30 \times 10^6$ psi.)

**Solution 9.10-3 Cantilever beam**

DATA: $L = 6$ ft = 72 in. $W = 1500$ lb

$h = 0.25$ in. $E = 30 \times 10^6$ psi

$W 8 \times 21$ $I = 75.3$ in.⁴ $S = 18.2$ in.³

MAXIMUM DEFLECTION (Eq. 9-131)

Equation (9-131) may be used for any linearly elastic structure by substituting $\delta_{\text{st}} = W/k$, where k is the stiffness of the particular structure being considered.

Simple beam with load at midpoint:

$$k = \frac{48EI}{L^3}$$

Cantilever beam with load at the free end: $k = \frac{3EI}{L^3}$

For the cantilever beam in this problem,

$$\begin{aligned} \delta_{\text{st}} &= \frac{WL^3}{3EI} = \frac{(1500 \text{ lb})(72 \text{ in.})^3}{3(30 \times 10^6 \text{ psi})(75.3 \text{ in.}^4)} \\ &= 0.08261 \text{ in.} \end{aligned}$$

Equation (9-131):

$$\delta_{\max} = \delta_{\text{st}} + (\delta_{\text{st}}^2 + 2h\delta_{\text{st}})^{1/2} = 0.302 \text{ in.} \quad \leftarrow$$

MAXIMUM BENDING STRESS

Consider a cantilever beam with load P at the free end:

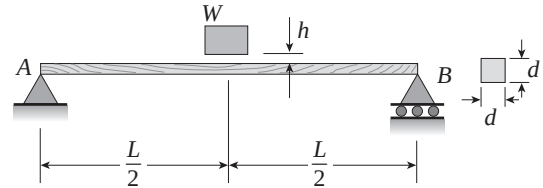
$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{PL}{S} \quad \delta_{\max} = \frac{PL^3}{3EI}$$

$$\text{Ratio: } \frac{\sigma_{\max}}{\delta_{\max}} = \frac{3EI}{SL^2}$$

$$\therefore \sigma_{\max} = \frac{3EI}{SL^2} \delta_{\max} = 21,700 \text{ psi} \quad \leftarrow$$

Problem 9.10-4 A weight $W = 20 \text{ kN}$ falls through a height $h = 1.0 \text{ mm}$ onto the midpoint of a simple beam of length $L = 3 \text{ m}$ (see figure). The beam is made of wood with square cross section (dimension d on each side) and $E = 12 \text{ GPa}$.

If the allowable bending stress in the wood is $\sigma_{\text{allow}} = 10 \text{ MPa}$, what is the minimum required dimension d ?



Solution 9.10-4 Simple beam with falling weight W

DATA: $W = 20 \text{ kN}$ $h = 1.0 \text{ mm}$ $L = 3.0 \text{ m}$
 $E = 12 \text{ GPa}$ $\sigma_{\text{allow}} = 10 \text{ MPa}$

CROSS SECTION OF BEAM (SQUARE)

d = dimension of each side

$$I = \frac{d^4}{12} \quad S = \frac{d^3}{6}$$

MAXIMUM DEFLECTION (EQ. 9-131)

$$\delta_{\text{max}} = \delta_{\text{st}} + (\delta_{\text{st}}^2 + 2h\delta_{\text{st}})^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ .

$$\therefore \frac{\sigma_{\text{max}}}{\sigma_{\text{st}}} = \frac{\delta_{\text{max}}}{\delta_{\text{st}}} = 1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2} \quad (1)$$

STATIC TERMS σ_{st} AND δ_{st}

$$\sigma_{\text{st}} = \frac{M}{S} = \left(\frac{WL}{4}\right)\left(\frac{6}{d^3}\right) = \frac{3WL}{2d^3} \quad (2)$$

$$\delta_{\text{st}} = \frac{WL^3}{48EI} = \frac{WL^3}{48E}\left(\frac{12}{d^4}\right) = \frac{WL^3}{4Ed^4} \quad (3)$$

SUBSTITUTE EQS. (2) AND (3) INTO EQ. (1)

$$\frac{2\sigma_{\text{max}}d^3}{3WL} = 1 + \left(1 + \frac{8hEd^4}{WL^3}\right)^{1/2}$$

SUBSTITUTE NUMERICAL VALUES:

$$\frac{2(10 \text{ MPa})d^3}{3(20 \text{ kN})(3.0 \text{ m})} = 1 + \left[1 + \frac{8(1.0 \text{ mm})(12 \text{ GPa})d^4}{(20 \text{ kN})(3.0 \text{ m})^3}\right]^{1/2}$$

$$\frac{1000}{9}d^3 - 1 = \left[1 + \frac{1600}{9}d^4\right]^{1/2} \quad (d = \text{meters})$$

SQUARE BOTH SIDES, REARRANGE, AND SIMPLIFY

$$\left(\frac{1000}{9}\right)^2 d^3 - \frac{1600}{9}d - \frac{2000}{9} = 0$$

$$2500d^3 - 36d - 45 = 0 \quad (d = \text{meters})$$

SOLVE NUMERICALLY

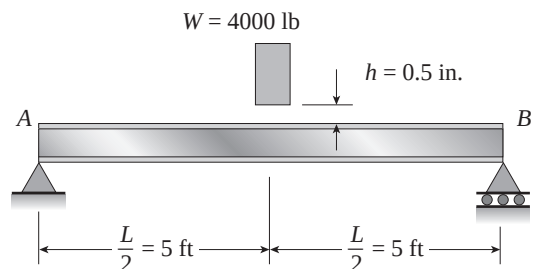
$$d = 0.2804 \text{ m} = 280.4 \text{ mm}$$

For minimum value, round upward.

$$\therefore d = 281 \text{ mm} \quad \leftarrow$$

Problem 9.10-5 A weight $W = 4000 \text{ lb}$ falls through a height $h = 0.5 \text{ in.}$ onto the midpoint of a simple beam of length $L = 10 \text{ ft}$ (see figure).

Assuming that the allowable bending stress in the beam is $\sigma_{\text{allow}} = 18,000 \text{ psi}$ and $E = 30 \times 10^6 \text{ psi}$, select the lightest wide-flange beam listed in Table F-1 in Appendix F that will be satisfactory.



Solution 9.10-5 Simple beam of wide-flange shapeDATA: $W = 4000 \text{ lb}$ $h = 0.5 \text{ in.}$ $L = 10 \text{ ft} = 120 \text{ in.}$ $\sigma_{\text{allow}} = 18,000 \text{ psi}$ $E = 30 \times 10^6 \text{ psi}$

MAXIMUM DEFLECTION (Eq. 9-131)

$$\delta_{\text{max}} = \delta_{\text{st}} + (\delta_{\text{st}}^2 + 2h\delta_{\text{st}})^{1/2}$$

$$\text{or } \frac{\delta_{\text{max}}}{\delta_{\text{st}}} = 1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2}$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ .

$$\therefore \frac{\sigma_{\text{max}}}{\sigma_{\text{st}}} = \frac{\delta_{\text{max}}}{\delta_{\text{st}}} = 1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2} \quad (1)$$

STATIC TERMS σ_{st} AND δ_{st}

$$\sigma_{\text{st}} = \frac{M}{S} = \frac{WL}{4S} \quad \delta_{\text{st}} = \frac{WL^3}{48EI}$$

$$\frac{\sigma_{\text{max}}}{\sigma_{\text{st}}} = \sigma_{\text{allow}} \left(\frac{4S}{WL} \right) = \frac{4\sigma_{\text{allow}}S}{WL} \quad (2)$$

$$\frac{2h}{\delta_{\text{st}}} = 2h \left(\frac{48EI}{WL^3} \right) = \frac{96hEI}{WL^3} \quad (3)$$

SUBSTITUTE EQS. (2) AND (3) INTO EQ. (1):

$$\frac{4\sigma_{\text{allow}}S}{WL} = 1 + \left(1 + \frac{96hEI}{WL^3}\right)^{1/2}$$

REQUIRED SECTION MODULUS

$$S = \frac{WL}{4\sigma_{\text{allow}}} \left[1 + \left(1 + \frac{96hEI}{WL^3}\right)^{1/2} \right]$$

SUBSTITUTE NUMERICAL VALUES

$$S = \left(\frac{20}{3} \text{ in.}^3 \right) \left[1 + \left(1 + \frac{5I}{24}\right)^{1/2} \right] \quad (4)$$

 $(S = \text{in.}^3; I = \text{in.}^4)$

PROCEDURE

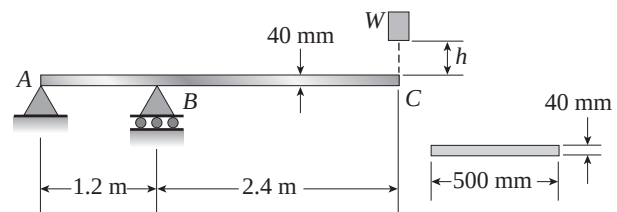
1. Select a trial beam from Table F-1.
2. Substitute I into Eq. (4) and calculate required S .
3. Compare with actual S for the beam.
4. Continue until the lightest beam is found.

Trial beam	Actual		Required S
	I	S	
W 8 $\times$ 35	127	31.2	41.6 (NG)
W 10 $\times$ 45	248	49.1	55.0 (NG)
W 10 $\times$ 60	341	66.7	63.3 (OK)
W 12 $\times$ 50	394	64.7	67.4 (NG)
W 14 $\times$ 53	541	77.8	77.8 (OK)
W 16 $\times$ 31	375	47.2	66.0 (NG)

Lightest beam is W 14 $\times$ 53

Problem 9.10-6 An overhanging beam ABC of rectangular cross section has the dimensions shown in the figure. A weight $W = 750 \text{ N}$ drops onto end C of the beam.

If the allowable normal stress in bending is 45 MPa , what is the maximum height h from which the weight may be dropped? (Assume $E = 12 \text{ GPa}$.)



Solution 9.10-6 Overhanging beamDATA $W = 750 \text{ N}$ $L_{AB} = 1.2 \text{ m}$ $L_{BC} = 2.4 \text{ m}$ $E = 12 \text{ GPa}$ $\sigma_{\text{allow}} = 45 \text{ MPa}$

$$I = \frac{bd^3}{12} = \frac{1}{12}(500 \text{ mm})(40 \text{ mm})^3$$

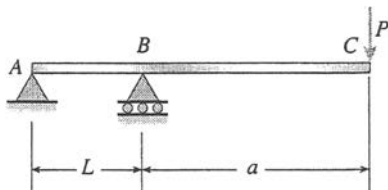
$$= 2.6667 \times 10^6 \text{ mm}^4$$

$$= 2.6667 \times 10^{-6} \text{ m}^4$$

$$S = \frac{bd^2}{6} = \frac{1}{6}(500 \text{ mm})(40 \text{ mm})^2$$

$$= 133.33 \times 10^3 \text{ mm}^3$$

$$= 133.33 \times 10^{-6} \text{ m}^3$$

DEFLECTION δ_C AT THE END OF THE OVERHANG $P =$ load at end C $L =$ length of span AB $a =$ length of overhang BC

From the answer to Prob. 9.8-5 or Prob. 9.9-3:

$$\delta_C = \frac{Pa^2(L + a)}{3EI}$$

$$\text{Stiffness of the beam: } k = \frac{P}{\delta_C} = \frac{3EI}{a^2(L + a)} \quad (1)$$

MAXIMUM DEFLECTION (EQ. 9-131)

Equation (9-131) may be used for any linearly elastic structure by substituting $\delta_{\text{st}} = W/k$, where k is the stiffness of the particular structure being considered. For instance:

$$\text{Simple beam with load at midpoint: } k = \frac{48EI}{L^3}$$

$$\text{Cantilever beam with load at free end: } k = \frac{3EI}{L^3} \text{ Etc.}$$

For the overhanging beam in this problem [see Eq. (1)],

$$\delta_{\text{st}} = \frac{W}{k} = \frac{Wa^2(L + a)}{3EI} \quad (2)$$

in which $a = L_{BC}$ and $L = L_{AB}$:

$$\delta_{\text{st}} = \frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{3EI} \quad (3)$$

EQUATION (9-131):

$$\delta_{\text{max}} = \delta_{\text{st}} + (\delta_{\text{st}}^2 + 2h\delta_{\text{st}})^{1/2}$$

or

$$\frac{\delta_{\text{max}}}{\delta_{\text{st}}} = 1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2} \quad (4)$$

MAXIMUM BENDING STRESS

For a linearly elastic beam, the bending stress σ is proportional to the deflection δ .

$$\therefore \frac{\sigma_{\text{max}}}{\sigma_{\text{st}}} = \frac{\delta_{\text{max}}}{\delta_{\text{st}}} = 1 + \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2} \quad (5)$$

$$\sigma_{\text{st}} = \frac{M}{S} = \frac{WL_{BC}}{S} \quad (6)$$

MAXIMUM HEIGHT h Solve Eq. (5) for h :

$$\begin{aligned} \frac{\sigma_{\text{max}}}{\sigma_{\text{st}}} - 1 &= \left(1 + \frac{2h}{\delta_{\text{st}}}\right)^{1/2} \\ \left(\frac{\sigma_{\text{max}}}{\sigma_{\text{st}}}\right)^2 - 2\left(\frac{\sigma_{\text{max}}}{\sigma_{\text{st}}}\right) + 1 &= 1 + \frac{2h}{\delta_{\text{st}}} \\ h &= \frac{\delta_{\text{st}}}{2} \left(\frac{\sigma_{\text{max}}}{\sigma_{\text{st}}} - 2\right) \end{aligned} \quad (7)$$

Substitute δ_{st} from Eq. (3), σ_{st} from Eq. (6), and σ_{allow} for σ_{max} :

$$h = \frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{6EI} \left(\frac{\sigma_{\text{allow}}S}{WL_{BC}}\right) \left(\frac{\sigma_{\text{allow}}S}{WL_{BC}} - 2\right) \quad (8)$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (8):

$$\frac{W(L_{BC}^2)(L_{AB} + L_{BC})}{6EI} = 0.08100 \text{ m}$$

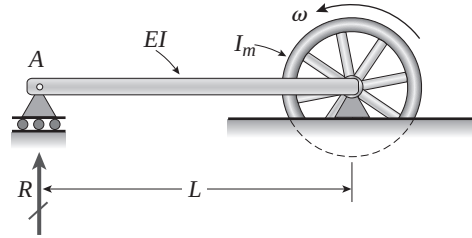
$$\frac{\sigma_{\text{allow}}S}{WL_{BC}} = \frac{10}{3} = 3.3333$$

$$h = (0.08100 \text{ m}) \left(\frac{10}{3}\right) \left(\frac{10}{3} - 2\right) = 0.36 \text{ m}$$

$$\text{or } h = 360 \text{ mm} \quad \leftarrow$$

Problem 9.10-7 A heavy flywheel rotates at an angular speed ω (radians per second) around an axle (see figure). The axle is rigidly attached to the end of a simply supported beam of flexural rigidity EI and length L (see figure). The flywheel has mass moment of inertia I_m about its axis of rotation.

If the flywheel suddenly freezes to the axle, what will be the reaction R at support A of the beam?



Solution 9.10-7 Rotating flywheel

NOTE: We will disregard the mass of the beam and all energy losses due to the sudden stopping of the rotating flywheel. Assume that *all* of the kinetic energy of the flywheel is transformed into strain energy of the beam.

KINETIC ENERGY OF ROTATING FLYWHEEL

$$KE = \frac{1}{2} I_m \omega^2$$

STRAIN ENERGY OF BEAM
$$U = \int \frac{M^2 dx}{2EI}$$

$M = Rx$, where x is measured from support A .

$$U = \frac{1}{2EI} \int_0^L (Rx)^2 dx = \frac{R^2 L^3}{6EI}$$

CONSERVATION OF ENERGY

$$KE = U \quad \frac{1}{2} I_m \omega^2 = \frac{R^2 L^3}{6EI}$$

$$R = \sqrt{\frac{3EI I_m \omega^2}{L^3}} \quad \leftarrow$$

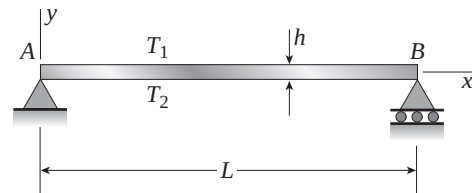
Note: The moment of inertia I_m has units of $\text{kg} \cdot \text{m}^2$ or $\text{N} \cdot \text{m} \cdot \text{s}^2$

Temperature Effects

The beams described in the problems for Section 9.11 have constant flexural rigidity EI . In every problem, the temperature varies linearly between the top and bottom of the beam.

Problem 9.11-1 A simple beam AB of length L and height h undergoes a temperature change such that the bottom of the beam is at temperature T_2 and the top of the beam is at temperature T_1 (see figure).

Determine the equation of the deflection curve of the beam, the angle of rotation θ_A at the left-hand support, and the deflection $\delta_{\max}$ at the midpoint.



Solution 9.11-1 Simple beam with temperature differential

$$\text{Eq. (9-136): } v'' = \frac{d^2v}{dx^2} = \frac{\alpha(T_2 - T_1)}{h}$$

$$v' = \frac{dv}{dx} = \frac{\alpha(T_2 - T_1)x}{h} + C_1$$

$$\text{B.C. 1 (Symmetry) } v'\left(\frac{L}{2}\right) = 0$$

$$\therefore C_1 = -\frac{\alpha L(T_2 - T_1)}{2h}$$

$$v = \frac{\alpha(T_2 - T_1)x^2}{2h} - \frac{\alpha L(T_2 - T_1)x}{2h} + C_2$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_2 = 0$$

$$v = -\frac{\alpha(T_2 - T_1)(x)(L - x)}{2h} \quad \leftarrow$$

(positive v is upward deflection)

$$v' = -\frac{\alpha(T_2 - T_1)(L - 2x)}{2h}$$

$$\theta_A = -v'(0) = \frac{\alpha L(T_2 - T_1)}{2h} \quad \leftarrow$$

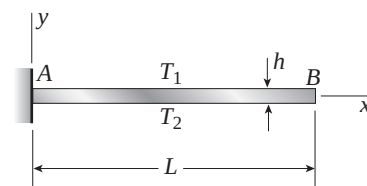
(positive θ_A is clockwise rotation)

$$\delta_{\max} = -v\left(\frac{L}{2}\right) = \frac{\alpha L^2(T_2 - T_1)}{8h} \quad \leftarrow$$

(positive $\delta_{\max}$ is downward deflection)

Problem 9.11-2 A cantilever beam AB of length L and height h (see figure) is subjected to a temperature change such that the temperature at the top is T_1 and at the bottom is T_2 .

Determine the equation of the deflection curve of the beam, the angle of rotation θ_B at end B , and the deflection δ_B at end B .

**Solution 9.11-2 Cantilever beam with temperature differential**

$$\text{Eq. (9-136): } v'' = \frac{d^2v}{dx^2} = \frac{\alpha(T_2 - T_1)}{h}$$

$$v' = \frac{dv}{dx} = \frac{\alpha(T_2 - T_1)}{h}x + C_1$$

$$\text{B.C. 1 } v'(0) = 0 \quad \therefore C_1 = 0$$

$$v' = \frac{\alpha(T_2 - T_1)}{h}x$$

$$v = \frac{\alpha(T_2 - T_1)}{h}\left(\frac{x^2}{2}\right) + C_2$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_2 = 0$$

$$v = \frac{\alpha(T_2 - T_1)x^2}{2h} \quad \leftarrow$$

(positive v is upward deflection)

$$\theta_B = v'(L) = \frac{\alpha L(T_2 - T_1)}{h} \quad \leftarrow$$

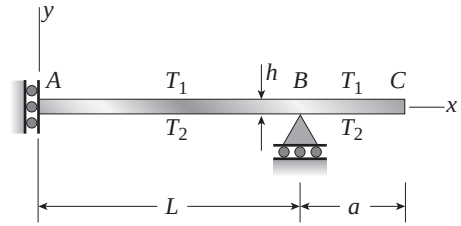
(positive θ_B is counterclockwise rotation)

$$\delta_B = v(L) = \frac{\alpha L^2(T_2 - T_1)}{2h} \quad \leftarrow$$

(positive δ_B is upward deflection)

Problem 9.11-3 An overhanging beam ABC of height h has a guided support at A and a roller at B . The beam is heated to a temperature T_1 on the top and T_2 on the bottom (see figure).

Determine the equation of the deflection curve of the beam, the angle of rotation θ_C at end C , and the deflection δ_C at end C .



Solution 9.11-3

$$v'' = \frac{d^2}{dx^2}v = \frac{\alpha(T_2 - T_1)}{h}$$

$$v' = \frac{\alpha(T_2 - T_1)x}{h} + C_1$$

$$v = \frac{\alpha(T_2 - T_1)x^2}{2h} + C_1x + C_2$$

$$\text{B.C. } v'(0) = 0 \quad C_1 = 0$$

$$\text{B.C. } v(L) = 0 \quad C_2 = -\frac{\alpha(T_2 - T_1)L^2}{2h}$$

$$v(x) = \frac{\alpha(T_2 - T_1)(x^2 - L^2)}{2h}$$

$$\delta_C = v(L + a) = \frac{\alpha(T_2 - T_1)[(L + a)^2 - L^2]}{2h}$$

$$= \frac{\alpha(T_2 - T_1)(2La + a^2)}{2h} \quad \leftarrow$$

(upward)

$$\theta_C = v'(L + a) = \frac{\alpha(T_2 - T_1)(L + a)}{h} \quad \leftarrow$$

(counterclockwise)

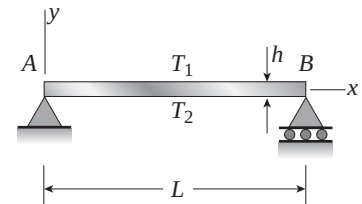
Problem 9.11-4 A simple beam AB of length L and height h (see figure) is heated in such a manner that the temperature difference $T_2 - T_1$ between the bottom and top of the beam is proportional to the distance from support A ; that is, assume the temperature difference varies *linearly* along the beam:

$$T_2 - T_1 = T_0x$$

in which T_0 is a constant having units of temperature (degrees) per unit distance.

(a) Determine the maximum deflection $\delta_{\max}$ of the beam.

(b) Repeat for *quadratic* temperature variation along the beam, $T_2 - T_1 = T_0x^2$.



Solution 9.11-4

$$(a) \quad (T_2 - T_1) = T_0x$$

$$v'' = \frac{d^2}{dx^2}v = \frac{\alpha T_0x}{h}$$

$$v' = \frac{\alpha T_0x^2}{2h} + C_1$$

$$v = \frac{\alpha T_0x^3}{6h} + C_1x + C_2$$

$$\text{B.C. } v(0) = 0 \quad C_2 = 0$$

$$\text{B.C. } v(L) = 0 \quad C_1 = -\frac{\alpha T_0L^2}{6h}$$

$$v(x) = \frac{\alpha T_0(x^3 - L^2x)}{6h}$$

$$v'(x) = \frac{\alpha T_0}{2h} \left(x^2 - \frac{L^2}{3} \right)$$

MAXIMUM DEFLECTION

Set $v'(x) = 0$ and solve for x .

$$0 = \frac{\alpha T_0}{2h} \left(x^2 - \frac{L^2}{3} \right) \quad x = \frac{L}{\sqrt{3}}$$

$$\begin{aligned} \delta_{\max} &= -v \left(\frac{L}{\sqrt{3}} \right) \\ &= -\frac{\alpha T_0 \left[\left(\frac{L}{\sqrt{3}} \right)^3 - L^2 \frac{L}{\sqrt{3}} \right]}{6h} \end{aligned}$$

$$\delta_{\max} = \frac{\alpha T_0 L^3}{9\sqrt{3}h} \quad (\text{downward}) \quad \leftarrow$$

$$(b) (T_2 - T_1) = T_0 x^2$$

$$v'' = \frac{d^2}{dx^2} v = \frac{\alpha T_0 x^2}{h}$$

$$v' = \frac{\alpha T_0 x^3}{3h} + C_1$$

$$v = \frac{\alpha T_0 x^4}{12h} + C_1 x + C_2$$

$$\text{B.C. } v(0) = 0 \quad C_2 = 0$$

$$\text{B.C. } v(L) = 0 \quad C_1 = -\frac{\alpha T_0 L^3}{12h}$$

$$v(x) = \frac{\alpha T_0 (x^4 - L^3 x)}{12h}$$

$$v'(x) = \frac{\alpha T_0}{3h} \left(x^3 - \frac{L^3}{4} \right)$$

MAXIMUM DEFLECTION

Set $v'(x) = 0$ and solve for x .

$$0 = \frac{\alpha T_0}{3h} \left(x^3 - \frac{L^3}{4} \right) \quad x = \frac{L}{\sqrt{2}}$$

$$\begin{aligned} \delta_{\max} &= -v \left(\frac{L}{\sqrt{2}} \right) \\ &= -\frac{\alpha T_0 \left[\left(\frac{L}{\sqrt{2}} \right)^4 - L^3 \frac{L}{\sqrt{2}} \right]}{12h} \end{aligned}$$

$$\delta_{\max} = \frac{\alpha T_0 L^4}{48h} (2\sqrt{2} - 1) \quad \leftarrow$$

(downward)

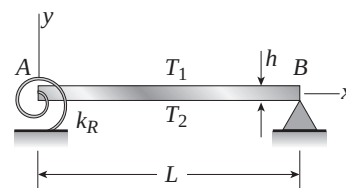
Problem 9.11-5 Beam AB , with elastic support k_R at A and pin support at B , of length L and height h (see figure) is heated in such a manner that the temperature difference $T_2 - T_1$ between the bottom and top of the beam is proportional to the distance from support A ; that is, assume the temperature difference varies *linearly* along the beam:

$$T_2 - T_1 = T_0 x$$

in which T_0 is a constant having units of temperature (degrees) per unit distance.

Assume the spring at A is unaffected by the temperature change.

- Determine the maximum deflection $\delta_{\max}$ of the beam.
- Repeat for *quadratic* temperature variation along the beam, $T_2 - T_1 = T_0 x^2$.
- What is $\delta_{\max}$ for parts (a) and (b) if k_R goes to infinity?



Solution 9.11-5

(a) $(T_2 - T_1) = T_0 x$

$$v'' = \frac{d^2}{dx^2} v = \frac{\alpha T_0 x}{h}$$

$$v' = \frac{\alpha T_0 x^2}{2h} + C_1$$

$$v = \frac{\alpha T_0 x^3}{6h} + C_1 x + C_2$$

B.C. $v'(0) = 0 \quad C_1 = 0$

B.C. $v(L) = 0 \quad C_2 = -\frac{\alpha T_0 L^3}{6h}$

$$v(x) = \frac{\alpha T_0 (x^3 - L^3)}{6h}$$

$$v'(x) = \frac{\alpha T_0 x^2}{2h}$$

MAXIMUM DEFLECTION

Set $v'(x) = 0$ and solve for x .

$$0 = \frac{\alpha T_0 x^2}{2h} \quad x = 0$$

$$\delta_{\max} = -v(0) = \frac{\alpha T_0 L^3}{6h} \quad (\text{downward}) \quad \leftarrow$$

(b) $(T_2 - T_1) = T_0 x^2$

$$v'' = \frac{d^2}{dx^2} v = \frac{\alpha T_0 x^2}{h}$$

$$v' = \frac{\alpha T_0 x^3}{3h} + C_1$$

$$v = \frac{\alpha T_0 x^4}{12h} + C_1 x + C_2$$

B.C. $v'(0) = 0 \quad C_1 = 0$

B.C. $v(L) = 0 \quad C_2 = -\frac{\alpha T_0 L^4}{12h}$

$$v(x) = \frac{\alpha T_0 (x^4 - L^4)}{12h}$$

$$v'(x) = \frac{\alpha T_0 x^3}{3h}$$

MAXIMUM DEFLECTION

Set $v'(x) = 0$ and solve for x .

$$0 = \frac{\alpha T_0 x^3}{3h} \quad x = 0$$

$$\delta_{\max} = -v(0) = \frac{\alpha T_0 L^4}{12h} \quad (\text{downward}) \quad \leftarrow$$

(c) Changing k_R does not change $\delta_{\max}$ in both cases.

10

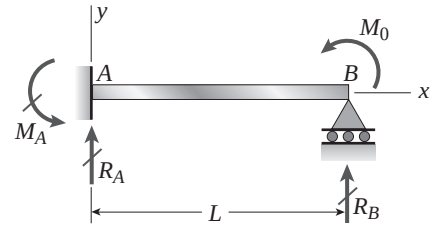
Statically Indeterminate Beams

Differential Equations of the Deflection Curve

The problems for Section 10.3 are to be solved by integrating the differential equations of the deflection curve. All beams have constant flexural rigidity EI . When drawing shear-force and bending-moment diagrams, be sure to label all critical ordinates, including maximum and minimum values.

Problem 10.3-1 A propped cantilever beam AB of length L is loaded by a counterclockwise moment M_0 acting at support B (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), obtain the reactions, shear forces, bending moments, slopes, and deflections of the beam. Construct the shear-force and bending-moment diagrams, labeling all critical ordinates.



Solution 10.3-1 Propped cantilever beam

M_0 = applied load

Select M_A as the redundant reaction.

REACTIONS (FROM EQUILIBRIUM)

$$R_A = \frac{M_A}{L} + \frac{M_0}{L} \quad (1) \quad R_B = -R_A \quad (2)$$

BENDING MOMENT (FROM EQUILIBRIUM)

$$M = R_A x - M_A = \frac{M_A}{L}(x - L) + \frac{M_0 x}{L} \quad (3)$$

DIFFERENTIAL EQUATIONS

$$EIv'' = M = \frac{M_A}{L}(x - L) + \frac{M_0 x}{L}$$

$$EIv' = \frac{M_A}{L}\left(\frac{x^2}{2} - Lx\right) + \frac{M_0 x^2}{2L} + C_1 \quad (4)$$

$$\text{B.C. 1 } v'(0) = 0 \quad \therefore C_1 = 0$$

$$EIv = \frac{M_A}{L}\left(\frac{x^3}{6} - \frac{Lx^2}{2}\right) + \frac{M_0 x^3}{6L} + C_2 \quad (5)$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_2 = 0$$

$$\text{B.C. 3 } v(L) = 0 \quad \therefore M_A = \frac{M_0}{2}$$

REACTIONS [SEE EQS. (1) AND (2)]

$$M_A = \frac{M_0}{2} \quad R_A = \frac{3M_0}{2L} \quad R_B = -\frac{3M_0}{2L} \quad \leftarrow$$

SHEAR FORCE (FROM EQUILIBRIUM)

$$V = R_A = \frac{3M_0}{2L} \quad \leftarrow$$

BENDING MOMENT [FROM EQ. (3)]

$$M = \frac{M_0}{2L} (3x - L) \quad \leftarrow$$

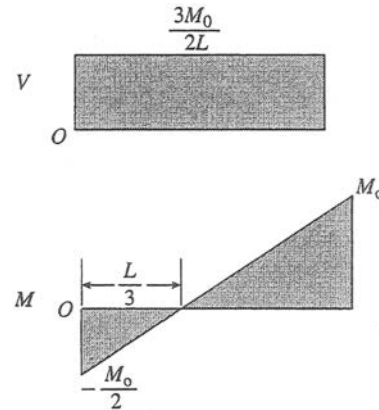
SLOPE [FROM EQ. (4)]

$$v' = -\frac{M_0 x}{4LEI} (2L - 3x) \quad \leftarrow$$

DEFLECTION [FROM EQ. (5)]

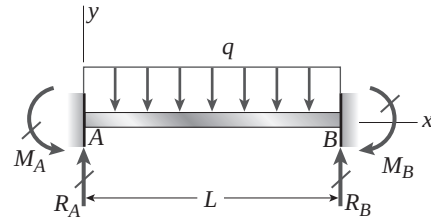
$$v = -\frac{M_0 x^2}{4LEI} (L - x) \quad \leftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAM



Problem 10.3-2 A fixed-end beam AB of length L supports a uniform load of intensity q (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), obtain the reactions, shear forces, bending moments, slopes, and deflections of the beam. Construct the shear-force and bending-moment diagrams, labeling all critical ordinates.



Solution 10.3-2 Fixed-end beam (uniform load)

Select M_A as the redundant reaction.

REACTIONS (FROM SYMMETRY AND EQUILIBRIUM)

$$R_A = R_B = \frac{qL}{2} \quad M_B = M_A$$

BENDING MOMENT (FROM EQUILIBRIUM)

$$M = R_A x - M_A - \frac{qx^2}{2} = -M_A + \frac{q}{2} (Lx - x^2) \quad (1)$$

DIFFERENTIAL EQUATIONS

$$EIv'' = M = -M_A + \frac{q}{2} (Lx - x^2)$$

$$EIv' = -M_A x + \frac{q}{2} \left(\frac{Lx^2}{2} - \frac{x^3}{3} \right) + C_1 \quad (2)$$

$$\text{B.C. 1 } v'(0) = 0 \quad \therefore C_1 = 0$$

$$EIv = -\frac{M_A x^2}{2} + \frac{q}{2} \left(\frac{Lx^3}{6} - \frac{x^4}{12} \right) + C_2 \quad (3)$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_2 = 0$$

$$\text{B.C. 3 } v(L) = 0 \quad \therefore M_A = \frac{qL^2}{12}$$

REACTIONS

$$R_A = R_B = \frac{qL}{2} \quad M_A = M_B = \frac{qL^2}{12} \quad \leftarrow$$

SHEAR FORCE (FROM EQUILIBRIUM)

$$V = R_A - qx = \frac{q}{2} (L - 2x) \quad \leftarrow$$

BENDING MOMENT [FROM EQ. (1)]

$$M = -\frac{q}{12}(L^2 - 6Lx + 6x^2) \quad \leftarrow$$

SLOPE [FROM EQ. (2)]

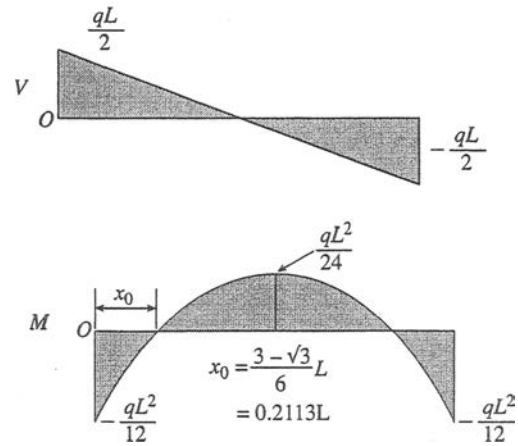
$$v' = -\frac{qx}{12EI}(L^2 - 3Lx + 2x^2) \quad \leftarrow$$

DEFLECTION [FROM EQ. (3)]

$$v = -\frac{qx^2}{24EI}(L - x)^2 \quad \leftarrow$$

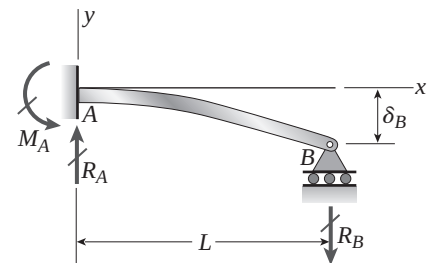
$$\delta_{\max} = -v\left(\frac{L}{2}\right) = \frac{qL^4}{384EI}$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



Problem 10.3-3 A cantilever beam AB of length L has a fixed support at A and a roller support at B (see figure). The support at B is moved downward through a distance δ_B .

Using the fourth-order differential equation of the deflection curve (the load equation), determine the reactions of the beam and the equation of the deflection curve. (Note: Express all results in terms of the imposed displacement δ_B .)



Solution 10.3-3 Cantilever beam with imposed displacement δ_B

REACTIONS (FROM EQUILIBRIUM)

$$R_A = R_B \quad (1)$$

$$M_A = R_B L \quad (2)$$

$$\text{B.C. 3 } v''(L) = 0 \quad \therefore C_1 L + C_2 = 0 \quad (8)$$

$$\text{B.C. 4 } v(L) = -\delta_B$$

$$\therefore C_1 L + 3C_2 = -6EI\delta_B/L^2 \quad (9)$$

DIFFERENTIAL EQUATIONS

$$EIv'''' = -q = 0 \quad (3)$$

$$EIv''' = V = C_1 \quad (4)$$

$$EIv'' = M = C_1 x + C_2 \quad (5)$$

$$EIv' = C_1 x^2/2 + C_2 x + C_3 \quad (6)$$

$$EIv = C_1 x^3/6 + C_2 x^2/2 + C_3 x + C_4 \quad (7)$$

$$\text{B.C. 1 } v(0) = 0 \quad \therefore C_4 = 0$$

$$\text{B.C. 2 } v'(0) = 0 \quad \therefore C_3 = 0$$

SOLVE EQS. (8) AND (9):

$$C_1 = \frac{3EI\delta_B}{L^3} \quad C_2 = -\frac{3EI\delta_B}{L^2}$$

SHEAR FORCE FROM EQ. (4)

$$V = \frac{3EI\delta_B}{L^3} \quad R_A = V(0) = \frac{3EI\delta_B}{L^3}$$

REACTIONS [EQS. (1) AND (2)]

$$R_A = R_B = \frac{3EI\delta_B}{L^3}$$

$$M_A = R_B L = \frac{3EI\delta_B}{L^2} \quad \leftarrow$$

DEFLECTION FROM EQ. (7):

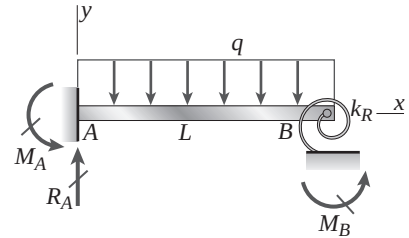
$$v = -\frac{\delta_B x^2}{2L^3} (3L - x) \quad \leftarrow$$

SLOPE FROM EQ. (6):

$$v' = -\frac{3\delta_B x}{2L^3} (2L - x)$$

Problem 10.3-4 A cantilever beam of length L and loaded by uniform load of intensity q has a fixed support at A and spring support at B with rotational stiffness k_R . A rotation at B , θ_B , results in a reaction moment $M_B = k_R \times \theta_B$.

Find rotation θ_B and displacement δ_B at end B . Use the second-order differential equation of the deflection curve to solve for displacements at end B .

**Solution 10.3-4**

q = intensity of uniform load

EQUILIBRIUM

$$R_A = qL$$

$$M_A = \frac{qL^2}{2} - M_B$$

$$M_B = k_R \theta_B$$

BENDING MOMENT

$$M = R_A x - M_A - \frac{qx^2}{2}$$

DIFFERENTIAL EQUATION

$$EIv'' = M = R_A x - M_A - \frac{qx^2}{2}$$

$$EIv' = R_A \frac{x^2}{2} - M_A x - \frac{qx^3}{6} + C_1$$

$$EIv = R_A \frac{x^3}{6} - M_A \frac{x^2}{2} - \frac{qx^4}{24} + C_1 x + C_2$$

$$\text{B.C. 1 } v'(0) = 0 \quad \therefore C_1 = 0$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_2 = 0$$

$$\text{B.C. 3 } v'(L) = \theta_B$$

Substitute R_A and M_A from Eqs. (1) and (2):

$$\therefore EI\theta_B = qL \frac{L^2}{2} - \left(\frac{qL^2}{2} - k_R \theta_B \right) L - \frac{qL^3}{6}$$

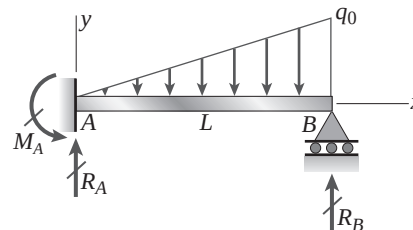
$$\therefore \theta_B = \frac{qL^3}{6(k_R L - EI)} \quad \leftarrow$$

$$\therefore EI\delta_B = qL \frac{L^3}{6} - \left(\frac{qL^2}{2} - k_R \theta_B \right) \frac{L^2}{2} - \frac{qL^4}{24}$$

$$\therefore \delta_B = \frac{1}{EI} \left(-\frac{1}{8} qL^4 + \frac{k_R qL^5}{12(k_R L - EI)} \right) \quad \leftarrow$$

Problem 10.3-5 A cantilever beam of length L and loaded by a triangularly distributed load of maximum intensity q_0 at B .

Use the fourth-order differential equation of the deflection curve to solve for reactions at A and B and also the equation of the deflection curve.



Solution 10.3-5

Triangular load $q = q_0 \frac{x}{L}$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0 \frac{x}{L} \quad (1)$$

$$EIv''' = -q_0 \frac{x^2}{2L} + C_1 \quad (2)$$

$$EIv'' = M = -q_0 \frac{x^3}{6L} + C_1x + C_2 \quad (3)$$

$$EIv' = -q_0 \frac{x^4}{24L} + C_1 \frac{x^2}{2} + C_2x + C_3 \quad (4)$$

$$EIv = -q_0 \frac{x^5}{120L} + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3x + C_4 \quad (5)$$

$$\text{B.C. 1 } v''(L) = 0 \quad \therefore C_1L + C_2 = q_0 \frac{L^2}{6} \quad (6)$$

$$\text{B.C. 2 } v'(0) = 0 \quad \therefore C_3 = 0$$

$$\text{B.C. 3 } v(0) = 0 \quad \therefore C_4 = 0$$

$$\text{B.C. 4 } v(L) = 0 \quad \therefore C_1 \frac{L}{3} + C_2 = q_0 \frac{L^2}{60} \quad (7)$$

Solve Eqs. (6) and (7):

$$C_1 = \frac{9}{40} q_0 L$$

$$C_2 = -\frac{7}{120} q_0 L^2$$

SHEAR FORCE FROM EQ. (2)

$$V = -q_0 \frac{x^2}{2L} + \frac{9}{40} q_0 L$$

REACTIONS

$$R_A = V(0) = \frac{9}{40} q_0 L \quad \leftarrow$$

$$R_B = -V(L) = \frac{11}{40} q_0 L \quad \leftarrow$$

FROM EQUILIBRIUM

$$M_A = \frac{7}{120} q_0 L^2 \quad \leftarrow$$

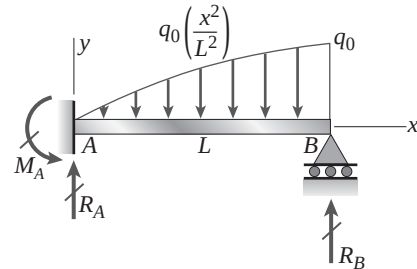
DEFLECTION CURVE FROM EQ. (5)

$$EIv = -q_0 \frac{x^5}{120L} + \frac{9}{40} q_0 L \frac{x^3}{6} - \frac{7}{120} q_0 L^2 \frac{x^2}{2} \quad \text{or}$$

$$v = \frac{1}{240LEI} (-2q_0 x^5 + 9q_0 L^2 x^3 - 7q_0 L^3 x^2) \quad \leftarrow$$

Problem 10.3-6 A propped cantilever beam of length L is loaded by a parabolically distributed load with maximum intensity q_0 at B .

- (a) Use the fourth-order differential equation of the deflection curve to solve for reactions at A and B and also the equation of the deflection curve.
 (b) Repeat part (a) if the parabolic load is replaced by $q_0 \sin(\pi x/2L)$.



SOLUTION 10.3-6

(a) Parabolic load $q = q_0 \frac{x^2}{L^2}$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0 \frac{x^2}{L^2} \quad (1)$$

$$EIv''' = -q_0 \frac{x^3}{3L^2} + C_1 \quad (2)$$

$$EIv'' = M = -q_0 \frac{x^4}{12L^2} + C_1x + C_2 \quad (3)$$

$$EIv' = -q_0 \frac{x^5}{60L^2} + C_1 \frac{x^2}{2} + C_2x + C_3 \quad (4)$$

$$EIv = -q_0 \frac{x^6}{360L^2} + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3x + C_4 \quad (5)$$

B.C. 1 $v''(L) = 0 \quad \therefore C_1L + C_2 = q_0 \frac{L^2}{12} \quad (6)$

B.C. 2 $v'(0) = 0 \quad \therefore C_3 = 0$

B.C. 3 $v(0) = 0 \quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0 \quad \therefore C_1L + 3C_2 = q_0 \frac{L^2}{60} \quad (7)$

Solve Eqs. (6) and (7):

$$C_1 = \frac{7}{60} q_0 L \quad C_2 = -\frac{1}{30} q_0 L^2$$

SHEAR FORCE FROM EQ. (2)

$$V = -q_0 \frac{x^3}{3L^2} + \frac{7}{60} q_0 L$$

REACTIONS

$$R_A = V(0) = \frac{7}{60} q_0 L \quad \leftarrow$$

$$R_B = -V(L) = \frac{13}{60} q_0 L \quad \leftarrow$$

FROM EQUILIBRIUM

$$M_A = \frac{1}{30} q_0 L^2 \quad \leftarrow$$

DEFLECTION CURVE FROM EQ. (5)

$$EIv = -q_0 \frac{x^6}{360L^2} + \frac{7}{60} q_0 L \frac{x^3}{6} - \frac{1}{30} q_0 L^2 \frac{x^2}{2} \quad \text{or}$$

$$v = \frac{q_0}{360L^2EI} (-x^6 + 7L^3x^3 - 6q_0L^4x^2) \quad \leftarrow$$

(b) Loading $q = q_0 \sin\left(\frac{\pi x}{2L}\right)$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0 \sin\left(\frac{\pi x}{2L}\right) \quad (1)$$

$$EIv''' = q_0 \frac{2L}{\pi} \cos\left(\frac{\pi x}{2L}\right) + C_1 \quad (2)$$

$$EIv'' = M = q_0 \left(\frac{2L}{\pi}\right)^2 \sin\left(\frac{\pi x}{2L}\right) + C_1x + C_2 \quad (3)$$

$$EIv' = -q_0 \left(\frac{2L}{\pi} \right)^3 \cos \left(\frac{\pi x}{2L} \right) + C_1 \frac{x^2}{2} + C_2 x + C_3 \quad (4)$$

$$EIv = -q_0 \left(\frac{2L}{\pi} \right)^4 \sin \left(\frac{\pi x}{2L} \right) + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \quad (5)$$

$$\text{B.C. 1 } v''(L) = 0$$

$$\therefore C_1 L + C_2 = -q_0 \frac{4L^2}{\pi^2} \quad (6)$$

$$\text{B.C. 2 } v'(0) = 0 \quad \therefore C_3 = q_0 \left(\frac{2L}{\pi} \right)^3$$

$$\text{B.C. 3 } v(0) = 0 \quad \therefore C_4 = 0$$

$$\text{B.C. 4 } v(L) = 0$$

$$\begin{aligned} \therefore C_1 L + 3C_2 &= -q_0 \left(\frac{2L}{\pi} \right)^3 L \frac{6}{L^2} \\ &\quad + q_0 \left(\frac{2L}{\pi} \right)^4 \frac{6}{L^2} \end{aligned} \quad (7)$$

Solve Eqs. (6) and (7):

$$C_1 = -6q_0 L \frac{\pi^2 - 4\pi + 8}{\pi^4}$$

$$C_2 = 2q_0 L^2 \frac{\pi^2 - 12\pi + 24}{\pi^4}$$

SHEAR FORCE FROM EQ. (2)

$$V = q_0 \frac{2L}{\pi} \cos \left(\frac{\pi x}{2L} \right) - 6q_0 L \frac{\pi^2 - 4\pi + 8}{\pi^4}$$

REACTIONS

$$\begin{aligned} R_A &= V(0) = 0.31q_0 L \\ &= \left(\frac{2}{\pi} - 6 \frac{\pi^2 - 4\pi + 8}{\pi^4} \right) q_0 L \quad \leftarrow \end{aligned}$$

$$\begin{aligned} R_B &= -V(L) = 0.327q_0 L \\ &= \left(6 \frac{\pi^2 - 4\pi + 8}{\pi^4} \right) q_0 L \quad \leftarrow \end{aligned}$$

From equilibrium,

$$M_A = -C_2 = -2q_0 L^2 \frac{\pi^2 - 12\pi + 24}{\pi^4} \quad \leftarrow$$

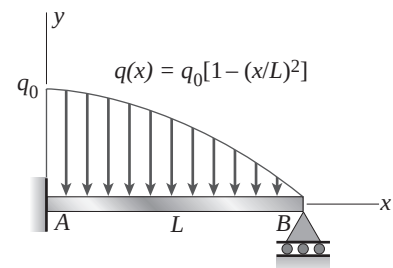
DEFLECTION CURVE FROM EQ. (5)

$$\begin{aligned} EIv &= -q_0 \left(\frac{2L}{\pi} \right)^4 \sin \left(\frac{\pi x}{2L} \right) \\ &\quad + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4, \text{ or} \end{aligned}$$

$$\begin{aligned} v &= \frac{1}{EI} \left[-q_0 \left(\frac{2L}{\pi} \right)^4 \sin \left(\frac{\pi x}{2L} \right) \right. \\ &\quad - 6q_0 L \frac{\pi^2 - 4\pi + 8}{\pi^4} \frac{x^3}{6} \\ &\quad + 2q_0 L^2 \frac{\pi^2 - 12\pi + 24}{\pi^4} \frac{x^2}{2} \\ &\quad \left. + q_0 \left(\frac{2L}{\pi} \right)^3 x \right] \quad \leftarrow \end{aligned}$$

Problem 10.3-7 A propped cantilever beam of length L is loaded by a parabolically distributed load with maximum intensity q_0 at A .

- Use the fourth-order differential equation of the deflection curve to solve for reactions at A and B and also the equation of the deflection curve.
- Repeat part (a) if the parabolic load is replaced by $q_0 \cos(\pi x/2L)$.



Solution 10.3-7

(a) PARABOLIC LOAD $q(x) = q_0 \left[1 - \left(\frac{x}{L} \right)^2 \right]$

$$EI \frac{d^4}{dx^4} v(x) = -q(x) = q_0 \left(\frac{x^2}{L^2} - 1 \right)$$

$$EI \frac{d^3}{dx^3} v(x) = \int -q(x) dx + C_1 = C_1 - q_0 x + \frac{q_0 x^3}{3L^2}$$

B.C. 1 $v(0) = R_A$ so $C_1 = R_A$

$$EI \frac{d^2}{dx^2} v(x) = \int R_A - q_0 x + \frac{q_0 x^3}{3L^2} dx + C_2 = C_2 - \frac{q_0 x^2}{2} + R_A x + \frac{q_0 x^4}{12L^2}$$

B.C. 2 $M(L) = 0$ so $C_2 = - \left(-\frac{q_0 L^2}{2} + R_A L + \frac{q_0 L^4}{12L^2} \right) = \frac{5L^2 q_0}{12} - L R_A$

$$EI \left(\frac{d}{dx} v(x) \right) = \int \frac{5L^2 q_0}{12} - L R_A - \frac{q_0 x^2}{2} = R_A x + \frac{q_0 x^4}{12L^2} dx + C_3 = C_3 + \frac{R_A x^2}{2} - \frac{q_0 x^3}{6} - x \left(L R_A - \frac{5L^2 q_0}{12} \right) + \frac{q_0 x^5}{60L^2}$$

B.C. 3 $\frac{d}{dx} v(0) = 0$ so $C_3 = 0$

$$EI v(x) = \int \frac{R_A x^2}{2} - \frac{q_0 x^3}{6} - x \left(L R_A - \frac{5L^2 q_0}{12} \right) + \frac{q_0 x^5}{60L^2} dx + C_4 = C_4 + x^2 \left(\frac{5L^2 q_0}{24} - \frac{L R_A}{2} \right) + \frac{R_A x^3}{6} - \frac{q_0 x^4}{24} + \frac{q_0 x^6}{360L^2}$$

B.C. 4 $v(0) = 0$ so $C_4 = 0$

EQUATION OF THE ELASTIC CURVE $v(x) = \frac{1}{EI} \left[x^2 \left(\frac{5L^2 q_0}{24} - \frac{L R_A}{2} \right) + \frac{R_A x^3}{6} - \frac{q_0 x^4}{24} + \frac{q_0 x^6}{360L^2} \right]$

$v(L) = 0$ Solve, $R_A = \frac{61L q_0}{120}$ $R_A = \frac{61L q_0}{120}$

so

substitute $R_A = \frac{61L q_0}{120}$ to get $v(x) = \frac{61 q_0 L^3 x^3 - 33 q_0 L^4 x^2 - 30 q_0 L^2 x^4 + 2 q_0 x^6}{720 EIL^2}$

$$v(x) = \frac{61 q_0 L^3 x^3 - 33 q_0 L^4 x^2 - 30 q_0 L^2 x^4 + 2 q_0 x^6}{720 EIL^2} = - \frac{q_0 x^2 (33L^4 - 61L^3 x + 30L^2 x^2 - 2x^4)}{720 EIL^2}$$

$$v(x) = - \frac{q_0 x^2 (33L^4 - 61L^3 x + 30L^2 x^2 - 2x^4)}{720 EIL^2} \quad v(0) = 0 \quad v(L) = 0$$

$$\frac{d}{dx} v(x) = \frac{q_0 x^2 (61L^3 - 60L^2 x + 8x^3)}{720 EIL^2} - \frac{q_0 x (33L^4 - 61L^3 x + 30L^2 x^2 - 2x^4)}{360 EIL^2}$$

STATICS

$$R_A = \frac{61L q_0}{120} \quad R_B = \int_0^L q(x) dx - R_A = \frac{19L q_0}{120} \quad M_A = \int_0^L q(x) x dx - R_B L = \frac{11L^2 q_0}{120}$$

(b) COSINE LOAD $q(x) = q_0 \cos\left(\frac{\pi x}{2L}\right)$

$$EI \frac{d^4}{dx^4} v(x) = -q(x) = -q_0 \cos\left(\frac{\pi x}{2L}\right)$$

$$EI \frac{d^3}{dx^3} v(x) = \int -q(x) dx + C_1 = C_1 - \frac{2L q_0 \sin\left(\frac{\pi x}{2L}\right)}{\pi}$$

B.C. 1 $V(0) = R_A$ so $C_1 = R_A$

$$EI \frac{d^2}{dx^2} v(x) = \int C_1 - \frac{2L q_0 \sin\left(\frac{\pi x}{2L}\right)}{\pi} dx + C_2 = C_2 + \frac{\pi^2 R_A x + 4L^2 q_0 \cos\left(\frac{\pi x}{2L}\right)}{\pi^2}$$

B.C. 2 $M(L) = 0$ so $C_2 = -\left(\frac{\pi^2 R_A L + 4L^2 q_0 \cos\left(\frac{\pi x}{2L}\right)}{\pi^2}\right) = -LR_A$

$$EI \left(\frac{d}{dx} v(x)\right) = \int -R_A L + \frac{\pi^2 R_A x + 4L^2 q_0 \cos\left(\frac{\pi x}{2L}\right)}{\pi^2} dx + C_3 = C_3 + \frac{\pi^2 R_A x^2 + 16L^2 q_0 \sin\left(\frac{\pi x}{2L}\right) - 2\pi^3 L R_A x}{2\pi^3}$$

B.C. 3 $\frac{d}{dx} v(0) = 0$ so $C_3 = 0$

$$EI v(x) = \int \frac{\pi^3 R_A x^2 + 16L^3 q_0 \sin\left(\frac{\pi x}{2L}\right) - 2\pi^3 L R_A x}{2\pi^3} dx + C_4 = C_4 - \frac{96L^4 q_0 \cos\left(\frac{\pi x}{2L}\right) - \pi^4 R_A x^3 + 3\pi^4 L R_A x^2}{6\pi^4}$$

B.C. 4 $v(0) = 0$ so $C_4 = \frac{96L^4 q_0 \cos\left(\frac{\pi 0}{2L}\right) - \pi^4 R_A 0^3 + 3\pi^4 L R_A 0^2}{6\pi^4} = \frac{16L^4 q_0}{\pi^4}$

EQUATION OF THE ELASTIC CURVE $v(x) = \frac{1}{EI} \left(\frac{16L^4 q_0}{\pi^4} - \frac{96L^4 q_0 \cos\left(\frac{\pi x}{2L}\right) - \pi^4 R_A x^3 + 3\pi^4 L R_A x^2}{6\pi^4} \right)$

$v(L) = 0$ solve, $R_A = \frac{48L q_0}{\pi^4}$

$v(x)$ substitute, $R_A = \frac{48L q_0}{\pi^4} = \frac{16L^4 q_0 - 24L^2 q_0 x^2 + 8L q_0 x^3 - 16L^4 q_0 \cos\left(\frac{\pi x}{2L}\right)}{\pi^4 EI}$

$$v(x) = \frac{16L^4 q_0 - 24L^2 q_0 x^2 + 8L q_0 x^3 - 16L^4 q_0 \cos\left(\frac{\pi x}{2L}\right)}{\pi^4 EI} \quad v(0) = 0 \quad v(L) = 0$$

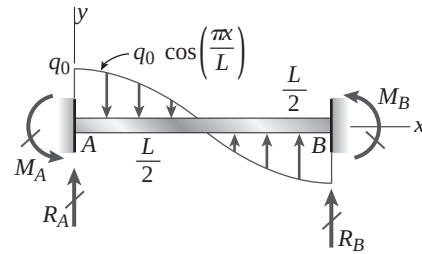
STATICS

$$R_A = \frac{48 L q_0}{\pi^4} \quad R_B = \int_0^L q(x) dx - R_A = \frac{2 L q_0}{\pi} - \frac{48 L q_0}{\pi^4}$$

$$M_A = \int_0^L q(x) x dx - R_B L = \frac{2 L^2 q_0 (\pi - 2)}{\pi^2} - L \left(\frac{2 L q_0}{\pi} - \frac{48 L q_0}{\pi^4} \right)$$

Problem 10.3-8 A fixed-end beam of length L is loaded by distributed load in the form of a cosine curve with maximum intensity q_0 at A .

- (a) Use the fourth-order differential equation of the deflection curve to solve for reactions at A and B and also the equation of the deflection curve.
 (b) Repeat part (a) using the distributed load $q_0 \sin(\pi x/L)$.

**Solution 10.3-8**

(a) Loading $q = q_0 \cos\left(\frac{\pi x}{L}\right)$

DIFFERENTIAL EQUATION

$$EI v'''' = -q = -q_0 \cos\left(\frac{\pi x}{L}\right) \quad (1)$$

$$EI v''' = -q_0 \frac{L}{\pi} \sin\left(\frac{\pi x}{L}\right) + C_1 \quad (2)$$

$$EI v'' = M = q_0 \left(\frac{L}{\pi}\right)^2 \cos\left(\frac{\pi x}{L}\right) + C_1 x + C_2 \quad (3)$$

$$EI v' = q_0 \left(\frac{L}{\pi}\right)^3 \sin\left(\frac{\pi x}{L}\right) + C_1 \frac{x^2}{2} + C_2 x + C_3 \quad (4)$$

$$EI v = -q_0 \left(\frac{L}{\pi}\right)^4 \cos\left(\frac{\pi x}{L}\right) + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \quad (5)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_3 = 0$

B.C. 2 $v(0) = 0 \quad \therefore C_4 = q_0 \left(\frac{L}{\pi}\right)^4$

B.C. 3 $v'(L) = 0 \quad \therefore C_1 \frac{L}{2} + C_2 = 0$

B.C. 4 $v(L) = 0 \quad \therefore C_1 \frac{L^3}{6} + \left(-C_1 \frac{L}{2}\right) \frac{L^2}{2} = -2q_0 \left(\frac{L}{\pi}\right)^4 \quad (6)$

Solve Eqs. (6):

$$C_1 = \frac{24}{\pi^4} q_0 L$$

$$C_2 = -\frac{12}{\pi^4} q_0 L^2$$

SHEAR FORCE FROM EQ. (2)

$$V = -\frac{q_0 L}{\pi} \sin\left(\frac{\pi x}{L}\right) + \frac{24}{\pi^4} q_0 L$$

REACTIONS

$$R_A = V(0) = \frac{24}{\pi^4} q_0 L \quad \leftarrow$$

$$R_B = -V(L) = -\frac{24}{\pi^4} q_0 L \quad \leftarrow$$

From equilibrium,

$$M_A = \left(\frac{12}{\pi^4} - \frac{1}{\pi^2}\right) q_0 L^2 \quad \leftarrow$$

(counterclockwise)

$$M_B = \left(\frac{12}{\pi^4} - \frac{1}{\pi^2} \right) q_0 L^2$$

(counterclockwise) ←

DEFLECTION CURVE FROM EQ. (5)

$$EIv = -q_0 \left(\frac{L}{\pi} \right)^4 \cos \left(\frac{\pi x}{L} \right) + \frac{24}{\pi^4} q_0 L \frac{x^3}{6}$$

$$- \frac{12}{\pi^4} q_0 L^2 \frac{x^2}{2} + q_0 \left(\frac{L}{\pi} \right)^4, \text{ or}$$

$$v = \frac{1}{\pi^4 EI} \left[-q_0 L^4 \cos \left(\frac{\pi x}{L} \right) + 4q_0 L x^3 \right.$$

$$\left. - 6q_0 L^2 x^2 + q_0 L^4 \right] \quad \leftarrow$$

(b) Loading $q = q_0 \sin \pi x/L$ From symmetry: $R_A = R_B \quad M_A = M_B$

DIFFERENTIAL EQUATIONS

$$EIv'''' = -q = -q_0 \sin \pi x/L \quad (1)$$

$$EIv''' = V = \frac{q_0 L}{\pi} \cos \frac{\pi x}{L} + C_1 \quad (2)$$

$$EIv'' = M = \frac{q_0 L^2}{\pi^2} \sin \frac{\pi x}{L} + C_1 x + C_2 \quad (3)$$

$$EIv' = -\frac{q_0 L^3}{\pi^3} \cos \frac{\pi x}{L} + C_1 \frac{x^2}{2} + C_2 x + C_3 \quad (4)$$

$$EIv = -\frac{q_0 L^4}{\pi^4} \sin \frac{\pi x}{L} + C_1 \frac{x^3}{6}$$

$$+ C_2 \frac{x^2}{2} + C_3 x + C_4 \quad (5)$$

$$\text{B.C. 1 From symmetry, } v \left(\frac{L}{2} \right) = 0 \quad \therefore C_1 = 0$$

$$\text{B.C. 2 } v'(0) = 0 \quad \therefore C_3 = q_0 \left(\frac{L}{\pi} \right)^3$$

$$\text{B.C. 3 } v'(L) = 0 \quad \therefore C_2 = -2q_0 \frac{L^2}{\pi^3}$$

$$\text{B.C. 4 } v(0) = 0 \quad \therefore C_4 = 0$$

SHEAR FORCE FROM EQ. (2)

$$V = \frac{q_0 L}{\pi} \cos \frac{\pi x}{L} \quad R_A = V(0) = \frac{q_0 L}{\pi} \quad \leftarrow$$

$$R_B = R_A = \frac{q_0 L}{\pi} \quad \leftarrow$$

BENDING MOMENT FROM EQ. (3)

$$M = \frac{q_0 L^2}{\pi^3} \left(\pi \sin \frac{\pi x}{L} - 2 \right)$$

$$M_A = -M(0) = \frac{2q_0 L^2}{\pi^3}$$

$$M_B = M_A = \frac{2q_0 L^2}{\pi^3} \quad \leftarrow$$

DEFLECTION CURVE FROM EQ. (5)

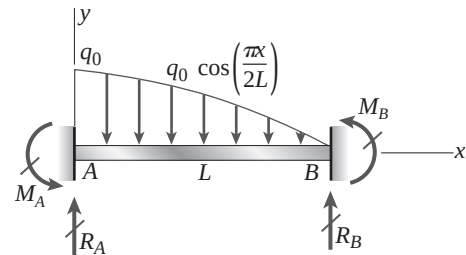
$$EIv = -\frac{q_0 L^2}{\pi^4} \sin \frac{\pi x}{L} - \frac{q_0 L^2 x^2}{\pi^3} + \frac{q_0 L^3 x}{\pi^3}$$

or

$$v = -\frac{q_0 L^2}{\pi^4 EI} \left(L^2 \sin \frac{\pi x}{L} + \pi x^2 - \pi L x \right) \quad \leftarrow$$

Problem 10.3-9 A fixed-end beam of length L is loaded by a distributed load in the form of a cosine curve with maximum intensity q_0 at A.

- (a) Use the fourth-order differential equation of the deflection curve to solve for reactions at A and B and also the equation of the deflection curve.
- (b) Repeat part (a) if the distributed load is now $q_0 (1 - x^2/L^2)$.



Solution 10.3-9

(a) Loading $q = q_0 \cos\left(\frac{\pi x}{2L}\right)$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0 \cos\left(\frac{\pi x}{2L}\right) \quad (1)$$

$$EIv''' = -q_0 \frac{2L}{\pi} \sin\left(\frac{\pi x}{2L}\right) + C_1 \quad (2)$$

$$EIv'' = M = q_0 \left(\frac{2L}{\pi}\right)^2 \cos\left(\frac{\pi x}{2L}\right) + C_1 x + C_2 \quad (3)$$

$$EIv' = q_0 \left(\frac{2L}{\pi}\right)^3 \sin\left(\frac{\pi x}{2L}\right) + C_1 \frac{x^2}{2} + C_2 x + C_3 \quad (4)$$

$$EIv = -q_0 \left(\frac{2L}{\pi}\right)^4 \cos\left(\frac{\pi x}{2L}\right) + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \quad (5)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_3 = 0$

B.C. 2 $v(0) = 0 \quad \therefore C_4 = q_0 \left(\frac{2L}{\pi}\right)^4$

B.C. 3 $v'(L) = 0$

$$\therefore C_1 \frac{L^2}{2} + C_2 L = -q_0 \left(\frac{2L}{\pi}\right)^3 \quad (6)$$

B.C. 4 $v(L) = 0$

$$\therefore C_1 \frac{L^3}{6} + C_2 \frac{L^2}{2} = -q_0 \left(\frac{2L}{\pi}\right)^4 \quad (7)$$

Solve Eqs. (6) and (7):

$$C_1 = \frac{48(4 - \pi)}{\pi^4} q_0 L$$

$$C_2 = -\frac{16(6 - \pi)}{\pi^4} q_0 L^2$$

SHEAR FORCE FROM EQ. (2)

$$V = -q_0 \frac{2L}{\pi} \sin\left(\frac{\pi x}{2L}\right) + \frac{48(4 - \pi)}{\pi^4} q_0 L$$

REACTIONS

$$R_A = V(0) = \frac{48(4 - \pi)}{\pi^4} q_0 L \quad \leftarrow$$

$$R_B = -V(L) = \left(\frac{2}{\pi} - \frac{48(4 - \pi)}{\pi^4}\right) q_0 L \quad \leftarrow$$

From equilibrium,

$$M_A = -q_0 \left(\frac{2L}{\pi}\right)^2 + \frac{16(6 - \pi)}{\pi^4} q_0 L^2 \quad \leftarrow$$

$$M_B = -\frac{32(\pi - 3)}{\pi^4} q_0 L^2 \quad \leftarrow$$

DEFLECTION CURVE FROM EQ. (5)

$$EIv = -q_0 \left(\frac{2L}{\pi}\right)^4 \cos\left(\frac{\pi x}{2L}\right) + \frac{48(4 - \pi)}{\pi^4}$$

$$\times q_0 L \frac{x^3}{6} - \frac{16(6 - \pi)}{\pi^4}$$

$$\times q_0 L^2 \frac{x^2}{2} + q_0 \left(\frac{2L}{\pi}\right)^4, \text{ or}$$

$$v = \frac{1}{\pi^4 EI} \left[-16 q_0 L^4 \cos\left(\frac{\pi x}{2L}\right) + 8(4 - \pi) q_0 L x^3 - 8(6 - \pi) q_0 L^2 x^2 + 16 q_0 L^4 \right] \quad \leftarrow$$

(b) Loading $q = q_0 \left(1 - \frac{x^2}{L^2}\right)$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0 \left(1 - \frac{x^2}{L^2}\right) \quad (1)$$

$$EIv''' = -q_0 \left(x - \frac{x^3}{3L^2}\right) + C_1 \quad (2)$$

$$EIv'' = M = -q_0 \left(\frac{x^2}{2} - \frac{x^4}{12L^2}\right) + C_1 x + C_2 \quad (3)$$

$$EIv' = -q_0 \left(\frac{x^3}{6} - \frac{x^5}{60L^2}\right) + C_1 \frac{x^2}{2} + C_2 x + C_3 \quad (4)$$

$$EIv = -q_0 \left(\frac{x^4}{24} - \frac{x^6}{360L^2}\right) + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \quad (5)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_3 = 0$

B.C. 2 $v(0) = 0 \quad \therefore C_4 = 0$

$$\text{B.C. 3 } v'(L) = 0$$

$$\therefore C_1 L + 2C_2 = \frac{3}{10} q_0 L^2 \quad (6)$$

$$\text{B.C. 4 } v(L) = 0$$

$$\therefore C_1 L + 3C_2 = \frac{7}{30} q_0 L^2 \quad (7)$$

Solve Eqs. (6) and (7):

$$C_1 = \frac{13}{30} q_0 L \quad C_2 = -\frac{1}{15} q_0 L^2$$

SHEAR FORCE FROM EQ. (2)

$$V = -q_0 \left(x - \frac{x^3}{3L^2} \right) + \frac{13}{30} q_0 L$$

REACTIONS

$$R_A = V(0) = \frac{13}{30} q_0 L \quad \leftarrow$$

$$R_B = -V(L) = \frac{7}{30} q_0 L \quad \leftarrow$$

From equilibrium,

$$M_A = -q_0 \left(\frac{x^2}{2} - \frac{x^4}{12L^2} \right) + \frac{13}{30} q_0 Lx - \frac{1}{15} q_0 L^2$$

$$M_A = \frac{1}{15} q_0 L^2 \quad (\text{counterclockwise}) \quad \leftarrow$$

$$M_B = -\frac{1}{20} q_0 L^2 \quad (\text{clockwise}) \quad \leftarrow$$

DEFLECTION CURVE FROM EQ. (5)

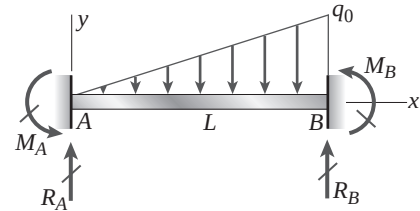
$$EIv = -q_0 \left(\frac{x^4}{24} - \frac{x^6}{360L^2} \right) + \frac{13}{30} q_0 L \frac{x^3}{6} - \frac{1}{15} q_0 L^2 \frac{x^2}{2}$$

or

$$v = \frac{q_0}{360L^2 EI} [x^6 - 15L^2 x^4 + 26L^3 x^3 - 12L^4 x^2] \quad \leftarrow$$

Problem 10.3-10 A fixed-end beam of length L is loaded by triangularly distributed load of maximum intensity q_0 at B .

Use the fourth-order differential equation of the deflection curve to solve for reactions at A and B and also the equation of the deflection curve.



