

Partial Differential Equations Notebook

2013-Second Semester

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(This file contains the full material
+ Exams with their solutions)

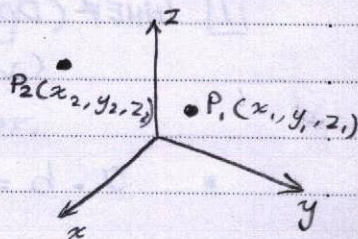
CHAPTER 9

VECTOR DIFFERENTIAL CALCULUS

- $\overrightarrow{P_1 P_2} = \mathbf{a} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$

↓ vector
 from P_1 toward P_2
 الأصل المقادير

for vector: []
for point: ()



- Unit vectors i, j , and k .

$$\begin{aligned}
 i &= [1, 0, 0] \\
 j &= [0, 1, 0] \\
 k &= [0, 0, 1]
 \end{aligned}
 \left. \vphantom{\begin{aligned} i &= [1, 0, 0] \\ j &= [0, 1, 0] \\ k &= [0, 0, 1] \end{aligned}} \right\} \text{Ex: } \mathbf{v} = 4\mathbf{i} + 7\mathbf{j} + \mathbf{k}$$

$$= 4[1, 0, 0] + 7[0, 1, 0] + [0, 0, 1]$$

$$= [4, 7, 1]$$

- Magnitude of a vector:

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

- constant (scalar) * vector:

$$c * \mathbf{v} = [cv_x, cv_y, cv_z]$$

- Sum of two vectors:

$$\mathbf{v} + \mathbf{a} = [v_x + a_x, v_y + a_y, v_z + a_z]$$

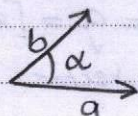
* VECTOR PRODUCT.

two types:

[1] INNER (DOT) PRODUCT

(Vector • Vector = scalar)

$$a \cdot b = |a||b| \cos \alpha$$



OR $a \cdot b = a_x b_x + a_y b_y + a_z b_z$

$$a \cdot b = 0 \iff \alpha = \frac{\pi}{2}, a \perp b$$

[2] OUTER (CROSS) PRODUCT.

(Vector x Vector = vector)

$$a \times b = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

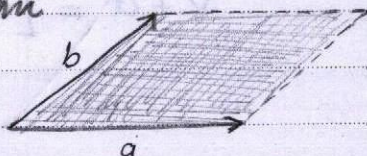
النَّيِّبُ (مِصْنُوع)
(مِصْنُوع)

$$|v| = |a \times b| = |a||b| \sin \alpha \quad \text{"as a magnit."}$$

$$a \times b = 0 \iff a \parallel b \quad \text{نفس الاتجاه أو عكسه}$$

area of the parallelogram
is length of $a \times b$.

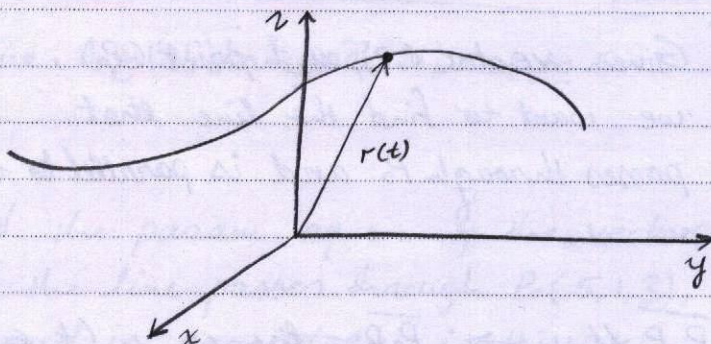
Direction obtained by
the right hand Rule.



9.5 FAMILIES OF CURVES.

$$r(t) = [x(t), y(t), z(t)]$$

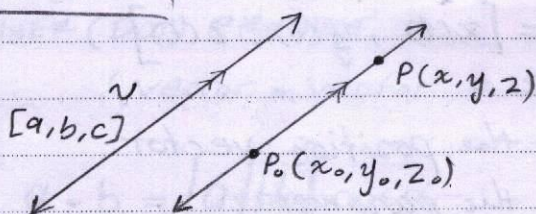
- r is the position vector.
- t is the parameter.



