Partial

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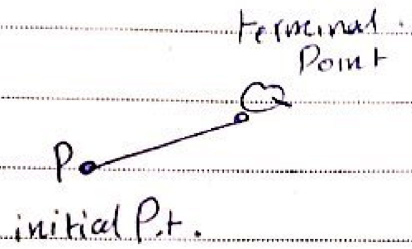
1st semester 2019

Mech Family
كن أنت التغيير
Notebooks

"Differantial calculus"

- Scalars
- Vectors
(magnitude + Direction)
(not m, length), Velocity



$\vec{a} = \overrightarrow{PQ}$ Vector PQ

$P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$
cordinate

$P(\ )$

$\overrightarrow{PQ} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$ components

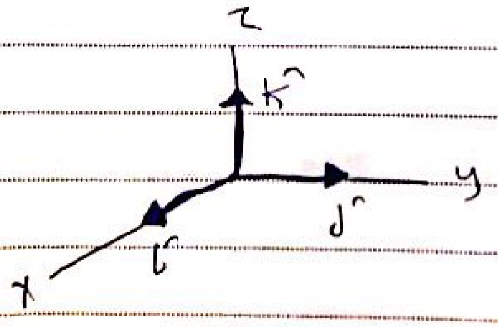
$= \langle x_2 - x_1, \ldots \rangle$

$\overrightarrow{PQ} = \vec{a} = [a_1, a_2, a_3] = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$a_1 = x_2 - x_1, \ldots$

$\hat{i}, \hat{j}, k$ is a basis in R^3

$\hat{i} = [1, 0, 0]$ $\hat{j} = [0, 1, 0]$ $\hat{k} = [0, 0, 1]$



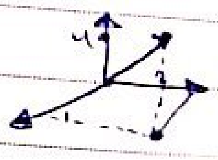
length of $|\vec{a}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

$= \sqrt{a_1^2 + a_2^2 + a_3^2}$

$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$

$\hat{a} = [1, 2, 4]$ Position Vector

يعني بيبدأ من نقطة الأصل



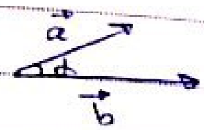
$\vec{a} = [a_1, a_2, a_3]$ $\vec{b} = [b_1, b_2, b_3]$

$\vec{a} \pm \vec{b} = [a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3]$

لو بدي اجمع vectors 2

Multiplication ① Scalar . (Dot Product)

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ ← مابدنا نشتق هذا

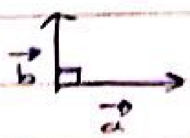


$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

If $\vec{a} \cdot \vec{b} = 0$

① $\vec{a}$ or $\vec{b}$ is 0 (zero)

② $\cos\alpha = 0 \Rightarrow \alpha = 90° = \pi/2$



chapter 9

Given $\vec{a} = [1, 0, -1]$

$\vec{b} = [2, 1, 3]$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 2 & 1 & 3 \end{vmatrix} = \hat{i} \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$$

$$+ \hat{k} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = \hat{i} - 5\hat{j} + \hat{k} = [1, -5, 1]$$

Properties

① $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

② $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

§4 :-

scalar function

$y = f(x)$ (values are scalers)

$y = x^2$

Distance function $d = f(x, y, z) =$

$$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

$P_0(x_0, y_0, z_0)$

$\bullet$———$\bullet$ $P(x, y, z)$

$R^3 \rightarrow R$ ٣ ارقام تحت الجذر ، الجواب رقم .

Vector functions

$\vec{r} = \vec{r}(t)$ (Values are vectors)

$\vec{r}(t) = [t, t^2, t^3] = t\hat{i} + t^2\hat{j} + t^3\hat{k}$

$\vec{r}'(t) = [1, 2t, 3t^2]$

$\vec{F}(x, y, z) = [F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)]$

لانه في 3 متغيرات نشتق اشتقاقا جزئي

$\vec{F}(x, y, z) = x^2yz\hat{i} + (x^2+y^2)\hat{j} + (z^3+xy)\hat{k}$

$\frac{\partial F}{\partial x} = [2xyz, 2x, y]$

$\frac{\partial F}{\partial y} = x^2z, 0, 2y, x$ $\frac{\partial F}{\partial z} = x^2y, 0, 3z^2$

9.5) curves (ex 1-7, 11-19, 24-28) From the Book

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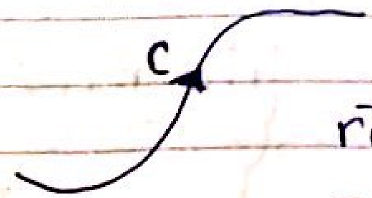
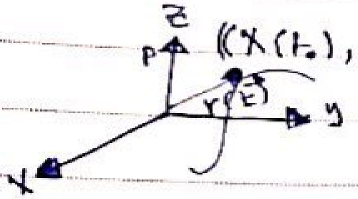


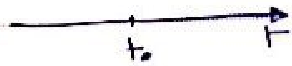
Parametric representation

$\vec{r}(t) = [x(t), y(t), z(t)]$

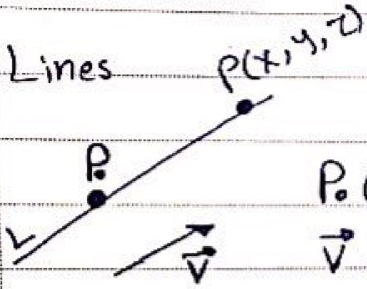
$= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

(t) is a parameter (x, y, z) are cartesian coordinates





Lines



$P_0(x_0, y_0, z_0)$

$\vec{V} = [a, b, c]$

$\overrightarrow{P_0P} \parallel \vec{V}$, $\overrightarrow{P_0P} = t\vec{V}$

$[x-x_0, y-y_0, z-z_0] = t[a,b,c]$

$= [ta, tb, tc]$

- Parametric equations

$x - x_0 = ta$

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

- vector equation:

$[x, y, z] = [x_0, y_0, z_0] + [ta, tb, tc]$

$\vec{r}(t) = \vec{r}(t_0) + t\vec{V}$

$\vec{r}(t) = \vec{r}(t_0) + t\vec{V}$

$\vec{r}(t) = \vec{r}_0 + t\vec{V}$

ex ① Find a parametric representation of the stright line through $P_0(3,1,2)$ in the direction $\hat{i}+4\hat{k}$

$\rightarrow r_0 = P_0$

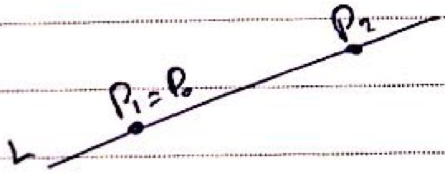
$\vec{V} = [1,0,4]$

$$\left.\begin{aligned} x &= 3+t \\ y &= 1 \\ z &= 2+4t \end{aligned}\right\} \text{Parametric equation}$$

$\vec{r}(t) = \vec{r}(0) + t\vec{V} = [3,1,2] + t[1,0,4]$

$\vec{r}(t) = [\underbrace{3+t}_{x(t)}, \underbrace{1}_{y(t)}, \underbrace{2+4t}_{z(t)}]$ vector equation

ex ② Given $P_1(2,4,-1)$ and $P_2(5,0,7)$ Find a vector eq-



$\vec{P_1P_2} = [3,-4,8]$ Parametric

$x = 2+3t$

$y = 4-4t \quad -\infty, t < \infty$

$z = -1+8t$

(9.5) NO. Homework 1

(1) $[3+2\cos t,\ 2\sin t,\ 0]$

$x = 3+2\cos t$

$y = 2\sin t$ Circle

$z = 0$

(7) $[4\cos t,\ 4\sin t,\ 3t]$

$x = 4\cos t$

$y = 4\sin t$ Helix

$z = 3t$

(11) $z = 1$ $c(3,2)$ Passing through the origin

$x = 3+\cos t$

$y = 2+\sin t$

$z = 1$

(19) $4x^2 - 3y^2 = 4$, $z = -2$

$x = \cosh(t)$ $\cosh^2 t - \sinh^2 t = 1$

$y = \frac{2}{\sqrt{3}}\sinh(t)$

Para.: $\left[\cosh t,\ \frac{2}{\sqrt{3}}\sinh t,\ -2\right]$

$z = -2$

NO Homework

9.1

Find the components of the vector (v) with initial Point (P) and terminal Point Q. Find |v|. Sketch |v|

Find the unit vector (u) in the direction of (v):

P(1,1,0) Q(6,2,0)

$\vec{V} = \vec{PQ} = [5,1,0]$

$|V| = \sqrt{5^2 + 1^2 + 0^2}$ $|V| = 5.099$

$u = \frac{V}{|V|} = \left[\frac{5}{5.099}, \frac{1}{5.099}, 0\right]$ $u = [0.98, 0.196, 0]$

P(1,1,1) Q(2,2,0)

$\vec{V} = [1,1,-1]$ $|V| = \sqrt{1^2 + 1^2 + (-1)^2}$ $|V| = \sqrt{3}$

$u = \frac{V}{|V|} = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right]$

P(-3.0, 4.0, -0.5) Q(5.5, 0, 1.2)

$|\vec{V}| = \sqrt{(8.5)^2 + (-4)^2 + (1.7)^2}$ $\vec{V} = [8.5, -4, 1.7]$

P: (1,4,2) Q (-1,-4,-2)

$\vec{PQ} = \vec{V} = [-2, -8, -4]$

$|V| = \sqrt{(-2)^2 + (-8)^2 + (-4)^2}$ $|V| = \sqrt{84} = 9.2$

$U = \left[\frac{-2}{\sqrt{84}}, \frac{-8}{\sqrt{84}}, \frac{-4}{\sqrt{84}}\right]$

P(0, 0,0) Q (2,1,-2)

$\vec{PQ} = [2,1,-2] = \vec{V}$

$|\vec{V}| = \sqrt{4+1+4} = 3$

$U = \frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$

(4,00) P (0,2,13) Find the terminal Point Q
P, nd |V|

$Q = [0+4, 2+0, 13+0] = [4, 2, 3]$

$|V| = \sqrt{16 + 0 + 100}$ $|V| = 10.77$

$\frac{1}{2}, 3, \frac{-1}{4}$ $P(\frac{7}{2}, -3, \frac{3}{4})$

$Q = [4, 0, \frac{1}{2}]$

$|V| = \sqrt{0.25 + 9 + 0.0625}$ $\boxed{|V| = 3}$

Let $a = [3,2,0] = 3\hat{i} + 2\hat{j}$
$b = [-4, 6, 0] = -4\hat{i} + 6\hat{j}$
$c = [5, -1, 8] = 5\hat{i} - \hat{j} + 8\hat{k}$
$d = [0, 0, 4] = 4\hat{k}$ Find :-

$2\vec{a} = 6\hat{i} + 4\hat{j}$ $[6, 4, 0]$

$\frac{1}{2}\vec{a} = \frac{3}{2}\hat{i} + \hat{j}$ $[\frac{3}{2}, 1, 0]$

$-\vec{a} = -3\hat{i} - 2\hat{j}$ $[-3, -2, 0]$

(2) $(a+b)+c = [4\hat{i}, 7\hat{j}, 8\hat{k}]$
$a+(b+c)$
$= [4, 7, 8]$

(3) $b+c = c+b = [1, 5, 8]$

(14) $3c - 6d = [15, -3, 0]$
$3(c-2d) = [15, -3, 0]$

(15) $7(c-b) = 7c - 7b = [7, 35, 56]$

(16) $\frac{a}{2}a - 3c = [-1,5, 12, -24]$

(17) $(7-3)a = 7a - 3a = [12, 8, 0]$
(18) $4a + 3b = [0, 26, 0]$
$-4a - 3b = [0, 26, 0]$

Find the resultant in terms of components and its magnitude

(24) $P[-1,2,-3]$ $q[1,1,1]$ $u[1,-2,2]$

resultant $R = P + q + u = [1, 1, 0]$

$|R| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$

(25) $u = [3, 1, -6]$ $V = [0, 2, 5]$ $w = [3, -1, -13]$

$R = [6, 2, -14]$ $|R| = \sqrt{36 + 4 + 196}$ $|R| = 15.3$

(26) Equilibrium. Find V such that P, q, u in Prop 21 and V are in equilibrium.

$P + q + u + V = 0$

$[4, 9, -3] + [V_1, V_2, V_3] = [0, 0, 0]$

$[V_1, V_2, V_3] = [-4, -9, 3]$

(27) Find V such that P, q, u in Prop 23 and P are in equilibrium.

$R [P + q + u + V = 0$

$[0, 0, 5] + [V_1, V_2, V_3] = [0, 0, 0]$

$[V_1, V_2, V_3] = [0, 0, -5]$

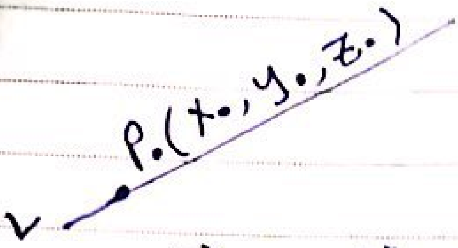
NO Homework

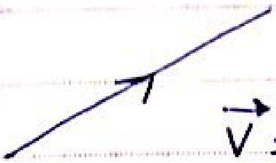
(28) Find the Unit vector in the direction of the resultant in Prop [24]

$R = P + q + u = [1, 1, 0]$

$|R| = \sqrt{2}$

$\vec{u} = \frac{R}{|R|} = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$





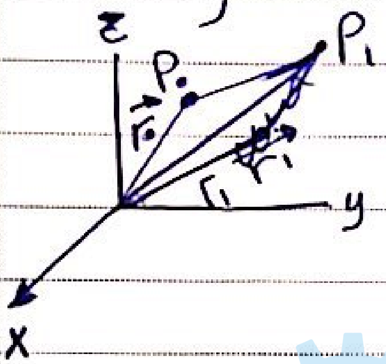
$\vec{V} = [a, b, c]$

$L: \vec{r}(t) = \vec{r}_0 + \vec{V} t$

or $x = x_0 + at$ / $y = y_0 + bt$ / $z = z_0 + ct$

$-\infty < t < \infty$

Line Segment



$P_0(x_0, y_0, z_0) \Rightarrow \vec{r}_0 = [x_0, y_0, z_0]$

$P_1(x_1, y_1, z_1)$

$\overrightarrow{P_0 P_1} = \vec{V} = \vec{r}_1 - \vec{r}_0$

$* \ \boxed{\vec{r}(t) = \vec{r}_0 + (\vec{r}_1 - \vec{r}_0)\, t}$, $0 \le t \le 1$

$t = 0 \Rightarrow \vec{r} = \vec{r}_0$

$t = 1 \Rightarrow \vec{r} = \vec{r}_1$

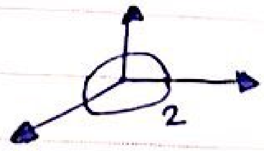
3 Find parametric equations of the $\vec{r}_0 = [2, 4, -1]$ line segment joining the points $P_0(2, 4, -1)$

$P_1(5, 0, 7)$ $\vec{r}_1\,[5, 0, 7]$

$\vec{r}(t) = [2, 4, -1] + ([5, 0, 7] - [2, 4, -1])\, t$

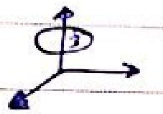
$= [2, 4, -1] + [-3, -4, 8]\, t = [\underbrace{2 + 3t}_{x(t)}, \underbrace{4 - 4t}_{y(t)}, \underbrace{-1 + 8t}_{z(t)}]$

$x^2 + y^2 = 4$, $z = 0$ $\longrightarrow$ 2^2

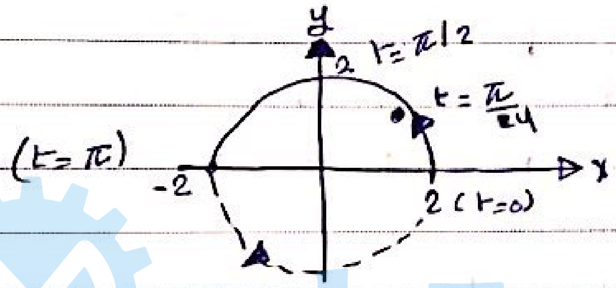


$$\left[\begin{array}{l} x(t) = x = 2\cos t \\ y(t) = y = 2\sin t \\ z = 0 \end{array}\right. \qquad 0 \le t \le 2\pi$$

$z = 3$ لو كانت



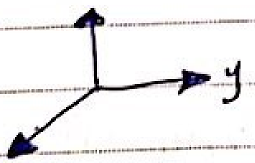
$\vec{r}(t) = [2\cos t, 2\sin t, 0]$



Counter clockwise direction.

$x^2 + z^2 = 1$ مثلها حكي احكي عن y بتكون
cylinder

$x^2 + z^2 = 1$, $y = 2$



$$\left.\begin{array}{l} x = \cos t \\ z = \sin t \\ y = 2 \end{array}\right] \qquad 0 \le t \le 2\pi$$

ex $y^2 + 4y + z^2 = 5$, $x = 3$

$(y^2 + 4y + 4) - 4 + z^2 = 5$

$(y+2)^2 + z^2 = 9$

$x = 3$
$2+y = 3\cos t$
$z = 3\sin t$

$x = 3$
$y = -2 + 3\cos t$
$z = 3\sin t$

$0 \le t \le 2\pi$

in General $x = x_0 + R\cos t$

$y = y_0 + R\sin t$ $\quad (x_0, y_0)$

$z = 0$

Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \;, \quad z = 0$$

Parametric equations

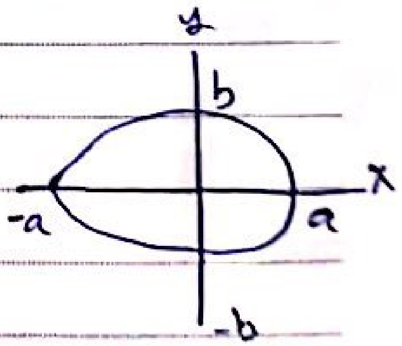
$$\boxed{\begin{aligned} x &= a\cos t \\ y &= b\sin t \\ z &= 0 \end{aligned}} \qquad 0 \leq t \leq 2\pi$$

$$\frac{x^2}{4} + \frac{y^2}{5} = 1 \;, \quad z = -5$$

$x = 2\cos t \qquad 0 \leq t \leq 2\pi$

$y = \sqrt{5}\sin t$

$z = -5$



$$\frac{(x-2)^2}{36} + \frac{9(y+1)^2}{36} = \frac{36}{36} \;, \quad z = 0$$

$$\frac{(x-2)^2}{36} + \frac{(y+1)^2}{4} = 1 \;,$$

$x - 2 = 6\cos t$

$y + 1 = 2\sin t$

$z = 0$

$x = 2 + 6\cos t$

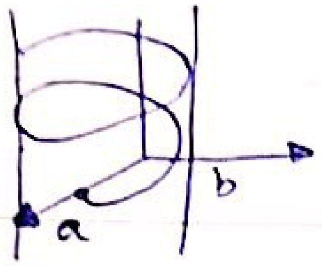
$y = -1 + 2\sin t$

Helix

$$\vec{r}(t) = [\underbrace{a\cos t}_{x}, \underbrace{b\sin t}_{y}, \underbrace{ct}_{z}]$$

If $c = 0 \rightarrow$ ellipse

شي مطلوب بس مرتفع



$$\vec{r}(t) = [2\cos t, 3\sin t, t]$$

$$\left.\begin{aligned} x &= 2\cos t \\ y &= 3\sin t \\ z &= t \end{aligned}\right] \rightarrow \text{Helix}$$

$y = x^2$, $z = 0$

$\vec{r}(t) = [t, t^2, 0]$

$x = t$

$y = t^2$

$z = 0$

$x = 0$, $z = \cos y$

$x = 0$

$y = t$

$z = \cos t$

9.7) Gradient of a scalar field ex(1-6, 11-15, 30-35)

A scalar function difines a scalar field.