Solution 10.3-10

$$\text{Triangular load } q = q_0 \frac{x}{L}$$

DIFFERENTIAL EQUATION

$$EIv'''' = -q = -q_0 \frac{x}{L} \quad (1)$$

$$EIv''' = -q_0 \frac{x^2}{2L} + C_1 \quad (2)$$

$$EIv'' = M = -q_0 \frac{x^3}{6L} + C_1 x + C_2 \quad (3)$$

$$EIv' = -q_0 \frac{x^4}{24L} + C_1 \frac{x^2}{2} + C_2 x + C_3 \quad (4)$$

$$EIv = -q_0 \frac{x^5}{120L} + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \quad (5)$$

$$\text{B.C. 1 } v'(0) = 0 \quad \therefore C_3 = 0$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_4 = 0$$

$$\text{B.C. 3 } v'(L) = 0 \quad \therefore C_1 L + 2C_2 = \frac{q_0 L^2}{12} \quad (6)$$

$$\text{B.C. 4 } v(L) = 0 \quad \therefore C_1 L + 3C_2 = \frac{q_0 L^2}{20} \quad (7)$$

Solve Eqs. (6) and (7):

$$C_1 = \frac{3}{20} q_0 L$$

$$C_2 = -\frac{1}{30} q_0 L^2$$

SHEAR FORCE FROM EQ. (2)

$$V = -q_0 \frac{x^2}{2L} + \frac{3}{20} q_0 L$$

REACTIONS

$$R_A = V(0) = \frac{3}{20} q_0 L \quad \leftarrow$$

$$R_B = -V(L) = \frac{7}{20} q_0 L \quad \leftarrow$$

From equilibrium,

$$M_A = \frac{1}{30} q_0 L^2 \quad \leftarrow$$

DEFLECTION CURVE FROM EQ. (5)

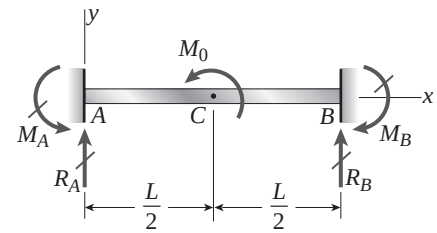
$$EIv = -q_0 \frac{x^5}{120L} + \frac{3}{20} q_0 L \frac{x^3}{6} - \frac{1}{30} q_0 L^2 \frac{x^2}{2} \quad \text{or}$$

$$v = \frac{1}{120LEI} (-q_0 x^5 + 3q_0 L^2 x^3 - 2q_0 L^3 x^2) \quad \leftarrow$$

Problem 10.3-11 A counterclockwise moment M_0 acts at the midpoint of a fixed-end beam ACB of length L (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), determine all reactions of the beam and obtain the equation of the deflection curve for the left-hand half of the beam.

Then construct the shear-force and bending-moment diagrams for the entire beam, labeling all critical ordinates. Also, draw the deflections curve for the entire beam.



Solution 10.3-11 Fixed-end beam (M_0 = applied load)

Beam is symmetric; load is antisymmetric.

$$\text{Therefore, } R_A = -R_B \quad M_A = -M_B \quad \delta_C = 0$$

DIFFERENTIAL EQUATION ($0 \leq x \leq L/2$)

$$EIv'' = M = R_A x - M_A \quad (1)$$

$$EIv' = R_A \frac{x^2}{2} - M_A x + C_1 \quad (2)$$

$$EIv = R_A \frac{x^3}{6} - M_A \frac{x^2}{2} + C_1 x + C_2 \quad (3)$$

$$\text{B.C. 1 } v'(0) = 0 \quad \therefore C_1 = 0$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_2 = 0$$

$$\text{B.C. 3 } v\left(\frac{L}{2}\right) = 0$$

$$\therefore M_A = \frac{R_A L}{6} \quad \text{Also, } M_B = \frac{-R_A L}{6}$$

EQUILIBRIUM (OF ENTIRE BEAM)

$$\sum M_B = 0 \quad M_A + M_0 - M_B - R_A L = 0$$

$$\text{or } \frac{R_A L}{6} + M_0 + \frac{R_A L}{6} - R_A L = 0$$

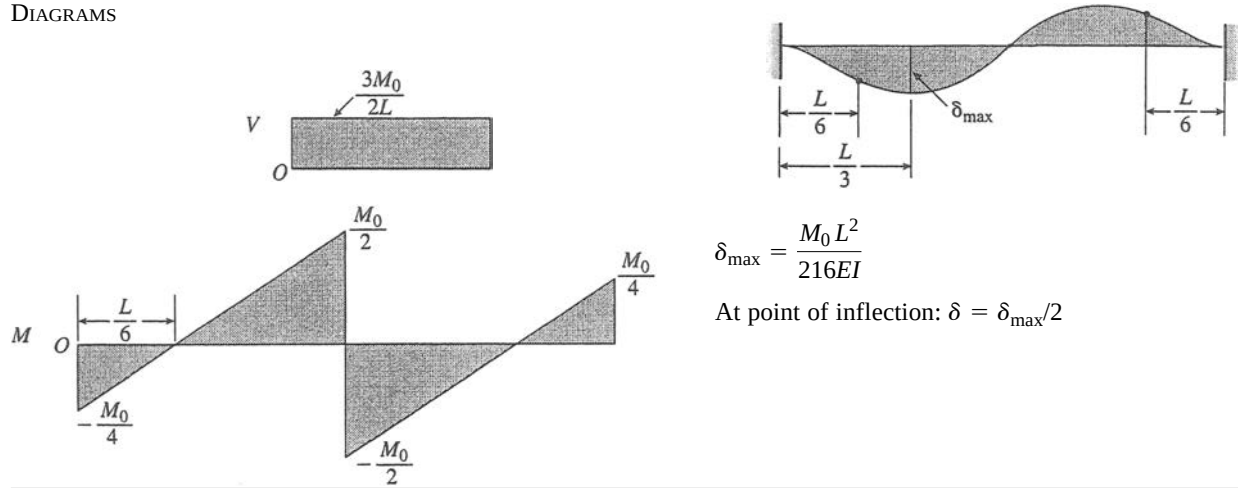
$$\therefore R_A = -R_B = \frac{3M_0}{2L} \quad \leftarrow$$

$$M_A = \frac{R_A L}{6} \quad \therefore M_A = -M_B = \frac{M_0}{4} \quad \leftarrow$$

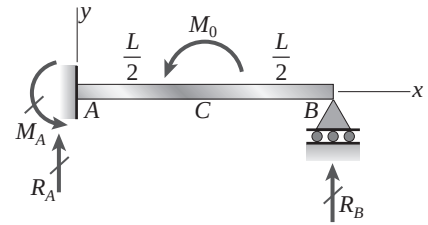
DEFLECTION CURVE FROM EQ. (3)

$$v = -\frac{M_0 x^2}{8LEI} (L - 2x) \quad \left(0 \leq x \leq \frac{L}{2}\right) \quad \leftarrow$$

DIAGRAMS



Problem 10.3-12 A propped cantilever beam of length L is loaded by a concentrated moment M_0 at midpoint C . Use the second-order differential equation of the deflection curve to solve for reactions at A and B . Draw shear-force and bending-moment diagrams for the entire beam. Also find the equations of the deflection curves for both halves of the beam, and draw the deflection curve for the entire beam.

**Solution 10.3-12**

EQUILIBRIUM

$$R_A = -R_B$$

$$M_A = -M_0 - R_B L$$

BENDING MOMENTS (FROM EQUILIBRIUM)

$$M = R_A x - M_A = -R_B x + M_0 + R_B L \quad \left(0 \leq x \leq \frac{L}{2}\right)$$

$$M = R_B (L - x) \quad \left(\frac{L}{2} \leq x \leq L\right)$$

DIFFERENTIAL EQUATIONS ($0 \leq x \leq L/2$)

$$EIv'' = M = -R_B x + M_0 + R_B L \quad (3)$$

$$EIv' = -R_B \frac{x^2}{2} + M_0 x + R_B L x + C_1 \quad (4)$$

$$EIv = -R_B \frac{x^3}{6} + M_0 \frac{x^2}{2} + R_B L \frac{x^2}{2} + C_1 x + C_2 \quad (5)$$

$$\text{B.C. 1 } v'(0) = 0 \quad \therefore C_1 = 0$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_2 = 0$$

DIFFERENTIAL EQUATIONS ($L/2 \leq x \leq L$)

$$(1) \quad EIv'' = M = R_B (L - x) \quad (6)$$

$$(2) \quad EIv' = R_B Lx - R_B \frac{x^2}{2} + C_3 \quad (7)$$

$$EIv = -R_B L \frac{x^2}{2} + R_B \frac{x^3}{6} + C_3 x + C_4 \quad (8)$$

$$\text{B.C. 3 } v(L) = 0 \quad \therefore C_3 L + C_4 = -\frac{R_B L^3}{3} \quad (9)$$

B.C. 4 continuity condition at point C

$$\text{At } x = \frac{L}{2}: (v')_{\text{left}} = (v')_{\text{right}}$$

$$-R_B \frac{L^2}{8} + M_0 \frac{L}{2} + R_B L \frac{L}{2}$$

$$= R_B L \frac{L}{2} - R_B \frac{L^2}{8} + C_3$$

$$C_3 = M_0 \frac{L}{2}$$

$$\text{From Eq. (9): } C_4 = -\frac{R_B L^3}{3} - M_0 \frac{L^2}{2}$$

B.C. 5 Continuity condition at point C

At $x = \frac{L}{2}$: $(v)_{\text{left}} = (v)_{\text{right}}$

$$EIv = -R_B \frac{x^3}{6} + M_0 \frac{x^2}{2} + R_B L \frac{x^2}{2} + C_1 x + C_2$$

$$-R_B \frac{L^3}{48} + M_0 \frac{L^2}{8} + R_B L \frac{L^2}{8} = R_B L \frac{L^2}{8}$$

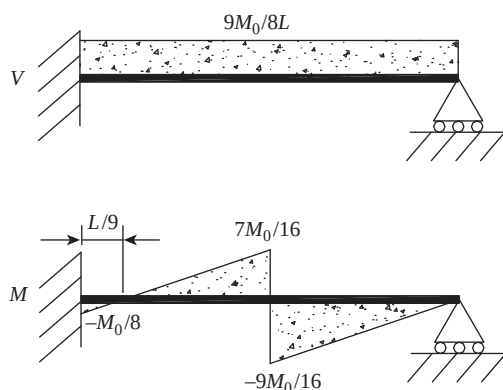
$$-R_B \frac{L^3}{48} + M_0 \frac{L}{2} \frac{L}{2} - R_B \frac{L^3}{3} - M_0 \frac{L^2}{2}$$

$$R_B = -\frac{9}{8} \frac{M_0}{L} \quad \leftarrow$$

$$\text{From Eq. (1)} \quad R_A = \frac{9}{8} \frac{M_0}{L} \quad \leftarrow$$

$$\text{From Eq. (2)} \quad M_A = -M_0 + \frac{9}{8} \frac{M_0}{L} L = \frac{1}{8} \frac{M_0}{L} \quad \leftarrow$$

SHEAR FORCE AND BENDING MOMENT DIAGRAMS



DEFLECTION CURVE FOR $(0 \leq x \leq L/2)$

$$EIv = -R_B \frac{x^3}{6} + M_0 \frac{x^2}{2} + R_B L \frac{x^2}{2} + C_1 x + C_2$$

$$v = \frac{1}{EI} \left(\frac{9M_0}{48L} x^3 - \frac{M_0}{16} x^2 \right) \quad \left(0 \leq x \leq \frac{L}{2} \right) \quad \leftarrow$$

DEFLECTION CURVE FOR $(L/2 \leq x \leq L)$

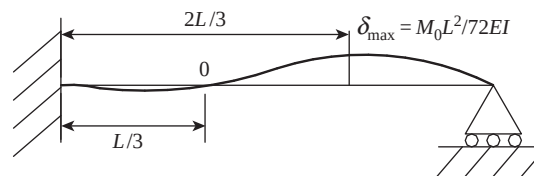
$$EIv = R_B L \frac{x^2}{2} - R_B \frac{x^3}{6} + C_3 x + C_4$$

$$EIv = R_B L \frac{x^2}{2} - R_B \frac{x^3}{6} + M_0 \frac{L}{2} x$$

$$+ \left(-\frac{R_B L^3}{3} - M_0 \frac{L^2}{2} \right)$$

$$v = \frac{1}{EI} \left(\frac{9M_0}{48L} x^3 - \frac{9M_0}{16} x^2 + \frac{M_0 L}{2} x - \frac{M_0 L^2}{8} \right) \quad \left(\frac{L}{2} \leq x \leq L \right) \quad \leftarrow$$

DEFLECTION CURVE

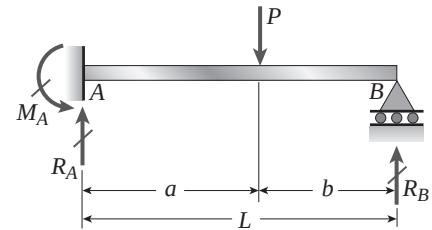


Method of Superposition

The problems for Section 10.4 are to be solved by the method of superposition. All beams have constant flexural rigidity EI unless otherwise stated. When drawing shear-force and bending-moment diagrams, be sure to label all critical ordinates, including maximum and minimum values.

Problem 10.4-1 A propped cantilever beam AB of length L carries a concentrated load P acting at the position shown in the figure.

Determine the reactions R_A , R_B , and M_A for this beam. Also, draw the shear-force and bending-moment diagrams, labeling all critical ordinates.



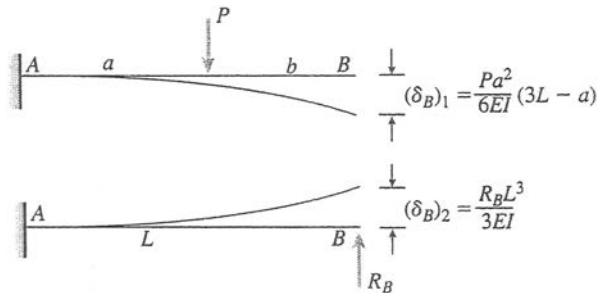
Solution 10.4-1 Propped cantilever beam

Select R_B as redundant.

EQUILIBRIUM

$$R_A = P - R_B \quad M_A = Pa - R_B L$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



COMPATIBILITY

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$$

$$\delta_B = \frac{Pa^2}{6EI} (3L - a) - \frac{R_B L^3}{3EI} = 0$$

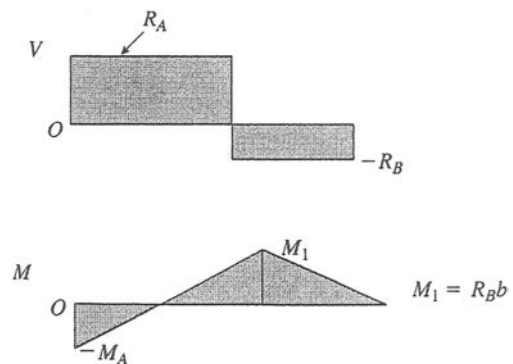
$$R_B = \frac{Pa^2}{2L^3} (3L - a) \quad \leftarrow$$

OTHER REACTIONS (FROM EQUILIBRIUM)

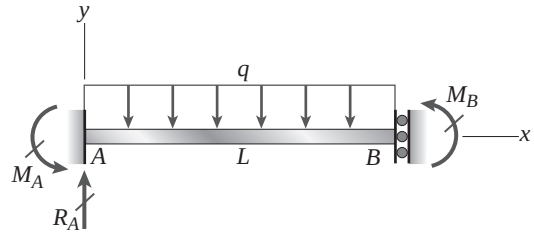
$$R_A = \frac{Pb}{2L^3} (3L^2 - b^2)$$

$$M_A = \frac{Pab}{2L^2} (L + b) \quad \leftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAM



Problem 10.4-2 A beam with a guided support at B is loaded by a uniformly distributed load with intensity q . Use the method of superposition to solve for all reactions. Also draw shear-force and bending-moment diagrams, labeling all critical ordinates.



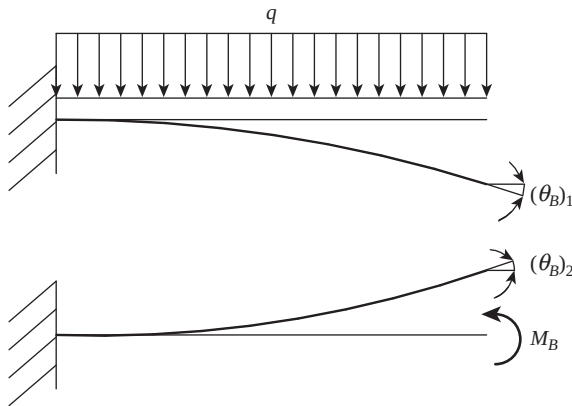
Solution 10.4-2

Select M_B as redundant.

EQUILIBRIUM

$$R_A = qL \quad (1) \quad M_A = \frac{qL^2}{2} - M_B \quad (2)$$

RELEASED STRUCTURE AND FORCE OR MOMENT-ROTATION RELATIONS



$(\theta_B)_1$ = Rotation at B due to uniform load q

$$(\theta_B)_1 = \frac{qL^3}{6EI}$$

$(\theta_B)_2$ = Rotation at B due to moment M_B

$$(\theta_B)_2 = \frac{M_B L}{EI}$$

FROM COMPATIBILITY EQUATION

$$(\theta_B)_1 = (\theta_B)_2 \quad \therefore M_B = \frac{qL^2}{6}$$

FROM EQUILIBRIUM EQUATIONS,

$$R_A = qL \quad M_A = \frac{qL^2}{2} - M_B = \frac{qL^2}{3}$$

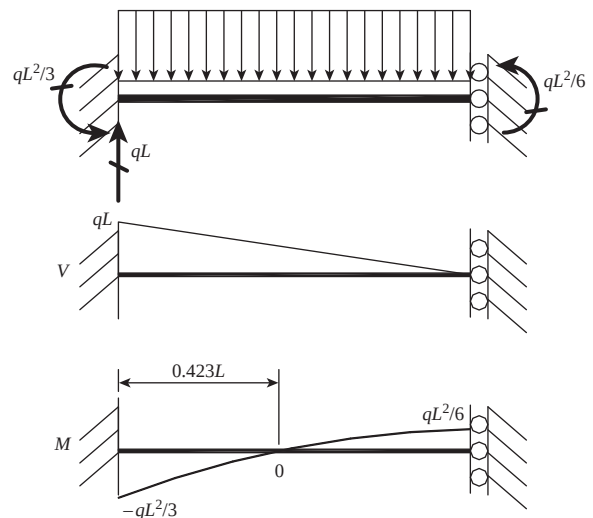
SHEAR FORCE

$$V = R_A - qx = q(L - x)$$

BENDING MOMENTS

$$M = R_A x - M_A - \frac{qx^2}{2} = -\frac{qx^2}{2} + qLx - \frac{qL^2}{3}$$

SHEAR FORCE AND BENDING MOMENT DIAGRAMS



Another solution by the second-order differential equation.

DIFFERENTIAL EQUATIONS

$$EIv'' = M = R_A x - M_A - \frac{qx^2}{2}$$

$$EIv' = R_A \frac{x^2}{2} - M_A x - \frac{qx^3}{6} + C_1$$

$$EIv = R_A \frac{x^3}{6} - M_A \frac{x^2}{2} - \frac{qx^4}{24} + C_1 x + C_2$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

B.C. 2 $v'(0) = 0 \quad \therefore C_2 = 0$

B.C. 3 $v'(L) = 0 \quad M_A L = qL \frac{L^2}{2} - \frac{qL^3}{6} = \frac{qL^3}{3}$

$$\therefore M_A = \frac{qL^2}{3} \quad \leftarrow$$

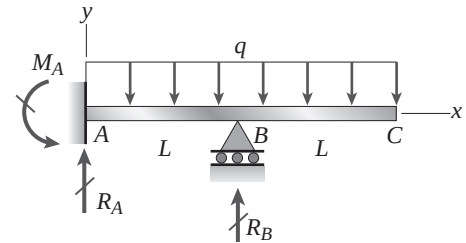
$$M_B = \frac{qL^2}{2} - \frac{qL^2}{3} = \frac{qL^2}{6} \quad \leftarrow$$

DEFLECTION CURVE

$$EIv = R_A \frac{x^3}{6} - M_A \frac{x^2}{2} - \frac{qx^4}{24} = qL \frac{x^3}{6} - \frac{qL^2}{3} \frac{x^2}{2} - \frac{qx^4}{24}$$

$$\text{or } v = \frac{q}{24EI} (-x^4 + 4Lx^3 - 4L^2x^2) \quad \leftarrow$$

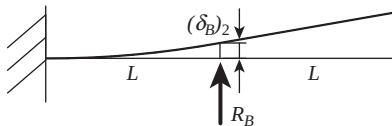
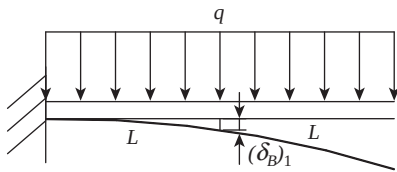
Problem 10.4-3 A propped cantilever beam of length $2L$ with support at B is loaded by a uniformly distributed load with intensity q . Use the method of superposition to solve for all reactions. Also draw shear-force and bending-moment diagrams, labeling all critical ordinates.



Solution 10.4-3

Select R_B as redundant.

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$(\delta_B)_1$ = Deflection at B due to uniform load q

$$(\delta_B)_1 = \frac{17qL^4}{24EI}$$

$(\delta_B)_2$ = Deflection at B due to force R_B

$$(\delta_B)_2 = \frac{R_B L^3}{3EI}$$

FROM COMPATIBILITY EQUATION

$$(\delta_B)_1 = (\delta_B)_2 \quad \therefore R_B = \frac{17}{8}qL$$

EQUILIBRIUM

$$R_A = 2qL - R_B = -\frac{1}{8}qL \quad (1)$$

$$M_A = 2qL^2 - R_B L = -\frac{1}{8}qL^2 \quad (2)$$

SHEAR FORCE

$$V = R_A - qx = -\frac{qL}{8} - qx \quad (0 \leq x \leq L)$$

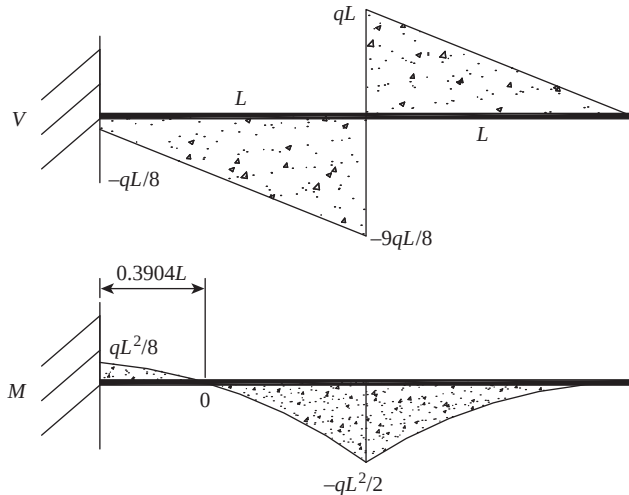
$$V = 2qL - qx \quad (L \leq x \leq 2L)$$

BENDING MOMENTS

$$M = R_A x - M_A - q \frac{x^2}{2} = -\frac{qx^2}{2} - \frac{qLx}{8} + \frac{qL^2}{8} \quad (0 \leq x \leq L)$$

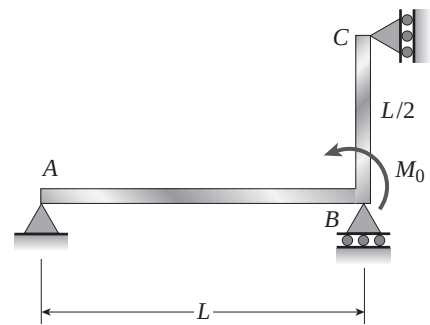
$$M = R_A x - M_A - q \frac{x^2}{2} + R_B x = -\frac{qx^2}{2} + 2qLx - 2qL^2 \quad (L \leq x \leq 2L)$$

SHEAR FORCE AND BENDING MOMENT DIAGRAMS



Problem 10.4-4 The continuous frame ABC has a pin support at A , roller supports at B and C , and a rigid corner connection at B (see figure). Members AB and BC each have flexural rigidity EI . A moment M_0 acts counterclockwise at B . (Note: Disregard axial deformations in member AB and consider only the effects of bending.)

- Find all reactions of the frame.
- Find joint rotations θ at A , B , and C .
- Find the required new length of member BC in terms of L , so that θ_B in part (b) is doubled in size.

**Solution 10.4-4**

- (a) SUPERPOSITION TO FIND REDUNDANT THEN STATICS TO FIND REACTIONS

One degree statically indeterminate; select R_C as the redundant; remove RC to create stat-determinate “released” structure

- (1) Remove R_C , find displ. in released structure at C due to moment M_0 .

$$\text{Table H-2, Case 7: } \theta_{B1} = \frac{M_0 L}{3EI} \quad \text{so} \quad \delta_{C1} = -\theta_{B1} \frac{L}{2} = -\frac{L^2 M_0}{6EI}$$

- (2) Apply R_C as a load on released structure, find displ. at C

Use Table H-1, Case 4, and Table H-2, Case 7.

$$\delta_{C2} = R_C \left[\frac{\left(\frac{L}{2}\right)^3}{3EI} \right] + \left(R_C \frac{L}{2} \right) \left(\frac{L}{3EI} \right) \left(\frac{L}{2} \right) = \frac{L^3 R_C}{8EI}$$

- (3) Solve compatibility equation to find R_C , where $\delta_{C1} + \delta_{C2} = 0$.

$$R_C = \frac{8EI}{L^3} \left(\frac{M_0 L^2}{6EI} \right) = \frac{4M_0}{3L} \quad \text{to the right}$$

- (4) STATICS TO FIND REMAINING REACTIONS

$$\sum F_H = 0 \quad H_A = -R_C = \frac{4M_0}{3L}$$

$$\sum M_A = 0 \quad R_B = \frac{1}{L} \left(-M_0 + R_C \frac{L}{2} \right) = \frac{M_0}{3L}$$

$$\sum F_V = 0 \quad R_A = -R_B = \frac{M_0}{3L}$$

- (b) SUPERPOSITION TO FIND JOINT ROTATIONS AT A, B, AND C; SUM ROTATIONS IN RELEASED STRUCTURE UNDER LOAD M_0 TO THOSE DUE TO R_C

$$\theta_{A1} = \frac{-M_0 L}{6EI} \quad \theta_{A2} = \frac{\left(R_C \frac{L}{2} \right) L}{6EI} \quad \theta_A = \theta_{A1} + \theta_{A2} = -\frac{LM_0}{18EI} \quad (\text{clockwise})$$

$$\theta_{B1} = \frac{M_0 L}{3EI} \quad \theta_{B2} = \frac{-\left(R_C \frac{L}{2} \right) L}{3EI} \quad \theta_B = \theta_{B1} + \theta_{B2} = \frac{LM_0}{9EI} \quad (\text{counterclockwise})$$

$$\theta_{C1} = \theta_{B1} \quad \theta_{C2} = \frac{-\left(R_C \frac{L}{2} \right) L}{3EI} - \frac{R_C \left[\left(\frac{L}{2} \right)^2 \right]}{2EI} = -\frac{7LM_0}{18EI} \quad \theta_C = \theta_{C1} + \theta_{C2} = -\frac{LM_0}{18EI} \quad (\text{clockwise})$$

^ superpose Cases 7 (Table H-2) and 4 (Table H-1)

- (c) USE θ_B EXPRESSIONS TO FIND NEW LENGTH OF BC

$$\theta_{\text{Bold}} = \frac{M_0 L}{9EI} < \text{solution from part (b)}$$

Must revise solution for R_C in part (a) to include L_{BC} rather than $L/2$.

$$\delta_{C1} = -\theta_{B1} L_{BC} \quad \delta_{C2} = R_C \left[\frac{(L_{BC})^3}{3EI} \right] + (R_C L_{BC}) \left(\frac{L}{3EI} \right) (L_{BC}) = \frac{L_{BC}^3 R_C}{3EI} + \frac{L L_{BC}^2 R_C}{3EI}$$

$$\text{Compatibility: } \delta_{C1} + \delta_{C2} = 0 \quad \text{Solve: } R_C = \frac{LM_0}{L_{BC}^2 + L L_{BC}} \quad R_C = \frac{LM_0}{L_{BC}^2 + L L_{BC}}$$

So new expression for θ_B , which includes length variable L_{BC} is

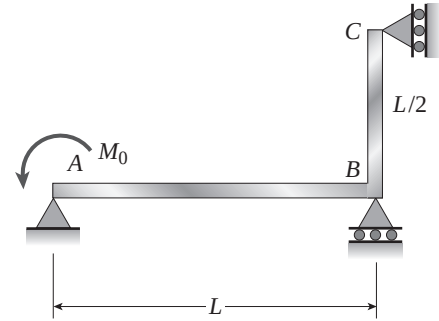
$$\theta_{\text{Bnew}} = \frac{M_0 L}{3EI} + \frac{(-R_C L_{BC}) L}{3EI} = \frac{LM_0}{3EI} - \frac{L^2 L_{BC} M_0}{3EI (L_{BC}^2 + L L_{BC})}$$

Finally, equate expressions for θ_B and solve for required length of BC.

$$\theta_{\text{Bnew}} = 2\theta_{\text{Bold}} \quad \text{Solve: } L_{BC} = 2L$$

Problem 10.4-5 The continuous frame ABC has a pin support at A , roller supports at B and C , and a rigid corner connection at B (see figure). Members AB and BC each have flexural rigidity EI . A moment M_0 acts counterclockwise at A . (Note: Disregard axial deformations in member AB and consider only the effects of bending.)

- Find all reactions of the frame.
- Find joint rotations θ at A , B , and C .
- Find the required new length of member AB in terms of L , so that θ_A in part (b) is doubled in size.



Solution 10.4-5

- (a) USE R_C AS THE REDUNDANT, SOLVE USING SUPERPOSITION

$$\delta_{C1} = \frac{M_0 L}{6EI} \left(\frac{L}{2} \right) = \frac{L^2 M_0}{12EI} \quad \delta_{C2} = \frac{R_C \left(\frac{L}{2} \right)^3}{3EI} + \frac{\left(R_C \frac{L}{2} \right) L}{3EI} \left(\frac{L}{2} \right) = \frac{L^3 R_C}{8EI}$$

Compatibility equation: $\delta_{C1} + \delta_{C2} = 0$ Solve: $R_C = -\frac{2M_0}{3L}$ $R_C = -\frac{2M_0}{3L}$

STATICS

$$H_A = -R_C \quad R_B = \frac{1}{L} \left(-M_0 + R_C \frac{L}{2} \right) = \frac{4M_0}{3L} \quad R_A = -R_B$$

$R_A = \frac{4M_0}{3L}$	$H_A = \frac{2M_0}{3L}$	$R_B = -\frac{4M_0}{3L}$	$R_C = -\frac{2M_0}{3L}$

- (b) USE SUPERPOSITION TO FIND JOINT ROTATIONS FROM ROTATIONS IN EACH RELEASED STRUCTURE IN PART (a)

$$\theta_{A1} = \frac{M_0 L}{3EI} \quad \theta_{A2} = \frac{R_C \left(\frac{L}{2} \right) L}{6EI} \quad \theta_A = \theta_{A1} + \theta_{A2} = \frac{5LM_0}{18EI}$$

$$\theta_{B1} = \frac{-M_0 L}{6EI} \quad \theta_{B2} = \frac{-R_C \left(\frac{L}{2} \right) L}{3EI} \quad \theta_B = \theta_{B1} + \theta_{B2} = -\frac{LM_0}{18EI}$$

$$\theta_{C1} = \theta_{B1} \quad \theta_{C2} = \theta_{B2} - \frac{R_C \left(\frac{L}{2} \right)^2}{2EI} \quad \theta_C = \theta_{C1} + \theta_{C2} = \frac{LM_0}{36EI}$$

$\theta_A = \frac{5}{18} \frac{M_0 L}{EI}$	$\theta_B = \frac{-M_0 L}{18EI}$	$\theta_C = \frac{M_0 L}{36EI}$

- (c) REVISE ABOVE EXPRESSIONS TO INCLUDE L_{AB} EXPLICITLY, THEN FIND NEW L_{AB} SO THAT θ_A IS DOUBLED IN SIZE

$$\theta_{A\text{old}} = \theta_A = \frac{5LM_0}{18EI}$$

$$\theta_{A1} = \frac{M_0 L_{AB}}{3EI} = \frac{L_{AB} M_0}{3EI}$$

Must resolve compatibility equation to find R_C in terms of L_{AB}

$$\delta_{C1} = \frac{M_0 L_{AB}}{6EI} \left(\frac{L}{2} \right) = \frac{L L_{AB} M_0}{12EI} \quad \delta_{C2} = \frac{R_C \left(\frac{L}{2} \right)^3}{3EI} + \frac{\left(R_C \frac{L}{2} \right) L_{AB}}{3EI} \left(\frac{L}{2} \right) = \frac{L^3 R_C}{24EI} + \frac{L^2 L_{AB} R_C}{12EI}$$

Compatibility equation: $\delta_{C1} + \delta_{C2} = 0$ Solve: $R_C = -\frac{2L_{AB}M_0}{L^2 + 2L_{AB}L}$ $R_C = -\frac{2L_{AB}M_0}{L^2 + 2L_{AB}L}$

Recompute rotation at A due to redundant R_C .

$$\theta_{A2} = \frac{R_C \left(\frac{L}{2} \right) L_{AB}}{6EI} = -\frac{L L_{AB}^2 M_0}{6EI(L^2 + 2L_{AB}L)} \quad \theta_{A2} = -\frac{L L_{AB}^2 M_0}{6EI(L^2 + 2L_{AB}L)}$$

and

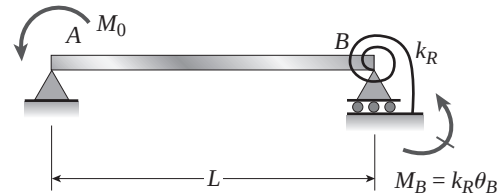
$$\theta_{A\text{new}} = \theta_{A1} + \theta_{A2} = \frac{L_{AB}M_0}{3EI} - \frac{L L_{AB}^2 M_0}{6EI(L^2 + 2L_{AB}L)}$$

Finally, equate expressions for θ_A and solve for required length of AB.

$$\theta_{A\text{new}} = 2 \frac{5}{18} \frac{M_0 L}{EI} \quad \text{Solve: } L_{AB} = \left[\frac{L(\sqrt{139} + 7)}{9} \right] \quad L_{AB} = \frac{L(\sqrt{139} + 7)}{9} = 2.088L$$

Problem 10.4-6 Beam AB has a pin support at A and a roller support at B. Joint B is also restrained by a linearly elastic rotational spring with stiffness K_R , which provides a resisting moment M_B due to rotation B. Member AB has flexural rigidity EI . A moment M_0 acts counterclockwise at A.

- Use the method of superposition to solve for all reactions.
- Find an expression for joint rotation θ_A in terms of spring stiffness k_R . What is θ_A when $k_R \rightarrow 0$? What is θ_A when $k_R \rightarrow \infty$? What is θ_A when $k_R = 6EI/L$?



Solution 10.4-6

- (a) SELECT REACTION MOMENT M_B AS THE REDUNDANT; REMOVE M_B TO CREATE A RELEASED STRUCTURE

Rotation at end of spring due to moment M_0 at A: $\theta_{B1} = \frac{-M_0 L}{6EI}$

Rotation at end of spring due to application M_B : $\theta_{B2} = \frac{M_B}{k_R} + \frac{M_B L}{3EI}$

Solve compatibility equation to find M_B .

$$\theta_{B1} + \theta_{B2} = 0 \quad \text{Solve: } M_B = \frac{L M_0 k_R}{6EI + 2Lk_R} \quad M_B = \frac{L M_0 k_R}{6EI + 2Lk_R} \text{ (counterclockwise)}$$

USE STATICS TO FIND REMAINING REACTIONS

$$\sum M_B = 0 \quad R_A = \frac{1}{L} (M_0 + M_B) = \frac{M_0}{L} + \frac{M_0 k_R}{2(3EI + LK_R)} \quad R_A = \frac{M_0}{L} + \frac{M_0 k_R}{2(3EI + LK_R)}$$

$$\sum F_V = 0 \quad R_B = -R_A$$

(b) EXPRESSIONS FOR ROTATION AT A, θ_A AS SPRING STIFFNESS k_R VARIES

Use superposition of rotations in two released structures in part (a).

$$\theta_{A1} = \frac{M_0 L}{3EI} \quad \theta_{A2} = \frac{-M_B L}{6EI} \quad \theta_A = \theta_{A1} + \theta_{A2}$$

$$\theta_A = \frac{LM_0}{3EI} - \frac{L^2 M_0 k_R}{6EI(6EI + 2Lk_R)} = \frac{LM_0}{4EI} + \frac{LM_0}{4(3EI + Lk_R)}$$

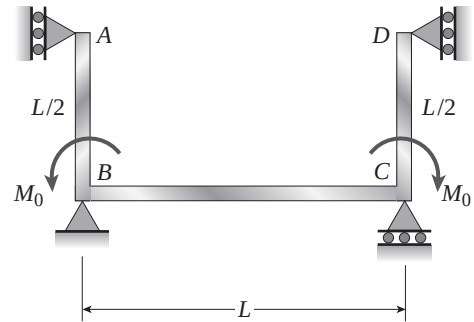
$$\text{For } k_R \text{ goes to zero,} \quad \theta_A = \frac{LM_0}{4EI} + \frac{LM_0}{4(3EI)} = \frac{LM_0}{3EI}$$

$$\text{For } k_R \text{ goes to infinity,} \quad \theta_A = \frac{M_0 L}{4EI}$$

$$\text{For } k_R \text{ goes to } 6EI/L, \quad \theta_A = \frac{LM_0}{4EI} + \frac{LM_0}{4\left[3EI + L\left(\frac{6EI}{L}\right)\right]} = \frac{5LM_0}{18EI} < \text{same as Prob. 10.4-5(b)}$$

Problem 10.4-7 The continuous frame ABCD has a pin support at B, roller supports at A, C, and D, and rigid corner connections at B and C (see figure). Members AB, BC, and CD each have flexural rigidity EI . Moment M_0 acts counterclockwise at B and clockwise at C. (Note: Disregard axial deformations in member AB and consider only the effects of bending.)

- Find all reactions of the frame.
- Find joint rotations θ at A, B, C, and D.
- Repeat parts (a) and (b) if both moments M_0 are counterclockwise.



Solution 10.4-7

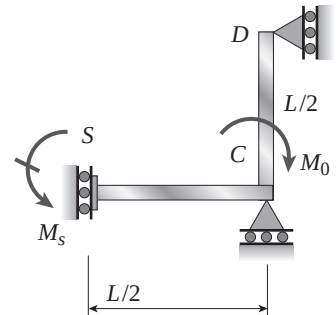
(a) SUPPORT REACTIONS FOR SYMMETRIC STRUCTURE

The frame is two degrees statically indeterminate but is symmetric and is acted upon by a symmetric loading. Solution options include (1) Select reactions at A and D as the two redundants, then use a superposition analysis (flexibility method). (2) Take advantage of symmetry; cut structure in half, use a sliding support at $x = L/2$ (a sliding support is required so that only symmetric distortions can occur), then analyze the one-degree statically indeterminate right half of the structure (see figure). Use symmetry to find reactions and joint rotations in left half of structure

USE OPTION 2 BASED ON SYMMETRY; SELECT M_S AS THE REDUNDANT

$$\theta_{S1} = \frac{-M_0 \frac{L}{2}}{3EI} = -\frac{LM_0}{6EI}$$

$$\theta_{S2} = \frac{M_S \frac{L}{2}}{EI} + \frac{M_S \frac{L}{2}}{3EI} = \frac{2LM_S}{3EI}$$



COMPATIBILITY EQUATION

$$\theta_{S1} + \theta_{S2} = 0 \text{ solve, } M_S = \frac{M_0}{4} \quad M_S = \frac{M_0}{4}$$

STATICS

$$\sum M_C = 0 \quad M_S - M_0 - H_D \frac{L}{2} = 0 \quad \text{Solve: } H_D = -\frac{3M_0}{2L} \quad \sum F_V = 0 \quad V_C = 0$$

$$\sum F_H = 0 \quad H_S = -H_D$$

SYMMETRY $H_A = -H_D \quad V_B = V_C$

$$\boxed{H_A = \frac{3}{2} \frac{M_0}{L}} \quad \boxed{H_B = 0} \quad \boxed{V_B = 0} \quad \boxed{V_C = 0} \quad \boxed{H_D = -H_A}$$

- (b) JOINT ROTATIONS FOR SYMMETRY CASE; USE SUPERPOSITION OF ROTATIONS FOR EACH RELEASED STRUCTURE ABOVE, THEN USE SYMMETRY

SUPERPOSITION using Tables H-1 and H-2

$$\theta_{C1} = \frac{-M_0 \frac{L}{2}}{3EI} = -\frac{LM_0}{6EI} \quad \theta_{C2} = \frac{M_S \frac{L}{2}}{3EI} = \frac{LM_0}{24EI} \quad \theta_C = \theta_{C1} + \theta_{C2} = -\frac{LM_0}{8EI}$$

$$\theta_{D1} = \frac{M_0 \frac{L}{2}}{6EI} = \frac{LM_0}{12EI} \quad \theta_{D2} = \frac{-M_S \frac{L}{2}}{6EI} \quad \theta_D = \theta_{D1} + \theta_{D2} = \frac{LM_0}{16EI}$$

SYMMETRY $\theta_A = -\theta_D \quad \theta_B = -\theta_C$

$$\boxed{\theta_A = \frac{-M_0 L}{16EI}} \quad \boxed{\theta_D = -\theta_A} \quad \boxed{\theta_B = \frac{M_0 L}{8EI}} \quad \boxed{\theta_C = -\theta_B} \quad (\text{counterclockwise is positive})$$

- (c) REACTIONS AND ALSO ROTATIONS FOR ANTI-SYMM. CASE

The frame is 2 degrees statically indeterminate but is symmetric and is acted upon by an **anti-symmetric** loading. Solution options include: (1) Select reactions at A and D as the two redundants, then use a superposition analysis (Flexibility Method); or (2) Take advantage of anti-symmetry. Cut structure in half, use a roller support at $x = L/2$ (a roller support is required so that only anti-symmetric distortions can occur), then analyze the one-degree statically indeterminate right half of the structure (see figure). Use anti-symmetry to find reactions and joint rotations in left half of structure

USE OPTION 2 BASED ON ANTI-SYMMETRY; SELECT V_S AS THE REDUNDANT

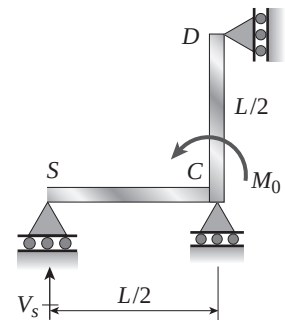
$$\delta_{VS1} = \frac{-M_0 \frac{L}{2}}{3EI} \left(\frac{L}{2} \right) = -\frac{L^2 M_0}{12EI}$$

$$\delta_{VS2} = \frac{V_S \left(\frac{L}{2} \right)^3}{3EI} + V_S \frac{L}{2} \left(\frac{\frac{L}{2}}{3EI} \right) \frac{L}{2} = \frac{L^3 V_S}{12EI}$$

COMPATIBILITY

$$\delta_{VS1} + \delta_{VS2} = 0 \quad \text{Solve: } V_S = \frac{M_0}{L} \quad V_S = \frac{M_0}{L}$$

$$\text{STATICS} \quad \sum F_V = 0 \quad V_C = -V_S \quad \sum M_C = 0 \quad H_D = \frac{1}{L} \left(M_0 - V_S \frac{L}{2} \right) = \frac{M_0}{L}$$



ANTI-SYMMETRY $V_B = -V_C$ $H_A = H_D$

So reactions are

$$H_A = \frac{M_0}{L} \quad H_B = -2\frac{M_0}{L} \quad V_B = \frac{M_0}{L} \quad V_C = \frac{-M_0}{L} \quad H_D = H_A$$

Joint rotations using superposition of rotations for each released structure in the anti-symmetry model.

$$\theta_{C1} = \frac{M_0 \frac{L}{2}}{3EI} = \frac{LM_0}{6EI} \quad \theta_{C2} = -V_S \frac{L}{2} \left(\frac{L}{3EI} \right) = \frac{LM_0}{12EI} \quad \theta_C = \theta_{C1} + \theta_{C2} = \frac{LM_0}{12EI}$$

$$\theta_{D1} = \frac{-M_0 \frac{L}{2}}{6EI} = -\frac{LM_0}{12EI} \quad \theta_{D2} = V_S \frac{L}{2} \left(\frac{L}{6EI} \right) = \frac{LM_0}{24EI} \quad \theta_D = \theta_{D1} + \theta_{D2} = -\frac{LM_0}{24EI}$$

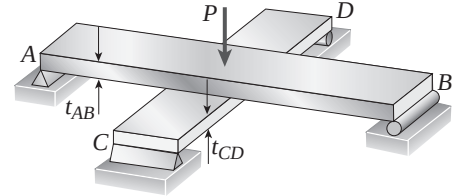
ANTI-SYMMETRY $\theta_A = \theta_D$ $\theta_B = \theta_C$

So joint rotations are

$$\boxed{\theta_A = \frac{-M_0 L}{24EI}} \quad \boxed{\theta_D = \theta_A} \quad \boxed{\theta_B = \frac{M_0 L}{12EI}} \quad \boxed{\theta_C = \theta_B} \quad (\text{counterclockwise is positive})$$

Problem 10.4-8 Two flat beams AB and CD , lying in horizontal planes, cross at right angles and jointly support a vertical load P at their midpoints (see figure). Before the load P is applied, the beams just touch each other. Both beams are made of the same material and have the same widths. Also, the ends of both beams are simply supported. The lengths of beams AB and CD are L_{AB} and L_{CD} , respectively.

What should be the ratio t_{AB}/t_{CD} of the thicknesses of the beams if all four reactions are to be the same?



Solution 10.4-8 Two beams supporting a load P

For all four reactions to be the same, each beam must support one-half of the load P .

DEFLECTIONS

$$\delta_{AB} = \frac{(P/2) L_{AB}^3}{48EI_{AB}} \quad \delta_{CD} = \frac{(P/2) L_{CD}^3}{48EI_{CD}}$$

COMPATIBILITY

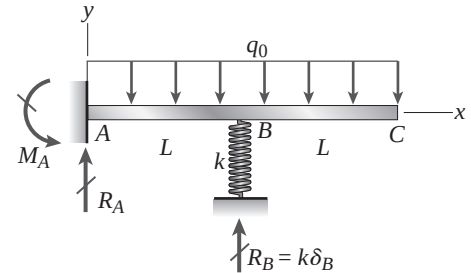
$$\delta_{AB} = \delta_{CD} \quad \text{or} \quad \frac{L_{AB}^3}{I_{AB}} = \frac{L_{CD}^3}{I_{CD}}$$

MOMENT OF INERTIA

$$I_{AB} = \frac{1}{12} b t_{AB}^3 \quad I_{CD} = \frac{1}{12} b t_{CD}^3$$

$$\therefore \frac{L_{AB}^3}{t_{AB}^3} = \frac{L_{CD}^3}{t_{CD}^3} \quad \frac{t_{AB}}{t_{CD}} = \frac{L_{AB}}{L_{CD}} \quad \leftarrow$$

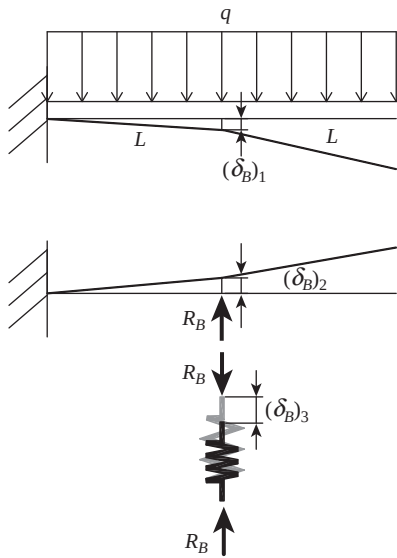
Problem 10.4-9 A propped cantilever beam of length $2L$ is loaded by a uniformly distributed load with intensity q . The beam is supported at B by a linearly elastic spring with stiffness k . Use the method of superposition to solve for all reactions. Also draw shear-force and bending-moment diagrams, labeling all critical ordinates. Let $k = 6EI/L^3$.



Solution 10.4-9

Select R_B as the redundant.

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$(\delta_B)_1$ = deflection at B due to uniform load q

$$(\delta_B)_1 = \frac{17qL^4}{24EI}$$

$(\delta_B)_2$ = deflection at B due to force R_B

$$(\delta_B)_2 = \frac{R_B L^3}{3EI}$$

$(\delta_B)_3$ = shortening in spring due to force R_B

$$(\delta_B)_3 = \frac{R_B}{k} = \frac{R_B L^3}{6EI}$$

FROM COMPATIBILITY EQUATION

$$(\delta_B)_1 - (\delta_B)_2 = (\delta_B)_3 \quad \therefore R_B = \frac{17}{12} qL$$

EQUILIBRIUM

$$R_A = 2qL - R_B = \frac{7}{12} qL \quad (1)$$

$$M_A = 2qL^2 - R_B L = \frac{7}{12} qL^2 \quad (2)$$

SHEAR FORCE

$$V = R_A - qx = \frac{7}{12} qL - qx \quad (0 \leq x \leq L)$$

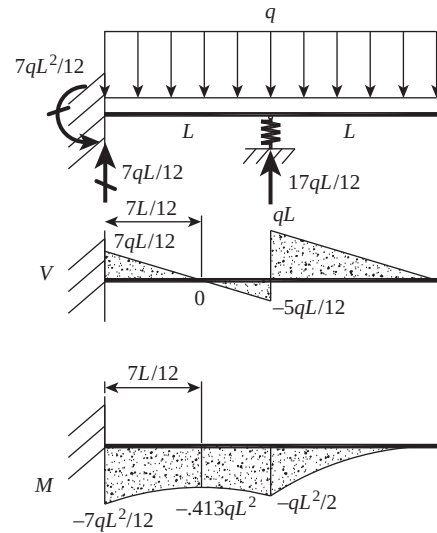
$$V = R_A - qx + R_B = 2qL - qx \quad (L \leq x \leq 2L)$$

BENDING MOMENTS

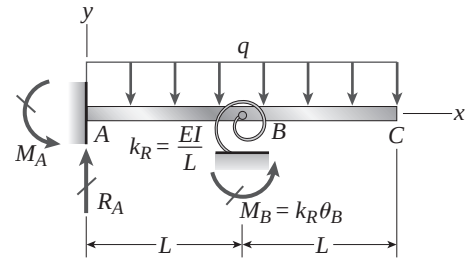
$$M = R_A x - M_A - q \frac{x^2}{2} = -q \left[\frac{1}{2} x^2 - \frac{7L}{12} x + \frac{7L^2}{12} \right] \quad (0 \leq x \leq L)$$

$$M = R_A x - M_A - q \frac{x^2}{2} + R_B (x - L) = -q \left[\frac{1}{2} x^2 - 2Lx + 2L^2 \right] \quad (L \leq x \leq 2L)$$

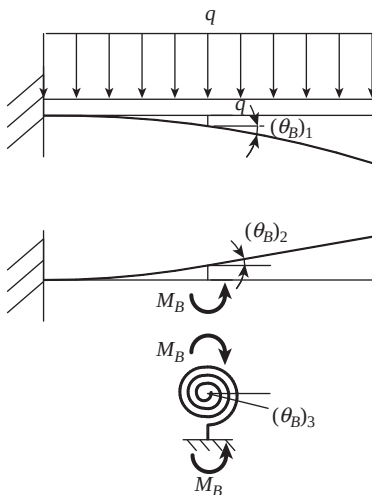
SHEAR FORCE AND BENDING MOMENT DIAGRAMS



Problem 10.4-10 A propped cantilever beam of length $2L$ is loaded by a uniformly distributed load with intensity q . The beam is supported at B by a linearly elastic rotational spring with stiffness k_R , which provides a resisting moment M_B due to rotation θ_B . Use the method of superposition to solve for all reactions. Also draw shear-force and bending-moment diagrams, labeling all critical ordinates. Let $k_R = EI/L$.



Solution 10.4-10



Select M_B as the redundant.

RELEASED STRUCTURE AND FORCE-SLOPE RELATIONS

$(\theta_B)_1$ = slope at B due to uniform load q

$$(\theta_B)_1 = \frac{7qL^3}{6EI}$$

$(\theta_B)_2$ = slope at B due to moment

$$(\theta_B)_2 = \frac{M_B L}{EI}$$

$(\theta_B)_3$ = spring rotation at B due to moment

$$(\theta_B)_3 = \frac{M_B}{k_R} = \frac{M_B L}{EI}$$

FROM COMPATIBILITY EQUATION

$$(\theta_B)_1 - (\theta_B)_2 = (\theta_B)_3 \quad \therefore M_B = \frac{7}{12} qL^2$$

FROM EQUILIBRIUM EQUATIONS

$$R_A = 2qL \quad (1)$$

$$M_A = 2qL^2 - M_B = \frac{17}{12} qL^2 \quad (2)$$

SHEAR FORCE

$$V = R_A - qx = 2qL - qx \quad (0 \leq x \leq 2L)$$

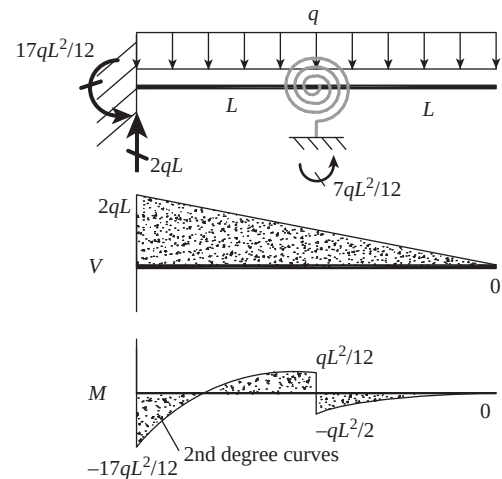
BENDING MOMENTS

$$M = R_A x - M_A - q \frac{x^2}{2} = -q \left[\frac{1}{2} x^2 - 2Lx + \frac{17L^2}{12} \right] \quad (0 \leq x \leq L)$$

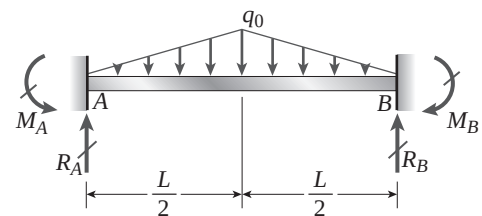
$$M = R_A x - M_A - q \frac{x^2}{2} - M_B$$

$$= -q \left[\frac{1}{2} x^2 - 2Lx + 2L^2 \right] \quad (L \leq x \leq 2L)$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



Problem 10.4-11 Determine the fixed-end moments (M_A and M_B) and fixed-end forces (R_A and R_B) for a beam of length L supporting a triangular load of maximum intensity q_0 (see figure). Then draw the shear-force and bending-moment diagrams, labeling all critical ordinates.



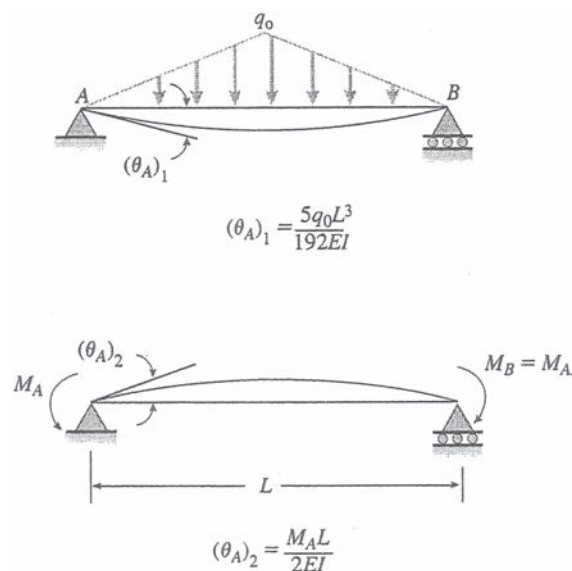
Solution 10.4-11 Fixed-end beam (triangular load)

Select M_A and M_B as redundants.

SYMMETRY $M_A = M_B$ $R_A = R_B$

EQUILIBRIUM $R_A = R_B = q_0 L/4$ ←

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS

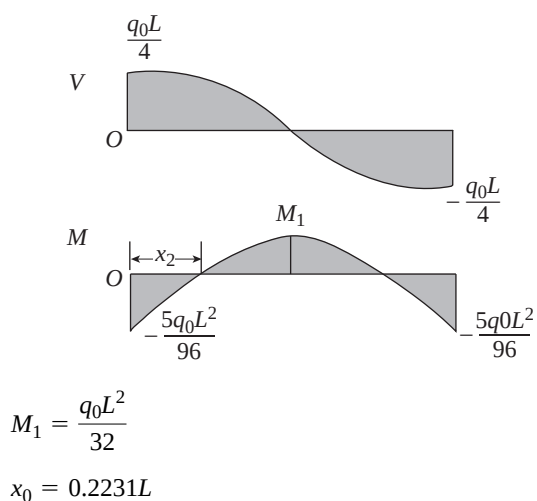


COMPATIBILITY $\theta_A = (\theta_A)_1 - (\theta_A)_2 = 0$

Substitute for $(\theta_A)_1$ and $(\theta_A)_2$ and solve for M_A :

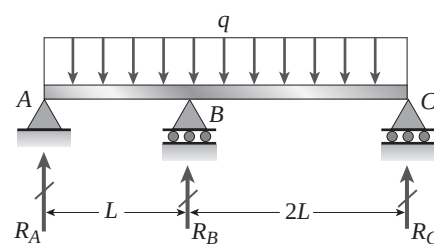
$$M_A = M_B = \frac{5q_0 L^2}{96} \quad \leftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAM



Problem 10.4-12 A continuous beam ABC with two unequal spans, one of length L and one of length $2L$, supports a uniform load of intensity q (see figure).

Determine the reactions R_A , R_B , and R_C for this beam. Also, draw the shear-force and bending-moment diagrams, labeling all critical ordinates.

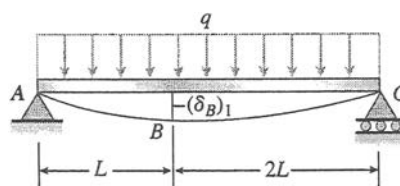
**Solution 10.4-12 Continuous beam with two spans**

Select R_B as the redundant.

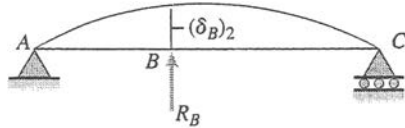
EQUILIBRIUM

$$R_A = \frac{3qL}{2} - \frac{2}{3}R_B \quad R_C = \frac{3qL}{2} - \frac{1}{3}R_B$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$$(\delta_B)_1 = \frac{11qL^4}{12EI}$$



$$(\delta_B)_2 = \frac{4R_B L^3}{9EI}$$

COMPATIBILITY

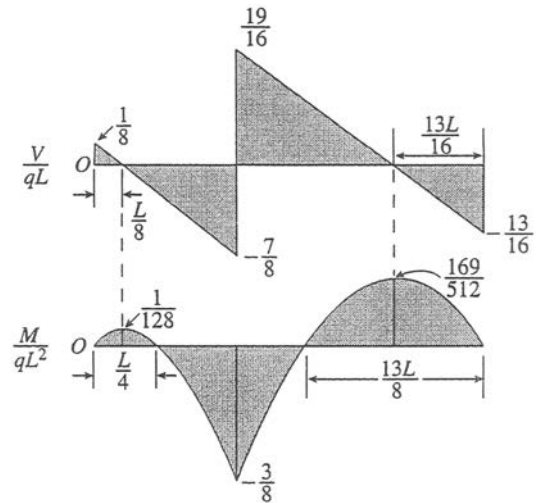
$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$$

$$\frac{11qL^4}{12EI} - \frac{4R_B L^3}{9EI} = 0 \quad R_B = \frac{33qL}{16} \quad \leftarrow$$

OTHER REACTIONS (FROM EQUILIBRIUM)

$$R_A = \frac{qL}{8} \quad R_C = \frac{13qL}{16} \quad \leftarrow$$

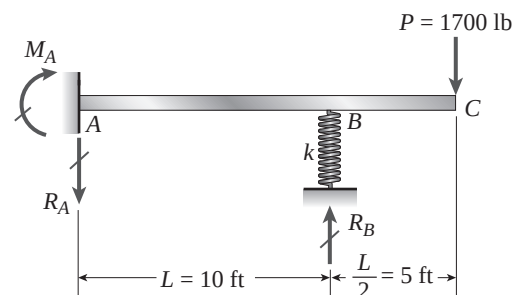
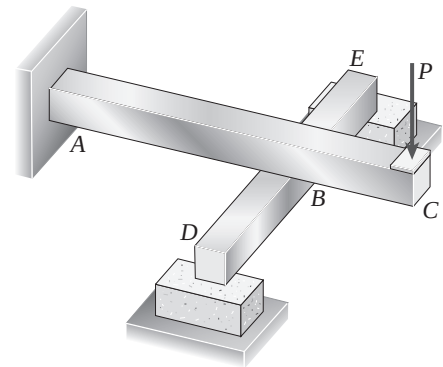
SHEAR-FORCE AND BENDING-MOMENT DIAGRAM



Problem 10.4-13 Beam ABC is fixed at support A and rests (at point B) upon the midpoint of beam DE (see the first part of the figure). Thus, beam ABC may be represented as a propped cantilever beam with an overhang BC and a linearly elastic support of stiffness k at point B (see the second part of the figure).

The distance from A to B is $L = 10$ ft, the distance from B to C is $L/2 = 5$ ft, and the length of beam DE is $L = 10$ ft. Both beams have the same flexural rigidity EI . A concentrated load $P = 1700$ lb acts at the free end of beam ABC .

Determine the reactions R_A , R_B , and M_A for beam ABC . Also, draw the shear-force and bending-moment diagrams for beam ABC , labeling all critical ordinates.

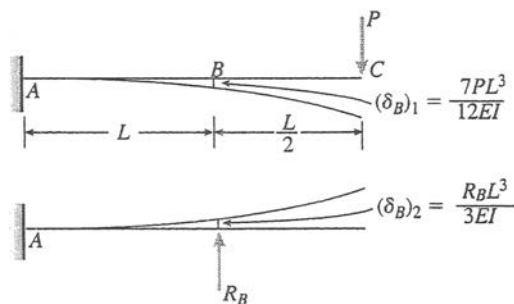


Solution 10.4-13 Beam with spring supportSelect R_B as the redundant.

EQUILIBRIUM

$$R_A = R_B - P \quad M_A = R_B L - 3PL/2$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT EQUATIONS



$$\text{COMPATIBILITY } \delta_B = (\delta_B)_1 - (\delta_B)_2 = \frac{R_B}{k}$$

$$\text{Beam DE: } k = \frac{48EI}{L^3}$$

$$\frac{7PL^3}{12EI} - \frac{R_B L^3}{3EI} = \frac{R_B L^3}{48EI} \quad R_B = \frac{28P}{17} \quad \leftarrow$$

OTHER REACTIONS (FROM EQUILIBRIUM)

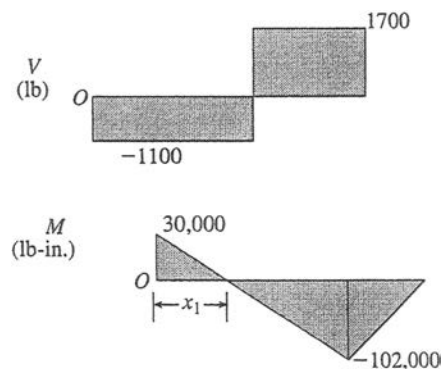
$$R_A = \frac{11P}{17} \quad M_A = \frac{5PL}{34} \quad \leftarrow$$

NUMERICAL VALUES

$$P = 1700 \text{ lb} \quad L = 10 \text{ ft} = 120 \text{ in.}$$

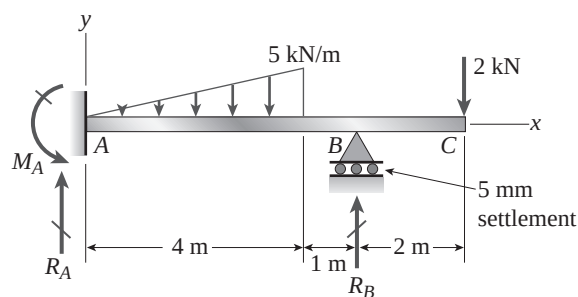
$$\left. \begin{aligned} R_A &= 1100 \text{ lb} & R_B &= 2800 \text{ lb} \\ M_A &= 30,000 \text{ lb-in.} \end{aligned} \right\} \quad \leftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAM



$$\begin{aligned} x_1 &= \frac{300}{11} \text{ in.} \\ &= 27.27 \text{ in.} \end{aligned}$$

Problem 10.4-14 A propped cantilever beam has flexural rigidity $EI = 4.5 \text{ MN} \cdot \text{m}^2$. When the loads shown are applied to the beam, it settles at joint B by 5 mm. Find the reaction at joint B.

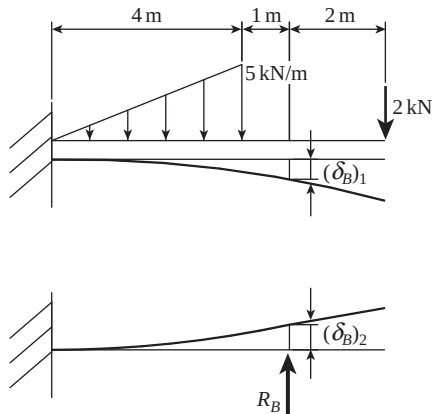


Solution 10.4-14

$$EI = 4.5 \text{ MN}\cdot\text{m}^2 = 4500 \text{ kN}\cdot\text{m}^2$$

Select R_B as the redundant.

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$(\delta_B)_1$ = Deflection at B due to distributed and concentrated loads

$$\begin{aligned} (\delta_B)_1 &= - \int_{2\text{ m}}^{7\text{ m}} 2x \frac{(x-2)}{EI} dx \\ &\quad - \int_{1\text{ m}}^{5\text{ m}} \left[\frac{5}{2}(x-1)^2 - \frac{5}{24}(x-1)^3 \right] \frac{x}{EI} dx \\ &= -(29.63 + 34.963) \times 10^3 \\ &= -64.593 \times 10^3 = -64.593 \text{ mm} \end{aligned}$$

$(\delta_B)_2$ = Deflection at B due to Force R_B

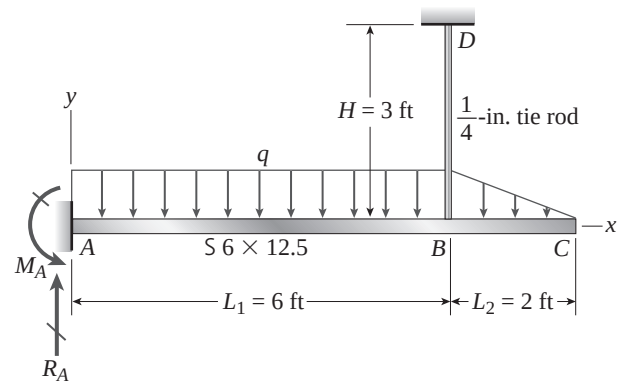
$$\begin{aligned} (\delta_B)_2 &= \int_0^{5\text{ m}} R_B x \frac{x}{EI} dx = R_B 9.259 \times 10^{-3} \\ &= R_B 9.259 \text{ mm} \end{aligned}$$

COMPATIBILITY (SETTLEMENT AT $B = 5 \text{ mm}$)

$$(\delta_B)_1 = (\delta_B)_2 - 5 \text{ mm} \quad \therefore R_B = 6.44 \text{ kN} \quad \leftarrow$$

Problem 10.4-15 A cantilever beam is supported by a tie rod at B as shown. Both the tie rod and the beam are steel with $E = 30 \times 10^6 \text{ psi}$. The tie rod is just taut before the distributed load $q = 200 \text{ lb/ft}$ is applied.

- Find the tension force in the tie rod.
- Draw shear-force and bending-moment diagrams for the beam, labeling all critical ordinates.



Solution 10.4-15

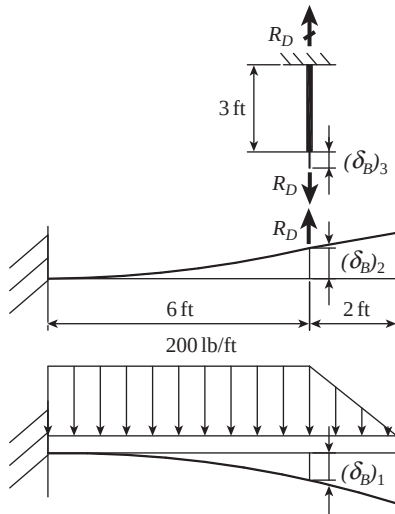
$$E = 30 \times 10^6 \text{ psi}$$

$$A = \frac{(1/4)^2 \pi}{4} = 0.0491 \text{ in.}^2$$

$$I = 22.1 \text{ in.}^4 \quad \text{From S } 6 \times 12.5$$

Select R_D as the redundant.

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$(\delta_B)_1$ = Deflection at B due to distributed load q

$$q = 200 \text{ lb/ft} = \frac{200}{12} \text{ lb/in.}$$

$$(\delta_B)_1 = \int_0^{72} \left(\frac{200}{12} \frac{x^2}{2} + \frac{200}{12} \frac{2 \times 12}{2} \frac{2 \times 12}{3} \right) \frac{x}{EI} dx$$

$$= 0.1282 \text{ in.}$$

$(\delta_B)_2$ = Deflection at B due to Force R_D

$$(\delta_B)_2 = \int_0^{72} \frac{R_D x^2}{EI} dx = R_D 1.877 \times 10^{-4} \text{ in.}$$

$(\delta_B)_3$ = Extension in tie rod due to Force R_D

$$(\delta_B)_3 = \frac{R_D L}{AE} = \frac{R_D (3 \times 12)}{(0.0491 \times 30) \times 10^6}$$

$$= R_D 2.445 \times 10^{-5} \text{ in.}$$

COMPATIBILITY

$$(\delta_B)_1 - (\delta_B)_2 = (\delta_B)_3$$

$$0.1282 - R_D 1.877 \times 10^{-4} = R_D 2.445 \times 10^{-5}$$

$$\therefore R_D = 604.3 \text{ lb}$$

(a) THE TENSION FORCE IN THE TIE ROD = R_D
 $= 604.3 \text{ lb}$ ←

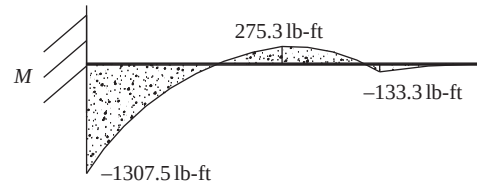
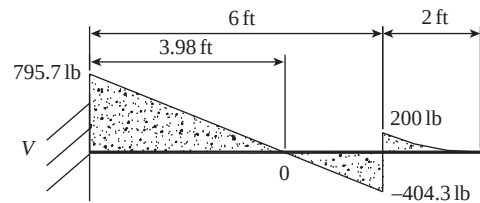
FROM EQUILIBRIUM EQUATIONS

$$R_A = (200 \times 6) + 200 \frac{2}{2} - R_D = 795.7 \text{ lb} \quad (1)$$

$$M_A = \frac{200 \times 2}{2} \left(6 + \frac{2}{3} \right) + 200 \frac{6^2}{2} - R_D 6$$

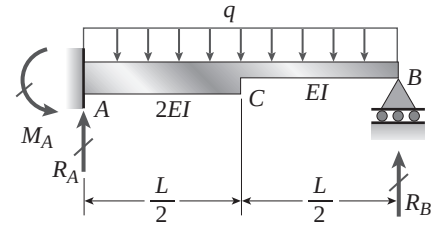
$$= 1308 \text{ lb-ft} = 1.569 \times 10^4 \text{ lb-in.} \quad \leftarrow$$

(b) SHEAR FORCE AND BENDING MOMENT DIAGRAMS



Problem 10.4-16 The figure shows a nonprismatic, propped cantilever beam AB with flexural rigidity $2EI$ from A to C and EI from C to B .