* TYPES OF CURVES WE WILL DISCUSS:

1. lines. (and line segment)
2. circles.
3. Ellipse.
4. Helix
 - circular.
 - elliptical.
5. Plane Curves.

1. LINES



{ Given vector (v) and point (P_0),
we want to find the line that
passes through P_0 and is parallel to v .

$$\overline{P_0P} \parallel v \Rightarrow \overline{P_0P} = tv \quad (t \in \mathbb{R})$$

$$[x - x_0, y - y_0, z - z_0] = [at, bt, ct]$$

$$x - x_0 = at$$

$$y - y_0 = bt$$

$$z - z_0 = ct$$

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \\ t \in \mathbb{R} \end{cases}$$

These are called the
parametrical Equations
of a line.

However, we can find the vector equation for a line.

→ From the param. eq.:

$$[x, y, z] = [x_0, y_0, z_0] + t[a, b, c]$$

$$\boxed{r(t) = r_0 + tv} \quad \begin{array}{l} \text{vector} \\ \text{equation.} \end{array}$$

→ line r_0 نقطة على الخط $r(t)$ و v متجه الاتجاه

Ex Find the param. eq.s & the vector eq. of the line passes through $P_0(5, 1, 3)$, and is parallel to $i + 4j - 2k$.

→ Par. eq. : $x = 5 + t$

$$y = 1 + 4t$$

$$z = 3 - 2t$$

$$t \in \mathbb{R}$$

→ r_0 نقطة على الخط

→ vector eq. : $r(t) = [5, 1, 3] + t[1, 4, -2]$
 $= [5 + t, 1 + 4t, 3 - 2t]$

or we can write it in this way:

$$r(t) = (5+t)i + (1+4t)j + (3-2t)k$$

Ex Find the param. eq.s and the vector eq. of the line passing through the points $P_1(2, 4, -1)$, $P_2(5, 0, 7)$

P_2 و P_1 نختار P_0 :
 (new point) vector v : $v = \overrightarrow{P_1 P_2}$

$$v = \overrightarrow{P_1 P_2} = [3, -4, 8]$$

$$P_0 = P_1 = (2, 4, -1)$$

نقطة P_0 ونختار v

→ Par. equ.

$$x = 2 + 3t$$

$$y = 4 - 4t$$

$$z = -1 + 8t$$

$$t \in \mathbb{R}$$

→ vector equ.

$$r(t) = [2 + 3t, 4 - 4t, 8t - 1]$$

$$= (2 + 3t)i + (4 - 4t)j + (8t - 1)k$$

NOTE | vector equation and parametrical equations are not unique for a line.

LINE SEGMENT

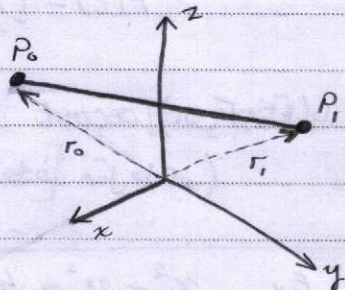
يعني جزء من line .. من نقطة الى أخرى فقط (مقطع خط)

$$r = r_0 + vt$$

But $v = r_1 - r_0$

$$\boxed{r = r_0 + (r_1 - r_0)t}$$

$$0 \leq t \leq 1$$



النسبة t : إذا عوضنا $t=0$ يعطينا r_0 التي يمثل بداية ال segment

وإذا عوضنا $t=1$ يعطينا r_1 التي يمثل نهاية ال segment

وإذا فعلنا أي شيء بين r_0 و r_1 حصلنا أكثر أو أقل

فيخرجها بين 0 و 1

$$0 \leq t \leq 1$$

for the previous Example:

find the Equation of line segment $[P_1, P_2]$

$$r(t) = [2+3t, 4-4t, 8t-1]$$

$$t \in [0, 1]$$

إذا ما سبقتنا t ..