A vector function difines a vector field.

$\vec{r}(t) = [t, t^2, 0]$

Def "1" Given a scalar function $f(x, y, z)$ that is defined and differaniable in some domain.

$\nabla \rightarrow$ del

The gradient of $f(x, y, z)$ in defined as a vector function.

$$\boxed{\text{grad } f = \nabla f = [f_x, f_y, f_z] = f_x i + f_y j + f_z k}$$

save

Ex:- Given $f(x, y, z) = 3y^2 + 5xz - 3x$

$\text{grad } f = \nabla f = [5z - 3, 6y, 5x]$

$\nabla f(1, 0, -2) = [-13, 0, 5]$

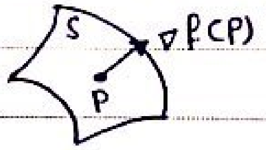
$$\nabla = \frac{\partial \square}{\partial x}\hat{i} + \frac{\partial \square}{\partial y}\hat{j} + \frac{\partial \square}{\partial z}\hat{k} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$$

Space

$$\nabla (x^2, y^2, z^2) = [2x, 2y, 2z]$$

* Scalar field $\xrightarrow{\nabla}$ vector field

Let f be a scalar function, let $f(x,y,z) = \text{Constant}$ represent a surface S.

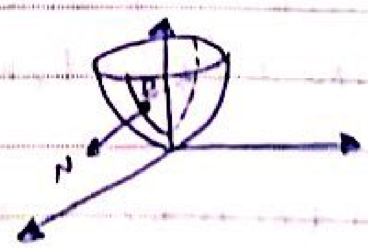


If the grad f at a point P of S is not the zero vector, it is a normal vector of S at P

ex: Given a paraboloid $z = 4x^2 + y^2$, Find a normal vector at the point $P(1,0,4)$

$\vec{N} = \nabla(z - 4x^2 - y^2) = [-8x, 2y, 1]$

$\vec{N}(P) = [-8, 0, 1]$



or

$4x^2 + y^2 - z = 0$

$\nabla f = [8x, 2y, -1]$

$\nabla f(1,0,4) = [8, 0, -1]$

9.7}

$\nabla^2 = \nabla \cdot \nabla$

$\nabla^2 = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$

$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Scalar field $\xrightarrow{\nabla^2}$ Scalar field

ex:- Given $f(x, y, z) = xy^2 + z^3$ Find $\nabla^2 f$

$\nabla^2(xy^2 + z^3)$

$f_x = y^2 \quad f_y = 2xy \quad f_z = 3z^2$

$f_{xx} = 0 \quad f_{yy} = 2x \quad f_{zz} = 6z$

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz}$$

9.8} Divergence of a vector field (ex 1-6, 9, 15, 20)

Given a vector function $\vec{F}(x, y, z) = [F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)]$

$$\boxed{\operatorname{div} \vec{F} = \nabla \cdot \vec{F}}$$

$\operatorname{div} \vec{F} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [F_1, F_2, F_3] = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

vector field $\xrightarrow{\operatorname{div} \vec{F}}$ Scalar field

1 $\boxed{\text{div}(\text{grad } f) = \nabla^2 f}$ نظري

$\checkmark$ Proof : Given f is a scalar f.

$\nabla f = [f_x, f_y, f_z]$

$\text{div}([f_x, f_y, f_z]) = f_{xx} + f_{yy} + f_{zz}$

$= \nabla^2 f$

If $\underline{k}$ is constant, $\vec{F}$ is a vector function, f is a scalar function, then

2 $\text{div}(k\vec{F}) = k \text{ div } \vec{F}$

*3 $\text{div}(f\vec{F}) = f \text{ div } \vec{F} + \vec{F} \cdot \nabla f$

*4 $\text{div}(f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g$

$\boxed{9.9}$ Curl of vector field (ex 4-8, 15, 20)

Review (32-37, 40)

Given a vector function $\vec{F} = [F_1, F_2, F_3]$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$\text{vector field} \xrightarrow{\text{Curl } \vec{F}} \text{vector field}$

Given $\vec{F} = [x, -xy, z^2]$, Find $\nabla \times \vec{F}$

$$= \text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -xy & z^2 \end{vmatrix}$$

$$\hat{i}\left(\frac{\partial(z^2)}{\partial y} - \frac{\partial(-xy)}{\partial z}\right) - \hat{j}\left(\frac{\partial z^2}{\partial x} - \frac{\partial x}{\partial z}\right)$$

$$+ \hat{k}\left(\frac{\partial(-xy)}{\partial x} - \frac{\partial(x)}{\partial y}\right)$$

$$= 0\hat{i} - 0\hat{j} - y\hat{k} = [0, 0, -y]$$

* Properties

Given : $\vec{F}, \vec{G}$ are vector functions, f is a Salar fn.

1) $\text{Curl}(\vec{F} + \vec{G}) = \text{Curl } \vec{F} + \text{curl } \vec{G}$ "* Proof"

2) $\text{Curl}(\text{grad } f) = \vec{0}$

Sc. f $\underline{\text{of}}$ vector

3) $\text{div}(\text{curl } \vec{F}) = 0$

ex: State whether each expression is meaningful

a) Curl f ✗ beause it is scalar function.

b) ∇f ✓

c) $\text{div } \vec{F}$ ✓

d) $\nabla \times (\nabla f) =$ ✓

e) $\text{Grad}(\text{div } \vec{F}) = \nabla(\nabla \cdot \vec{F})$ ✓

f) $\text{div}(\underbrace{\text{grad } f}_{V})$ ✓

g) $\text{grad}(\text{div } f)$ ✗

h) $\text{Curl}(\text{curl } \vec{F})$ ✓

i) $\underbrace{\text{div}}_{\text{scalar}}(\text{curl}(\underbrace{\text{grad } f}_{\text{vector}}))$ ✓

ex If $\vec{r} = [x, y, z]$, $\vec{a}$ is a constant Vector

Show that $\nabla \cdot (\vec{a} \times \vec{r}) = 0$

$\operatorname{div}(\vec{a} \times \vec{r}) = 0$

let $\vec{a} = [a_1, a_2, a_3]$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(a_2 z - a_3 y) - \hat{j}(a_1 z - a_3 x) + \hat{k}(a_1 y - a_2 x)$$

$$\nabla \cdot (\vec{a} \times \vec{r}) = \operatorname{div}(\vec{a} \times \vec{r})$$
$$= 0 + 0 + 0 = 0$$

* $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$
$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$

$$\operatorname{Curl}(gV) = (\nabla g \times V) + g \operatorname{curl} V$$

$$\operatorname{div}(\vec{u} \times \vec{V}) = V \cdot \operatorname{curl} u - u \cdot \operatorname{curl} V$$

$$\operatorname{curl}(gu + v) = \operatorname{curl}(gu) + \operatorname{curl} V$$

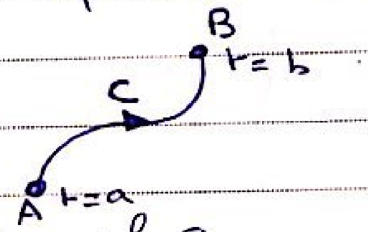
Vector integral calculus

NO. Cha (10)

10.1) Line integrals (ex 2-11)

$$\int_a^b f(x)\,dx$$

Line integral: We integrate a given function along a curve C in space (in the plane)



Parametric representation of C.

$$\vec{r}(t) = [x(t), y(t), z(t)] = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$a \leq t \leq b$$

The direction from A to B, in which t increases is called the Positive direction of C.

Assumption:- Every curve C is assumed to be Piecewise smooth.

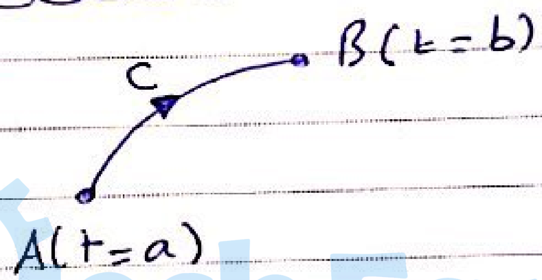
($\vec{r}'(t)$ is Continious $\vec{r}'(t) \neq 0$)

Def. A line integral of a vector function $\vec{F}$ along a curve C is defined as

$$\int_C \vec{F} \cdot d\vec{r}$$

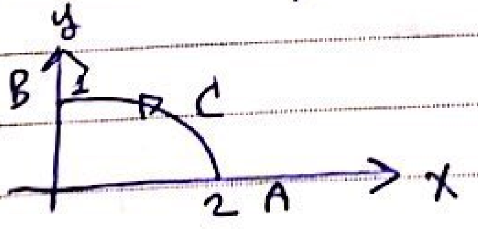
$\vec{F} = [F_1, F_2, F_3]$, $C : \vec{r}(t)$, $d\vec{r} = \vec{r}'(t)\,dt$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt}$$



Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = [x^2, -xy]$

and C is the quarter ellipse from A to B



Sol:- ① Parametrize C . $\frac{x^2}{4} + \frac{y^2}{1} = 1$

$x = 2\cos t$

$y = \sin t$

$(0 \leq t \leq \pi/2)$

A(2,0) $2 = 2\cos t$

B(0,1) $0 = 2\cos t$

$1 = \sin t$

t $(t = 0)$

$\vec{r}(t) = [2\cos t, \sin t]$ or $x = 2\cos t$, $y = \sin t$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} [\underbrace{4\cos^2 t}_{①}, \underbrace{-2\cos t \sin t}_{②}] \cdot [\underbrace{-2\sin t}_{①},$$

$$= \int_0^{\pi/2} (\overset{①\times①}{-8\cos^2 t \sin t} - \overset{②\times②}{2\cos^2 t \sin t})\, dt$$

$$= -10\int_0^{\pi/2} \cos^2 t \sin t\, dt = 10 \left.\frac{\cos^3 t}{3}\right|_0^{\pi/2} = -\frac{10}{3}$$

$u = \cos t$ $\quad du = -\sin t\, dt$

$-\int u^2\, du = \frac{u^3}{3}$

Q2) Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y,z) = [xy]\hat{i} + (yz)\hat{j} + ($

$C: \vec{r}(t) = [\underbrace{t}_{x}, \underbrace{t^2}_{y}, \underbrace{t^3}_{z}]$, $0 \le t \le 1$

Sol:

$\vec{F} = t^3\hat{i} + t^5\hat{j} + t^4\hat{k}$

$\vec{r}'(t) = [1, 2t, 3t^2]$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 [t^3, t^5, t^4]\cdot[1, 2t, 3t^2]\, dt$$

$$= \int_0^1 (t^3 + 2t^6 + 3t^6)\, dt$$

$$= \left.\frac{t^4}{4} + \frac{2t^7}{7} + \frac{3t^7}{7}\right|_0^1 = \frac{27}{28}.$$

If k is constant $\Rightarrow$

$$\int_C k\vec{F}\cdot d\vec{r} = k\int_C \vec{F}\cdot d\vec{r}$$

If $$\int_C (\vec{F}+\vec{G})\, d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_C \vec{G}\cdot d\vec{r}$$

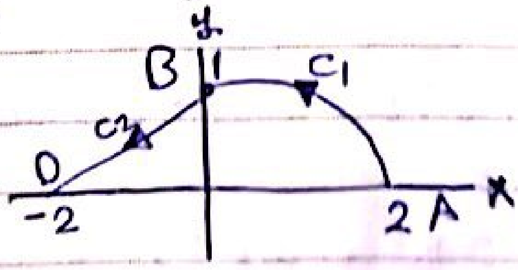
$$\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$$

ملاحظة لو كان الخط مع عقارب الساعة ، بحله حل عادي كأنه
عكس عقارب الساعة وبكتب بخطه سالب وبعود التكامل بتقلب

If $C = C_1 + C_2 + \dots + C_n$, then $\int_C \vec{F}\cdot d\vec{r}$

$$= \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \dots + \int_{C_n} \vec{F}\cdot d\vec{r}$$

Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = [x^3, -xy]$



C: from A to B is $\frac{x^2}{4} + y^2 = 1$ and from B to D is the line segment $C = C_1 + C_2$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \frac{-10}{3} \quad (\text{ex } 1)$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} =$$

$C_2 : (0,1) \rightarrow (-2,0)$; $\vec{r}(t) = r_0 + (\vec{r}_1 - \vec{r}_0)t$

$, \ 0 \leq t \leq 1$

$B\,(0,1) \Rightarrow \vec{r}_0 = [0,1]$

$D\,(-2,0) \Rightarrow \vec{r}_1 = [-2,0]$

$0 \leq t \leq 1$

$$\vec{r}(t) = [0,1] + ([-2,0] - [0,1])\,t = [\underbrace{-2t}_{x}, \underbrace{1-t}_{y}]$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 [4t^2, 2t(1-t)]\cdot[-2,-1]\,dt$$

$$= \int_0^1 \{-8t^2 - 2t(1-t)\}\,dt$$

$$= \frac{-8}{3}t^3 - t^2 + \frac{2t^3}{3}\Big|_0^1$$

$$= \frac{-8}{3} - 1 + \frac{2}{3} = -3$$

If $\vec{F} = [F_1, F_2, F_3]$; $C: \vec{r}(t)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot r'(t)\, dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C F_1\, dx + F_2\, dy + F_3\, dz$$

ex4) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where

$\vec{F} = [y, x]$ along $y = \ln x$ from $(1, 0)$ to $(e, 1)$

① Parametrize C: $\underset{y(t)}{y} = \ln \underset{x(t)}{x}$;

$r(\vec{t}) = [t, \ln t]$

$A(1, 0)$ $\quad x = 1 \quad x = t = 1 \Rightarrow \boxed{t = 1}$

$B(e, 1)$ $\quad y = 0 \quad y = \ln(1) = 0$

$\boxed{t = e}$

$1 \leq t \leq e$

$r'(t) = [1, \frac{1}{t}]$

$$\int_C \vec{F} \cdot d\vec{r} = \int_1^e [\ln t, t] \cdot [1, \frac{1}{t}]\, dt$$

$$= \int_1^e (\ln t + 1)\, dt = t \ln t \Big|_1^e - \cancel{t}\Big|_1^e$$

(2a) $\quad + \cancel{t} \quad - e \ln e$

لا تمسح الحل بأي طريقة.
تكون فسرت المسألة ، الحل بالطريقة 2 (...)

Evaluate $\int_C y\,dx + x\,dy$, where C: $y = \ln x$

from $(1, 0)$ to $(e, 1)$

$$\int_C y\,dx + x\,dy = \int_1^e \ln t\,dt + t\left(\frac{1}{t}\right)dt$$

$$= \int_1^e (\ln t + 1)\,dt = \dots = e$$

Ex 2: $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = [\underbrace{xy}_{F_1}, \underbrace{zy}_{F_2}, \underbrace{zx}_{F_3}]$

$\vec{r}(t) = [\underbrace{t}_{x}, \underbrace{t^2}_{y}, \underbrace{t^3}_{z}] \quad 0 \le t \le 1$

$\vec{r}'(t) = [1, 2t, 3t^2]$

$$\int_C \vec{F}\,d\vec{r} \cdot \int_0^1 [t^3, t^5, t^4]\cdot[1, 2t, 3t^2]\,dt$$

$$\int_0^1 t, 2t^3, 3t^5\,dt$$

$$\frac{t^2 + t^4 + t^6}{2}\Bigg|_0^1 = \frac{3}{2}.$$

$$= \frac{1}{4} + \frac{2}{7} + \frac{3}{7} = \frac{21}{28}$$

NO Cha 10

We define a work W done by a variable force $\vec{F} = [F_1, F_2, F_3]$ in moving a particle along a curve C as

$$\boxed{W = \int_C \vec{F} \cdot d\vec{r}}$$

ex 5) Find the work done by the force $\vec{F} = [x^2, y^2, z^2]$ in moving a particle from $(1,0,1)$ to $(1,0,e^{2\pi})$ along the exponential helix.

$$\vec{r}(t) = [\cos t, \sin t, e^t]$$

Parametrize اول خطوة لازم نعرف

① $C: \vec{r}(t) = [\underbrace{\cos t}_{x}, \underbrace{\sin t}_{y}, \underbrace{e^t}_{z}]$

$\vec{F} = [\cos^2 t, \sin^2 t, e^{2t}]$; $A(1,0,1) \to B(1,0,e^{2\pi})$

$$w = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} [\cos^2 t, \sin^2 t, e^{2t}] \cdot [-\sin t, \cos t, e^t]\, dt$$

$$= \int_0^{2\pi} (-\cos^2 t \sin t + \sin^2 t \cos t + e^{3t})\, dt$$

$x = \cos t \to 1 = \cos t$

$0 = \sin t$

$1 = e^t$

$\boxed{t = 0}$

$e^{2\pi} = e^t$

$\boxed{t = 2\pi}$

$u = \cos t$

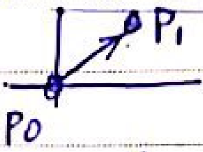
$du = -\sin t \, dt$

$$\int u^2 \, du = \frac{u^3}{3} = \frac{\cos^3 t}{3}\Big|_0^{2\pi} + 0 + \frac{1}{3}e^{3t}\Big|_0^{2\pi} = \frac{1}{3}\left(e^{6\pi} - 1\right)$$

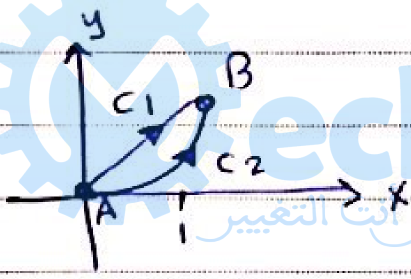
1-1 تكاملات $0 \to 2\pi$ من $\sin^2 t \cos t$, $\cos^2 t \sin t$ بيساوي صفر

6) Given $\vec{F} = [0, xy]$. Evaluate.

$\int_{C_1} \vec{F} \cdot d\vec{r}$, where C_1 is the line segment from $A(0,0)$ to $B(1,1)$



$\int_{C_2} \vec{F} \cdot d\vec{r}$, where C_2 is $y = x^2$ from $A(0,0)$ to $B(1,1)$



a) C_1: $\vec{r}(t) = \vec{r_0} + (\vec{r_1} - \vec{r_0})t$, $0 \le t \le 1$

$\vec{r}(t) = [0,0] + ([1,1] - [0,0])t = [\underset{x}{t}, \underset{y}{t}]$

$$\int_0^1 [0, t^2] \cdot [1,1] \, dt = \int_0^1 t^2 \, dt = \boxed{\frac{1}{3}}$$

C_2 is $y = x^2$ $\quad \vec{r}(t) = [t, t^2] \quad 0 \le t \le 1$

$$\int_0^1 [0, t^3] \cdot [1, 2t] \, dt = \int_0^1 2t^4 \, dt = \boxed{\frac{2}{5}}$$

Path dependence!