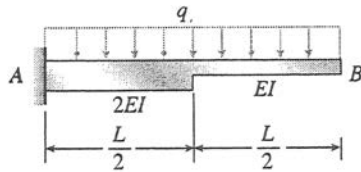
Determine all reactions of the beam due to the uniform load of intensity q . (Hint: Use the results of Problems 9.7-1 and 9.7-2.)



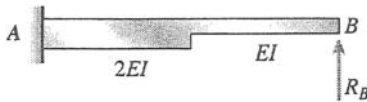
Solution 10.4-16 Nonprismatic beam

Select R_B as the redundant.

RELEASED STRUCTURE



$(\delta_B)_1$ = downward deflection of end B due to load q



$(\delta_B)_2$ = upward deflection due to reaction R_B

FORCE-DISPLACEMENT RELATIONS

$$\text{From Prob. 9.7-2, } \delta_B = \frac{qL^4}{128EI_1} \left(1 + 15 \frac{I_1}{I_2} \right)$$

$$I_1 = I \quad I_2 = 2I \quad \therefore (\delta_B)_1 = \frac{17qL^4}{256EI}$$

From Prob. 9.7-1,

$$\delta_B = \frac{PL^3}{24EI_1} \left(1 + 7 \frac{I_1}{I_2} \right) \quad \therefore (\delta_B)_2 = \frac{3R_B L^3}{16EI}$$

COMPATIBILITY

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$$

or

$$\frac{17qL^4}{256EI} - \frac{3R_B L^3}{16EI} = 0 \quad R_B = \frac{17qL}{48} \quad \leftarrow$$

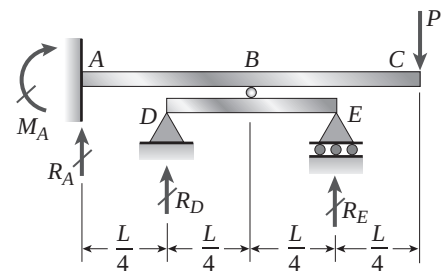
EQUILIBRIUM

$$R_A = qL - R_B = \frac{31qL}{48}$$

$$M_A = \frac{qL^2}{2} - R_B L = \frac{7qL^2}{48} \quad \leftarrow$$

Problem 10.4-17 A beam ABC is fixed at end A and supported by beam DE at point B (see figure). Both beams have the same cross section and are made of the same material.

- Determine all reactions due to the load P .
- What is the numerically largest bending moment in either beam?

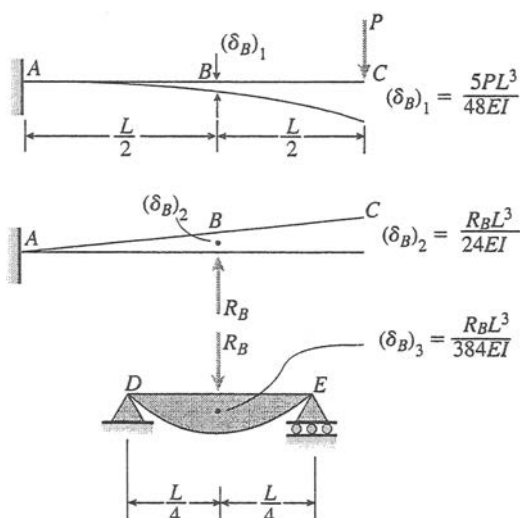


Solution 10.4-17 Beam supported by a beam

Let R_B = interaction force between beams

Select R_B as the redundant.

RELEASED STRUCTURE AND FORCE-DISPLACEMENT EQUATIONS



COMPATIBILITY $(\delta_B)_1 - (\delta_B)_2 = (\delta_B)_3$

Substitute and solve: $R_B = \frac{40P}{17}$ ←

SYMMETRY AND EQUILIBRIUM

$R_D = R_E = \frac{R_B}{2} = \frac{20P}{17}$ ←

$R_A = P - R_D - R_E = -\frac{23P}{17}$ ←

(minus means downward)

$M_A = R_B \left(\frac{L}{2} \right) - PL = \frac{3PL}{17}$ ←

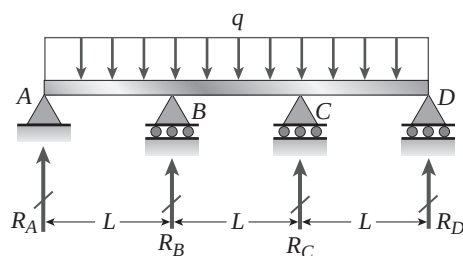
BEAM ABC: $M_{\max} = M_B = -\frac{PL}{2}$

BEAM DE: $M_{\max} = M_B = \frac{5PL}{17}$

$|M_{\max}| = \frac{PL}{2}$ ←

Problem 10.4-18 A three-span continuous beam $ABCD$ with three equal spans supports a uniform load of intensity q (see figure).

Determine all reactions of this beam and draw the shear-force and bending-moment diagrams, labeling all critical ordinates.

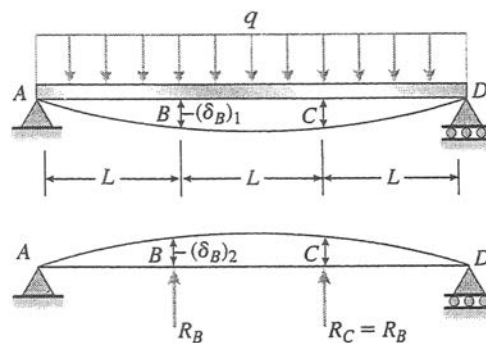
**Solution 10.4-18 Three-span continuous beam**

SELECT R_B AND R_C AS THE REDUNDANTS.

RELEASED STRUCTURE

SYMMETRY AND EQUILIBRIUM

$R_C = R_B$ $R_A = R_D = \frac{3qL}{2} - R_B$



FORCE-DISPLACEMENT RELATIONS

$$(\delta_B)_1 = \frac{11qL^4}{12EI} \quad (\delta_B)_2 = \frac{5R_B L^3}{6EI}$$

COMPATIBILITY

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0 \quad \therefore R_B = \frac{11qL}{10} \quad \leftarrow$$

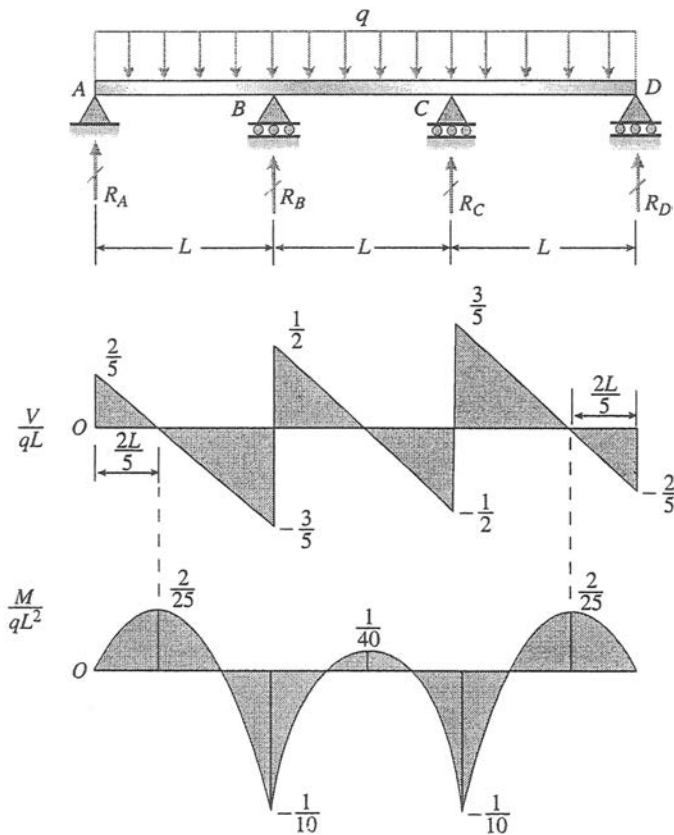
OTHER REACTIONS

From symmetry and equilibrium,

$$R_C = R_B = \frac{11qL}{10} \quad \leftarrow$$

$$R_A = R_D = \frac{2qL}{5} \quad \leftarrow$$

LOADING, SHEAR-FORCE, AND BENDING-MOMENT DIAGRAMS

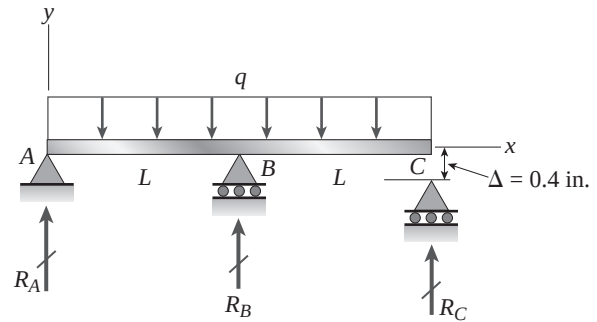


$$M_B = M_C = -\frac{qL^2}{10}$$

$$M_{\max} = \frac{2qL^2}{25}$$

Problem 10.4-19 A beam rests on supports at A and B and is loaded by a distributed load with intensity q as shown. A small gap Δ exists between the unloaded beam and the support at C . Assume that span length $L = 40$ in. and flexural rigidity of the beam $EI = 0.4 \times 10^9$ lb-in.². Plot a graph of the bending moment at B as a function of the load intensity q .

(Hint: See Example 9-9 for guidance on computing the deflection at C .)



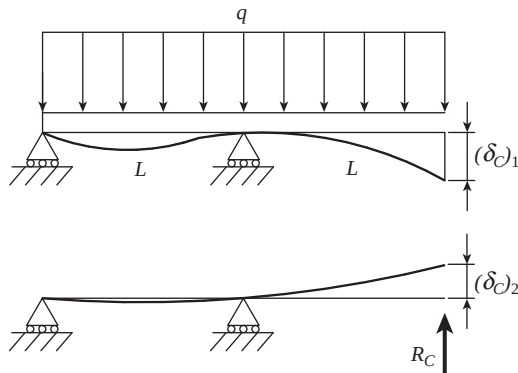
Solution 10.4-19

Select R_C as the redundant.

EQUILIBRIUM

$$R_C = 2qL - R_A - R_B$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$$(\delta_C)_1 = \frac{qL^4}{4EI}$$

$$(\delta_C)_2 = \frac{2L^3}{3EI} R_C$$

COMPATIBILITY

$$1) \delta_C = (\delta_C)_1 \quad \text{for } (\delta_C)_1 < 0.4 \text{ in.} \quad \therefore R_C(q) = 0$$

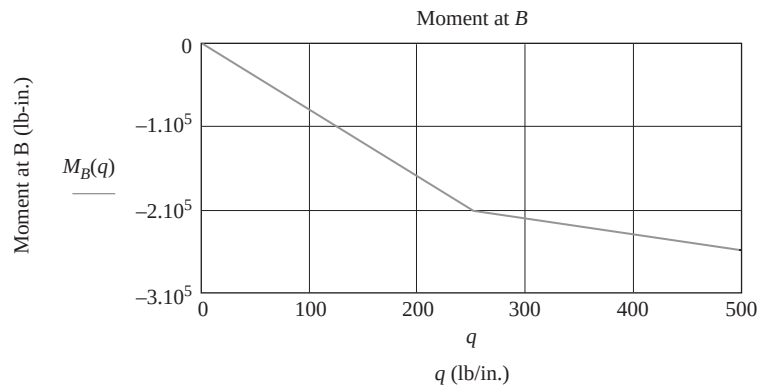
$$M_B(q) = -\frac{qL^2}{2} = -800q \text{ lb-in. for } q < 250 \text{ lb/in.}$$

$$2) \delta_C = (\delta_C)_1 - (\delta_C)_2 = 0.4 \text{ in. for } (\delta_C)_1 = 0.4 \text{ in.}$$

$$\therefore R_C(q) = 15q - 3750$$

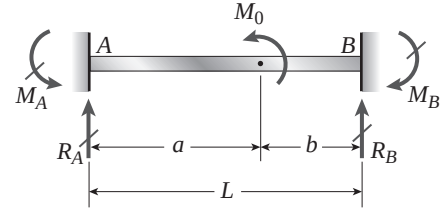
$$M_B(q) = R_AL - \frac{qL^2}{2} = -200q - 150000 \text{ lb-in.}$$

$$\text{for } q \geq 250 \text{ lb/in.}$$



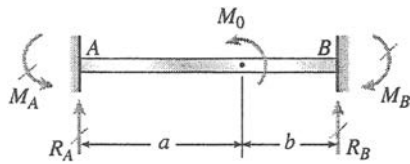
Problem 10.4-20 A fixed-end beam AB of length L is subjected to a moment M_0 acting at the position shown in the figure.

- Determine all reactions for this beam.
- Draw shear-force and bending-moment diagrams for the special case in which $a = b = L/2$.



Solution 10.4-20 Fixed-end beam ($M_0 =$ applied load)

Select R_B and M_B as redundants.

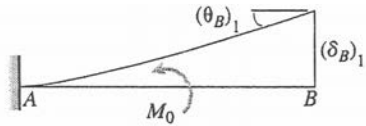


$$L = a + b$$

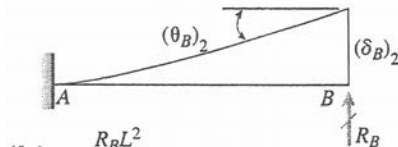
EQUILIBRIUM

$$R_A = -R_B \quad M_A = M_B - R_B L - M_0$$

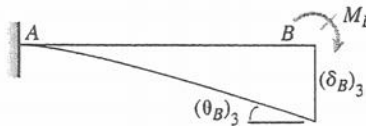
RELEASED STRUCTURE AND FORCE-DISPLACEMENT EQUATIONS



$$(\theta_B)_1 = \frac{M_0 a}{EI} \quad (\delta_B)_1 = \frac{M_0 a}{2EI} (a + 2b)$$



$$(\theta_B)_2 = \frac{R_B L^2}{2EI} \quad (\delta_B)_2 = \frac{R_B L^3}{3EI}$$



$$(\theta_B)_3 = \frac{M_B L}{EI} \quad (\delta_B)_3 = \frac{M_B L^2}{2EI}$$

$$(\theta_B)_1 = \frac{M_0 a}{EI} \quad (\delta_B)_1 = \frac{M_0 a}{2EI} (a + 2b)$$

$$(\theta_B)_2 = \frac{R_B L^2}{2EI} \quad (\delta_B)_2 = \frac{R_B L^3}{3EI}$$

$$(\theta_B)_3 = \frac{M_B L}{EI} \quad (\delta_B)_3 = \frac{M_B L^2}{2EI}$$

COMPATIBILITY

$$\delta_B = -(\delta_B)_1 - (\delta_B)_2 + (\delta_B)_3 = 0$$

$$\text{or } 2R_B L^3 - 3M_B L^2 = -3M_0 a(a + 2b) \quad (1)$$

$$\theta_B = (\theta_B)_1 + (\theta_B)_2 - (\theta_B)_3 = 0$$

$$\text{or } R_B L^2 - 2M_B L = -2M_0 a \quad (2)$$

SOLVE EQS. (1) AND (2)

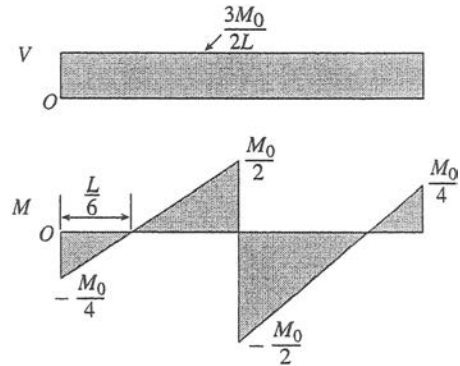
$$R_B = -\frac{6M_0 ab}{L^3} \quad M_B = -\frac{M_0 a}{L^2} (3b - L) \quad \leftarrow$$

FROM EQUILIBRIUM

$$R_A = \frac{6M_0 ab}{L^3} \quad M_A = \frac{M_0 b}{L^2} (3a - L) \quad \leftarrow$$

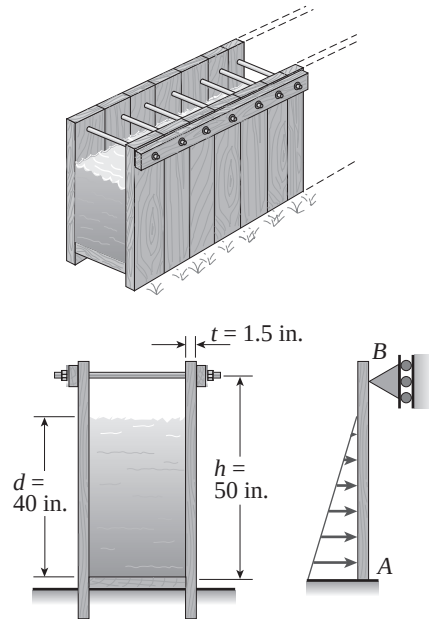
SPECIAL CASE $a = b = L/2$

$$R_A = -R_B = \frac{3M_0}{2L} \quad M_A = -M_B = \frac{M_0}{4}$$

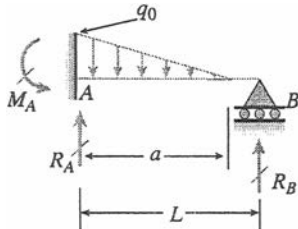


Problem 10.4-21 A temporary wood flume serving as a channel for irrigation water is shown in the figure. The vertical boards forming the sides of the flume are sunk in the ground, which provides a fixed support. The top of the flume is held by tie rods that are tightened so that there is no deflection of the boards at that point. Thus, the vertical boards may be modeled as a beam AB , supported and loaded as shown in the last part of the figure.

Assuming that the thickness t of the boards is 1.5 in., the depth d of the water is 40 in., and the height h to the tie rods is 50 in., what is the maximum bending stress σ in the boards? (Hint: The numerically largest bending moment occurs at the fixed support.)



Solution 10.4-21 Side wall of a wood flume



Select R_B as the redundant.

$$\text{Equilibrium: } M_A = \frac{q_0 a^2}{6} - R_B L$$

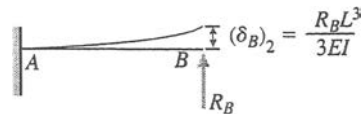
RELEASED STRUCTURE AND FORCE-DISPLACEMENT EQUATIONS



From Table H-1, Case 8:

$$(\delta_B)_1 = \frac{q_0 a^4}{30EI} + \frac{q_0 a^3}{24EI} (L-a) = \frac{q_0 a^3}{120EI} (5L-a)$$

$$(\delta_B)_2 = \frac{R_B L^3}{3EI}$$



COMPATIBILITY

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0 \quad \therefore R_B = \frac{q_0 a^3 (5L-a)}{40L^3}$$

MAXIMUM BENDING MOMENT

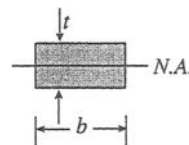
$$M_{\max} = M_A = \frac{1}{6} q_0 a^2 - R_B L$$

$$= \frac{q_0 a^2}{120L^2} (20L^2 - 15aL + 3a^2)$$

NUMERICAL VALUES

$$a = 40 \text{ in.} \quad L = 50 \text{ in.} \quad t = 1.5 \text{ in.}$$

b = width of beam



$$S = \frac{bt^2}{6} \quad \sigma = \frac{M_{\max}}{S}$$

$$\gamma = 62.4 \text{ lb/ft}^3 = 0.03611 \text{ lb/in.}^3$$

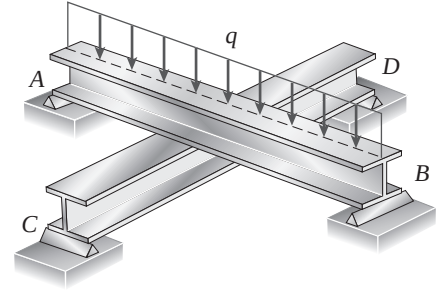
$$\text{Pressure } p = \gamma a \quad q_0 = pb = \gamma ab$$

$$M_{\max} = \frac{\gamma a^3 b}{120L^2} (20L^2 - 15aL + 3a^2) = 191.05 b$$

$$S = \frac{bt^2}{6} = 0.3750 b \quad \sigma = \frac{M_{\max}}{S} = 509 \text{ psi} \quad \leftarrow$$

Problem 10.4-22 Two identical, simply supported beams AB and CD are placed so that they cross each other at their midpoints (see figure). Before the uniform load is applied, the beams just touch each other at the crossing point.

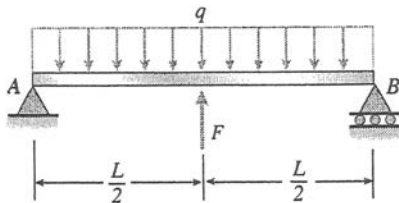
Determine the maximum bending moments $(M_{AB})_{\max}$ and $(M_{CD})_{\max}$ in beams AB and CD , respectively, due to the uniform load if the intensity of the load is $q = 6.4 \text{ kN/m}$ and the length of each beam is $L = 4 \text{ m}$.



Solution 10.4-22 Two beams that cross

F = interaction force between the beams

UPPER BEAM



$(\delta_B)_1$ = downward deflection due to q

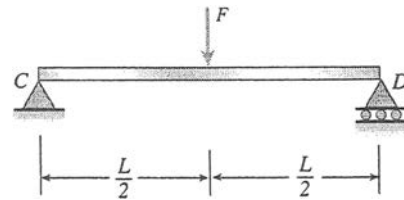
$$= \frac{5qL^4}{384EI}$$

$(\delta_B)_2$ = downward deflection due to F

$$= \frac{FL^3}{48EI}$$

$$\begin{aligned} \delta_{AB} &= (\delta_B)_1 - (\delta_B)_2 \\ &= \frac{5qL^4}{384EI} - \frac{FL^3}{48EI} \end{aligned}$$

LOWER BEAM

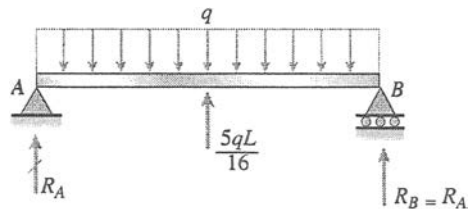


$$\delta_{CD} = \frac{FL^3}{48EI}$$

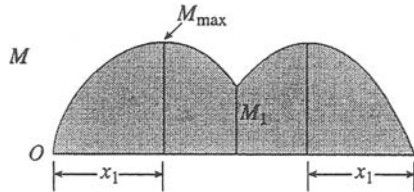
COMPATIBILITY $\delta_{AB} = \delta_{CD}$

$$\frac{5qL^4}{384EI} - \frac{FL^3}{48EI} = \frac{FL^3}{48EI} \quad \therefore F = \frac{5qL}{16}$$

UPPER BEAM



$$R_A = \frac{11qL}{32}$$



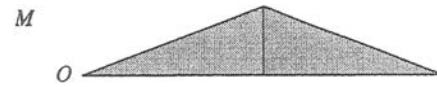
$$x_1 = \frac{11L}{32}$$

$$M_{\max} = \frac{121qL^2}{2048}$$

$$M_1 = \frac{3qL^2}{64} \quad (M_{AB})_{\max} = \frac{121qL^2}{2048} \quad \leftarrow$$

LOWER BEAM

$$M_{\max} = \frac{FL}{4} = \frac{5qL^2}{64}$$



$$(M_{CD})_{\max} = \frac{5qL^2}{64} \quad \leftarrow$$

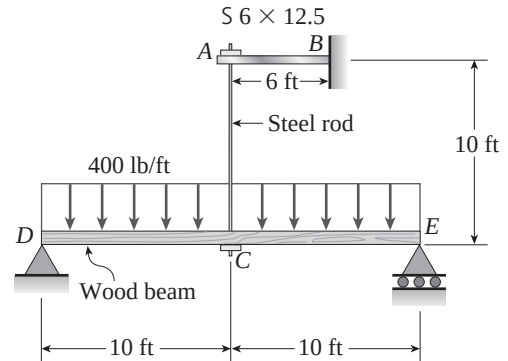
NUMERICAL VALUES

$$q = 6.4 \text{ kN/m} \quad (M_{AB})_{\max} = 6.05 \text{ kN}\cdot\text{m} \quad \leftarrow$$

$$L = 4 \text{ m} \quad (M_{CD})_{\max} = 8.0 \text{ kN}\cdot\text{m} \quad \leftarrow$$

Problem 10.4-23 The cantilever beam AB shown in the figure is an $S 6 \times 12.5$ steel I-beam with $E = 30 \times 10^6$ psi. The simple beam DE is a wood beam 4 in. $\times$ 12 in. (normal dimension) in cross section with $E = 1.5 \times 10^6$ psi. A steel rod AC of diameter 0.25 in., length 10 ft, and $E = 30 \times 10^6$ psi serves as a hanger joining the two beams. The hanger fits snugly between the beam before the uniform load is applied to beam DE .

Determine the tensile force F in the hanger and the maximum bending moments M_{AB} and M_{DE} in the two beams due to the uniform load, which has intensity $q = 400$ lb/ft. (Hint: To aid in obtaining the maximum bending moment in beam DE , draw the shear-force and bending-moment diagrams.)

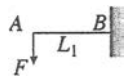


Solution 10.4-23 Beams joined by a hanger

F = tensile force in hanger

Select F as redundant.

(1) CANTILEVER BEAM AB



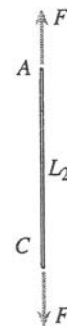
$$S 6 \times 12.5 \quad I_1 = 22.1 \text{ in.}^4$$

$$L_1 = 6 \text{ ft} = 72 \text{ in.}$$

$$E_1 = 30 \times 10^6 \text{ psi}$$

$$(\delta_A)_1 = \frac{FL_1^3}{3E_1I_1} = 187.66 \times 10^{-6} F \quad \begin{cases} F = \text{lb} \\ \delta = \text{in.} \end{cases}$$

(2) HANGER AC



$$d = 0.25 \text{ in.} \quad L_2 = 10 \text{ ft} = 120 \text{ in.}$$

$$E_2 = 30 \times 10^6 \text{ psi}$$

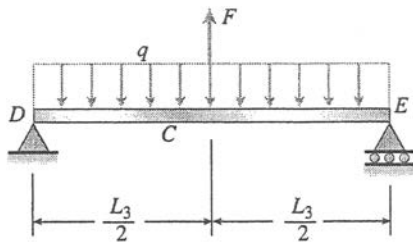
$$A_2 = \frac{\pi d^2}{4} = 0.049087 \text{ in.}^2$$

Δ = elongation of AC

$$\Delta = \frac{FL_2}{E_2 A_2} = 81.488 \times 10^{-6} F$$

(F = lb, Δ = in.)

(3) BEAM DCE



$$L_3 = 20 \text{ ft} = 240 \text{ in.}$$

$$q = 400 \text{ lb/ft}$$

$$= 33.333 \text{ lb/in.}$$

$$E_3 = 1.5 \times 10^6 \text{ psi}$$

$$4 \text{ in.} \times 12 \text{ in. (nominal)}$$

$$I_3 = 415.28 \text{ in.}^4$$

$$(\delta_C)_3 = \frac{5qL_3^4}{384E_3I_3} - \frac{FL_3^3}{48E_3I_3}$$

$$= 2.3117 \text{ in.} - 462.34 \times 10^{-6} F \quad \begin{cases} F = \text{lb} \\ \delta = \text{in.} \end{cases}$$

COMPATIBILITY

$$(\delta_A)_1 + \Delta = (\delta_C)_3$$

$$187.66 \times 10^{-6} F + 81.488 \times 10^{-6} F$$

$$= 2.3117 - 462.34 \times 10^{-6} F$$

$$F = 3160 \text{ lb} \quad \leftarrow$$

(1) MAX. MOMENT IN AB

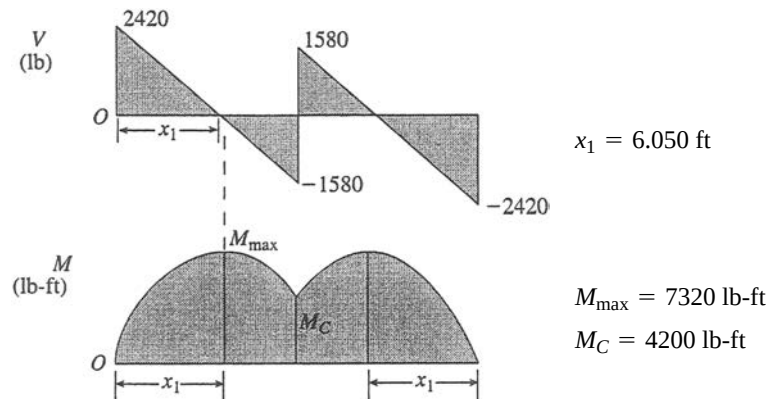
$$M_{AB} = FL_1 = (3160 \text{ lb})(6 \text{ ft})$$

$$= 18,960 \text{ lb-ft} \quad \leftarrow$$

(3) MAX. MOMENT IN DCE

$$R_D = \frac{qL_3}{2} - \frac{F}{2} = 2420 \text{ lb}$$

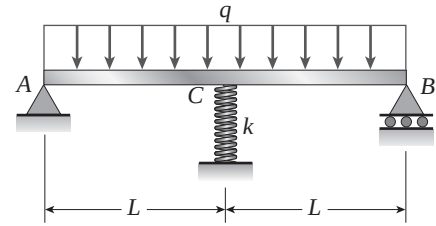
SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



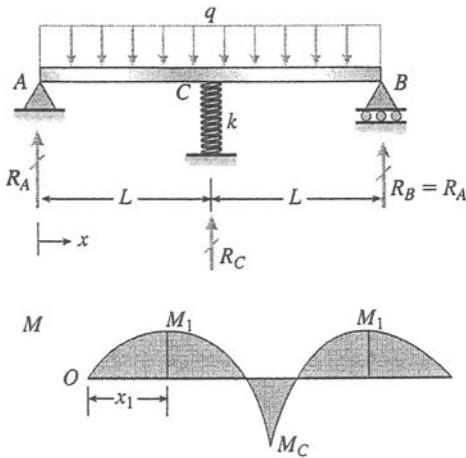
$$M_{DE} = 7320 \text{ lb-ft} \quad \leftarrow$$

Problem 10.4-24 The beam AB shown in the figure is simply supported at A and B and supported on a spring of stiffness k at its midpoint C . The beam has flexural rigidity EI and length $2L$.

What should be the stiffness k of the spring in order that the maximum bending moment in the beam (due to the uniform load) will have the smallest possible value?



Solution 10.4-24 Beam supported by a spring



BENDING MOMENT $M = R_A x - \frac{qx^2}{2}$

LOCATION OF MAXIMUM POSITIVE MOMENT

$$\frac{dM}{dx} = 0 \quad R_A - qx = 0 \quad x_1 = \frac{R_A}{q}$$

MAXIMUM POSITIVE MOMENT

$$M_1 = (M)_{x=x_1} = \frac{R_A^2}{2q}$$

MAXIMUM NEGATIVE MOMENT

$$M_C = (M)_{x=L} = R_A L - \frac{qL^2}{2}$$

FOR THE SMALLEST MAXIMUM MOMENT:

$$|M_1| = |M_C| \text{ or } M_1 = -M_C$$

$$\frac{R_A^2}{2q} = -R_A L + \frac{qL^2}{2}$$

Solve for R_A :

$$R_A = qL(\sqrt{2} - 1)$$

EQUILIBRIUM

$$\sum F_{\text{vert}} = 0 \quad 2R_A + R_C - 2qL = 0$$

$$R_C = 2qL(2 - \sqrt{2})$$

DOWNWARD DEFLECTION OF BEAM

$$(\delta_C)_1 = \frac{5qL^4}{24EI} - \frac{R_C L^3}{6EI} = \frac{qL^4}{24EI}(8\sqrt{2} - 11)$$

DOWNWARD DISPLACEMENT OF SPRING

$$(\delta_C)_2 = \frac{R_C}{k} = \frac{2qL}{k}(2 - \sqrt{2})$$

COMPATIBILITY $(\delta_C)_1 = (\delta_C)_2$

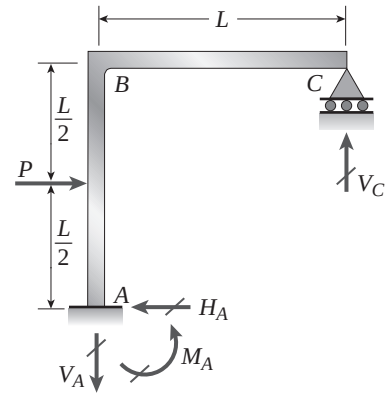
Solve for k :

$$k = \frac{48EI}{7L^3}(6 + 5\sqrt{2})$$

$$= 89.63 \frac{EI}{L^3} \quad \leftarrow$$

Problem 10.4-25 The continuous frame ABC has a fixed support at A , a roller support at C , and a rigid corner connection at B (see figure). Members AB and BC each have length L and flexural rigidity EI . A horizontal force P acts at midheight of member AB .

- Find all reactions of the frame.
- What is the largest bending moment $M_{\max}$ in the frame? (Note: Disregard axial deformations in member AB and consider only the effects of bending.)



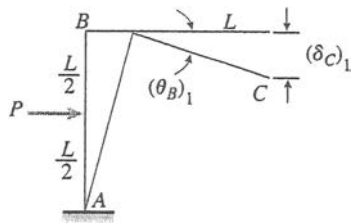
Solution 10.4-25 Frame ABC with fixed support

Select V_C as the redundant.

EQUILIBRIUM $V_A = V_C$ $H_A = P$

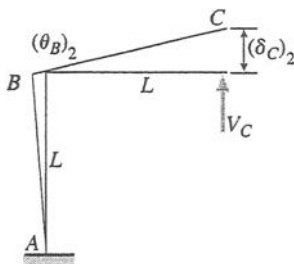
$$M_A = PL/2 - V_C L$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT EQUATIONS



$$(\theta_B)_1 = \frac{PL^2}{8EI}$$

$$(\delta_C)_1 = (\theta_B)_1 L = \frac{PL^3}{8EI}$$



$$(\theta_B)_2 = \frac{V_C L^2}{EI}$$

$$(\delta_C)_2 = (\theta_B)_2 L + \frac{V_C L^3}{3EI} = \frac{4V_C L^3}{3EI}$$

COMPATIBILITY $(\delta_C)_1 = (\delta_C)_2$

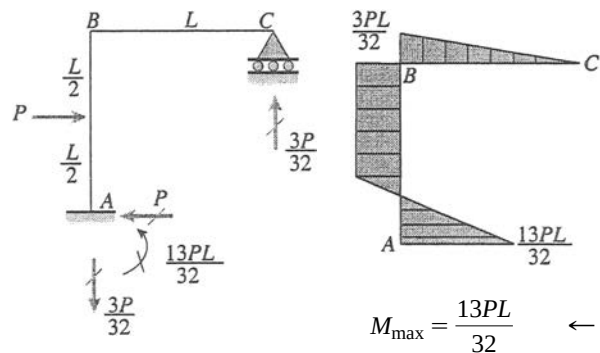
Substitute for $(\delta_C)_1$ and $(\delta_C)_2$ and solve:

$$V_C = \frac{3P}{32} \quad \leftarrow$$

FROM EQUILIBRIUM:

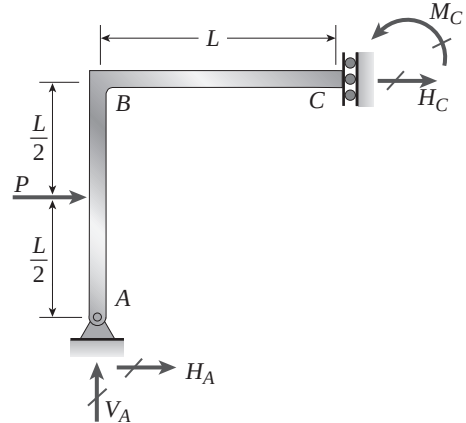
$$V_A = \frac{3P}{32} \quad H_A = P \quad M_A = \frac{13PL}{32} \quad \leftarrow$$

REACTIONS AND BENDING MOMENTS



Problem 10.4-26 The continuous frame ABC has a pinned support at A , a guided support at C , and a rigid corner connection at B (see figure). Members AB and BC each have length L and flexural rigidity EI . A horizontal force P acts at midheight of member AB .

- Find all reactions of the frame.
- What is the largest bending moment $M_{\max}$ in the frame? (Note: Disregard axial deformations in members AB and BC and consider only the effects of bending.)



Solution 10.4-26

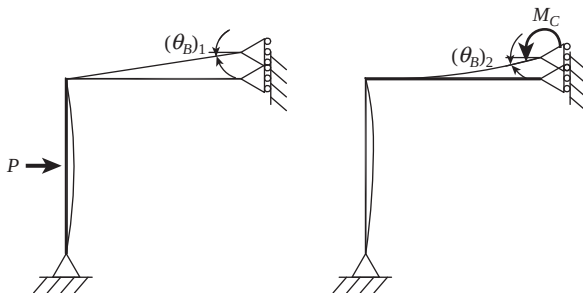
Select M_C as the redundant.

EQUILIBRIUM

$$V_A = 0$$

$$H_A = -\frac{P}{2} - \frac{M_C}{L} \quad H_C = -\frac{P}{2} + \frac{M_C}{L}$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



$$(\theta_C)_1 = \frac{PL^2}{16EI}$$

$$(\theta_C)_1 = \frac{4L}{3EI} M_C$$

COMPATIBILITY

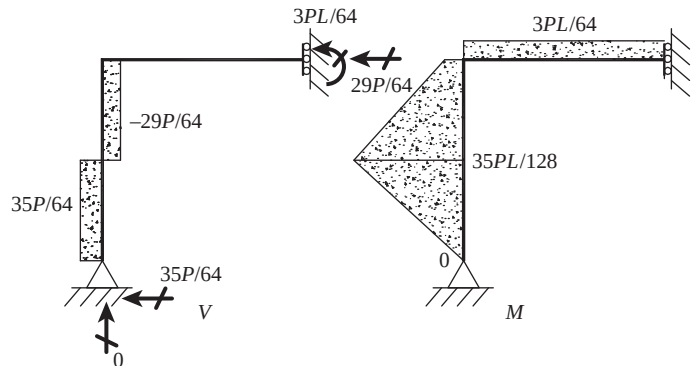
$$\theta_C = (\theta_C)_1 - (\theta_C)_2 = 0 \quad \therefore M_C = \frac{3}{64} PL$$

FROM EQUILIBRIUM

$$H_A = -\frac{P}{2} - \frac{M_C}{L} = -\frac{35}{64} P$$

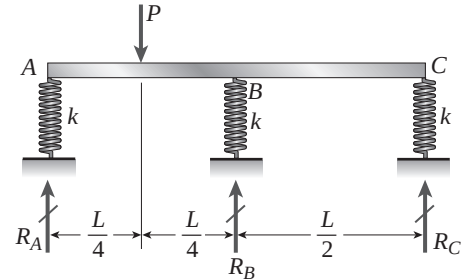
$$H_C = -\frac{P}{2} + \frac{M_C}{L} = -\frac{29}{64} P$$

$$M_{\max} = \frac{35}{128} PL$$

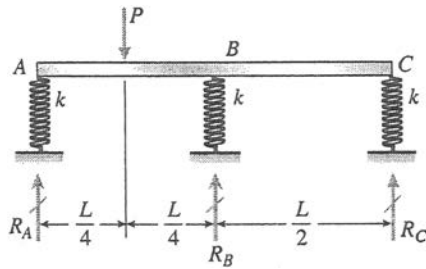


Problem 10.4-27 A wide-flange beam ABC rests on three identical spring supports at points A , B , and C (see figure). The flexural rigidity of the beam is $EI = 6912 \times 10^6 \text{ lb-in.}^2$, and each spring has stiffness $k = 62,500 \text{ lb/in.}$ The length of the beam is $L = 16 \text{ ft.}$

If the load P is 6000 lb, what are the reactions R_A , R_B , and R_C ? Also, draw the shear-force and bending-moment diagrams for the beam, labeling all critical ordinates.



Solution 10.4-27 Beam on three springs

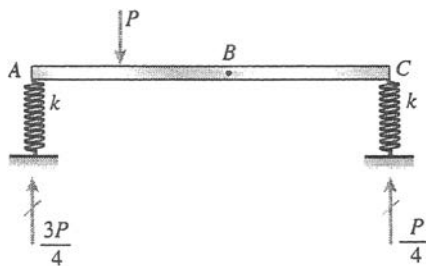


Select R_B as the redundant.

EQUILIBRIUM

$$R_A = \frac{3P}{4} - \frac{R_B}{2} \quad R_C = \frac{P}{4} - \frac{R_B}{2}$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT EQUATIONS



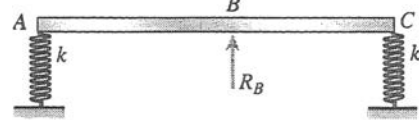
$$(\delta_A)_1 = \frac{3P}{4k}$$

$$(\delta_C)_1 = \frac{P}{4k}$$

$$(\delta_B)_1 = \frac{1}{2}[(\delta_A)_1 + (\delta_C)_1] + \frac{P\left(\frac{L}{4}\right)\left[3L^2 - 4\left(\frac{L}{4}\right)^2\right]}{48EI}$$

(Case 5, Table H-2)

$$(\delta_B)_1 = \frac{P}{2k} + \frac{11PL^3}{768EI} \quad (\text{downward})$$



$$(\delta_A)_2 = \frac{R_B}{2k}$$

$$(\delta_C)_2 = \frac{R_B}{2k}$$

$$\begin{aligned} (\delta_B)_2 &= \frac{1}{2}[(\delta_A)_2 + (\delta_C)_2] + \frac{R_B L^3}{48EI} \\ &= \frac{R_B}{2k} + \frac{R_B L^3}{48EI} \quad (\text{upward}) \end{aligned}$$

$$\text{COMPATIBILITY } (\delta_B)_1 - (\delta_B)_2 = \frac{R_B}{k}$$

Substitute and solve:

$$R_B = P \left(\frac{384EI + 11kL^3}{1152EI + 16kL} \right)$$

$$\text{Let } k^* = \frac{kL^3}{EI} \quad (\text{nondimensional}) \quad \leftarrow$$

$$R_B = \frac{P}{16} \left(\frac{384 + 11k^*}{72 + k^*} \right) \quad \leftarrow$$

FROM EQUILIBRIUM

$$R_A = \frac{P}{32} \left(\frac{1344 + 13k^*}{72 + k^*} \right) \quad \leftarrow$$

$$R_C = \frac{3P}{32} \left(\frac{64 - k^*}{72 + k^*} \right) \quad \leftarrow$$

NUMERICAL VALUES

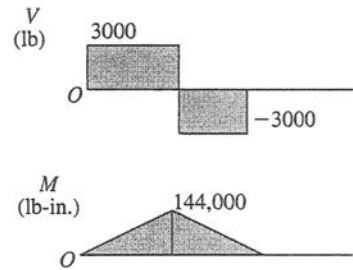
$$EI = 6912 \times 10^6 \text{ lb-in.}^2 \quad k = 62,500 \text{ lb/in.}$$

$$L = 16 \text{ ft} = 192 \text{ in.} \quad P = 6000 \text{ lb}$$

$$k^* = \frac{kL^3}{EI} = 64 \quad R_B = 3000 \text{ lb} \quad \leftarrow$$

$$R_A = 3000 \text{ lb} \quad R_C = 0 \quad \leftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAM



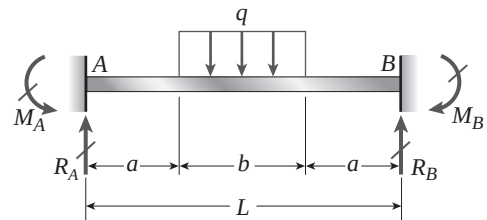
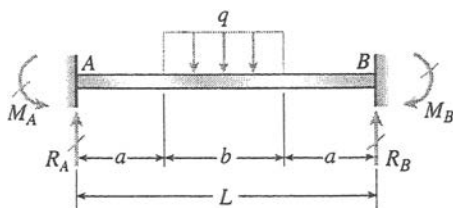
Problem 10.4-28 A fixed-end beam AB of length L is subjected to a uniform load of intensity q acting over the middle region of the beam (see figure).

- Obtain a formula for the fixed-end moments M_A and M_B in terms of the load q , the length L , and the length b of the loaded part of the beam.
- Plot a graph of the fixed-end moment M_A versus the length b of the loaded part of the beam. For convenience, plot the graph in the following nondimensional form:

$$\frac{M_A}{qL^2/12} \quad \text{versus} \quad \frac{b}{L}$$

with the ratio b/L varying between its extreme values of 0 and 1.

- For the special case in which $a = b = L/3$, draw the shear-force and bending-moment diagrams for the beam, labeling all critical ordinates.

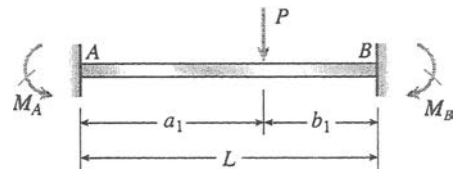
**Solution 10.4-28 Fixed-end Beam**

$$M_A = M_B$$

$$R_A = R_B = \frac{qb}{2}$$

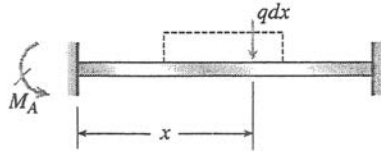
$$a = \frac{L-b}{2}$$

FROM EXAMPLE 10-4, EQ. (10-30a):



$$M_A = \frac{Pa_1b_1^2}{L^2}$$

FOR THE PARTIAL UNIFORM LOAD



$$dM_A = \frac{(qdx)(x)(L-x)^2}{L^2}$$

$$M_A = \int_a^{a+b} dM_A = \int_{(L-b)/2}^{(L+b)/2} dM_A$$

$$= \frac{q}{L^2} \int_{(L-b)/2}^{(L+b)/2} x(L-x)^2 dx$$

$$= \frac{q}{L^2} \int_{(L-b)/2}^{(L+b)/2} (L^2x - 2Lx^2 + x^3) dx$$

$$= \frac{q}{L^2} \left[\frac{L^2x^2}{2} - \frac{2Lx^3}{3} + \frac{x^4}{4} \right]_{(L-b)/2}^{(L+b)/2}$$

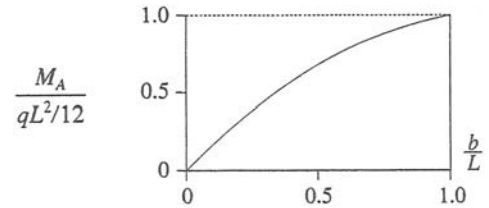
... (lengthy substitution) ...

$$= \frac{qb}{24L} (3L^2 - b^2)$$

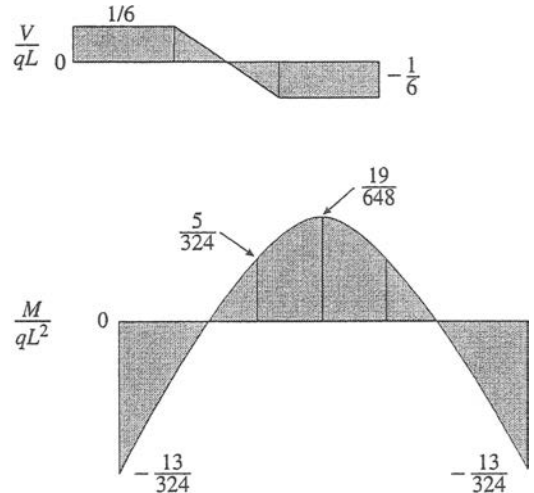
$$(a) \quad M_A = M_B = \frac{qb}{24L} (3L^2 - b^2) \quad \leftarrow$$

(b) GRAPH OF FIXED-END MOMENT

$$\frac{M_A}{qL^2/12} = \frac{b}{2L} \left(3 - \frac{b^2}{L^2} \right)$$

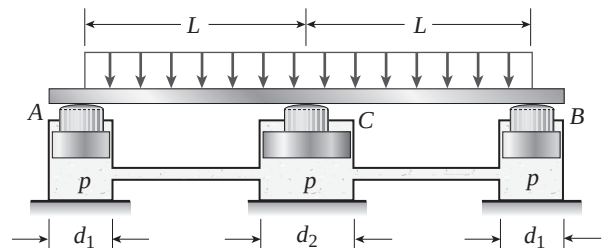
(c) SPECIAL CASE $a = b = L/3$

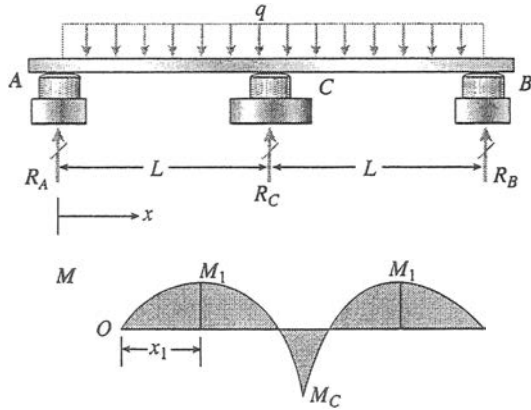
$$R_A = R_B = \frac{qL}{6} \quad M_A = M_B = \frac{13qL^2}{324}$$



Problem 10.4-29 A beam supporting a uniform load of intensity q throughout its length rests on pistons at points A , C , and B (see figure). The cylinders are filled with oil and are connected by a tube so that the oil pressure on each piston is the same. The pistons at A and B have diameter d_1 , and the pistons at C has diameter d_2 .

- Determine the ratio of d_2 to d_1 so that the largest bending moment in the beam is as small as possible.
- Under these optimum conditions, what is the largest bending moment $M_{\max}$ in the beam?
- What is the difference in elevation between point C and the end supports?



Solution 10.4-29 Beam supported by pistons


$$\text{BENDING MOMENT } M = R_A x - \frac{qx^2}{2}$$

LOCATION OF MAXIMUM POSITIVE MOMENT

$$\frac{dM}{dx} = 0 \quad R_A - qx = 0 \quad x_1 = \frac{R_A}{q}$$

MAXIMUM POSITIVE MOMENT

$$M_1 = (M)_{x=x_1} = \frac{R_A^2}{2q}$$

MAXIMUM NEGATIVE MOMENT

$$M_C = (M)_{x=L} = R_A L - \frac{qL^2}{2}$$

FOR THE SMALLEST MAXIMUM MOMENT:

$$|M_1| = |M_C| \text{ or } M_1 = -M_C$$

$$\frac{R_A^2}{2q} = -R_A L + \frac{qL^2}{2}$$

$$\text{Solve for } R_A: R_A = qL(\sqrt{2} - 1)$$

EQUILIBRIUM

$$\sum F_{\text{vert}} = 0 \quad 2R_A + R_C - 2qL = 0$$

$$R_C = 2qL(2 - \sqrt{2})$$

REACTIONS BASED UPON PRESSURE

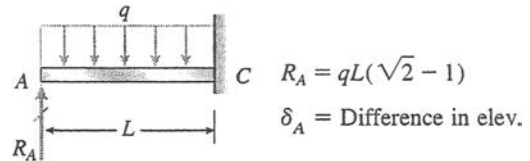
$$R_A = R_B = p \left(\frac{\pi d_1^2}{4} \right) \quad R_C = p \left(\frac{\pi d_2^2}{4} \right)$$

$$(a) \therefore \frac{d_2}{d_1} = \sqrt{\frac{R_C}{R_A}} = \sqrt{\frac{2(2 - \sqrt{2})}{\sqrt{2} - 1}} = \sqrt[4]{8} = 1.682 \quad \leftarrow$$

$$(b) M_{\max} = M_1 = \frac{R_A^2}{2q} = \frac{qL^2}{2}(3 - 2\sqrt{2}) = 0.08579 qL^2 \quad \leftarrow$$

(c) DIFFERENCE IN ELEVATION

By symmetry, beam has zero slope at C.

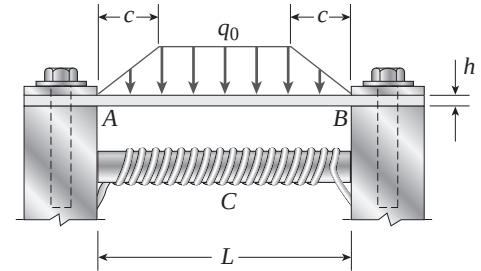


$$\delta_A = \frac{R_A L^3}{3EI} - \frac{qL^4}{8EI} = \frac{qL^4}{24EI}(8\sqrt{2} - 11) = 0.01307 qL^4/EI \quad \leftarrow$$

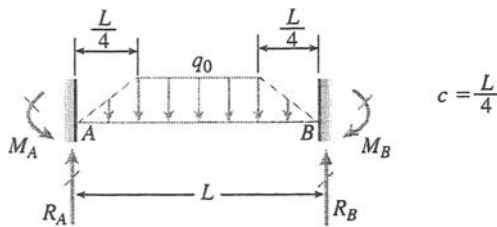
 Point C is below points A and B by the amount $0.01307 qL^4/EI$.

Problem 10.4-30 A thin steel beam AB used in conjunction with an electromagnet in a high-energy physics experiment is securely bolted to rigid supports (see figure). A magnetic field produced by coils C results in a force acting on the beam. The force is trapezoidally distributed with maximum intensity $q_0 = 18 \text{ kN/m}$. The length of the beam between supports is $L = 200 \text{ mm}$ and the dimension c of the trapezoidal load is 50 mm . The beam has a rectangular cross section with width $b = 60 \text{ mm}$ and height $h = 20 \text{ mm}$.

Determine the maximum bending stress $\sigma_{\max}$ and the maximum deflection $\delta_{\max}$ for the beam. (Disregard any effects of axial deformations and consider only the effects of bending. Use $E = 200 \text{ GPa}$.)



Solution 10.4-30 Fixed-end beam (trapezoidal load)

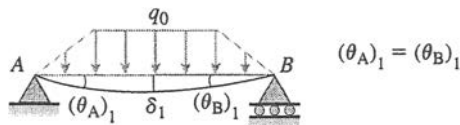


FROM SYMMETRY AND EQUILIBRIUM

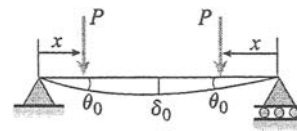
$$M_A = M_B \quad R_A = R_B = \frac{3q_0L}{8}$$

SELECT M_A AND M_B AS REDUNDANTS

RELEASED STRUCTURE WITH APPLIED LOAD



Consider the following beam from Case 6, Table H-2:



$$\theta_0 = \frac{Px(L-x)}{2EI} \quad \delta_0 = \frac{Px}{24EI}(3L^2 - 4x^2)$$

Consider the load P as an element of the distributed load.

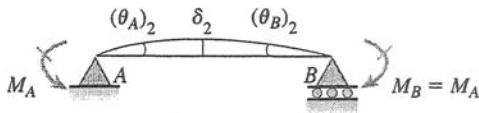
Replace P by qdx , where

$$q = \frac{4q_0x}{L} \quad x \text{ from } 0 \text{ to } L/4$$

$$q = q_0 \quad x \text{ from } L/4 \text{ to } L/2$$

$$\begin{aligned} (\theta_A)_1 &= \frac{1}{2EI} \int_0^{L/2} \left(\frac{4q_0x}{L} \right) (x)(L-x) dx \\ &\quad + \frac{1}{2EI} \int_{L/4}^{L/2} q_0x(L-x) dx \\ &= \frac{13q_0L^3}{1536EI} + \frac{11q_0L^3}{384EI} = \frac{19q_0L^3}{512EI} \end{aligned}$$

$$\begin{aligned}
 \delta_1 &= \frac{1}{24EI} \int_0^{L/4} \left(\frac{4q_0x}{L} \right) (x)(3L^2 - 4x^2) dx \\
 &\quad + \frac{1}{24EI} \int_{L/4}^{L/2} q_0x(3L^2 - 4x^2) dx \\
 &= \frac{19q_0L^4}{7680EI} + \frac{19q_0L^4}{2048EI} = \frac{361q_0L^4}{30,720EI}
 \end{aligned}$$



RELEASED STRUCTURE WITH REDUNDANTS

$$(\theta_A)_2 = (\theta_B)_2 \quad M_B = M_A$$

From Case 10, Table H-2,

$$(\theta_A)_2 = \frac{M_AL}{2EI} \quad \delta_2 = \frac{M_AL^2}{8EI}$$

COMPATIBILITY

$$\theta_A = (\theta_A)_1 - (\theta_A)_2 = 0$$

$$\frac{19q_0L^3}{512EI} - \frac{M_AL}{2EI} = 0 \quad M_A = \frac{19q_0L^2}{256}$$

DEFLECTION AT THE MIDPOINT

$$\begin{aligned}
 \delta_{\max} &= \delta_1 - \delta_2 = \frac{361q_0L^4}{30,720EI} - \frac{M_AL^2}{8EI} \\
 &= \frac{361q_0L^4}{30,720EI} - \left(\frac{19q_0L^2}{256} \right) \left(\frac{L^2}{8EI} \right) \\
 &= \frac{19q_0L^4}{7680EI}
 \end{aligned}$$

BENDING MOMENT AT THE MIDPOINT

$$\begin{aligned}
 M_C &= R_A \left(\frac{L}{2} \right) - M_A - \frac{q_0L^2}{24} - \frac{q_0L^2}{32} \\
 &= \frac{3q_0L}{8} \left(\frac{L}{2} \right) - \frac{19q_0L^2}{256} - \frac{7q_0L^2}{96} = \frac{31q_0L^2}{768}
 \end{aligned}$$

MAXIMUM BENDING MOMENT

$$M_A > M_C \quad \therefore M_{\max} = M_A = \frac{19q_0L^2}{256}$$

NUMERICAL VALUES

$$q_0 = 18 \text{ kN/m} \quad L = 200 \text{ mm} \quad b = 60 \text{ mm}$$

$$h = 20 \text{ mm} \quad E = 200 \text{ GPa}$$

$$S = \frac{bh^2}{6} = 4.0 \times 10^{-6} \text{ m}^3$$

$$I = \frac{bh^3}{12} = 40 \times 10^{-9} \text{ m}^4$$

$$M_{\max} = \frac{19q_0L^2}{256} = 53.44 \text{ N}\cdot\text{m}$$

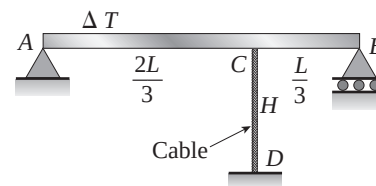
$$\sigma_{\max} = \frac{M_{\max}}{S} = 13.4 \text{ MPa} \quad \leftarrow$$

$$\delta_{\max} = \frac{19q_0L^4}{7680EI} = 0.00891 \text{ mm} \quad \leftarrow$$

Temperature Effects

The beams described in the problems for Section 10.5 have constant flexural rigidity EI .

Problem 10.5-1 A cable CD of length H is attached to the third point of a simple beam AB of length L (see figure). The moment of inertia of the beam is I , and the effective cross-sectional area of the cable is A . The cable is initially taut but without any initial tension.



- Obtain a formula for the tensile force S in the cable when the temperature drops uniformly by ΔT degrees, assuming that the beam and cable are made of the same material (modulus of elasticity E and coefficient of thermal expansion α). Use the method of superposition in the solution.
- Repeat part (a) assuming a wood beam and steel cable.

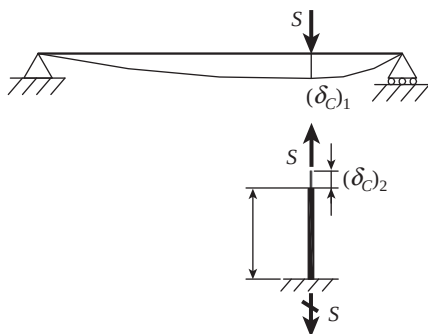
Solution 10.5-1

ΔT = Decrease in temperature. Use method of superposition. Select tensile force S in the cable as redundant.

I = Moment of inertia of beam

A = Cross-sectional area of cable

RELEASED STRUCTURE



- BEAM AND CABLE ARE SAME MATERIAL

$$(\delta_C)_1 = \frac{4SL^3}{243EI} \quad (\text{downward})$$

$$\text{CABLE} \quad (\delta_C)_2 = \alpha H(\Delta T) - \frac{SH}{EA} \quad (\text{downward})$$

COMPATIBILITY

$$(\delta_C)_1 = (\delta_C)_2 \quad \frac{4SL^3}{243EI} = \alpha H(\Delta T) - \frac{SH}{EA}$$

$$\text{SOLVE FOR } S: S = \frac{243EIAH\alpha(\Delta T)}{4AL^3 + 243IH} \quad \leftarrow$$

- WOOD BEAM, STEEL CABLE

$$(\delta_C)_1 = \frac{4SL^3}{243E_W I} \quad (\text{downward})$$

$$\text{CABLE} \quad (\delta_C)_2 = \alpha_S H(\Delta T) - \frac{SH}{E_S A} \quad (\text{downward})$$

COMPATIBILITY

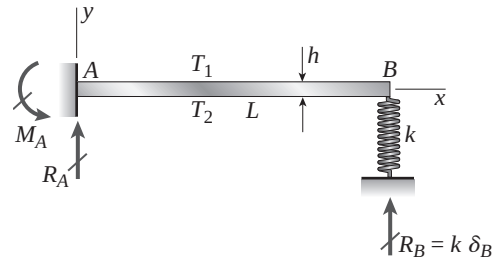
$$(\delta_C)_1 = (\delta_C)_2 \quad \frac{4SL^3}{243E_W I} = \alpha_S H(\Delta T) - \frac{SH}{E_S A}$$

SOLVE FOR S :

$$S = \frac{243E_S E_W IAH\alpha_S(\Delta T)}{4AL^3E_S + 243IHE_W} \quad \leftarrow$$

Problem 10.5-2 A propped cantilever beam, fixed at the left-hand end A and simply supported at the right-hand end B , is subjected to a temperature differential with temperature T_1 on its upper surface and T_2 on its lower surface (see figure).

- Find all reactions for this beam. Use the method of superposition in the solution. Assume the spring support is unaffected by temperature.
- What are the reactions when $k \rightarrow \infty$?



Probs. 10.5-2 and 10.5-3

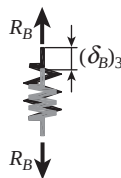
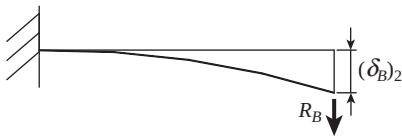
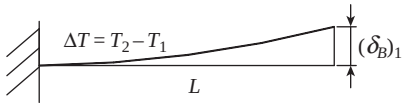
Solution 10.5-2

- (a) REACTIONS ASSUMING AN ELASTIC SPRING AT B

Use the method of superposition.

Select R_B as the redundant.

RELEASED STRUCTURE



$$(\delta_B)_1 = \frac{\alpha(T_2 - T_1)L^2}{2h}$$

$$(\delta_B)_2 = \frac{R_B L^3}{3EI}$$

$$(\delta_B)_3 = (\delta_B)_1 - (\delta_B)_2 = \frac{R_B}{k}$$

$$R_B = \frac{\alpha(T_2 - T_1)L^2}{2h} \left(\frac{3EI}{3EI + L^3k} \right) \quad (\text{downward})$$

FROM EQUILIBRIUM

$$R_A = -R_B = -\frac{\alpha(T_2 - T_1)L^2}{2h} \left(\frac{3EI}{3EI + L^3k} \right) \quad (\text{upward})$$

$$M_A = R_B L = \frac{\alpha(T_2 - T_1)L^3}{2h} \left(\frac{3EI}{3EI + L^3k} \right) \quad (\text{counterclockwise}) \quad \leftarrow$$

- (b) REACTIONS ASSUMING AN SPRING AT B IS RIGID

$$R_B = \frac{3EI\alpha(T_2 - T_1)}{2hL} \quad (\text{downward})$$

$$R_A = -R_B = -\frac{3EI\alpha(T_2 - T_1)}{2hL} \quad (\text{upward})$$

$$M_A = R_B L = \frac{3EI\alpha(T_2 - T_1)}{2h} \quad (\text{counterclockwise}) \quad \leftarrow$$

Problem 10.5-3 Solve the preceding problem by integrating the differential equation of the deflection curve.

Solution 10.5-3

(a) DIFFERENTIAL EQUATION (Eq. 10-45b)

$$EIv'' = M + \frac{\alpha EI(T_2 - T_1)}{h}$$

$$EIv'' = -R_B(L - x) + \frac{\alpha EI(T_2 - T_1)}{h}$$

$$EIv' = -R_B Lx + R_B \frac{x^2}{2} + \frac{\alpha EI(T_2 - T_1)}{h}x + C_1$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv = -R_B L \frac{x^2}{2} + R_B \frac{x^3}{6} + \frac{\alpha EI(T_2 - T_1)}{2h}x^2 + C_2$$

B.C. 2 $v(0) = 0 \quad \therefore C_2 = 0$

B.C. 3 $v(L) = \delta_B = \frac{R_B}{k}$

$$\therefore R_B = \frac{\alpha EI(T_2 - T_1)L^2}{2h} \left(\frac{3k}{3EI + L^3k} \right)$$

(downward) ←

FROM EQUILIBRIUM

$$R_A = -R_B = -\frac{\alpha EI(T_2 - T_1)L^2}{2h} \left(\frac{3k}{3EI + L^3k} \right)$$

(upward) ←

$$M_A = R_B L = \frac{\alpha EI(T_2 - T_1)L^3}{2h} \left(\frac{3k}{3EI + L^3k} \right)$$

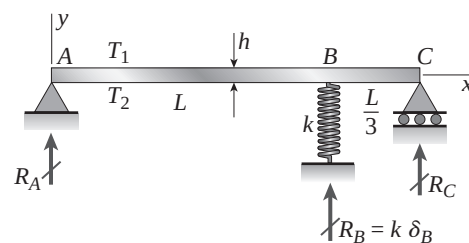
(counterclockwise) ←

(b) SAME REACTIONS AS IN 10.5-2(b) WHEN

$$k \rightarrow \infty$$

Problem 10.5-4 A two-span beam with spans of lengths L and $L/3$ is subjected to a temperature differential with temperature T_1 on its upper surface and T_2 on its lower surface (see figure).

- Determine all reactions for this beam. Use the method of superposition in the solution. Assume the spring support is unaffected by temperature.
- What are the reactions when $k \rightarrow \infty$?



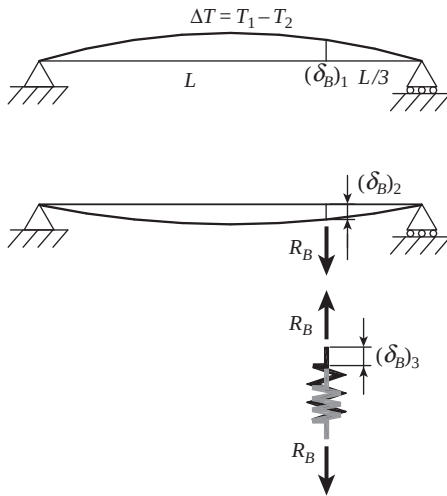
Probs. 10.5-4 and 10.5-5

Solution 10.5-4

(a) Use the method of superposition.

 Select R_B as the redundant.

RELEASED STRUCTURE



$$(\delta_B)_1 = \frac{\alpha(T_1 - T_2)L^2}{6h}$$

$$(\delta_B)_2 = \frac{R_B L^3}{36EI}$$

COMPATIBILITY

$$(\delta_B)_3 = (\delta_B)_1 - (\delta_B)_2 = \frac{R_B}{k}$$

$$R_B = -\frac{\alpha(T_1 - T_2)L^2}{h} \left(\frac{6EI k}{36EI + L^3 k} \right) \quad (\text{downward})$$

FROM EQUILIBRIUM

$$R_A + R_B + R_C = 0$$

$$\sum M_C = 0:$$

$$R_A = \frac{1}{4} R_B = \frac{\alpha(T_1 - T_2)L^2}{2h} \left(\frac{3EI k}{36EI + L^3 k} \right) \quad (\text{upward})$$

$$R_C = \frac{3}{4} R_B = \frac{\alpha(T_1 - T_2)L^2}{2h} \left(\frac{9EI k}{36EI + L^3 k} \right) \quad (\text{upward})$$

$$(b) R_B = -\frac{6EI\alpha(T_1 - T_2)}{Lh} \quad (\text{downward})$$

$$R_A = \frac{3EI\alpha(T_1 - T_2)}{2Lh} \quad (\text{upward})$$

$$R_C = \frac{9EI\alpha(T_1 - T_2)}{2Lh} \quad (\text{upward})$$

Problem 10.5-5 Solve the preceding problem by integrating the differential equation of the deflection curve.