2. Circles

$$r(t) = [\underline{x}, \underline{y}, \underline{z}]$$

تسمى قيم $a \cos t$ و $a \sin t$ (التي تكون بالمعادلة) و $constant$ (مقداراً ثابتاً)

Ex $x^2 + y^2 = 4, z = 0$

التي تكون بالمعادلة $\left\{ \begin{array}{l} x = 2 \cos t \\ y = 2 \sin t \\ z = 0 \end{array} \right.$ parametrical Equations

$r(t) = [2 \cos t, 2 \sin t, 0]$ vector Equation.

NOTE t is the angle that draws the circle,

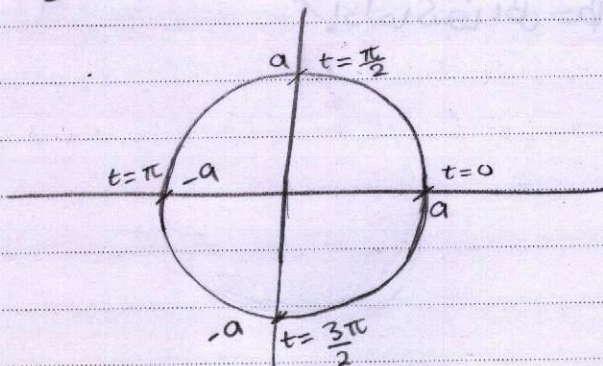
نستخدم t إذا اطلبنا بالزاوية t أو t دائرة

نصف دائرة

$$t \in [0, \pi]$$

دائرة كاملة

$$t \in [\pi, \frac{3\pi}{2}]$$



$$\underline{\text{Ex}} \quad x^2 + y^2 = 16, \quad z = -3$$

$$t \in [0, 2\pi]$$

$$r(t) = [4\cos t, 4\sin t, -3]$$

$$\underline{\text{Ex}} \quad x^2 + z^2 = 9, \quad y = 1$$

$$t \in [0, 2\pi]$$

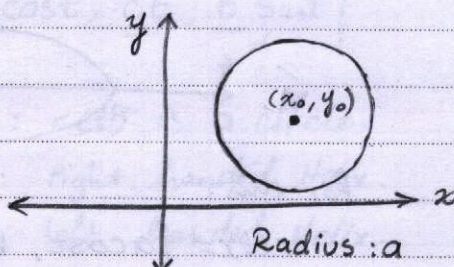
$$r(t) = [3\cos t, 1, 3\sin t]$$

In General:

$$x = x_0 + a\cos t$$

$$y = y_0 + a\sin t$$

$$z = z$$



$$\text{Its Equation: } (x - x_0)^2 + (y - y_0)^2 = a^2$$

$$\underline{\text{Ex}} \quad x^2 + 4x + y^2 = 5, \quad z = 2.$$

المكتمل مربع

$$(x + 2)^2 + y^2 = 9$$

$$\hookrightarrow x_0 = -2$$

$$x = -2 + 3\cos t$$

$$t \in [0, 2\pi]$$

$$y = 3\sin t$$

$$z = 2.$$

أو يمكننا إكمالها فها: الذي كان المربع

$$x + 2 = 3\cos t$$

$$x = -2 + 3\cos t$$

نفس الشيء ..

3. Ellipse

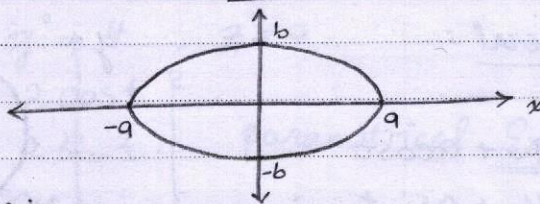
$$r(t) = [x, y, z]$$

• constant ω , $b \sin t$, $a \cos t$

• $a = b$: circle

• Its Equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0$$



For this:

$$r(t) = [a \cos t, b \sin t, 0]$$

Ex $\frac{y^2}{4} + \frac{z^2}{5} = 1, \quad x = 3$

$$r(t) = [3, 2 \cos t, \sqrt{5} \sin t]$$

Ex $(x-2)^2 + 9(y+1)^2 = 36, \quad z = -1$

$$\frac{(x-2)^2}{36} + \frac{(y+1)^2}{4} = 1, \quad z = -1$$

$$x-2 = 6 \cos t, \quad y+1 = 2 \sin t$$

$$r(t) = [2 + 6 \cos t, 2 \sin t - 1, -1]$$

4. Helix

↳ circular: like the circle but with ct instead of c .