Path independence of line integral (ex 3-19 | 3-19)

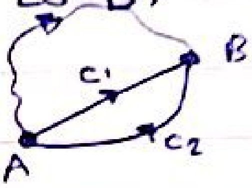
Given $\vec{F} = [F_1, F_2, F_3]$: F_1, F_2, F_3 are continuous

A line integral

$$\int_C \vec{F}\cdot d\vec{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz \qquad (1) \quad (d\vec{r} = [dx, dy, dz])$$

is called path independent if it has the same value for all paths C (from A to B)

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_3} \vec{F}\cdot d\vec{r}$$



Theorem "1" A line integral (1) with continuous F_1, F_2, F_3 in some domain is path independent if and only if $\vec{F}$ is the gradient of some f.

That is $\boxed{\vec{F} = \nabla f}$

$$\vec{F} = [F_1, F_2, F_3] = [f_x, f_y, f_z]$$

$$F_1 = f_x \qquad F_2 = f_y \qquad F_3 = f_z$$

The field (function) $\vec{F}$ is called a conservative field, f is called a potential (function)

The value of (1) is

$$\int_C \vec{F}\cdot d\vec{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz = f(B) - f(A)$$

1) Evaluate $\int_{(0,0,0)}^{(2,2,2)} 2x\,dx + 2y\,dy + 4z\,dz$

$\vec{F} = \nabla f \longrightarrow F_1 = 2x = f_x \qquad F_2 = f_y = 2y \qquad F_3 = f_z = 4z$

$f(x,y,z) = \int 2x\,dx = \boxed{x^2 + g(y,z)}$

$f_y = 0 + \dfrac{\partial g(y,z)}{\partial y} = f_y = 2y$

$\dfrac{\partial g}{\partial y} = 2y \Rightarrow g(y,z) = \int 2y\,dy = \boxed{y^2 + h(z)}$

$f(x,y,z) = x^2 + y^2 + h(z)$ Find $h(z)$

$f_z = 0 + 0 + h'(z) = 4z \rightarrow h'(z) = 4z$

$h(z) = 4\int z\,dz = 2z^2 + k$, $\boxed{f(x,y,z) = x^2 + y^2 + 2z^2 + k}$

$\int_{(0,0,0)}^{(2,2,2)} 2x\,dx + 2y\,dy + 4z\,dz = f(2,2,2) - f(0,0,0)$

$= 4 + 4 + 8 + k - k = 16$

If $\circ$ is a closed curve, denote $\oint_C \vec{F} \cdot d\vec{r}$

Therm 2. The line integral is path independent if, and only if its value around every closed path C is zero.

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

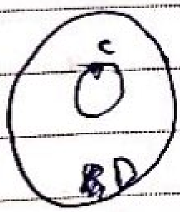
ex 2. $\oint_C 2x\,dx + 2y\,dy + 4z\,dz$, where $C: [\cos t, \sin t, 0]$

(circle)

$\vec{F} = [2x, 2y, 4z]$ is conservative

$$\oint_C 2x\,dx + 2y\,dy + 4z\,dz = \underline{f(B) - f(A)} = \text{zero}$$

* Therm 3. Simply connected region



multiply connected region

no holes

* اللي فوق ال simply

* let F_1, F_2, F_3 be continuous functions and have continuous partial derivatives in (D) in space

Then

(a) If $\vec{F} = [F_1, F_2, F_3]$ is conservative, then

$$\boxed{\text{curl } \vec{F} = \vec{0}}$$

If Curl $\vec{F} = \vec{0}$ and D is simply connected

connected $\vec{F}$ is conservative.

This is test for a conservative field.

$\vec{F} = [2x, 2y, 4z]$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & 4z \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(0) = \vec{0}$$

$\vec{F}$ is conserv. field $\rightarrow$ $\vec{F} = \nabla f$

check dot

شلون اعرف conservative ولا لا

$\vec{F} = [F_1, F_2, F_3]$, F_1, F_2, F_3 are continuous in (D)

$$\int_C \vec{F} \cdot d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz \quad (1)$$

1- A line integral (1) is Path independent

2- $\vec{F} = \nabla f$ [$\hookrightarrow \vec{F}$ is conservative field f is a Potential]

3. $\oint_C \vec{F} \cdot d\vec{r} = 0$ (C is any closed Path).

4. If Curl $\vec{F} = 0$ and region (D) is simply connected

$\rightarrow$ $F = \nabla f$ (line integral (1) is Path independent.

$d\vec{r} = [dx, dy, dz]$

$\vec{F}\cdot d\vec{r} = [F_1, F_2, F_3]\cdot[dx, dy, dz] = F_1 dx + F_2 dy + F_3 dz$

$= \underbrace{f_x\,dx + f_y\,dy + f_z\,dz = df}_{\text{exact differantion}}$

If $\vec{F} = \nabla f \Rightarrow F_1 = f_x,\ F_2 = f_y,\ F_3 = f_z$

If $\vec{F} = \nabla f$ ($\vec{F}$ is conservative f.) $\Rightarrow$ $\vec{F}\cdot d\vec{r}$ is exact diff.

check (curl $\vec{F} = \vec{0}$)

Example. 2: Show that the line integral is path independent (or) ($\vec{F} = \nabla f$) (or) exact diff.

(or) form under the integral is exact)

$$\int_{(0,0,0)}^{(1,1,1)} e^y\,dx + (xe^y - e^z)\,dy - ye^z\,dz$$

, Then evaluate the integral.

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & (xe^y - e^z) & -ye^z \end{vmatrix} = \hat{i}(-e^z + e^z) - \hat{j}(0) + \hat{k}(e^y - e^y)$$

$= \vec{0}$

$\vec{F} = \nabla f$ (or the line integral is Path ind, form is exact diff, $\vec{F}$ is cons. field.

find f : $[F_1, F_2, F_3] = [f_x, f_y, f_z]$ (Partial)

$F_1 = f_x = e^y$

$F_2 = f_y = xe^y - e^z \Rightarrow f(x,y,z) = \int e^y dx = e^y x + h(y,z)$

$F_3 = f_z = -y e^z$

Find $h(y,z)$

$$f_y = xe^y + \frac{\partial h}{\partial y} = xe^y - e^z$$

$$\frac{\partial h}{\partial y} = -e^z \to h(y,z) = \int -e^z dz = -y e^z + g(z)$$

$$f(x,y,z) = e^y x \, - y e^z + g(z)$$

Find $g(z)$ $\quad f_z = \cancel{-ye^z} + g'(z) = \cancel{-ye^z}$ يلزم في تشطيب

اذا ما تشطب يكون في غلط.

$$g'(z) = 0 \Rightarrow g(z) = k \text{ const}$$

$$\boxed{f(x,y,z) = e^y x - y e^z + k}$$

check $\frac{\partial}{\partial x}$ $\quad \frac{\partial}{\partial y}$ $\quad \frac{\partial}{\partial z}$

$$\int_{(0,0,0)}^{(1,1,1)} e^y dx + (xe^y - e^z)\,dy - ye^z dz = f(1,1,1) - f(0,0,0)$$

$$= e - e + k - k$$

$$= 0$$

NO.

طريقة ثانية :

$F_1 = f_x = e^y \Rightarrow x e^y + h(y,z)$

$F_2 = f_y = x e^y - e^z \Rightarrow x e^y - e^z y + g(x,z)$

$F_3 = f_z = -y e^z \Rightarrow -y e^z + p(x,y)$

تكامل بالنسبة لـ x ، وبالنسبة لـ y وبالنسبة لـ z

وبعدها بنوخذ الـ Intersection (اشي ما كان موجود بالكل فيه)

يعني مرة $(x_2, y, z / x, y)$

$$x e^y - e^z y + k = f(x,y,z)$$

Line integral in the plane.

$$\vec{F}(x,y) = [F_1(x,y), F_2(x,y)]$$

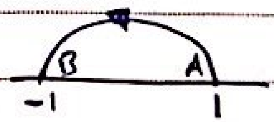
$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(x,y) & F_2(x,y) & 0 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \vec{0}$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$ Non zero. stop.

ex. 3] let $\vec{F}(x,y) = e^y \hat{i} + x e^y \hat{j}$ denote a force

a) Verify that $\vec{F}$ is conservative

b) find the work done by $\vec{F}$ on a particle that moves from $(1,0)$ to $(-1,0)$

$\frac{0}{2}$ along the semicircular path 

(a) $\frac{\partial F_2}{\partial x} = e^y = \frac{\partial F_1}{\partial y} = e^y \Rightarrow \vec{F} = \nabla f$

$$\int_{(1,0)}^{(-1,0)} e^y dx + x e^y dy = f(-1,0) - f(1,0) = -1 + k - 1 - k$$
$$= -2$$

$F_1 = f_x = e^y \Rightarrow f(x,y) = \int e^y \, dx = x e^y + g(y)$

$f_y = x e^y + g'(y) = x e^y$

$g'(y) = 0 \quad g(y) = k$

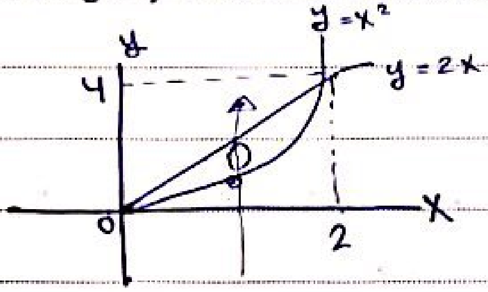
$\therefore f(x,y) = x e^y + k$

Double Integrals ex (1-9)

$$\iint_D f(x,y)\,dA$$

dA = $dx\,dy$ = $dy\,dx$

1) $\iint_D (x^2+y^2)\,dA$, where



$$\iint_D (x^2+y^2)\,dA = \int_0^2\int_{x^2}^{2x} (x^2+y^2)\,dy\,dx$$

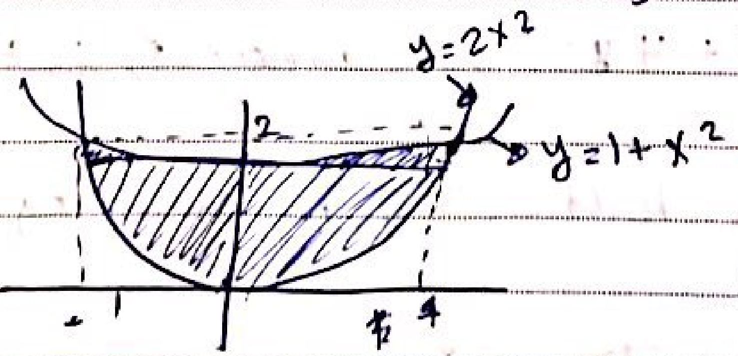
$$= \int_0^2 \left. x^2y+\frac{y^3}{3}\right|_{x^2}^{2x} dx = \int_0^2 2x^3+\frac{8x^3}{3}-x^4-\frac{x^6}{3}\,dx = \dots = \frac{216}{35}$$

$y=x^2 \rightarrow x=\sqrt{y}$

or $\iint_D (x^2+y^2)\,dA = \int_0^4\int_{y/2}^{\sqrt{y}} (x^2+y^2)\,dx\,dy = \int_0^4 \left.\left(\frac{x^3}{3}+y^2x\right)\right|_{y/2}^{\sqrt{y}} dy$

$$= \int_0^4 \left(\frac{(\sqrt{y})^3}{3} + y^2\sqrt{y} - \frac{y^3}{8\cdot 3} - \frac{y^3}{2}\right) dy = \frac{216}{35}$$

2)



$$\int_{2x^2}^{1+x^2} dy\,dx$$

Application of a double integral.

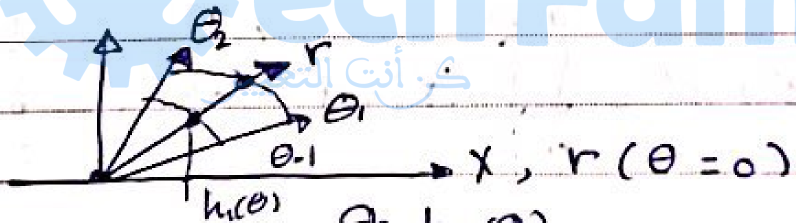
1. The area of the region D

$$\boxed{\text{Area of } D = \iint_D dx\,dy} \quad (f(x,y)=1)$$

Polar coordinates

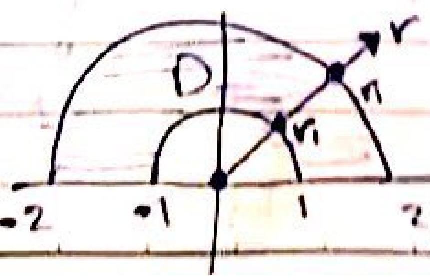
$x = r\cos\theta$
$y = r\sin\theta$, $\boxed{dx\,dy = r\,dr\,d\theta}$

$$\iint_D f(x,y)\,dx\,dy = \int_{\theta_1}^{\theta_2}\int f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$



$$\iint_D f(x,y)\,dx\,dy = \int_{\theta_1}^{\theta_2}\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Evaluate $\iint_D (3x+4y^2)\,dA$ where D is the region in upper half_Plane bounded by $x^2+y^2=1$ and $x^2+y^2=4$



$\rightarrow$

$$\iint_D (3x+4y^2)\,dA = \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta$$
$$= \dots \frac{15}{2}\pi$$

$x^2+y^2=1 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 1$
$\boxed{r=1}$

$x^2+y^2=4 \Rightarrow \boxed{r=2}$

$* \ \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\approx \frac{15\pi}{2}$

* Section 10.4

Green's theorem in the plane (ex (1–10))

Th. Let R be a closed founded region in the x-y plane → (Green's [illegible]), whose boundary C is a positively oriented piecewise smooth closed curve. If $F_1(x,y)$ and $F_2(x,y)$ have continious partial derivatives on an open region containing R, then

Green's Th
$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy = \oint_C F_1\,dx + F_2\,dy$$

R is on the left!

11'4

NO. Green's Theorem

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy = \oint F_1\,dx + F_2\,dy$$

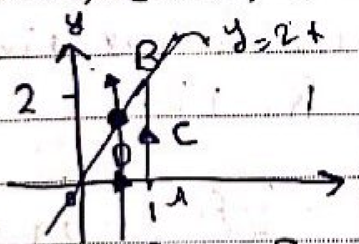
* $\dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y}$: Test for conservative field

$$\iint_D = \oint_C \vec{F} \cdot d\vec{r} = \vec{0}$$

Let $\vec{F} = [F_1, F_2]$; the vector form is

$$\iint_D (\text{curl}\,\vec{F}) \cdot \hat{k}\,dx\,dy = \oint \vec{F} \cdot d\vec{r}$$

ex1 Evaluate $\oint_C xy\,dx + x^2y^3\,dy$, where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ (ccw oriented)



الرسمة مطلوبة.

لما $x=1 \leftarrow y=2$

معناها $y=2x$

1) method $\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$

$C_1 : OA \quad C_2 : AB \quad C_3 : BO$

$\dfrac{\partial F_2}{\partial x} = 2xy^3 \qquad \dfrac{\partial F_1}{\partial y} = x$

$$\oint_C = \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$$

NO Examples.

$$= \int_0^1 \frac{xy^4}{2} - xy \Big|_0^2 \, dx = \int_0^1 8x^5 - 2x^2 \, dx = \frac{8x^6}{6} - \frac{2x^3}{3} \Big|_0^1$$

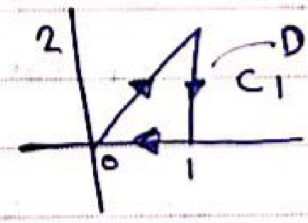
$$= \frac{4}{3} - \frac{2}{3} = \frac{2}{3} .$$

Ex 2:- Evaluate $\oint_{C_1} xy\,dx + x^2y^3\,dy$, where C:-

Green Theorem

<u>close</u> $\oint$ لازم

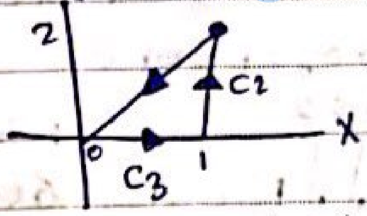
لازم D inside نعكس الإشارة .



$$\oint_{C_1} = -\int_C = -\frac{2}{3}$$

Ex 3:- Evaluate $\int_{C_2} xy\,dx + x^2y^3\,dy$

C_2: $(0,1) \to (1,2) \to (0,0)$



Introduce the line segment C_3: from $(0,0)$ to $(1,0)$

$C_2 + C_3 = C$ "The last step"

$$\oint_C = \int_{C_2} + \int_{C_3} \Rightarrow \int_{C_2} = \underbrace{\oint}_{ex1} - \int_{C_3} = \frac{2}{3} - 0 = \frac{2}{3}$$

Evaluate $\int_{C_3} xy\,dx + x^2y^3\,dy$

C_3: $\vec{r}(t) = \vec{r}_0 + (\vec{r}_1 - \vec{r}_0)t = [0,0] + ([1,0] - [0,0])t$

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FIVE APPLE

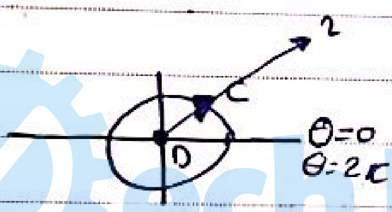
Scanned with CamScanner

$= [\underbrace{t}_{x}, \underbrace{0}_{y}], \quad 0 \le t \le 1$

$= \int_0^1 0 = 0$

ex4) Evaluate $\oint_C y^3\,dx - x^3\,dy$, where C is
ccw → a positively oriented circle of radius 2
centered at the origin. (Polar جي O لأنو)

Sol:



$C: x^2 + y^2 = 2^2$

$\frac{\partial F_2}{\partial x} = -3x^2 \qquad \frac{\partial f_1}{\partial y} = 3y^2$

$\oint_C = \iint_D (\underbrace{-3x^2 - 3y^2}_{-3(x^2+y^2)})\,dx\,dy$

$x = r\cos\theta \quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$= -3 \int_0^{2\pi}\int_0^2 r^2\cos^2\theta + r^2\sin^2\theta \quad r\,dr\,d\theta$

$= -3 \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta$

$\frac{r^4}{4}\Big|_0^2$

$\int 4$

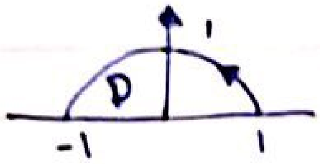
$4\theta\Big|_0^{2\pi}$

$-3 \times 8\pi \rightarrow = -24\pi$

Examples

ex5) Let $\vec{F}(x,y) = [\, e^{x^2} + 4y \,,\ \cos y^4 + 5x \,]$

$$\oint_C \vec{F} \cdot d\vec{r}$$



* المشتقة بـ e ومتاها.

$$\frac{\partial F_2}{\partial x} = 5 \qquad \frac{\partial F_1}{\partial y} = 4$$

$$\oint_C = \iint_D (5-4)\, dx\, dy = \iint_D dx\, dy = \text{Area of } D$$

$$= \frac{1}{2}\pi (1)^2$$

$$= \frac{1}{2}\pi .$$

10.5 : Surfaces (ex 1-8, 14-19)

$z = g(x,y)$ or $g(x,y,z) = 0$

$z = \sqrt{a^2 - x^2 - y^2}$ $\qquad$ $x^2 + y^2 + z^2 = a^2$

upper semisphere $\qquad$ sphere

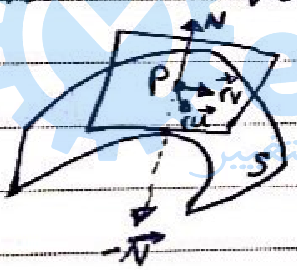
A parametric representation of a surfaces in space is of the form.