Solution 10.5-5

(a) EQUILIBRIUM

$$R_A + R_B + R_C = 0$$

$$\sum M_C = 0: R_A = -\frac{1}{4} R_B$$

$$\sum M_A = 0: R_C = -\frac{3}{4} R_B$$

DIFFERENTIAL EQUATION (Eq. 10-45b)

 For $0 \leq x \leq L$,

$$EIv'' = R_A x + \frac{\alpha EI(T_1 - T_2)}{h}$$

$$EIv' = \frac{1}{2} R_A x^2 + \frac{\alpha EI(T_1 - T_2)}{h} x + C_1 \quad (1)$$

$$EIv = \frac{1}{6} R_A x^3 + \frac{\alpha EI(T_1 - T_2)}{2h} x^2 + C_1 x + C_2 \quad (2)$$

$$\text{B.C.1 } v(0) = 0 \quad \therefore C_2 = 0$$

For $L \leq x \leq \frac{4}{3}L$,

$$EIv'' = R_A x + \frac{\alpha EI(T_1 - T_2)}{h} - 4R_A(x - L)$$

$$EIv'' = -3R_A x + \frac{\alpha EI(T_1 - T_2)}{h} + 4R_A L$$

$$EIv' = -\frac{3}{2}R_A x^2 + \frac{\alpha EI(T_1 - T_2)}{h}x + 4R_A Lx + C_3 \quad (3)$$

$$EIv = -\frac{1}{2}R_A x^3 + \frac{\alpha EI(T_1 - T_2)}{2h}x^2 + 2R_A Lx^2 + C_3x + C_4 \quad (4)$$

B.C. 2 Continuity condition at point B:

At $x = L$: $(v')_{\text{left}} = (v')_{\text{right}}$

$$C_1 = 2R_A L^2 + C_3$$

B.C. 3 Continuity condition at point B:

At $x = L$: $(v)_{\text{left}} = (v)_{\text{right}}$

$$\text{B.C. 4 } v\left(\frac{4}{3}L\right) = 0$$

$$\text{B.C. 5 } v(L)_{\text{left}} = v(L)_{\text{right}} = \frac{R_B}{k}$$

$$R_A = \frac{\alpha(T_1 - T_2)L^2}{2h} \left(\frac{3EI k}{36EI + L^3 k} \right) \quad (\text{upward})$$

$$R_B = -\frac{\alpha(T_1 - T_2)L^2}{h} \left(\frac{6EI k}{36EI + L^3 k} \right) \quad (\text{downward})$$

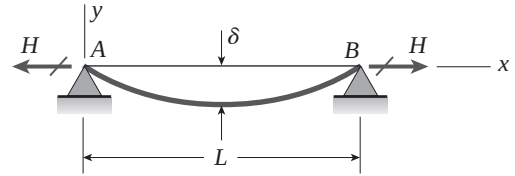
$$R_C = \frac{\alpha(T_1 - T_2)L^2}{2h} \left(\frac{9EI k}{36EI + L^3 k} \right) \quad (\text{upward})$$

(b) SAME REACTIONS AS IN 10.5-4(b) WHEN $k \rightarrow \infty$

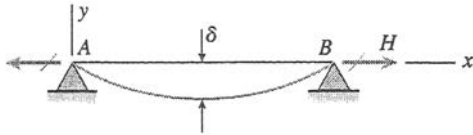
Longitudinal Displacements at the Ends of Beams

Problem 10.6-1 Assume that the deflected shape of a beam AB with *immovable* pinned supports (see figure) is given by the equation $v = -\delta \sin \pi x/L$, where δ is the deflection at the midpoint of the beam and L is the length. Also, assume that the beam has constant axial rigidity EA .

- Obtain formulas for the longitudinal force H at the ends of the beam and the corresponding axial tensile stress σ_t .
- For an aluminum-alloy beam with $E = 10 \times 10^6$ psi, calculate the tensile stress σ_t when the ratio of the deflection δ to the length L equals 1/200, 1/400, and 1/600.



Solution 10.6-1 Beam with immovable supports



$$(a) \quad v = -\delta \sin \frac{\pi x}{L} \quad \frac{dv}{dx} = -\frac{\pi \delta}{L} \cos \frac{\pi x}{L}$$

$$\text{Eq. (10-55): } \lambda = \frac{1}{2} \int_0^L \left(\frac{dv}{dx} \right)^2 dx = \frac{\pi^2 \delta^2}{4L}$$

$$\text{Eq. (10-58): } H = \frac{EA\lambda}{L} = \frac{\pi^2 EA \delta^2}{4L^2} \quad \leftarrow$$

$$\text{Eq. (10-59): } \sigma_t = \frac{H}{A} = \frac{\pi^2 E \delta^2}{4L^2} \quad \leftarrow$$

(b) ALUMINUM ALLOY

$$E = 10 \times 10^6 \text{ psi}$$

$$\sigma_t = 24.67 \times 10^6 \left(\frac{\delta}{L} \right)^2 \text{ (psi)}$$

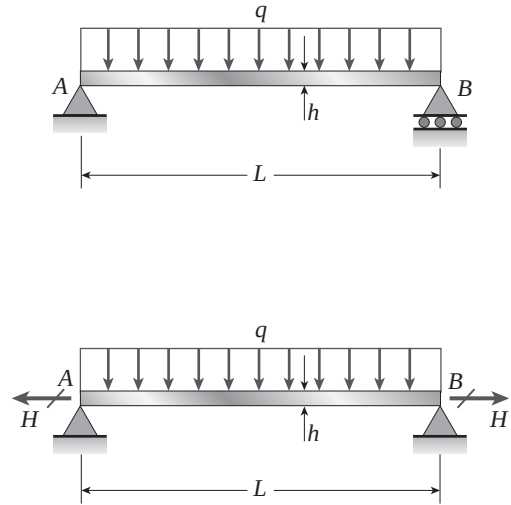
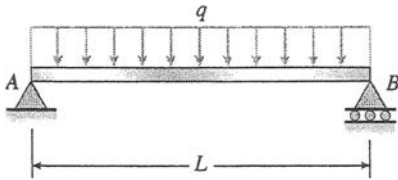
$\frac{\delta}{L}$	$\frac{1}{200}$	$\frac{1}{400}$	$\frac{1}{600}$
σ_t (psi)	617	154	69

Note: The axial stress increases as the deflection increases.

Problem 10.6-2

- (a) A simple beam AB with length L and height h supports a uniform load of intensity q (see the *first part* of the figure). Obtain a formula for the curvature shortening λ of this beam. Also, obtain a formula for the maximum bending stress σ_b in the beam due to the load q .
- (b) Now assume that the ends of the beam are pinned so that curvature shortening is prevented and a horizontal force H develops at the supports (see the *second part* of the figure). Obtain a formula for the corresponding axial tensile stress σ_t .
- (c) Using the formulas in parts (a) and (b), calculate the curvature shortening λ , the maximum bending stress σ_b , and the tensile stress σ_t for the following steel beam: length $L = 3$ m, height $h = 300$ mm, modulus of elasticity $E = 200$ GPa, and moment of inertia $I = 36 \times 10^6 \text{ mm}^4$. Also, the load on the beam has intensity $q = 25$ kN/m.

Compare the tensile stress σ_t produced by the axial forces with the maximum bending stress σ_b produced by the uniform load.

**Solution 10.6-2 Beam with uniform load**

h = height of beam

(a) CURVATURE SHORTENING

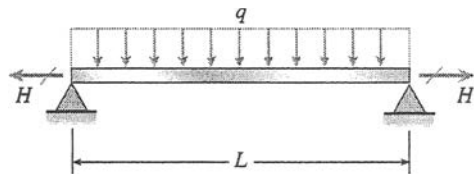
From Case 1, Table H-2,

$$\frac{dv}{dx} = -\frac{q}{24EI}(L^3 - 6Lx^2 - 4x^3)$$

$$\begin{aligned} \text{Eq. (10-55): } \lambda &= \frac{1}{2} \int_0^L \left(\frac{dv}{dx} \right)^2 dx \\ &= \frac{17q^2L^7}{40,320E^2I^2} \quad \leftarrow \end{aligned}$$

BENDING STRESS

$$\begin{aligned} M_{\max} &= \frac{qL^2}{8} \quad c = \frac{h}{2} \\ \sigma_b &= \frac{Mc}{I} = \frac{qhL^2}{16I} \quad \leftarrow \end{aligned}$$

(b) IMMOVABLE SUPPORTS

$$\text{Eq. (10-58): } H = \frac{EA\lambda}{L}$$

$$\text{Eq. (10-59): } \sigma_t = \frac{H}{A} = \frac{E\lambda}{L} = \frac{17q^2L^6}{40,320EI^2} \quad \leftarrow$$

(c) NUMERICAL VALUES $q = 25$ kN/m

$$L = 3 \text{ m} \quad h = 300 \text{ mm} \quad E = 200 \text{ GPa}$$

$$I = 36 \times 10^6 \text{ mm}^4 \quad \lambda = 0.01112 \text{ mm} \quad \leftarrow$$

$$\sigma_b = 117.2 \text{ MPa} \quad \sigma_t = 0.7411 \text{ MPa} \quad \leftarrow$$

The bending stress is much larger than the axial tensile stress due to curvature shortening.

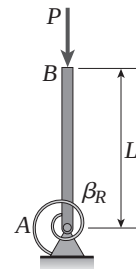
11

Columns

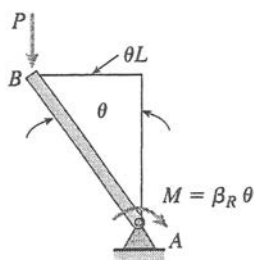
Idealized Buckling Models

Problem 11.2-1 The figure shows an idealized structure consisting of one or more **rigid bars** with pinned connections and linearly elastic springs. Rotational stiffness is denoted β_R , and translational stiffness is denoted β .

Determine the critical load P_{cr} for the structure.



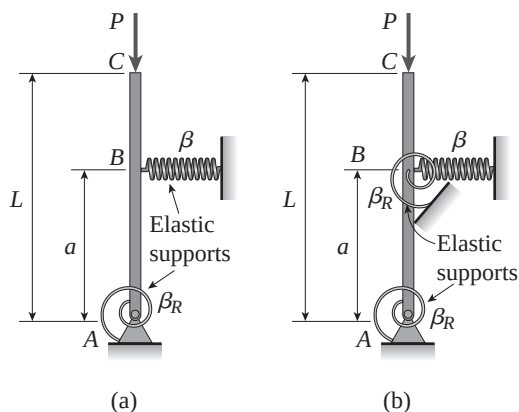
Solution 11.2-1 Rigid bar AB



$$\begin{aligned}\sum M_A &= 0 \\ P(\theta L) - \beta_R \theta &= 0 \\ P_{cr} &= \frac{\beta_R}{L} \quad \leftarrow\end{aligned}$$

Problem 11.2-2 The figure shows an idealized structure consisting of one or more rigid bars with pinned connections and linearly elastic springs. Rotational stiffness is denoted β_R , and translational stiffness is denoted β .

- Determine the critical load P_{cr} for the structure from the figure part a.
- Find P_{cr} if another rotational spring is added at B from the figure part b.



Solution 11.2-2

$$(a) \sum M_A = 0$$

$$P\theta L - \beta\theta a^2 - \beta_R\theta = 0$$

$$P_{cr} = \frac{\beta a^2 + \beta_R}{L} \quad \leftarrow$$

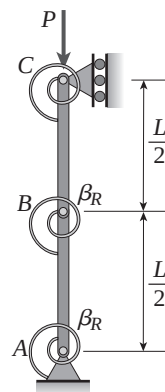
$$(b) \sum M_A = 0$$

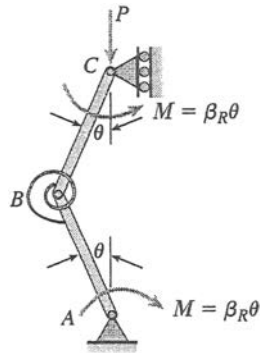
$$P\theta L - \beta\theta a^2 - 2\beta_R\theta = 0$$

$$P_{cr} = \frac{\beta a^2 + 2\beta_R}{L} \quad \leftarrow$$

Problem 11.2-3 The figure shows an idealized structure consisting of one or more **rigid bars** with pinned connections and linearly elastic springs. Rotational stiffness is denoted β_R , and translational stiffness is denoted β .

Determine the critical load P_{cr} for the structure.



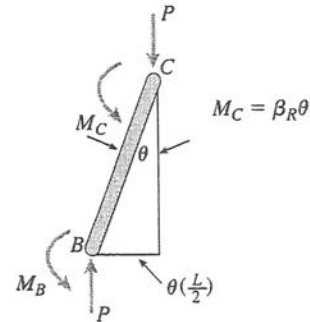
Solution 11.2-3 Two rigid bars with a pin connection

$\sum M_A = 0$ Shows that there are no horizontal reactions at the supports.

FREE-BODY DIAGRAM OF BAR BC

$$M_C = \beta_R \theta$$

$$M_B = \beta_R (2\theta)$$



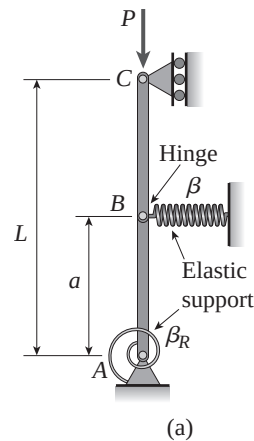
$$\sum M_B = 0 \quad M_B + M_C - P\theta\left(\frac{L}{2}\right) = 0$$

$$\beta_R(2\theta) + \beta_R\theta = \frac{PL\theta}{2}$$

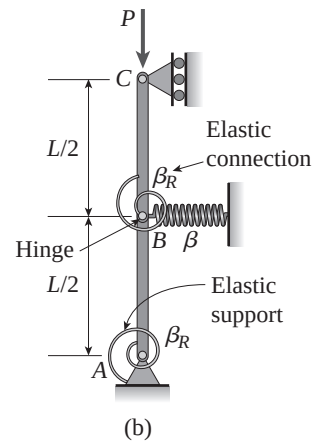
$$P_{cr} = \frac{6\beta_R}{L} \quad \leftarrow$$

Problem 11.2-4 The figure shows an idealized structure consisting of bars AB and BC which are connected using a hinge at B and linearly elastic springs at A and B . Rotational stiffness is denoted β_R and translational stiffness is denoted β .

- Determine the critical load P_{cr} for the structure from the figure part a.
- Find P_{cr} if an elastic connection is now used to connect bar segments AB and BC from the figure part b.



(a)



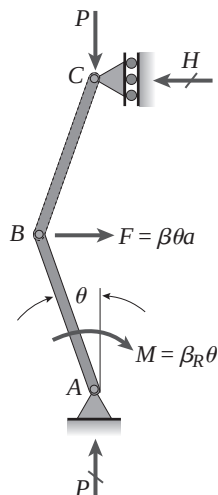
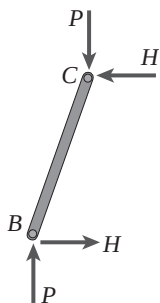
(b)

Solution 11.2-4

(a) $\sum M_A = 0$

$$\beta\theta a^2 + \beta_R\theta - HL = 0$$

$$H = \frac{\beta\theta a^2 + \beta_R\theta}{L}$$

FREE-BODY DIAGRAM OF BAR BC 

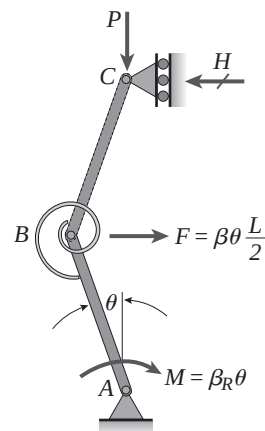
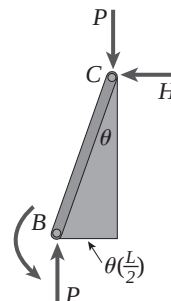
$$\sum M_B = 0 \quad H(L - a) - P(\theta a) = 0$$

$$P_{cr} = \frac{(\beta a^2 + \beta_R)(L - a)}{aL} \quad \leftarrow$$

(b) $\sum M_A = 0$

$$\beta\theta\left(\frac{L}{2}\right)^2 + \beta_R\theta - HL = 0$$

$$H = \frac{\beta\theta L^2 + 4\beta_R\theta}{4L}$$

FREE-BODY DIAGRAM OF BAR BC 

$$M_B = \beta_R(2\theta)$$

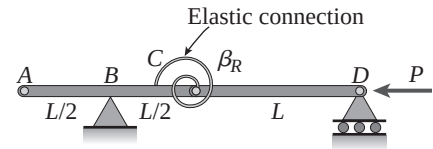
$$\sum M_B = 0 \quad H\left(\frac{L}{2}\right) - P\left(\theta\frac{L}{2}\right) + M_B = 0$$

$$\frac{\beta\theta L^2 + 4\beta_R\theta}{4L}\left(\frac{L}{2}\right)$$

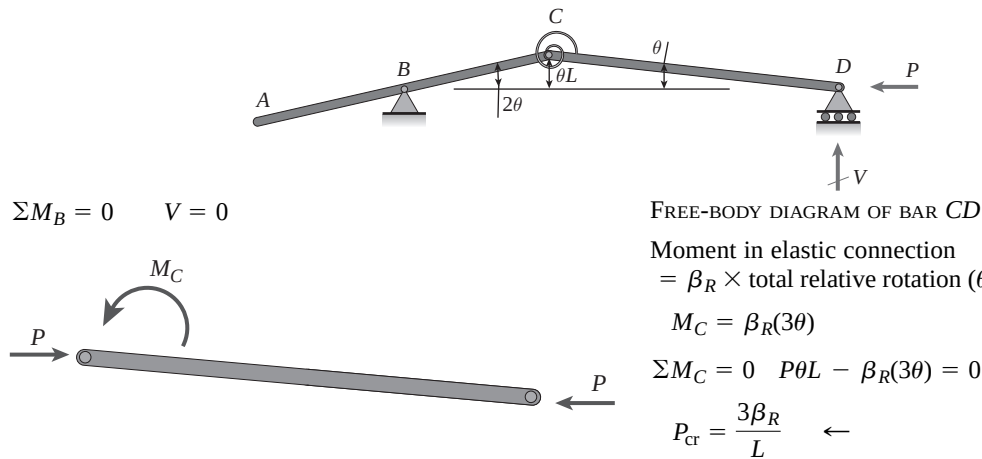
$$- P\left(\theta\frac{L}{2}\right) + \beta_R(2\theta) = 0$$

$$P_{cr} = \frac{\beta L^2 + 20\beta_R}{4L} \quad \leftarrow$$

Problem 11.2-5 The figure shows an idealized structure consisting of two rigid bars joined by an elastic connection with rotational stiffness β_R . Determine the critical load P_{cr} for the structure.

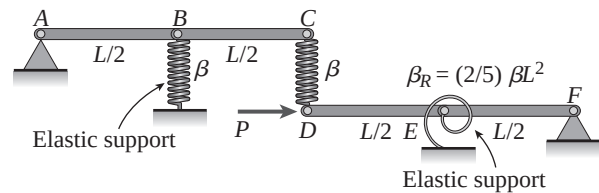


Solution 11.2-5



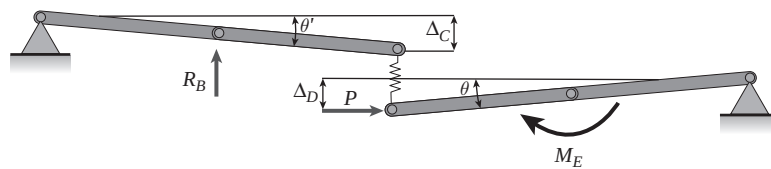
Problem 11.2-6 The figure shows an idealized structure consisting of rigid bars ABC and DEF joined by linearly elastic spring β between C and D . The structure is also supported by translational elastic support β at B and rotational elastic support at β_R at E .

Determine the critical load P_{cr} for the structure.



Solution 11.2-6

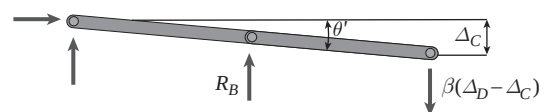
FREE-BODY DIAGRAM OF DEFORMED STRUCTURE



$$\Delta_C = L\theta' \quad \Delta_D = L\theta \quad R_B = \beta \left(\theta' \frac{L}{2} \right)$$

$$M_E = \beta_R \theta = \frac{2}{5} \beta L^2 \theta$$

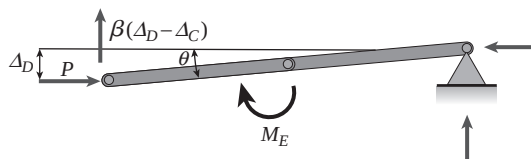
FREE-BODY DIAGRAM OF MEMBER ABC



$$\Sigma M_A = 0 \quad R_B \frac{L}{2} - \beta(\Delta_D - \Delta_C)L = 0$$

$$\beta \left(\theta' \frac{L}{2} \right) \frac{L}{2} - \beta(L\theta - L\theta')L = 0$$

FREE-BODY DIAGRAM OF MEMBER DEF



$$\theta' = \frac{4}{5} \theta$$

$$\Sigma M_F = 0$$

$$P\Delta_D - \beta(\Delta_D - \Delta_C)L - M_E = 0$$

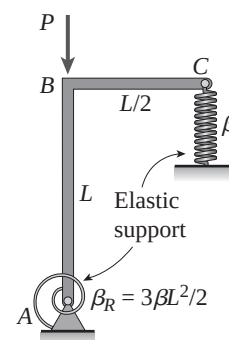
$$P(L\theta) - \beta \left[L\theta - L \left(\frac{4}{5} \theta \right) \right]$$

$$L - \frac{2}{5} \beta L^2 \theta = 0$$

$$P_{cr} = \frac{3}{5} \beta L \quad \leftarrow$$

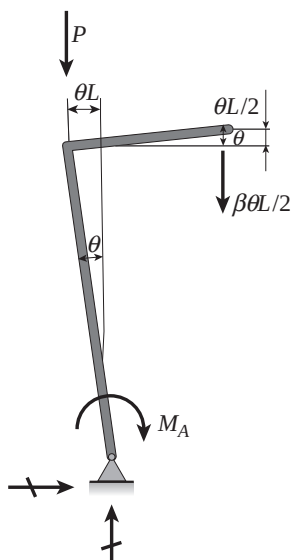
Problem 11.2-7 The figure shows an idealized structure consisting of an L-shaped rigid bar structure supported by linearly elastic springs at A and C. Rotational stiffness in denoted β_R and translational stiffness is denoted β .

Determine the critical load P_{cr} for the structure.



Solution 11.2-7

FREE-BODY DIAGRAM OF DEFORMED STRUCTURE



$$M_A = \beta_R \theta = \frac{3}{2} \beta L^2 \theta$$

$$\Sigma M_A = 0$$

$$M_A - P(\theta L) + \beta \theta \left(\frac{L}{2} \right)^2 = 0$$

$$\frac{3}{2} \beta L^2 \theta - P(\theta L) + \beta \theta \left(\frac{L}{2} \right)^2 = 0$$

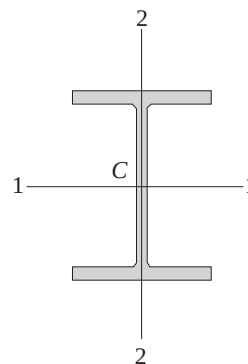
$$P_{cr} = \frac{7}{2} \beta L \quad \leftarrow$$

Critical Loads of Columns with Pinned Supports

The problems for Section 11.3 are to be solved using the assumptions of ideal, slender, prismatic, linearly elastic columns (Euler buckling). Buckling occurs in the plane of the figure unless stated otherwise.

Problem 11.3-1 Calculate the critical load P_{cr} for a W 8 × 35 steel column (see figure) having length $L = 24$ ft and $E = 30 \times 10^6$ psi under the following conditions:

(a) The column buckles by bending about its strong axis (axis 1–1), (b) The column buckles by bending about its weak axis (axis 2–2). In both cases, assume that the column has pinned ends.



Probs. 11.3.1 through 11.3-3

Solution 11.3-1 Column with pinned supports

W 8 × 35 steel column

$$L = 24 \text{ ft} = 288 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$$

$$I_1 = 127 \text{ in.}^4 \quad I_2 = 42.6 \text{ in.}^4 \quad A = 10.3 \text{ in.}^2$$

(a) BUCKLING ABOUT STRONG AXIS

$$P_{cr} = \frac{\pi^2 EI_1}{L^2} = 453 \text{ k} \quad \leftarrow$$

(b) BUCKLING ABOUT WEAK AXIS

$$P_{cr} = \frac{\pi^2 EI_2}{L^2} = 152 \text{ k} \quad \leftarrow$$

$$\text{Note: } \sigma_{cr} = \frac{P_{cr}}{A} = \frac{453 \text{ k}}{10.3 \text{ in.}^2} = 44 \text{ ksi}$$

$\therefore$ Solution is satisfactory if $\sigma_{PL} \geq 44 \text{ ksi}$.

Problem 11.3-2 Solve the preceding problem for a W 250 × 89 steel column having length $L = 10$ m. Let $E = 200$ GPa

Solution 11.3-2

W 250 × 89

$$E = 200 \text{ GPa} \quad L = 10.0 \text{ m} \quad I_1 = 142 \times 10^6 \text{ mm}^4$$

$$I_2 = 48.3 \times 10^6 \text{ mm}^4 \quad A = 11,400 \text{ mm}^2$$

BUCKLING ABOUT STRONG AXIS

$$P_{cr1} = \frac{\pi^2 EI_1}{L^2} \quad P_{cr1} = 2803 \text{ kN} \quad \leftarrow$$

BUCKLING ABOUT WEAK AXIS

$$P_{cr2} = \frac{\pi^2 EI_2}{L^2} \quad P_{cr2} = 953 \text{ kN} \quad \leftarrow$$

$$\text{Note: } \sigma_{cr} = \frac{P_{cr1}}{A} \quad \sigma_{cr} = 246 \text{ MPa}$$

$\therefore$ Solution is satisfactory if $\sigma_{PL} \geq 246 \text{ MPa}$.

Problem 11.3-3 Solve Problem 11.3-1 for a W 10 × 45 steel column having length $L = 28$ ft.

Solution 11.3-3 Column with pinned supports

W 10 × 45 steel column

$$L = 28 \text{ ft} = 336 \text{ in.} \quad E = 30 \times 10^6 \text{ psi}$$

$$I_1 = 248 \text{ in.}^4 \quad I_2 = 53.4 \text{ in.}^4 \quad A = 13.3 \text{ in.}^2$$

(a) BUCKLING ABOUT STRONG AXIS

$$P_{\text{cr}} = \frac{\pi^2 EI_1}{L^2} = 650 \text{ k} \quad \leftarrow$$

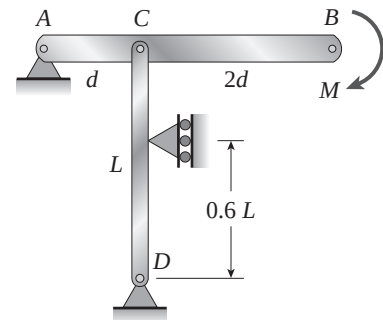
(b) BUCKLING ABOUT WEAK AXIS

$$P_{\text{cr}} = \frac{\pi^2 EI_2}{L^2} = 140 \text{ k} \quad \leftarrow$$

$$\text{NOTE: } \sigma_{\text{cr}} = \frac{P_{\text{cr}}}{A} = \frac{650 \text{ k}}{13.3 \text{ in.}^2} = 49 \text{ ksi}$$

 $\therefore$ Solution is satisfactory if $\sigma_{PL} \geq 49 \text{ ksi}$.

Problem 11.3-4 A horizontal beam AB is pin-supported at end A and carries a clockwise moment M at joint B , as shown in the figure. The beam is also supported at C by a pinned-end column of length L ; the column is restrained laterally at $0.6L$ from the base at D . Assume the column can only buckle in the plane of the frame. The column is a solid steel bar ($E = 200 \text{ GPa}$) of square cross section having length $L = 2.4 \text{ m}$ and side dimensions $b = 70 \text{ mm}$. Let dimensions $d = L/2$. Based upon the critical load of the column, determine the allowable moment M if the factor of safety with respect to buckling is $n = 2.0$.

**Solution 11.3-4**COLUMN CD (STEEL)

$$E = 200 \text{ GPa} \quad L = 2.4 \text{ m} \quad d = \frac{L}{2} \quad d = 1.2 \text{ m}$$

Square cross section: $b = 70 \text{ mm}$ Factor of safety: $n = 2.0$

$$I = \frac{b^4}{12} \quad I = 2.00 \times 10^6 \text{ mm}^4$$

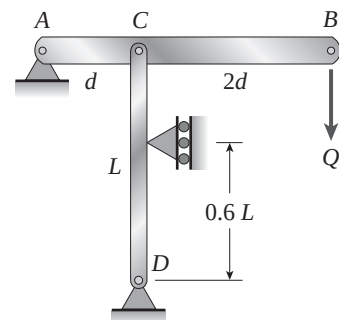
$$P_{\text{cr}} = \frac{\pi^2 EI}{(0.6L)^2} \quad P_{\text{cr}} = 1905 \text{ kN}$$

$$P_{\text{allow}} = \frac{P_{\text{cr}}}{n} \quad P_{\text{allow}} = 952.3 \text{ kN}$$

$$\text{Beam } ACB \quad \Sigma M_A = 0 \quad M = Pd$$

$$M_{\text{allow}} = P_{\text{allow}} d \quad M_{\text{allow}} = 1143 \text{ kN}\cdot\text{m} \quad \leftarrow$$

Problem 11.3-5 A horizontal beam AB is pin-supported at end A and carries a load Q at joint B , as shown in the figure. The beam is also supported at C by a pinned-end column of length L ; the column is restrained laterally at $0.6L$ from the base at D . Assume the column can only buckle in the plane of the frame. The column is a solid aluminium bar ($E = 10 \times 10^6 \text{ psi}$) of square cross section having length $L = 30 \text{ in.}$ and side dimensions $b = 1.5 \text{ in.}$ Let dimension $d = L/2$. Based upon the critical load of the column, determine the allowable force Q if the factor of safety with respect to buckling is $n = 1.8$.



Solution 11.3-5COLUMN *CD* (ALUMINUM)

$$E = 10 \times 10^6 \text{ psi} \quad L = 30 \text{ in.}$$

$$d = \frac{L}{2} \quad d = 15 \text{ in.}$$

Square cross section: $b = 1.5 \text{ in.}$ Factor of safety: $n = 1.8$

$$I = \frac{b^4}{12} \quad I = 0.422 \text{ in.}^4$$

$$P_{\text{cr}} = \frac{\pi^2 EI}{(0.6L)^2} \quad P_{\text{cr}} = 129 \text{ k}$$

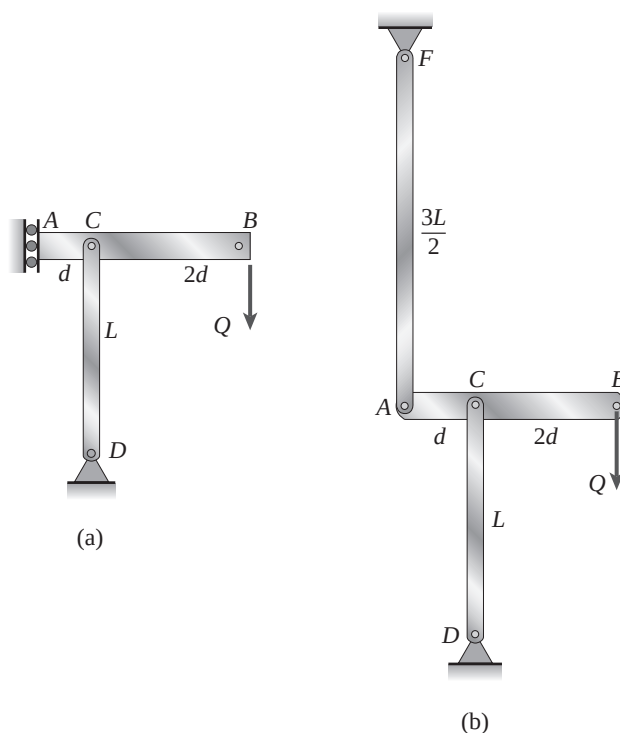
$$P_{\text{allow}} = \frac{P_{\text{cr}}}{n} \quad P_{\text{allow}} = 71 \text{ k}$$

$$\text{Beam } ACB \quad \Sigma M_A = 0 \quad Q = \frac{P}{3}$$

$$Q_{\text{allow}} = \frac{P_{\text{allow}}}{3} \quad Q_{\text{allow}} = 23.8 \text{ k} \quad \leftarrow$$

Problem 11.3-6 A horizontal beam *AB* is pin-supported at end *A* and carries a load *Q* at joint *B*, as shown in the figure part a. The beam is also supported at *C* by a pinned-end column of length *L*. The column has flexural rigidity *EI*.

- (a) For the case of a guided support at *A* (figure part a), what is the critical load Q_{cr} ? (In other words, at what load Q_{cr} does the system collapse because of Euler buckling of the column *DC*?)
- (b) Repeat part (a) if the guided support at *A* is replaced by column *AF* with length $3L/2$ and flexural rigidity *EI* (see figure part b).

**Solution 11.3-6**

$$(a) \quad P_{\text{cr}} = \frac{\pi^2 EI}{L^2}$$

FROM FREE-BODY DIAGRAM OF THE SYSTEM

$$\Sigma F_y = 0 \quad Q = P$$

$$\text{Therefore, } Q_{\text{cr}} = P_{\text{cr}} = \frac{\pi^2 EI}{L^2} \quad \leftarrow$$

(b) Q_{cr} BASED UPON P_{cr} IN COLUMN *AF*

$$P_{\text{cr}, AF} = \frac{\pi^2 EI}{\left(\frac{3L}{2}\right)^2} = \frac{4\pi^2 EI}{9L^2}$$

FROM FREE-BODY DIAGRAM OF BEAM ACB

$$\Sigma M_C = 0 \quad Q = \frac{P_{AF}}{2}$$

$$\text{Therefore, } Q_{cr, AF} = \frac{P_{cr, AF}}{2} = \frac{2\pi^2 EI}{9L^2}$$

 Q_{cr} based upon P_{cr} in column CD:

$$P_{cr, CD} = \frac{\pi^2 EI}{L^2} \leftarrow$$

FROM FREE-BODY DIAGRAM OF BEAM ACB

$$\Sigma M_A = 0 \quad Q = \frac{P_{CD}}{3}$$

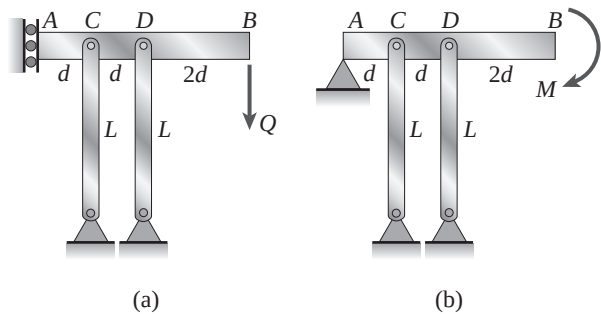
$$\text{Therefore, } Q_{cr, CD} = \frac{P_{cr, CD}}{2} = \frac{\pi^2 EI}{3L^2}$$

$$Q_{cr} = Q_{cr, AF} = \frac{2\pi^2 EI}{9L^2} \leftarrow$$

Column AF governs.

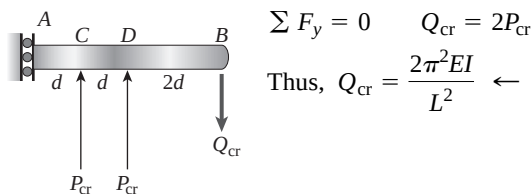
Problem 11.3-7 A horizontal beam AB has a guided support at end A and carries a load Q at end B, as show in the figure part a. The beam is supported at C and D by two identical pinned-end columns of length L. Each column has flexural rigidity EI.

- Find an expression for the critical load Q_{cr} . (In other words, at what load Q_{cr} does the system collapse because of Euler buckling of the columns?)
- Repeat part (a), but assume spin support at A. Find an expression for the critical moment M_{cr} (i.e., find the moment M at B at which the system collapses because of Euler buckling of the columns).

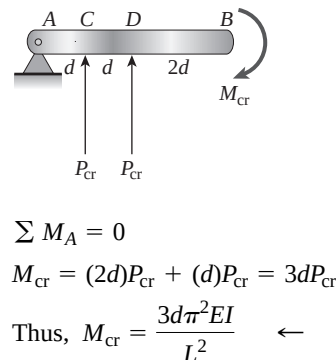
**Solution 11.3-7**

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

- (a) COLLAPSE OCCURS WHEN BOTH COLUMNS REACH THE CRITICAL LOAD

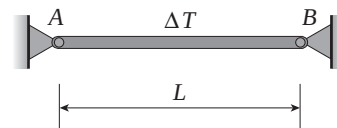


- (b) COLLAPSE OCCURS WHEN BOTH COLUMNS REACH THE CRITICAL LOAD



Problem 11.3-8 A slender bar AB with pinned ends and length L is held between immovable supports (see figure).

What increase ΔT in the temperature of the bar will produce buckling at the Euler load?



Solution 11.3-8 Bar with immovable pin supports

L = length A = cross-sectional area
 I = moment of inertia E = modulus of elasticity
 α = coefficient of increase in temperature
 ΔT = uniform increase in temperature

AXIAL COMPRESSIVE FORCE IN BAR

$$P = EA\alpha (\Delta T)$$

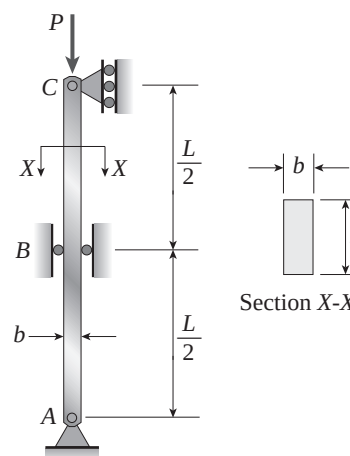
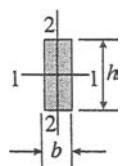
$$\text{EULER LOAD } P_{cr} = \frac{\pi^2 EI}{L^2}$$

INCREASE IN TEMPERATURE TO PRODUCE BUCKLING

$$P = P_{cr} \quad EA\alpha (\Delta T) = \frac{\pi^2 EI}{L^2} \quad \Delta T = \frac{\pi^2 I}{\alpha AL^2} \quad \leftarrow$$

Problem 11.3-9 A rectangular column with cross-sectional dimensions b and h is pin-supported at ends A and C (see figure). At midheight, the column is restrained in the plane of the figure but is free to deflect perpendicularly to the plane of the figure.

Determine the ratio h/b such that the critical load is the same for buckling in the two principal planes of the column.

**Solution 11.3-9 Column with restraint at midheight**

Critical loads for buckling about axes 1-1 and 2-2:

$$P_1 = \frac{\pi^2 EI_1}{L^2} \quad P_2 = \frac{\pi^2 EI_2}{(L/2)^2} = \frac{4\pi^2 EI_2}{L^2}$$

FOR EQUAL CRITICAL LOADS

$$P_1 = P_2 \quad \therefore I_1 = 4I_2$$

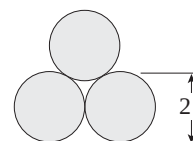
$$I_1 = \frac{bh^3}{12} \quad I_2 = \frac{hb^3}{12}$$

$$bh^3 = 4hb^3 \quad \frac{h}{b} = 2 \quad \leftarrow$$

Problem 11.3-10 Three identical, solid circular rods, each of radius r and length L , are placed together to form a compression member (see the cross section shown in the figure).

Assuming pinned-end conditions, determine the critical load P_{cr} as follows:
 (a) The rods act independently as individual columns, and (b) the rods are bonded by epoxy throughout their lengths so that they function as a single member.

What is the effect on the critical load when the rods act as a single member?

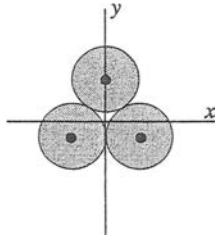


Solution 11.3-10 Three solid circular rods r = radius L = length

(A) RODS ACT INDEPENDENTLY

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad (3) \quad I = \frac{\pi r^4}{4}$$

$$P_{cr} = \frac{3\pi^3 Er^4}{4L^2} \quad \leftarrow$$



(b) RODS ARE BONDED TOGETHER

The x and y axes have their origin at the centroid of the cross section. Because there are three different

centroidal axes of symmetry, all centroidal axes are principal axes and all centroidal moments of inertia are equal (see Section 12.9).

From Case 9, Appendix E,

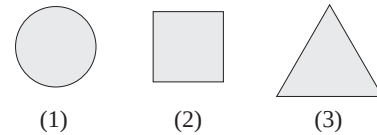
$$I = I_Y = \frac{\pi r^4}{4} + 2 \left(\frac{5\pi r^4}{4} \right) = \frac{11\pi r^4}{4}$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{11\pi^3 Er^4}{4L^2} \quad \leftarrow$$

Note: Joining the rods so that they act as a single member increases the critical load by a factor of $11/3$ or 3.67. $\leftarrow$

Problem 11.3-11 Three pinned-end columns of the same material have the same length and the same cross-sectional area (see figure). The columns are free to buckle in any direction. The columns have cross sections as follows: (1) a circle, (2) a square, and (3) an equilateral triangle.

Determine the ratios $P_1 : P_2 : P_3$ of the critical loads for these columns.

**Solution 11.3-11 Three pinned-end columns**

E, L and A are the same for all three columns.

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad \therefore P_1 : P_2 : P_3 = I_1 : I_2 : I_3$$

(1) CIRCLE Case 9, Appendix E:

$$I = \frac{\pi d^4}{64} \quad A = \frac{\pi d^2}{4} \quad \therefore I_1 = \frac{A^2}{4\pi}$$

(2) SQUARE Case 1, Appendix E:

$$I = \frac{b^4}{12} \quad A = b^2 \quad \therefore I_2 = \frac{A^2}{12}$$

(3) EQUILATERAL TRIANGLE Case 5, Appendix E:

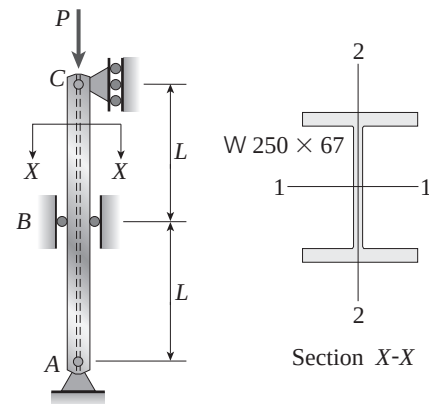
$$I = \frac{b^4 \sqrt{3}}{96} \quad A = \frac{b^2 \sqrt{3}}{4} \quad \therefore I_3 = \frac{A^2 \sqrt{3}}{18}$$

$$P_1 : P_2 : P_3 = I_1 : I_2 : I_3 = 1 : \frac{\pi}{3} : \frac{2\pi \sqrt{3}}{9} \\ = 1.000 : 1.047 : 1.209 \quad \leftarrow$$

Note: For each of the cross sections, every centroidal axis has the same moment of inertia (see Section 12.9).

Problem 11.3-12 A long slender column ABC is pinned at ends A and C and compressed by an axial force P (see figure). At the midpoint B , lateral support is provided to prevent deflection in the plane of the figure. The column is a steel wide-flange section (W 250 $\times$ 67) with $E = 200$ GPa. The distance between lateral supports is $L = 55$ m.

Calculate the allowable load P using a factor of safety $n = 2.4$, taking into account the possibility of Euler buckling about either principal centroidal axis (i.e., axis 1–1 or axis 2–2).



Solution 11.3-12W 250 $\times$ 67 $E = 200$ GPa $L = 5.5$ m $I_1 = 103 \times 10^6$ mm⁴ $I_2 = 22.2 \times 10^6$ mm⁴ $n = 2.4$

BUCKLING ABOUT AXIS 1-1

$$P_{cr1} = \frac{\pi^2 EI_1}{(2L)^2} \quad P_{cr1} = 1680 \text{ kN}$$

BUCKLING ABOUT AXIS 2-2

$$P_{cr2} = \frac{\pi^2 EI_2}{(L)^2} \quad P_{cr2} = 1449 \text{ kN}$$

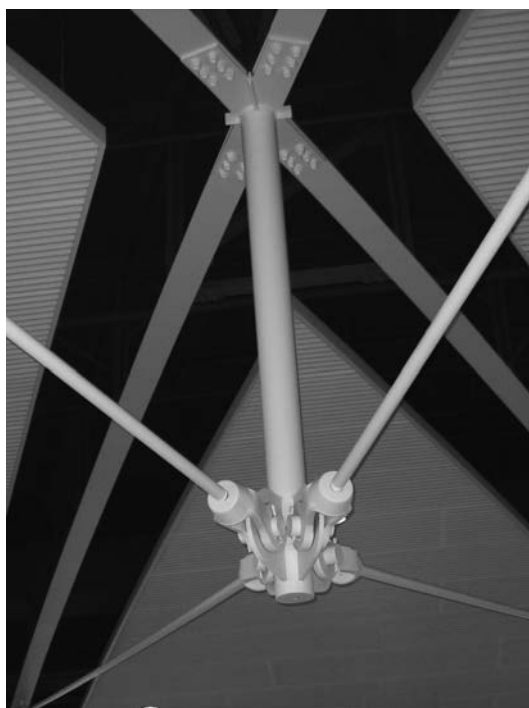
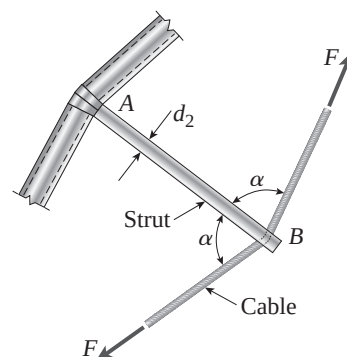
$$P_{cr} = P_{cr2} \quad (\text{axis 2-2 governs})$$

ALLOWABLE LOAD

$$P_{allow} = \frac{P_{cr}}{n} \quad P_{allow} = 604 \text{ kN} \quad \leftarrow$$

Problem 11.3-13 The roof over a concourse at an airport is supported by the use of pretensioned cables. At a typical joint in the roof structure, a strut AB is compressed by the action of tensile forces F in a cable that makes an angle $\alpha = 75^\circ$ with the strut (see figure and photo). The strut is a circular tube of steel ($E = 30,000$ ksi) with outer diameter $d_2 = 2.5$ in. and inner diameter $d_1 = 2.0$ in. The strut is 5.75 ft long and is assumed to be pin-connected at both ends.

Using a factor of safety $n = 2.5$ with respect to the critical load, determine the allowable force F in the cable.



Solution 11.3-13

$$E = 30,000 \text{ ksi} \quad d_2 = 2.5 \text{ in.} \quad d_1 = 2.0 \text{ in.}$$

$$L = 5.75 \text{ ft} \quad n = 2.5 \quad \alpha = 75^\circ$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) \quad I = 1.132 \text{ in.}^4$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad P_{cr} = 70.40 \text{ k}$$

ALLOWABLE LOAD

$$P_{allow} = \frac{P_{cr}}{n} \quad P_{allow} = 28.16 \text{ k}$$

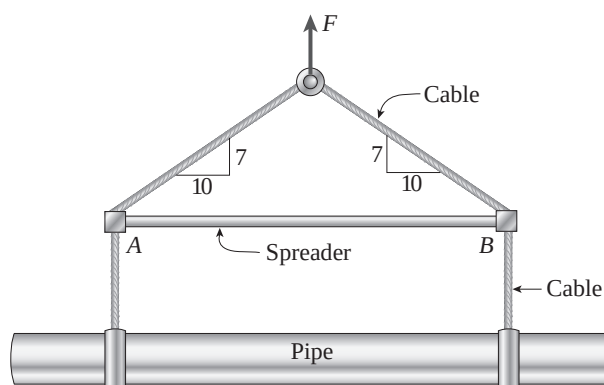
EQUILIBRIUM OF JOINT B

$$P = 2F \cos(\alpha)$$

$$\text{Thus, } F_{allow} = \frac{P_{allow}}{2 \cos(\alpha)} \quad F_{allow} = 54.4 \text{ k} \quad \leftarrow$$

Problem 11.3-14 The hoisting arrangement for lifting a large pipe is shown in the figure. The spreader is a steel tubular section with outer diameter 70 mm and inner diameter 57 mm. Its length is 2.6 m and its modulus of elasticity is 200 GPa.

Based upon a factor of safety of 2.25 with respect to Euler buckling of the spreader, what is the maximum weight of pipe that can be lifted? (Assume pinned conditions at the ends of the spreader.)

**Solution 11.3-14**

$$E = 200 \text{ GPa} \quad d_2 = 70 \text{ mm}$$

$$d_1 = 57 \text{ mm} \quad L = 2.6 \text{ m}$$

$$n = 2.25 \quad \alpha = \arctan\left(\frac{7}{10}\right)$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) \quad I = 660.4 \times 10^3 \text{ mm}^4$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad P_{cr} = 199 \text{ kN}$$

ALLOWABLE LOAD

$$P_{allow} = \frac{P_{cr}}{n} \quad P_{allow} = 88.6 \text{ kN}$$

EQUILIBRIUM OF JOINT A

$$\Sigma F_{horiz} = 0 \quad -P + T \cos(\alpha) = 0$$

$$\Sigma F_{vert} = 0 \quad T \sin(\alpha) - \frac{W}{2} = 0$$

SOLVE THE EQUATION

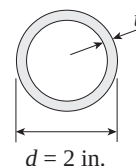
$$W = 2P \tan(\alpha)$$

MAXIMUM WEIGHT OF PIPE

$$W_{max} = 2P_{allow} \tan(\alpha) \quad W_{max} = 124 \text{ kN} \quad \leftarrow$$

Problem 11.3-15 A pinned-end strut of aluminium ($E = 10,400$ ksi) with length $L = 6$ ft is constructed of circular tubing with outside diameter $d = 2$ in. (see figure). The strut must resist an axial load $P = 4$ kips with a factor of safety $n = 2.0$ with respect to the critical load.

Determine the required thickness t of the tube.



Solution 11.3-15

$$E = 10,400 \text{ ksi} \quad L = 6 \text{ ft}$$

$$d = 2 \text{ in.} \quad n = 2.0 \quad P = 4 \text{ k}$$

$$P_{cr} = nP \quad P_{cr} = 8.0 \text{ k}$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad I = \frac{P_{cr} L^2}{\pi^2 E} \quad I = 0.404 \text{ in.}^4$$

MOMENT OF INERTIA

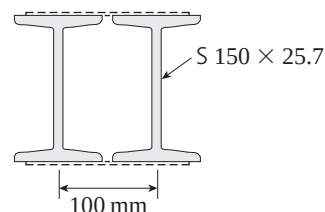
$$I = \frac{\pi}{64} [d^4 - (d - 2t)^4]$$

$$d^4 - (d - 2t)^4 = I \frac{64}{\pi}$$

$$t_{\min} = 0.165 \text{ in.}$$

Problem 11.3-16 The cross section of a column built up of two steel I-beams (S 150 $\times$ 25.7 sections) is shown in the figure. The beams are connected by spacer bars, or *lacing*, to ensure that they act together as a single column. (The lacing is represented by dashed lines in the figure.)

The column is assumed to have pinned ends and may buckle in any direction. Assuming $E = 200$ GPa and $L = 8.5$ m, calculate the critical load P_{cr} for the column.



Solution 11.3-16

$$S 150 \times 25.7 \quad E = 200 \text{ GPa}$$

$$L = 8.5 \text{ m} \quad I_1 = 10.9 \times 10^6 \text{ mm}^4$$

$$I_2 = 0.953 \times 10^6 \text{ mm}^4$$

$$A = 3260 \text{ mm}^2 \quad d = \frac{100 \text{ mm}}{2}$$

COMPOSITE COLUMN

$$I_x = 2I_1 \quad I_x = 21.80 \times 10^6 \text{ mm}^4$$

$$I_y = 2(I_2 + Ad^2) \quad I_y = 18.21 \times 10^6 \text{ mm}^4$$

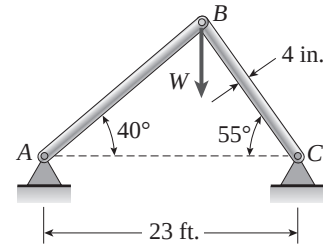
Buckling occurs about the y axis, since $I_y < I_x$.

$$\text{Critical load: } P_{cr} = \frac{\pi^2 EI_y}{L^2}$$

$$P_{cr} = 497 \text{ kN} \quad \leftarrow$$

Problem 11.3-17 The truss ABC shown in the figure supports a vertical load W at joint B . Each member is a slender circular steel pipe ($E = 30,000$ ksi) with outside diameter 4 in. and wall thickness 0.25 in. The distance between supports is 23 ft. Joint B is restrained against displacement perpendicular to the plane of the truss.

Determine the critical value W_{cr} of the load.



Solution 11.3-17

$$E = 30,000 \text{ ksi} \quad L = 23 \text{ ft}$$

$$d_2 = 4 \text{ in.} \quad t = 0.25 \text{ in.}$$

$$d_1 = d_2 - 2t \quad d_1 = 3.50 \text{ in.}$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) \quad I = 5.200 \text{ in.}^4$$

$$\theta_1 = 40^\circ \quad \theta_2 = 55^\circ$$

$$L_{AB} = L \left(\frac{\sin(\theta_2)}{\sin(180^\circ - \theta_1 - \theta_2)} \right)$$

$$L_{AB} = 18.912 \text{ ft}$$

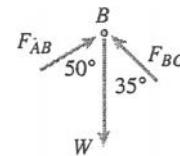
$$L_{BC} = L \left(\frac{\sin(\theta_1)}{\sin(180^\circ - \theta_1 - \theta_2)} \right)$$

$$L_{BC} = 14.841 \text{ ft}$$

$$\text{Critical loads: } P_{crAB} = \frac{\pi^2 EI}{L_{AB}^2} \quad P_{crAB} = 29.89 \text{ k}$$

$$P_{crBC} = \frac{\pi^2 EI}{L_{BC}^2} \quad P_{crBC} = 48.55 \text{ k}$$

FREE-BODY DIAGRAM OF JOINT B



$$\Sigma F_{\text{horiz}} = 0 \quad F_{AB} \cos(\theta_1) - F_{BC} \cos(\theta_2) = 0$$

$$\Sigma F_{\text{vert}} = 0 \quad F_{AB} \sin(\theta_1) - F_{BC} \sin(\theta_2) - W = 0$$

SOLVE THE TWO EQUATIONS

$$W = 1.7361 F_{AB} \quad W = 1.3004 F_{BC}$$

CRITICAL VALUE OF THE LOAD W

Based on member AB :

$$W_{crAB} = 1.7361 P_{crAB} \quad W_{crAB} = 51.90 \text{ k}$$

Based on member BC :

$$W_{crBC} = 1.3004 P_{crBC} \quad W_{crBC} = 63.13 \text{ k}$$

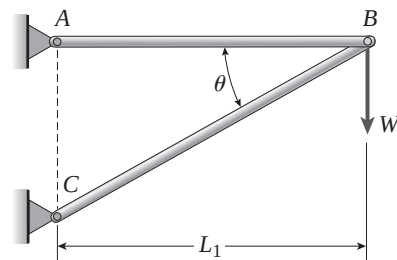
$$W_{cr} = \min(W_{crAB}, W_{crBC})$$

$$W_{cr} = 51.9 \text{ k} \quad \leftarrow$$

Member AB governs.

Problem 11.3-18 A truss ABC supports a load W at joint B , as shown in the figure. The length L_1 of member AB is fixed, but the length of strut BC varies as the angle θ is changed. Strut BC has a solid circular cross section. Joint B is restrained against displacement perpendicular to the plane of the truss.

Assuming that collapse occurs by Euler buckling of the strut, determine the angle θ for minimum weight of the strut.



Solution 11.3-18 Truss ABC (minimum weight)

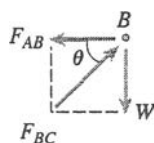
LENGTHS OF MEMBERS

$$L_{AB} = L_1 \text{ (a constant)}$$

$$L_{BC} = \frac{L_1}{\cos \theta} \text{ (angle } \theta \text{ is variable)}$$

Strut BC may buckle.

FREE-BODY DIAGRAM OF JOINT B



$$\sum F_{\text{vert}} = 0 \quad F_{BC} \sin \theta - W = 0$$

STRUT BC (SOLID CIRCULAR BAR)

$$A = \frac{\pi d^2}{4} \quad I = \frac{\pi d^4}{64} \quad \therefore I = \frac{A^2}{4\pi}$$

$$P_{cr} = \frac{\pi^2 EI}{L_{BC}^2} = \frac{\pi EA^2 \cos^2 \theta}{4 L_1^2}$$

$$F_{BC} = P_{cr} \quad \text{or} \quad \frac{W}{\sin \theta} = \frac{\pi EA^2 \cos^2 \theta}{4 L_1^2}$$

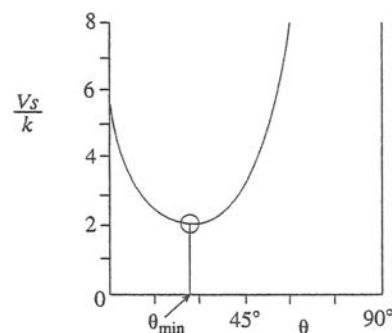
$$\text{Solve for area } A: A = \frac{2L_1}{\cos \theta} \left(\frac{W}{\pi E \sin \theta} \right)^{1/2}$$

For minimum weight, the volume V_s of the strut must be a minimum.

$$V_s = AL_{BC} = \frac{AL_1}{\cos \theta} = \frac{2L_1^2}{\cos^2 \theta} \left(\frac{W}{\pi E \sin \theta} \right)^{1/2}$$

All the terms are constants except $\cos \theta$ and $\sin \theta$. Therefore, we can write V_s in the following form:

$$V_s = \frac{k}{\cos^2 \theta \sqrt{\sin \theta}} \text{ where } k \text{ is a constant.}$$

GRAPH OF $\frac{V_s}{k}$  $\theta_{\min}$ = angle for minimum volume (and minimum weight)For minimum weight, the term $\cos^2 \theta \sqrt{\sin \theta}$ must be a maximum.For minimum value, the derivative with respect to θ equals zero.

$$\text{Therefore, } \frac{d}{d\theta} (\cos^2 \theta \sqrt{\sin \theta}) = 0$$

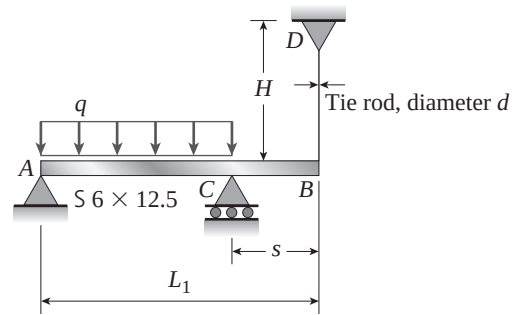
Taking the derivative and simplifying, we get $\cos^2 \theta - 4 \sin^2 \theta = 0$

$$\text{or } 1 - 4 \tan^2 \theta = 0 \quad \text{and} \quad \tan \theta = \frac{1}{2}$$

$$\therefore \theta_{\min} = \arctan \frac{1}{2} = 26.57^\circ \quad \leftarrow$$

Problem 11.3-19 An S 6 × 12.5 steel cantilever beam AB is supported by a steel tie rod at B as shown. The tie rod is just taut when a roller support is added at C at a distance s to the left of B, then the distributed load q is applied to beam segment AC. Assume $E = 30 \times 10^6$ psi and neglect the self weight of the beam and tie rod. See Table F-2(a) in Appendix F for the properties of the S-shape beam.

- What value of uniform load q will, if exceeded, result in buckling of the tie rod if $L_1 = 6$ ft, $s = 2$ ft, $H = 3$ ft, and $d = 0.25$ in.?
- What minimum beam moment of inertia I_b is required to prevent buckling of the tie rod if $q = 200$ lb/ft, $L_1 = 6$ ft, $H = 3$ ft, $d = 0.25$ in., and $s = 2$ ft?
- For what distance s will the tie rod be just on the verge of buckling if $q = 200$ lb/ft, $L_1 = 6$ ft, $H = 3$ ft, and $d = 0.25$ in.?



Solution 11.3-19

$$E = 30 \times 10^6 \text{ psi} \quad L_1 = 6 \text{ ft} \quad d = \frac{1}{4} \text{ in.}$$

$$H = 3 \text{ ft} \quad q = 200 \text{ lb/ft}$$

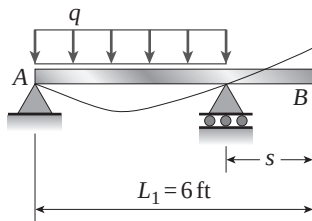
$$I_r = \frac{\pi}{64} d^4 \quad I_r = 191.7 \times 10^{-6} \text{ in.}^4$$

$$A_r = \frac{\pi}{4} d^2 \quad A_r = 0.049 \text{ in.}^2$$

$$P_{cr} = \frac{\pi^2 EI_r}{H^2} \quad P_{cr} = 43.81 \text{ lb}$$

$$\text{S } 6 \times 12.5 \quad I_{\text{beam}} = 22.0 \text{ in.}^4$$

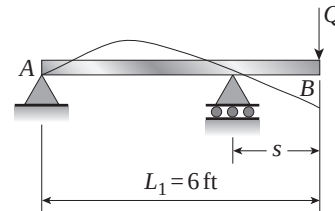
Analyze first-degree statically indeterminate beam by letting force in tie rod be a redundant released beam AB with the uniform load q .



Released beam AB with the redundant Q .

From Appendix H,

$$\delta_B = \frac{q(L_1 - s)^3}{24EI_{\text{beam}}}$$



FROM APPENDIX H

$$\delta_B'' = \frac{Qs^3}{3EI_{\text{beam}}} + \frac{Qs(L_1 - s)s}{3EI_{\text{beam}}}$$

$$\text{Shortening in tie rod: } \delta_r = \frac{QH}{EA_r}$$

$$\text{Compatibility equation: } \delta_B' - \delta_B'' = \delta_r$$

$$\frac{q(L_1 - s)^3}{24EI_{\text{beam}}} - \left[\frac{Qs^3}{3EI_{\text{beam}}} + \frac{Qs(L_1 - s)s}{3EI_{\text{beam}}} \right] = \frac{QH}{EA_r}$$

$$Q = \frac{sqA_r(L_1 - s)^3}{8(s^2A_rL_1 + 3HI_b)} \quad (1)$$

$$(a) \text{ FOR } Q = P_{cr}, \quad s = 2.0 \text{ ft}$$

From Eq. (1),

$$q_{\text{max}} = \frac{8Q}{sA_r(L_1 - s)^3} \times (s^2A_rL_1 + 3HI_b)$$

$$q_{\text{max}} = 142.4 \text{ lb/ft} \quad \leftarrow$$

(b) FOR $q = 200 \text{ lb/ft}$,

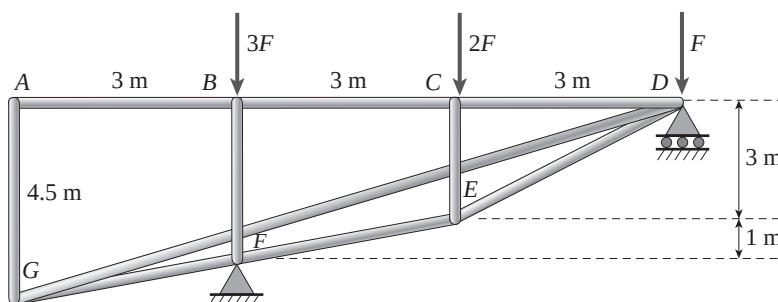
$$\text{From Eq. (1), } I_{\text{beammin}} = \frac{-1}{24} s A_r \frac{8QsL_1 - qL_1^3 + 3sqL_1^2 - 3s^2qL_1 + s^3q}{QH}$$

$$I_{\text{beammin}} = 38.5 \text{ in.}^4 \quad \leftarrow$$

(c) FROM EQ. (1), NUMERICALLY SOLVE FOR s WHEN $Q = P_{\text{cr}} \rightarrow$ TWO SOLUTIONS ARE POSSIBLE:

$$s = 0.264 \text{ ft} \quad \text{and} \quad s = 2.42 \text{ ft} \quad \leftarrow$$

Problem 11.3-20 The plane truss shown in the figure supports vertical loads F at joint D , $2F$ at joint C , and $3F$ at joint B . Each member is a slender circular pipe ($E = 70 \text{ GPa}$) with outside diameter 60 mm and wall thickness 5 mm . Joint B is restrained against displacement perpendicular to the plane of the truss. Determine the critical value of load variable $F(\text{kN})$ at which member BF fails by Euler buckling.



Solution 11.3-20

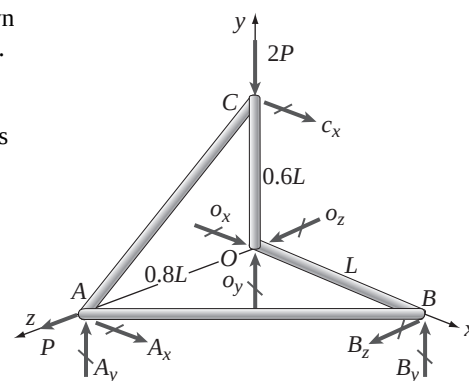
$$E = 70 \text{ GPa} \quad L = 4 \text{ m} \quad d = 60 \text{ mm} \quad t = 5 \text{ mm} \quad I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 3.294 \times 10^{-7} \text{ m}^4$$

Plane truss analysis gives (see Prob. 1.2-6) $F_{BF} = -4F$

$$F_{BF\text{cr}} = \frac{\pi^2 EI}{L^2} = 14.22 \text{ kN} \quad \text{So critical value of applied load variable } F \text{ is}$$

$$F_{\text{cr}} = \frac{F_{BF\text{cr}}}{4} = 3.56 \text{ kN}$$

Problem 11.3-21 A space truss is restrained at joints O , A , B , and C , as shown in the figure. Load P is applied at joint A and load $2P$ acts downward at joint C . Each member is a slender circular pipe ($E = 10,600 \text{ ksi}$) with outside diameter of 3.5 in. and wall thickness of 0.25 in. Length variable $L = 11 \text{ ft}$. Determine the critical value of load variable P (kips) at which member OB fails by Euler buckling.



Solution 11.3-21

$$E = 10,600 \text{ ksi} \quad L = 11 \text{ ft} \quad d = 3.5 \text{ in.} \quad t = 0.25 \text{ in.} \quad I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 1.635 \times 10^{-4} \text{ ft}^4$$

$$\text{Space truss analysis gives (see Prob. 1.2-8)} \quad F_{OB} = \frac{-5}{4} P$$

$$F_{OBcr} = \frac{\pi^2 EI}{L^2} = 20.35 \text{ k}$$

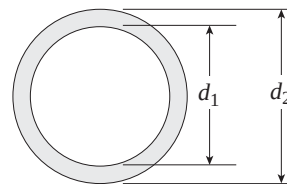
So critical value of applied load variable P is

$$P_{cr} = \frac{F_{OBcr}}{\frac{5}{4}} = 16.28 \text{ kip}$$

Columns with Other Support Conditions

The problems for Section 11.4 are to be solved using the assumptions of ideal, slender, prismatic, linearly elastic columns (Euler buckling). Buckling occurs in the plane of the figure unless stated otherwise.

Problem 11.4-1 An aluminum pipe column ($E = 10,400 \text{ ksi}$) with length $L = 10.0 \text{ ft}$ has inside and outside diameters $d_1 = 5.0 \text{ in.}$ and $d_2 = 6.0 \text{ in.}$, respectively (see figure). The column is supported only at the ends and may buckle in any direction.



Probs. 11.4-1 and 11.4-2

Calculate the critical load P_{cr} for the following end conditions:
(1) pinned-pinned, (2) fixed-free, (3) fixed-pinned, and (4) fixed-fixed.

Solution 11.4-1 Aluminum pipe column

$$d_2 = 6.0 \text{ in.} \quad d_1 = 5.0 \text{ in.} \quad E = 10,400 \text{ ksi}$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) = 32.94 \text{ in.}^4$$

$$L = 10.0 \text{ ft} = 120 \text{ in.}$$

(1) PINNED-PINNED

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (10,400 \text{ ksi}) (32.94 \text{ in.}^4)}{(120 \text{ in.})^2} = 235 \text{ k} \quad \leftarrow$$

$$(2) \text{ FIXED-FREE} \quad P_{cr} = \frac{\pi^2 EI}{4L^2} = 58.7 \text{ k} \quad \leftarrow$$

$$(3) \text{ FIXED-PINNED} \quad P_{cr} = \frac{2.046 \pi^2 EI}{L^2} = 480 \text{ k} \quad \leftarrow$$

$$(4) \text{ FIXED-FIXED} \quad P_{cr} = \frac{4 \pi^2 EI}{L^2} = 939 \text{ k} \quad \leftarrow$$

Problem 11.4-2 Solve the preceding problem for a steel pipe column ($E = 210 \text{ GPa}$) with length $L = 1.2 \text{ m}$, inner diameter $d_1 = 36 \text{ mm}$, and outer diameter $d_2 = 40 \text{ mm}$.

Solution 11.4-2 Steel pipe column

$$d_2 = 40 \text{ mm} \quad d_1 = 36 \text{ mm} \quad E = 210 \text{ GPa}$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) = 43.22 \times 10^3 \text{ mm}^4 \quad L = 1.2 \text{ m}$$

$$(1) \text{ PINNED-PINNED} \quad P_{\text{cr}} = \frac{\pi^2 EI}{L^2} = 62.2 \text{ kN} \quad \leftarrow$$

$$(2) \text{ FIXED-FREE} \quad P_{\text{cr}} = \frac{\pi^2 EI}{4L^2} = 15.6 \text{ kN} \quad \leftarrow$$

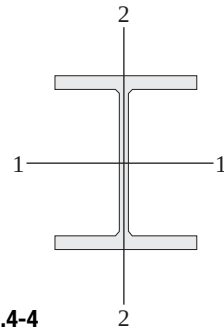
$$(3) \text{ FIXED-PINNED} \quad P_{\text{cr}} = \frac{2.046\pi^2 EI}{L^2} = 127 \text{ kN} \quad \leftarrow$$

$$(4) \text{ FIXED-FIXED} \quad P_{\text{cr}} = \frac{4\pi^2 EI}{L^2} = 249 \text{ kN} \quad \leftarrow$$

Problem 11.4-3 A wide-flange steel column ($E = 30 \times 10^6$ psi) of W 12 $\times$ 87 shape (see figure) has length $L = 28$ ft. It is supported only at the ends and may buckle in any direction.

Calculate the allowable load P_{allow} based upon the critical load with a factor of safety $n = 2.5$. Consider the following end conditions:

(1) pinned-pinned, (2) fixed-free, (3) fixed-pinned, and (4) fixed-fixed.



Probs. 11.4-3 and 11.4-4

Solution 11.4-3 Wide-flange column

$$\text{W12} \times 87 \quad E = 30 \times 10^6 \text{ psi}$$

$$L = 28 \text{ ft} = 336 \text{ in.} \quad n = 2.5 \quad I_2 = 241 \text{ in.}^4$$

(1) PINNED-PINNED

$$P_{\text{allow}} = \frac{P_{\text{cr}}}{n} = \frac{\pi^2 EI_2}{nL^2} = 253 \text{ k} \quad \leftarrow$$

(2) FIXED-FREE

$$P_{\text{allow}} = \frac{\pi^2 EI_2}{4nL^2} = 63.2 \text{ k} \quad \leftarrow$$

(3) FIXED-PINNED

$$P_{\text{allow}} = \frac{2.046\pi^2 EI_2}{nL^2} = 517 \text{ k} \quad \leftarrow$$

(4) FIXED-FIXED

$$P_{\text{allow}} = \frac{4\pi^2 EI_2}{nL^2} = 1011 \text{ k} \quad \leftarrow$$

Problem 11.4-4 Solve the preceding problem for a W 250 $\times$ 89 shape with length $L = 7.5$ m and $E = 200$ GPa.