$$\text{so } r(t) = [\underline{a} \cos t, \underline{a} \sin t, \underline{ct}]$$

↳ elliptical: like the elliptic but with ct instead of c .

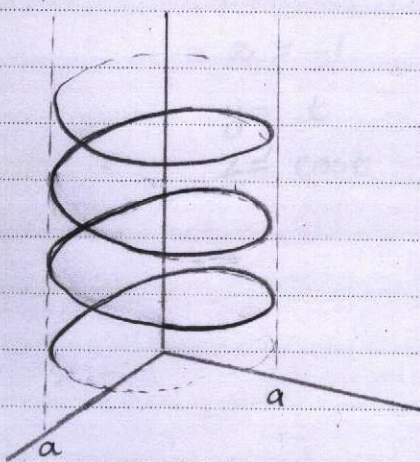
$$r(t) = [\underline{a} \cos t, \underline{ct}, \underline{b} \sin t]$$

value of c | $c=0$: $r(t)$ is a circle.

$c > 0$: right handed Helix.

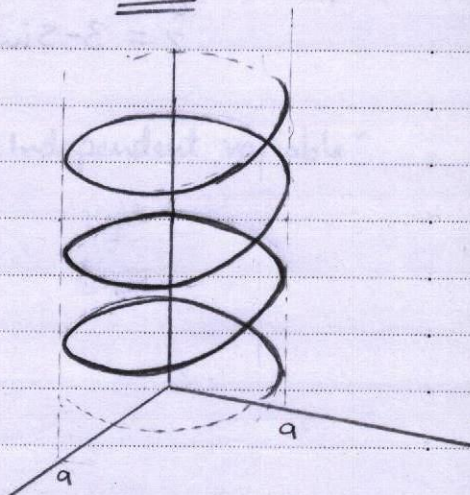
$c < 0$: left handed Helix.

$c < 0$



left handed.

$c > 0$



right handed

Ex Find the vector equation of :

$$x^2 + y^2 = 9, \quad z = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r(t) = [3 \cos t, 3 \sin t, t]$$

$$\hookrightarrow \tan^{-1} \frac{3 \sin t}{3 \cos t} = \tan^{-1} \tan t = t$$

that was a circular helix.

Ex Find the parametric equations of the curve results from intersection of :

$$x^2 + y^2 = 1 \quad \& \quad y + z = 3$$

$$\hookrightarrow z = 3 - y$$

$$x = \cos t$$

$$y = \sin t$$

$$z = 3 - \sin t$$

5. Plane Curves

بعض المتغيرات المستقلة، بشروط متساوية
تفرقة المتغيرات المستقلة
و يجب الباقين بناءً على ذلك ..

Ex $y = x^2$, $z = 0$

$$x = t$$

"The indep. variable"

$$y = t^2$$

$$z = 0$$

Ex $z = \cos y$, $x = -1$

$$x = -1$$

$$y = t$$

"Independent variable"

$$z = \cos t$$

Subtitle

Equation of Tangent Line

$$\boxed{r(t) = r(t_0) + r'(t_0)t}$$

Ex Find parametrical Equations of the tangent line to the circular helix $r(t) = [\cos t, \sin t, t]$ at $t = \frac{\pi}{2}$.