$\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$

$(u,v) \in D$

S is oriented, it has two normal vectors. $+, -$

A normal vector $\vec{N} = \vec{r_u} \times \vec{r_v}$, if $(\vec{r_u} \times \vec{r_v} \neq \vec{0})$

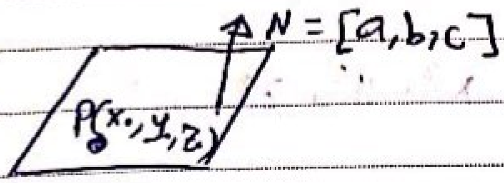


+ The Unit normal vector $\hat{n} = \dfrac{\vec{N}}{|\vec{N}|}$

$\vec{N} = \text{grad}\,[g(x,y,z)]$

– Planes

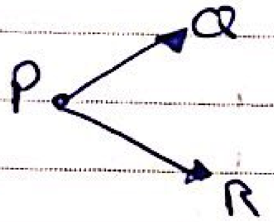
$ax + by + cz + d = 0$ (given equ of the Plane)



or $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

Ex:- Find an equation of the plane passes through the points $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$

Assume this Point



$\vec{PQ} = [2,-4,4]$

$\vec{PR} = [4,-1,-2]$

$$\vec{N} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$= 12i + 20j + 14k$$

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

$$6x + 10y + 7z = 50 \qquad N_1 \ (+)$$

$$-6x - 10y - 7z = -50 \qquad N_2 \ (-)$$

Ex2:- Find a parametric representation of the plane.

let $x = x$, $y = y$, $z = 6x + 5y - 4$

$$\vec{r}(x,y) = [x, y, 6x+5y-4]$$

Ex 3:-

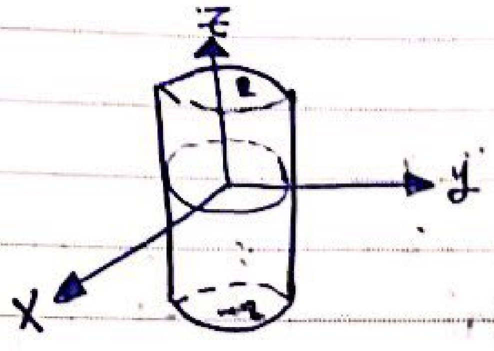
$6x + 10y + 7z = 50$, $x = x$, $y = y$

$$z = \frac{1}{7}(50 - 6x - 10y)$$

$$\vec{r}(x,y) = [x, y, \frac{1}{7}(50 - 6x - 10y)]$$

* Cylinder

$x^2+y^2=a^2$, $-2\le z\le 2$



* use cylindrical coordinates

$x=r\cos\theta$

$y=r\sin\theta$

$z=z$

let $x=a\cos v$

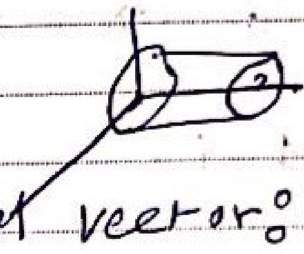
$y=a\sin v$

$z=u$

$\vec{r}(u,v)=[a\cos v,\ a\sin v,\ u]$, $0\le v\le 2\pi$

$-2\le u\le 2$

$\frac{x^2}{4}+\frac{z^2}{9}=1$, $0\le y\le 2$



$\vec{r}(u,v)=[\underbrace{2\cos u}_{x},\ \underbrace{v}_{y},\ \underbrace{3\sin u}_{z}]$

Find a normal vector:

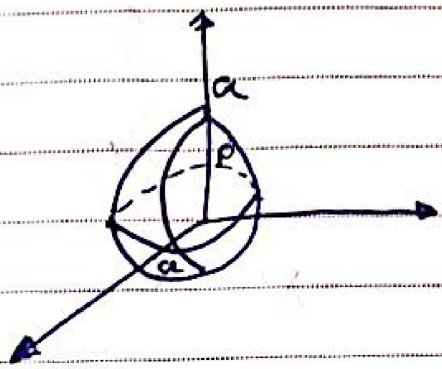
$\vec{r}_u=[-3\sin u,0,3\cos u]$ $\vec{r}_v=[0,1,0]$

$\vec{N}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix} i & j & k\\ -2\sin u & 0 & 3\cos u\\ 0 & 1 & 0\end{vmatrix}=(-3\cos u)\hat{i}+\hat{k}(-2\sin u)$

$=[-3\cos u,0,-2\sin u]$

Sphere

$x^2 + y^2 + z^2 = a^2$



Use Spherical coordinate

$x = \rho \cos\theta \sin\phi$

$y = \rho \sin\theta \sin\phi$

$z = \rho \cos\phi$

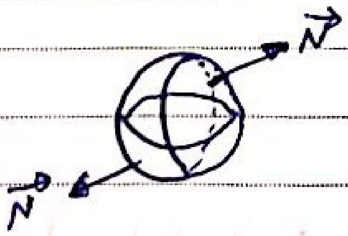
Parametrical representation of a sphere:

$\vec{r}(u,v) = [a \cos v \sin u, a \sin v \sin u, a \cos u]$

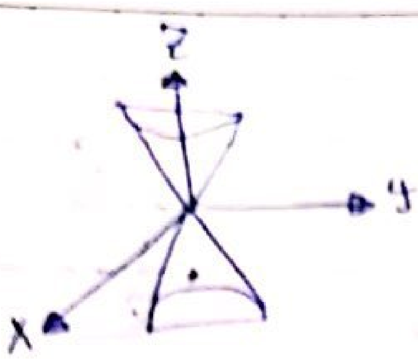
$0 \leq v \leq 2\pi$

$0 \leq u \leq \pi$ لو لـ upper بتصير $\frac{\pi}{2}$

If S is a closed surface, then a Positive Orientation is Outward orient.



Cone!



$z^2 = x^2 + y^2$

$-2 \le z \le 2$

$z = \sqrt{x^2+y^2}$, $0 \le z \le 2$

$$\vec{r}(u,v) = [\underbrace{u\cos v}_{x}, \underbrace{u\sin v}_{y}, \underbrace{u}_{z}]$$

$0 \le v \le 2\pi$, $0 \le u \le 2$

or let $x = x$

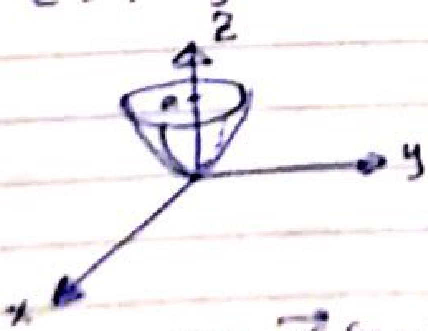
$y = y$

$z = \sqrt{x^2+y^2}$

$\vec{r}(x,y) = [x, y, \sqrt{x^2+y^2}]$

Paraboloid

$z = x^2+y^2$, $0 \le z \le 9$



$$\vec{r}(u,v) = [\underbrace{u\cos v}_{x}, \underbrace{u\sin v}_{y}, \underbrace{u^2}_{z}]$$

$0 \le v \le 2\pi$, $0 \le u \le 3$

or $\vec{r}(x,y) = [x, y, x^2+y^2]$.

Surface Integral.

(10.6) ex [1–10, 12–6]

Given a surface S : $\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$

S is a piecewise smooth surface, so that it has a normal vector $\vec{N} = \vec{r}_u \times \vec{r}_v$ $(\vec{N} \neq \vec{0})$

→ a normal unit vector $\hat{n} = \dfrac{\vec{N}}{|\vec{N}|}$

A surface integral of a vector function $\vec{F}$ over S is of the form.

$$\iint_S \vec{F} \cdot \hat{n}\, dA \longrightarrow dA \text{ is element of } S$$

$$dA \approx |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

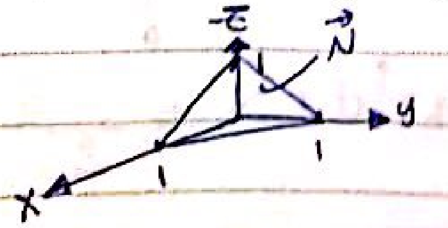
$$\vec{F} \cdot \hat{n}\, dA = \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \cancel{|\vec{r}_u \times \vec{r}_v|}\, du\, dv$$

$$\boxed{\iint_S \vec{F} \cdot \hat{n}\, dA = \iint_R \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv}$$

R is the projection of S on to the u-v plane.

ex 1 : Evaluate surface integral (flux) $\iint_S \vec{F} \cdot \hat{n}\, dA$, where $\vec{F} = [x^2, 0, 3y^2]$ and S is the portion of the plane $x + y + z = 1$ in the first octant

$(+x, +y)$



$0 = z, y$ plane

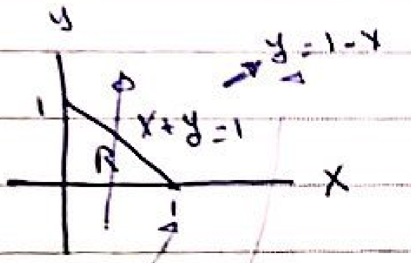
$x = 1$ [illegible]

Parameterize S

مجموع معامل $\boxed{1 = z}$

let $x = x \quad y = y \quad z = 1 - x - y$

$\vec{r}(x,y) = [x, y, 1-x-y]$



بفصل z عشان نطلع x,y
بعمل Projection لـ $\underline{S}$ عشان نطلع
$\underline{R}$ نطلع.

$0 \le x \le 1 \rightarrow 0 \le y \le 1-x$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = i(1) - \hat{j}(-1) + \hat{k}(1) = [1,1,1]$$

x معامل y معامل z معامل

Note: or $\vec{N} = \operatorname{grad}(x+y+z-1) = [1,1,1]$

*

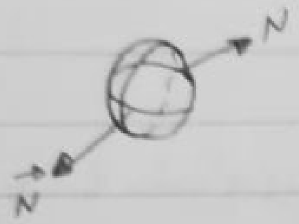
$$\iint_S \vec{F} \cdot n \, dA = \int_0^1 \int_0^{1-x} [x^2, 0, 3y^2] \cdot [1,1,1] \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} [x^2 + 3y^2] \, dy \, dx = \int_0^1 \left[x^2 y + y^3\right]\Big|_0^{1-x} dx$$

$$= \int_0^1 x^2(1-x) + (1-x)^3 \, dx \cdots = \frac{1}{3}$$

ex 2. Find $\iint_S \vec{F}\cdot\hat{n}\, dA$; where $\vec{F} = z\hat{k} = [0,0,z]$

S is the outward sphere $x^2+y^2+z^2=a^2$



1. $\vec{r}(u,v) = [a\cos v\sin u,\ a\sin v\sin u,\ a\cos u]$

$0 \le v \le 2\pi$ $\quad 0 \le u \le \pi$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (a\cos v\cos u) & (a\sin v\cos u) & (-a\sin u) \\ (-a\sin v\sin u) & (a\cos v\sin u) & (0) \end{vmatrix}$$

$$\hat{i}(a^2\cos v\sin^2 u) - \hat{j}(-a^2\sin(v)\sin^2 u) + \hat{k}(a^2\cos^2 v\cos u\sin u + a^2\sin^2 v\cos u\sin u)$$

$a^2\cos u\sin u$ عامل مشترك

$$= a\sin u\,[\hat{i}(a\cos v\sin u) + \hat{j}(a\sin v\sin u) + \hat{k}(a\cos u)]$$

$$= \underset{\ge 0}{a}\ \underset{\ge 0}{\sin u}\,[\hat{i}(a\cos v\sin u) + \hat{j}(a\sin v\sin u) + \hat{k}(a\cos u)]$$

$\vec{N} = a\sin u\,[x, y, z]$

$\vec{N}$ is in the direction of the Position vector $[x, y, z] \Rightarrow \vec{N}$ is Outward normal.

$$\iint_S \vec{F}\cdot\hat{n}\,dA = \int_0^{2\pi}\int_0^{\pi} (0, 0, a\cos u)\cdot[a^2\sin^2 u\sin v, a^2\sin^2 u\cos v, a^2\sin u\cos u]\,du\,dv$$

$$\int_0^{2\pi}\int_0^{\pi} a^3\cos^2 u\sin u\,du\,dv = \frac{4\pi a^3}{3}$$

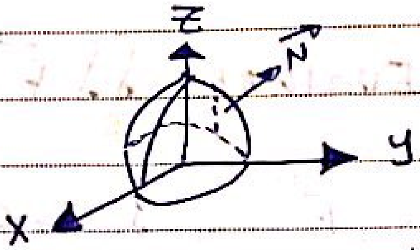
نهاية مادة ال $\boxed{\text{First}}$

ex 3)) Evaluate $\iint_S \vec{F} \cdot \hat{n} \, dA$, where

$\vec{F}(x, y, z) = [y, x, z]$ and S is the Paraboloid $z = 1 - x^2 - y^2$ that lines above the x-y-Plane. $z = 0$

Intersection $z = 1 - x^2 - y^2$, $z = 0$

$\Rightarrow x^2 + y^2 = 1$



Paraboloid, Cone } x = r

sphere, cylind.] u v

(grad) ← $\vec{N}$ لازم أستخدم بدلالة x,y ← Parametrization أول اشي

① $x = x$, $y = y$; $z = 1 - x^2 - y^2$

Cross. Prd. ↑ لو (u,v)

$\vec{r}(x, y) = [x, y, 1 - x^2 - y^2]$

$$\iint_S \vec{F} \cdot \hat{n} \, dA = \iint_R \vec{F}(\vec{r}(x,y)) \cdot \vec{N} \, dx\,dy$$

dx dy   grad

$\vec{N} = \text{grad}(z + x^2 + y^2 - 1) = [2x, 2y, 1]$

$$\iint_S \vec{F} \cdot \hat{n} \, dA = \iint_R [y, x, 1 - x^2 - y^2] \cdot [2x, 2y, 1] \, dx\,dy$$

$$= \iint_{R: x^2+y^2 \le 1} (4xy + 1 - \underbrace{x^2 - y^2}) \, dx\,dy = \int_0^{2\pi}\int_0^1 (4r^2 \underbrace{\cos\theta \sin\theta}_{2\sin 2\theta = 0} + 1 - r^2) r \, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r - r^3)\,dr\,d\theta = (2\pi)\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}$$

$x = r\cos\theta$ $\quad y = r\sin\theta$ $\quad dx\,dy = r\,dr\,d\theta$

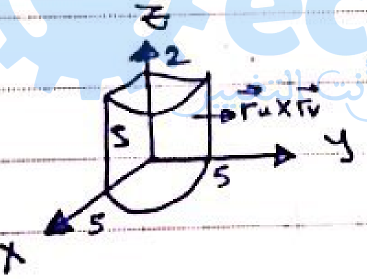
$0 \le r \le 1$

$0 \le \theta \le 2\pi$

Ex 4) Evaluate $\iint_S \vec F \cdot \hat n \, dA$

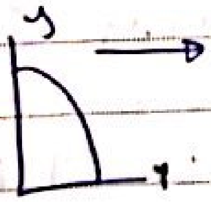
where $\vec F = [y, z, e^x]$ and $S: x^2 + y^2 = 25$ is the part of Cylinder

, $x \ge 0$, $y \ge 0$, $0 \le z \le 2$



① $\vec r(u,v) = [\underbrace{5\cos u}_{x}, \underbrace{5\sin u}_{y}, \underbrace{v}_{z}]$

(u,v) بال Projection على x



$0 \le u \le \frac{\pi}{2}$ $\quad , V = z \ (0 \le V \le 2)$

$$\vec r_u \times \vec r_v = \begin{vmatrix} \hat i & \hat j & \hat k \\ -5\sin u & 5\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat i(5\cos u) - \hat j(-5\sin u)$$

$= [5\cos u, 5\sin u, 0]$

$$\iint_S \vec{F}\cdot\hat{n}\,dA = \int_0^{\frac{\pi}{2}}\int_0^{2} [5\sin u, v, e^{5\cos u}] \cdot [5\cos u, 5\sin u, 0]\, dv\, du$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2} (25\sin u\cos u + 5v\sin u)\,dv\,du = 2\cdot\left(\frac{25}{2}\int_0^{\frac{\pi}{2}} \sin 2u\right.$$

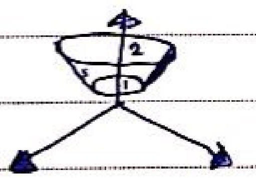
$$+ 5\int_0^{\frac{\pi}{2}} \sin u\,du \int_0^{2} v\,dv = \ldots$$

Surface integral without regard to Orientation

$$\iint_S G(\vec{r})\,dA = \iint_R G(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv$$

$G(\vec{r})$ is a scalar function $\longrightarrow$ Scalar fun.

ex 5)) Evaluate $\iint_S y^2 z^2\,dA$, where S is the Part of the cone $z=\sqrt{x^2+y^2}$ that lies between the Planes $z=1$ and $z=2$



. vector مش Product (.) لاحظ

$$\vec{r}(u,v) = [\underbrace{u\cos v}_{x}, \underbrace{u\sin v}_{y}, \underbrace{u}_{z}]$$

$$1 \le u \le 2 \qquad 0 \le v \le 2\pi$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \hat{i}(-u\cos v) - \hat{j}(u\sin v) + \hat{k}(\underbrace{u\cos^2 v + u\sin^2 v}_{u})$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = u\sqrt{2}$$

Divergence theorem (9-18)

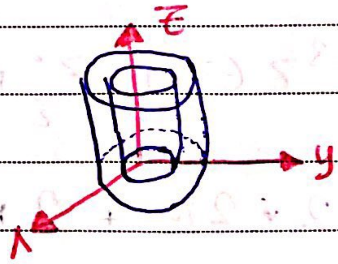
Th: let (T) be closed bounded region in the space whose boundary is a piecewise smooth orientable surfaces.
{S is outward oriented}, let $\vec{F}(x, y, z)$ be a continious vector function and it has continious

first partial derivatives is some domain, then

$$\iint_S \vec{F} \cdot \hat{n} \, dA = \iiint_T \operatorname{div} \vec{F} \, dV$$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Ex 1) Find $\iint_S \vec{F} \cdot \vec{n} \, dA$, where $\vec{F} = [x^3, y^3, z^2]$
(S) is the surface of the region closed by the cylinder $x^2 + y^2 = 9$, and planes $z = 0$ and $z = 2$