Solution 11.4-4

$$W 250 \times 89 \quad E = 200 \text{ GPa}$$

$$L = 7.5 \text{ m} \quad n = 2.5$$

$$I_2 = 48.3 \times 10^6 \text{ mm}^4$$

(1) PINNED-PINNED

$$P_{\text{allow}} = \frac{\pi^2 EI_2}{nL^2}$$

$$P_{\text{allow}} = 678 \text{ kN} \quad \leftarrow$$

(2) FIXED-FREE

$$P_{\text{allow}} = \frac{\pi^2 EI_2}{4nL^2}$$

$$P_{\text{allow}} = 169.5 \text{ kN} \quad \leftarrow$$

(3) FIXED-PINNED

$$P_{\text{allow}} = \frac{2.046\pi^2 EI_2}{nL^2}$$

$$P_{\text{allow}} = 1387 \text{ kN} \quad \leftarrow$$

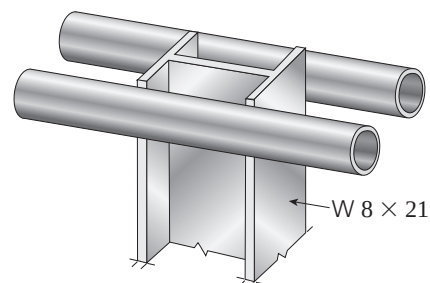
(4) FIXED-FIED

$$P_{\text{allow}} = \frac{4\pi^2 EI_2}{nL^2}$$

$$P_{\text{allow}} = 2712 \text{ kN} \quad \leftarrow$$

Problem 11.4-5 The upper end of a W 8 × 21 wide-flange steel column ($E = 30 \times 10^3$ ksi) is supported laterally between two pipes (see figure). The pipes are not attached to the column, and friction between the pipes and the column is unreliable. The base of the column provides a fixed support, and the column is 13 ft long.

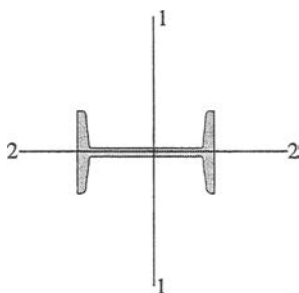
Determine the critical load for the column, considering Euler buckling in the plane of the web and also perpendicular to the plane of the web.

**Solution 11.4-5 Wide-flange steel column**

$$W 8 \times 21 \quad E = 30 \times 10^3 \text{ ksi}$$

$$L = 13 \text{ ft} = 156 \text{ in.} \quad I_1 = 75.3 \text{ in.}^4$$

$$I_2 = 9.77 \text{ in.}^4$$



AXIS 1-1 (FIXED-FREE)

$$P_{\text{cr}} = \frac{\pi^2 EI_1}{4L^2} = 229 \text{ k}$$

AXIS 2-2 (FIXED-PINNED)

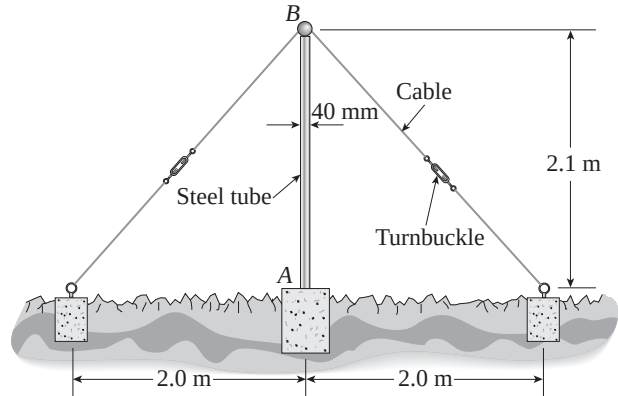
$$P_{\text{cr}} = \frac{2.046\pi^2 EI_2}{L^2} = 243 \text{ k}$$

Buckling about axis 1-1 governs.

$$P_{\text{cr}} = 229 \text{ k} \quad \leftarrow$$

Problem 11.4-6 A vertical post AB is embedded in a concrete foundation and held at the top by two cables (see figure). The post is a hollow steel tube with modulus of elasticity 200 GPa, outer diameter of 40 mm, and thickness of 5 mm. The cables are tightened equally by turnbuckles.

If a factor of safety of 3.0 against Euler buckling in the plane of the figure is desired, what is the maximum allowable tensile force T_{allow} in the cables?



Solution 11.4-6 Steel tube

$$E = 200 \text{ GPa} \quad d_2 = 40 \text{ mm} \quad d_1 = 30 \text{ mm}$$

$$L = 2.1 \text{ m} \quad n = 3.0$$

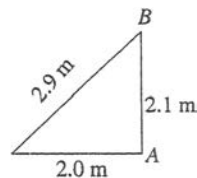
$$I = \frac{\pi}{64} (d_2^4 - d_1^4) = 85,903 \text{ mm}^4$$

Buckling in the plane of the figure means fixed-pinned end conditions.

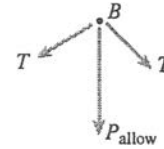
$$P_{\text{cr}} = \frac{2.046\pi^2 EI}{L^2} = 78.67 \text{ kN}$$

$$P_{\text{allow}} = \frac{P_{\text{cr}}}{n} = \frac{78.67 \text{ kN}}{3.0} = 26.22 \text{ kN}$$

DIMENSIONS



FREE-BODY DIAGRAM OF JOINT B



T = tensile force in each cable

P_{allow} = compressive force in tube

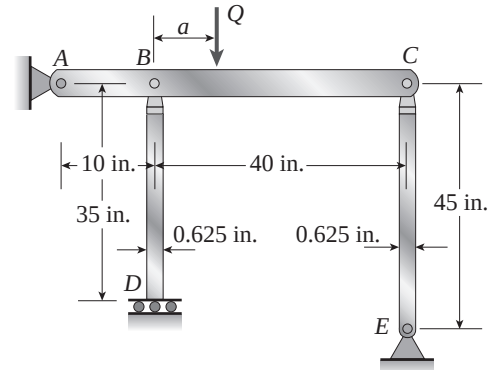
EQUILIBRIUM

$$\sum F_{\text{vert}} = 0 \quad P_{\text{allow}} - 2T \left(\frac{2.1 \text{ m}}{2.9 \text{ m}} \right) = 0$$

ALLOWABLE FORCE IN CABLES

$$T_{\text{allow}} = (P_{\text{allow}}) \left(\frac{1}{2} \right) \left(\frac{2.9 \text{ m}}{2.1 \text{ m}} \right) = 18.1 \text{ kN} \quad \leftarrow$$

Problem 11.4-7 The horizontal beam ABC shown in the figure is supported by column BD and CE . The beam is prevented from moving horizontally by the pin support at end A . Each column is pinned at its upper end to the beam, but at the lower ends, support D is a guided support and support E is pinned. Both columns are solid steel bars ($E = 30 \times 10^6$ psi) of square cross section with width equal to 0.625 in. A load Q acts at distance a from column BD .



- If the distance $a = 12$ in., what is the critical value Q_{cr} of the load?
- If the distance a can be varied between 0 and 40 in., what is the maximum possible value of Q_{cr} ? What is the corresponding value of the distance a ?

Solution

$$E = 30 \times 10^6 \text{ psi} \quad b = 0.625 \text{ in.}$$

$$I = \frac{b^4}{12} \quad I = 0.01272 \text{ in.}^4$$

$$\text{Column BD} \quad L = 35 \text{ in.} \quad P_{crBD} = \frac{\pi^2 EI}{4L^2}$$

$$P_{crBD} = 768.4 \text{ lb}$$

$$\text{Column CE} \quad L = 45 \text{ in.} \quad P_{crCE} = \frac{\pi^2 EI}{L^2}$$

$$P_{crCE} = 1859 \text{ lb}$$

- FIND Q_{cr} IF $a = 12$ in.

The system collapses when both columns buckle.

$$\sum M_A = 0 \quad P_{crBD}(10 \text{ in.}) + P_{crCE}(50 \text{ in.}) - Q_{cr}(a + 10 \text{ in.}) = 0$$

$$Q_{cr} = \frac{P_{crBD}(10 \text{ in.}) + P_{crCE}(50 \text{ in.})}{a + 10 \text{ in.}}$$

$$Q_{cr} = 4575 \text{ lb} \quad \leftarrow$$

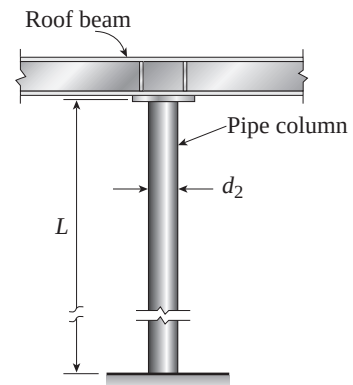
- Q_{cr} IS MAXIMUM WHEN $a = 0$ in.

$$Q_{cr} = \frac{P_{crBD}(10 \text{ in.}) + P_{crCE}(50 \text{ in.})}{a + 10 \text{ in.}}$$

$$Q_{cr} = 10,065 \text{ lb} \quad \leftarrow$$

Problem 11.4-8 The roof beams of a warehouse are supported by pipe columns (see figure on the next page) having outer diameter $d_2 = 100$ mm and inner diameter $d_1 = 90$ mm. The columns have length $L = 4.0$ m, modulus $E = 210$ GPa, and fixed supports at the base.

Calculate the critical load P_{cr} of one of the columns using the following assumptions: (1) the upper end is pinned and the beam prevents horizontal displacement; (2) the upper end is fixed against rotation and the beam prevents horizontal displacement; (3) the upper end is pinned but the beam is free to move horizontally; and (4) the upper end is fixed against rotation, but the beam is free to move horizontally.



Solution 11.4-8 Pipe column (with fixed base)

$$E = 210 \text{ GPa} \quad L = 4.0 \text{ m}$$

$$d_2 = 100 \text{ mm} \quad I = \frac{\pi}{64} (d_2^4 - d_1^4) = 1688 \times 10^3 \text{ mm}^4$$

$$d_1 = 90 \text{ mm}$$

- (1) UPPER END IS PINNED (WITH NO HORIZONTAL DISPLACEMENT)



$$P_{cr} = \frac{2.046\pi^2 EI}{L^2} = 447 \text{ kN} \quad \leftarrow$$

- (2) UPPER END IS FIXED (WITH NO HORIZONTAL DISPLACEMENT)



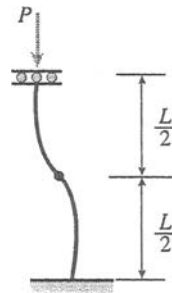
$$P_{cr} = \frac{4\pi^2 EI}{L^2} = 875 \text{ kN} \quad \leftarrow$$

- (3) UPPER END IS PINNED (BUT NO HORIZONTAL RESTRAINT)



$$P_{cr} = \frac{\pi^2 EI}{4L^2} = 54.7 \text{ kN} \quad \leftarrow$$

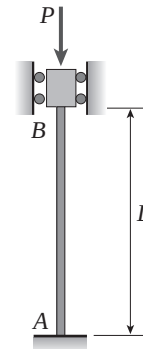
- (4) UPPER END IS GUIDED (NO ROTATION; NO HORIZONTAL RESTRAINT)



The lower half of the column is in the same condition as in part (c).

$$P_{cr} = \frac{\pi^2 EI}{4(L/2)^2} = \frac{\pi^2 EI}{L^2} = 219 \text{ kN} \quad \leftarrow$$

Problem 11.4-9 Determine the critical load P_{cr} and the equation of the buckled shape for an ideal column with ends fixed against rotation (see figure) by solving the differential equation of the deflection curve. (See also Fig. 11-18.)



Solution 11.4-9 Fixed-end column

v = deflection in the y direction

DIFFERENTIAL EQUATION (Eq. 11-4)

$$EIv'' = M = M_0 - Pv \quad k^2 = \frac{P}{EI}$$

$$v'' + k^2v = \frac{M_0}{EI}$$

GENERAL SOLUTION

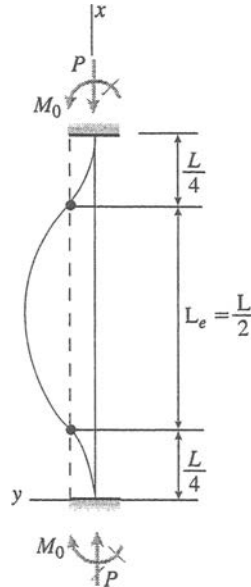
$$v = C_1 \sin kx + C_2 \cos kx + \frac{M_0}{P}$$

$$\text{B.C. 1 } v(0) = 0 \quad \therefore C_2 = -\frac{M_0}{P}$$

$$v = C_1 k \cos kx - C_2 k \sin kx$$

$$\text{B.C. 2 } v'(0) = 0 \quad \therefore C_1 = 0$$

$$v = \frac{M_0}{P}(1 - \cos kx)$$



BUCKLING EQUATION

$$\text{B.C. 3 } v(L) = 0 \quad 0 = \frac{M_0}{P}(1 - \cos kL)$$

$$\therefore \cos kL = 1 \quad \text{and} \quad kL = 2\pi$$

CRITICAL LOAD

$$k^2 = \left(\frac{2\pi}{L}\right)^2 = \frac{4\pi^2}{L^2} \quad \frac{P}{EI} = \frac{4\pi^2}{L^2}$$

$$P_{cr} = \frac{4\pi^2 EI}{L^2} \quad \leftarrow$$

BUCKLED MODE SHAPE

Let δ = deflection at midpoint $\left(x = \frac{L}{2}\right)$

$$v\left(\frac{L}{2}\right) = \delta = \frac{M_0}{P}\left(1 - \cos \frac{kL}{2}\right)$$

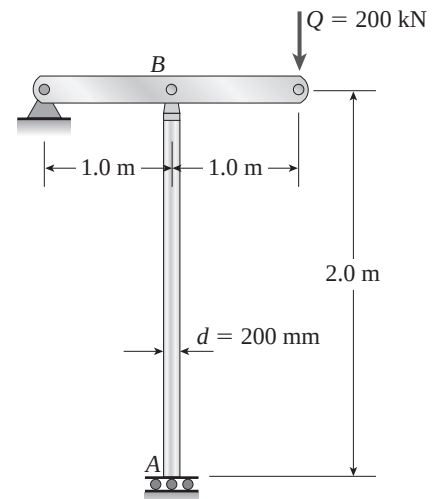
$$\frac{kL}{2} = \pi \quad \therefore \delta = \frac{M_0}{P}(1 - \cos \pi)$$

$$= \frac{2M_0}{P} \quad \frac{M_0}{P} = \frac{\delta}{2}$$

$$v = \frac{\delta}{2}\left(1 - \cos \frac{2\pi x}{L}\right) \quad \leftarrow$$

Problem 11.4-10 An aluminum tube AB of circular cross section has a guided support at the base and is pinned at the top to a horizontal beam supporting a load $Q = 200$ kN (see figure).

Determine the required thickness t of the tube if its outside diameter d is 200 mm and the desired factor of safety with respect to Euler buckling is $n = 3.0$. (Assume $E = 72$ GPa.)



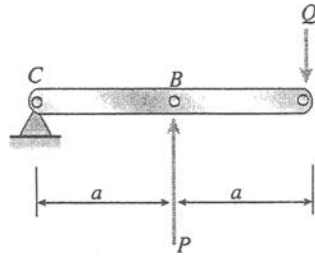
Solution 11.4-10

$$E = 72 \text{ GPa} \quad a = 1.0 \text{ m}$$

$$L = 2.0 \text{ m} \quad n = 3.0$$

$$Q = 200 \text{ kN} \quad d = 200 \text{ mm}$$

FREE-BODY DIAGRAM OF THE BEAM



$$\Sigma M_C = 0 \quad P = 2Q$$

$$P = 400 \text{ kN}$$

CRITICAL LOAD

$$P_{cr} = P \cdot n \quad P_{cr} = 1200 \text{ kN}$$

$$P_{cr} = \frac{\pi^2 EI}{4L^2}$$

$$I = \frac{4P_{cr}L^2}{\pi^2 E} \quad I = 27.019 \times 10^6 \text{ mm}^4$$

MOMENT OF INERTIA

$$I = \frac{\pi}{64} \left[d^4 - (d - 2t)^4 \right]$$

$$t_{\min} = \frac{d - \sqrt[4]{d^4 - I \frac{64}{\pi}}}{2}$$

$$t_{\min} = 10.0 \text{ mm} \quad \leftarrow$$

Problem 11.4-11 The frame ABC consists of two members AB and BC that are rigidly connected at joint B, as shown in part a of the figure. The frame has pin supports at A and C. A concentrated load P acts at joint B, thereby placing member AB in direct compression.

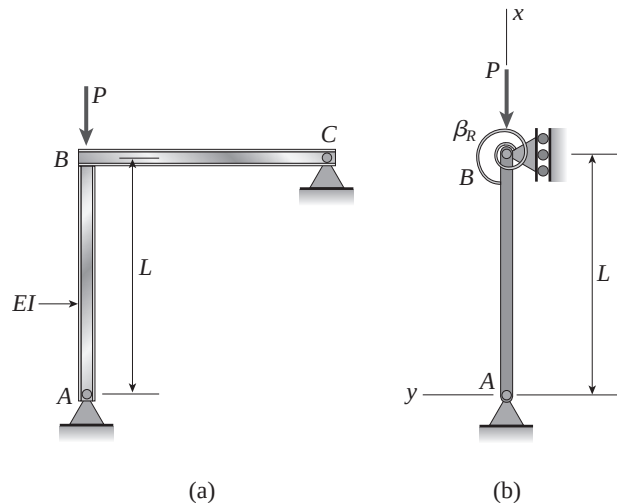
To assist in determining the buckling load for member AB, we represent it as a pinned-end column, as shown in part b of the figure. At the top of the column, a rotational spring of stiffness β_R represents the restraining action of the horizontal beam BC on the column (note that the horizontal beam provides resistance to rotation of joint B when the column buckles). Also, consider only bending effects in the analysis (i.e., disregard the effects of axial deformations).

- (a) By solving the differential equation of the deflection curve, derive the following buckling equation for this column:

$$\frac{\beta_R L}{EI} (kL \cot kL - 1) - k^2 L^2 = 0$$

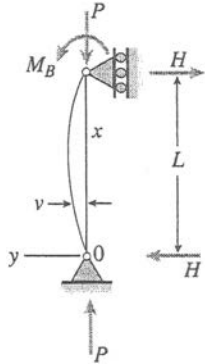
in which L is the length of the column and EI is its flexural rigidity.

- (b) For the particular case when member BC is identical to member AB, the rotation stiffness β_R equals $3EI/L$ (see Case 7, Table H-2, Appendix H). For this special case, determine the critical load P_{cr} .



Solution 11.4-11 Column AB with elastic support at B

FREE-BODY DIAGRAM OF COLUMN

 v = deflection in the y direction M_B = moment at end B θ_B = angle of rotation at end B (positive clockwise) $M_B = \beta_R \theta_B$ H = horizontal reactions at ends A and B

EQUILIBRIUM

$$\sum M_0 = \sum M_A = 0 \quad M_B - HL = 0$$

$$H = \frac{M_B}{L} = \frac{\beta_R \theta_B}{L}$$

DIFFERENTIAL EQUATION (Eq. 11-3)

$$EIv'' = M = Hx - Pv \quad k^2 = \frac{P}{EI}$$

$$v'' + k^2v = \frac{\beta_R \theta_B}{LEI}x$$

GENERAL SOLUTION

$$v = C_1 \sin kx + C_2 \cos kx + \frac{\beta_R \theta_B}{PL}x$$

$$\text{B.C. 1 } v(0) = 0 \quad \therefore C_2 = 0$$

$$\text{B.C. 2 } v(L) = 0 \quad \therefore C_1 = -\frac{\beta_R \theta_B}{P \sin kL}$$

$$v = C_1 \sin kx + \frac{\beta_R \theta_B}{PL}x$$

$$v' = C_1 k \cos kx + \frac{\beta_R \theta_B}{PL}$$

(a) BUCKLING EQUATION

$$\text{B.C. 3 } v'(L) = -\theta_B$$

$$\therefore -\theta_B = -\frac{\beta_R \theta_B}{P \sin kL} (k \cos kL) + \frac{\beta_R \theta_B}{PL}$$

Cancel θ_B and multiply by PL :

$$-PL = -\beta_R kL \cot kL + \beta_R$$

Substitute $P = k^2 EI$ and rearrange:

$$\frac{\beta_R L}{EI} (kL \cot kL - 1) - k^2 L^2 = 0 \quad \leftarrow$$

(b) CRITICAL LOAD FOR $\beta_R = 3EI/L$

$$3(kL \cot kL - 1) - (kL)^2 = 0$$

Solve numerically for kL : $kL = 3.7264$

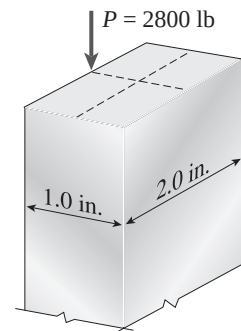
$$P_{cr} = k^2 EI = (kL)^2 \left(\frac{EI}{L^2} \right) = 13.89 \frac{EI}{L^2} \quad \leftarrow$$

Columns with Eccentric Axial Loads

When solving the problems for Section 11.5, assume that bending occurs in the principal plane containing the eccentric axial load.

Problem 11.5-1 An aluminum bar having a rectangular cross section (2.0 in. $\times$ 1.0 in.) and length $L = 30$ in. is compressed by axial loads that have a resultant $P = 2800$ lb acting at the midpoint of the long side of the cross section (see figure).

Assuming that the modulus of elasticity E is equal to 10×10^6 psi and that the ends of the bar are pinned, calculate the maximum deflection δ and the maximum bending moment $M_{\max}$.



Solution 11.5-1 Bar with rectangular cross section

$$b = 2.0 \text{ in.} \quad h = 1.0 \text{ in.} \quad L = 30 \text{ in.}$$

$$P = 2800 \text{ lb} \quad e = 0.5 \text{ in.} \quad E = 10 \times 10^6 \text{ psi}$$

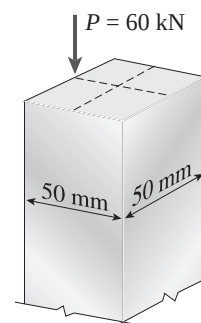
$$I = \frac{bh^3}{12} = 0.1667 \text{ in.}^4 \quad kL = L\sqrt{\frac{P}{EI}} = 1.230$$

$$\text{Eq. (11-59): } \delta = e \left(\sec \frac{kL}{2} - 1 \right) = 0.112 \text{ in.} \quad \leftarrow$$

$$\begin{aligned} \text{Eq. (11-64): } M_{\max} &= Pe \sec \frac{kL}{2} \\ &= 1710 \text{ lb-in.} \quad \leftarrow \end{aligned}$$

Problem 11.5-2 A steel bar having a square cross section (50 mm $\times$ 50 mm) and length $L = 2.0$ m is compressed by axial loads that have a resultant $P = 60$ kN acting at the midpoint of one side of the cross section (see figure).

Assuming that the modulus of elasticity E is equal to 210 GPa and that the ends of the bar are pinned, calculate the maximum deflection δ and the maximum bending moment $M_{\max}$.

**Solution 11.5-2 Bar with square cross section**

$$b = 50 \text{ mm} \quad L = 2 \text{ m} \quad P = 60 \text{ kN} \quad e = 25 \text{ mm}$$

$$E = 210 \text{ GPa} \quad I = \frac{b^4}{12} = 520.8 \times 10^3 \text{ mm}^4$$

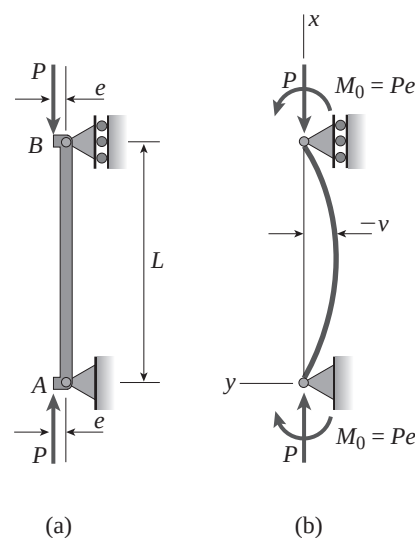
$$kL = L\sqrt{\frac{P}{EI}} = 1.481$$

$$\text{Eq. (11-59): } \delta = e \left(\sec \frac{kL}{2} - 1 \right) = 8.87 \text{ mm} \quad \leftarrow$$

$$\text{Eq. (11-64): } M_{\max} = Pe \sec \frac{kL}{2} = 2.03 \text{ kN}\cdot\text{m} \quad \leftarrow$$

Problem 11.5-3 Determine the bending moment M in the pinned-end column with eccentric axial loads shown in the figure. Then plot the bending-moment diagram for an axial load $P = 0.3P_{cr}$.

Note: Express the moment as a function of the distance x from the end of the column, and plot the diagram in nondimensional form with M/Pe as ordinate and x/L as abscissa.



Probs. 11.5-3 through 11.5-5

Solution 11.5-3 Column with eccentric loads

Column has pinned ends.

Use Eq. (11-57):

$$v = -e \left(\tan \frac{kL}{2} \sin kx + \cos kx - 1 \right)$$

From Eq. (11-53): $M = Pe - Pv$

$$\therefore M = Pe \left(\tan \frac{kL}{2} \sin kx + \cos kx \right) \quad \leftarrow$$

FOR $P = 0.3 P_{cr}$:

$$\begin{aligned} \text{From Eq. (11-61): } kL &= \pi \sqrt{\frac{P}{P_{cr}}} = \pi \sqrt{0.3} \\ &= 1.7207 \end{aligned}$$

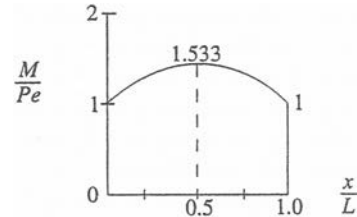
$$\frac{M}{Pe} = \left(\tan \frac{1.7207}{2} \right) \left(\sin 1.7207 \frac{x}{L} \right) + \cos 1.7207 \frac{x}{L}$$

or

$$\frac{M}{Pe} = 1.162 \left(\sin 1.721 \frac{x}{L} \right) + \cos 1.721 \frac{x}{L} \quad \leftarrow$$

(Note: kL and kx are in radians.)

BENDING-MOMENT DIAGRAM FOR $P = 0.3 P_{cr}$



Problem 11.5-4 Plot the load-deflection diagram for a pinned-end column with eccentric axial loads (see figure) if the eccentricity e of the load is 5 mm and the column has length $L = 3.6$ m, moment of inertia $I = 9.0 \times 10^6 \text{ mm}^4$, and modulus of elasticity $E = 210 \text{ GPa}$.

Note: Plot the axial load as ordinate and the deflection at the midpoint as abscissa.

Solution 11.5-4 Column with eccentric loads

Column has pinned ends.

Use Eq. (11-62) for the deflection at the midpoint (maximum deflection):

$$\delta = e \left[\sec \left(\frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} \right) - 1 \right] \quad (1)$$

DATA

$$e = 5.0 \text{ mm} \quad L = 3.6 \text{ m} \quad E = 210 \text{ GPa}$$

$$I = 9.0 \times 10^6 \text{ mm}^4$$

CRITICAL LOAD

$$P_{cr} = \frac{\pi^2 EI}{L^2} = 1439.3 \text{ kN}$$

MAXIMUM DEFLECTION (FROM EQ. 1)

$$\delta = (5.0) [\sec (0.041404 \sqrt{P}) - 1] \quad (2)$$

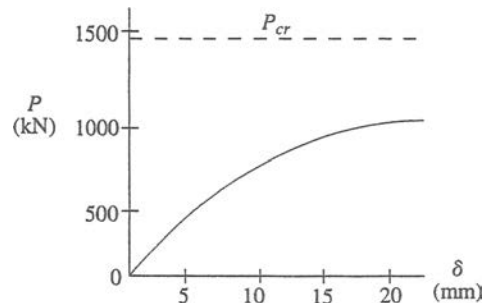
Units: $P = \text{kN}$ $\delta = \text{mm}$

Angles are in radians.

SOLVE EQ. (2) FOR P :

$$P = 583.3 \left[\arccos \left(\frac{5.0}{5.0 + \delta} \right) \right]^2 \quad \leftarrow$$

LOAD-DEFLECTION DIAGRAM



Problem 11.5-5 Solve the preceding problem for a column with $e = 0.20$ in., $L = 12$ ft, $I = 21.7$ in.⁴, and $E = 30 \times 10^6$ psi.

Solution 11.5-5 Column with eccentric loads

Column has pinned ends.

Use Eq. (11-62) for the deflection at the midpoint (maximum deflection):

$$\delta = e \left[\sec \left(\frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} \right) - 1 \right] \quad (1)$$

DATA

$$e = 0.20 \text{ in.} \quad L = 12 \text{ ft} = 144 \text{ in.}$$

$$E = 30 \times 10^6 \text{ psi}$$

$$I = 21.7 \text{ in.}^4$$

CRITICAL LOAD

$$P_{cr} = \frac{\pi^2 EI}{L^2} = 309.9 \text{ k}$$

MAXIMUM DEFLECTION (FROM EQ. 1)

$$\delta = (0.20) [\sec (0.08924 \sqrt{P}) - 1] \quad (2)$$

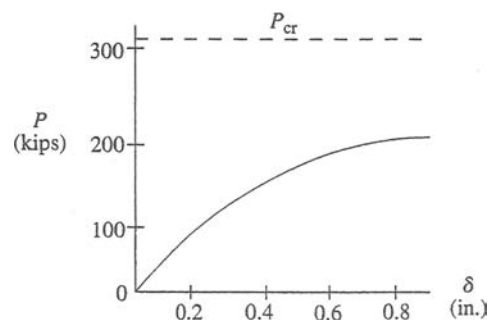
Units: P = kips δ = inches

Angles are in radians.

SOLVE EQ. (2) FOR P :

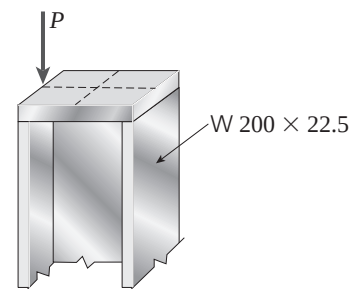
$$P = 125.6 \left[\arccos \left(\frac{0.2}{0.2 + \delta} \right) \right]^2 \quad \leftarrow$$

LOAD-DEFLECTION DIAGRAM



Problem 11.5-6 A wide-flange member (W 200 $\times$ 22.5) is compressed by axial loads that have a resultant P acting at the point shown in the figure. The member has modulus of elasticity $E = 200$ GPa and pinned conditions at the ends. Lateral supports prevent any bending about the weak axis of the cross section.

If the length of the member is 6.2 m, and the deflection is limited to 6.5 mm, what is the maximum allowable load P_{allow} ?



Solution 11.5-6

$$W 200 \times 22.5 \quad E = 200 \text{ GPa} \quad L = 6.2 \text{ m}$$

$$\delta = 6.5 \text{ mm} \quad I = 20 \times 10^6 \text{ mm}^4 \quad d = 206 \text{ mm}$$

$$e = \frac{d}{2} \quad e = 103 \text{ mm}$$

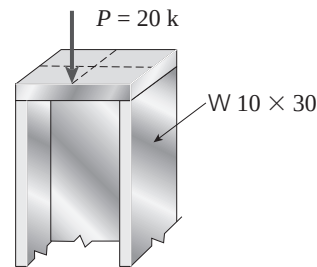
CRITICAL LOAD

$$P_{cr} = \frac{\pi^2 EI}{L^2} \quad P_{cr} = 1027 \text{ kN}$$

$$\text{Maximum deflection } \delta = e \left(\sec \left(\frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} \right) - 1 \right)$$

$$\text{Solve for } P: \quad P_{allow} = 49.9 \text{ kN} \quad \leftarrow$$

Problem 11.5-7 A wide-flange member (W 10 × 30) is compressed by axial loads that have a resultant $P = 20$ k acting at the point shown in the figure. The material is steel with modulus of elasticity $E = 29,000$ ksi. Assuming pinned-end conditions, determine the maximum permissible length $L_{\max}$ if the deflection is not to exceed 1/400th of the length.



Solution 11.5-7 Column with eccentric axial load

Wide-flange member: W 10 × 30

Pinned-end conditions.

Bending occurs about the weak axis (axis 2–2).

$P = 20$ k $E = 29,000$ ksi $L =$ length (inches)

Maximum allowable deflection $= \frac{L}{400}$ ($= \delta$)

From Table F-1: $I = 16.7$ in.⁴

$$e = \frac{5.810 \text{ in.}}{2} = 2.905 \text{ in.}$$

$$k = \sqrt{\frac{P}{EI}} = 0.006426 \text{ in.}^{-1}$$

Deflection at midpoint (Eq. 11-59):

$$\delta = e \left(\sec \frac{kL}{2} - 1 \right)$$

$$\frac{L}{400} = (2.905 \text{ in.}) \left[\sec (0.003213 L) - 1 \right]$$

Rearrange terms and simplify:

$$\sec (0.003213 L) - 1 - \frac{L}{1162 \text{ in.}} = 0$$

(Note: Angles are in radians)

Solve the equation numerically for the length L :

$$L = 150.5 \text{ in.}$$

MAXIMUM ALLOWABLE LENGTH

$$L_{\max} = 150.5 \text{ in.} = 12.5 \text{ ft} \quad \leftarrow$$

Problem 11.5-8 Solve the preceding problem (W 250 × 44.8) if the resultant force P equals 110 kN and $E = 200$ GPa.

Solution 11.5-8

W 250 × 44.8 $E = 200$ GPa

$$P = 110 \text{ kN} \quad \delta = \frac{L}{400}$$

Bending occurs about the weak axis (axis 2–2).

$$I = 6.95 \times 10^6 \text{ mm}^4 \quad b = 148 \text{ mm}$$

$$e = \frac{b}{2} \quad e = 74 \text{ mm}$$

$$k = \sqrt{\frac{P}{EI}} \quad k = 0.000281 \text{ mm}^{-1}$$

Deflection at midpoint:

$$\delta = e \left(\sec \left(\frac{kL}{2} \right) - 1 \right)$$

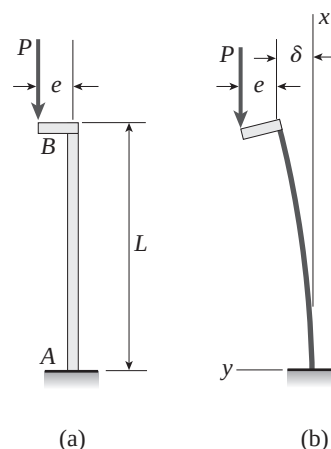
$$\frac{L}{400} = e \left(\sec \left(\frac{kL}{2} \right) - 1 \right)$$

Solve for the length L :

$$L_{\max} = 3.14 \text{ m} \quad \leftarrow$$

Problem 11.5-9 The column shown in the figure is fixed at the base and free at the upper end. A compressive load P acts at the top of the column with an eccentricity e from the axis of the column.

Beginning with the differential equation of the deflection curve, derive formulas for the maximum deflection δ of the column and the maximum bending moment $M_{\max}$ in the column.



Solution 11.5-9 Fixed-free column

e = eccentricity of load P

δ = deflection at the end of the column

v = deflection of the column at distance x from base

DIFFERENTIAL EQUATION (EQ. 11-4)

$$EIv'' = M = P(e + \delta - v) \quad k^2 = \frac{P}{EI}$$

$$v'' = k^2(e + \delta - v)$$

$$v'' + k^2v = k^2(e + \delta)$$

GENERAL SOLUTION

$$v = C_1 \sin kx + C_2 \cos kx + e + \delta$$

$$v' = C_1 k \cos kx - C_2 k \sin kx$$

$$\text{B.C. 1} \quad v(0) = 0 \quad \therefore C_2 = -e - \delta$$

$$\text{B.C. 2} \quad v'(0) = 0 \quad \therefore C_1 = 0$$

$$v = (e + \delta)(1 - \cos kx)$$

$$\text{B.C. 3} \quad v(L) = \delta \quad \therefore \delta = (e + \delta)(1 - \cos kL)$$

$$\text{or} \quad \delta = e(\sec kL - 1)$$

$$\text{MAXIMUM DEFLECTION} \quad \delta = e(\sec kL - 1) \quad \leftarrow$$

MAXIMUM BENDING MOMENT (AT BASE OF COLUMN)

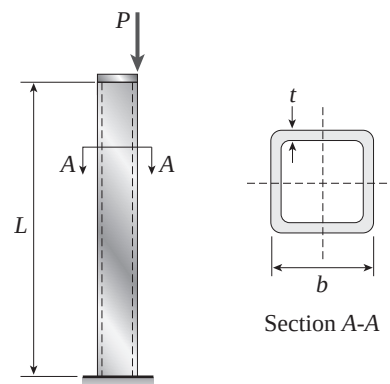
$$M_{\max} = P(e + \delta) = Pe \sec kL \quad \leftarrow$$

$$\text{Note:} \quad v = (e + \delta)(1 - \cos kx)$$

$$= e(\sec kL)(1 - \cos kx)$$

Problem 11.5-10 An aluminum box column of square cross section is fixed at the base and free at the top (see figure). The outside dimension b of each side is 100 mm and the thickness t of the wall is 8 mm. The resultant of the compressive loads acting on the top of the column is a force $P = 50$ kN acting at the outer edge of the column at the midpoint of one side.

What is the longest permissible length $L_{\max}$ of the column if the deflection at the top is not to exceed 30 mm? (Assume $E = 73$ GPa.)



Probs. 11.5-10 and 11.5-11

Solution 11.5-10 Fixed-free column δ = deflection at the topUse Eq. (11-59) with $L/2$ replaced by L :

$$\delta = e (\sec kL - 1) \quad (1)$$

(This same equation is obtained in Prob. 11.5-9.)

SOLVE FOR L FROM EQ. (1)

$$\sec kL = 1 + \frac{\delta}{e} = \frac{e + \delta}{e}$$

$$\cos kL = \frac{e}{e + \delta} \quad kL = \arccos \frac{e}{e + \delta}$$

$$L = \frac{1}{k} \arccos \frac{e}{e + \delta} \quad k = \sqrt{\frac{P}{EI}}$$

$$L = \sqrt{\frac{EI}{P}} \arccos \frac{e}{e + \delta} \quad (2)$$

NUMERICAL DATA

$$E = 73 \text{ GPa} \quad b = 100 \text{ mm} \quad t = 8 \text{ mm}$$

$$P = 50 \text{ kN} \quad \delta = 30 \text{ mm} \quad e = \frac{b}{2} = 50 \text{ mm}$$

$$I = \frac{1}{12} [b^4 - (b - 2t)^4] = 4.1844 \times 10^6 \text{ mm}^4$$

MAXIMUM ALLOWABLE LENGTH

Substitute numerical data into Eq.(2).

$$\sqrt{\frac{EI}{P}} = 2.4717 \text{ m} \quad \frac{e}{e + \delta} = 0.625$$

$$\arccos \frac{e}{e + \delta} = 0.89566 \text{ radians}$$

$$L_{\max} = (2.4717 \text{ m})(0.89566) = 2.21 \text{ m} \quad \leftarrow$$

Problem 11.5-11 Solve the preceding problem for an aluminum column with $b = 6.0$ in., $t = 0.5$ in., $P = 30$ k, and $E = 10.6 \times 10^3$ ksi. The deflection at the top is limited to 2.0 in.

Solution 11.5-11 Fixed-free column δ = deflection at the topUse Eq. (11-59) with $L/2$ replaced by L :

$$\delta = e (\sec kL - 1) \quad (1)$$

(This same equation is obtained in Prob. 11.5-9.)

SOLVE FOR L FROM EQ. (1)

$$\sec kL = 1 + \frac{\delta}{e} = \frac{e + \delta}{e}$$

$$\cos kL = \frac{e}{e + \delta} \quad kL = \arccos \frac{e}{e + \delta}$$

$$L = \frac{1}{k} \arccos \frac{e}{e + \delta} \quad k = \sqrt{\frac{P}{EI}}$$

$$L = \sqrt{\frac{EI}{P}} \arccos \frac{e}{e + \delta}$$

NUMERICAL DATA

$$E = 10.6 \times 10^3 \text{ ksi} \quad b = 6.0 \text{ in.} \quad t = 0.5 \text{ in.}$$

$$P = 30 \text{ k} \quad \delta = 2.0 \text{ in.} \quad e = \frac{b}{2} = 3.0 \text{ in.}$$

$$I = \frac{1}{12} [b^4 - (b - 2t)^4] = 55.917 \text{ in.}^4$$

MAXIMUM ALLOWABLE LENGTH

Substitute numerical data into Eq. (2).

$$\sqrt{\frac{EI}{P}} = 140.56 \text{ in.} \quad \frac{e}{e + \delta} = 0.60$$

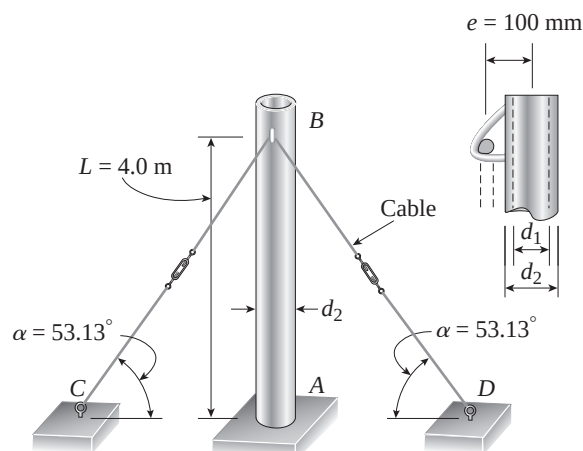
$$\arccos \frac{e}{e + \delta} = 0.92730 \text{ rad}$$

$$L_{\max} = (140.56 \text{ in.})(0.92730) \\ = 130.3 \text{ in.} = 10.9 \text{ ft} \quad \leftarrow$$

Problem 11.5-12 A steel post AB of hollow circular cross section is fixed at the base and free at the top (see figure). The inner and outer diameters are $d_1 = 96$ mm and $d_2 = 110$ mm, respectively, and the length $L = 4.0$ m.

A cable CBD passes through a fitting that is welded to the side of the post. The distance between the plane of the cable (plane CBD) and the axis of the post is $e = 100$ mm, and the angles between the cable and the ground are $\alpha = 53.13^\circ$. The cable is pretensioned by tightening the turnbuckles.

If the deflection at the top of the post is limited to $\delta = 20$ mm, what is the maximum allowable tensile force T in the cable? (Assume $E = 205$ GPa.)



Solution 11.5-12 Fixed-free column

δ = deflection at the top

P = compressive force in post $k = \sqrt{\frac{P}{EI}}$

Use Eq. (11-59) with $L/2$ replaced by L :

$$\delta = e(\sec kL - 1) \quad (1)$$

(This same equation is obtained in Prob. 11.5-9.)

Solve for P from Eq. (1)

$$\sec kL = 1 + \frac{\delta}{e} = \frac{e + \delta}{e}$$

$$\cos kL = \frac{e}{e + \delta} \quad kL = \arccos \frac{e}{e + \delta}$$

$$kL = \sqrt{\frac{PL^2}{EI}} \quad \sqrt{\frac{PL^2}{EI}} = \arccos \frac{e}{e + \delta}$$

Square both sides and solve for P :

$$P = \frac{EI}{L^2} \left(\arccos \frac{e}{e + \delta} \right)^2 \quad (2)$$

NUMERICAL DATA

$$E = 205 \text{ GPa} \quad L = 4.0 \text{ m} \quad e = 100 \text{ mm}$$

$$\delta = 20 \text{ mm} \quad d_2 = 110 \text{ mm} \quad d_1 = 96 \text{ mm}$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) = 3.0177 \times 10^6 \text{ mm}^4$$

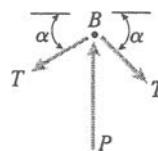
MAXIMUM ALLOWABLE COMPRESSIVE FORCE P

Substitute numerical data into Eq. (2).

$$P_{\text{allow}} = 13,263 \text{ N} = 13.263 \text{ kN}$$

MAXIMUM ALLOWABLE TENSILE FORCE T IN THE CABLE

Free-body diagram of joint B :



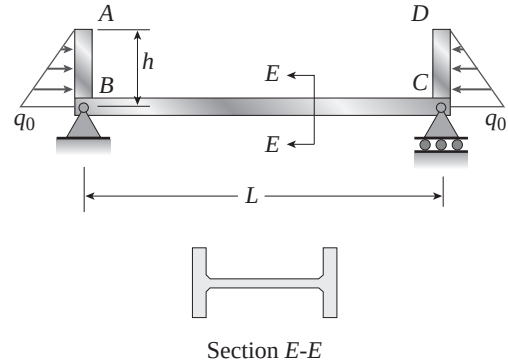
$$\alpha = 53.13^\circ$$

$$\sum F_{\text{vert}} = 0 \quad P - 2T \sin \alpha = 0$$

$$T = \frac{P}{2 \sin \alpha} = \frac{5P}{8} = 8289 \text{ N}$$

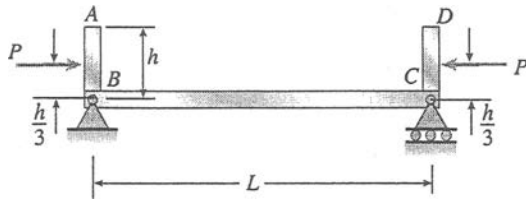
$$\therefore T_{\text{max}} = 8.29 \text{ kN} \quad \leftarrow$$

Problem 11.5-13 A frame $ABCD$ is constructed of steel wide-flange members ($W 8 \times 21$; $E = 30 \times 10^6$ psi) and subjected to triangularly distributed loads of maximum intensity q_0 acting along the vertical members (see figure). The distance between supports is $L = 20$ ft and the height of the frame is $h = 4$ ft. The members are rigidly connected at B and C .



- Calculate the intensity of load q_0 required to produce a maximum bending moment of 80 k-in. in the horizontal member of BC .
- If the load q_0 is reduced to one-half of the value calculated in part (a), what is the maximum bending moment in member BC ? What is the ratio of this moment to the moment of 80 k-in. in part (a)?

Solution 11.5-13 Frame with triangular loads



P = resultant force

e = eccentricity

$$P = \frac{q_0 h}{2} \quad e = \frac{h}{3}$$

MAXIMUM BENDING MOMENT IN BEAM BC

From Eq. (11-64): $M_{\max} = Pe \sec \frac{kL}{2}$

$$k = \sqrt{\frac{P}{EI}} \quad \therefore M_{\max} = Pe \sec \sqrt{\frac{PL^2}{4EI}} \quad (1)$$

NUMERICAL DATA

$W 8 \times 21 \quad I = I_2 = 9.77 \text{ in.}^4$ (from Table F-1a)

$E = 30 \times 10^6 \text{ psi} \quad L = 20 \text{ ft} = 240 \text{ in.}$

$h = 4 \text{ ft} = 48 \text{ in.}$

$e = \frac{h}{3} = 16 \text{ in.}$

- (a) LOAD q_0 TO PRODUCE $M_{\max} = 80 \text{ k-in.}$

Substitute numerical values into Eq. (1).

Units: pounds and inches

$$M_{\max} = 80,000 \text{ lb-in.} = \sqrt{\frac{PL^2}{4EI}} \\ = 0.1170093 \sqrt{P} \quad (\text{radians})$$

$$80,000 = P(16 \text{ in.}) [\sec (0.0070093 \sqrt{P})]$$

$$5,000 = P \sec (0.0070093 \sqrt{P})$$

$$P - 5,000 [\cos (0.0070093 \sqrt{P})] = 0 \quad (2)$$

SOLVE EQ. (2) NUMERICALLY

$$P = 4461.9 \text{ lb}$$

$$q_0 = \frac{2P}{h} = 186 \text{ lb/in.} = 2230 \text{ lb/ft} \quad \leftarrow$$

- (b) LOAD q_0 IS REDUCED TO ONE-HALF ITS VALUE

$\therefore P$ is reduced to one-half its value.

$$P = \frac{1}{2} (4461.9 \text{ lb}) = 2231.0 \text{ lb}$$

Substitute numerical values into Eq. (1) and solve for $M_{\max}$.

$$M_{\max} = 37.75 \text{ k-in.} \quad \leftarrow$$

$$\text{Ratio: } \frac{M_{\max}}{80 \text{ k-in.}} = \frac{37.7}{80} = 0.47 \quad \leftarrow$$

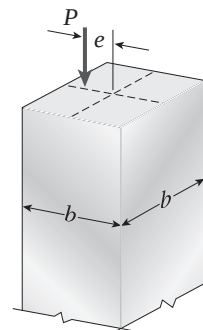
This result shows that the bending moment varies nonlinearly with the load.

The Secant Formula

When solving the problems for Section 11.6, assume that bending occurs in the principal plane containing the eccentric axial load.

Problem 11.6-1 A steel bar has a square cross section of width $b = 2.0$ in. (see figure). The bar has pinned supports at the ends and is 3.0 ft long. The axial forces acting at the end of the bar have a resultant $P = 20$ k located at distance $e = 0.75$ in. from the center of the cross section. Also, the modulus of elasticity of the steel is 29,000 ksi.

- Determine the maximum compressive stress $\sigma_{\max}$ in the bar.
- If the allowable stress in the steel is 18,000 psi, what is the maximum permissible length $L_{\max}$ of the bar?



Probs. 11.6-1 through 11.6-3

Solution 11.6-1 Bar with square cross section

Pinned supports.

DATA

$$b = 2.0 \text{ in.} \quad L = 3.0 \text{ ft} = 36 \text{ in.} \quad P = 20 \text{ k}$$

$$e = 0.75 \text{ in.} \quad E = 29,000 \text{ ksi}$$

- (a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-67):

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right] \quad (1)$$

$$\frac{P}{A} = \frac{P}{b^2} = 5.0 \text{ ksi} \quad c = \frac{b}{2} = 1.0 \text{ in.}$$

$$I = \frac{b^4}{12} = 1.333 \text{ in.}^4 \quad r^2 = \frac{I}{A} = 0.3333 \text{ in.}^2$$

$$\frac{ec}{r^2} = 2.25 \quad \frac{L}{r} = 62.354 \quad \frac{P}{EA} = 0.00017241$$

Substitute into Eq. (1):

$$\sigma_{\max} = 17.3 \text{ ksi} \quad \leftarrow$$

- (b) MAXIMUM PERMISSIBLE LENGTH

$$\sigma_{\text{allow}} = 18,000 \text{ psi}$$

Solve Eq. (1) for the length L :

$$L = 2 \sqrt{\frac{EI}{P}} \arccos \left[\frac{P(ec/r^2)}{\sigma_{\max} A - P} \right] \quad (2)$$

Substitute numerical values:

$$L_{\max} = 46.2 \text{ in.} \quad \leftarrow$$

Problem 11.6-2 A brass bar ($E = 100$ GPa) with a square cross section is subjected to axial force having a resultant P acting at distance e from the center (see figure). The bar is pin supported at the ends and is 0.6 m in length. The side dimension b of the bar is 30 mm and the eccentricity e of the load is 10 mm.

If the allowable stress in the brass is 150 MPa, what is the allowable axial force P_{allow} ?

Solution 11.6-2 Bar with square cross section

Pinned supports.

DATA

$$b = 30 \text{ mm} \quad L = 0.6 \text{ m} \quad \sigma_{\text{allow}} = 150 \text{ MPa}$$

$$e = 10 \text{ mm} \quad E = 100 \text{ GPa}$$

SECANT FORMULA (Eq. 11-67):

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right] \quad (1)$$

Units: Newtons and meters

$$\sigma_{\max} = 150 \times 10^6 \text{ N/m}^2$$

$$A = b^2 = 900 \times 10^{-6} \text{ m}^2$$

$$c = \frac{b}{2} = 0.015 \text{ m} \quad r^2 = \frac{I}{A} = \frac{b^2}{12} = 75 \times 10^{-6} \text{ m}^2$$

$$\frac{ec}{r^2} = 2.0 \quad P = \text{newtons} \quad \frac{L}{2r} \sqrt{\frac{P}{EA}} = 0.0036515 \sqrt{P}$$

SUBSTITUTE NUMERICAL VALUES INTO EQ. (1):

$$150 \times 10^6 = \frac{P}{900 \times 10^{-6}} [1 + 2 \sec(0.0036515 \sqrt{P})]$$

or

$$P[1 + 2 \sec(0.0036515 \sqrt{P})] - 135,000 = 0 \quad (2)$$

SOLVE EQ. (2) NUMERICALLY:

$$P_{\text{allow}} = 37,200 \text{ N} = 37.2 \text{ kN} \quad \leftarrow$$

Problem 11.6-3 A square aluminum bar with pinned ends carries a load $P = 25 \text{ k}$ acting at distance $e = 2.0 \text{ in.}$ from the center (see figure). The bar has length $L = 54 \text{ in.}$ and modulus of elasticity $E = 10,600 \text{ ksi.}$

If the stress in the bar is not exceed 6 ksi, what is the minimum permissible width $b_{\min}$ of the bar?

Solution 11.6-3 Square aluminum bar

Pinned ends.

DATA

Units: pounds and inches

$$P = 25 \text{ k} = 25,000 \text{ lb} \quad e = 2.0 \text{ in.}$$

$$L = 54 \text{ in.} \quad E = 10,600 \text{ ksi} = 10,600,000 \text{ psi}$$

$$\sigma_{\max} = 6.0 \text{ ksi} = 6,000 \text{ psi}$$

SECANT FORMULA (EQ. 11-67):

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right] \quad (1)$$

$$A = b^2 \quad c = \frac{b}{2} \quad r^2 = \frac{I}{A} = \frac{b^2}{12}$$

$$\frac{ec}{r^2} = \frac{12}{b} \quad \frac{L}{2r} \sqrt{\frac{P}{EA}} = \frac{4.5423}{b^2}$$

SUBSTITUTE TERMS INTO EQ. (1):

$$6,000 = \frac{25,000}{b^2} \left[1 + \frac{12}{b} \sec \left(\frac{4.5423}{b^2} \right) \right]$$

or

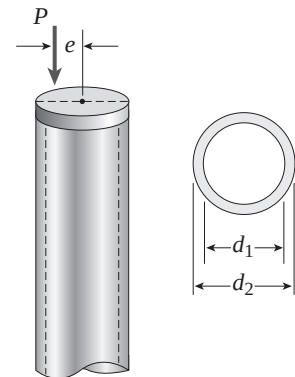
$$1 + \frac{12}{b} \sec \left(\frac{4.5423}{b^2} \right) - 0.24 b^2 = 0 \quad (2)$$

SOLVE EQ. (2) NUMERICALLY:

$$b_{\min} = 4.10 \text{ in.} \quad \leftarrow$$

Problem 11.6-4 A pinned-end column of length $L = 2.1 \text{ m}$ is constructed of steel pipe ($E = 210 \text{ GPa}$) having inside diameter $d_1 = 60 \text{ mm}$ and outside diameter $d_2 = 68 \text{ mm}$ (see figure). A compressive load $P = 10 \text{ kN}$ acts with eccentricity $e = 30 \text{ mm.}$

- What is the maximum compressive stress $\sigma_{\max}$ in the column?
- If the allowable stress in the steel is 50 MPa, what is the maximum permissible length $L_{\max}$ of the column?

**Probs. 11.6-4 through 11.6-6**

Solution 11.6-4 Steel pipe column

Pinned ends.

DATA Units: Newtons and meters

$$\begin{aligned} L &= 2.1 \text{ m} & E &= 210 \text{ GPa} = 210 \times 10^9 \text{ N/m}^2 \\ d_1 &= 60 \text{ mm} = 0.06 \text{ m} & d_2 &= 68 \text{ mm} = 0.068 \text{ m} \\ P &= 10 \text{ kN} = 10,000 \text{ N} & e &= 30 \text{ mm} = 0.03 \text{ m} \end{aligned}$$

TUBULAR CROSS SECTION

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 804.25 \times 10^{-6} \text{ m}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 413.38 \times 10^{-9} \text{ m}^4$$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-67):

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right] \quad (1)$$

$$\frac{P}{A} = 12.434 \times 10^6 \text{ N/m}^2$$

$$r^2 = \frac{I}{A} = 513.99 \times 10^{-6} \text{ m}^2$$

$$r = 22.671 \times 10^{-3} \text{ m} \quad c = \frac{d_2}{2} = 0.034 \text{ m}$$

$$\frac{ec}{r^2} = 1.9845 \quad \frac{L}{2r} \sqrt{\frac{P}{EA}} = 0.35638$$

Substitute into Eq. (1):

$$\sigma_{\max} = 38.8 \times 10^6 \text{ N/m}^2 = 38.8 \text{ MPa} \quad \leftarrow$$

(b) MAXIMUM PERMISSIBLE LENGTH

$$\sigma_{\text{allow}} = 50 \text{ MPa}$$

Solve Eq. (1) for the length L :

$$L = 2 \sqrt{\frac{EI}{P}} \arccos \left[\frac{P(ec/r^2)}{\sigma_{\max}A - P} \right] \quad (2)$$

Substitute numerical values:

$$L_{\max} = 5.03 \text{ m} \quad \leftarrow$$

Problem 11.6-5 A pinned-end strut of length $L = 5.2 \text{ ft}$ is constructed of steel pipe ($E = 30 \times 10^3 \text{ ksi}$) having inside diameter $d_1 = 2.0 \text{ in.}$ and outside diameter $d_2 = 2.2 \text{ in.}$ (see figure). A compressive load $P = 2.0 \text{ k}$ is applied with eccentricity $e = 1.0 \text{ in.}$

- What is the maximum compressive stress $\sigma_{\max}$ in the strut?
- What is the allowable load P_{allow} if a factor of safety $n = 2$ with respect to yielding is required? (Assume that the yield stress σ_Y of the steel is 42 ksi.)

Solution 11.6-5 Pinned-end strut

Steel pipe.

DATA Units: kips and inches

$$\begin{aligned} L &= 5.2 \text{ ft} = 62.4 \text{ in.} & E &= 30 \times 10^3 \text{ ksi} \\ d_1 &= 2.0 \text{ in.} & d_2 &= 2.2 \text{ in.} \\ P &= 2.0 \text{ k} & e &= 1.0 \text{ in.} \end{aligned}$$

TUBULAR CROSS SECTION

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = 0.65973 \text{ in.}^2$$

$$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 0.36450 \text{ in.}^4$$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-67):

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right] \quad (1)$$

$$\frac{P}{A} = 3.0315 \text{ ksi} \quad c = \frac{d_2}{2} = 1.1 \text{ in.}$$

$$r^2 = \frac{I}{A} = 0.55250 \text{ in.}^2 \quad \frac{ec}{r^2} = 1.9910$$

$$r = 0.74330 \text{ in.} \quad \frac{L}{2r} \sqrt{\frac{P}{EA}} = 0.42195$$

Substitute into Eq. (1):

$$\sigma_{\max} = 9.65 \text{ ksi} \quad \leftarrow$$

(b) ALLOWABLE LOAD

$$\sigma_Y = 42 \text{ ksi} \quad n = 2 \quad \text{Find } P_{\text{allow}}.$$

Substitute numerical values into Eq. (1):

$$42 = \frac{P}{0.65973} [1 + 1.9910 \sec(0.29836 \sqrt{P})] \quad (2)$$

Solve Eq. (2) numerically: $P = P_Y = 7.184 \text{ k}$

$$P_{\text{allow}} = \frac{P_Y}{n} = 3.59 \text{ k} \quad \leftarrow$$

Problem 11.6-6 A circular aluminum tube with pinned ends supports a load $P = 18 \text{ kN}$ acting at distance $e = 50 \text{ mm}$ from the center (see figure). The length of the tube is 3.5 m and its modulus of elasticity is 73 GPa .

If the maximum permissible stress in the tube is 20 MPa , what is the required outer diameter d_2 if the ratio of diameter is to be $d_1/d_2 = 0.9$?

Solution 11.6-6 Aluminum tube

Pinned ends.

$$\text{DATA} \quad P = 18 \text{ kN} \quad e = 50 \text{ mm}$$

$$L = 3.5 \text{ m} \quad E = 73 \text{ GPa}$$

$$\sigma_{\max} = 20 \text{ MPa} \quad d_1/d_2 = 0.9$$

SECANT FORMULA (Eq. 11-67)

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right]$$

$$A = \frac{\pi}{4} (d_2^2 - d_1^2) = \frac{\pi}{4} [d_2^2 - (0.9d_2)^2] = 0.14923d_2^2$$

$$(d_2 = \text{mm}; A = \text{mm}^2)$$

$$\frac{P}{A} = \frac{18,000 \text{ N}}{0.14923d_2^2} = \frac{120,620}{d_2^2} \left(\frac{P}{A} = \text{MPa} \right)$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) = \frac{\pi}{64} [d_2^4 - (0.9d_2)^4] = 0.016881d_2^4$$

$$(d_2 = \text{mm}; I = \text{mm}^4)$$

$$r^2 = \frac{I}{A} = 0.11313d_2^2 \quad (d_2 = \text{mm}; r^2 = \text{mm}^2)$$

$$r = 0.33634 d_2 \quad (r = \text{mm})$$

$$c = \frac{d_2}{2} \quad \frac{ec}{r^2} = \frac{(50 \text{ mm})(d_2/2)}{0.11313 d_2^2} = \frac{220.99}{d_2}$$

$$\frac{L}{2r} = \frac{3500 \text{ mm}}{2(0.33634 d_2)} = \frac{5,203.1}{d_2}$$

$$\frac{P}{EA} = \frac{18,000 \text{ N}}{(73,000 \text{ N/mm}^2)(0.14923 d_2^2)} = \frac{1.6524}{d_2^2}$$

$$\frac{L}{2r} \sqrt{\frac{P}{EA}} = \frac{5,203.1}{d_2} \sqrt{\frac{1.6524}{d_2^2}} = \frac{6688.2}{d_2^2}$$

SUBSTITUTE THE ABOVE EXPRESSIONS INTO EQ. (1):

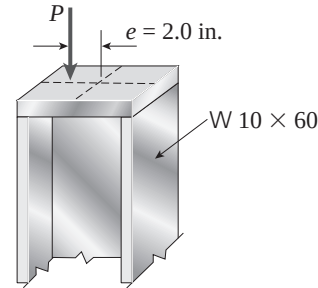
$$\begin{aligned} \sigma_{\max} = 20 \text{ MPa} &= \frac{120,620}{d_2^2} \\ &+ \left[1 + \frac{220.99}{d_2} \sec \left(\frac{6688.2}{d_2^2} \right) \right] \end{aligned} \quad (2)$$

SOLVE EQ. (2) NUMERICALLY:

$$d_2 = 131 \text{ mm} \quad \leftarrow$$

Problem 11.6-7 A steel column ($E = 30 \times 10^3$ ksi) with pinned ends is constructed of a W 10 $\times$ 60 wide-flange shape (see figure). The column is 24 ft long. The resultant of the axial loads acting on the column is a force P acting with eccentricity $e = 2.0$ in.

- If $P = 120$ k, determine the maximum compressive stress $\sigma_{\max}$ in the column.
- Determine the allowable load P_{allow} if the yield stress is $\sigma_Y = 42$ ksi and the factor of safety with respect to yielding of the material is $n = 2.5$.



Solution 11.6-7 Steel column with pinned ends

$$E = 30 \times 10^3 \text{ ksi} \quad L = 24 \text{ ft} = 288 \text{ in.}$$

$$e = 2.0 \text{ in.}$$

W 10 $\times$ 60 wide-flange shape

$$A = 17.6 \text{ in.}^2 \quad I = 341 \text{ in.}^4 \quad d = 10.22 \text{ in.}$$

$$r^2 = \frac{I}{A} = 19.38 \text{ in.}^2 \quad r = 4.402 \text{ in.} \quad c = \frac{d}{2} = 5.11 \text{ in.}$$

$$\frac{L}{r} = 65.42 \quad \frac{ec}{r^2} = 0.5273$$

- (a) MAXIMUM COMPRESSIVE STRESS ($P = 120$ k)

Secant formula (Eq. 11-67):

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right] \quad (1)$$

$$\frac{P}{A} = 6.818 \text{ ksi} \quad \frac{L}{2r} \sqrt{\frac{P}{EA}} = 0.4931$$

$$\text{Substitute into Eq. (1): } \sigma_{\max} = 10.9 \text{ ksi} \quad \leftarrow$$

- (b) ALLOWABLE LOAD

$$\sigma_Y = 42 \text{ ksi} \quad n = 2.5 \quad \text{Find } P_{\text{allow}}$$

Substitute into Eq. (1):

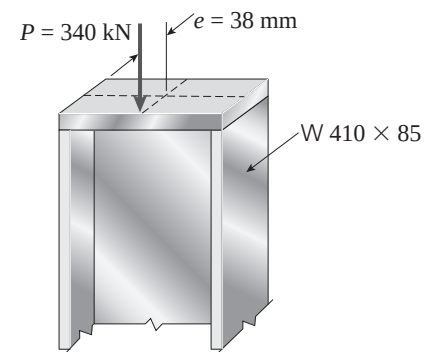
$$42 = \frac{P}{17.6} [1 + 0.5273 \sec (0.04502 \sqrt{P})]$$

Solve numerically: $P = P_Y = 399.9$ k

$$P_{\text{allow}} = P_Y/n = 160 \text{ k} \quad \leftarrow$$

Problem 11.6-8 A W 410 $\times$ 85 steel column is compressed by a force $P = 340$ kN acting with an eccentricity $e = 38$ mm., as shown in the figure. The column has pinned ends and length L . Also, the steel has modulus of elasticity $E = 200$ GPa and yield stress $\sigma_Y = 250$ MPa.

- If the length $L = 3$ m, what is the maximum compressive stress $\sigma_{\max}$ in the column?
- If a factor of safety $n = 2.0$ is required with respect to yielding, what is the longest permissible length $L_{\max}$ of the column?



Solution 11.6-8

$$W\ 410 \times 85 \quad A = 10800\text{ mm}^2 \quad I = I_2$$

$$I = 17.9 \times 10^6\text{ mm}^4 \quad b = 181\text{ mm} \quad c = \frac{b}{2}$$

$$c = 90.5\text{ mm}$$

$$e = 38\text{ mm} \quad r = \sqrt{\frac{I}{A}} \quad r = 40.711\text{ mm}$$

$$P = 340\text{ kN} \quad E = 200\text{ GPa} \quad L = 3\text{ m}$$

(a) MAXIMUM COMPRESSION STRESS

$$\sigma_{\max} = \frac{P}{A} \left(1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right)$$

$$\sigma_{\max} = 104.5\text{ MPa} \quad \leftarrow$$

(b) MAXIMUM LENGTH FOR

$$\sigma_y = 250\text{ MPa} \quad n = 2.0$$

$$P_y = nP \quad P_y = 680\text{ kN}$$

$$\text{from} \quad \sigma_y = \frac{P_y}{A} \left(1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P_y}{EA}} \right) \right)$$

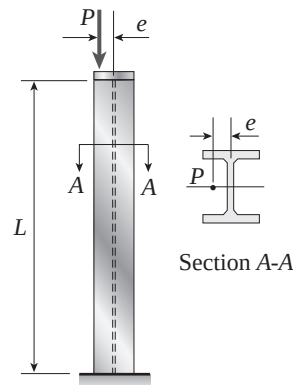
Solve for the length L :

$$L_{\max} = 2 \sqrt{\frac{EI}{P_y}} \arccos \left[\frac{P_y \left(\frac{ec}{r^2} \right)}{\sigma_y A - P_y} \right]$$

$$L_{\max} = 3.66\text{ m} \quad \leftarrow$$

Problem 11.6-9 A steel column ($E = 30 \times 10^3$ ksi) that is fixed at the base and free at the top is constructed of a W 8 $\times$ 35 wide-flange member (see figure). The column is 9.0 ft long. The force P acting at the top of the column has an eccentricity $e = 1.25$ in.

- (a) If $P = 40$ k, what is the maximum compressive stress in the column?
 (b) If the yield stress is 36 ksi and the required factor of safety with respect to yielding is 2.1, what is allowable load P_{allow} ?

**Probs. 11.6-9 and 11.6-10****Solution 11.6-9 Steel column (fixed-free)**

$$E = 30 \times 10^3\text{ ksi} \quad e = 1.25\text{ in.}$$

$$L_e = 2L = 2(9.0\text{ ft}) = 18\text{ ft} = 216\text{ in.}$$

W 8 $\times$ 35 WIDE-FLANGE SHAPE

$$A = 10.3\text{ in.}^2 \quad I = I_2 = 42.6\text{ in.}^4 \quad b = 8.020\text{ in.}$$

$$r^2 = \frac{I}{A} = 4.136\text{ in.}^2 \quad r = 2.034\text{ in.}$$

$$c = \frac{b}{2} = 4.010\text{ in.} \quad \frac{L_e}{r} = 106.2 \quad \frac{ec}{r^2} = 1.212$$

(a) MAXIMUM COMPRESSION STRESS ($P = 40$ k)

Secant formula (Eq. 11-67):

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{L_e}{2r} \sqrt{\frac{P}{EA}} \right) \right] \quad (1)$$

$$\frac{P}{A} = 3.883\text{ ksi} \quad \frac{L_e}{2r} \sqrt{\frac{P}{EA}} = 0.6042$$

$$\text{Substitute into Eq. (1): } \sigma_{\max} = 9.60\text{ ksi} \quad \leftarrow$$

(b) ALLOWABLE LOAD

$$\sigma_y = 36\text{ ksi} \quad n = 2.1 \quad \text{Find } P_{\text{allow}}.$$

Substitute into Eq. (1):

$$36 = \frac{P}{10.3} [1 + 1.212 \sec (0.09552 \sqrt{P})]$$

$$\text{Solve numerically: } P = P_y = 112.6\text{ k}$$

$$P_{\text{allow}} = P_y/n = 53.6\text{ k} \quad \leftarrow$$

Problem 11.6-10 A W 310 × 74 wide-flange steel column with length $L = 3.8$ m is fixed at the base and free at the top (see figure). The load P acting on the column is intended to be centrally applied, but because of unavoidable discrepancies in construction, an eccentricity ratio of 0.25 is specified. Also, the following data are supplied: $E = 200$ GPa, $\sigma_Y = 290$ MPa and $P = 310$ kN.

- (a) What is the maximum compressive stress $\sigma_{\max}$ in the column?
 (b) What is the factor of safety n with respect to yielding of the steel?