Sol: I need $r(\pi)$ and $r'(\pi)$.

$$= r(\pi) = [-1, 0, \pi]$$

$$r'(t) = [-\sin t, \cos t, 1]$$

$$= r'(\pi) = [0, -1, 1]$$

$$r(t) = [-1, 0, \pi] + [0, -1, 1]t$$

$$= [-1, -t, t + \pi]$$

$$\left. \begin{array}{l} x = -1 \\ y = -t \\ z = t + \pi \end{array} \right\} \text{param. equ. of line}$$

9.7 GRADIENT (Ex: 1-6, 11-15, 30-35)

* There are two types of functions:

1. Scalar functions. (no i, j, k)

For Example:

$$f(x, y, z) = x^2 + xyz + zy$$

2. Vector functions. (with i, j, k)

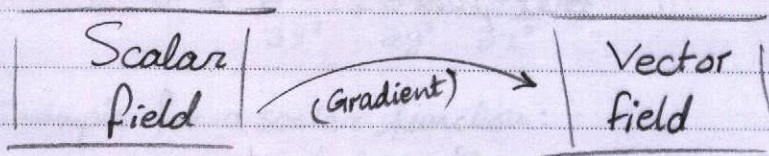
For Example:

$$f(x, y, z) = xyi + 3j - zk$$

$$\text{or } r(t) = r(t_0) + r'(t_0)t$$

Gradient exists only for scalar functions.

It converts them to vector functions.



Def

let $f(x, y, z)$ be a differentiable function in D .

Gradient of f is:

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$\nabla f = f_x i + f_y j + f_z k$$

$$\text{or } \nabla f = [f_x, f_y, f_z]$$

Ex $f(x, y, z) = 3y^2 + 5xz - 3x$. Find:

[1] ∇f

[2] $\nabla f(1, 0, -1)$

Sol:

Ans: [1] $\nabla f = (5z - 3)\mathbf{i} + 6y\mathbf{j} + 5x\mathbf{k}$
 $= [5z - 3, 6y, 5x]$

Therefore [2] $\nabla f(1, 0, -1) = [-8, 0, 5] = -8\mathbf{i} + 5\mathbf{k}$.

$f(x, y, z) = c$ is a surface in $\mathbb{R}^3$

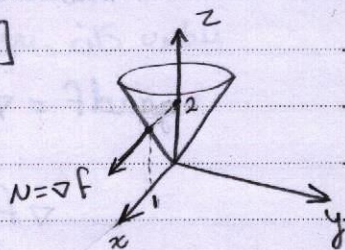
↳ The Gradient ∇f at any point p is
the normal vector to the surface at
that point.

Ex Given a cone $z^2 = 4(x^2 + y^2)$
find the normal vector at the point $p(1, 0, 2)$

$$f(x, y, z) = 4x^2 + 4y^2 - z^2$$

$$\nabla f = [8x, 8y, -2z]$$

$$\nabla f(1, 0, 2) = [8, 0, -4]$$



Def

A vector field F is called conservative field if it is the gradient of some scalar function f . and f is called potential function.

$$\begin{array}{ccc} F & = & \nabla f \\ \uparrow & & \uparrow \\ \text{conservative} & & \text{potential} \end{array}$$

LAPLACIAN OPERATOR ∇^2 or Δ

Scalar $\xrightarrow{\nabla^2}$ Scalar

Vector $\xrightarrow{\nabla^2}$ Vector

$$\nabla^2 = \nabla \cdot \nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- Example for a scalar function:

$$f(x, y, z) = xy^2 + z^3$$

$$\nabla^2 f = 0 + 2x + 6z$$

- Example for a vector function:

$$F(x, y, z) = 3z^2 i + xyz j + x^2 z^2 k$$

$$\begin{aligned} \nabla^2 F &= \nabla^2(3z^2) i + \nabla^2(xyz) j + \nabla^2(x^2 z^2) k \\ &= 6 i + (2z^2 + 2x^2) k \end{aligned}$$

9.8 DIVERGENCE

(Ex: 1-6, 9, 15-20)

$$\boxed{\text{Vector field}} \xrightarrow{(\text{div})} \boxed{\text{scalar field}}$$

$$\text{div } F = \nabla \cdot F = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot [F_x, F_y, F_z]$$

$$\Rightarrow \nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \text{scalar}$$