Solution:-

$$\operatorname{div} \vec{F} = 3x^2 + 3y^2 + 2z$$

$R = x^2 + y^2 \leq 9$ $\qquad 0 \leq r \leq 3$

$x = r\cos\theta$ $\qquad 0 \leq \theta \leq 2\pi$

$y = r\sin\theta$ $\qquad dxdy = rdrd\theta$

$$= \int_0^{2\pi}\int_0^{3}\int_0^{2}(3r^2\cos^2\theta + 3r^2\sin^2\theta + 2z)\, r\, dz\,dr\,d\theta$$

$$* \int_0^{2\pi}\int_0^{3}\int_0^{2}(3r^2 + 2zr)\, dz\,dr\,d\theta$$

$$= 3r^2z + z^2r \Big|_0^2$$

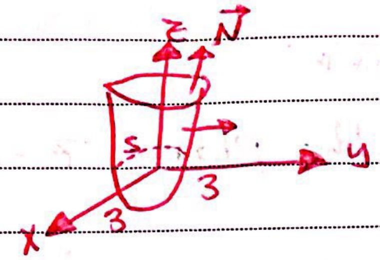
$$= \frac{3r^3}{3}z + \frac{z^2r^2}{2} \Big|_0^3$$

$$r^3z\theta + \frac{z^2}{2}r^2\theta \Big|_0^{2\pi}$$

$$(27 * 2 * 2\pi + 2 * 9 * 2\pi) = (279\pi)$$

Ex 2 Evaluate $\iint_{S_1} \vec{F} \cdot d\vec{r}$

$S_1 : x^2 + y^2 = 9 \quad , \quad z = 0$



is not closed

Introduce $S_2 : z = 2$

$S = S_1 + S_2$ (from ex1)

$$\iint_{S_1} = \iint_{\underbrace{S}_{\underbrace{(S_1+S_2)}_{ex_1}}} - \iint_{S_2} = 279\pi -$$

$\iint_{S_2} \vec{F} \cdot \hat{n} \, dA$, Parametrize

$\vec{r}(x,y) = [x, y, 2]$; $\vec{N} = [0,0,1]$

$x^2 + y^2 = 9$

or $\vec{N} = \text{grad}(z-2) = [0,0,1]$

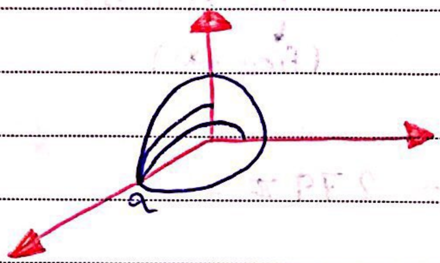
$$\iint_{R: x^2+y^2 \le 9} [x^3, y^3, 4] \cdot [0,0,1] \, dx\,dy = 4 \iint_R dx\,dy$$

$= 4$ Area of $R = 4(\pi 9) = 36\pi$

Ex3 Evaluate $\iint_S \vec{F}\cdot\hat{n}\, dA$, where

$\vec{F} = [x^3, y^3, z^3]$ S is the surface of the region that is enclosed by the hemisphere.
div Th

Sol: $z = \sqrt{a^2 - x^2 - y^2}$ and the plane $z = 0$



① $\text{div}\,\vec{F} = 3x^2 + 3y^2 + 3z^2$

$$\iint_S \vec{F}\cdot\hat{n}\, dA = \iiint_T (3x^2 + 3y^2 + 3z^2)\, dV \quad (*)$$

because himesphere, use the spherical coord.

$x = \rho\cos\theta\sin\phi$

$y = \rho\sin\theta\sin\phi$ $\qquad 0 \le \rho \le a$

$z = \rho\cos\phi$ $\qquad 0 \le \theta \le 2\pi$

$dV = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$ $\qquad 0 \le \phi \le \pi/2$

$$(*) \rightarrow = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{a} 3\left((\rho^2\cos^2\theta\sin^2\phi) + (\rho^2\sin^2\theta\sin^2\phi) + (\rho^2\cos^2\phi)\right)$$
$$\times\cos^2\phi\quad \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a} \rho^2(\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \qquad (G)$$

$$= 3(2\pi)(-\cos\phi)\Big|_0^{\frac{\pi}{2}} \frac{\rho^5}{5}\Big|_0^{a}$$

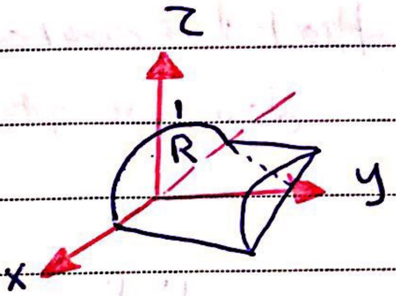
$$= \frac{6\pi a^5}{5}$$

Evaluate $\iint_S \vec{F}\cdot\hat{n}\,dA$, where $\vec{F} = [xy,\ y^2 + e^{xz^2},\ \sin xy]$

S is the surface of the region that is enclosed by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, $y + z = 2$.

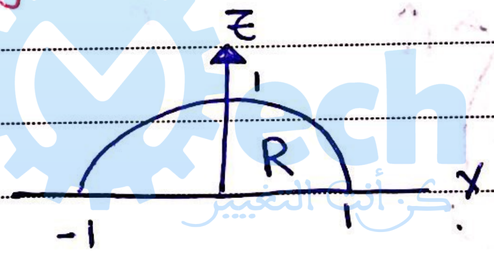
* اذا الفنكشن صعب و معقد ، كله على طريقة div

$\operatorname{div}\vec{F} = y + 2y = 3y$



$$\iint_S \vec{F}\cdot\hat{n}\,dA = \iint_R \int_0^{2-y} 3y\,dy \quad (*)$$

R is projection on to the x-z plane.



$$(*) = \int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{2-z} 3y\,dy\,dz\,dx = 3\int_{-1}^{1}\int_{1}^{1-x^2} \frac{3}{2}(2-z)^2\,dz\,dx$$

Stoke's Theorem (ex 1-10, 13-20)

Theorem : let S be a Picewise smooth Oriented surface in space and let the boundary of S be a Picewise smooth simple closed curve C

let $\vec{F}(x,y,z)$ be contain, Vector function that has contain first Partial derivative is some domain then

$$\iint_S (\operatorname{curl} \vec{F}) \cdot \hat{n} \, dA = \oint_C \vec{F} \cdot d\vec{r}$$

Sol First exam

$$\frac{\partial F_2}{\partial x} = -4xy = \frac{\partial F_1}{\partial y}$$

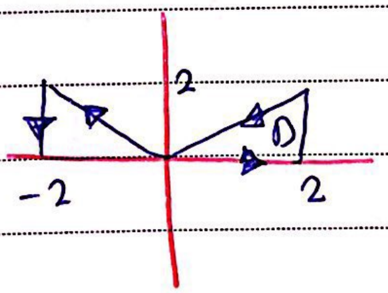
$$f(x,y) = 3x^5 - x^2y^2 + e^y + k$$

$$\int_C \vec{F}\cdot d\vec{r} = f(1,0) - f(-1,0)$$
$$= 3+1+k-(-3+1+k) = 6$$

$$\vec{r}(t) = \left[\frac{t^2}{3}, t, 1\right], \quad 3 \le t \le \sqrt{3}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_3^{\sqrt{3}} \left[-t, \frac{t^2}{3}, \dots\right]\cdot\left[\frac{2t}{3}, 1, 0\right] dt$$

$$= \int_3^{\sqrt{3}} \frac{-1}{3}t^2\, dt = \frac{-1}{9}\left(3\sqrt{3} - 27\right)$$



$$\iint_D (8-5)\, dx\, dy = 3\underbrace{\iint_D dx\, dy}_{\text{Area of } D} = 3\cdot 4$$

$$\iint_S (z + x^2y)\, dA = \iint_R G(r(u,v))\,|r_u \times r_v|\, du\, dv$$

$$\vec{r}(u,v) = (u\cos v, u\sin v, u) \qquad 0 \le u \le 2$$
$$0 \le v \le 2\pi$$

$$|r_u \times r_v| = u\sqrt{2}$$

$$\iint_S (z + x^2 y)\, dA = \int_0^{2\pi}\int_0^{2} \left(u + u^2\cos^2 v\,(u\sin v)\right) u\sqrt{2}\, du\, dv$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^{2} u^2\, du\, dv$$

Sol First exam

$$= \sqrt{2}\,(2\pi)\cdot\frac{1}{2}\cdot 8$$

$$= 8\sqrt{2}\,\pi$$

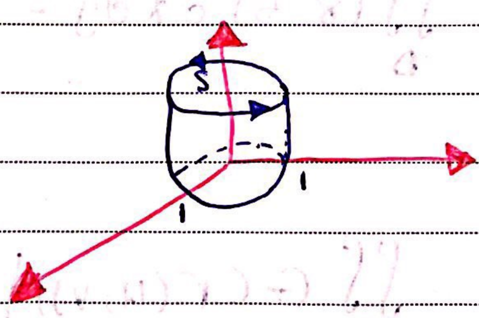
* Stockés theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl } \vec{F})\cdot \hat{n}\, dA$$

Evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (-y^2, x, z^2)$

C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$

C oriented ccw.



NO.

$$\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (-y^2) & x & z^2 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(1+2y)$$

$$= [0, 0, 1+2y]$$

S is the portion of the plane $y + z = 2$

$z = 2 - y$ $\quad \vec{r}(x,y) = [x, y, 2-y]$

$z - 2 + y = 0$

$\vec{N} = \text{grad}(z - 2 + y) = [0, 1, 1]$

R is the Projection of S on to the x-y Plane

$x^2 + y^2 = 1$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \hat{n} \, dA = \iint_R [0, 0, 1+2y] \cdot [0, 1, 1] \, dx \, dy$$

$$= \iint_R (1 + 2y) \, dx \, dy = \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta) \, r \, dr \, d\theta$$

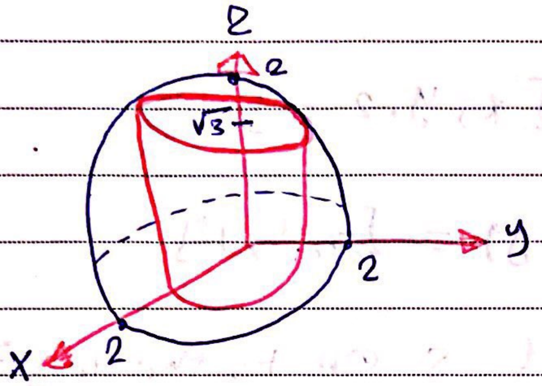
$$= \int_0^{2\pi} \int_0^1 r \, dr \, d\theta + \int_0^{2\pi} \int_0^1 2r^2 \sin\theta \, dr \, d\theta$$

$x = r\cos\theta$ $\qquad = 2\pi \times 1/2 = \pi$

$y = r\sin\theta$

Evaluate $\iint_S (\text{curl}\,\vec{F})\,\hat{n}\,dA$ where $\vec{F} = [-yz, xz, xy]$

and S is the part of the sphere $x^2+y^2+z^2=4$
that lies inside the cylinder $x^2+y^2=1$
and above the x-y plane



Stocke's theorem $\oint_C \vec{F}\,d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,dA$

Find an equation of C

$x^2+y^2+z^2=4 \Rightarrow 1+z^2=4$
$x^2+y^2=1 \qquad z^2=3 \qquad z=\pm\sqrt{3}$

$z=\sqrt{3}\quad , x^2+y^2=1$

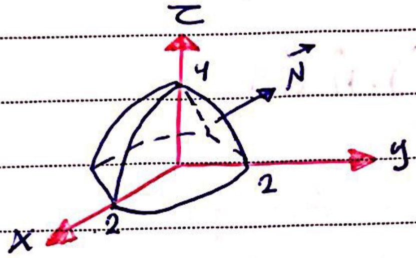
$\iint_S (\text{curl}\,\vec{F})\,\hat{n}\,dA = \oint_C \vec{F}\,d\vec{r} = \int_0^{2\pi} [-\sqrt{3}\sin t, \sqrt{3}\cos t, \cos t\sin t]\cdot[-\sin t, \cos t, 0]\,dt$

$= \int_0^{2\pi} (\sqrt{3}\sin^2 t + \sqrt{3}\cos^2 t)\,dt = \sqrt{3}\int_0^{2\pi} dt = 2\sqrt{3}\pi$

Method 1

$$\oint_C \vec{F}\cdot d\vec{r}$$

$$\vec{F} = [2z, 3x, 5y]$$



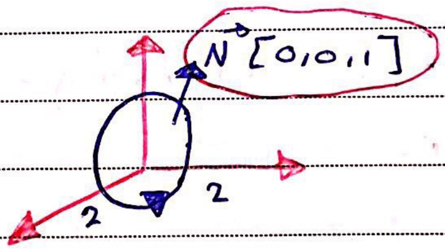
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl } \vec{F})\cdot \vec{N}\, dA = 12\pi$$

$S: z = 4 - x^2 - y^2$, curl $\vec{F} = [5, 2, 3]$

Method 2

Introduce the surface S. S is $z = 0$

$\vec{r}(x, y) = [x, y, 0]$ ccw $\uparrow \vec{N}$



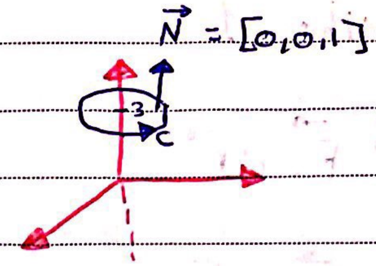
grad(z) $= [0, 0, 1]$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_{S_1} (\text{curl } \vec{F})\cdot \hat{n}\, dA = \iint [5, 2, 3]\cdot[0, 0, 1]\, dx\, dy$$

$R: x^2 + y^2 \le 4$

$$= 3\iint_R dx\, dy = 3 \text{ Area of } R = 3\cdot 4\pi = 12\pi$$

NO. STOCK'S

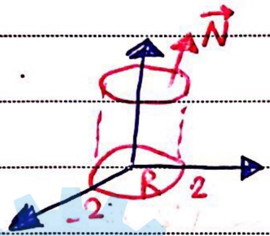


Evaluate $\oint_C \vec{F} \cdot d\vec{r}$

$\vec{F} = [\,y,\ xz^3,\ -zy^3\,]$

$C:\ x^2 + y^2 = 4\ ,\ z = 3$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F}) \cdot \hat{n}\, dA$$



S is the plane $z = 3$

$\vec{N} = [0,\ 0,\ 1]$

$R:\ x^2 + y^2 \leq 4$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz^3 & -zy^3 \end{vmatrix} = (m)\hat{i} - \hat{j}(m) + \hat{k}(z^3 - 1)$$

$$\iint [\,m,\ m,\ z^3 - 1\,] \cdot \boxed{[0,0,1]}\, dx\,dy = \iint_R 26\, dx\,dy = 26 \cdot 4\pi$$

$R:\ x^2 + y^2 \leq 4$

✓

Fourier Analysis

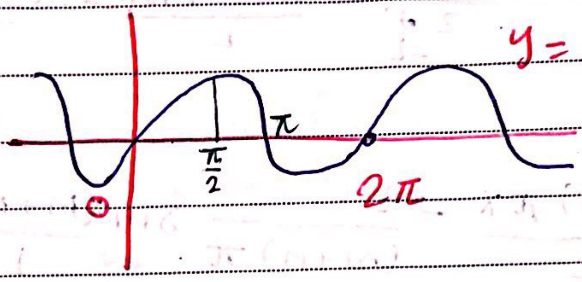
Def A function $f(x)$ is a periodic function if

$$f(x+p) = f(x)$$

دورية لها زمن دوري محدد

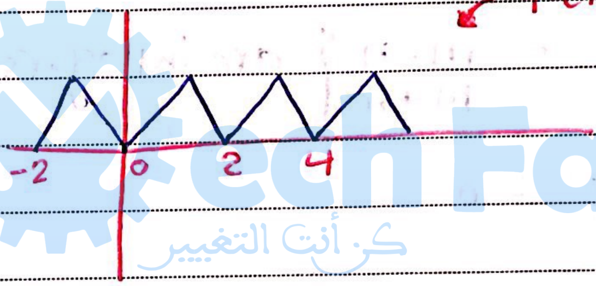
where $p > 0$ is a constant

$P = 2\pi$



$P = 2\pi$

$f(x+p) = f(x)$



If $f(x)$ has Period P, it has Period's $n\pi$, $n = 1, 2, \ldots$

Th. The functions $\sin \frac{n\pi x}{L}$, $\cos \frac{n\pi x}{L}$, where $n = 0, 1, 2$ form an Orthogonal Set on $-L \le x \le L$, that is

$$\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx = \begin{cases} 0, & \text{if } n \ne m \\ L, & \text{if } n = m \end{cases}$$

$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx = \begin{cases} 0, & \text{if } n \ne m \\ L, & \text{if } n = m \end{cases}$$

$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx = 0 \quad \text{for all } n, m$$

1 $\cos\frac{\pi x}{L}$, $\sin\frac{\pi x}{L}$, $\cos\frac{2\pi x}{L}$, $\sin\frac{2\pi x}{L}$, ...