Solution 11.6-10

$$\text{W } 310 \times 74 \quad A = 9420 \text{ mm}^2 \quad I = I_2$$

$$I = 23.4 \times 10^6 \text{ mm}^4 \quad r = \sqrt{\frac{I}{A}} \quad r = 49.841 \text{ mm}$$

$$\frac{ec}{r^2} = 0.25 \quad P = 310 \text{ kN} \quad E = 200 \text{ GPa}$$

$$L = 3.8 \text{ m} \quad L_e = 2L \quad L_e = 7.6 \text{ m}$$

- (a) MAXIMUM COMPRESSION STRESS

$$\sigma_{\max} = \frac{P}{A} \left(1 + \frac{ec}{r^2} \sec \left(\frac{L_e}{2r} \sqrt{\frac{P}{EA}} \right) \right)$$

$$\sigma_{\max} = 47.6 \text{ MPa} \quad \leftarrow$$

- (b) FACTOR OF SAFETY WITH RESPECT TO
YIELDING $\sigma_Y = 290$ MPa

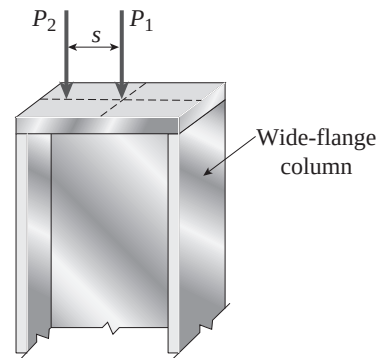
$$\text{From} \quad \sigma_Y = \frac{P_y}{A} \left(1 + \frac{ec}{r^2} \sec \left(\frac{L_e}{2r} \sqrt{\frac{P_y}{EA}} \right) \right)$$

Solve numerically for P_y :

$$P_y = 712 \text{ kN} \quad n = \frac{P_y}{P} \quad n = 2.30 \quad \leftarrow$$

Problem 11.6-11 A pinned-end column with length $L = 18$ ft is constructed from a W 12 × 87 wide-flange shape (see figure). The column is subjected to a centrally applied load $P_1 = 180$ k and an eccentrically applied load $P_2 = 75$ k. The load P_2 acts at distance $s = 5.0$ in. from the centroid of the cross section. The properties of the steel are $E = 29,000$ ksi and $\sigma_Y = 36$ ksi.

- (a) Calculate the maximum compressive stress in the column.
 (b) Determine the factor of safety with respect to yielding.



Probs. 11.6.11 and 11.6.12

Solution 11.6-11 Column with two loads

Pinned-end column. W 12 × 87

DATA

$$L = 18 \text{ ft} = 216 \text{ in.}$$

$$P_1 = 180 \text{ k} \quad P_2 = 75 \text{ k} \quad s = 5.0 \text{ in.}$$

$$E = 29,000 \text{ ksi} \quad \sigma_Y = 36 \text{ ksi}$$

$$P = P_1 + P_2 = 255 \text{ k} \quad e = \frac{P_2 s}{P} = 1.471 \text{ in.}$$

$$A = 25.6 \text{ in.}^2 \quad I = I_1 = 740 \text{ in.}^4 \quad d = 12.53 \text{ in.}$$

$$r^2 = \frac{I}{A} = 28.91 \text{ in.}^2 \quad r = 5.376 \text{ in.}$$

$$c = \frac{d}{2} = 6.265 \text{ in.} \quad \frac{ec}{r^2} = 0.3188$$

$$\frac{P}{A} = 9.961 \text{ ksi} \quad \frac{L}{2r} \sqrt{\frac{P}{EA}} = 0.3723$$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-67):

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right] \quad (1)$$

$$\text{Substitute into Eq. (1):} \quad \sigma_{\max} = 13.4 \text{ ksi} \quad \leftarrow$$

(b) FACTOR OF SAFETY WITH RESPECT TO YIELDING

$$\sigma_{\max} = \sigma_Y = 36 \text{ ksi} \quad P = P_Y$$

Substitute into Eq. (1):

$$36 = \frac{P_Y}{25.6} [1 + 0.3188 \sec(0.02332 \sqrt{P_Y})]$$

$$\text{Solve numerically:} \quad P_Y = 664.7 \text{ k}$$

$$P = 255 \text{ k} \quad n = \frac{P_Y}{P} = \frac{664.7 \text{ k}}{255 \text{ k}} = 2.61 \quad \leftarrow$$

Problem 11.6-12 The wide-flange pinned-end column shown in the figure carries two loads, a force $P_1 = 450 \text{ kN}$ acting at the centroid and a force $P_2 = 270 \text{ kN}$ acting at distance $s = 100 \text{ mm}$, from the centroid. The column is a W 250 × 67 shape with $L = 4.2 \text{ m}$, $E = 200 \text{ GPa}$, and $\sigma_Y = 290 \text{ MPa}$.

- (a) What is the maximum compressive stress in the column?
 (b) If the load P_1 remains at 450 kN, what is the largest permissible value of the load P_2 in order to maintain a factor of safety of 2.0 with respect to yielding?

Solution 11.6-12W 250 × 67 $L = 4.2 \text{ m}$

$$P_1 = 450 \text{ kN} \quad P_2 = 270 \text{ kN}$$

$$s = 100 \text{ mm} \quad E = 200 \text{ GPa}$$

$$\sigma_Y = 290 \text{ MPa} \quad P = P_1 + P_2$$

$$P = 720 \text{ kN} \quad e = \frac{P_2 s}{P} \quad e = 37.5 \text{ mm}$$

$$A = 8580 \text{ mm}^2 \quad I = I_1 \quad I = 103 \times 10^6 \text{ mm}^4$$

$$d = 257 \text{ mm} \quad r = \sqrt{\frac{I}{A}} \quad r = 109.6 \text{ mm}$$

$$c = \frac{d}{2} \quad c = 128.5 \text{ mm}$$

(a) MAXIMUM COMPRESSION STRESS

$$\sigma_{\max} = \frac{P}{A} \left(1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P}{EA}} \right) \right)$$

$$\sigma_{\max} = 120.4 \text{ MPa} \quad \leftarrow$$

(b) LARGEST VALUE OF LOAD P_2 WHEN

$$P_1 = 450 \text{ kN} \quad n = 2.0$$

$$P_Y = n(P_1 + P_2)$$

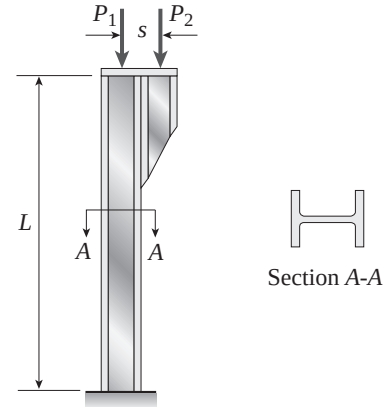
$$\text{From} \quad \sigma_Y = \frac{P_Y}{A} \left(1 + \frac{ec}{r^2} \sec \left(\frac{L}{2r} \sqrt{\frac{P_Y}{EA}} \right) \right)$$

$$\sigma_Y = \frac{n(P_1 + P_2)}{A} \times \left[1 + \frac{ec}{r^2} \sec \left[\frac{L}{2r} \sqrt{\frac{n(P_1 + P_2)}{EA}} \right] \right]$$

$$\text{Solve for } P_2: \quad P_2 = 387 \text{ kN} \quad \leftarrow$$

Problem 11.6-13 A W 14 × 53 wide-flange column of length $L = 15$ ft is fixed at the base and free at the top (see figure). The column supports a centrally applied load $P_1 = 120$ k and a load $P_2 = 40$ k supported on a bracket. The distance from the centroid of the column to the load P_2 is $s = 12$ in. Also, the modulus of elasticity is $E = 29,000$ ksi and yield stress is $\sigma_Y = 36$ ksi.

- Calculate the maximum compressive stress in the column.
- Determine the factor of safety with respect to yielding.



Probs. 11.6-13 and 11.6-14

Solution 11.6-13 Column with two loads

Fixed-free column. W 14 × 53

DATA

$$\begin{aligned}
 L &= 15 \text{ ft} = 180 \text{ in.} & L_e &= 2L = 360 \text{ in.} \\
 P_1 &= 120 \text{ k} & P_2 &= 40 \text{ k} & s &= 12 \text{ in.} \\
 E &= 29,000 \text{ ksi} & \sigma_Y &= 36 \text{ ksi} \\
 P &= P_1 + P_2 = 160 \text{ k} & e &= \frac{P_2 s}{P} = 3.0 \text{ in.} \\
 A &= 15.6 \text{ in.}^2 & I &= I_1 = 541 \text{ in.}^4 & d &= 13.92 \text{ in.} \\
 r^2 &= \frac{I}{A} = 34.68 \text{ in.}^2 & r &= 5.889 \text{ in.} \\
 c &= \frac{d}{2} = 6.960 \text{ in.} & \frac{ec}{r^2} &= 0.6021 \\
 \frac{P}{A} &= 10.26 \text{ ksi} & \frac{L_e}{2r} \sqrt{\frac{P}{EA}} &= 0.5748
 \end{aligned}$$

(a) MAXIMUM COMPRESSIVE STRESS

Secant formula (Eq. 11-67):

$$\sigma_{\max} = \frac{P}{A} \left[1 + \frac{ec}{r^2} \sec \left(\frac{L_e}{2r} \sqrt{\frac{P}{EA}} \right) \right] \quad (1)$$

Substitute into Eq. (1): $\sigma_{\max} = 17.6 \text{ ksi} \quad \leftarrow$

(b) FACTOR OF SAFETY WITH RESPECT TO YIELDING

$$\sigma_{\max} = \sigma_Y = 36 \text{ ksi} \quad P = P_Y$$

Substitute into Eq. (1):

$$36 = \frac{P_Y}{15.6} [1 + 0.6021 \sec (0.04544 \sqrt{P_Y})]$$

Solve numerically: $P_Y = 302.6 \text{ k}$

$$P = 160 \text{ k} \quad n = \frac{P_Y}{P} = \frac{302.6 \text{ k}}{160 \text{ k}} = 1.89 \quad \leftarrow$$

Problem 11.6-14 A wide-flange column with a bracket is fixed at the base and free at the top (see figure). The column supports a load $P_1 = 340$ kN acting at the centroid and a load $P_2 = 110$ kN acting on the bracket at distance $s = 250$ mm, from the load P_1 . The column is a W 310 × 52 shape with $L = 5$ m, $E = 200$ GPa, and $\sigma_Y = 290$ MPa.

- What is the maximum compressive stress in the column?
- If the load P_1 remains at 340 kN, what is the largest permissible value of the load P_2 in order to maintain a factor of safety of 1.8 with respect to yielding?

Solution 11.6-14

$$W\ 310 \times 52 \quad L = 5.0\text{ m}$$

$$P_1 = 340\text{ kN} \quad P_2 = 110\text{ kN}$$

$$s = 250\text{ mm} \quad E = 200\text{ GPa}$$

$$\sigma_y = 290\text{ MPa} \quad P = P_1 + P_2$$

$$P = 450\text{ kN} \quad e = \frac{P_2 s}{P} \quad e = 61.1\text{ mm}$$

$$A = 6650\text{ mm}^2 \quad I = I_1 \quad I = 119 \times 10^6\text{ mm}^4$$

$$d = 318\text{ mm} \quad r = \sqrt{\frac{I}{A}} \quad r = 133.8\text{ mm}$$

$$c = \frac{d}{2} \quad c = 159.0\text{ mm}$$

$$L_e = 2L \quad L_e = 10.0\text{ m}$$

(a) MAXIMUM COMPRESSION STRESS

$$\sigma_{\max} = \frac{P}{A} \left(1 + \frac{ec}{r^2} \sec \left(\frac{L_e}{2r} \sqrt{\frac{P}{EA}} \right) \right)$$

$$\sigma_{\max} = 115.2\text{ MPa} \quad \leftarrow$$

(b) LARGEST VALUE OF LOAD P_2 WHEN

$$P_1 = 340\text{ kN} \quad n = 1.8$$

$$P_y = n(P_1 + P_2)$$

$$\text{From } \sigma_y = \frac{P_y}{A} \left(1 + \frac{ec}{r^2} \sec \left(\frac{L_e}{2r} \sqrt{\frac{P_y}{EA}} \right) \right)$$

$$\sigma_y = \frac{n(P_1 + P_2)}{A} \times \left[1 + \frac{ec}{r^2} \sec \left[\frac{L_e}{2r} \sqrt{\frac{n(P_1 + P_2)}{EA}} \right] \right]$$

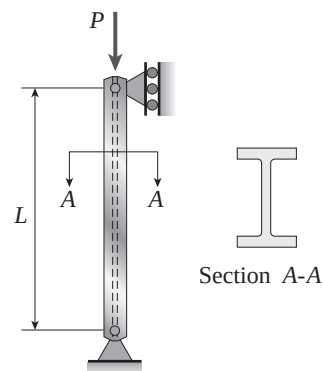
$$\text{Solve for } P_2: \quad P_2 = 193\text{ kN}$$

Design Formulas for Columns

The problems for Section 11.9 are to be solved assuming that the axial loads are centrally applied at the ends of the columns. Unless otherwise stated, the columns may buckle in any direction.

STEEL COLUMNS

Problem 11.9-1 Determine the allowable axial load P_{allow} for a W 10 $\times$ 45 steel wide-flange column with pinned ends (see figure) for each of the following lengths: $L = 8\text{ ft}$, 16 ft , 24 ft , and 32 ft . (Assume $E = 29,000\text{ ksi}$ and $\sigma_y = 36\text{ ksi}$.)

**Probs 11.9-1 through 11.9-6**

Solution 11.9-1 Steel wide-flange columnPinned ends $K = 1$

Buckling about axis 2-2

$$W 10 \times 45 \quad A = 13.3 \text{ in.}^2 \quad r_2 = 2.01 \text{ in.} \quad r = r_2 \quad E = 29,000 \text{ ksi} \quad \sigma_y = 36 \text{ ksi} \quad L_{\max} = 200r$$

$$L_{\max} = 33.5 \text{ ft} \quad L_c = 4.71r\sqrt{\frac{E}{\sigma_y}} \quad L_c = 22.392 \text{ ft}$$

$$L = \begin{pmatrix} 8 \text{ ft} \\ 16 \text{ ft} \\ 24 \text{ ft} \\ 32 \text{ ft} \end{pmatrix} \quad \frac{L}{r} = \begin{pmatrix} 47.761 \\ 95.522 \\ 143.284 \\ 191.045 \end{pmatrix}$$

 $i = 1 \text{ to } 4$

$$\sigma_{e_i} = \frac{\pi^2 E}{\left(\frac{KL_i}{r}\right)^2}$$

$$\sigma_{\text{allow}_i} = \sigma_y \begin{cases} \frac{\sigma_y}{\sigma_{e_i}} \\ \frac{0.658}{1.67} \quad \left(\text{if } \frac{KL_i}{r} \leq \frac{KL_c}{r} \right) \\ \frac{0.877 \sigma_{e_i}}{1.67 \sigma_y} \quad (\text{otherwise}) \end{cases} \quad \sigma_{\text{allow}} = \begin{pmatrix} 19.118 \\ 13.334 \\ 7.321 \\ 4.118 \end{pmatrix} \text{ ksi}$$

$$P_{\text{allow}_i} = A \sigma_{\text{allow}_i} \quad \boxed{P_{\text{allow}} = \begin{pmatrix} 254 \\ 177 \\ 97 \\ 55 \end{pmatrix} \text{ k}} \quad \text{for} \quad L = \begin{pmatrix} 8 \\ 16 \\ 24 \\ 32 \end{pmatrix} \text{ ft}$$

Problem 11.9-2 Determine the allowable axial load P_{allow} for a W 310 $\times$ 129 steel wide-flange column with pinned ends (see figure) for each of the following lengths: $L = 3$ m, 6 m, 9 m, and 12 m. (Assume $E = 200$ GPa and $\sigma_Y = 340$ MPa.)

Solution 11.9-2

Pinned ends $K = 1$

Buckling about axis 2–2

$$\text{W } 310 \times 129 \quad A = 16,500 \text{ mm}^2 \quad r_2 = 78.0 \text{ mm} \quad r = r_2 \quad E = 200 \text{ GPa} \quad \sigma_y = 340 \text{ MPa} \quad L_{\text{max}} = 200r$$

$$L_{\text{max}} = 15.6 \text{ m} \quad L_c = 4.71r \sqrt{\frac{E}{\sigma_y}} \quad L_c = 8.910 \text{ m}$$

$$L = \begin{pmatrix} 3 \text{ m} \\ 6 \text{ m} \\ 9 \text{ m} \\ 12 \text{ m} \end{pmatrix} \quad \frac{L}{r} = \begin{pmatrix} 38.462 \\ 76.923 \\ 115.385 \\ 153.846 \end{pmatrix}$$

$i = 1$ to 4

$$\sigma_{e_i} = \frac{\pi^2 E}{\left(\frac{KL_i}{r}\right)^2}$$

$$\sigma_{\text{allow}_i} = \sigma_y \begin{cases} \frac{\sigma_y}{\sigma_{e_i}} & \left(\text{if } \frac{KL_i}{r} \leq \frac{KL_c}{r} \right) \\ \frac{0.658}{1.67} & \\ \frac{0.877 \sigma_{e_i}}{1.67 \sigma_y} & \text{(otherwise)} \end{cases} \quad \sigma_{\text{allow}} = \begin{pmatrix} 182.998 \\ 132.891 \\ 77.860 \\ 43.797 \end{pmatrix} \text{ MPa}$$

$$P_{\text{allow}_i} = A \sigma_{\text{allow}_i} \quad \boxed{P_{\text{allow}} = \begin{pmatrix} 3019 \\ 2193 \\ 1285 \\ 723 \end{pmatrix} \text{ kN}} \quad \text{for} \quad \begin{pmatrix} 3 \text{ m} \\ 6 \text{ m} \\ 9 \text{ m} \\ 12 \text{ m} \end{pmatrix}$$

Problem 11.9-3 Determine the allowable axial load P_{allow} for a W 10 $\times$ 60 steel wide-flange column with pinned ends (see figure) for each of the following lengths: $L = 10$ ft, 20 ft, 30 ft, and 40 ft. (Assume $E = 29,000$ ksi and $\sigma_Y = 36$ ksi.)

Solution 11.9-3 Steel wide-flange column

Pinned ends $K = 1$

Buckling about axis 2–2

$$\text{W } 10 \times 60 \quad A = 17.6 \text{ in.}^2 \quad r_2 = 2.57 \text{ in.} \quad r = r_2 \quad E = 29,000 \text{ ksi} \quad \sigma_y = 36 \text{ ksi} \quad L_{\text{max}} = 200r$$

$$L_{\text{max}} = 42.8 \text{ ft} \quad L_c = 4.71r \sqrt{\frac{E}{\sigma_y}} \quad L_c = 28.630 \text{ ft}$$

$$L = \begin{pmatrix} 10 \text{ ft} \\ 20 \text{ ft} \\ 30 \text{ ft} \\ 40 \text{ ft} \end{pmatrix} \quad \frac{L}{r} = \begin{pmatrix} 46.693 \\ 93.385 \\ 140.078 \\ 186.77 \end{pmatrix}$$

$i = 1$ to 4

$$\sigma_{e_i} = \frac{\pi^2 E}{\left(\frac{KL_i}{r}\right)^2}$$

$$\sigma_{\text{allow}_i} = \sigma_y \begin{cases} \frac{0.658}{1.67} \frac{\sigma_y}{\sigma_{e_i}} & \left(\text{if } \frac{KL_i}{r} \leq \frac{KL_c}{r} \right) \\ \frac{0.877 \sigma_{e_i}}{1.67 \sigma_y} & \text{(otherwise)} \end{cases} \quad \sigma_{\text{allow}} = \begin{pmatrix} 19.219 \\ 13.621 \\ 7.660 \\ 4.309 \end{pmatrix} \text{ ksi}$$

$$P_{\text{allow}_i} = A \sigma_{\text{allow}_i} \quad \boxed{P_{\text{allow}} = \begin{pmatrix} 338 \\ 240 \\ 135 \\ 76 \end{pmatrix} \text{ k}} \quad \text{for } L = \begin{pmatrix} 10 \\ 20 \\ 30 \\ 40 \end{pmatrix} \text{ ft}$$

Problem 11.9-4 Select a steel wide-flange column of nominal depth 250 mm. (W 250 shape) to support an axial load $P = 800$ kN (see figure). The column has pinned ends and length $L = 4.25$ m. Assume $E = 200$ GPa and $\sigma_Y = 250$ MPa. (Note: The selection of columns is limited to those listed in Table F-1(b), Appendix F.)

Solution 11.9-4

$$K = 1 \quad P = 800 \text{ kN} \quad L = 4.25 \text{ m} \quad \sigma_Y = 250 \text{ MPa} \quad E = 200 \text{ GPa}$$

$$\frac{L_c}{r} = 4.71 \sqrt{\frac{E}{\sigma_Y}} \quad 4.71 \sqrt{\frac{E}{\sigma_Y}} = 133.219$$

(1) TRIAL VALUE OF σ_{allow}

$$\text{Upper limit: With } \frac{L}{r} = 0 \quad \text{FS} = \frac{5}{3}$$

$$\sigma_{\text{allow\_max}} = \frac{\sigma_Y}{\text{FS}} \quad \sigma_{\text{allow\_max}} = 150 \text{ MPa}$$

$$\text{Try: } \sigma_{\text{allow}} = 110 \text{ MPa}$$

(2) TRIAL VALUE OF AREA

$$A = \frac{P}{\sigma_{\text{allow}}} \quad A = 7273 \text{ mm}^2$$

(3) TRIAL COLUMN W 250 $\times$ 67

$$A = 8580 \text{ mm}^2 \quad r = 51.1 \text{ mm}$$

(4) ALLOWABLE STRESS FOR TRIAL COLUMN

$$\frac{L}{r} = 83.170 \quad \frac{L}{r} < \frac{L_c}{r} \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 285.360 \text{ MPa}$$

$$\sigma_{\text{allow}} = \sigma_Y \frac{1}{\text{FS}} \left(0.658^{\frac{\sigma_Y}{\sigma_e}} \right) \quad \sigma_{\text{allow}} = 104.0 \text{ MPa}$$

(5) ALLOWABLE LOAD FOR TRIAL COLUMN

$$P_{\text{allow}} = \sigma_{\text{allow}} A \quad P_{\text{allow}} = 891.9 \text{ kN} \quad P_{\text{allow}} > P \quad (\text{OK})$$

(W 250 $\times$ 67) Use W 250 $\times$ 67

(6) NEXT SMALLER SIZE COLUMN

$$\text{W 250} \times 44.8 \quad A = 5700 \text{ mm}^2 \quad r = 34.8 \text{ mm}$$

$$\frac{L}{r} = 122.126 \quad \frac{L}{r} < \frac{L_c}{r}$$

$$\sigma_{\text{allow}} = \sigma_Y \frac{1}{\text{FS}} \left(0.658^{\frac{\sigma_Y}{\sigma_e}} \right) \quad \sigma_{\text{allow}} = 103.95 \text{ MPa}$$

$$P_{\text{allow}} = A \sigma_{\text{allow}} \quad P_{\text{allow}} = 592.5 \text{ kN} \quad P_{\text{allow}} < P \quad (\text{Not satisfactory})$$

Problem 11.9-5 Select a steel wide-flange column of nominal depth 12 in. (W 12 shape) to support an axial load $P = 175$ k (see figure). The column has pinned ends and length $L = 35$ ft. Assume $E = 29,000$ ksi and $\sigma_y = 36$ ksi. (Note: The selection of columns is limited to those listed in Table F-1, Appendix F.)

Solution 11.9-5

$$K = 1 \quad P = 175 \text{ k} \quad L = 35 \text{ ft} \quad \sigma_y = 36 \text{ ksi} \quad E = 29,000 \text{ ksi}$$

$$\frac{L_c}{r} = 4.71 \sqrt{\frac{E}{\sigma_y}} \quad 4.71 \sqrt{\frac{E}{\sigma_y}} = 133.681$$

(1) TRIAL VALUE OF σ_{allow}

$$\text{Upper limit: With } \frac{L}{r} = 0 \quad \text{FS} = \frac{5}{3}$$

$$\sigma_{\text{allow\_max}} = \frac{\sigma_y}{\text{FS}} \quad \sigma_{\text{allow\_max}} = 22 \text{ ksi}$$

$$\text{Try: } \sigma_{\text{allow}} = 10 \text{ ksi}$$

(2) TRIAL VALUE OF AREA

$$A = \frac{P}{\sigma_{\text{allow}}} \quad A = 17 \text{ in.}^2$$

(3) TRIAL COLUMN W 12 $\times$ 87

$$A = 25.6 \text{ in.}^2 \quad r = 3.07 \text{ in.}$$

(4) ALLOWABLE STRESS FOR TRIAL COLUMN

$$\frac{L}{r} = 136.808 \quad \frac{L}{r} > \frac{L_c}{r} \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 105.437 \text{ MPa}$$

$$\sigma_{\text{allow}} = \frac{1}{\text{FS}} (0.877 \sigma_e) \quad \sigma_{\text{allow}} = 8.0 \text{ ksi}$$

(5) ALLOWABLE LOAD FOR TRIAL COLUMN

$$P_{\text{allow}} = \sigma_{\text{allow}} A \quad P_{\text{allow}} = 206.0 \text{ k} \quad P_{\text{allow}} > P \quad (\text{OK})$$

(W 12 $\times$ 87) Use W 12 $\times$ 87

(6) NEXT SMALLER SIZE COLUMN

$$\text{W 12} \times 50 \quad A = 14.6 \text{ in.}^2 \quad r = 1.96 \text{ in.}$$

$$\frac{L}{r} = 214.286 \quad \frac{L}{r} > \frac{L_c}{r} \quad (\text{Since the maximum permissible value of } L/r \text{ is 200, this section is not satisfactory.})$$

Problem 11.9-6 Select a steel wide-flange column of nominal depth 360 mm. (W 360 shape) to support an axial load $P = 1100$ kN (see figure). The column has pinned ends and length $L = 6$ m. Assume $E = 200$ GPa and $\sigma_Y = 340$ MPa. (Note: The selection of columns is limited to those listed in Table F-1(b), Appendix F.

Solution 11.9-6

$$K = 1 \quad P = 1100 \text{ kN} \quad L = 6 \text{ m} \quad \sigma_Y = 340 \text{ MPa} \quad E = 200 \text{ GPa}$$

$$\frac{L_c}{r} = 4.71 \sqrt{\frac{E}{\sigma_Y}} \quad 4.71 \sqrt{\frac{E}{\sigma_Y}} = 114.234$$

(1) TRIAL VALUE OF σ_{allow}

$$\text{Upper limit: With } \frac{L}{r} = 0 \quad FS = \frac{5}{3}$$

$$\sigma_{\text{allow\_max}} = \frac{\sigma_Y}{FS} \quad \sigma_{\text{allow\_max}} = 204 \text{ MPa}$$

$$\text{Try } \sigma_{\text{allow}} = 100 \text{ MPa}$$

(2) TRIAL VALUE OF AREA

$$A = \frac{P}{\sigma_{\text{allow}}} \quad A = 11,000 \text{ mm}^2$$

(3) TRIAL COLUMN W 360 $\times$ 122

$$A = 15,500 \text{ mm}^2 \quad r = 63 \text{ mm}$$

(4) ALLOWABLE STRESS FOR TRIAL COLUMN

$$\frac{L}{r} = 95.238 \quad \frac{L}{r} < \frac{L_c}{r} \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 217.625 \text{ MPa}$$

$$\sigma_{\text{allow}} = \sigma_Y \frac{0.658^{\frac{\sigma_Y}{\sigma_e}}}{1.67} \quad \sigma_{\text{allow}} = 105.9 \text{ MPa}$$

(5) ALLOWABLE LOAD FOR TRIAL COLUMN

$$P_{\text{allow}} = \sigma_{\text{allow}} A \quad P_{\text{allow}} = 1641.0 \text{ kN} \quad P_{\text{allow}} > P \quad (\text{Satisfactory})$$

$$(\text{W } 360 \times 122) \quad \boxed{\text{Use W } 360 \times 122}$$

(6) NEXT SMALLER SIZE COLUMN

$$\text{W } 360 \times 79 \quad A = 10,100 \text{ mm}^2 \quad r = 48.8 \text{ mm}$$

$$\frac{L}{r} = 122.951 \quad \frac{L}{r} > \frac{L_c}{r} \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 130.577 \text{ MPa}$$

$$\sigma_{\text{allow}} = \frac{0.877 \sigma_e}{FS} \quad \sigma_{\text{allow}} = 68.71 \text{ MPa}$$

$$P_{\text{allow}} = A \sigma_{\text{allow}} \quad P_{\text{allow}} = 694.0 \text{ kN} \quad P_{\text{allow}} < P \quad (\text{Unsatisfactory})$$

Problem 11.9-7 Determine the allowable axial load P_{allow} for a steel *pipe column with pinned ends* having an outside diameter of 4.5 in. and wall thickness of 0.237 in. for each of the following lengths: $L = 6$ ft, 12 ft, 18 ft, and 24 ft. (Assume $E = 29,000$ ksi and $\sigma_Y = 36$ ksi.)

Solution 11.9-7 Steel pipe column

Pinned ends $K = 1$

$d = 4.5$ in.

$t = 0.237$ in.

$$A = \frac{\pi}{4} [d^2 - (d - 2t)^2] = 3.174 \text{ in.}^2$$

$E = 29,000$ ksi

$\sigma_y = 36$ ksi

$$I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 7.233 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.510 \text{ in.}$$

$$L_{\text{max}} = 200r$$

$$L_{\text{max}} = 25.2 \text{ ft}$$

$$L_c = 4.71r \sqrt{\frac{E}{\sigma_y}}$$

$$L_c = 16.816 \text{ ft}$$

$$L = \begin{pmatrix} 6 \text{ ft} \\ 12 \text{ ft} \\ 18 \text{ ft} \\ 24 \text{ ft} \end{pmatrix} \quad \frac{L}{r} = \begin{pmatrix} 47.697 \\ 95.394 \\ 143.091 \\ 190.788 \end{pmatrix}$$

$i = 1$ to 4

$$\sigma_{e_i} = \frac{\pi^2 E}{\left(\frac{KL_i}{r}\right)^2}$$

$$\sigma_{\text{allow}_i} = \sigma_y \begin{cases} \frac{0.658 \frac{\sigma_y}{\sigma_{e_i}}}{1.67} & \left(\text{if } \frac{KL_i}{r} \leq \frac{KL_c}{r} \right) \\ \frac{0.877 \sigma_{e_i}}{1.67 \sigma_y} & \text{(otherwise)} \end{cases} \quad \sigma_{\text{allow}} = \begin{pmatrix} 19.124 \\ 13.351 \\ 7.341 \\ 4.129 \end{pmatrix} \text{ ksi}$$

$$P_{\text{allow}_i} = A \sigma_{\text{allow}_i} \quad \boxed{P_{\text{allow}} = \begin{pmatrix} 60.7 \\ 42.4 \\ 23.3 \\ 13.1 \end{pmatrix} \text{ k}} \quad \text{for } L = \begin{pmatrix} 6 \\ 12 \\ 18 \\ 24 \end{pmatrix} \text{ ft}$$

Problem 11.9-8 Determine the allowable axial load P_{allow} for a steel *pipe column with pinned ends* having an outside diameter of 220 mm and wall thickness of 12 mm for each of the following lengths: $L = 2.5$ m, 5 m, 7.5 m, and 10 m. (Assume $E = 200$ GPa and $\sigma_Y = 250$ MPa.)

Solution 11.9-8 Steel pipe column

Pinned ends $K = 1$

$d = 220$ mm

$t = 12$ mm

$$A = \frac{\pi}{4} [d^2 - (d - 2t)^2] = 7841.415 \text{ mm}^2$$

$E = 200$ GPa

$\sigma_Y = 250$ MPa

$$I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 42,547,519.219 \text{ mm}^4$$

$$r = \sqrt{\frac{I}{A}} = 73.661 \text{ mm}$$

$$L_{\text{max}} = 200r$$

$$L_{\text{max}} = 14.7 \text{ m}$$

$$L_c = 4.71r \sqrt{\frac{E}{\sigma_Y}}$$

$$L_c = 9.813 \text{ m}$$

$$L = \begin{pmatrix} 2.5 \text{ m} \\ 5 \text{ m} \\ 7.5 \text{ m} \\ 10 \text{ m} \end{pmatrix} \quad \frac{L}{r} = \begin{pmatrix} 33.939 \\ 67.878 \\ 101.817 \\ 135.756 \end{pmatrix}$$

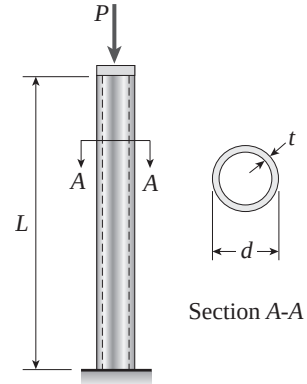
$i = 1$ to 4

$$\sigma_{e_i} = \frac{\pi^2 E}{\left(\frac{KL_i}{r}\right)^2}$$

$$\sigma_{\text{allow}_i} = \sigma_Y \begin{cases} \frac{0.658}{1.67} \frac{\sigma_Y}{\sigma_{e_i}} & \left(\text{if } \frac{KL_i}{r} \leq \frac{KL_c}{r} \right) \\ \frac{0.877\sigma_{e_i}}{1.67\sigma_Y} & (\text{otherwise}) \end{cases} \quad \sigma_{\text{allow}} = \begin{pmatrix} 140.833 \\ 117.260 \\ 86.409 \\ 56.246 \end{pmatrix} \text{ MPa}$$

$$P_{\text{allow}_i} = A \sigma_{\text{allow}_i} \quad \boxed{P_{\text{allow}} = \begin{pmatrix} 1104 \\ 919 \\ 678 \\ 441 \end{pmatrix} \text{ kN}} \quad \text{for } L = \begin{pmatrix} 2.5 \\ 5.0 \\ 7.5 \\ 10.0 \end{pmatrix} \text{ m}$$

Problem 11.9-9 Determine the allowable axial load P_{allow} for a steel pipe column that is fixed at the base and free at the top (see figure) for each of the following lengths: $L = 6$ ft, 9 ft, 12 ft, and 15 ft. The column has outside diameter $d = 6.625$ in. and wall thickness $t = 0.280$ in. (Assume $E = 29,000$ ksi and $\sigma_Y = 36$ ksi.)



Probs 11.9-9 through 11.9-12

Solution 11.9-9 Steel pipe column

Fixed-free ends $K = 2$

$d = 6.625$ in.

$t = 0.280$ in.

$$A = \frac{\pi}{4} [d^2 - (d - 2t)^2] = 5.581 \text{ in.}^2$$

$E = 29,000$ ksi

$\sigma_Y = 36$ ksi

$$I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 28.142 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 2.245 \text{ in.}$$

$$L_{\text{max}} = 200 \frac{r}{K}$$

$$L_{\text{max}} = 18.7 \text{ ft}$$

$$L_c = 4.71 \frac{r}{K} \sqrt{\frac{E}{\sigma_Y}}$$

$$L_c = 12.507 \text{ ft}$$

$$L = \begin{pmatrix} 6 \text{ ft} \\ 9 \text{ ft} \\ 12 \text{ ft} \\ 15 \text{ ft} \end{pmatrix} \quad \frac{L}{r} = \begin{pmatrix} 32.064 \\ 48.097 \\ 64.129 \\ 80.161 \end{pmatrix}$$

$i = 1$ to 4

$$\sigma_{e_i} = \frac{\pi^2 E}{\left(\frac{KL_i}{r}\right)^2}$$

$$\sigma_{\text{allow}_i} = \sigma_Y \begin{cases} \frac{\sigma_Y}{\sigma_{e_i}} & \left(\text{if } \frac{KL_i}{r} \leq \frac{KL_c}{r} \right) \\ \frac{0.877\sigma_{e_i}}{1.67\sigma_Y} & (\text{otherwise}) \end{cases} \quad \sigma_{\text{allow}} = \begin{pmatrix} 17.360 \\ 13.244 \\ 9.067 \\ 5.848 \end{pmatrix} \text{ ksi}$$

$$P_{\text{allow}_i} = A \sigma_{\text{allow}_i}$$

$$P_{\text{allow}} = \begin{pmatrix} 96.9 \\ 73.9 \\ 50.6 \\ 32.6 \end{pmatrix} \text{ k} \quad \text{for} \quad L = \begin{pmatrix} 6.0 \\ 9.0 \\ 12.0 \\ 15.0 \end{pmatrix} \text{ ft}$$

Problem 11.9-10 Determine the allowable axial load P_{allow} for a steel pipe column that is fixed at the base and free at the top (see figure) for each of the following lengths: $L = 2.6$ m, 2.8 m, 3.0 m, and 3.2 m. The column has outside diameter $d = 140$ mm and wall thickness $t = 7$ mm. (Assume $E = 200$ GPa and $\sigma_Y = 250$ MPa.)

Solution 11.9-10

Fixed-free ends $K = 2$

$d = 140$ mm

$t = 7$ mm

$$A = \frac{\pi}{4} [d^2 - (d - 2t)^2] = 2924.823 \text{ mm}^2$$

$E = 200$ GPa

$\sigma_Y = 250$ MPa

$$I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 6,485,063.266 \text{ mm}^4 \quad r = \sqrt{\frac{I}{A}} = 47.088 \text{ mm} \quad L_{\text{max}} = 200 \frac{r}{K}$$

$$L_{\text{max}} = 4.7 \text{ m}$$

$$L_c = 4.71 \frac{r}{K} \sqrt{\frac{E}{\sigma_Y}} \quad L_c = 3.136 \text{ m}$$

$$L = \begin{pmatrix} 2.6 \text{ m} \\ 2.8 \text{ m} \\ 3 \text{ m} \\ 3.2 \text{ m} \end{pmatrix} \quad \frac{L}{r} = \begin{pmatrix} 55.216 \\ 59.464 \\ 63.711 \\ 67.958 \end{pmatrix}$$

$i = 1$ to 4

$$\sigma_{e_i} = \frac{\pi^2 E}{\left(\frac{KL_i}{r}\right)^2}$$

$$\sigma_{\text{allow}_i} = \sigma_Y \begin{cases} \frac{\sigma_Y}{\sigma_{e_i}} & \left(\text{if } \frac{KL_i}{r} \leq \frac{KL_c}{r} \right) \\ \frac{0.877\sigma_{e_i}}{1.67\sigma_Y} & (\text{otherwise}) \end{cases} \quad \sigma_{\text{allow}} = \begin{pmatrix} 78.427 \\ 70.731 \\ 63.304 \\ 56.114 \end{pmatrix} \text{ MPa}$$

$$P_{\text{allow}_i} = A\sigma_{\text{allow}_i} \quad P_{\text{allow}} = \begin{pmatrix} 229 \\ 207 \\ 185 \\ 164 \end{pmatrix} \text{ kN} \quad \text{for } L = \begin{pmatrix} 2.6 \\ 2.8 \\ 3.0 \\ 3.2 \end{pmatrix} \text{ m}$$

Problem 11.9-11 Determine the maximum permissible length $L_{\max}$ for a steel pipe column that is fixed at the base and free at the top and must support an axial load $P = 40$ k (see figure). The column has outside diameter $d = 4.0$ in., wall thickness $t = 0.226$ in., $E = 29,000$ ksi, and $\sigma_Y = 42$ ksi.

Solution 11.9-11 Steel pipe column

$$\text{Fixed-free ends } K = 2 \quad P = 40 \text{ k}$$

$$d = 4 \text{ in.}$$

$$t = 0.226 \text{ in.}$$

$$A = \frac{\pi}{4} [d^2 - (d - 2t)^2] = 2.680 \text{ in.}^2$$

$$E = 29,000 \text{ ksi}$$

$$\sigma_Y = 42 \text{ ksi}$$

$$I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 4.788 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.337 \text{ in.}$$

$$L_{\max} = 200 \frac{r}{K}$$

$$L_{\max} = 11.1 \text{ ft}$$

$$L_c = 4.71 \frac{r}{K} \sqrt{\frac{E}{\sigma_Y}}$$

$$L_c = 6.893 \text{ ft}$$

Select trial values of length L (see column of values below) and compute corresponding values of P_{allow} . Interpolate between trial values to find that value of L that produces $P_{\text{allow}} = P$.

$$i = 1 \text{ to } 4 \quad L = \begin{pmatrix} 5 \text{ ft} \\ 5.13 \text{ ft} \\ 5.2 \text{ ft} \\ 5.3 \text{ ft} \end{pmatrix} \quad \sigma_{e_i} = \frac{\pi^2 E}{\left(\frac{KL_i}{r}\right)^2} \quad \sigma_e = \begin{pmatrix} 35.514 \\ 33.737 \\ 32.835 \\ 31.608 \end{pmatrix} \text{ ksi}$$

$$\sigma_{\text{allow}_i} = \sigma_Y \begin{cases} \frac{\sigma_Y}{\sigma_{e_i}} \\ \frac{0.658 \sigma_{e_i}}{1.67} \left(\text{if } \frac{KL_i}{r} \leq \frac{KL_c}{r} \right) \\ \frac{0.877 \sigma_{e_i}}{1.67 \sigma_Y} \text{ (otherwise)} \end{cases} \quad \sigma_{\text{allow}} = \begin{pmatrix} 105.702 \\ 102.981 \\ 101.517 \\ 99.429 \end{pmatrix} \text{ MPa}$$

$$P_{\text{allow}_i} = A \sigma_{\text{allow}_i} \quad P_{\text{allow}} = \begin{pmatrix} 41.1 \\ 40.0 \\ 39.5 \\ 38.6 \end{pmatrix} \text{ k} \quad \text{for } L = \begin{pmatrix} 5.00 \\ 5.13 \\ 5.20 \\ 5.30 \end{pmatrix} \text{ ft}$$

$$\boxed{L_{\max} = 5.13 \text{ ft}}$$

Problem 11.9-12 Determine the maximum permissible length $L_{\max}$ for a steel pipe column that is fixed at the base and free at the top and must support an axial load $P = 500$ kN (see figure). The column has outside diameter $d = 200$ mm, wall thickness $t = 10$ mm, $E = 200$ GPa, and $\sigma_Y = 250$ MPa.

Solution 11.9-12 Steel pipe column

Fixed-free ends $K = 2$ $P = 500$ kN

$d = 200$ mm

$t = 10$ mm

$$A = \frac{\pi}{4} [d^2 - (d - 2t)^2] = 5969.026 \text{ mm}^2$$

$E = 200$ GPa

$\sigma_Y = 250$ MPa

$$I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 2.701 \times 10^7 \text{ mm}^4$$

$$r = \sqrt{\frac{I}{A}} = 67.268 \text{ mm}$$

$$L_{\max} = 200 \frac{r}{K}$$

$$L_{\max} = 6.7 \text{ m}$$

$$L_c = 4.71 \frac{r}{K} \sqrt{\frac{E}{\sigma_Y}}$$

$$L_c = 4.481 \text{ m}$$

Select trial values of length L (see column of values below) and compute corresponding values of P_{allow} . Interpolate between trial values to find that value of L that produces $P_{\text{allow}} = P$.

$$i = 1 \text{ to } 4 \quad L = \begin{pmatrix} 3.5 \text{ m} \\ 3.52 \text{ m} \\ 3.53 \text{ m} \\ 3.54 \text{ m} \end{pmatrix} \quad \sigma_{e_i} = \frac{\pi^2 E}{\left(\frac{KL_i}{r}\right)^2} \quad \sigma_e = \begin{pmatrix} 182.286 \\ 180.220 \\ 179.200 \\ 178.189 \end{pmatrix} \text{ MPa}$$

$$\sigma_{\text{allow}_i} = \sigma_Y \begin{cases} \frac{\sigma_Y}{\sigma_{e_i}} \\ \frac{0.658}{1.67} \left(\text{if } \frac{KL_i}{r} \leq \frac{KL_c}{r} \right) \\ \frac{0.877\sigma_{e_i}}{1.67\sigma_Y} \text{ (otherwise)} \end{cases} \quad \sigma_{\text{allow}} = \begin{pmatrix} 84.319 \\ 83.766 \\ 83.490 \\ 83.214 \end{pmatrix} \text{ MPa}$$

$$P_{\text{allow}_i} = A\sigma_{\text{allow}_i} \quad P_{\text{allow}} = \begin{pmatrix} 503.3 \\ 500.0 \\ 498.4 \\ 496.7 \end{pmatrix} \text{ kN} \quad \text{for } L = \begin{pmatrix} 3.50 \\ 3.52 \\ 3.53 \\ 3.54 \end{pmatrix} \text{ m}$$

$$L_{\max} = 3.52 \text{ m}$$

Problem 11.9-13 A steel pipe column with *pinned ends* supports an axial load $P = 21$ k. The pipe has outside and inside diameters of 3.5 in. and 2.9 in., respectively. What is the maximum permissible length $L_{\max}$ of the column if $E = 29,000$ ksi and $\sigma_Y = 36$ ksi?

Solution 11.9-13 Steel pipe column

$$\text{Pinned ends } K = 1 \quad P = 21 \text{ k} \quad d_2 = 3.5 \text{ in.} \quad d_1 = 2.9 \text{ in.}$$

$$A = \frac{\pi}{4} (d_2^2 - d_1^2) = 3.016 \text{ in.}^2 \quad E = 29000 \text{ ksi} \quad \sigma_Y = 36 \text{ ksi}$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) = 3.894 \text{ in.}^4 \quad r = \sqrt{\frac{I}{A}} = 1.136 \text{ in.} \quad L_{\max} = 200 \frac{r}{K}$$

$$L_{\max} = 18.9 \text{ ft} \quad L_c = 4.71 \frac{r}{K} \sqrt{\frac{E}{\sigma_Y}} \quad L_c = 12.659 \text{ ft}$$

Select trial values of length L (see column of values below) and compute corresponding values of P_{allow} . Interpolate between trial values to find that value of L that produces $P_{\text{allow}} = P$.

$$i = 1 \text{ to } 4 \quad L = \begin{pmatrix} 13 \text{ ft} \\ 13.2 \text{ ft} \\ 14 \text{ ft} \\ 13.9 \text{ ft} \end{pmatrix} \quad \sigma_{e_i} = \frac{\pi^2 E}{\left(\frac{KL_i}{r}\right)^2} \quad \sigma_e = \begin{pmatrix} 104.708 \\ 101.559 \\ 90.284 \\ 19.587 \end{pmatrix} \text{ MPa}$$

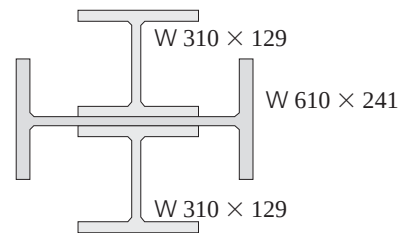
$$\sigma_{\text{allow}_i} = \sigma_Y \begin{cases} \frac{\sigma_Y}{\sigma_{e_i}} \\ \frac{0.658}{1.67} \left(\text{if } \frac{KL_i}{r} \leq \frac{KL_c}{r} \right) \\ \frac{0.877\sigma_{e_i}}{1.67\sigma_Y} \quad (\text{otherwise}) \end{cases} \quad \sigma_{\text{allow}} = \begin{pmatrix} 54.987 \\ 53.333 \\ 47.412 \\ 48.097 \end{pmatrix} \text{ MPa}$$

$$P_{\text{allow}_i} = A\sigma_{\text{allow}_i} \quad P_{\text{allow}} = \begin{pmatrix} 24.1 \\ 23.3 \\ 20.7 \\ 21.0 \end{pmatrix} \text{ k} \quad \text{for } L = \begin{pmatrix} 13.00 \\ 13.20 \\ 14.00 \\ 13.90 \end{pmatrix} \text{ ft}$$

$$\boxed{L_{\max} = 13.9 \text{ ft}}$$

Problem 11.9-14 The steel columns used in a college recreation center are 16.75 m long and are formed by welding three wide-flange sections (see figure). The columns are pin-supported at the ends and may buckle in any direction.

Calculate the allowable load P_{allow} for one column, assuming $E = 200 \text{ GPa}$ and $\sigma_Y = 250 \text{ MPa}$.



Solution 11.9-14

$$L = 16.75 \text{ m} \quad E = 200 \text{ GPa} \quad \sigma_Y = 250 \text{ MPa} \quad K = 1$$

$$\text{W } 310 \times 129 \quad A_1 = 16,500 \text{ mm}^2 \quad d_1 = 318 \text{ mm} \quad I_{11} = 308 \times 10^6 \text{ mm}^4 \quad I_{21} = 100 \times 10^6 \text{ mm}^4$$

$$\text{W } 610 \times 241 \quad A_2 = 30,800 \text{ mm}^2 \quad t_w = 17.9 \text{ mm} \quad I_{12} = 2150 \times 10^6 \text{ mm}^4 \quad I_{22} = 184 \times 10^6 \text{ mm}^4$$

For built-up column:

$$A = 2A_1 + A_2 \quad A = 63,800 \text{ mm}^2 \quad I_y = 2I_{21} + I_{12} \quad I_y = 2.350 \times 10^9 \text{ mm}^4$$

$$h = \frac{d_1}{2} + \frac{t_w}{2} \quad h = 168 \text{ mm} \quad I_z = I_{22} + 2(I_{11} + A_1 h^2) \quad I_z = 1.731 \times 10^9 \text{ mm}^4$$

$$r = \sqrt{\frac{I_z}{A}} \quad r = 165 \text{ mm}$$

$$\frac{L_c}{r} = 4.71 \sqrt{\frac{E}{\sigma_Y}} \quad 4.71 \sqrt{\frac{E}{\sigma_Y}} = 133.219$$

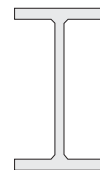
$$\frac{L}{r} = 101.694 \quad \frac{L}{r} < \frac{L_c}{r} \quad \text{FS} = 1.67$$

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 190.869 \text{ MPa}$$

$$\sigma_{\text{allow}} = \sigma_Y \frac{1}{\text{FS}} \left(\frac{\frac{\sigma_Y}{\sigma_e}}{0.658} \right) = 86.524 \text{ MPa}$$

$$P_{\text{allow}} = A \sigma_{\text{allow}} \quad P_{\text{allow}} = 5520 \text{ kN}$$

Problem 11.9-15 A W 8 × 28 steel wide-flange column with pinned ends carries an axial load P . What is the maximum permissible length $L_{\max}$ of the column if (a) $P = 50$ k, and (b) $P = 100$ k? (Assume $E = 29,000$ ksi and $\sigma_Y = 36$ ksi.)



Probs. 11.9-15 and 11.9-16

Solution 11.9-15

$$E = 29,000 \text{ ksi} \quad \sigma_Y = 36 \text{ ksi} \quad K = 1$$

$$\text{W } 8 \times 28 \quad A = 8.24 \text{ in.}^2 \quad r_2 = 1.62 \text{ in.} \quad r = r_2 \quad L_{\max} = 200r$$

$$L_c = 4.71r\sqrt{\frac{E}{\sigma_Y}} \quad L_c = 18.047 \text{ ft} \quad \frac{L_c}{r} = 133.681$$

For each load, select trial values of the length L and calculate the corresponding values of P_{allow} . This solution shows only the successful trial.

(a) $P = 50$ k

$$\text{Try } L = 21.25 \text{ ft} \quad \frac{L}{r} = 157.407 \quad \frac{L}{r} > \frac{L_c}{r} \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 11.552 \text{ ksi}$$

$$\text{FS} = 1.67$$

$$\sigma_{\text{allow}} = \frac{0.877\sigma_e}{\text{FS}} \quad \sigma_{\text{allow}} = 6.07 \text{ ksi}$$

$$P_{\text{allow}} = A\sigma_{\text{allow}} \quad P_{\text{allow}} = 50.0 \text{ k}$$

$$\text{Therefore, } L_{\max} = L: \quad \boxed{L_{\max} = 21.25 \text{ ft}}$$

(b) $P = 100$ k

$$\text{Try } L = 14.1 \text{ ft} \quad \frac{L}{r} = 104.444 \quad \frac{L}{r} < \frac{L_c}{r} \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 26.238 \text{ ksi}$$

$$\sigma_{\text{allow}} = \sigma_Y \frac{1}{\text{FS}} \left(\frac{\frac{\sigma_Y}{\sigma_e}}{0.658} \right) \quad \sigma_{\text{allow}} = 12.14 \text{ ksi}$$

$$P_{\text{allow}} = A\sigma_{\text{allow}} \quad P_{\text{allow}} = 100 \text{ k}$$

$$\text{Therefore, } L_{\max} = L \quad \boxed{L_{\max} = 14.10 \text{ ft}}$$

Problem 11.9-16 A W 250 × 67 steel wide-flange column with pinned ends carries an axial load P . What is the maximum permissible length $L_{\max}$ of the column if (a) $P = 560$ kN, and (b) $P = 890$ kN? (Assume $E = 200$ GPa and $\sigma_Y = 290$ MPa.)

Solution 11.9-16

$$E = 200 \text{ GPa} \quad \sigma_Y = 290 \text{ MPa} \quad K = 1$$

$$\text{W } 250 \times 67 \quad A = 8580 \text{ mm}^2 \quad r_2 = 51.1 \text{ mm} \quad r = r_2 \quad L_{\max} = 200r$$

$$L_c = 4.71r\sqrt{\frac{E}{\sigma_Y}} \quad L_c = 6.321 \text{ m} \quad \frac{L_c}{r} = 123.691$$

For each load, select trial values of the length L and calculate the corresponding values of P_{allow} . This solution shows only the successful trial.

(a) $P = 560$ kN

$$\text{Try } L = 6.44 \text{ m} \quad \frac{L}{r} = 126.027 \quad \frac{L}{r} > \frac{L_c}{r} \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 124.280 \text{ MPa}$$

$$\text{FS} = 1.67$$

$$\sigma_{\text{allow}} = \frac{0.877 \sigma_e}{\text{FS}} \quad \sigma_{\text{allow}} = 65.27 \text{ MPa}$$

$$P_{\text{allow}} = A \sigma_{\text{allow}} \quad P_{\text{allow}} = 560.0 \text{ kN}$$

$$\text{Therefore, } L_{\max} = L \quad \boxed{L_{\max} = 6.44 \text{ m}}$$

(b) $P = 890$ kN

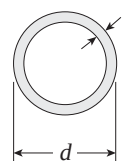
$$\text{Try } L = 4.68 \text{ m} \quad \frac{L}{r} = 91.585 \quad \frac{L}{r} < \frac{L_c}{r} \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 235.331 \text{ MPa}$$

$$\sigma_{\text{allow}} = \sigma_Y \frac{1}{\text{FS}} \left(\frac{\sigma_Y}{\sigma_e} \right) \quad \sigma_{\text{allow}} = 103.68 \text{ MPa}$$

$$P_{\text{allow}} = A \sigma_{\text{allow}} \quad P_{\text{allow}} = 890 \text{ kN}$$

$$\text{Therefore, } L_{\max} = L \quad \boxed{L_{\max} = 4.68 \text{ m}}$$

Problem 11.9-17 Find the required outside diameter d for a steel pipe column (see figure) of length $L = 20$ ft that is pinned at both ends and must support an axial load $P = 25$ k. Assume that the wall thickness t is equal $d/20$. (Use $E = 29,000$ ksi and $\sigma_Y = 36$ ksi.)



Probs. 11.9-17 through 11.9-20

Solution 11.9-17 Pipe column

$$E = 29,000 \text{ ksi} \quad \sigma_Y = 36 \text{ ksi} \quad K = 1 \quad P = 25 \text{ k} \quad L = 20 \text{ ft}$$

For each load, select trial values of the diameter d , compute cross-sectional properties, then find corresponding values of P_{allow} until $P_{\text{allow}} = P = 25$ k.

TRY $d = 5$ in.

$$A = \frac{\pi}{4} \left[d^2 - \left(d - 2 \frac{d}{20} \right)^2 \right] = 3.731 \text{ in.}^2 \quad I = \frac{\pi}{64} \left[d^4 - \left(d - 2 \frac{d}{20} \right)^4 \right] = 10.551 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.682 \text{ in.} \quad L_c = 4.71r \sqrt{\frac{E}{\sigma_Y}} \quad L_c = 18.734 \text{ ft} \quad \frac{L_c}{r} = 133.681 \quad \frac{L}{r} = 142.712$$

$$L_{\text{max}} = 200r = 28.028 \text{ ft}$$

$$\frac{L}{r} > \frac{L_c}{r} \quad \text{FS} = 1.67 \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r} \right)^2} = 14.053 \text{ ksi}$$

$$\sigma_{\text{allow}} = \frac{0.877 \sigma_e}{\text{FS}} = 7.380 \text{ ksi} \quad P_{\text{allow}} = A \sigma_{\text{allow}} = 27.532 \text{ k}$$

AFTER SEVERAL MORE TRIALS, TRY $d = 4.88$ in.

$$A = \frac{\pi}{4} \left[d^2 - \left(d - 2 \frac{d}{20} \right)^2 \right] = 3.555 \text{ in.}^2 \quad I = \frac{\pi}{64} \left[d^4 - \left(d - 2 \frac{d}{20} \right)^4 \right] = 9.582 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.642 \text{ in.} \quad L_c = 4.71r \sqrt{\frac{E}{\sigma_Y}} \quad L_c = 18.288 \text{ ft} \quad \frac{L_c}{r} = 133.681 \quad \frac{L}{r} = 146.192$$

$$L_{\text{max}} = 200r = 27.361 \text{ ft}$$

$$\frac{L}{r} > \frac{L_c}{r} \quad \text{FS} = 1.67 \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r} \right)^2} = 13.392 \text{ ksi}$$

$$\sigma_{\text{allow}} = \frac{0.877 \sigma_e}{\text{FS}} = 7.033 \text{ ksi} \quad P_{\text{allow}} = A \sigma_{\text{allow}} = 25.003 \text{ k} \quad \boxed{\text{Use } d = 4.88 \text{ in.}}$$

Problem 11.9-18 Find the required outside diameter d for a steel pipe column (see figure) of length $L = 3.5$ m that is pinned at both ends and must support an axial load $P = 130$ kN. Assume that the wall thickness t is equal to $d/20$. (Use $E = 200$ GPa and $\sigma_Y = 275$ MPa.)

Solution 11.9-18 Pipe column

$$E = 200 \text{ GPa} \quad \sigma_Y = 275 \text{ MPa} \quad K = 1 \quad P = 130 \text{ kN} \quad L = 3.5 \text{ m}$$

For each load, select trial values of the diameter d , compute cross-sectional properties, then find corresponding values of P_{allow} until $P_{\text{allow}} = P = 130$ kN.

TRY $d = 100$ mm.

$$A = \frac{\pi}{4} \left[d^2 - \left(d - 2 \frac{d}{20} \right)^2 \right] = 1492.257 \text{ mm}^2 \quad I = \frac{\pi}{64} \left[d^4 - \left(d - 2 \frac{d}{20} \right)^4 \right] = 1.688 \times 10^6 \text{ mm}^4$$

$$r = \sqrt{\frac{I}{A}} = 33.634 \text{ mm} \quad L_c = 4.71 r \sqrt{\frac{E}{\sigma_Y}} \quad L_c = 4.272 \text{ m} \quad \frac{L_c}{r} = 127.019 \quad \frac{L}{r} = 104.061$$

$$L_{\text{max}} = 200 r = 6.727 \text{ m}$$

$$\frac{L}{r} < \frac{L_c}{r} \quad \text{FS} = 1.67 \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r} \right)^2} = 182.286 \text{ MPa}$$

$$\sigma_{\text{allow}} = \frac{\frac{\sigma_Y}{\text{FS}}}{0.658 \frac{\sigma_e}{\sigma_Y}} = 87.577 \text{ MPa} \quad P_{\text{allow}} = A \sigma_{\text{allow}} = 130.687 \text{ kN}$$

AFTER SEVERAL MORE TRIALS, TRY $d = 99.8$ mm

$$A = \frac{\pi}{4} \left[d^2 - \left(d - 2 \frac{d}{20} \right)^2 \right] = 1487.485 \text{ mm}^2 \quad I = \frac{\pi}{64} \left[d^4 - \left(d - 2 \frac{d}{20} \right)^4 \right] = 1.677 \times 10^6 \text{ mm}^4$$

$$r = \sqrt{\frac{I}{A}} = 33.580 \text{ mm} \quad L_c = 4.71 r \sqrt{\frac{E}{\sigma_Y}} \quad L_c = 4.265 \text{ m} \quad \frac{L_c}{r} = 127.019 \quad \frac{L}{r} = 104.228$$

$$L_{\text{max}} = 200 r = 6.716 \text{ m}$$

$$\frac{L}{r} < \frac{L_c}{r} \quad \text{FS} = 1.67 \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r} \right)^2} = 181.703 \text{ MPa}$$

$$\sigma_{\text{allow}} = \frac{\frac{\sigma_Y}{\text{FS}}}{0.658 \frac{\sigma_e}{\sigma_Y}} = 87.399 \text{ MPa} \quad P_{\text{allow}} = A \sigma_{\text{allow}} = 130.005 \text{ kN} \quad \boxed{\text{Use } d = 99.8 \text{ mm}}$$

Problem 11.9-19 Find the required outside diameter d for a steel pipe column (see figure) of length $L = 11.5$ ft that is pinned at both ends and must support an axial load $P = 80$ k. Assume that the wall thickness t is 0.30 in. (Use $E = 29,000$ ksi and $\sigma_Y = 42$ ksi.)

Solution 11.9-19 Pipe column

$$E = 29,000 \text{ ksi} \quad \sigma_Y = 42 \text{ ksi} \quad K = 1 \quad P = 80 \text{ k} \quad L = 11.5 \text{ ft} \quad t = 0.30 \text{ in.}$$

For each load, select trial values of the diameter d , compute cross-sectional properties, then find corresponding values of P_{allow} until $P_{\text{allow}} = P = 80$ k.

TRY $d = 5$ in.

$$A = \frac{\pi}{4} [d^2 - (d - 2t)^2] = 4.430 \text{ in.}^2 \quad I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 12.281 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.665 \text{ in.} \quad L_c = 4.71r \sqrt{\frac{E}{\sigma_Y}} \quad L_c = 17.173 \text{ ft} \quad \frac{L_c}{r} = 123.764 \quad \frac{L}{r} = 82.879$$

$$L_{\max} = 200r = 27.751 \text{ ft}$$

$$\frac{L}{r} < \frac{L_c}{r} \quad \text{FS} = 1.67 \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 41.669 \text{ ksi}$$

$$\sigma_{\text{allow}} = \frac{0.658 \frac{\sigma_Y}{\sigma_e}}{\text{FS}} = 16.494 \text{ ksi} \quad P_{\text{allow}} = A \sigma_{\text{allow}} = 73.061 \text{ kip} < \text{low}$$

AFTER SEVERAL MORE TRIALS, TRY $d = 5.24$ in.

$$A = \frac{\pi}{4} [d^2 - (d - 2t)^2] = 4.659 \text{ in.}^2 \quad I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 14.281 \text{ in.}^4$$

$$r = \sqrt{\frac{I}{A}} = 1.751 \text{ in.} \quad L_c = 4.71r \sqrt{\frac{E}{\sigma_Y}} \quad L_c = 18.058 \text{ ft} \quad \frac{L_c}{r} = 123.764 \quad \frac{L}{r} = 78.820$$

$$L_{\max} = 200r = 29.181 \text{ ft}$$

$$\frac{L}{r} < \frac{L_c}{r} \quad \text{FS} = 1.67 \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 46.071 \text{ ksi}$$

$$\sigma_{\text{allow}} = \frac{0.658 \frac{\sigma_Y}{\sigma_e}}{\text{FS}} = 17.172 \text{ ksi} \quad P_{\text{allow}} = A \sigma_{\text{allow}} = 79.999 \text{ k} < \text{OK} \quad \boxed{\text{Use } d = 5.25 \text{ in.}}$$

Problem 11.9-20 Find the required outside diameter d for a steel pipe column (see figure) of length $L = 3.0$ m that is pinned at both ends and must support an axial load $P = 800$ kN. Assume that the wall thickness t is 9 mm. (Use $E = 200$ GPa and $\sigma_Y = 300$ MPa.)

Solution 11.9-20 Pipe column

$$E = 200 \text{ GPa} \quad \sigma_Y = 300 \text{ MPa} \quad K = 1 \quad P = 800 \text{ kN} \quad L = 3 \text{ m} \quad t = 9 \text{ mm}$$

For each load, select trial values of the diameter d , compute cross-sectional properties, then find corresponding values of P_{allow} until $P_{\text{allow}} = P = 800$ kN.

TRY $d = 200$ mm

$$A = \frac{\pi}{4} [d^2 - (d - 2t)^2] = 5400.398 \text{ mm}^2 \quad I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 2.468 \times 10^7 \text{ mm}^4$$

$$r = \sqrt{\frac{I}{A}} = 67.604 \text{ mm} \quad L_c = 4.71r \sqrt{\frac{E}{\sigma_Y}} \quad L_c = 8.221 \text{ m} \quad \frac{L_c}{r} = 121.612 \quad \frac{L}{r} = 44.376$$

$$L_{\text{max}} = 200r = 13.521 \text{ m}$$

$$\frac{L}{r} < \frac{L_c}{r} \quad \text{FS} = 1.67 \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 1002.368 \text{ MPa}$$

$$\sigma_{\text{allow}} = \frac{0.658 \frac{\sigma_Y}{\sigma_e} \sigma_Y}{\text{FS}} = 158.490 \text{ MPa} \quad P_{\text{allow}} = A \sigma_{\text{allow}} = 855.908 \text{ kN} < \text{high}$$

AFTER SEVERAL MORE TRIALS, TRY $d = 190$ mm

$$A = \frac{\pi}{4} [d^2 - (d - 2t)^2] = 5117.654 \text{ mm}^2 \quad I = \frac{\pi}{64} [d^4 - (d - 2t)^4] = 2.101 \times 10^7 \text{ mm}^4$$

$$r = \sqrt{\frac{I}{A}} = 64.072 \text{ mm} \quad L_c = 4.71r \sqrt{\frac{E}{\sigma_Y}} \quad L_c = 7.792 \text{ m} \quad \frac{L_c}{r} = 121.612 \quad \frac{L}{r} = 46.822$$

$$L_{\text{max}} = 200r = 12.814 \text{ m}$$

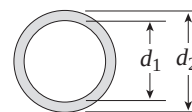
$$\frac{L}{r} < \frac{L_c}{r} \quad \text{FS} = 1.67 \quad \sigma_e = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2} = 900.382 \text{ MPa}$$

$$\sigma_{\text{allow}} = \frac{0.658 \frac{\sigma_Y}{\sigma_e} \sigma_Y}{\text{FS}} = 156.257 \text{ MPa} \quad P_{\text{allow}} = A \sigma_{\text{allow}} = 799.669 \text{ kN} < \text{OK} \quad \boxed{\text{Use } d = 190 \text{ mm}}$$

Aluminum Columns

Problem 11.9-21 An aluminium pipe column (alloy 2014-T6) with pinned ends has outside diameter $d_2 = 5.60$ in. and inside diameter $d_1 = 4.80$ in. (see figure).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 6$ ft, 8 ft, 10 ft, and 12 ft.



Probs. 11.9-21 through 11.9-24

Solution 11.9-21 Aluminum pipe column

$$K = 1 \quad L = \begin{pmatrix} 6 \text{ ft} \\ 8 \text{ ft} \\ 10 \text{ ft} \\ 12 \text{ ft} \end{pmatrix} \quad d_2 = 5.6 \text{ in.} \quad d_1 = 4.8 \text{ in.}$$

$$\text{CROSS-SECTIONAL PROPERTIES} \quad A = \frac{\pi}{4} (d_2^2 - d_1^2) = 6.535 \text{ in.}^2$$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) = 22.217 \text{ in.}^4 \quad r = \sqrt{\frac{I}{A}} = 1.844 \text{ in.} \quad \frac{KL}{r} = \begin{pmatrix} 39.047 \\ 52.063 \\ 65.079 \\ 78.095 \end{pmatrix}$$

ALLOWABLE STRESSES FOR ALUMINUM ALLOY 2014-T6

$i = 1$ to 4

$$\sigma_{\text{allow}_i} = \begin{cases} \left[30.9 - 0.229 \left(\frac{KL_i}{r} \right) \right] \text{ ksi} & \left(\text{if } \frac{KL_i}{r} \leq 55 \right) \\ \frac{55400 \text{ ksi}}{\left(\frac{KL_i}{r} \right)^2} & \text{(otherwise)} \end{cases} \quad \sigma_{\text{allow}} = \begin{pmatrix} 21.958 \\ 18.978 \\ 13.081 \\ 9.084 \end{pmatrix} \text{ ksi}$$

ALLOWABLE AXIAL LOADS

$$P_{\text{allow}} = \sigma_{\text{allow}} A = \begin{pmatrix} 143.5 \\ 124.0 \\ 85.5 \\ 59.4 \end{pmatrix} \text{ k} \quad P_{\text{allow}} = \begin{pmatrix} 143.5 \\ 124.0 \\ 85.5 \\ 59.4 \end{pmatrix} \text{ k} \quad \text{for } L = \begin{pmatrix} 6.0 \\ 8.0 \\ 10.0 \\ 12.0 \end{pmatrix} \text{ ft}$$

Problem 11.9-22 An aluminum pipe column (alloy 2014-T6) with pinned ends has outside diameter $d_2 = 120$ mm and inside diameter $d_1 = 110$ mm (see figure).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 1.0$ m, 2.0 m, 3.0 m, and 4.0 m.

Solution 11.9-22 Aluminum pipe column

$$K = 1 \quad L = \begin{pmatrix} 1 \text{ m} \\ 2 \text{ m} \\ 3 \text{ m} \\ 4 \text{ m} \end{pmatrix} \quad d_2 = 120 \text{ mm} \quad d_1 = 110 \text{ mm}$$

CROSS-SECTIONAL PROPERTIES $A = \frac{\pi}{4} (d_2^2 - d_1^2) = 1806.416 \text{ mm}^2$

$$I = \frac{\pi}{64} (d_2^4 - d_1^4) = 2.992 \times 10^6 \text{ mm}^4 \quad r = \sqrt{\frac{I}{A}} = 40.697 \text{ mm} \quad \frac{KL}{r} = \begin{pmatrix} 24.572 \\ 49.144 \\ 73.715 \\ 98.287 \end{pmatrix}$$

ALLOWABLE STRESSES FOR ALUMINUM ALLOY 2014-T6

$i = 1$ to 4

$$\sigma_{\text{allow}_i} = \begin{cases} \left[213 - 1.557 \left(\frac{KL_i}{r} \right) \right] \text{ MPa} & \left(\text{if } \frac{KL_i}{r} \leq 55 \right) \\ \frac{3.81 (10^5) \text{ MPa}}{\left(\frac{KL_i}{r} \right)^2} & (\text{otherwise}) \end{cases} \quad \sigma_{\text{allow}} = \begin{pmatrix} 174.250 \\ 135.501 \\ 70.115 \\ 39.439 \end{pmatrix} \text{ MPa}$$

ALLOWABLE AXIAL LOADS

$$P_{\text{allow}} = \sigma_{\text{allow}} A = \begin{pmatrix} 315 \\ 245 \\ 127 \\ 71 \end{pmatrix} \text{ kN} \quad P_{\text{allow}} = \begin{pmatrix} 315 \\ 245 \\ 127 \\ 71 \end{pmatrix} \text{ kN} \quad \text{for} \quad L = \begin{pmatrix} 1.0 \\ 2.0 \\ 3.0 \\ 4.0 \end{pmatrix} \text{ m}$$

Problem 11.9-23 An aluminum pipe column (alloy 6061-T6) that is fixed at the base and free at the top has outside diameter $d_2 = 3.25$ in. and inside diameter $d_1 = 3.00$ in. (see figure).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 2$ ft, 3 ft, 4 ft, and 5 ft.