Ex Find $\text{div } F$ for $F = [x, -xy, z^2]$

$$\nabla \cdot F = 1 - x + 2z$$

9.9 CURL OF VECTOR FIELDS

$$\boxed{\text{Vector field}} \xrightarrow{(\text{curl})} \boxed{\text{Vector field}}$$

$$\text{for } F = [F_x, F_y, F_z]$$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \text{vector}$$

Ex $F = xi - xyj + z^2k$. find $\text{curl } F$.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -xy & z^2 \end{vmatrix} = 0i + 0j - yk = -yk$$

$$= [0, 0, -y]$$

NOTES

- $\text{div}(\text{curl } F) = 0$ "zero as a scalar"
 - $\text{curl}(\nabla f) = 0$ "zero vector"
- ↳ This means that gradient field is irrotational.

Ex $r = xi + yj + zk$

Verify: a $\nabla \cdot r = 3$

b $\nabla \cdot (|r|r) = 4|r|$

Sol a $\nabla \cdot r = 1 + 1 + 1 = 3$

b $R = |r| = \sqrt{x^2 + y^2 + z^2}$

$|r|r = Rr = Rx i + Ry j + Rz k$

$\nabla \cdot (|r|r) = \nabla \cdot (Rx i + Ry j + Rz k)$

كل مرة: الأول $\nabla \cdot (Rx i) + \nabla \cdot (Ry j) + \nabla \cdot (Rz k)$ * متعة الأول

$= R + x R_x + R + y R_y + R + z R_z$

$= 3R + \frac{2x^2}{2R} + \frac{2y^2}{2R} + \frac{2z^2}{2R}$

$= 3R + \frac{1}{R} (x^2 + y^2 + z^2) = 4R$

$= 4|r|$

و هو المطلوب

Ex Given $F = \sin x \, i + \cos(x-y) \, j + z \, k$
Find $\nabla(\nabla \times F)$.

Sol $\nabla(\text{curl } F) = 0$ for any F .

Ex Given that $u = [u_1, u_2, u_3]$
Find $(u \cdot \nabla) f$

ملاحظة: هذا النوع من المسألة عموماً يكون في
تبريد فيزياء. $\times$ cross $\cdot$ dot
أيكل كل المعادلات

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]$$

Sol $(u \cdot \nabla) f = ([u_1, u_2, u_3] \cdot [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}]) f$

$$= u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} + u_3 \frac{\partial f}{\partial z}$$

Ex Given: $F = (x^2 z) i + (y z^2) j + (-3xy) k$
 $G = y^2 i + (-yz) j + 2x k$
 $f = 2x^2 + yz$.

Find:

[a] $(F \cdot \nabla) f = x^2 z (4x) + y z^2 (z) - 3xy (y)$

[b] $(F \cdot \nabla) G = x^2 z [0, 0, 2] + y z^2 [2y, -z, 0]$
 $- 3xy [0, -y, 0]$
 $= [y z^2, 3xy^2 - y z^3, 2x^2 z]$

[c] $(F \times \nabla) f$.

Ex u and v are vector fields.
and f is a scalar function.

Find:

$$\begin{aligned} \boxed{1} \quad (u \cdot \nabla) v &= \left(u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + u_3 \frac{\partial}{\partial z} \right) v \\ &= \left[u_1 \frac{\partial v_1}{\partial x} + u_2 \frac{\partial v_1}{\partial y} + u_3 \frac{\partial v_1}{\partial z}, u_1 \frac{\partial v_2}{\partial x} + u_2 \frac{\partial v_2}{\partial y} + u_3 \frac{\partial v_2}{\partial z}, \right. \\ &\quad \left. u_1 \frac{\partial v_3}{\partial x} + u_2 \frac{\partial v_3}{\partial y} + u_3 \frac{\partial v_3}{\partial z} \right] \end{aligned}$$

$$\begin{aligned} \boxed{2} \quad (u \times \nabla) f &= \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} * f \\ &= [u_2 f_z - u_3 f_y, u_3 f_x - u_1 f_z, u_1 f_y - u_2 f_x] \end{aligned}$$