Proof

Let $m \neq n$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(\cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}\right)dx$$

$$= \frac{1}{2}\left[\frac{L}{(m-n)\pi}\sin\frac{(m-n)\pi x}{L} - \frac{L}{(m+n)\pi}\sin\frac{(m+n)\pi x}{L}\right]_{-L}^{L} = 0$$

m, n are integers : $\left.\begin{matrix} m-n \\ m+n \end{matrix}\right\}$ are integers

$$\sin\frac{(m-n)\pi \cancel{L}}{\cancel{L}} = 0$$

$m = n$

(Part 2)

$$\int_{-L}^{L}\left(\sin\frac{n\pi x}{L}\right)^2 dx = \int_{-L}^{L}\frac{1-\cos\frac{2\pi x}{L}}{2}\,dx$$

$$= \frac{1}{2}(L+L) = L$$

A series of the form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}$$

is called a forier Series (F.S).

a_0, a_n, b_n are coefficients of (F.S)

A Series of the form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

is called a Fourier Series for some function $f(x)$

"$\sim$" can be represented

Find coefficient of a Fourier series : a_n, b_n

Multiply (1) by $\cos\frac{m\pi x}{L}$ and integrate from $-L$ to L

$$\sin\frac{m\pi x}{L}\Big|_{-L}^{L} = \sin m\pi$$

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \underbrace{a_0\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx}_{=0} + \sum_{n=1}^{\infty} a_n \int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$$

$$= L$$

$$+ \underbrace{b_n\int \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx}_{=0}$$

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L} = a_n L \quad (n=m)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx = \begin{cases} 0 , & n \neq m \\ L , & n = m \end{cases}$$

where $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \quad (2)$

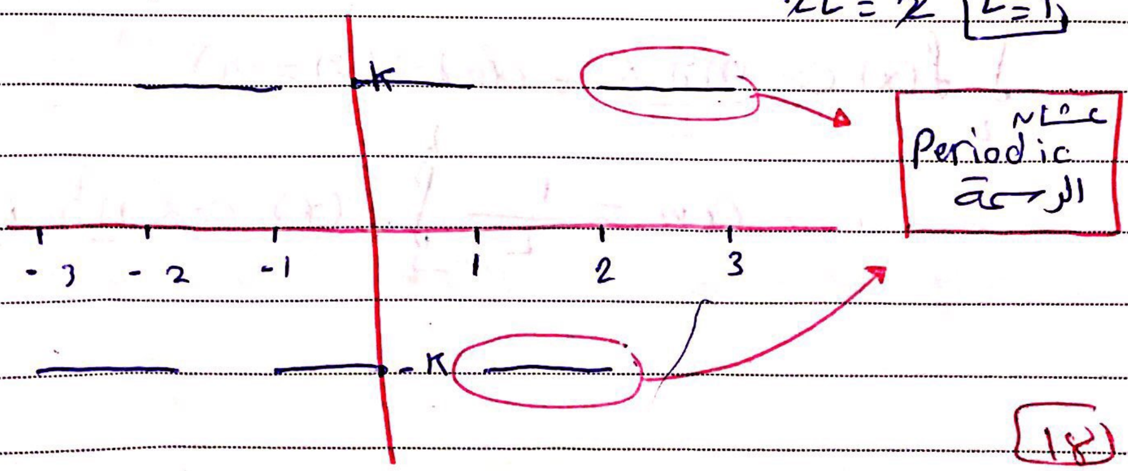
$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \quad (3)$$

$$* \; b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \quad (4)$$

ex 1. Find the Fourier series of $f(x)$

$$f(x) = \begin{cases} -k , & -1 < x < 0 \\ k , & 0 < x < 1 \end{cases}$$

$f(x+2L) = f(x)$ ∴ $f(x)$ is Periodic with Period $2L = 2$ $\boxed{L = 1}$



$$a_0 = \frac{1}{2 \cdot 1} \int_{-1}^{1} f(x)\, dx = \frac{1}{2}\left[\int_{-1}^{0} (-k)\, dx + \int_{0}^{1} k\, dx\right]$$

$$= \frac{1}{2}(-k + k) = 0$$

$$-kx \Big|_{-1}^{0} = 0 \qquad (\text{تكملة التكامل})$$

$$a_n = \frac{1}{1} \int_{-1}^{0} (-k) \cos n\pi x\, dx \int_{0}^{1} k \cos n\pi x\, dx$$

$$= -k \frac{1}{n\pi} \sin n\pi x \Big|_{-1}^{0} + k \frac{1}{n\pi} \sin n\pi x \Big|_{0}^{1} = 0$$

$$b_n = \frac{1}{1}\left[\int_{-1}^{0} (-k) \sin n\pi x\, dx + \int_{0}^{1} k \sin n\pi x\, dx\right.$$

$$= k \frac{1}{n\pi} \cos n\pi x \Big|_{-1}^{0} - k \frac{1}{n\pi} \cos n\pi x \Big|_{0}^{1}$$

$$= \frac{k}{n\pi}\left[1 - \cos n\pi - \cos n\pi + 1\right] = \frac{2k}{n\pi}\left(1 - (-1)^n\right)$$

$$= \frac{2k}{n\pi} \begin{cases} 1 - (1) = 0 & , \text{if } n \text{ is even} \\ 1 - (-1) = 2 & , \text{if } n \text{ is odd} \end{cases}$$

(*) $\cos \pi = -1$, $\cos 2\pi = 1$, $\cos 2\pi = -1$

even $2n$, odd $(2n-1)$

Note: $\sin n\pi = 0$, $n = 0, 1, \dots$;
$\cos(0) = 1$, $\cos n\pi = (-1)^n$, $n = 0, 1, \dots$

The final solution $= \frac{4k}{(2n-1)\pi}$.

$$f(x) = \sum_{n=1}^{\infty} \frac{4k}{(2n-1)\pi} \sin(2n-1)\pi x.$$

$$= \frac{4k}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \dots\right)$$

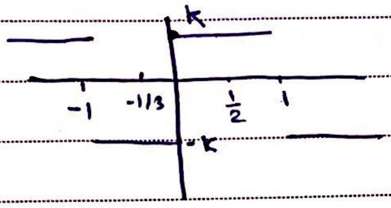
Theorem (convergence)
Suppose that f and f' are peacewise continious on the interval $-L \leq x < L$

$f(x+2L) = f(x)$ is periodic with a period $2L$
Then $(f(x))$ has fourier series (1)

The Fourier series converges to $f(x)$ at all point, where $f(x)$ is continious and it converges

to $\dfrac{f(x^+) + f(x^-)}{2}$ at all points where $f(x)$ is discontinious.

$$f(x) = \begin{cases} -k \,, & -1 < x < 0 \\ k \,, & 0 < x < 1 \end{cases} \qquad f(x+2) = f(x)$$



at the point $x = \frac{1}{2}$ the F.S converges to $f(1/2) = k$

$x = -1/3$ 〃 $f(-1/3) = -k$

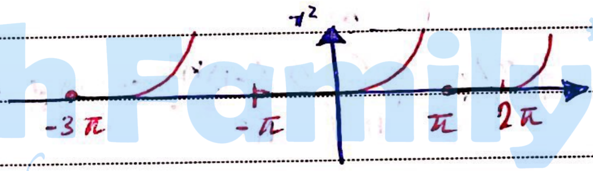
$x = 0$ 〃 $\frac{-k+k}{2} = 0$

limit @ the Right and left. $x = -1$ 〃 $\frac{k-k}{2} = 0$

$x = 1$ 〃 $\frac{k-k}{2} = 0$

Ex: Given $f(x) = \begin{cases} 0 \,, & -\pi < x < 0 \\ x^2 \,, & 0 \leq x < \pi \end{cases}$, $f(x+2\pi) = f(x)$

@ Find the F.S



$$a_0 = \frac{1}{2\pi}\int_0^{\pi} x^2\,dx = \frac{1}{6\pi}\pi^3 = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \underbrace{x^2\cos nx\,dx}_{\text{By part}} = \frac{1}{\pi}\left[\underbrace{\frac{x^2}{n}\sin nx}_{0} + \frac{2x}{n^2}\cos nx\Big|^{\pi} - \underbrace{\frac{2}{n^3}\sin nx}_{0}\right]$$

Der	Int
x^2	$\cos nx$
$2x$	$\frac{1}{n}\sin nx$
2	$\frac{-1}{n^2}\cos nx$
0	$\frac{-1}{n^3}\sin nx$
	$\frac{1}{n^3}\cos nx$

$$= \frac{1}{\pi}\,\frac{2\pi}{n^2}\cos n\pi = \frac{2(-1)^n}{n^2}$$

$\cancel{b_n = \frac{+}{\pi}}$

21

$* \cos n\pi = (-1)^n \;,\; n = 1, 2, \dots$ $\qquad\qquad -(-1)^n = (-1)^{n+1}$

bn

Der		Int
x^2	+	$\sin nx$
$2x$	−	$\frac{-1}{n}\cos nx$
2	+	$\frac{-1}{n^2}\sin nx$
0		$\frac{1}{n^3}\cos nx$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2 \sin nx \, dx = \frac{1}{\pi}\left[\frac{-x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{-\pi^2}{n}\cos nx\,(-1)^n + \frac{2}{n^3}\left(\overset{(-1)^n}{\cos n\pi} - 1\right)\right]$$

$$f(x) \sim \pi^2/6 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}\cos nx + \left[\frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}\left((-1)^n - 1\right)\right] * \sin nx$$

B) At $x = 1$ the F.S converges to ... (أكمل فراغ)

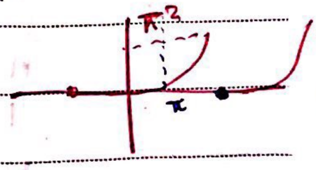
$f(1) = 1$

$x = 2$ the F.s converges $f(2) = 2^2 = 4$

$x = -3$ " $f(-3) = 0$

$x = 0$ " $f(0) = 0$

$x = \pi$ " $\frac{\pi^2 + 0}{2} = \frac{\pi^2}{2}$ من الرسمة



(c) show that

$$\pi^2/12 = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

or Find the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

$$f(x) \sim \pi^2/6 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos nx + \left[\frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}\left((-1)^n - 1\right)\right] \sin nx$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

$\sin nx = 0$

$x = 0, \pi, 2\pi, \dots$

If $x = 0 \rightarrow \cos(0) = 1$

@ $x=0$ the F.S Conv

$f(0) = \text{Zero}$

$$@\ x=0 \rightarrow 0 = \pi^2/6 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}$$

$(f(0)=0)$

$$-\pi^2/6 = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\pi^2/12 = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

show that $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\sin nx = 0 \rightarrow x = 0, \pi, 2\pi$

$$\text{If } x = \pi \quad \frac{\pi^2 + 0}{2} = \frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n \cos n\pi}{n^2} + 0$$

$$\frac{\pi^2}{2} - \frac{\pi^2}{6} = 2\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}$$

$$\frac{2\pi^2}{6 \cdot 2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$*\ \cos n\pi = (-1)^n$

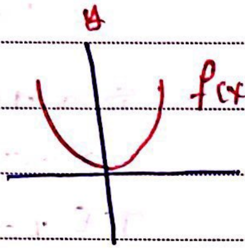
$*\ (-1)^n(-1)^n = (-1)^{2n} = 1$

Even and odd functions

Even $f(x)$

$f(-x) = f(x)$

const, x^2, x^{12}, $\cos x$

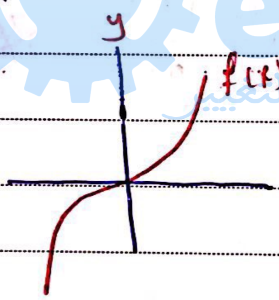


Symmetry with respect to the y-axis

Odd $f(x)$

$f(-x) = -f(x)$

x^3, x, $\sin x$, ...



Symmetry with respect to the Origin

Properties

① odd $\times$ or $\div$ odd = Even

Even $\times$ or $\div$ Even = Even

Even $\times$ or $\div$ odd = odd

② $\displaystyle\int_{-L}^{L} \underbrace{f(x)}_{\text{Even}}\, dx = 2\int_{0}^{L} f(x)\, dx$ / $\displaystyle\int_{-L}^{L} \underbrace{f(x)\, dx}_{\text{odd}} = 0$

Cosine Fourier Series

If $f(x)$ is even , $f(x+2L) = f(x)$, then

$$a_0 = \frac{1}{2L}\int_{-L}^{L} \underbrace{f(x)}_{\text{even}} dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx \quad (2)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\cos\frac{n\pi x}{L}}_{\text{even}}}_{\text{even}} dx = \frac{2}{L}\int_{0}^{L} f(x) \cos\frac{n\pi x}{L}\,dx \quad (3)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\sin\frac{n\pi x}{L}}_{\text{odd}}}_{\text{odd}} dx = 0$$

$$\boxed{f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}} \quad (1) \text{ Cosine Fourier Series}$$

where a_0, a_n are given by (2, 3)

If $f(x)$ is odd , $f(x+2L) = f(x)$, then it can be represented as Fourier Sine Series

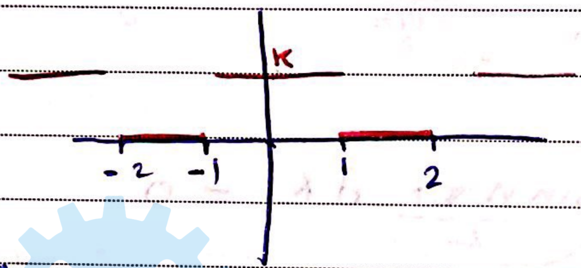
$$\boxed{f(x) \sim a_0 + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}} \quad (4)$$

where $b_n = \frac{2}{L}\int_{0}^{L} f(x) \sin\frac{n\pi x}{L}\,dx \quad (5)$

ex1: Find the Fourier series for:-

$$f(x) = \begin{cases} 0, & \text{if } -2 < x < -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

$$f(x+4) = f(x) \qquad 2L = 4 \qquad \boxed{L = 2}$$



$f(x)$ is even $\Rightarrow$ use the F. cosine series

$$a_0 = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\int_0^1 \underbrace{f(x)}_{k}\,dx + \frac{1}{2}\int_1^2 \underbrace{f(x)}_{0}\,dx$$

$$= \frac{k}{2}$$

$$a_n = \frac{2}{2}\int_0^2 k \cos\frac{n\pi x}{2}\,dx = \int_0^1 k\cos\frac{n\pi x}{2}\,dx + \cancel{\int_1^2 0 \cos\frac{n\pi x}{2}\,dx}$$

$$a_n = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

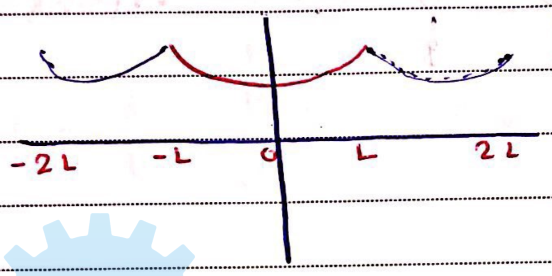
$$f(x) \sim \underbrace{a_0}_{\left(\frac{k}{2}\right)} + \sum_{n=1}^{\infty}\left(\frac{2k}{n\pi}\sin\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}$$

At $x=1$ the F.c.s converges to $\frac{K+0}{2}=\frac{K}{2}$

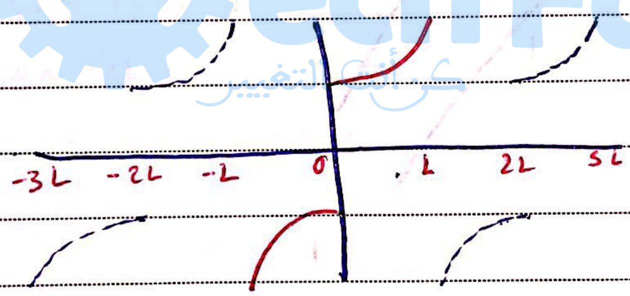
$x=\frac{1}{2}$ the F.c.s converges to $\rightarrow f(\frac{1}{2})=K$

If $f(x)$ is defined on $[0,L]$ it is not Periodic

we use half-Range periodic expansions.



even Periodic expansion of $f(x)$, Period 2L use the F. Cosine series



Odd Periodic expansion of $f(x)$, period 2L use the F. Sine Series.

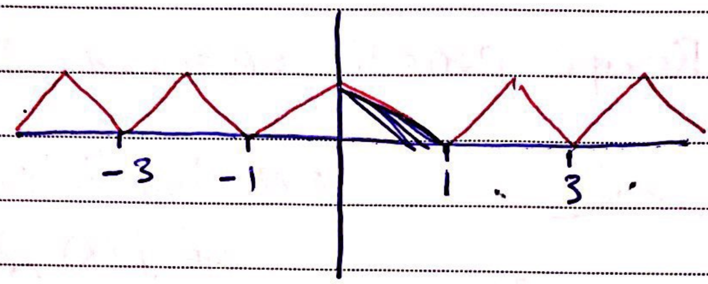
ex 2 let $f(x) = 1 - x$, $0 < x < 1$

Find

ⓐ the F. cosine S. $\boxed{L = 1}$

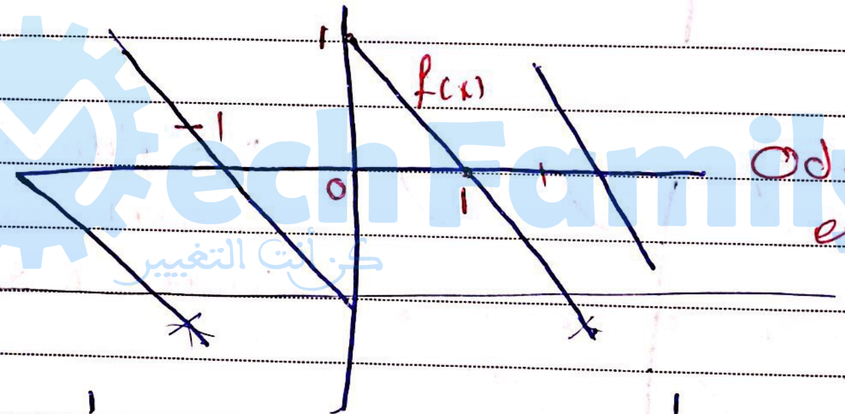
ⓑ the F. sine S $2L = L$

ⓐ



Even Periodic expansion

ⓑ



Odd Periodic expansion

ⓐ $a_0 = \frac{1}{1}\int_0^1 (1-x)\,dx = x - \frac{x^2}{2}\Big|_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$

$a_n = \frac{2}{1}\int_0^1 (1-x)\cos n\pi x\,dx = 2\left[\underbrace{\frac{(1-x)\sin n\pi x}{n\pi}}_{0} - \frac{1}{n^2\pi^2}\cos n\pi x\right]_0^1$

$\cos n\pi x$

Der	Int
$1-x$	$\cos n\pi x$
-1	$\frac{1}{n\pi}\sin n\pi x$
0	$-\frac{1}{n^2\pi^2}\cos n\pi x$

$= \frac{-2}{n^2\pi^2}\left[\cos n\pi - 1\right]$ (28)

$= \frac{-2}{n^2\pi^2}(-1)^n - 1$ →

FIVE APPLE

$$= -\frac{2}{n^2\pi^2}\begin{cases} -2, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases} = \frac{4}{(2n-1)^2\pi^2}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi^2} \cos(2n-1)\pi x$$

(b) F. Sine Series

$$b_n = \frac{2}{1}\int_0^1 (1-x)\sin n\pi x \, dx$$

$$= 2\left[-\frac{(1-x)}{n\pi}\cos n\pi x - \underbrace{\frac{1}{n^2\pi^2}\sin n\pi x}_{0}\right]_0^1$$

$1-x$		$\sin n\pi x$
-1	$+$	$\frac{-1}{n\pi}\cos n\pi x$
0	$-$	$\frac{-1}{n^2\pi^2}\sin n\pi x$

$$= +\frac{2}{n\pi}$$

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin n\pi x$$

11 Fourier Integrals (ex 1-12, 16-20)

$f(x)$ $x \in [0, L]$
(find expansion)
a) F.C.S
b) F.S.S

Find Fourier Series.
check اولا شيء ① قبل
$\underbrace{\text{even}}$ or $\underbrace{\text{odd}}$
F.C.S F.S.S

- $f(x)$ $\underline{-\infty < x < \infty}$ and $f(x)$ is not Periodic

Def A representation $f(x) = \int_0^\infty (A(w)\cos wx + B(w)\sin wx)\,dw$ (1)
where:

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos wx\,dx \quad (2)$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin wx\,dx \quad (3)$$

is called a Fourier integral for $f(x)$.

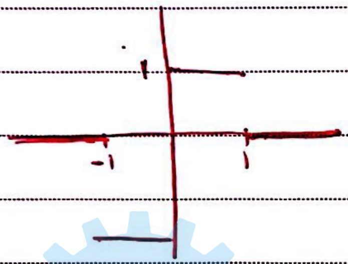
Th. If $f(x)$ is piecewise continious in every finite interval and has a right-hand and left-hand derivatives at every point ($\lim^+$, $\lim^-$)

$\int_{-\infty}^{\infty} |f(x)|\,dx$ exists ($f(x)$ is absolutely integrable functione) the $f(x)$ can be represented by Fourier integral (1). At points where $f(x)$ is continious the F. integral converges to $f(x)$. At points where $f(x)$ is discontinious, the F. integral converges to $\frac{f(x^-) + f(x^+)}{2}$

In example 1 f(x) is even $\Rightarrow$ we can use F. Cosine integral.