Solution 11.9-23 Aluminum pipe column

Use $K = 2$ for fixed-free column end conditions $L = \begin{pmatrix} 2 \text{ ft} \\ 3 \text{ ft} \\ 4 \text{ ft} \\ 5 \text{ ft} \end{pmatrix}$

$d_2 = 3.25$ in. $d_1 = 3.00$ in.

CROSS-SECTIONAL PROPERTIES $A = \frac{\pi}{4}(d_2^2 - d_1^2) = 1.227 \text{ in.}^2$

$I = \frac{\pi}{64}(d_2^4 - d_1^4) = 1.500 \text{ in.}^4$ $r = \sqrt{\frac{I}{A}} = 1.106 \text{ in.}$ $\frac{KL}{r} = \begin{pmatrix} 43.410 \\ 65.115 \\ 86.820 \\ 108.525 \end{pmatrix}$

ALLOWABLE STRESSES FOR ALUMINUM ALLOY 6061-T6

$i = 1$ to 4

$\sigma_{\text{allow}_i} = \begin{cases} \left[20.3 - 0.127 \left(\frac{KL_i}{r} \right) \right] \text{ ksi} & \left(\text{if } \frac{KL_i}{r} \leq 66 \right) \\ \frac{51400 \text{ ksi}}{\left(\frac{KL_i}{r} \right)^2} & (\text{otherwise}) \end{cases}$ $\sigma_{\text{allow}} = \begin{pmatrix} 14.79 \\ 12.03 \\ 6.82 \\ 4.36 \end{pmatrix} \text{ ksi}$

ALLOWABLE AXIAL LOADS

$P_{\text{allow}} = \sigma_{\text{allow}} A = \begin{pmatrix} 18.1 \\ 14.8 \\ 8.4 \\ 5.4 \end{pmatrix} \text{ k}$ $P_{\text{allow}} = \begin{pmatrix} 18.1 \\ 14.8 \\ 8.4 \\ 5.4 \end{pmatrix} \text{ k}$ for $L = \begin{pmatrix} 2.0 \\ 3.0 \\ 4.0 \\ 5.0 \end{pmatrix} \text{ ft}$

Problem 11.9-24 An aluminum pipe column (alloy 6061-T6) that is fixed at the base and free at the top has outside diameter $d_2 = 80$ mm and inside diameter $d_1 = 72$ mm (see figure).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 0.6$ m, 0.8 m, 1.0 m, and 1.2 m.

Solution 11.9-24 Aluminum pipe column

Use $K = 2$ for fixed-free column end conditions $L = \begin{pmatrix} 0.6 \text{ m} \\ 0.8 \text{ m} \\ 1 \text{ m} \\ 1.2 \text{ m} \end{pmatrix}$

$d_2 = 80$ mm $d_1 = 72$ mm

CROSS-SECTIONAL PROPERTIES $A = \frac{\pi}{4}(d_2^2 - d_1^2) = 955.044 \text{ mm}^2$

$I \frac{\pi}{64}(d_2^4 - d_1^4) = 6.915 \times 10^5 \text{ mm}^4$ $r = \sqrt{\frac{I}{A}} = 26.907 \text{ mm}$ $\frac{KL}{r} = \begin{pmatrix} 44.598 \\ 59.464 \\ 74.329 \\ 89.195 \end{pmatrix}$

ALLOWABLE STRESSES FOR ALUMINUM ALLOY 6061-T6

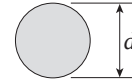
$i = 1.4$

$\sigma_{\text{allow}_i} = \begin{cases} \left[140 - 0.874 \left(\frac{KL_i}{r} \right) \right] \text{ MPa} & \left(\text{if } \frac{KL_i}{r} \leq 66 \right) \\ \frac{3.53 (10^5) \text{ MPa}}{\left(\frac{KL_i}{r} \right)^2} & \text{(otherwise)} \end{cases}$ $\sigma_{\text{allow}} = \begin{pmatrix} 101.02 \\ 88.03 \\ 63.89 \\ 44.37 \end{pmatrix} \text{ ksi}$

ALLOWABLE AXIAL LOADS

$\sigma_{\text{allow}} = \sigma_{\text{allow}_i} A = \begin{pmatrix} 96.5 \\ 84.1 \\ 61.0 \\ 42.4 \end{pmatrix} \text{ kN}$ $P_{\text{allow}} = \begin{pmatrix} 96.5 \\ 84.1 \\ 61.0 \\ 42.4 \end{pmatrix} \text{ kN}$ for $L = \begin{pmatrix} 0.6 \\ 0.8 \\ 1.0 \\ 1.2 \end{pmatrix} \text{ m}$

Problem 11.9-25 A solid round bar of aluminum having diameter d (see figure) is compressed by an axial force $P = 60$ k. The bar has pinned supports and is made of alloy 2014-T6.



Probs. 11.9-25 through 11.9-28

- (a) If the diameter $d = 2.0$ in., what is the maximum allowable length $L_{\max}$ of the bar?
 (b) If the length $L = 30$ in., what is the minimum required diameter $d_{\min}$?

Solution 11.9-25 Aluminum bar

$$K = 1 \quad P = 60 \text{ k}$$

(a) **MAXIMUM ALLOWABLE LENGTH** $d = 2$ in. $A = \frac{\pi}{4}(d^2) = 3.142 \text{ in.}^2$

$$I = \frac{\pi}{64}(d^4) = 0.785 \text{ in.}^4 \quad r\sqrt{\frac{I}{A}} = 0.500 \text{ in.} \quad \frac{P}{A} = 19.099 \text{ ksi}$$

Assume that $KL/r < 55$; rearrange allowable stress equation to solve for length L .

$$L_{\max} = \left(30.9 \text{ ksi} - \frac{P}{A} \right) \frac{r}{0.229 \text{ ksi}(K)} = 25.8 \text{ in.} \quad \text{So,} \quad \boxed{L_{\max} = 25.2 \text{ in.}}$$

Check: $\frac{KL_{\max}}{r} = 50.4 < \text{OK, is less than 55 as assumed}$

(b) **MINIMUM REQUIRED DIAMETER** $L = 30$ in.

$$r = \sqrt{\frac{\frac{\pi}{64}d^4}{\frac{\pi}{4}d^2}} \quad r = \frac{d}{4}$$

Assume that $KL/r > 55$; substitute $r = d/4$ and $\sigma = P/A$ into allowable stress equation and solve for $d_{\min}$.

$$\sigma_{\text{allow}} = \frac{55,400 \text{ ksi}}{\left(\frac{KL}{r}\right)^2} \quad \text{So,} \quad \frac{60}{\left(\frac{\pi d^2}{4}\right)} = \frac{55,400}{\left(\frac{30}{\frac{d}{4}}\right)^2} \quad \text{or} \quad \frac{60}{\left(\frac{\pi d^2}{4}\right)} - \frac{55,400}{\left(\frac{30}{\frac{d}{4}}\right)^2} = 0$$

Solving: $\boxed{d_{\min} = 2.11 \text{ in.}}$

Check that $KL/r > 55$: $\frac{KL}{r} = \frac{(1)30 \text{ in.}}{\frac{d_{\min}}{4}} = 56.872 < \text{OK}$

Problem 11.9-26 A solid round bar of aluminum having diameter d (see figure) is compressed by an axial force $P = 175$ kN. The bar has pinned supports and is made of alloy 2014-T6.

- (a) If the diameter $d = 40$ mm, what is the maximum allowable length $L_{\max}$ of the bar?
 (b) If the length $L = 0.6$ m, what is the minimum required diameter $d_{\min}$?

Solution 11.9-26 Aluminum bar

$$K = 1 \quad P = 175 \text{ kN}$$

(a) MAXIMUM ALLOWABLE LENGTH $d = 40 \text{ mm}$ $A \frac{\pi}{4} (d^2) = 1256.637 \text{ mm}^2$

$$I = \frac{\pi}{64} (d^4) = 1.257 \times 10^5 \text{ mm}^4 \quad r = \sqrt{\frac{I}{A}} = 10.000 \text{ mm} \quad \frac{P}{A} = 139.261 \text{ MPa}$$

Assume that $KL/r < 55$; rearrange allowable stress equation to solve for length L .

$$L_{\max} = \left(213 \text{ MPa} - \frac{P}{A} \right) \frac{r}{1.577 \text{ MPa}(K)} = 46.8 \text{ mm} \quad \text{so} \quad \boxed{L_{\max} = 468 \text{ mm}}$$

Check: $\frac{KL_{\max}}{r} = 46.8 < \text{OK, is less than } 55 \text{ as assumed}$

(b) MINIMUM REQUIRED DIAMETER $L = 0.6 \text{ m}$

$$r = \sqrt{\frac{\frac{\pi}{64} d^2}{\frac{\pi}{4} d^2}} \quad r = \frac{d}{4}$$

Assume that $KL/r > 55$; substitute $r = d/4$ and $\sigma = P/A$ into allowable stress equation and solve for $d_{\min}$.

$$\sigma_{\text{allow}} = \frac{3.81 (10^5)}{\left(\frac{KL}{r} \right)^2} \quad \text{So,} \quad \frac{175}{\left(\frac{\pi d^2}{4} \right)} = \frac{3.81 (10^5)}{\left(\frac{600}{\frac{d}{4}} \right)^2} \quad \text{or} \quad \frac{17,5000}{\left(\frac{\pi d^2}{4} \right)} - \frac{3.81 (10^5)}{\left(\frac{600}{\frac{d}{4}} \right)^2} = 0$$

Solving, $\boxed{d_{\min} = 42.8 \text{ mm}}$

Check that $KL/r > 55$: $\frac{KL}{r} = \frac{(1)0.6 \text{ m}}{\frac{d_{\min}}{4}} = 56.1 < \text{OK}$

Problem 11.9-27 A solid round bar of aluminum having diameter d (see figure) is compressed by an axial force $P = 10$ k. The bar has pinned supports and is made of alloy 6061-T6.

- (a) If the diameter $d = 1.0$ in., what is the maximum allowable length $L_{\max}$ of the bar?
 (b) If the length $L = 20$ in., what is the minimum required diameter $d_{\min}$?

Solution 11.9-27 Aluminum bar

$$K = 1 \quad P = 10 \text{ k}$$

- (a) Maximum allowable length $d = 1$ in.

$$A \frac{\pi}{4} (d^2) = 0.785 \text{ in.}^2$$

$$I = \frac{\pi}{64} (d^4) = 0.049 \text{ in.}^4 \quad r = \sqrt{\frac{I}{A}} = 0.250 \text{ in.} \quad \frac{P}{A} = 12.732 \text{ ksi}$$

Assume that $KL/r < 66$; rearrange allowable stress equation to solve for length L .

$$L_{\max} = \left(20.3 \text{ ksi} - \frac{P}{A} \right) \frac{r}{0.127 \text{ ksi}(K)} = 14.90 \text{ in.} \quad \text{so} \quad \boxed{L_{\max} = 14.9 \text{ in.}}$$

$$\text{Check:} \quad \frac{KL_{\max}}{r} = 59.6 < \text{OK, is less than 66 as assumed}$$

- (b) MINIMUM REQUIRED DIAMETER $L = 20$ in.

$$r = \sqrt{\frac{\frac{\pi}{64} d^4}{\frac{\pi}{4} d^2}} \quad r = \frac{d}{4}$$

Assume that $KL/r > 66$; substitute $r = d/4$ and $\sigma = P/A$ into allowable stress equation and solve for $d_{\min}$.

$$\sigma_{\text{allow}} = \frac{51,400}{\left(\frac{KL}{r}\right)^2} \quad \text{So,} \quad \frac{10}{\left(\frac{\pi d^2}{4}\right)} = \frac{51,400}{\left(\frac{20}{\frac{d}{4}}\right)^2} \quad \text{or} \quad \frac{10}{\left(\frac{\pi d^2}{4}\right)} - \frac{51,400}{\left(\frac{20}{\frac{d}{4}}\right)^2} = \frac{40}{\pi d^2} - \frac{257 d^2}{32} = 0$$

$$\text{Solving:} \quad \boxed{d_{\min} = 1.12 \text{ in.}}$$

$$\text{Check that } KL/r > 66 \quad \frac{KL}{r} = \frac{(1)20 \text{ in.}}{\frac{d_{\min}}{4}} = 71.4 < \text{OK}$$

Problem 11.9-28 A solid round bar of aluminum having diameter d (see figure) is compressed by an axial force $P = 60$ kN. The bar has pinned supports and is made of alloy 6061-T6.

- (a) If the diameter $d = 30$ mm, what is the maximum allowable length $L_{\max}$ of the bar?
 (b) If the length $L = 0.6$ m, what is the minimum required diameter $d_{\min}$?

Solution 11.9-28 Aluminum bar

$$K = 1 \quad P = 60 \text{ kN}$$

- (a) MAXIMUM ALLOWABLE LENGTH $d = 30$ mm

$$A = \frac{\pi}{4} (d^2) = 706.858 \text{ mm}^2$$

$$I = \frac{\pi}{64} (d^4) = 3.976 \times 10^4 \text{ mm}^4 \quad r = \sqrt{\frac{I}{A}} = 7.500 \text{ mm} \quad \frac{P}{A} = 84.883 \text{ MPa}$$

Assume that $KL/r < 66$; rearrange allowable stress equation to solve for length L .

$$L_{\max} = \left(140 \text{ MPa} - \frac{P}{A} \right) \frac{r}{0.874 \text{ MPa} (K)} = 473 \text{ mm} \quad \text{so} \quad \boxed{L_{\max} = 473 \text{ mm}}$$

$$\text{Check: } \frac{KL_{\max}}{r} = 63.1 < \text{OK, is less than 66 as assumed}$$

- (b) MINIMUM REQUIRED DIAMETER $L = 0.6$ m

$$r = \sqrt{\frac{\frac{\pi}{64} d^4}{\frac{\pi}{4} d^2}} \quad r = \frac{d}{4}$$

Assume that $KL/r > 66$; substitute $r = d/4$ and $\sigma = P/A$ into allowable stress equation and solve for $d_{\min}$.

$$\sigma_{\text{allow}} = \frac{3.53 (10^5)}{\left(\frac{KL}{r} \right)^2} \quad \text{So, } \frac{60,000}{\left(\frac{\pi d^2}{4} \right)} = \frac{3.53 (10^5)}{\left(\frac{600}{\frac{d}{4}} \right)^2} \quad \text{or} \quad \frac{24,0000}{\pi d^2} - 0.0612847 d^2 = 0$$

$$\text{Solving: } \boxed{d_{\min} = 33.4 \text{ mm}}$$

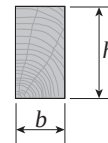
$$\text{Check that } KL/r > 66: \quad \frac{KL}{r} = \frac{(1) 0.6 \text{ m}}{\frac{d_{\min}}{4}} = 71.9 < \text{OK}$$

Wood Columns

When solving the problems for wood columns, assume that the columns are constructed of sawn lumber ($c = 0.8$) and have pinned-end conditions. Also, buckling may occur about either principal axis of the cross section.

Problem 11.9-29 A wood post of rectangular cross section (see figure) is constructed of 4 in. $\times$ 6 in. structural grade, Douglas fir lumber ($F_c = 2,000$ psi, $E = 1,800,000$ psi). The net cross-sectional dimensions of the post are $b = 3.5$ in. and $h = 5.5$ in. (see Appendix G).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 5.0$ ft, 7.5 ft and 10.0 ft.



Probs. 11.9-29 through 11.9-32

Solution 11.9-29 Wood post (rectangular cross section)

$$F_c = 2000 \text{ psi} \quad E = 1.8(10^6) \text{ psi} \quad b = 3.5 \text{ in.} \quad h = 5.5 \text{ in.} \quad L = \begin{pmatrix} 5 \text{ ft} \\ 7.5 \text{ ft} \\ 10 \text{ ft} \end{pmatrix}$$

$$c = 0.8 \quad d = b \quad L_e = L$$

WOOD POST SLENDERNESS RATIOS FOR EACH COLUMN LENGTH

$$\frac{L_e}{d} = \begin{pmatrix} 17.143 \\ 25.714 \\ 34.286 \end{pmatrix}$$

RATIO OF EULER BUCKLING COEFFICIENT TO COMPRESSIVE DESIGN STRESS FOR EACH COLUMN LENGTH

$$\phi = \frac{0.822E}{F_c \left(\frac{L_e}{d} \right)^2} = \begin{pmatrix} 2.517 \\ 1.119 \\ 0.629 \end{pmatrix}$$

COLUMN STABILITY FACTOR FOR EACH COLUMN LENGTH

$$C_p = \frac{1 + \phi}{2c} - \sqrt{\left(\frac{1 + \phi}{2c} \right)^2 - \frac{\phi}{c}} = \begin{pmatrix} 0.900 \\ 0.728 \\ 0.518 \end{pmatrix}$$

ALLOWABLE AXIAL LOAD ON COLUMN

$$P_{\text{allow}} = F_c C_p (bh) = \begin{pmatrix} 34.6 \\ 28.0 \\ 19.9 \end{pmatrix} \text{ k} \quad P_{\text{allow}} = \begin{pmatrix} 34.6 \\ 28.0 \\ 19.9 \end{pmatrix} \text{ k} \quad \text{for } L = \begin{pmatrix} 5.0 \\ 7.5 \\ 10.0 \end{pmatrix} \text{ ft}$$

Problem 11.9-30 A wood post of rectangular cross section (see figure) is constructed of structural grade, southern pine lumber ($F_c = 14$ MPa, $E = 12$ GPa). The cross-sectional dimensions of the post (actual dimensions) are $b = 100$ mm and $h = 150$ mm.

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 1.5$ m, 2.0 m, and 2.5 m.

Solution 11.9-30 Wood post (rectangular cross section)

$$F_c = 14 \text{ MPa} \quad E = 12 \text{ GPa} \quad b = 100 \text{ mm} \quad h = 150 \text{ mm} \quad L = \begin{pmatrix} 1.5 \text{ m} \\ 2 \text{ m} \\ 2.5 \text{ m} \end{pmatrix}$$

$$c = 0.8 \quad d = b \quad L_e = L$$

WOOD POST SLENDERNESS RATIOS FOR EACH COLUMN LENGTH

$$\frac{L_e}{d} = \begin{pmatrix} 15.000 \\ 20.000 \\ 25.000 \end{pmatrix}$$

RATIO OF EULER BUCKLING COEFFICIENT TO COMPRESSIVE DESIGN STRESS FOR EACH COLUMN LENGTH

$$\phi = \frac{0.822 E}{F_c \left(\frac{L_e}{d} \right)^2} = \begin{pmatrix} 3.131 \\ 1.761 \\ 1.127 \end{pmatrix}$$

COLUMN STABILITY FACTOR FOR EACH COLUMN LENGTH

$$C_P = \frac{1 + \phi}{2c} - \sqrt{\left(\frac{1 + \phi}{2c} \right)^2 - \frac{\phi}{c}} = \begin{pmatrix} 0.923 \\ 0.844 \\ 0.731 \end{pmatrix}$$

ALLOWABLE AXIAL LOAD ON COLUMN

$P_{\text{allow}} = F_c C_P (bh) = \begin{pmatrix} 193.8 \\ 177.3 \\ 153.5 \end{pmatrix} \text{ kN}$	$P_{\text{allow}} = \begin{pmatrix} 193.8 \\ 177.3 \\ 153.5 \end{pmatrix} \text{ kN} \quad \text{for } L = \begin{pmatrix} 1.5 \\ 2.0 \\ 2.5 \end{pmatrix} \text{ m}$

Problem 11.9-31 A wood post column of rectangular cross section (see figure) is constructed of 4 in. $\times$ 8 in. construction grade, western hemlock lumber ($F_c = 1,000$ psi, $E = 1,300,000$ psi). The net cross-sectional dimensions of the column are $b = 3.5$ in. and $h = 7.25$ in. (see Appendix G).

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 6$ ft, 8 ft, and 10 ft.

Solution 11.9-31 Wood column (rectangular cross section)

$$F_c = 1000 \text{ psi} \quad E = 1.3(10^6) \text{ psi} \quad b = 3.5 \text{ in.} \quad h = 7.25 \text{ in.} \quad L = \begin{pmatrix} 6 \text{ ft} \\ 8 \text{ ft} \\ 10 \text{ ft} \end{pmatrix}$$

$$c = 0.8 \quad d = b \quad L_e = L$$

WOOD POST SLENDERNESS RATIOS FOR EACH COLUMN LENGTH

$$\frac{L_e}{d} = \begin{pmatrix} 20.571 \\ 27.429 \\ 34.286 \end{pmatrix}$$

RATIO OF EULER BUCKLING COEFFICIENT TO COMPRESSIVE DESIGN STRESS FOR EACH COLUMN LENGTH

$$\phi = \frac{0.822 E}{F_c \left(\frac{L_e}{d} \right)^2} = \begin{pmatrix} 2.525 \\ 1.420 \\ 0.909 \end{pmatrix}$$

COLUMN STABILITY FACTOR FOR EACH COLUMN LENGTH

$$C_P = \frac{1 + \phi}{2c} - \sqrt{\left(\frac{1 + \phi}{2c} \right)^2 - \frac{\phi}{c}} = \begin{pmatrix} 0.900 \\ 0.797 \\ 0.657 \end{pmatrix}$$

ALLOWABLE AXIAL LOAD ON COLUMN

$$P_{\text{allow}} = F_c C_P (bh) = \begin{pmatrix} 22.8 \\ 20.2 \\ 16.7 \end{pmatrix} \text{ k} \quad P_{\text{allow}} = \begin{pmatrix} 22.8 \\ 20.2 \\ 16.7 \end{pmatrix} \text{ k} \quad \text{for } L = \begin{pmatrix} 6.0 \\ 8.0 \\ 10.0 \end{pmatrix} \text{ ft}$$

Problem 11.9-32 A wood column of rectangular cross section (see figure) is constructed of structural grade, Douglas fir lumber ($F_c = 12$ MPa, $E = 10$ GPa). The cross-sectional dimensions of the column (actual dimensions) are $b = 140$ mm and $h = 210$ mm.

Determine the allowable axial load P_{allow} for each of the following lengths: $L = 2.5$ m, 3.5 m, and 4.5 m.

Solution 11.9-32 Wood column (rectangular cross section)

$$F_c = 12 \text{ MPa} \quad E = 10 \text{ GPa} \quad b = 140 \text{ mm} \quad h = 210 \text{ mm} \quad L = \begin{pmatrix} 2.5 \text{ m} \\ 3.5 \text{ m} \\ 4.5 \text{ m} \end{pmatrix}$$

$$c = 0.8 \quad d = b \quad L_e = L$$

WOOD POST SLENDERNESS RATIOS FOR EACH COLUMN LENGTH

$$\frac{L_e}{d} = \begin{pmatrix} 17.857 \\ 25.000 \\ 32.143 \end{pmatrix}$$

RATIO OF EULER BUCKLING COEFFICIENT TO COMPRESSIVE DESIGN STRESS FOR EACH COLUMN LENGTH

$$\phi = \frac{0.822E}{F_c \left(\frac{L_e}{d} \right)^2} = \begin{pmatrix} 2.148 \\ 1.096 \\ 0.663 \end{pmatrix}$$

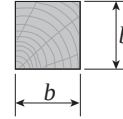
COLUMN STABILITY FACTOR FOR EACH COLUMN LENGTH

$$C_P = \frac{1 + \phi}{2c} - \sqrt{\left(\frac{1 + \phi}{2c} \right)^2 - \frac{\phi}{c}} = \begin{pmatrix} 0.878 \\ 0.722 \\ 0.538 \end{pmatrix}$$

ALLOWABLE AXIAL LOAD ON COLUMN

$$P_{\text{allow}} = F_c C_P (bh) = \begin{pmatrix} 310 \\ 255 \\ 190 \end{pmatrix} \text{ kN} \quad P_{\text{allow}} = \begin{pmatrix} 310 \\ 255 \\ 190 \end{pmatrix} \text{ kN} \quad \text{for } L = \begin{pmatrix} 2.5 \\ 3.5 \\ 4.5 \end{pmatrix} \text{ m}$$

Problem 11.9-33 A square wood column with side dimensions b (see figure) is constructed of a structural grade of Douglas fir for which $F_c = 1700$ psi and $E = 1,400,000$ psi. An axial force $P = 40$ k acts on the column.



Probs. 11.9-33 through 11.9-36

- (a) If the dimension $b = 5.5$ in., what is the maximum allowable length $L_{\max}$ of the column?
 (b) If the length $L = 11$ ft, what is the minimum required dimension $b_{\min}$?

Solution 11.9-33 Wood column (square cross section)

$$F_c = 1700 \text{ psi} \quad E = 1.4(10^6) \text{ psi} \quad P = 40 \text{ k} \quad c = 0.8$$

- (a) MAXIMUM ALLOWABLE LENGTH OF COLUMN $b = 5.5$ in., $h = b$, AND $d = b$

$$C_P = \frac{P}{F_c b^2} = 0.77783 \quad \text{so} \quad \frac{1 + \phi}{1.6} - \sqrt{\left(\frac{1 + \phi}{1.6}\right)^2 - \frac{\phi}{0.8}} = C_P$$

Solving for ϕ using trial-and-error gives $\phi = 1.32248$.

Solve for maximum length using ratio of Euler buckling coefficient to compressive design stress.

$$\frac{L}{d} = \sqrt{\frac{0.822E}{\phi F_c}} = 22.625 \quad \text{So,} \quad L_{\max} = d \sqrt{\frac{0.822E}{\phi F_c}} = 124.4 \text{ in.} \quad \text{or} \quad L_{\max} = 10.37 \text{ ft}$$

- (b) MINIMUM REQUIRED WIDTH $L = 11$ ft, $b = d$

$$b = \sqrt{\frac{P}{C_P F_c}} \quad \text{and} \quad b = \frac{L}{\sqrt{\frac{0.822E}{\phi F_c}}}$$

Substituting for C_P in terms of ϕ then equating expressions for b :

$$\frac{P}{\left[\frac{1 + \phi}{1.6} - \sqrt{\left(\frac{1 + \phi}{1.6}\right)^2 - \frac{\phi}{0.8}}\right] F_c} - \frac{L^2}{\frac{0.822E}{\phi F_c}} = 0$$

$$\text{Solving by trial-and-error: } \phi = 1.213 \quad \text{So,} \quad b_{\min} = \frac{L}{\sqrt{\frac{0.822E}{\phi F_c}}} = 5.59 \text{ in.}$$

$$\boxed{b_{\min} = 5.59 \text{ in.}}$$

Problem 11.9-34 A square wood column with side dimensions b (see figure) is constructed of a structural grade of southern pine for which $F_c = 10.5$ MPa and $E = 12$ GPa. An axial force $P = 200$ kN acts on the column.

- (a) If the dimension $b = 150$ mm, what is the maximum allowable length $L_{\max}$ of the column?
 (b) If the length $L = 4.0$ m, what is the minimum required dimension $b_{\min}$?

Solution 11.9-34 Wood column (square cross section)

$$F_c = 10.5 \text{ MPa} \quad E = 12 \text{ GPa} \quad P = 200 \text{ kN} \quad c = 0.8$$

- (a) MAXIMUM ALLOWABLE LENGTH OF COLUMN $b = 150$ mm, $h = b$, AND $d = b$

$$C_P = \frac{P}{F_c b^2} = 0.84656 \quad \text{So,} \quad \frac{1 + \phi}{1.6} - \sqrt{\left(\frac{1 + \phi}{1.6}\right)^2 - \frac{\phi}{0.8}} = C_P$$

Solving for ϕ using trial-and-error gives $\phi = 1.7807$.

Solve for maximum length using ratio of Euler buckling coefficient to compressive design stress.

$$\frac{L}{d} = \sqrt{\frac{0.822 E}{\phi F_c}} = 22.969 \quad \text{So,} \quad L_{\max} = d \sqrt{\frac{0.822 E}{\phi F_c}} = 3445.3 \text{ mm} \quad \text{or} \quad \boxed{L_{\max} = 3.45 \text{ m}}$$

- (b) MINIMUM REQUIRED WIDTH $L = 4$ m, $b = d$

$$b = \sqrt{\frac{P}{C_P F_c}} \quad \text{and} \quad b = \frac{L}{\sqrt{\frac{0.822 E}{\phi F_c}}}$$

Substituting for C_P in terms of ϕ then equating expressions for b :

$$\frac{P}{\left[\frac{1 + \phi}{1.6} - \sqrt{\left(\frac{1 + \phi}{1.6}\right)^2 - \frac{\phi}{0.8}}\right] F_c} - \frac{L^2}{\frac{0.822 E}{\phi F_c}} = 0$$

$$\text{Solving by trial-and-error: } \phi = 1.408 \quad \text{So,} \quad b_{\min} = \frac{L}{\sqrt{\frac{0.822 E}{\phi F_c}}} = 154.9 \text{ mm}$$

$$\boxed{b_{\min} = 154.9 \text{ mm}}$$

Problem 11.9-35 A square wood column with side dimensions b (see figure) is constructed of a structural grade of spruce for which $F_c = 900$ psi and $E = 1,500,000$ psi. An axial force $P = 8.0$ k acts on the column.

- (a) If the dimension $b = 3.5$ in., what is the maximum allowable length $L_{\max}$ of the column?
 (b) If the length $L = 10$ ft, what is the minimum required dimension $b_{\min}$?

Solution 11.9-35 Wood column (square cross section)

$$F_c = 900 \text{ psi} \quad E = 1.5 (10^6) \text{ psi} \quad P = 8 \text{ k} \quad c = 0.8$$

- (a) MAXIMUM ALLOWABLE LENGTH OF COLUMN $b = 3.5$ in., $h = b$, AND $d = b$

$$C_P = \frac{P}{F_c b^2} = 0.72562 \quad \text{So,} \quad \frac{1 + \phi}{1.6} - \sqrt{\left(\frac{1 + \phi}{1.6}\right)^2 - \frac{\phi}{0.8}} = C_P$$

Solving for ϕ using trial-and-error gives $\phi = 1.10942$.

Solve for maximum length using ratio of Euler buckling coefficient to compressive design stress.

$$\frac{L}{d} = \sqrt{\frac{0.822E}{\phi F_c}} = 35.141 \quad \text{So,} \quad L_{\max} = d \sqrt{\frac{0.822E}{\phi F_c}} = 123.0 \text{ in.} \quad \text{or} \quad \boxed{L_{\max} = 10.25 \text{ ft}}$$

- (b) MINIMUM REQUIRED WIDTH $L = 10$ ft, $b = d$

$$b = \sqrt{\frac{P}{C_P F_c}} \quad \text{and} \quad b = \frac{L}{\sqrt{\frac{0.822E}{\phi F_c}}}$$

Substituting for C_P in terms of ϕ then equating expressions for b :

$$\frac{P}{\left[\frac{1 + \phi}{1.6} - \sqrt{\left(\frac{1 + \phi}{1.6}\right)^2 - \frac{\phi}{0.8}}\right] F_c} - \frac{L^2}{\frac{0.822E}{\phi F_c}} = 0$$

$$\text{Solving by trial-and-error: } \phi = 1.148 \quad \text{So,} \quad b_{\min} = \frac{L}{\sqrt{\frac{0.822E}{\phi F_c}}} = 3.47 \text{ in.}$$

$$\boxed{b_{\min} = 3.47 \text{ in.}}$$

Problem 11.9-36 A square wood column with side dimensions b (see figure) is constructed of a structural grade of eastern white pine for which $F_c = 8.0$ MPa and $E = 8.5$ GPa. An axial force $P = 100$ kN acts on the column.

- (a) If the dimension $b = 120$ mm, what is the maximum allowable length $L_{\max}$ of the column?
 (b) If the length $L = 4.0$ m, what is the minimum required dimension $b_{\min}$?

Solution 11.9-36 Wood column (square cross section)

$$F_c = 8 \text{ MPa} \quad E = 8.5 \text{ GPa} \quad P = 100 \text{ kN} \quad c = 0.8$$

- (a) MAXIMUM ALLOWABLE LENGTH OF COLUMN $b = 120$ mm, $h = b$, AND $d = b$

$$C_P = \frac{P}{F_c b^2} = 0.86806 \quad \text{So,} \quad \frac{1 + \phi}{1.6} - \sqrt{\left(\frac{1 + \phi}{1.6}\right)^2 - \frac{\phi}{0.8}} = C_P$$

Solving for ϕ using trial-and-error gives $\phi = 2.0102$.

Solve for maximum length using ratio of Euler buckling coefficient to compressive design stress.

$$\frac{L}{d} = \sqrt{\frac{0.822 E}{\phi F_c}} = 20.844 \quad \text{So,} \quad L_{\max} = d \sqrt{\frac{0.822 E}{\phi F_c}} = 2501.3 \text{ mm} \quad \text{or} \quad \boxed{L_{\max} = 2.50 \text{ m}}$$

- (b) MINIMUM REQUIRED WIDTH $L = 4$ m, $b = d$

$$b = \sqrt{\frac{P}{C_P F_c}} \quad \text{and} \quad b = \frac{L}{\sqrt{\frac{0.822 E}{\phi F_c}}}$$

Substituting for C_P in terms of ϕ then equating expressions for b :

$$\frac{P}{\left[\frac{1 + \phi}{1.6} - \sqrt{\left(\frac{1 + \phi}{1.6}\right)^2 - \frac{\phi}{0.8}} \right] F_c} - \frac{L^2}{\frac{0.822 E}{\phi F_c}} = 0$$

$$\text{Solving by trial-and-error:} \quad \phi = 0.992 \quad \text{So,} \quad b_{\min} = \frac{L}{\sqrt{\frac{0.822 E}{\phi F_c}}} = 134.78 \text{ mm}$$

$$\boxed{b_{\min} = 134.8 \text{ mm}}$$

12

Review of Centroids and Moments of Inertia

Centroids of Plane Areas

The problems for Section 12.2 are to be solved by integration.

Problem 12.2-1 Determine the distances $\bar{x}$ and $\bar{y}$ to the centroid C of a right triangle having base b and altitude h (see Case 6, Appendix E).

Solution 12.2-1 Centroid of a right triangle

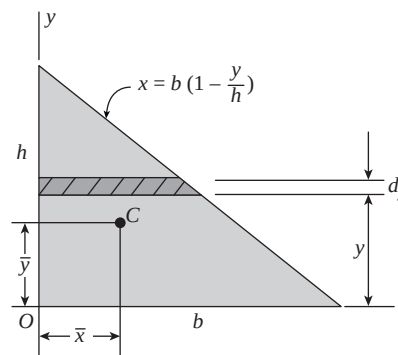
$$dA = xdy = b(1 - y/h)dy$$

$$A = \int dA = \int_0^h b(1 - y/h)dy = \frac{bh}{2}$$

$$Q_x = \int ydA = \int_0^h yb(1 - y/h)dy = \frac{bh^2}{6}$$

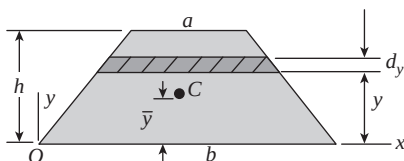
$$\bar{y} = \frac{Q_x}{A} = \frac{h}{3} \quad \leftarrow$$

$$\text{Similarly, } \bar{x} = \frac{b}{3} \quad \leftarrow$$



Problem 12.2-2 Determine the distance $\bar{y}$ to the centroid C of a trapezoid having bases a and b and altitude h (see Case 8, Appendix E).

Solution 12.2-2 Centroid of a trapezoid



Width of element $= b + (a - b)y/h$

$$dA = [b + (a - b)y/h]dy$$

$$A = \int dA = \int_0^h [b + (a - b)y/h]dy = \frac{h(a + b)}{2}$$

$$Q_x = \int ydA = \int_0^h y[b + (a - b)y/h]dy$$

$$= \frac{h^2}{6}(2a + b)$$

$$\bar{y} = \frac{Q_x}{A} = \frac{h(2a + b)}{3(a + b)} \quad \leftarrow$$

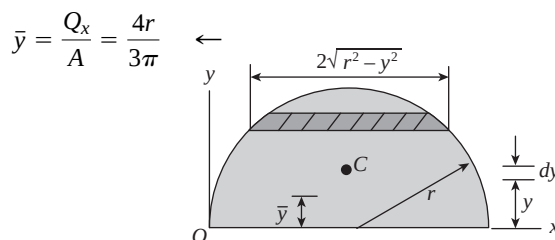
Problem 12.2-3 Determine the distance $\bar{y}$ to the centroid C of a semicircle of radius r (see Case 10, Appendix E).

Solution 12.2-3 Centroid of a semicircle

$$dA = 2\sqrt{r^2 - y^2}dy$$

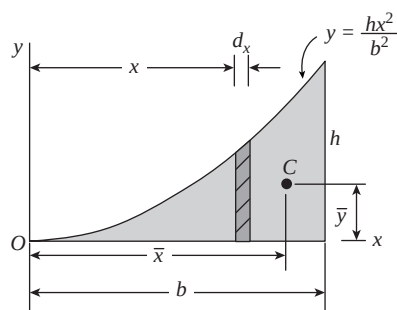
$$A = \int dA = \int_0^r 2\sqrt{r^2 - y^2}dy = \frac{\pi r^2}{2}$$

$$Q_x = \int ydA = \int_0^r 2y\sqrt{r^2 - y^2}dy = \frac{2r^3}{3}$$



Problem 12.2-4 Determine the distances $\bar{x}$ and $\bar{y}$ to the centroid C of a parabolic spandrel of base b and height h (see Case 18, Appendix E).

Solution 12.2-4 Centroid of a parabolic spandrel



$$dA = ydx = \frac{hx^2}{b^2}dx$$

$$A = \int dA = \int_0^b \frac{hx^2}{b^2}dx = \frac{bh}{3}$$

$$Q_y = \int x dA = \int_0^b \frac{hx^3}{b^2}dx = \frac{b^2h}{4}$$

$$\bar{x} = \frac{Q_y}{A} = \frac{3b}{4} \quad \leftarrow$$

$$Q_x = \int y/2 dA = \int_0^b \frac{1}{2} \left(\frac{hx^2}{b^2} \right) \left(\frac{hx^2}{b^2} \right) dx = \frac{bh^2}{10}$$

$$\bar{y} = \frac{Q_x}{A} = \frac{3h}{10} \quad \leftarrow$$

Problem 12.2-5 Determine the distances $\bar{x}$ and $\bar{y}$ to the centroid C of a semisegment of n th degree having base b and height h (see Case 19, Appendix E).

Solution 12.2-5 Centroid of a semisegment of n th degree

$$dA = ydx = h \left(1 - \frac{x^n}{b^n} \right) dx$$

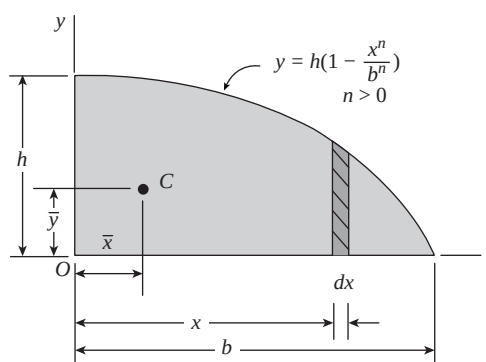
$$A = \int dA = \int_0^b h \left(1 - \frac{x^n}{b^n} \right) dx = bh \left(\frac{n}{n+1} \right)$$

$$Q_y = \int x dA = \int_0^b xh \left(1 - \frac{x^n}{b^n} \right) dx = \frac{hb^2}{2} \left(\frac{n}{n+2} \right)$$

$$\bar{x} = \frac{Q_y}{A} = \frac{b(n+1)}{2(n+2)} \quad \leftarrow$$

$$\begin{aligned} Q_x &= \int \frac{y}{2} dA = \int_0^b \frac{1}{2} h \left(1 - \frac{x^n}{b^n} \right) \left(h \left(1 - \frac{x^n}{b^n} \right) \right) dx \\ &= bh^2 \left[\frac{n^2}{(n+1)(2n+1)} \right] \end{aligned}$$

$$\bar{y} = \frac{Q_x}{A} = \frac{hn}{2n+1} \quad \leftarrow$$

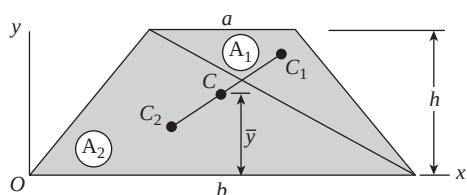


Centroids of Composite Areas

The problems for Section 12.3 are to be solved by using the formulas for composite areas.

Problem 12.3-1 Determine the distance $\bar{y}$ to the centroid C of a trapezoid having bases a and b and altitude h (see Case 8, Appendix E) by dividing the trapezoid into two triangles.

Solution 12.3-1 Centroid of a trapezoid



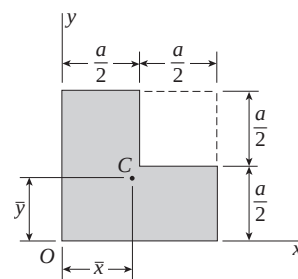
$$A_1 = \frac{ah}{2} \quad \bar{y}_1 = \frac{2h}{3} \quad A_2 = \frac{bh}{2} \quad \bar{y}_2 = \frac{h}{3}$$

$$A = \sum A_i = \frac{ah}{2} + \frac{bh}{2} = \frac{h}{2}(a + b)$$

$$Q_x = \sum \bar{y}_i A_i = \frac{2h}{3} \left(\frac{ah}{2} \right) + \frac{h}{3} \left(\frac{bh}{2} \right) = \frac{h^2}{6}(2a + b)$$

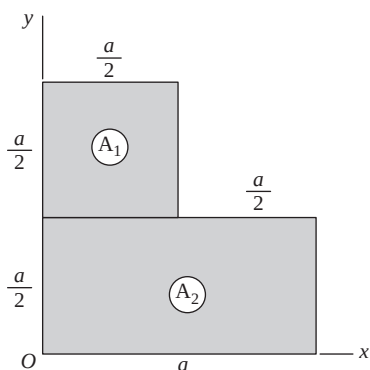
$$\bar{y} = \frac{Q_x}{A} = \frac{h(2a + b)}{3(a + b)} \quad \leftarrow$$

Problem 12.3-2 One quarter of a square of side a is removed (see figure). What are the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the remaining area?



PROBS. 12.3-2 and 12.5-2

Solution 12.3-2 Centroid of a composite area



$$A_1 = \frac{a^2}{4} \quad \bar{y}_1 = \frac{3a}{4}$$

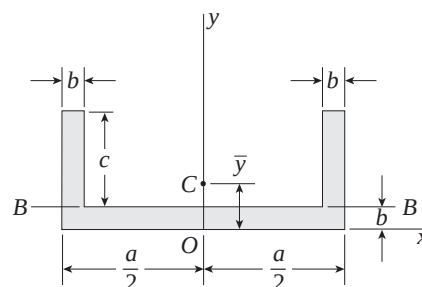
$$A_2 = \frac{a^2}{4} \quad \bar{y}_2 = \frac{a}{4}$$

$$A = \sum A_i = \frac{3a^2}{4}$$

$$Q_x = \sum \bar{y}_i A_i = \frac{3a}{4} \left(\frac{a^2}{4} \right) + \frac{a}{4} \left(\frac{a^2}{4} \right) = \frac{5a^3}{16}$$

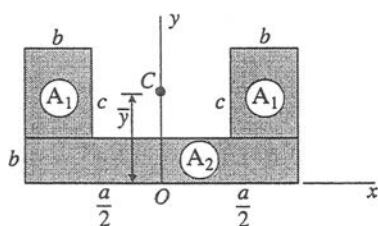
$$\bar{x} = \bar{y} = \frac{Q_x}{A} = \frac{5a}{12} \quad \leftarrow$$

Problem 12.3-3 Calculate the distance $\bar{y}$ to the centroid C of the channel section shown in the figure if $a = 6$ in., $b = 1$ in., and $c = 2$ in.



PROBS. 12.3-3, 12.3-4, and 12.5-3

Solution 12.3-3 Centroid of a channel section



$$a = 6 \text{ in.} \quad b = 1 \text{ in.} \quad c = 2 \text{ in.}$$

$$A_1 = bc = 2 \text{ in.}^2 \quad \bar{y}_1 = b + c/2 = 2 \text{ in.}$$

$$A_2 = ab = 6 \text{ in.}^2 \quad \bar{y}_2 = \frac{b}{2} = 0.5 \text{ in.}$$

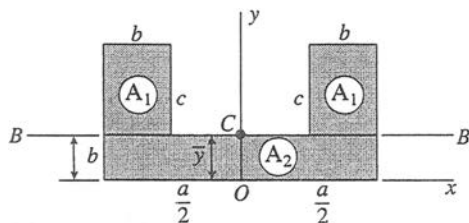
$$A = \sum A_i = 2A_1 + A_2 = 10 \text{ in.}^2$$

$$Q_x = \sum \bar{y}_i A_i = 2\bar{y}_1 A_1 + \bar{y}_2 A_2 = 11.0 \text{ in.}^3$$

$$\bar{y} = \frac{Q_x}{A} = 1.10 \text{ in.} \quad \leftarrow$$

Problem 12.3-4 What must be the relationship between the dimensions a , b , and c of the channel section shown in the figure in order that the centroid C will lie on line BB ?

Solution 12.3-4 Dimensions of channel section



$$A_1 = bc \quad \bar{y}_1 = b + c/2$$

$$A_2 = ab \quad \bar{y}_2 = b/2$$

$$A = \sum A_i = 2A_1 + A_2 = b(2c + a)$$

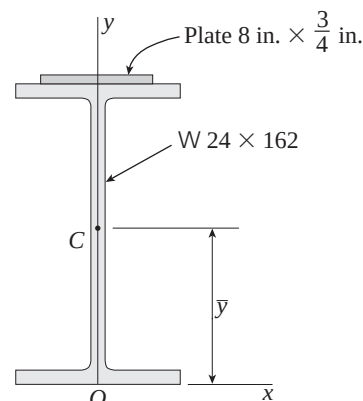
$$Q_x = \sum \bar{y}_i A_i = 2\bar{y}_1 A_1 + \bar{y}_2 A_2 = b/2(4bc + 2c^2 + ab)$$

$$\bar{y} = \frac{Q_x}{A} = \frac{4bc + 2c^2 + ab}{2(2c + a)}$$

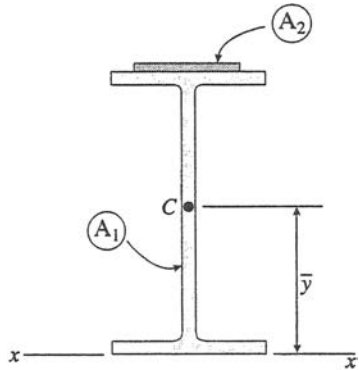
$$\text{Set } \bar{y} = b \text{ and solve: } 2c^2 = ab \quad \leftarrow$$

Problem 12.3-5 The cross section of a beam constructed of a W 24 $\times$ 162 wide-flange section with an 8 in. $\times$ 3/4 in. cover plate welded to the top flange is shown in the figure.

Determine the distance $\bar{y}$ from the base of the beam to the centroid C of the cross-sectional area.



PROBS. 12.3-5 and 12.5-5

Solution 12.3-5 Centroid of beam cross section

$$W\ 24 \times 162 \quad A_1 = 47.7\text{ in.}^2 \quad d = 25.00\text{ in.}$$

$$\bar{y}_1 = d/2 = 12.5\text{ in.}$$

$$\text{PLATE: } 8.0 \times 0.75\text{ in.} \quad A_2 = (8.0)(0.75) = 6.0\text{ in.}^2$$

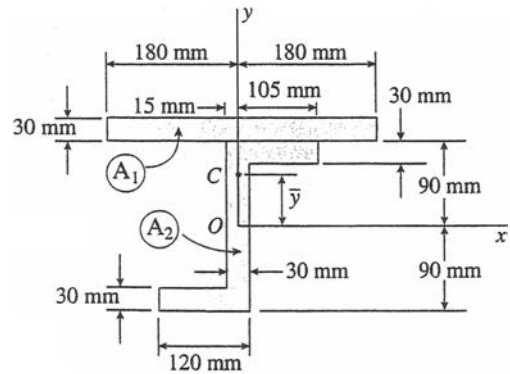
$$\bar{y}_2 = 25.00 + 0.75/2 = 25.375\text{ in.}$$

$$A = \sum A_i = A_1 + A_2 = 53.70\text{ in.}^2$$

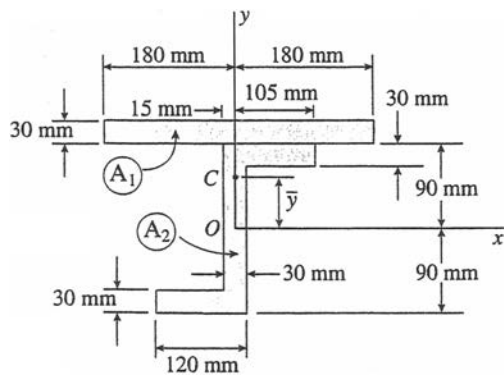
$$Q_x = \sum \bar{y}_i A_i = \bar{y}_1 A_1 + \bar{y}_2 A_2 = 748.5\text{ in.}^3$$

$$\bar{y} = \frac{Q_x}{A} = 13.94\text{ in.} \quad \leftarrow$$

Problem 12.3-6 Determine the distance $\bar{y}$ to the centroid C of the composite area shown in the figure.



PROBS. 12.3-6, 12.5-6, and 12.7-6

Solution 12.3-6 Centroid of composite area

$$A_1 = (360)(30) = 10,800\text{ mm}^2$$

$$\bar{y}_1 = 105\text{ mm}$$

$$A_2 = 2(120)(30) + (120)(30) = 10,800\text{ mm}^2$$

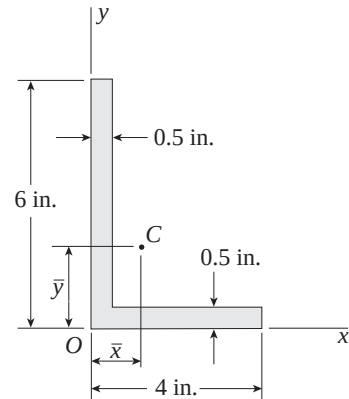
$$\bar{y}_2 = 0$$

$$A = \sum A_i = A_1 + A_2 = 21,600\text{ mm}^2$$

$$Q_x = \sum \bar{y}_i A_i = \bar{y}_1 A_1 + \bar{y}_2 A_2 = 1.134 \times 10^6\text{ mm}^3$$

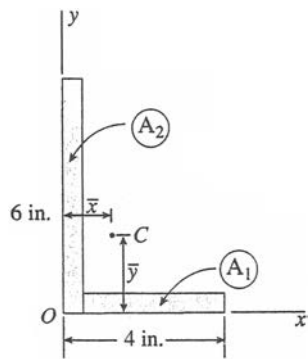
$$\bar{y} = \frac{Q_x}{A} = 52.5\text{ mm} \quad \leftarrow$$

Problem 12.3-7 Determine the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the L-shaped area shown in the figure.



PROBS. 12.3-7, 12.4-7, 12.5-7, and 12.7-7

Solution 12.3-7 Centroid of L-shaped area



Thickness $t = 0.5$ in.

$$A_1 = (3.5)(0.5) = 1.75 \text{ in.}^2$$

$$\bar{y}_1 = 0.25 \text{ in.} \quad \bar{x}_1 = 2.25 \text{ in.}$$

$$A_2 = (6)(0.5) = 3.0 \text{ in.}^2$$

$$\bar{y}_2 = 3.0 \text{ in.} \quad \bar{x}_2 = 0.25 \text{ in.}$$

$$A = \sum A_1 = A_1 + A_2 = 4.75 \text{ in.}^2$$

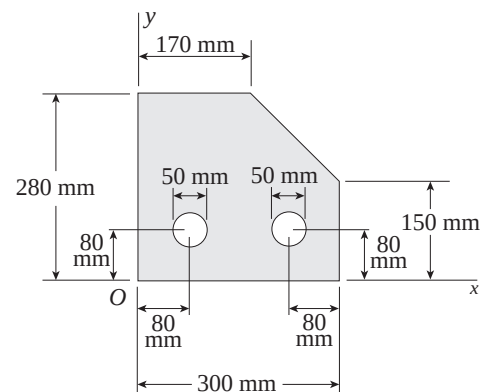
$$Q_y = \sum \bar{x}_1 A_1 = \bar{x}_1 A_1 + \bar{x}_2 A_2 = 4.688 \text{ in.}^3$$

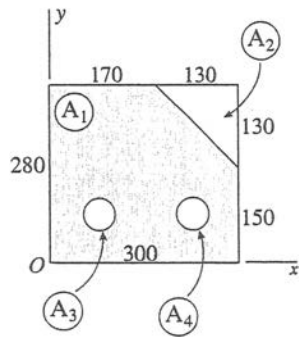
$$\bar{x} = \frac{Q_y}{A} = 0.99 \text{ in.} \quad \leftarrow$$

$$Q_x = \sum \bar{y}_1 A_1 = \bar{y}_1 A_1 + \bar{y}_2 A_2 = 9.438 \text{ in.}^3$$

$$\bar{y} = \frac{Q_x}{A} = 1.99 \text{ in.} \quad \leftarrow$$

Problem 12.3-8 Determine the coordinates $\bar{x}$ and $\bar{y}$ of the centroid C of the area shown in the figure.



Solution 12.3-8 Centroid of composite area

A_1 = large rectangle
 A_2 = triangular cutout
 $A_3 = A_4$ = circular holes
 All dimensions are in millimeters.
 Diameter of holes = 50 mm
 Centers of holes are 80 mm from edges.

$$A_1 = (280)(300) = 84,000 \text{ mm}^2$$

$$\bar{x}_1 = 150 \text{ mm} \quad \bar{y}_1 = 140 \text{ mm}$$

$$A_2 = 1/2(130)^2 = 8450 \text{ mm}^2$$

$$\bar{x}_2 = 300 - 130/3 = 256.7 \text{ mm}$$

$$\bar{y}_2 = 280 - 130/3 = 236.7 \text{ mm}$$

$$A_3 = \frac{\pi d^2}{4} = \frac{\pi}{4} (50)^2 = 1963 \text{ mm}^2$$

$$\bar{x}_3 = 80 \text{ mm} \quad \bar{y}_3 = 80 \text{ mm}$$

$$A_4 = 1963 \text{ mm}^2 \quad \bar{x}_4 = 220 \text{ mm} \quad \bar{y}_4 = 80 \text{ mm}$$

$$A = \sum A_i = A_1 - A_2 - A_3 - A_4 = 71,620 \text{ mm}^2$$

$$Q_y = \sum \bar{x}_i A_i = \bar{x}_1 A_1 - \bar{x}_2 A_2 - \bar{x}_3 A_3 - \bar{x}_4 A_4$$

$$= 9.842 \times 10^6 \text{ mm}^3$$

$$\bar{x} = \frac{Q_y}{A} = \frac{9.842 \times 10^6}{71,620} = 137 \text{ mm} \quad \leftarrow$$

$$Q_x = \sum \bar{y}_i A_i = \bar{y}_1 A_1 - \bar{y}_2 A_2 - \bar{y}_3 A_3 - \bar{y}_4 A_4$$

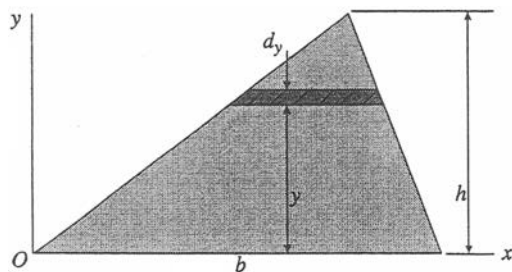
$$= 9.446 \times 10^5 \text{ mm}^3$$

$$\bar{y} = \frac{Q_x}{A} = \frac{9.446 \times 10^5}{71,620} = 132 \text{ mm} \quad \leftarrow$$

Moments of Inertia of Plane Areas

Problems 12.4-1 through 12.4-4 are to be solved by integration.

Problem 12.4-1 Determine the moment of inertia I_x of a triangle of base b and altitude h with respect to its base (see Case 4, Appendix E).

Solution 12.4-1 Moment of inertia of a triangle

Width of element

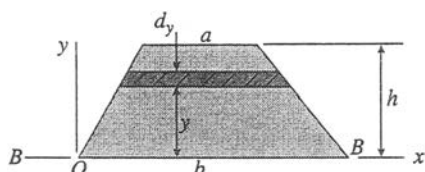
$$= b \left(\frac{h-y}{h} \right)$$

$$dA = \frac{b(h-y)}{h} dy$$

$$\begin{aligned}
 I_x &= \int y^2 dA = \int_0^h y^2 b \frac{(h-y)}{h} dy \\
 &= \frac{bh^3}{12} \quad \leftarrow
 \end{aligned}$$

Problem 12.4-2 Determine the moment of inertia I_{BB} of a trapezoid having bases a and b and altitude h with respect to its base (see Case 8, Appendix E).

Solution 12.4-2 Moment of inertia of a trapezoid

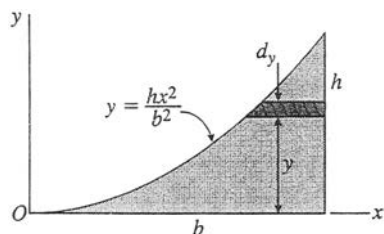


Width of element

$$\begin{aligned}
 &= a + (b - a)\left(\frac{h - y}{h}\right) \\
 dA &= \left[a + (b - a)\left(\frac{h - y}{h}\right) \right] dy \\
 I_{BB} &= \int y^2 dA = \int_0^h y^2 \left[a + (b - a)\left(\frac{h - y}{h}\right) \right] dy \\
 &= \frac{h^3(3a + b)}{12} \quad \leftarrow
 \end{aligned}$$

Problem 12.4-3 Determine the moment of inertia I_x of a parabolic spandrel of base b and height h with respect to its base (see Case 18, Appendix E).

Solution 12.4-3 Moment of inertia of a parabolic spandrel

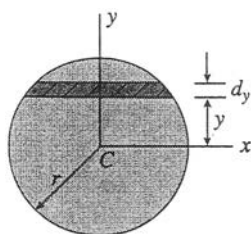


Width of element

$$\begin{aligned}
 &= b - x = b - b\sqrt{\frac{y}{h}} \\
 &= b(1 - \sqrt{y/h}) \\
 dA &= b(1 - \sqrt{y/h}) dy \\
 I_x &= \int y^2 dA = \int_0^h y^2 b(1 - \sqrt{y/h}) dy = \frac{bh^3}{21} \quad \leftarrow
 \end{aligned}$$

Problem 12.4-4 Determine the moment of inertia I_x of a circle of radius r with respect to a diameter (see Case 9, Appendix E).

Solution 12.4-4 Moment of inertia of a circle

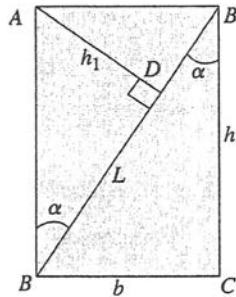


$$\begin{aligned}
 \text{Width of element} &= 2\sqrt{r^2 - y^2} \\
 dA &= 2\sqrt{r^2 - y^2} dy \\
 I_x &= \int y^2 dA = \int_{-r}^r y^2 (2\sqrt{r^2 - y^2}) dy \\
 &= \frac{\pi r^4}{4} \quad \leftarrow
 \end{aligned}$$

Problems 12.4-5 through 12.4-9 are to be solved by considering the area to be a composite area.

Problem 12.4-5 Determine the moment of inertia I_{BB} of a rectangle having sides of lengths b and h with respect to a diagonal of the rectangle (see Case 2, Appendix E).

Solution 12.4-5 Moment of inertia of a rectangle with respect to a diagonal



L = length of diagonal BB

$$L = \sqrt{b^2 + h^2}$$

h_1 = distance from A to diagonal BB

$$\text{Triangle } BBC: \sin \alpha = \frac{b}{L}$$

$$\text{Triangle } ADB: \sin \alpha = \frac{h_1}{h} \quad h_1 = h \sin \alpha = \frac{bh}{L}$$

I_1 = moment of inertia of triangle ABB with respect to its base BB

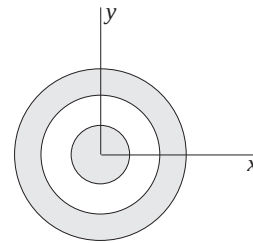
From Case 4, Appendix E:

$$I_1 = \frac{Lh_1^3}{12} = \frac{L}{12} \left(\frac{bh}{L} \right)^3 = \frac{b^3h^3}{12L^2}$$

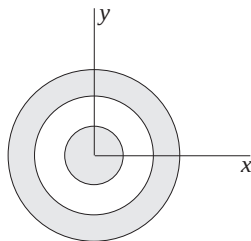
For the rectangle:

$$I_{BB} = 2I_1 = \frac{b^3h^3}{6(b^2 + h^2)} \quad \leftarrow$$

Problem 12.4-6 Calculate the moment of inertia I_x for the composite circular area shown in the figure. The origin of the axes is at the center of the concentric circles, and the three diameters are 20, 40, and 60 mm.



Solution 12.4-6 Moment of inertia of composite area



Diameters = 20, 40, and 60 mm

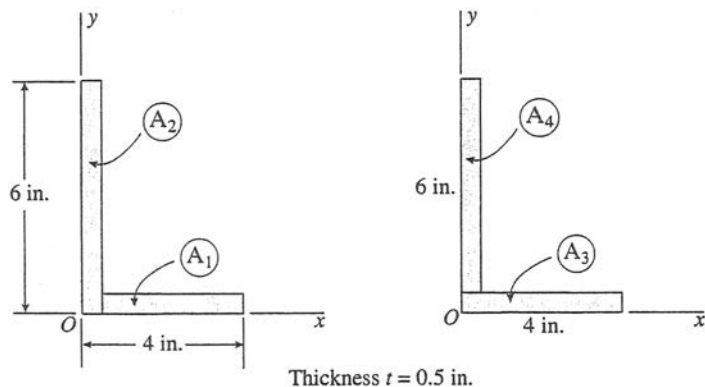
$$I_x = \frac{\pi d^4}{64} \quad (\text{for a circle})$$

$$I_x = \frac{\pi}{64} [(60)^4 - (40)^4 + (20)^4]$$

$$I_x = 518 \times 10^3 \text{ mm}^4 \quad \leftarrow$$

Problem 12.4-7 Calculate the moments of inertia I_x and I_y with respect to the x and y axes for the L-shaped area shown in the figure for Prob. 12.3-7.

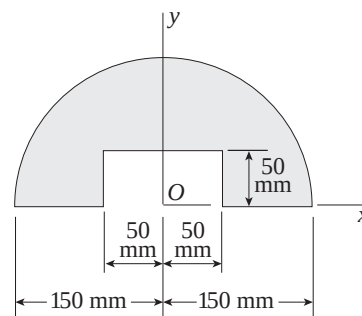
Solution 12.4-7 Moments of inertia of composite area



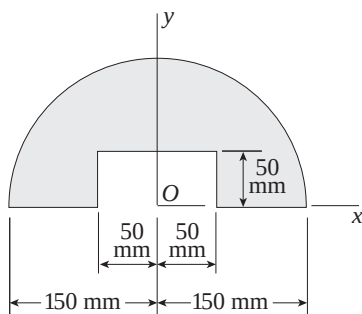
$$\begin{aligned}
 I_x &= I_1 + I_2 \\
 &= \frac{1}{3} (3.5)(0.5)^3 + \frac{1}{3} (0.5)(6)^3 \\
 &= 36.1 \text{ in.}^4 \leftarrow \\
 I_y &= I_3 + I_4 \\
 &= \frac{1}{3} (0.5)(4)^3 + \frac{1}{3} (5.5)(0.5)^3 \\
 &= 10.9 \text{ in.}^4 \leftarrow
 \end{aligned}$$

Problem 12.4-8 A semicircular area of radius 1.50 mm has a rectangular cutout of dimensions 50 mm $\times$ 100 mm (see figure).

Calculate the moments of inertia I_x and I_y with respect to the x and y axes. Also, calculate the corresponding radii of gyration r_x and r_y .



Solution 12.4-8 Moments of inertia of composite area

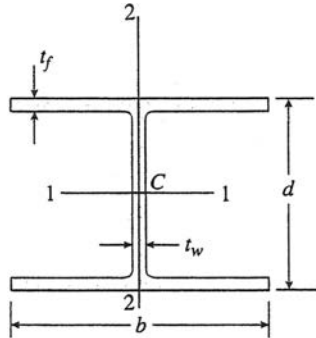


All dimensions in millimeters.

$$\begin{aligned}
 r &= 150 \text{ mm} \quad b = 100 \text{ mm} \quad h = 50 \text{ mm} \\
 I_x &= (I_x)_{\text{semicircle}} - (I_x)_{\text{rectangle}} = \frac{\pi r^4}{8} - \frac{bh^3}{3} \\
 &= 194.6 \times 10^6 \text{ mm}^4 \leftarrow \\
 I_y &= I_x \leftarrow \\
 A &= \frac{\pi r^2}{2} - bh = 30.34 \times 10^3 \text{ mm}^2 \\
 r_x &= \sqrt{I_x/A} = 80.1 \text{ mm} \leftarrow \\
 r_y &= r_x \leftarrow
 \end{aligned}$$

Problem 12.4-9 Calculate the moments of inertia I_1 and I_2 of a W 16 $\times$ 100 wide-flange section using the cross-sectional dimensions given in Table F-1, Appendix F. (Disregard the cross-sectional areas of the fillets.) Also, calculate the corresponding radii of gyration r_1 and r_2 , respectively.

Solution 12.4-9 Moments of inertia of a wide-flange section



W 16 $\times$ 100 $d = 16.97$ in.

$t_w = t_{\text{web}} = 0.585$ in.

$b = 10.425$ in.

$t_f = t_{\text{flange}} = 0.985$ in.

All dimensions in inches.

$$\begin{aligned} I_1 &= \frac{1}{12} b d^3 - \frac{1}{12} (b - t_w)(d - 2t_f)^3 \\ &= \frac{1}{12} (10.425)(16.97)^3 - \frac{1}{12} (9.840)(15.00)^3 \\ &= 1478 \text{ in.}^4 \quad \text{say, } I_1 = 1480 \text{ in.}^4 \quad \leftarrow \end{aligned}$$

$$\begin{aligned} I_2 &= 2 \left(\frac{1}{12} t_f b^3 + \frac{1}{12} (d - 2t_f) t_w^3 \right) \\ &= \frac{1}{6} (0.985)(10.425)^3 + \frac{1}{12} (15.00)(0.585)^3 \\ &= 186.3 \text{ in.}^4 \quad \text{say, } I_2 = 186 \text{ in.}^4 \quad \leftarrow \end{aligned}$$

$$\begin{aligned} A &= 2(bt_f) + (d - 2t_f)t_w \\ &= 2(10.425)(0.985) + (15.00)(0.585) \\ &= 29.31 \text{ in.}^2 \end{aligned}$$

$$r_1 = \sqrt{I_1/A} = 7.10 \text{ in.} \quad \leftarrow$$

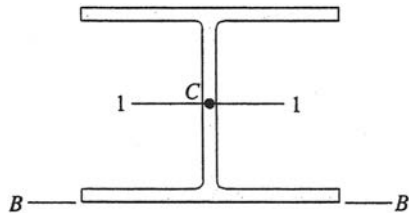
$$r_2 = \sqrt{I_2/A} = 2.52 \text{ in.} \quad \leftarrow$$

Note that these results are in close agreement with the tabulated values.

Parallel-Axis Theorem

Problem 12.5-1 Calculate the moment of inertia I_b of a W 12 $\times$ 50 wide-flange section with respect to its base. (Use data from Table F-1, Appendix F.)

Solution 12.5-1 Moment of inertia



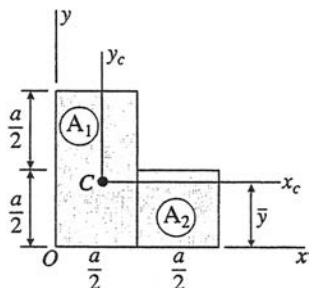
$$\text{W } 12 \times 50 \quad I_t = 394 \text{ in.}^4 \quad A = 14.7 \text{ in.}^2$$

$$d = 12.19 \text{ in.}$$

$$\begin{aligned} I_b &= I_1 + A \left(\frac{d}{2} \right)^2 \\ &= 394 + 14.7(6.095)^2 = 940 \text{ in.}^4 \quad \leftarrow \end{aligned}$$

Problem 12.5-2 Determine the moment of inertia I_c with respect to an axis through the centroid C and parallel to the x axis for the geometric figure described in Prob. 12.3-2.

Solution 12.5-2 Moment of inertia



From Prob. 12.3-2:

$$A = 3a^2/4$$

$$\bar{y} = 5a/12$$

$$I_x = \frac{1}{3} \left(\frac{a}{2} \right) (a^3) + \frac{1}{3} \left(\frac{a}{2} \right) \left(\frac{a}{2} \right)^3 = \frac{3a^4}{16}$$

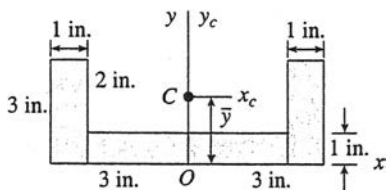
$$I_x = I_{x_c} + A\bar{y}^3$$

$$I_c = I_{x_c} = I_x - A\bar{y}^2 = \frac{3a^4}{16} - \frac{3a^2}{4} \left(\frac{5a}{12} \right)^2$$

$$= \frac{11a^4}{192} \quad \leftarrow$$

Problem 12.5-3 For the channel section described in Prob. 12.3-3, calculate the moment of inertia I_{x_c} with respect to an axis through the centroid C and parallel to the x axis.

Solution 12.5-3 Moment of inertia



From Prob. 12.3-3:

$$A = 10.0 \text{ in.}^2$$

$$\bar{y} = 1.10 \text{ in.}$$

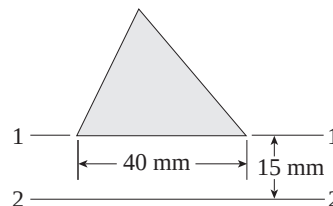
$$I_x = 1/3(4)(1)^3 + 2(1/3)(1)(3)^3 = 19.33 \text{ in.}^4$$

$$I_x = I_{x_c} + A\bar{y}^2$$

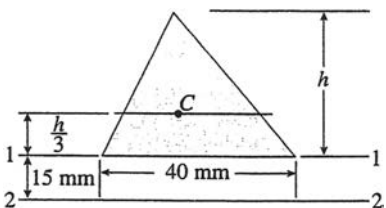
$$I_{x_c} = I_x - A\bar{y}^2 = 19.33 - (10.0)(1.10)^2$$

$$= 7.23 \text{ in.}^4 \quad \leftarrow$$

Problem 12.5-4 The moment of inertia with respect to axis 1-1 of the scalene triangle shown in the figure is $90 \times 10^3 \text{ mm}^4$. Calculate its moment of inertia I_2 with respect to axis 2-2.



Solution 12.5-4 Moment of inertia



$$b = 40 \text{ mm} \quad I_1 = 90 \times 10^3 \text{ mm}^4 \quad I_1 = bh^3/12$$

$$h = \sqrt[3]{\frac{12I_1}{b}} = 30 \text{ mm}$$

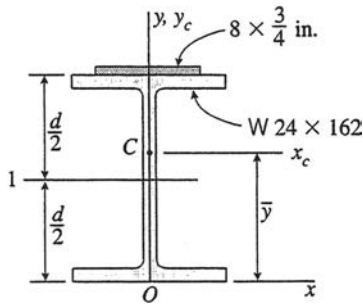
$$I_c = bh^3/36 = 30 \times 10^3 \text{ mm}^4$$

$$I_2 = I_c + Ad^2 = I_c + (bh/2)d^2 = 30 \times 10^3$$

$$+ \frac{1}{2} (40) (30) (25)^2 = 405 \times 10^3 \text{ mm}^4 \quad \leftarrow$$

Problem 12.5-5 For the beam cross section described in Prob. 12.3-5, calculate the centroidal moments of inertia I_{x_c} and I_{y_c} with respect to axes through the centroid C such that the x_c axis is parallel to the x axis and the y_c axis coincides with the y axis.

Solution 12.5-5 Moment of inertia



From Prob. 12.3-5:

$$\bar{y} = 13.94 \text{ in.}$$

$$W 24 \times 162 \quad d = 25.00 \text{ in.} \quad d/2 = 12.5 \text{ in.}$$

$$I_1 = 5170 \text{ in.}^4 \quad A = 47.7 \text{ in.}^2$$

$$I_2 = I_y = 443 \text{ in.}^4$$

$$I'_{x_c} = I_1 + A(\bar{y} - d/2)^2 = 5170 + (47.7)(1.44)^2 = 5269 \text{ in.}^4$$

$$I'_{y_c} = I_2 = 443 \text{ in.}^4$$

PLATE

$$\begin{aligned} I''_{x_c} &= 1/12(8)(3/4)^3 + (8)(3/4)(d + 3/8 - \bar{y})^2 \\ &= 0.2813 + 6(25.00 + 0.375 - 13.94)^2 \\ &= 0.2813 + 6(11.44)^2 = 785 \text{ in.}^4 \end{aligned}$$

$$I''_{y_c} = 1/12(3/4)(8)^3 = 32.0 \text{ in.}^4$$

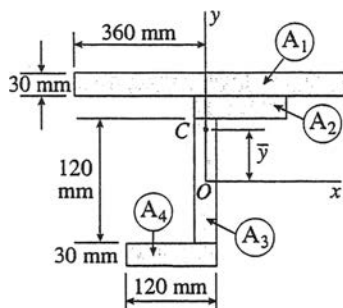
ENTIRE CROSS SECTION

$$I_{x_c} = I'_{x_c} + I''_{x_c} = 5269 + 785 = 6050 \text{ in.}^4 \quad \leftarrow$$

$$I_{y_c} = I'_{y_c} + I''_{y_c} = 443 + 32 = 475 \text{ in.}^4 \quad \leftarrow$$

Problem 12.5-6 Calculate the moment of inertia I_{x_c} with respect to an axis through the centroid C and parallel to the x axis for the composite area shown in the figure for Prob. 12.3-6.

Solution 12.5-6 Moment of inertia



From Prob. 12.3-6:

$$\bar{y} = 52.50 \text{ mm} \quad t = 30 \text{ mm} \quad A = 21.600 \text{ mm}^2$$

$$A_1: I_{x_c} = 1/12(360)(30)^3 + (360)(30)(105)^2 = 119.9 \times 10^6 \text{ mm}^4$$

$$A_2: I_{x_c} = 1/12(120)(30)^3 + (120)(30)(75)^2 = 20.52 \times 10^6 \text{ mm}^4$$

$$A_3: I_{x_c} = 1/12(30)(120)^3 = 4.32 \times 10^6 \text{ mm}^4$$

$$A_4: I_{x_c} = 20.52 \times 10^6 \text{ mm}^4$$

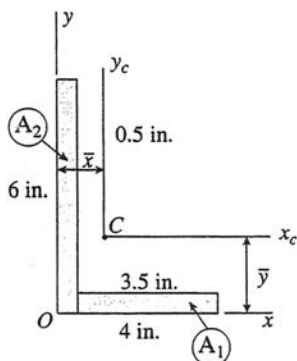
ENTIRE AREA:

$$I_x = \sum I_{x_c} = 165.26 \times 10^6 \text{ mm}^4$$

$$\begin{aligned} I_{x_c} &= I_x - A\bar{y}^2 = 165.26 \times 10^6 - (21,600)(52.50)^2 \\ &= 106 \times 10^6 \text{ mm}^4 \quad \leftarrow \end{aligned}$$

Problem 12.5-7 Calculate the centroidal moments of inertia I_{x_c} and I_{y_c} with respect to axes through the centroid C and parallel to the x and y axes, respectively, for the L-shaped area shown in the figure for Prob. 12.3-7.

Solution 12.5-7 Moments of inertia



From Prob. 12.3-7:

$$t = 0.5 \text{ in.} \quad A = 4.75 \text{ in.}^2$$

$$\bar{y} = 1.987 \text{ in.}$$

$$\bar{x} = 0.9869 \text{ in.}$$

From Prob. 12.4-7:

$$I_x = 36.15 \text{ in.}^4$$

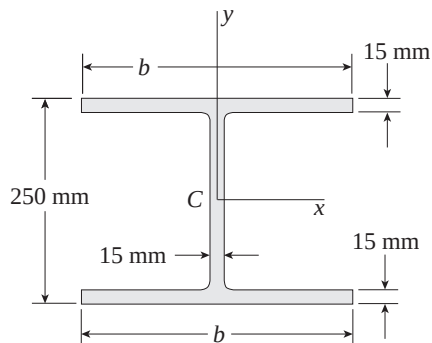
$$I_y = 10.90 \text{ in.}^4$$

$$I_{x_c} = I_x - A\bar{y}^2 = 36.15 - (4.75)(1.987)^2 \\ = 17.40 \text{ in.}^4 \quad \leftarrow$$

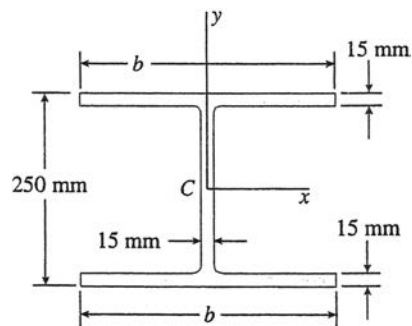
$$I_{y_c} = I_y - A\bar{x}^2 = 10.90 - (4.75)(0.9869)^2 \\ = 6.27 \text{ in.}^4 \quad \leftarrow$$

Problem 12.5-8 The wide-flange beam section shown in the figure has a total height of 250 mm and a constant thickness of 15 mm.

Determine the flange width b if it is required that the centroidal moments of inertia I_x and I_y be in the ratio 3 to 1, respectively.



Solution 12.5-8 Wide-flange beam



$$t = 15 \text{ mm} \quad b = \text{flange width}$$

All dimensions in millimeters.

$$I_x = \frac{1}{12}(b)(250)^3 - \frac{1}{12}(b - 15)(220)^3 \\ = 0.4147 \times 10^6 b + 13.31 \times 10^6 (\text{mm}^4)$$

$$I_y = 2\left(\frac{1}{12}\right)(15)(b)^3 + \frac{1}{12}(220)(15)^3 \\ = 2.5b^3 + 61,880 (\text{mm}^4)$$

Equate I_x to $3I_y$ and rearrange:

$$7.5b^3 - 0.4147 \times 10^6 b - 13.12 \times 10^6 = 0$$

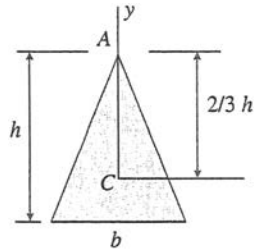
Solve numerically:

$$b = 250 \text{ mm} \quad \leftarrow$$

Polar Moments of Inertia

Problem 12.6-1 Determine the polar moment of inertia I_P of an isosceles triangle of base b and altitude h with respect to its apex (see Case 5, Appendix E)

Solution 12.6-1 Polar moment of inertia



POINT A (APEX):

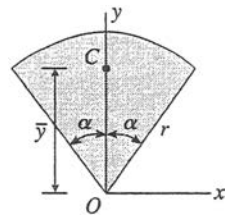
$$\begin{aligned} I_P &= (I_P)_C + A \left(\frac{2h}{3} \right)^2 \\ &= \frac{bh}{144} (4h^2 + 3b^2) + \frac{bh}{2} \left(\frac{2h}{3} \right)^2 \\ I_P &= \frac{bh}{48} (b^2 + 12h^2) \quad \leftarrow \end{aligned}$$

POINT C (CENTROID) FROM CASE 5:

$$(I_P)_C = \frac{bh}{144} (4h^2 + 3b^2)$$

Problem 12.6-2 Determine the polar moment of inertia $(I_P)_C$ with respect to the centroid C for a circular sector (see Case 13, Appendix E).

Solution 12.6-2 Polar moment of inertia



POINT O (ORIGIN) FROM CASE 13:

$$(I_P)_O = \frac{\alpha r^4}{2} \quad (\alpha = \text{radians})$$

$$A = \alpha r^2$$

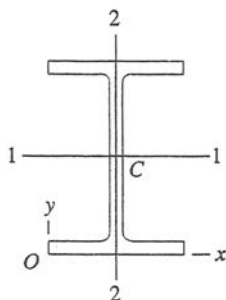
$$\bar{y} = \frac{2r \sin \alpha}{3\alpha}$$

POINT C (CENTROID):

$$\begin{aligned} (I_P)_C &= (I_P)_O - A \bar{y}^2 = \frac{\alpha r^4}{2} - \alpha r^2 \left(\frac{2r \sin \alpha}{3\alpha} \right)^2 \\ &= \frac{r^4}{18\alpha} (9\alpha^2 - 8 \sin^2 \alpha) \quad \leftarrow \end{aligned}$$

Problem 12.6-3 Determine the polar moment of inertia I_P for a W 8 × 21 wide-flange section with respect to one of its outermost corners.

Solution 12.6-3 Polar moment of inertia



$$\text{W } 8 \times 21 \quad I_1 = 75.3 \text{ in.}^4 \quad I_2 = 9.77 \text{ in.}^4$$

$$A = 6.16 \text{ in.}^2$$

$$\text{Depth } d = 8.28 \text{ in.}$$

$$\text{Width } b = 5.27 \text{ in.}$$

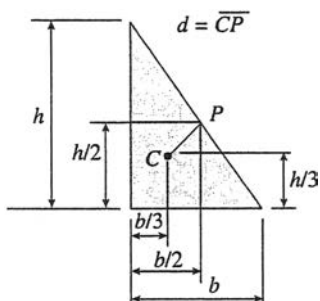
$$I_x = I_1 + A(d/2)^2 = 75.3 + 6.16(4.14)^2 = 180.9 \text{ in.}^4$$

$$I_y = I_2 + A(b/2)^2 = 9.77 + 6.16(2.635)^2 = 52.5 \text{ in.}^4$$

$$I_P = I_x + I_y = 233 \text{ in.}^4 \quad \leftarrow$$

Problem 12.6-4 Obtain a formula for the polar moment of inertia I_P with respect to the midpoint of the hypotenuse for a right triangle of base b and height h (see Case 6, Appendix E).

Solution 12.6-4 Polar moment of inertia



POINT C FROM CASE 6:

$$(I_P)_C = \frac{bh}{36} (h^2 + b^2)$$

POINT P:

$$I_P = (I_P)_C + Ad^2$$

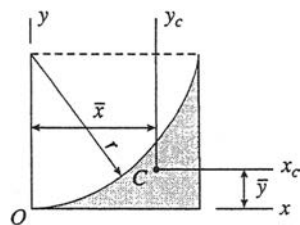
$$A = \frac{bh}{2}$$

$$\begin{aligned} d^2 &= \left(\frac{b}{2} - \frac{b}{3} \right)^2 + \left(\frac{h}{2} - \frac{h}{3} \right)^2 \\ &= \frac{b^2}{36} + \frac{h^2}{36} = \frac{b^2 + h^2}{36} \end{aligned}$$

$$\begin{aligned} I_P &= \frac{bh}{36} (h^2 + b^2) + \frac{bh}{2} \left(\frac{b^2 + h^2}{36} \right) \\ &= \frac{bh}{24} (b^2 + h^2) \quad \leftarrow \end{aligned}$$

Problem 12.6-5 Determine the polar moment of inertia $(I_P)_C$ with respect to the centroid C for a quarter-circular spandrel (see Case 12, Appendix E).

Solution 12.6-5 Polar moment of inertia



POINT O FROM CASE 12:

$$I_x = \left(1 - \frac{5\pi}{16} \right) r^4$$

$$\bar{y} = \frac{(10 - 3\pi)r}{3(4 - \pi)}$$

$$A = \left(1 - \frac{\pi}{4} \right) r^2$$

POINT C (CENTROID):

$$\begin{aligned} I_{x_c} &= I_x - A\bar{y}^2 = \left(1 - \frac{5\pi}{16} \right) r^4 \\ &\quad - \left(1 - \frac{\pi}{4} \right) (r^2) \left[\frac{(10 - 3\pi)r}{3(4 - \pi)} \right]^2 \end{aligned}$$

COLLECT TERMS AND SIMPLIFY:

$$I_{x_c} = \frac{r^4}{144} \left(\frac{176 - 84\pi + 9\pi^2}{4 - \pi} \right)$$

$$I_{y_c} = I_{x_c} \quad (\text{by symmetry})$$

$$(I_P)_C = 2I_{x_c} = \frac{r^4}{72} \left(\frac{176 - 84\pi + 9\pi^2}{4 - \pi} \right) \quad \leftarrow$$

Products of Inertia

Problem 12.7-1 Using integration, determine the product of inertia I_{xy} for the parabolic semisegment shown in Fig. 12-5 (see also Case 17 in Appendix E).

Solution 12.7-1 Product of Inertia

Product of inertia of element dA with respect to axes through its own centroid equals zero.

$$dA = y dx = h \left(1 - \frac{x^2}{b^2} \right) dx$$

dI_{xy} = product of inertia of element dA with respect to xy axes

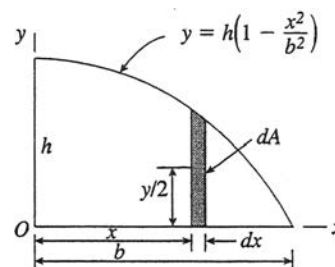
$$d_1 = x \quad d_2 = y/2$$

Parallel-axis theorem applied to element dA :

$$dI_{xy} = 0 + (dA)(d_1 d_2) = (y dx)(x)(y/2)$$

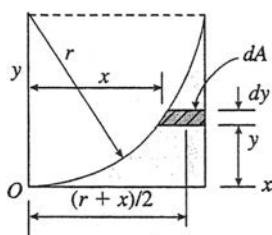
$$= \frac{h^2 x}{2} \left(1 - \frac{x^2}{b^2} \right)^2 dx$$

$$I_{xy} = \int dI_{xy} = \frac{h^2}{2} \int_0^b x \left(1 - \frac{x^2}{b^2} \right)^2 dx = \frac{b^2 h^2}{12} \quad \leftarrow$$



Problem 12.7-2 Using integration, determine the product of inertia I_{xy} for the quarter-circular spandrel shown in Case 12, Appendix E.

Solution 12.7-2 Product of inertia



EQUATION OF CIRCLE:

$$x^2 + (y - r)^2 = r^2$$

$$\text{or } r^2 - x^2 = (y - r)^2$$

ELEMENT dA :

d_1 = distance to its centroid in x direction

$$= (r + x)/2$$

d_2 = distance to its centroid in y direction = y

dA = area of element = $(r - x) dy$

Product of inertia of element dA with respect to axes through its own centroid equals zero.

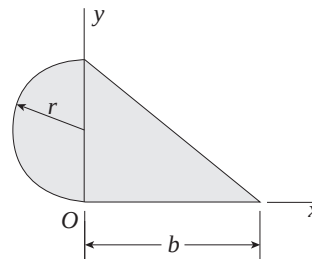
Parallel-axis theorem applied to element dA :

$$dI_{xy} = 0 + (dA)(d_1 d_2) = (r - x)(dy) \left(\frac{r + x}{2} \right) (y)$$

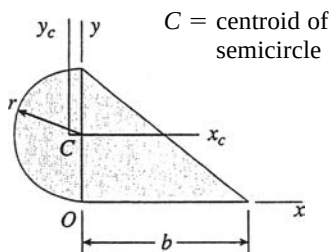
$$= \frac{1}{2} (r^2 - x^2) y dy = \frac{1}{2} (y - r)^2 y dy$$

$$I_{xy} = 1/2 \int_0^r y (y - r)^2 dy = \frac{r^4}{24} \quad \leftarrow$$

Problem 12.7-3 Find the relationship between the radius r and the distance b for the composite area shown in the figure in order that the product of inertia I_{xy} will be zero.



Solution 12.7-3 Product of Inertia



TRIANGLE (CASE 7):

$$I_{xy} = \frac{b^2 h^2}{24} = \frac{b^2 (2r)^2}{24} = \frac{b^2 r^2}{6}$$

SEMICIRCLE (CASE 10):

$$I_{xy} = I_{x_c y_c} + A d_1 d_2$$

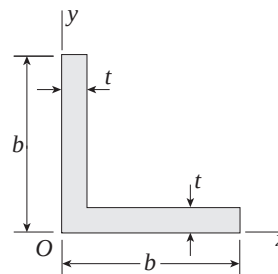
$$I_{x_c y_c} = 0 \quad A = \frac{\pi r^2}{2} \quad d_1 = r \quad d_2 = -\frac{4r}{3\pi}$$

$$I_{xy} = 0 + \left(\frac{\pi r^2}{2}\right)(r)\left(-\frac{4r}{3\pi}\right) = -\frac{2r^4}{3}$$

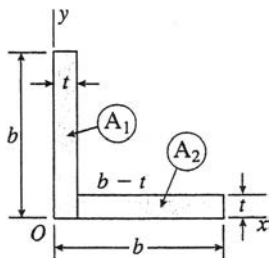
COMPOSITE AREA ($I_{xy} = 0$)

$$I_{xy} = \frac{b^2 r^2}{6} - \frac{2r^4}{3} = 0 \quad \therefore b = 2r \quad \leftarrow$$

Problem 12.7-4 Obtain a formula for the product of inertia I_{xy} of the symmetrical L-shaped area shown in the figure.



Solution 12.7-4 Product of inertia



AREA 1:

$$(I_{xy})_1 = \frac{t^2 b^2}{4}$$

AREA 2:

$$(I_{xy})_2 = I_{x_c y_c} + A_2 d_1 d_2$$

$$= 0 + (b-t)(t)(t/2) \left(\frac{b+t}{2}\right)$$

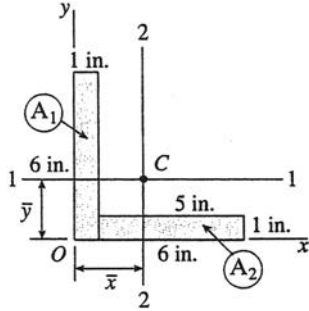
$$= \frac{t^2}{4} (b^2 - t^2)$$

COMPOSITE AREA:

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 = \frac{t^2}{4} (2b^2 - t^2) \quad \leftarrow$$

Problem 12.7-5 Calculate the product of inertia I_{12} with respect to the centroidal axes 1–1 and 2–2 for an L $6 \times 6 \times 1$ in. angle section (see Table F-4, Appendix F). (Disregard the cross-sectional areas of the fillet and rounded corners.)

Solution 12.7-5 Product of inertia



All dimensions in inches.

$$A_1 = (6)(1) = 6.0 \text{ in.}^2$$

$$A_2 = (5)(1) = 5.0 \text{ in.}^2$$

$$A = A_1 + A_2 = 11.0 \text{ in.}^2$$

With respect to the x axis:

$$Q_1 = (6.0 \text{ in.}^2) \left(\frac{6 \text{ in.}}{2} \right) = 18.0 \text{ in.}^3$$

$$Q_2 = (5.0 \text{ in.}^2) \left(\frac{1.0 \text{ in.}}{2} \right) = 2.5 \text{ in.}^3$$

$$\bar{y} = \frac{Q_1 + Q_2}{A} = \frac{20.5 \text{ in.}}{11.0 \text{ in.}^2} = 1.8636 \text{ in.}$$

$$\bar{x} = \bar{y} = 1.8636 \text{ in.}$$

Coordinates of centroid of area A_1 with respect to 1–2 axes:

$$d_1 = -(\bar{x} - 0.5) = -1.3636 \text{ in.}$$

$$d_2 = 3.0 - \bar{y} = 1.1364 \text{ in.}$$

Product of inertia of area A_1 with respect to 1–2 axes:

$$\begin{aligned} I'_{12} &= 0 + A_1 d_1 d_2 \\ &= (6.0 \text{ in.}^2)(-1.3636 \text{ in.})(1.1364 \text{ in.}) = -9.2976 \text{ in.}^4 \end{aligned}$$

Coordinates of centroid of area A_2 with respect to 1–2 axes:

$$d_1 = 3.5 - \bar{x} = 1.6364 \text{ in.}$$

$$d_2 = -(\bar{y} - 0.5) = -1.3636 \text{ in.}$$

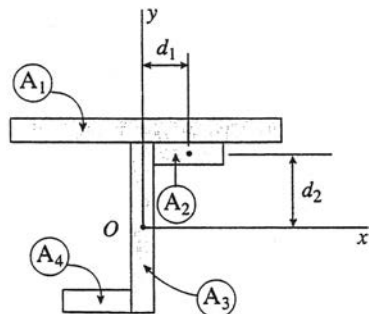
Product of inertia of area A_2 with respect to 1–2 axes:

$$\begin{aligned} I'_{12} &= 0 + A_2 d_1 d_2 \\ &= (5.0 \text{ in.}^2)(1.6364 \text{ in.})(-1.3636 \text{ in.}) \\ &= -11.1573 \text{ in.}^4 \end{aligned}$$

$$\text{ANGLE SECTION: } I_{12} = I'_{12} + I'_{12} = -20.5 \text{ in.}^4 \quad \leftarrow$$

Problem 12.7-6 Calculate the product of inertia I_{xy} for the composite area shown in Prob. 12.3-6.

Solution 12.7-6 Product of inertia



All dimensions in millimeters

$$A_1 = 360 \times 30 \text{ mm} \quad A_2 = 90 \times 30 \text{ mm}$$

$$A_3 = 180 \times 30 \text{ mm} \quad A_4 = 90 \times 30 \text{ mm}$$

$$d_1 = 60 \text{ mm} \quad d_2 = 75 \text{ mm}$$

$$\text{AREA } A_1: (I_{xy})_1 = 0 \quad (\text{by symmetry})$$

$$\begin{aligned} \text{AREA } A_2: (I_{xy})_2 &= 0 + A_2 d_1 d_2 = (90 \times 30)(60)(75) \\ &= 12.15 \times 10^6 \text{ mm}^4 \end{aligned}$$

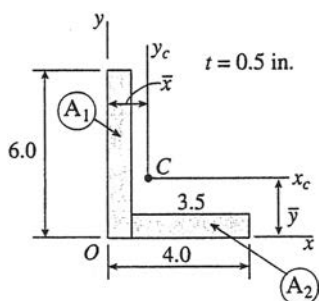
$$\text{AREA } A_3: (I_{xy})_3 = 0 \quad (\text{By symmetry})$$

$$\text{AREA } A_4: (I_{xy})_4 = (I_{xy})_2 = 12.15 \times 10^6 \text{ mm}^4$$

$$\begin{aligned} I_{xy} &= (I_{xy})_1 + (I_{xy})_2 + (I_{xy})_3 + (I_{xy})_4 \\ &= (2)(12.15 \times 10^6 \text{ mm}^4) \\ &= 24.3 \times 10^6 \text{ mm}^4 \quad \leftarrow \end{aligned}$$

Problem 12.7-7 Determine the product of inertia $I_{x_c y_c}$ with respect to centroidal axes x_c and y_c parallel to the x and y axes, respectively, for the L-shaped area shown in Prob. 12.3-7.

Solution 12.7-7 Product of inertia



All dimensions in inches.

$$A_1 = (6.0)(0.5) = 3.0 \text{ in.}^2$$

$$A_2 = (3.5)(0.5) = 1.75 \text{ in.}^2$$

$$A = A_1 + A_2 = 4.75 \text{ in.}^2$$

With respect to the x axis:

$$Q_1 = A_1 \bar{y}_1 = (3.0 \text{ in.}^2)(3.0 \text{ in.}) = 9.0 \text{ in.}^3$$

$$Q_2 = A_2 \bar{y}_2 = (1.75 \text{ in.}^2)(0.25 \text{ in.}) = 0.4375 \text{ in.}^3$$

$$\bar{y} = \frac{Q_1 + Q_2}{A} = \frac{9.4375 \text{ in.}^3}{4.75 \text{ in.}^2} = 1.9868 \text{ in.}$$

With respect to the y axis:

$$Q_1 = A_1 \bar{x}_1 = (3.0 \text{ in.}^2)(0.25 \text{ in.}) = 0.75 \text{ in.}^3$$

$$Q_2 = A_2 \bar{x}_2 = (1.75 \text{ in.}^2)(2.25 \text{ in.}) = 3.9375 \text{ in.}^3$$

$$\bar{x} = \frac{Q_1 + Q_2}{A} = \frac{4.6875 \text{ in.}^3}{4.75 \text{ in.}^2} = 0.98684 \text{ in.}$$

Product of inertia of area A_1 with respect to xy axes:

$$\begin{aligned} (I_{xy})_1 &= (I_{xy})_{\text{centroid}} + A_1 d_1 d_2 \\ &= 0 + (3.0 \text{ in.}^2)(0.25 \text{ in.})(3.0 \text{ in.}) = 2.25 \text{ in.}^4 \end{aligned}$$

Product of inertia of area A_2 with respect to xy axes:

$$\begin{aligned} (I_{xy})_2 &= (I_{xy})_{\text{centroid}} + A_2 d_1 d_2 \\ &= 0 + (1.75 \text{ in.}^2)(2.25 \text{ in.})(0.25 \text{ in.}) = 0.98438 \text{ in.}^4 \end{aligned}$$

ANGLE SECTION

$$I_{xy} = (I_{xy})_1 + (I_{xy})_2 = 3.2344 \text{ in.}^4$$

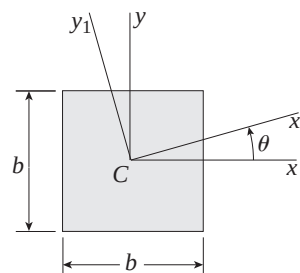
CENTROIDAL AXES

$$\begin{aligned} I_{x_c y_c} &= I_{xy} - A \bar{x} \bar{y} \\ &= 3.2344 \text{ in.}^4 - (4.75 \text{ in.}^2)(0.98684 \text{ in.})(1.9868 \text{ in.}) \\ &= -6.079 \text{ in.}^4 \quad \leftarrow \end{aligned}$$

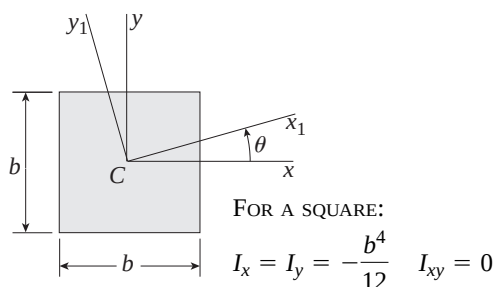
Rotation of Axes

The problems for Section 12.8 are to be solved by using the transformation equations for moments and products of inertia.

Problem 12.8-1 Determine the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1 y_1}$ for a square with sides b , as shown in the figure. (Note that the $x_1 y_1$ axes are centroidal axes rotated through an angle θ with respect to the xy axes.)



Solution 12.8-1 Rotation of axes



EQ. (12-33):

$$\begin{aligned} I_{x_1} &= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \\ &= \frac{I_x + I_y}{2} + 0 - 0 = \frac{b^4}{12} \quad \leftarrow \end{aligned}$$

EQ. (12-37):

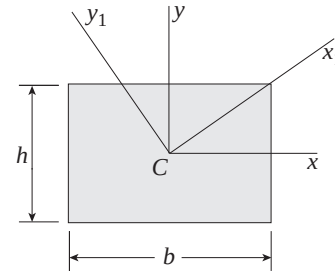
$$I_{x_1} + I_{y_1} = I_x + I_y \quad \therefore I_{y_1} = \frac{b^4}{12} \quad \leftarrow$$

EQ. (12-35):

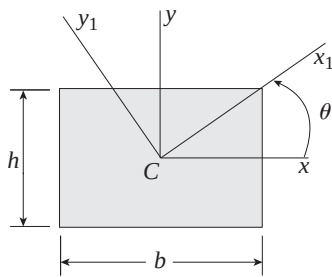
$$I_{x_1 y_1} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta = 0 \quad \leftarrow$$

Since θ may be any angle, we see that all moments of inertia are the same and the product of inertia is always zero (for axes through the centroid C).

Problem 12.8-2 Determine the moments and product of inertia with respect to the x_1y_1 axes for the rectangle shown in the figure. (Note that the x_1 axis is a diagonal of the rectangle.)



Solution 12.8-2 Rotation of axes (rectangle)



APPENDIX D, CASE 1:

$$I_x = \frac{bh^3}{12} \quad I_y = \frac{hb^3}{12} \quad I_{xy} = 0$$

ANGLE OF ROTATION:

$$\cos \theta = \frac{b}{\sqrt{b^2 + h^2}} \quad \sin \theta = \frac{h}{\sqrt{b^2 + h^2}}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \frac{b^2 - h^2}{b^2 + h^2}$$

$$\sin 2\theta = 2 \sin \theta \cos \theta = \frac{2bh}{b^2 + h^2}$$

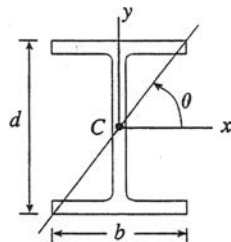
SUBSTITUTE INTO EQS. (12-33), (12-37), AND (12-35) AND SIMPLIFY:

$$I_{x1} = \frac{b^3h^3}{6(b^2 + h^2)} \quad \leftarrow \quad I_{y1} = \frac{bh(b^4 + h^4)}{12(b^2 + h^2)} \quad \leftarrow$$

$$I_{x1y1} = \frac{b^2h^2(h^2 - b^2)}{12(b^2 + h^2)} \quad \leftarrow$$

Problem 12.8-3 Calculate the moment of inertia I_d for a W 12 $\times$ 50 wide-flange section with respect to a diagonal passing through the centroid and two outside corners of the flanges. (Use the dimensions and properties given in Table F-1.)

Solution 12.8-3 Rotation of axes



W 12 $\times$ 50 $I_x = 394 \text{ in.}^4$

$I_y = 56.3 \text{ in.}^4 \quad I_{xy} = 0$

Depth $d = 12.19 \text{ in.}$

Width $b = 8.080 \text{ in.}$

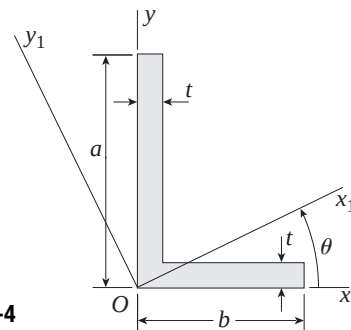
$$\tan \theta = \frac{d}{b} = \frac{12.19}{8.080} = 1.509$$

$$\theta = 56.46^\circ \quad 2\theta = 112.92^\circ$$

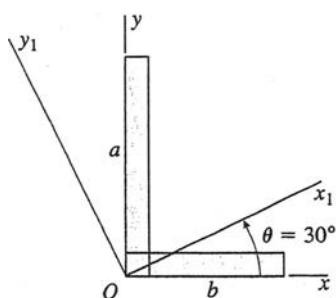
EQ. (12-33):

$$\begin{aligned} I_d &= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \\ &= \frac{394 + 56.3}{2} + \frac{394 - 56.3}{2} \cos (112.92^\circ) - 0 \\ &= 225 \text{ in.}^4 - 66 \text{ in.}^4 = 159 \text{ in.}^4 \quad \leftarrow \end{aligned}$$

Problem 12.8-4 Calculate the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1y_1}$ with respect to the x_1y_1 axes for the L-shaped area shown in the figure if $a = 150$ mm, $b = 100$ mm, $t = 15$ mm, and $\theta = 30^\circ$.



Probs. 12.8-4 and 12.9-4

Solution 12.8-4 Rotation of axes

All dimensions in millimeters.

$$a = 150 \text{ mm} \quad b = 100 \text{ mm} \\ t = 15 \text{ mm} \quad \theta = 30^\circ$$

$$I_x = \frac{1}{3}ta^3 + \frac{1}{3}(b-t)t^3 \\ = \frac{1}{3}(15)(150)^3 + \frac{1}{3}(85)(15)^3 \\ = 16.971 \times 10^6 \text{ mm}^4$$

$$I_y = \frac{1}{3}(a-t)t^3 + \frac{1}{3}tb^3 \\ = \frac{1}{3}(135)(15)^3 + \frac{1}{3}(15)(100)^3 \\ = 5.152 \times 10^6 \text{ mm}^4$$

$$I_{xy} = \frac{1}{4}t^2a^2 + Ad_1d_2 \quad A = (b-t)t \\ d_1 = t + \frac{b-t}{2} \quad d_2 = \frac{t}{2}$$

$$I_{xy} = \frac{1}{4}(15)^2(150)^2 + (85)(15)(57.5)(7.5) \\ = 1.815 \times 10^6 \text{ mm}^4$$

SUBSTITUTE into Eq. (12-33) with $\theta = 30^\circ$:

$$I_{xy} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \\ = 12.44 \times 10^6 \text{ mm}^4 \quad \leftarrow$$

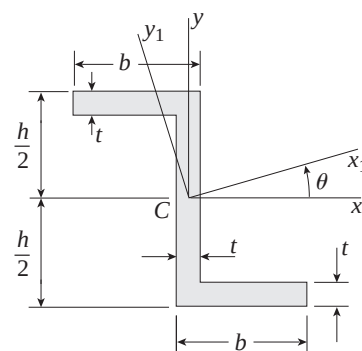
SUBSTITUTE into Eq. (12-33) with $\theta = 120^\circ$:

$$I_{y_1} = 9.68 \times 10^6 \text{ mm}^4 \quad \leftarrow$$

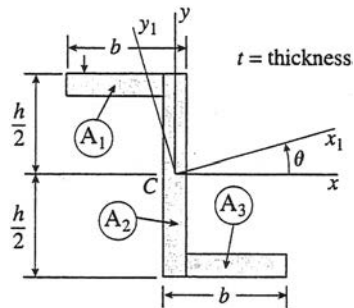
SUBSTITUTE into Eq. (12-35) with $\theta = 30^\circ$:

$$I_{x_1y_1} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta \\ = 6.03 \times 10^6 \text{ mm}^4 \quad \leftarrow$$

Problem 12.8-5 Calculate the moments of inertia I_{x_1} and I_{y_1} and the product of inertia $I_{x_1y_1}$ with respect to the x_1y_1 axes for the Z-section shown in the figure if $b = 3$ in., $h = 4$ in., $t = 0.5$ in., and $\theta = 60^\circ$.



Probs. 12.8-5, 12.8-6, 12.9-5, and 12.9-6

Solution 12.8-5 Rotation of axes

All dimensions in inches.

$$b = 3.0 \text{ in.} \quad h = 4.0 \text{ in.} \quad t = 0.5 \text{ in.} \quad \theta = 60^\circ$$

MOMENT OF INERTIA I_x

$$\begin{aligned} \text{Area } A_1: I'_x &= \frac{1}{12} (b - t)(t^3) + (b - t)(t) \left(\frac{h}{2} - \frac{t}{2} \right)^2 \\ &= 3.8542 \text{ in.}^4 \end{aligned}$$

$$\text{Area } A_2: I''_x = \frac{1}{12} (t)(h^3) = 2.6667 \text{ in.}^4$$

$$\text{Area } A_3: I'''_x = I'_x = 3.8542 \text{ in.}^4$$

$$I_x = I'_x + I''_x + I'''_x = 10.3751 \text{ in.}^4$$

MOMENT OF INERTIA I_y

$$\begin{aligned} \text{Area } A_1: I'_y &= \frac{1}{12} (t)(b - t)^3 + (b - t)(t) \left(\frac{b}{2} - \frac{t}{2} \right)^2 \\ &= 3.4635 \text{ in.}^4 \end{aligned}$$

$$\text{Area } A_2: I''_y = \frac{1}{12} (h)(t^3) = 0.0417 \text{ in.}^4$$

$$\text{Area } A_3: I'''_y = I'_y = 3.4635 \text{ in.}^4$$

$$I_y = I'_y + I''_y + I'''_y = 6.9688 \text{ in.}^4$$

PRODUCT OF INERTIA I_{xy}

$$\begin{aligned} \text{Area } A_1: I'_{xy} &= 0 + (b - t)(t) \left(-\frac{b}{2} \right) \left(\frac{h}{2} - \frac{t}{2} \right) \\ &= -\frac{1}{4} (bt)(b - t)(h - t) = -3.2813 \text{ in.}^4 \end{aligned}$$

$$\text{Area } A_2: I''_{xy} = 0 \quad \text{Area } A_3: I'''_{xy} = I'_{xy}$$

$$I_{xy} = I'_{xy} + I''_{xy} + I'''_{xy} = -6.5625 \text{ in.}^4$$

SUBSTITUTE into Eq. (12-33) with $\theta = 60^\circ$:

$$\begin{aligned} I_{x_1} &= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \\ &= 13.50 \text{ in.}^4 \quad \leftarrow \end{aligned}$$

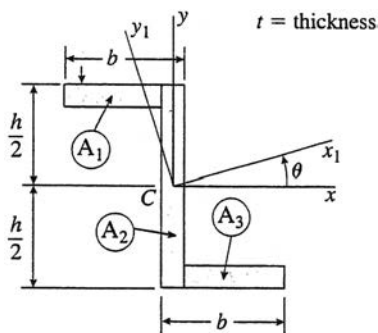
SUBSTITUTE into Eq. (12-33) with $\theta = 150^\circ$:

$$I_{y_1} = 3.84 \text{ in.}^4 \quad \leftarrow$$

SUBSTITUTE into Eq. (12-35) with $\theta = 60^\circ$:

$$I_{x_1 y_1} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta = 4.76 \text{ in.}^4 \quad \leftarrow$$

Problem 12.8-6 Solve the preceding problem if $b = 80 \text{ mm}$, $h = 120 \text{ mm}$, $t = 12 \text{ mm}$, and $\theta = 30^\circ$.

Solution 12.8-6 Rotation of axes

All dimensions in millimeters.

$$b = 80 \text{ mm} \quad h = 120 \text{ mm}$$

$$t = 12 \text{ mm} \quad \theta = 30^\circ$$

MOMENT OF INERTIA I_x

$$\begin{aligned} \text{Area } A_1: I'_x &= \frac{1}{12} (b - t)(t^3) + (b - t)(t) \left(\frac{h}{2} - \frac{t}{2} \right)^2 \\ &= 2.3892 \times 10^6 \text{ mm}^4 \end{aligned}$$

$$\text{Area } A_2: I''_x = \frac{1}{12} (t)(h^3) = 1.7280 \times 10^6 \text{ mm}^4$$

$$\text{Area } A_3: I'''_x = I'_x = 2.3892 \times 10^6 \text{ mm}^4$$

$$I_x = I'_x + I''_x + I'''_x = 6.5065 \times 10^6 \text{ mm}^4$$

(Continued)

MOMENT OF INERTIA I_y

$$\begin{aligned}\text{Area } A_1: I'_y &= \frac{1}{12}(t)(b-t)^3 + (b-t)(t)\left(\frac{b}{2}\right)^2 \\ &= 1.6200 \times 10^6 \text{ mm}^4\end{aligned}$$

$$\text{Area } A_2: I''_y = \frac{1}{12}(h)(t^3) = 0.01728 \times 10^6 \text{ mm}^4$$

$$\text{Area } A_3: I'''_y = I'_y = 1.6200 \times 10^6 \text{ mm}^4$$

$$I_y = I'_y + I''_y + I'''_y = 3.2573 \times 10^6 \text{ mm}^4$$

PRODUCT OF INERTIA I_{xy}

$$\begin{aligned}\text{Area } A_1: I'_{xy} &= 0 + (b-t)(t)\left(-\frac{b}{2}\right)\left(\frac{h}{2} - \frac{t}{2}\right) \\ &= -\frac{1}{4}(bt)(b-t)(h-t) = -1.7626 \times 10^6 \text{ mm}^4\end{aligned}$$

$$\text{Area } A_2: I''_{xy} = 0 \quad \text{Area } A_3: I'''_{xy} = I'_{xy}$$

$$I_{xy} = I'_{xy} + I''_{xy} + I'''_{xy} = -3.5251 \times 10^6 \text{ mm}^4$$

SUBSTITUTE into Eq. (12-33) with $\theta = 30^\circ$:

$$\begin{aligned}I_{x_1} &= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta \\ &= 8.75 \times 10^6 \text{ mm}^4 \quad \leftarrow\end{aligned}$$

SUBSTITUTE into Eq. (12-33) with $\theta = 120^\circ$:

$$I_{y_1} = 1.02 \times 10^6 \text{ mm}^4 \quad \leftarrow$$

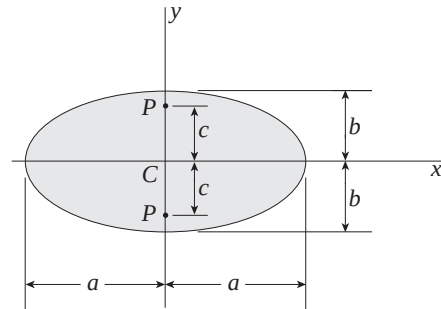
SUBSTITUTE into Eq. (12-35) with $\theta = 30^\circ$:

$$\begin{aligned}I_{x_1 y_1} &= \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta \\ &= -0.356 \times 10^6 \text{ mm}^4 \quad \leftarrow\end{aligned}$$

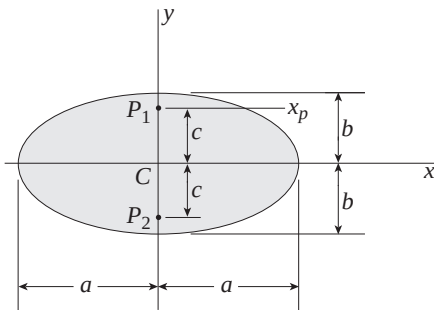
Principal Axes, Principal Points, and Principal Moments of Inertia

Problem 12.9-1 An ellipse with major axis of length $2a$ and minor axis of length $2b$ is shown in the figure.

- Determine the distance c from the centroid C of the ellipse to the principal points P on the minor axis (y axis).
- For what ratio a/b do the principal points lie on the circumference of the ellipse?
- For what ratios do they lie inside the ellipse?



Solution 12.9-1 Principal points of an ellipse



(a) LOCATION OF PRINCIPAL POINTS

At a principal point, all moments of inertia are equal.

At point P_1 : $I_{xp} = I_y$ (1)

$$\text{From Case 16: } I_y = \frac{\pi b a^3}{4}$$

$$I_x = \frac{\pi a b^3}{4} \quad A = \pi a b$$

Parallel-axis theorem:

$$I_{xp} = I_x + A c^2 = \frac{\pi a b^3}{4} + \pi a b c^2$$

Substitute into Eq. (1):

$$\frac{\pi a b^3}{4} + \pi a b c^2 = \frac{\pi b a^3}{4}$$

$$\text{Solve for } c: \quad c = \frac{1}{2} \sqrt{a^2 - b^2} \quad \leftarrow$$

(b) PRINCIPAL POINTS ON THE CIRCUMFERENCE

$$\therefore c = b \text{ and } b = \frac{1}{2} \sqrt{a^2 - b^2}$$

$$\text{Solve for ratio } \frac{a}{b}: \quad \frac{a}{b} = \sqrt{5} \quad \leftarrow$$

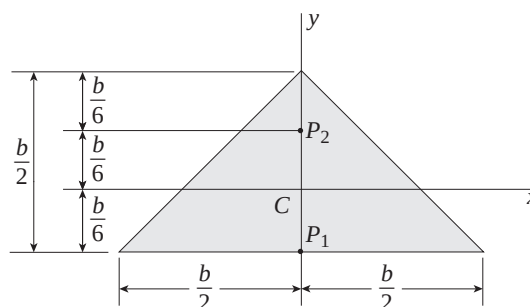
(c) PRINCIPAL POINTS INSIDE THE ELLIPSE

$$\therefore 0 \leq c \leq b \quad \text{For } c = 0: \quad a = b \text{ and } \frac{a}{b} = 1$$

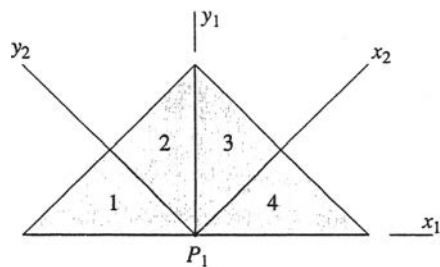
$$\text{For } c = b: \quad \frac{a}{b} = \sqrt{5}$$

$$\therefore 1 \leq \frac{a}{b} < \sqrt{5} \quad \leftarrow$$

Problem 12.9-2 Demonstrate that the two points P_1 and P_2 , located as shown in the figure, are the principal points of the isosceles right triangle.



Solution 12.9-2 Principal points of an isosceles right triangle



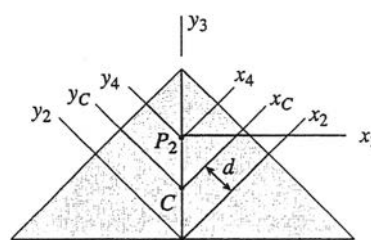
CONSIDER POINT P_1 :

$I_{x_1}I_{y_1} = 0$ because y_1 is an axis of symmetry.

$I_{x_2}I_{y_2} = 0$ because areas 1 and 2 are symmetrical about the y_2 axis and areas 3 and 4 are symmetrical about the x_2 axis.

Two different sets of principal axes exist at point P_1 .

$\therefore P_1$ is a principal point $\leftarrow$



CONSIDER POINT P_2 :

$I_{x_3}I_{y_3} = 0$ because y_3 is an axis of symmetry.

$I_{x_2}I_{y_2} = 0$ (see above).

Parallel-axis theorem:

$$I_{x_2y_2} = I_{x_cy_c} + Ad_1d_2 \quad A = \frac{b^2}{4} \quad d = d_1 = d_2 = \frac{b}{6\sqrt{2}}$$

$$I_{x_cy_c} = -\left(\frac{b^2}{4}\right)\left(\frac{b}{6\sqrt{2}}\right)^2 = -\frac{b^4}{288}$$

Parallel-axis theorem:

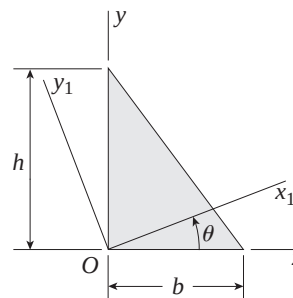
$$I_{x_4y_4} = I_{x_cy_c} + Ad_1d_2 \quad d_1 = d_2 = -\frac{b}{6\sqrt{2}}$$

$$I_{x_4y_4} = -\frac{b^4}{288} + \frac{b^2}{4}\left(\frac{b}{6\sqrt{2}}\right)^2 = 0$$

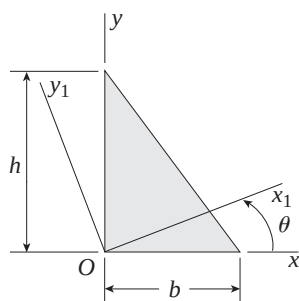
Two different sets of principal axes (x_3y_3 and x_4y_4) exist at point P_2 .

$\therefore P_2$ is a principal point $\leftarrow$

Problem 12.9-3 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the origin O for the right triangle shown in the figure if $b = 6$ in. and $h = 8$ in. Also, calculate the corresponding principal moments of inertia I_1 and I_2 .



Solution 12.9-3 Principal axes



RIGHT TRIANGLE

$$b = 6.0 \text{ in.} \quad h = 8.0 \text{ in.}$$

CASE 7:

$$I_x = \frac{bh^3}{12} = 256 \text{ in.}^4$$

$$I_y = \frac{hb^3}{12} = 144 \text{ in.}^4$$

$$I_{xy} = \frac{b^2h^2}{24} = 96 \text{ in.}^4$$

$$\text{Eq. (12-39): } \tan 2\theta_p = -\frac{2I_{xy}}{I_x - I_y} = -1.71429$$

$$2\theta_p = -59.744^\circ \quad \text{and} \quad 120.256^\circ$$

$$\theta_p = -29.872^\circ \quad \text{and} \quad 60.128^\circ$$

SUBSTITUTE into Eq. (12-33) with $\theta = -29.872^\circ$:

$$I_{x_1} = 311.1 \text{ in.}^4$$

SUBSTITUTE into Eq. (12-33) with $\theta = 60.128^\circ$:

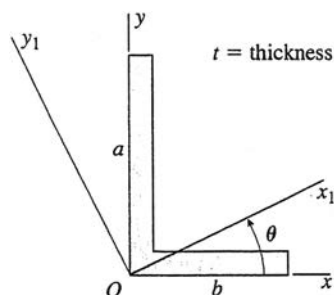
$$I_{x_2} = 88.9 \text{ in.}^4$$

$$\text{THEREFORE, } \left. \begin{aligned} I_1 &= 311.1 \text{ in.}^4 & \theta_{p_1} &= -29.87^\circ \\ I_2 &= 88.9 \text{ in.}^4 & \theta_{p_2} &= 60.13^\circ \end{aligned} \right\} \leftarrow$$

NOTE: The principal moments of inertia can be verified with Eqs. (12-42a and b) and Eq. (12-37).

Problem 12.9-4 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the origin O and the corresponding principal moments of inertia I_1 and I_2 for the L-shaped area described in Prob. 12.8-4 ($a = 150$ mm, $b = 100$ mm, and $t = 15$ mm).

Solution 12.9-4 Principal axes



ANGLE SECTION

$$a = 150 \text{ mm} \quad b = 100 \text{ mm} \quad t = 15 \text{ mm}$$

FROM PROB. 12.8-4:

$$I_x = 16.971 \times 10^6 \text{ mm}^4$$

$$I_y = 5.152 \times 10^6 \text{ mm}^4 \quad I_{xy} = 1.815 \times 10^6 \text{ mm}^4$$

$$\text{Eq. (12-39): } \tan 2\theta_p = -\frac{2I_{xy}}{I_x - I_y} = -0.3071$$

$$2\theta_p = -17.07^\circ \quad \text{and} \quad 162.93^\circ$$

$$\theta_p = -8.54^\circ \quad \text{and} \quad 81.46^\circ$$

SUBSTITUTE into Eq. (12-33) with $\theta = -8.54^\circ$:

$$I_{x_1} = 17.24 \times 10^6 \text{ mm}^4$$

SUBSTITUTE into Eq. (12-33) with $\theta = 81.46^\circ$:

$$I_{x_1} = 4.88 \times 10^6 \text{ mm}^4$$

THEREFORE,

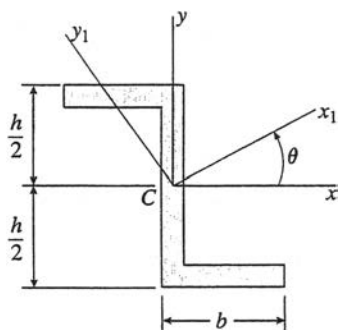
$$I_1 = 17.24 \times 10^6 \text{ mm}^4 \quad \theta_{p_1} = -8.54^\circ$$

$$I_2 = 4.88 \times 10^6 \text{ mm}^4 \quad \theta_{p_2} = 81.46^\circ$$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-42a and b) and Eq. (12-37).

Problem 12.9-5 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the centroid C and the corresponding principal centroidal moments of inertia I_1 and I_2 for the Z-section described in Prob. 12.8-5 ($b = 3$ in., $h = 4$ in., and $t = 0.5$ in.).

Solution 12.9-5 Principal axes



Z-SECTION

$t = \text{thickness} = 0.5$ in.

$b = 3.0$ in. $h = 4.0$ in.

FROM PROB. 12.8-5:

$$I_x = 10.3751 \text{ in.}^4 \quad I_y = 6.9688 \text{ in.}^4$$

$$I_{xy} = -6.5625 \text{ in.}^4$$

$$\text{Eq. (12-39):} \quad \tan 2\theta_p = -\frac{2I_{xy}}{I_x - I_y} = 3.8532$$

$$2\theta_p = 75.451^\circ \text{ and } 255.451^\circ$$

$$\theta_p = 37.726^\circ \text{ and } 127.726^\circ$$

SUBSTITUTE into Eq. (12-33) with $\theta = 37.726^\circ$:

$$I_{x_1} = 15.452 \text{ in.}^4$$

SUBSTITUTE into Eq. (12-33) with $\theta = 127.726^\circ$:

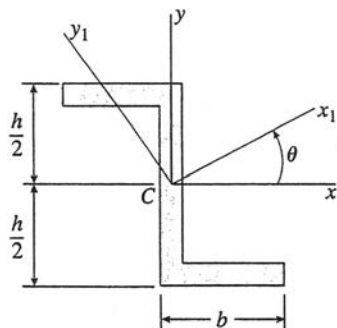
$$I_{x_1} = 1.892 \text{ in.}^4$$

$$\text{THEREFORE, } \left. \begin{array}{l} I_1 = 15.45 \text{ in.}^4 \quad \theta_{p_1} = 37.73^\circ \\ I_2 = 1.89 \text{ in.}^4 \quad \theta_{p_2} = 127.73^\circ \end{array} \right\} \leftarrow$$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-42a and b) and Eq. (12-37).

Problem 12.9-6 Solve the preceding problem for the Z-section described in Prob. 12.8-6 ($b = 80$ mm, $h = 120$ mm, and $t = 12$ mm).

Solution 12.9-6 Principal axes



Z-SECTION

$t = \text{thickness}$

$= 12$ mm

$b = 80$ mm

$h = 120$ mm

FROM PROB. 12.8-6:

$$I_x = 6.5065 \times 10^6 \text{ mm}^4 \quad I_y = 3.2573 \times 10^6 \text{ mm}^4$$

$$I_{xy} = -3.5251 \times 10^6 \text{ mm}^4$$

(Continued)

$$\text{Eq. (12-39): } \tan 2\theta_p = -\frac{2I_{xy}}{I_x - I_y} = 2.1698$$

$$2\theta_p = 65.257^\circ \text{ and } 245.257^\circ$$

$$\theta_p = 32.628^\circ \text{ and } 122.628^\circ$$

SUBSTITUTE into Eq. (12-33) with $\theta = 32.628^\circ$:

$$I_{x_1} = 8.763 \times 10^6 \text{ mm}^4$$

SUBSTITUTE into Eq. (12-33) with $\theta = 122.628^\circ$:

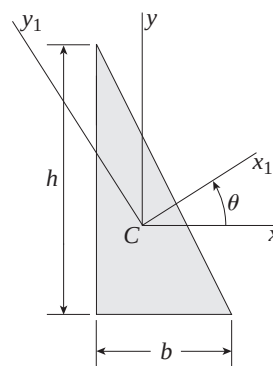
$$I_{x_1} = 1.000 \times 10^6 \text{ mm}^4$$

THEREFORE,

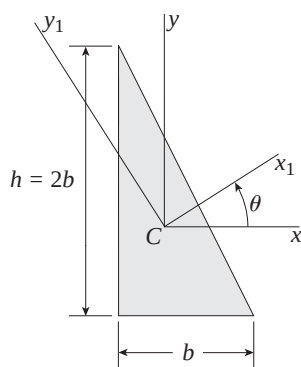
$$\left. \begin{aligned} I_1 &= 8.76 \times 10^6 \text{ mm}^4 & \theta_{p_1} &= 32.63^\circ \\ I_2 &= 1.00 \times 10^6 \text{ mm}^4 & \theta_{p_2} &= 122.63^\circ \end{aligned} \right\} \leftarrow$$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-42a and b) and Eq. (12-37).

Problem 12.9-7 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal axes through the centroid C for the right triangle shown in the figure if $h = 2b$. Also, determine the corresponding principal centroidal moments of inertia I_1 and I_2 .



Solution 12.9-7 Principal axes



RIGHT TRIANGLE

$$h = 2b$$

CASE 6

$$I_x = \frac{bh^3}{36} = \frac{2b^4}{9}$$

$$I_y = \frac{hb^3}{36} = \frac{b^4}{18}$$

$$I_{xy} = -\frac{b^2h^2}{72} = -\frac{b^4}{18}$$

$$\text{Eq. (12-39): } \tan 2\theta_p = -\frac{2I_{xy}}{I_x - I_y} = \frac{2}{3}$$

$$2\theta_p = 33.6901^\circ \text{ and } 213.6901^\circ$$

$$\theta_p = 16.8450^\circ \text{ and } 106.8450^\circ$$

SUBSTITUTE into Eq. (12-33) with $\theta = 16.8450^\circ$:

$$I_{x_1} = 0.23904b^4$$

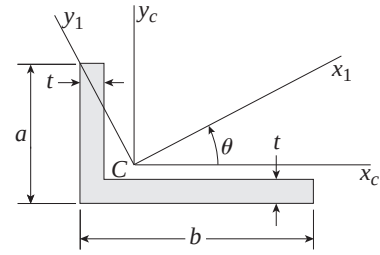
SUBSTITUTE into Eq. (12-33) with $\theta = 106.8450^\circ$:

$$I_{x_1} = 0.03873b^4$$

$$\left. \begin{aligned} \text{THEREFORE, } I_1 &= 0.2390b^4 & \theta_{p_1} &= 16.85^\circ \\ I_2 &= 0.0387b^4 & \theta_{p_2} &= 106.85^\circ \end{aligned} \right\} \leftarrow$$

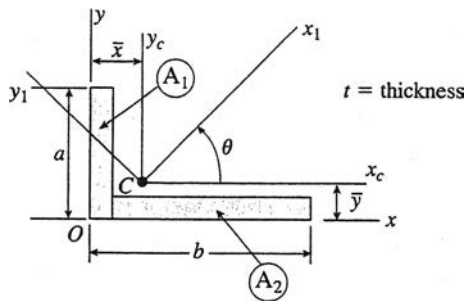
NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-42a and b) and Eq. (12-37).

Problem 12.9-8 Determine the angles θ_{p_1} and θ_{p_2} defining the orientations of the principal centroidal axes and the corresponding principal moments of inertia I_1 and I_2 for the L-shaped area shown in the figure if $a = 80$ mm, $b = 150$ mm, and $t = 16$ mm.



Probs. 12.9-8 and 12.9-9

Solution 12.9-8 Principal axes (angle section)



$$\begin{aligned} a &= 80 \text{ mm} \quad b = 150 \text{ mm} \quad t = 16 \text{ mm} \\ A_1 &= at = 1280 \text{ mm}^2 \\ A_2 &= (b - t)t = 2144 \text{ mm}^2 \\ A &= A_1 + A_2 = t(a + b - t) = 3424 \text{ mm}^2 \end{aligned}$$

LOCATION OF CENTROID C

$$\begin{aligned} Q_x &= \sum A_i \bar{y}_i = (at) \left(\frac{a}{2} \right) + (b - t)t \left(\frac{t}{2} \right) \\ &= 68,352 \text{ mm}^3 \\ \bar{y} &= \frac{Q_x}{A} = \frac{68,352 \text{ mm}^3}{3,424 \text{ mm}^2} = 19.9626 \text{ mm} \\ Q_y &= \sum A_i \bar{x}_i = (at) \left(\frac{t}{2} \right) + (b - t)t \left(\frac{b + t}{2} \right) \\ &= 188,192 \text{ mm}^3 \\ \bar{x} &= \frac{Q_y}{A} = \frac{188,192 \text{ mm}^3}{3,424 \text{ mm}^2} = 54.9626 \text{ mm} \end{aligned}$$

MOMENTS OF INERTIA (xy AXES)

Use parallel-axis theorem.

$$\begin{aligned} I_x &= \frac{1}{12} (t)(a^3) + A_1 \left(\frac{a}{2} \right)^2 + \frac{1}{12} (b - t)(t^3) + A_2 \left(\frac{b + t}{2} \right)^2 \\ &= \frac{1}{12} (16)(80)^3 + (1280)(40)^2 + \frac{1}{12} (134)(16)^3 \\ &\quad + (2144)(8)^2 \\ &= 2.91362 \times 10^6 \text{ mm}^4 \end{aligned}$$

$$\begin{aligned} I_y &= \frac{1}{12} (a)(t^3) + A_1 \left(\frac{t}{2} \right)^2 + \frac{1}{12} (t)(b - t^3) \\ &\quad + A_2 \left(\frac{b + t}{2} \right)^2 \\ &= \frac{1}{12} (80)(16)^3 + (1280)(8)^2 + \frac{1}{12} (16)(134)^3 \\ &\quad + (2144) \left(\frac{166}{2} \right)^2 \\ &= 18.08738 \times 10^6 \text{ mm}^4 \end{aligned}$$

MOMENTS OF INERTIA ($x_c y_c$ AXES)

Use parallel-axis theorem.

$$\begin{aligned} I_{x_c} &= I_x - A\bar{y}^2 = 2.91362 \times 10^6 - (3424)(19.9626)^2 \\ &= 1.54914 \times 10^6 \text{ mm}^4 \\ I_{y_c} &= I_y - A\bar{x}^2 = 18.08738 \times 10^6 - (3424)(54.9626)^2 \\ &= 7.74386 \times 10^6 \text{ mm}^4 \end{aligned}$$

PRODUCT OF INERTIA

Use parallel-axis theorem: $I_{xy} = I_{\text{centroid}} + A d_1 d_2$

$$\begin{aligned} \text{Area } A_1: I'_{x_c y_c} &= 0 + A_1 \left[-\left(\bar{x} - \frac{t}{2} \right) \right] \left[\frac{e}{2} - \bar{y} \right] \\ &= (1280)(8 - 54.9626)(40 - 19.9626) \\ &= -1,20449 \times 10^6 \text{ mm}^4 \end{aligned}$$

$$\begin{aligned} \text{Area } A_2: I''_{x_c y_c} &= 0 + A_2 \left[\frac{b + t}{2} - \bar{x} \right] \left[-\left(\bar{y} - \frac{t}{2} \right) \right] \\ &= (2144)(83 - 54.9626)(8 - 19.9626) \\ &= -0.71910 \times 10^6 \text{ mm}^4 \end{aligned}$$

$$I_{x_c y_c} = I'_{x_c y_c} + I''_{x_c y_c} = -1.92359 \times 10^6 \text{ mm}^4$$

SUMMARY

$$\begin{aligned} I_{x_c} &= 1.54914 \times 10^6 \text{ mm}^4 \quad I_{y_c} = 7.74386 \times 10^6 \text{ mm}^4 \\ I_{x_c y_c} &= -1.92359 \times 10^6 \text{ mm}^4 \end{aligned}$$

(Continued)

PRINCIPAL AXES

$$\text{Eq. (12-39): } \tan 2\theta_p = -\frac{2I_{xy}}{I_x - I_y} = -0.621041$$

$$2\theta_p = -31.8420^\circ \text{ and } 148.1580^\circ$$

$$\theta_p = -15.9210^\circ \text{ and } 74.0790^\circ$$

SUBSTITUTE into Eq. (12-33) with $\theta = -15.9210^\circ$

$$I_{x_1} = 1.0004 \times 10^6 \text{ mm}^4$$

SUBSTITUTE into Eq. (12-33) with $\theta = 74.0790^\circ$

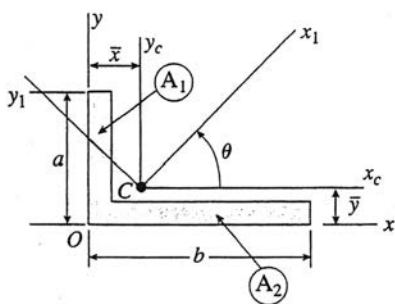
$$I_{x_1} = 8.2926 \times 10^6 \text{ mm}^4$$

THEREFORE,

$$\left. \begin{aligned} I_1 &= 8.29 \times 10^6 \text{ mm}^4 & \theta_{p_1} &= 74.08^\circ \\ I_2 &= 1.00 \times 10^6 \text{ mm}^4 & \theta_{p_2} &= -15.92^\circ \end{aligned} \right\} \leftarrow$$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-42a and b) and Eq. (12-37).

Problem 12.9-9 Solve the preceding problem if $a = 3$ in., $b = 6$ in., and $t = 5/8$ in.

Solution 12.9-9 Principal axes (angle section)

$$a = 3.0 \text{ in.}$$

$$b = 6.0 \text{ in.}$$

$$t = 5/8 \text{ in.}$$

$$A_1 = at = 1.875 \text{ in.}^2$$

$$A_2 = (b - t)t = 3.35938 \text{ in.}^2$$

$$A = A_1 + A_2 = t(a + b - t) = 5.23438 \text{ in.}^2$$

LOCATION OF CENTROID C

$$\begin{aligned} Q_x &= \sum A_i \bar{y}_i = (at)\left(\frac{a}{2}\right) + (b - t)t\left(\frac{t}{2}\right) \\ &= 3.86230 \text{ in.}^3 \end{aligned}$$

$$\bar{y} = \frac{Q_x}{A} = \frac{3.86230 \text{ in.}^3}{5.23438 \text{ in.}^2} = 0.73787 \text{ in.}$$

$$\begin{aligned} Q_y &= \sum A_i \bar{x}_i = (at)\left(\frac{t}{2}\right) + (b - t)t\left(\frac{b + t}{2}\right) \\ &= 11.71387 \text{ in.}^3 \end{aligned}$$

$$\bar{x} = \frac{Q_y}{A} = \frac{11.71387 \text{ in.}^3}{5.23438 \text{ in.}^2} = 2.23787 \text{ in.}$$

MOMENTS OF INERTIA (xy AXES)

Use parallel-axis theorem.

$$\begin{aligned} I_x &= \frac{1}{12}(t)(a^3) + A_1\left(\frac{a}{2}\right)^2 + \frac{1}{12}(b - t)(t^3) + A_2\left(\frac{t}{2}\right)^2 \\ &= \frac{1}{12}\left(\frac{5}{8}\right)(3.0)^3 + (1.875)(1.5)^2 + \frac{1}{12}(5.375)\left(\frac{5}{8}\right)^3 \\ &\quad + (3.35938)\left(\frac{5}{16}\right)^2 \\ &= 6.06242 \text{ in.}^4 \end{aligned}$$

$$\begin{aligned} I_y &= \frac{1}{12}(a)(t^3) + A_1\left(\frac{t}{2}\right)^2 + \frac{1}{12}(t)(b - t)^3 \\ &\quad + A_2\left(\frac{b + t}{2}\right)^2 \\ &= \frac{1}{12}(3.0)\left(\frac{5}{8}\right)^3 + (1.875)\left(\frac{5}{16}\right)^2 + \frac{1}{12}\left(\frac{5}{8}\right)(5.375)^3 \\ &\quad + (3.35938)\left(\frac{6.625}{2}\right)^2 \\ &= 45.1933 \text{ in.}^4 \end{aligned}$$

MOMENTS OF INERTIA ($x_c y_c$ AXES)

Use parallel-axis theorem.

$$\begin{aligned} I_{x_c} &= I_x - A\bar{y}^2 = 6.06242 - (5.23438)(0.73787)^2 \\ &= 3.21255 \text{ in.}^4 \end{aligned}$$

$$\begin{aligned} I_{y_c} &= I_y - A\bar{x}^2 = 45.1933 - (5.23438)(2.23787)^2 \\ &= 18.97923 \text{ in.}^4 \end{aligned}$$

PRODUCT OF INERTIA

Use parallel-axis theorem: $I_{xy} = I_{\text{centroid}} + Ad_1d_2$

$$\begin{aligned}\text{Area } A_1: I'_{x_c y_c} &= 0 + A_1 \left[-\left(\bar{x} - \frac{t}{2} \right) \right] \left[\frac{a}{2} - \bar{y} \right] \\ &= (1.875)(-1.92537)(0.76213) \\ &= -2.75134 \text{ in.}^4\end{aligned}$$

$$\begin{aligned}\text{Area } A_2: I''_{x_c y_c} &= 0 + A_2 \left[\frac{b+t}{2} - \bar{x} \right] \left[-\left(\bar{y} - \frac{t}{2} \right) \right] \\ &= (3.35938)(1.07463)(-0.42537) \\ &= -1.53562 \text{ in.}^4\end{aligned}$$

$$I_{x_c y_c} = I'_{x_c y_c} + I''_{x_c y_c} = -4.28696 \text{ in.}^4$$

SUMMARY

$$I_{x_c} = 3.21255 \text{ in.}^4 \quad I_{y_c} = 18.97923 \text{ in.}^4$$

$$I_{x_c y_c} = -4.28696 \text{ in.}^4$$

PRINCIPAL AXES

$$\text{Eq. (12-39): } \tan 2\theta_p = -\frac{2I_{xy}}{I_x - I_y} = -0.54380$$

$$2\theta_p = -28.5374^\circ \text{ and } 151.4626^\circ$$

$$\theta_p = -14.2687^\circ \text{ and } 75.7313^\circ$$

SUBSTITUTE into Eq. (12-33) with $\theta = -14.2687^\circ$

$$I_{x_1} = 2.1223 \text{ in.}^4$$

SUBSTITUTE into Eq. (12-33) with $\theta = -75.7313^\circ$

$$I_{x_1} = 20.0695 \text{ in.}^4$$

THEREFORE,

$$\left. \begin{aligned} I_1 &= 20.07 \text{ in.}^4 & \theta_{p_1} &= 75.73^\circ \\ I_2 &= 2.12 \text{ in.}^4 & \theta_{p_2} &= -14.27^\circ \end{aligned} \right\} \leftarrow$$

NOTE: The principal moments of inertia I_1 and I_2 can be verified with Eqs. (12-42a and b) and Eq. (12-37).