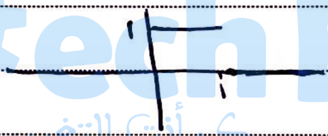
If f(x) we can use F-cosine integral

If $f(x) = \begin{cases} 1, & \text{if } 0<x<1 \\ -1, & \text{if } -1<x<0 \\ 0, & \text{otherwise} \end{cases}$



f(x) is odd func $\Rightarrow$ Use F. sine int.

If f(x)

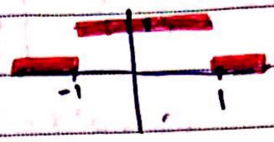


(Not sin, Not cos)
Fourier integral

* برسم وبيبين معي تعرفي، كل على أي طريقة.

ex1 Find the Fourier integral representation of

$$f(x) = \begin{cases} 1, & \text{when } |x| < 1 \\ 0, & \text{when } |x| > 1 \end{cases}$$



$$A(w) = \frac{1}{\pi}\int_{-1}^{1} 1 \cos wx \, dx = \frac{1}{\pi} \left. \frac{\sin wx}{w} \right|_{-1}^{1} = \frac{1}{\pi} \frac{\sin w - \sin(-w)}{w}$$

$$= \frac{2}{\pi} \frac{\sin w}{w}$$

(خطأي ↑)

$$B(w) = \frac{1}{\pi}\int_{-1}^{1} 1 \sin wx \, dx = 0$$

$$f(x) \sim \int_0^\infty \frac{2}{\pi} \frac{\sin w}{w} \cos wx \, dw$$

at $x=0$ the F.I converges to 1

$x=1$ the F.I converges to $\frac{1+0}{2} = 1/2$

$x=-1$ the F.I converges to $\frac{0+1}{2} = 1/2$

$$\int_0^\infty \frac{\sin w}{w} \cos wx \, dw = \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ \frac{\pi}{4} & x = \pm 1 \\ 0 & |x| > 1 \end{cases}$$

نقسم على $\frac{2}{2\pi}$

Application of F.I

Evaluate $\int_0^\infty \frac{\sin w \cos w}{w} \, dw \Rightarrow$ $x=1$ بالمقارنة مع الأصلي

ومنا $(\pi/4)$

$$\int_0^\infty \frac{\sin w \cos 5w}{w} \, dw = 0$$

Show that $\boxed{\int_0^\infty \frac{\sin w}{w}\, dw = \frac{\pi}{2}}$

$\cos wx = 1$ $\boxed{x = 0}$

* If $f(x)$ is even, then we use the Fourier cosine integral

$$f(x) = \int_0^\infty A(w) \cos wx \, dw \quad \text{--- (4)}$$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(x) \cos wx \, dx \quad \text{--- (5)}$$

If $f(x)$ is odd, The Fourier sine integral

$$f(x) = \int_0^\infty B(w) \sin wx \, dw \quad \text{--- (6)}$$

$$B(w) = \frac{2}{\pi} \int_0^\infty f(x) \sin wx \, dx \quad \text{--- (7)}$$

ex2 Given $f(x) = e^{-kx}$ ($x > 0$, $k > 0$ is a constant)

Find: a) Fourier cosine integral

b) Fourier sine integral

(use even or odd expansion)

NO. cha 12

Partial differential equations (PDEs)

Basic concepts in PDEs.

ODEs: $y = y(x)$ $\quad y'' + 2y' + y = e^x$

$u = u(x, y)$

$\frac{\partial u}{\partial x} = u_x \qquad \frac{\partial u}{\partial y} = u_y$

Def. A Partial Differential equation involving one or more partial derivatives of unknown function.

$u(x_1, \dots, x_n)$

1. $u_x + u_y = \underset{\approx}{1}$ $\longrightarrow$ First-order PDEs, linear non homo.

2. $u_{tt} = c^2 u_{xx}$ c is const, second order. Linear

3. $u_x + u_{xx} + u^2 = 0$ nonlinear PDEs.

4. $u_t = a^2 u_{xx}$ Heat eqn

5. $u_{xx} + u_{yy} = 0$ Laplace PDE

NO. Cha 12

$(u_t)^2 - u_{xx} = 0$ non liner

$u_x + u u_y = 1$ non lineer

$u_{xx} = 5$

$\int u_{xx} dx = \int 5 dx$

$u_x = 5x + g(y)$

$u = \int 5x dx + \int g(y) dx$

$u = \frac{5x^2}{2} + \int g(y) dx + h(y)$

ex 1) Solve $u_{xx} - 2u_x - 3u = 0$ where $u = (u(x, y))$

$u'' - 2u' - 3u = 0$

$k^2 - 2k - 3 = 0$

$(k + 1)(k - 3)$

$\boxed{k = -1}$ $\boxed{k = 3}$

$u(x, y) = c_1(y)\, e^{-x} + c_2(y)\, e^{3x}$

ex 2) solve $u_{yy} + 10u_y + 25u = 0$

$u'' + 10u' + 25 = 0$

$k^2 + 10k + 25 = 0$

$(k_1 + 5)(k_2 + 5)$

$k_1 = k_2 = -5$

$u(x, y) = c_1(x)\, e^{-5y} + c_2(x)\, y\, e^{-5y}$

ex 3) Solve $u_{xx} + 2u_x + 5u = 0$

$u'' + 2u' + 5u = 0$

$k^2 + 2k + 5 = 0$

$(\quad\quad)$

$k = \dfrac{-2 \pm \sqrt{4 - 20}}{2} = \dfrac{-2 \pm 4i}{2}$

2. Solve $u_{xy} = -u_x$

Denote $P = u_x \Rightarrow (u_x)_y = P_y$

$P_y = -P$ separaple d.e

$$\frac{dP}{dy} = -P$$

$$\int \frac{dP}{P} = -\int dy \quad = \ln P = -y + c_1(x)$$

$$P = e^{-y} e^{c_1(x)} = c_2(x) e^{-y}$$

$$u_x = c_2(x) e^{-y}$$

$$u(x,y) = e^{-y} \underbrace{\int c_2(x)\,dx}_{c_4(x)} + c_3(y)$$

Notes $u_{xy} = u_{yx}$

$$u(x,y) = c_4(x) e^{-y} + c_3(y)$$

the g. sol.

ex 2:- Solve $u_{xy} - 2u_y = e^{-x}$

let $P = u_y \Rightarrow u_{xy} = (u_y)_x = P_x$

$P_x - 2P = e^{-x}$ linear first order d.e

$$P = e^{-\int -2\,dx}\left[\int e^{\int -2\,dx} e^{-x}\,dx\right]$$

$$P = e^{2x}\left[\int e^{-3x}\,dx\right] =$$

$$e^{2x}\left[\frac{-1}{3} e^{-3x} + c_1(y)\right]$$

$$y' + a(x) = b(x)$$

$$y = e^{-\int a(x)\,dx}\left[\int e^{\int a(x)\,dx}\, b(x)\,dx + c\right]$$

$$y = e^{-h}\left[\int e^{h}\,dx + c\right]$$

$\boxed{\frac{-1}{3} \bar{e}^{x} \int dy}$

$u_y = \frac{-1}{3} \bar{e}^{x} + c_1(y) e^{2x} \longrightarrow u(x,y) = \frac{-1}{3} \bar{e}^{x} y + e^{2x} \int c_1(y)\, dy + C_2(x)$

The g. sol.

① Separation of Variables

Given a PDE in $u(x,y)$

* Let $u(x,y) = F(x)G(y)$ be a Product Sol.

ex1 Separate the PDE in to a system of O.D.Es.

$u_x + u_y = 0$ — Linear and homogeneous PDEs

$u(x,y) = F(x)\, G(y)$

$u_x = F'(x)\, G(y) \qquad u_y = F(x) G'(y)$

$u_x + u_y = F'G + FG' = 0$

$F'G = -FG'$

Separation constant

$\frac{F'(x)}{F(x)} = -\frac{G'(y)}{G(y)} = \lambda = \text{const.}$

$$\frac{F'(x)}{F(x)} = \lambda \longrightarrow F'(x) - \lambda F(x) = 0$$

$$-\frac{G'(y)}{G(y)} = \lambda \longrightarrow G'(y) + \lambda G(y) = 0$$

Ex Separate the PDE in to a system of ODEs

$u_{xx} - u_y + u = 0$

$u(x,y) = F(x)\,G(y)$

$u_y = FG' \qquad u_x = F'G \qquad u_{xx} = F''G$

$F''G - FG' + FG = 0$ (Devide by FG)

$$\frac{F''}{F} - \frac{G'}{G} + 1 = 0$$

$$\frac{F''}{F} = \frac{G'}{G} - 1 = \lambda$$

$$\frac{F''}{F} = \lambda$$

$$\frac{G'}{G} - 1 = \lambda \longrightarrow \boxed{\begin{array}{l} F'' - \lambda F = 0 \\ G' - G - \lambda G = 0 \end{array}}$$

NO. PDE

ex $u_{xx} + (x+y)\, u_{yy} = 0$

$u(x,y) = F(x)\,G(y)$

$F''G + (x+y)\, FG'' = 0 \qquad \div FG$

$$\frac{F''(x)}{F(x)} + (x+y)\,\frac{G''(y)}{G(y)} = 0$$

Non separable d.e

ex $x\, u_{xx} + u_x - x\, u_y = 0$

$x F''G + F'G - x FG' = 0$

$(x F'' + F')\, G = x FG'$

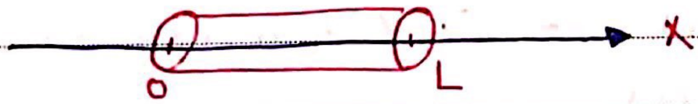
$$\frac{x F'' + F'}{xF} = \frac{G'}{G} = \lambda$$

$x F'' + F' - \lambda x F = 0$

$G' - \lambda G = 0$

Heat equation

Consider a rod of uniform cross-section and homogeneous material



$x=0$ and $x=L$ are the ends of the rod.
Sides are insulated.
Assume that the cross-section is small
Let $u=u(x,t)$ be a temperature in the rod.
Let t be a time variable

The variation of temperature in the rod is given by

$$\boxed{u_t = \underset{1 \text{ or } 4}{a^2} u_{xx}}$$ Heat equation (1)

a is a constant (thermal diffusivity)

Boundary conditions:

(2) $u(0,t)=0$ * بالامتحان ، احنا كالنا بنعرف على
$u(L,t)=0$ $0 \le x \le 1$ أي condition.

Initial condition:

$u(x,0)=f(x), t>0$ (3)

Step 1 Separation of d.e (1)

$u_t = a^2 u_{xx}$

$u(x,t) = F(x)G(t)$ (4)

$FG' = a^2 F'' G$

$\dfrac{F''(x)}{F(x)} = \dfrac{G'(t)}{a^2 G(t)} = -\lambda$ (separation const)

$F'' + \lambda F = 0$ (5)

$G' + \lambda a^2 G = 0$ (6)

Step 2 Separation of B.C. (2)

$u(0,t) = 0$

$u(0,t) = F(0)G(t) = 0 \rightarrow \boxed{F(0) = 0}$

$\rightarrow G(t) = 0 \rightarrow$ from (4)

$u(x,t) = F(x) \cdot 0 = 0$ is trivial sol (X)

$u(L,t) = F(L)G(t) = 0 \rightarrow F(L)G(t) = 0 \rightarrow \boxed{F(L) = 0}$

$\rightarrow G(t) = 0$ (X)

$$\boxed{\begin{aligned} F(0) &= 0 \\ F(L) &= 0 \end{aligned}} \quad \cdots (7)$$

Step 3 Solve the Boundary-Value Problem (BVP1).

(5) + (7) eqn

$F''(x) = \lambda F(x) = 0$

$r^2 + \lambda = 0$

$\lambda \to \cancel{\lambda = 0}$, $\lambda < 0$, $\lambda > 0$

→ No negative eignvalues!
The Non Trivial λ are called eigenvalues.

Case I

If $\lambda = 0 \Rightarrow F''(x) = 0 \Rightarrow F(x) = Ax + B$ $0 < x < L$

Find A and B from B.C (7)

$F(0) = A \cdot 0 + B \Rightarrow \boxed{B = 0}$; $F(L) = AL = 0 \Rightarrow \boxed{A = 0}$

$f(x) \equiv 0 \Rightarrow$ from (4) $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$

$\lambda = 0$ is not eigenValue. Trivial Solution.

Case II

ما معه — No بجوه الجوابة

If $\lambda < 0$ (let $\lambda = -k^2$, $k > 0$)

$F'' - k^2 F = 0$

$r^2 - k^2 = 0$ $r_1 = k$ $r_2 = -k$

$F(x) = c_1 e^{kx} + c_2 e^{-kx}$

$F(0) = c_1 + c_2 = 0$

$F(L) = c_1 e^{kL} + c_2 e^{-kL} = 0$

let $c_2 = -c_1$ $\to$ $c_1 (\underbrace{e^{kL} - e^{-kL}}_{\neq 0}) = 0$

$c_1 = 0 \to c_2 = 0$

$F(x) = 0 \Rightarrow u(x,t) = 0$ trivial soln.

Case III

If $\lambda > 0$ (let $\lambda = k^2$, $k > 0$)

$$F'' + k^2 F = 0$$

$r^2 + k = 0$ $\quad r_1 = ki \quad r_2 = -ki$

$$F(x) = c_1 \cos kx + c_2 \sin kx$$

$$F(0) = c_1 = 0$$

$F(L) = c_2 \sin kL = 0$ → $\cancel{c_2 = 0}$ → $F(x) = 0$ → trivial sol
→ $\sin kL = 0$ ⇒ $kL = n\pi$ $n = 1, 2, \dots$

Notes

* $k = \frac{n\pi}{L}$ $n = 1, 2, \dots$
* $\lambda_n = \frac{n^2 \pi^2}{L^2}$ (8) are eigenvalues

$F(x) = c_2 \sin \frac{n\pi x}{L}$, let $c_2 = 1$

$\boxed{F_n(x) = \sin \frac{n\pi x}{L}}$ (9) are eigen functions

$$e^{-n^2\pi^2a^2t/L^2}$$

Step 4 Solve $G' + a^2\lambda G = 0$ (6)

$$\lambda = \frac{n^2\pi^2}{L^2}$$

$$G' + \frac{a^2n^2\pi^2}{L^2} G = 0$$

$$\frac{G'}{G} = -\frac{a^2n^2\pi^2}{L^2}$$

$$\ln G = -\frac{a^2n^2\pi^2}{L^2} t \Rightarrow \boxed{G_n(t) = e^{-\frac{a^2n^2\pi^2 t}{L^2}}} \quad (10)$$

constant $= 1$

Step 5 Superposition Principle

$$U_n = \sin\frac{n\pi x}{L} e^{-n^2\pi^2a^2t/L^2} \qquad n = 1, 2, \dots$$

The general Sol is $u(x,t) = \sum_{n=1}^{\infty} C_n U_n$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} e^{-n^2\pi^2a^2t/L^2}$$

cot

ex1 solve without F.S. $u_t = 4u_{xx}$

$, 0<x<1 , t>0$

$u(0,t) = 0$

$u(1,t) = 0$

$u(x,0) = 3\sin 2\pi x - 4\sin 3\pi x$

$u(x,t) = F(x)G(t)$

$\vdots$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin n\pi x \, e^{-4n^2\pi^2 t}$$

Find C_n

$t=0 \quad u(x,0) = \sum_{n=1}^{\infty} C_n \sin n\pi x$

$= C_1 \sin \pi x + C_2 \sin 2\pi x + C_3 \sin 3\pi x + \dots$

$= 3\sin 2\pi x - 4\sin 3\pi x$

$\underset{n=2}{C_2 = 3} \quad , \underset{n=3}{C_3 = -4} \quad C_1 = C_4 = \dots = 0$

$$u(x,t) = 3\sin 2\pi x \, e^{-4\cdot 2^2\pi^2 t} - 4\sin 3\pi x \, e^{-4\cdot 3^2\pi^2 t}$$

Given $u_t = a^2 u_{xx} , 0<x<1 , t>0$

I.C $u(x,0) = f(x)$

Bc

① $u(0,t)=0$

$u(L,t)=0$

BVP

Boundary Value Prop. 1

② $u_x(0,t)=0$

$u_x(L,t)=0$

B.V.P 2

خلاصة.

③ $u_x(0,t)=0$

$u_L(L,t)=0$

BVP 3

④ $u(0,t)=0$

$u_x(L,t)=0$

BVP 4

3,4 Homework

Proplem 2

Solve $u_t = a^2 u_{xx}$, $0 < x < l$, $t > 0$ ...... (1)

I.c $u(x,0) = f(x)$ ...... (2)

B.c $u_x(0,t) = 0$ ...... (3)
$u_x(L,t) = 0$

Step 1

$u(x,t) = F(x)\,G(t)$ ...... (4)

$F'' + \lambda F = 0$ ...... (5)

$G' + \lambda a^2 G = 0$ ...... (6)

Step 2

B.c $u_x(0,t) = F'(0)\,G(t) = 0$ $F'(0) = 0$
$u_x(L,t) = F'(L)\,G(t) = 0$ $F'(L) = 0$ ...... (7)

Step 3

BVP 2 $F'' + \lambda F = 0$
$F'(0) = F'(L) = 0$

$\lambda \begin{cases} = 0 \\ \cancel{<0} \rightarrow \text{no Negative eigenval (only trivial Sol)} \\ > 0 \end{cases}$

(15) (8)

case 1

If $\lambda = 0 \Rightarrow F'' = 0 \Rightarrow F(x) = Ax + B$

$F'(x) = A \qquad F'(0) = A = 0 \qquad F'(L) = A = 0$

$F(x) = B = \text{const.}$

let $B = 1 \Rightarrow F(0) = 1 \quad \dots (8)$

eigenf. $\lambda(0) = 0$ eigenval.

case 2

If $\lambda > 0$ (let $\lambda = k^2$ $(k > 0)$)

$F'' + k^2 F = 0$

$F(x) = c_1 \cos kx + c_2 \sin kx$

$F'(x) = -k c_1 \sin kx + k c_2 \cos kx$

$F'(0) = +k c_2 = 0 \Rightarrow c_2 = 0$

$F'(L) = -k c_1 \sin kL = 0$

$\sin kL = 0 \qquad kL = n\pi \qquad n = 1, 2, \dots$

$k = \frac{n\pi}{L} \qquad \lambda_n = \frac{\pi^2 n^2}{L^2} \qquad n = 1, 2 \dots (9)$

$F = c_1 \cos \frac{n\pi x}{L}$ $\quad \boxed{F_n(x) = \cos \frac{n\pi x}{L} \quad \dots (10)}$

Step 4 Solve (6) $G' + a^2 \lambda G = 0$

If $\lambda = 0 \Rightarrow G' = 0 \rightarrow G_0(t) = 1$ (const) ...(11)

$$\lambda_n = \frac{n^2\pi^2}{L^2} \rightarrow G' + \frac{a^2 n^2 \pi^2}{L^2} G = 0$$

$$\Rightarrow \boxed{G_n(t) = e^{-a^2 n^2 \pi^2 t / L^2}} \quad \text{...(12)}$$

Step 5

The g.s $u(x,t) = \sum_{n=0}^{\infty} c_n u_n(x,t)$

$$= \sum_{n=0}^{\infty} c_n F_n(x) G_n(t)$$

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi x}{L} e^{-a^2 n^2 \pi^2 t / L^2} \quad \text{...(13)}$$

The General Solution

$$u(x,0) = c_0 + \sum_{n=1}^{\infty} \cos\frac{n\pi x}{L} = f(x)$$ f.cos.s for f(x)

$$c_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{...(14)}$$

17

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad \text{...(15)}$$

Wave equation

$$u_{tt} = c^2 u_{xx} \;,\; 0 < x < l \;,\; t > 0 \quad \sim (1)$$

B.C $\rightarrow$ $u(0,t) = 0 \quad \sim (2)$

$u(L,t) = 0$

I.C $u(x,0) = f(x)$

$u_t(x,0) = g(x) \quad \sim (3)$

$$u(x,t) = F(x)\,G(t) \quad \sim (4)$$

$$F G'' = c^2 F'' G$$

$$\frac{F''}{F} = \frac{G''}{c^2 G} = -\lambda$$

$F'' + \lambda F = 0 \quad \sim (5)$

$G'' + \lambda c^2 G = 0 \quad \sim (6)$

Step 2 $u(0,t) = F(0)\,G(t) = 0 \quad \therefore F(0) = 0$

$F(L) = 0 \quad \sim (7)$

Step 3 solve $F'' + \lambda F = 0$ ~ see heat eqn (BVP1)

$F(0) = F(L) = 0$

$$\lambda_n = \frac{n^2 \pi^2}{L^2} \quad \sim (8)$$

$$F(n) = \sin \frac{n\pi}{L} x \quad \sim (9)$$

Step 4

Solve (6)

$$G'' + \frac{n^2\pi^2}{L^2} c^2 G = 0$$

$$G_n(t) = \overset{A_n}{C_{1n}} \cos \frac{n\pi c}{L} t + \overset{B_n}{C_{2n}} \sin \frac{n\pi c}{L} t \quad n=1,2.. \; (10)$$

Step 5 $$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left[A_n \cos \frac{n\pi c}{L} t + B_n \sin \frac{n\pi c}{L} t \right] (11)$$

Step 6 find A_n and B_n from I.C

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = f(x) \quad F \sin S$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \quad (12)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(-A_n \frac{n\pi c}{L} \sin \frac{n\pi c}{L} t + B_n \frac{n\pi c}{L} \cos \frac{n\pi c t}{L} \right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L} = g(x)$$

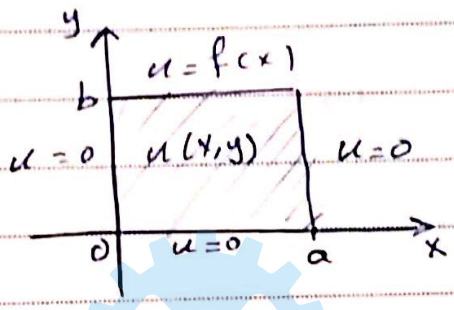
$$B_n \frac{n\pi c}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx = 0$$

F. sine Ser

Laplac's equation in a Rectangle

$u_{xx} + u_{yy} = 0 \quad , \; 0<x<a \quad , \; 0<y<b \qquad (1)$

$u(x,y)$ (Rectangle)



B.c $u(0,y) = 0 \qquad 0 \le y \le b$

$u(a,y) = 0$

→ بتغير احداثياته

$(2) \qquad u(x,0) = 0 \qquad 0 \le x \le a$

$u(x,b) = f(x)$

① $u(x,y) = F(x)G(y) \qquad (3)$

$F''G + FG'' = 0$

$$\frac{F''}{F} = -\frac{G''}{G} = -\lambda$$

$F'' + \lambda F = 0 \quad (4)$

$G'' - \lambda G = 0 \quad (5)$

② $u(0,y) = F(0)\,G(y) = 0 \Rightarrow F(0) = 0$
$u(a,y) = F(a)\,G(y) = 0 \Rightarrow F(a) = 0$ $\Big\}$ — (6)

$u(x,0) = F(x)\,G(0) = 0 \qquad G(0) = 0 \qquad (7)$

③ Solve the BVP (4) + (6)
eqn

$F'' + \lambda F = 0$

$F(0) = 0$

$F(a) = 0$

see BVP 1 for Heat equ

(L = a)

$\lambda \begin{cases} = 0 & \times \\ < 0 & \times \\ > 0 & \checkmark \end{cases} \longrightarrow \lambda_n = \dfrac{n^2 \pi^2}{a^2}$ (8)

$F_n(x) = \sin \dfrac{n\pi x}{a}$ (a)

(4) Solve $G'' - \dfrac{n^2\pi^2}{a^2} G = 0$

$r^2 - \dfrac{n^2\pi^2}{a^2} = 0$ characteristic equ

$r_1 = \dfrac{n\pi}{a}$, $r_2 = -\dfrac{n\pi}{a}$

$G(y) = A_1 e^{\frac{n\pi y}{a}} + A_2 e^{\frac{-n\pi y}{a}}$

use (7)

$G(0) = A_1 + A_2 = 0 \longrightarrow (A_2 = -A_1)$

بنقدر نوقف بدون sinh

$G_n(y) = 2A_1 \left(\dfrac{e^{n\pi y/a} - e^{-n\pi y/a}}{2} \right) = \underbrace{2A_1}_{b} \sinh\left(\frac{n\pi y}{a}\right)$

let $2A_1 = 1$

$G_n(y) = \sinh\left(\dfrac{n\pi y}{a}\right)$

⑤ The general Solution.

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}} \quad (11)$$

b Fn b Gn

⑥ Find An from $u(x,b) = f(x)$

$$u(x,b) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a} = f(x)$$

F. Sine Series for $f(x)$

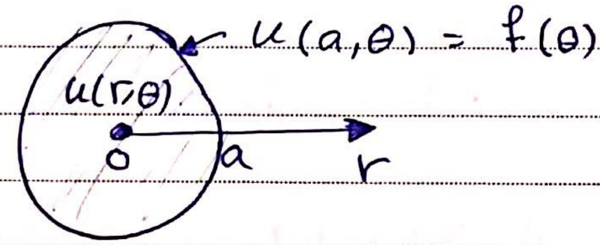
$$A_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a}\, dx \quad (12)$$

laplace's equation in a circle

$$u_{xx} + u_{yy} = 0$$

نادر يعمل

$$u = u(r, \theta)$$



Interior Proplem.

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad (1) \qquad \boxed{r<a}$$

لو معطى بالسؤال $-\pi < \theta < \pi$

B.C $u(a,\theta) = f(\theta) \quad (2)$

① let $u(r,\theta) = F(r)\,G(\theta) \quad (3)$

$$F''G + \frac{1}{r} F'G + \frac{1}{r^2} FG'' = 0$$

$\div FG$

seperoples:-

$$\frac{F''}{F} + \frac{1}{r}\frac{F'}{F} = -\frac{1}{r^2}\frac{G''}{G}$$

$$r^2\left[\frac{F''}{F} + \frac{1}{r}\frac{F'}{F}\right] = \frac{-G''}{G} = \lambda$$

$$r^2 F'' + rF' - \lambda F = 0 \quad (4)$$

$$G'' + \lambda G = 0 \quad (5)$$

B.C عوضو

(Boundary Conditions)

B.C

Function $u(r,\theta)$ must be

Periodic in θ with Period 2π

+ bounded for $r \leq a$

λ:
- <0 No eigenvalues
- $=0$
- >0

If $\lambda < 0$ (let $\lambda = -k^2$, $k > 0$)

From (5) $\Rightarrow$ $G'' - k^2 G = 0$

$r^2 - k^2 = 0$

$r_1 = k \qquad r_2 = -k$

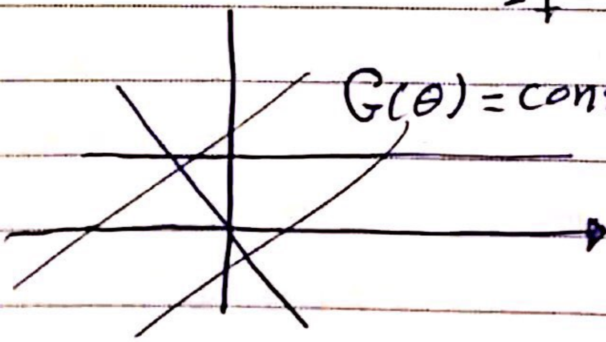
$G(\theta) = C_1 e^{k\theta} + C_2 e^{-k\theta}$ is not Periodic

$C_1 = C_2 = 0$

$G(\theta) \equiv 0 \Rightarrow$ triv. Sol.

If $\lambda = 0 \qquad G'' = 0$

$G(\theta) = A\theta + B$



$G(\theta) = \text{const}$

If $A = 0$, then $G(\theta) = B = \text{Const}$ is Periodic with any Period

$\boxed{G_0 = 1}$ "حل"

(6)

NO.

From (4)

$r^2F'' + rF' = 0$ Euler d.e

$m(m-1)+m=0$

$m^2=0$

$F(r) = c_1 \cancel{e^0} + c_2 r e^0$

$\boxed{F(r) = c_1 + c_2 \ln r \text{ Linear f.n must be bounded}}$

$c_2 = 0 \rightarrow F(r) = c_1 = \text{const}$

let $\boxed{F_0 = 1}$

$u(r,\theta) = F(r)\,G(\theta)$

$r^2F'' + rF' - \lambda F = 0$

$G'' + \lambda G = 0$

λ ① $\to$ $\cancel{<0}$
② $\to$ $=0$ $F_0 = 1, \quad G_0 = 1$
③ $\to$ >0

If $\lambda > 0$ (let $\lambda = +k^2,\ k > 0$)

$G'' + k^2 G = 0$

$G(\theta) = c_1 \cos k\theta + c_2 \sin k\theta$

must be Periodic with period 2π

$\boxed{k = n}$, $n = 1, 2, \dots$ integers

$\boxed{\lambda_n = n^2}$ eigenvalues from (3)

$\boxed{G_n(\theta)} = A_n \cos n\theta + B_n \sin n\theta$ from (3)

$r^2F'' + rF' - n^2F = 0$ Euler d.e

$m(m-1) + m - n^2 = 0$

$m^2 - n^2 = 0 \qquad m_1 = n \qquad m_2 = -n$

$F(r) = c_1 r^n + c_2 r^{-n}$

must be bounded :-

if $r \to 0 \quad \frac{1}{r^n} \to \infty \Rightarrow C_2 = 0$

$\boxed{F_n(r) = r^n}$ from (3)

The General Solution is

$r^{-n} \to$ لو كان exterior

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

Find A_n

$$u(a,\theta) = A_0 + \sum_{n=1}^{\infty} (a^n A_n \cos n\theta + a^n B_n \sin n\theta) = f(\theta)$$

F. Series for $f(\theta)$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

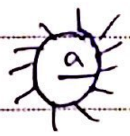
ext. لو كان a^{-n}

$$a^n A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta$$

a^{-n}

$$a^n B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$$

Exterior proplem $u(a,\theta) = f(\theta)$ $r > a$



If $r \to \infty$ $\boxed{C_1 = 0}$, $F_n(r) = r^{-n}$

$C_2 = 1$

NO.

Solve eqn use Fourier Transform :-

$$F_c\{f''(x)\} = -\omega^2 F_c\{f\} - \sqrt{\frac{2}{\pi}}\, f'(0)$$

$$F_s\{f''(x)\} = -\omega^2 F_s\{f\} + \sqrt{\frac{2}{\pi}}\, \omega f(0)$$

$$F\{f''(x)\} = -\omega^2 F\{f(x)\}$$

$$F_s\{u_{xx}(x,t)\} = -\omega^2 U_s(\omega,t) + \sqrt{\frac{2}{\pi}}\, \omega f(0)$$

تعطى هذه المعادلات في السؤال :

let $F_c\{u_{xx}(x,t)\} = -\omega^2 F\{u(x,t)\} - \sqrt{\frac{2}{\pi}}\, u_x(0,t)$

let $U_c(\omega,t) = F_c\{u(x,t)\}$

$$F_c\{u_{xx}(x,t)\} = -\omega^2 U_c(\omega,t) - \sqrt{\frac{2}{\pi}}\, u_x(0,t)$$

t مثل const & نتعامل معه Transform

$$F_c\{u_t(x,t)\} = \frac{d}{dt} U_c(\omega,t)$$

$$F_c\{u_{tt}\} = \frac{d^2}{dt^2}\left(U_c(\omega,t)\right)$$

علم

ا/جلال cos, sin الحالة المشابهة

Solve the Proplem $(\cos, \sin)$

$x > 0 \quad F_{c,s}$ transform

$u_t = u_{xx}, \quad 0 < x < \infty, \quad t > 0$ في القوانين التي تحل

B.C $u_x(0,t) = 0 \to$ Use Fcosine — لو $u(0,t) \to$ sine

$-\infty < x < \infty$ F.transform (General)

I.C $u(x,0) = \begin{cases} 2, & 0 < x < 1 \\ 0, & 1 \end{cases}$ لو كان مثلاً 9 non homo.

$$F_c\{u_t(x,t)\} = F_c\{u_{xx}(x,t)\}$$

$\frac{d}{dt}$

let $F_c\{u(x,t)\} = U_c(\omega,t)$ ملاحظة: (اسم الاقتران)

$$\frac{d}{dt} U_c(\omega,t) = -\omega^2 U_c(\omega,t) - \sqrt{\frac{2}{\pi}}\, u_x(0,t)$$

linear First-Order or separaple $\qquad \frac{dU_c}{dt} = -\omega^2 U_c \qquad$ from (B.C) $= 0$

$$\frac{dU_c}{U_c} = -\omega^2 \int dt$$

$$\ln U_c = -\omega^2 t + \ln A(\omega)$$

$$U_c(\omega,t) = A(\omega)\, e^{-\omega^2 t}$$

NO.

From I.C

Find $A(\omega)$

$$\text{If } t=0 \quad U_c(\omega,0) = A(\omega) \quad \text{and} \quad \underbrace{U_c(\omega,0)}_{\text{by def.}}$$

$$U_c(\omega,0) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} u(x,0)\cos\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}}\, 2\, \frac{\sin\omega x}{\omega}\Big|_0^{1} = 2\sqrt{\frac{2}{\pi}}\frac{\sin\omega}{\omega}$$

$$A(\omega) = 2\sqrt{\frac{2}{\pi}}\,\frac{\sin\omega}{\omega}$$

$$\boxed{U_c(\omega,t) = 2\sqrt{\frac{2}{\pi}}\,\frac{\sin\omega}{\omega}\, e^{-\omega^2 t}}$$

$$u(x,t) = F^{-1}\left\{U_c(\omega,t)\right\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} 2\sqrt{\frac{2}{\pi}}\,\frac{\sin\omega}{\omega}\, e^{-\omega^2 t}\, d\omega$$

NO. مم ا مركا مرزة

$$F_c\{f''(x)\} = -w^2 F_c\{f(x)\} - \sqrt{\frac{2}{\pi}}\boxed{f'(0)}$$

$$F_s\{f''(x)\} = -w^2 F_s\{f(x)\} + \sqrt{\frac{2}{\pi}}\, w\boxed{f(0)}$$

$$F\{f''(x)\} = -w^2 F\{f\}$$

ex: $u_{tt} = 4u_{xx}$, $0<x<\infty$, $x>0$, $t>0$

B.C $u(0,t) = 0$, $t \geq 0$ $f(u(0,t))$

I.C $u(x,0) = \begin{cases} 2 , & 0<x<1 \\ 0 , & x>1 \end{cases}$

$u_t(x,0) = 0$

F-sine transform

زي الثابت

let $F_s\{u(x,t)\} = U_s(w,t)$

Transform x

$$F_s\{u_{tt}\} = 4F_s\{u_{xx}\}$$

$$\frac{d^2}{dt^2}U_s(w,t) = 4(-w^2 U_s(w,t) + \sqrt{\frac{2}{\pi}}\, w\, \underbrace{u(0,t)}_{\text{Zero}})$$

Zero from B.C

$$\frac{d^2}{dt}U_s + 4w^2 U_s = 0$$

بشبه $y'' + 4y = 0$

complex

$$U_s(w,t) = A(w)\cos(2wt) + B(w)\sin(2wt)$$

find $A(w)$

$t=0$ $\quad \boxed{U_s(w,0)} \quad A(w)$

by def $\boxed{U_s(w,0)} = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\sin wx\,dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 2\sin wx\,dx$$

$$= 2\sqrt{\frac{2}{\pi}}\,\frac{1-\cos w}{w}$$

$$A(w) = 2\sqrt{\frac{2}{\pi}}\,\frac{1-\cos w}{w}$$

find $B(w)$

$$(U_s(w,t))_t = -2wA\sin(2wt) + 2wB\cos(2wt)$$

$t=0$ $\quad (U_s(w,0))_t = 2w\,B(w)$

By def $(U_s(w,t))_t = \frac{d}{dt}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty u(x,t)\sin wx\,dx\right)$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty u_t(x,t)\sin wx\,dx$$

NO. Questions

If $t=0$

$$\left(U_s(w,0)\right)_t = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \underbrace{u_t(x,0)}_{\substack{h \\ 0}} \sin wx \, dx = 0$$

$$\boxed{B(w) = 0}$$

$\downarrow$ Fourier cosine

$$u(x,t) = F^{-1}\left(U_s(w,t)\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \underbrace{2\sqrt{\frac{2}{\pi}} \frac{1-\cos w}{w}}_{A}$$

$\cos(2wt) \, dw$

NO. Question 3

ما عندي B.C فقط عندي I.C لانه $-\infty < x < \infty$

Fourier T.

$u_t = u_{xx} \quad , -\infty < x < \infty \quad , t > 0$

$$I.C \quad u(x,0) = \begin{cases} 2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

$$\text{let } F\{u(x,t)\} = \mathcal{U}(\omega,t)$$

$$F\{u_t(x,t)\} = F\{u_{xx}(x,t)\}$$

$$\frac{d\,\mathcal{U}(\omega,t)}{dt} = -\omega^2\,\mathcal{U}(\omega,t)$$

$$\int \frac{d\mathcal{U}}{dt} = \int -\omega^2\,\mathcal{U}$$: Linear or Separaple

$$\ln \mathcal{U} = -\omega^2 t + \ln A$$

$$\boxed{\mathcal{U} = A(\omega)\,e^{-\omega^2 t}}$$ Solution Linear equation

NO. Question

Find $A(\omega)$

$t=0 \quad \mathcal{U}(\omega,0) = A(\omega)$

$$A(\omega) = \sqrt{\frac{2}{\pi}}\ \frac{1-e^{i\omega}}{i\omega}$$

~~by def~~

~~$\mathcal{U}(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-i\omega x}\, dx$~~

<u>the last step</u>

$$u(x,t) = F^{-1}\{\mathcal{U}(\omega,t)\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\left(\sqrt{\frac{2}{\pi}}\ \frac{1-e^{-i\omega}}{\omega_i}\right)}_{A(\omega)} e^{-\omega^2 t}\, e^{i\omega x}\, d\omega$$

by def

$$\mathcal{U}(\omega,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{1} 2 e^{-i\omega x}\, dx = \frac{2}{\sqrt{2\pi}}\ \frac{1}{-i\omega}\, e^{-i\omega x}\Big|_{0}^{1}$$

$$= \frac{2}{\sqrt{2\pi}}\ \frac{e^{-i\omega} - 1}{(-i\omega)}$$