END OF CHAPTER 9

CHAPTER 10

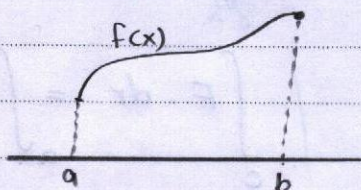
VECTOR INTEGRAL CALCULUS

10.1 LINE INTEGRAL.

(Ex: 2-11)

we used to integrate over a line segment of the x -axis.

$$\int_a^b f(x) dx$$

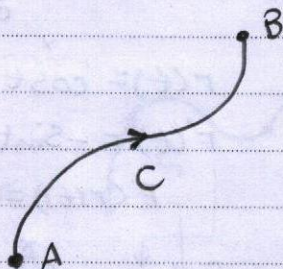


from now on, we will integrate functions along any line C , where:

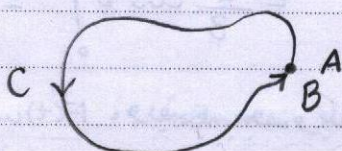
$$C: \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

- C is an oriented curve (from A to B).

- The direction from A to B , in which t increases, is called the positive direction of C .



- If $A = B$; C is a closed path.



Def

let C be a piecewise smooth curve,
 $C: r(t)$, and $r'(t)$ is continuous.

The line integral $\int_C$ of a vector function F
 along curve C is given by:

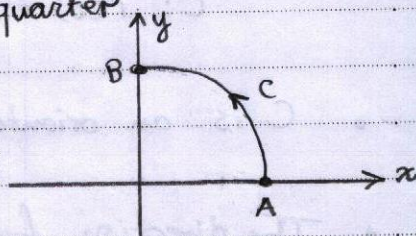
$$\int_C F \cdot dr = \int_a^b \underbrace{F(r(t))}_{\substack{\text{نقطة } F \text{ على } C \\ \text{بمجرد } t \text{ فقط}}} \cdot r'(t) dt$$

Ex $F = x^2 i - xy j$ along the quarter

$A(1,0)$, $B(0,1)$

← Evaluate $\int_C F \cdot dr$

$t = \frac{\pi}{2}$



$$r(t) = \cos t i + \sin t j$$

$$r'(t) = -\sin t i + \cos t j$$

$$F(r(t)) = \cos^2 t i - \cos t \sin t j$$

t هو x و y في F

$$\int_C F \cdot dr = \int_0^{\frac{\pi}{2}} -\sin t \cos^2 t - \sin t \cos^2 t dt = -2 \int_0^{\frac{\pi}{2}} \sin t \cos^2 t dt$$

$$= \frac{2}{3} \cos^3 t \Big|_0^{\frac{\pi}{2}} = -\frac{2}{3}$$

نقطة F على C بمجرد t فقط $r(t)$ و $r'(t)$ فقط t فقط

Ex $F = xyi + yzj + zxk$

$C: r(t) = \underbrace{t}_x i + \underbrace{t^2}_y j + \underbrace{t^3}_z k \quad 0 \leq t \leq 1$

[1] $r'(t) = 1i + 2tj + 3t^2k$

[2] $F(t) = t^3i + t^5j + t^4k$

$$\int_C F \cdot dr = \int_0^1 t^3 + 2t^6 + 3t^6 dt = \frac{27}{28}$$

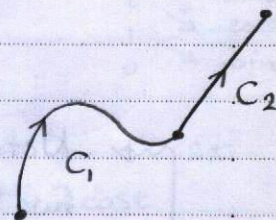
PROPERTIES

• $\int_C kF \cdot dr = k \int_C F \cdot dr$ "Linearity"

• $\int_C (F+G) \cdot dr = \int_C F \cdot dr + \int_C G \cdot dr$

• when C is piecewise:

$$\int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr$$



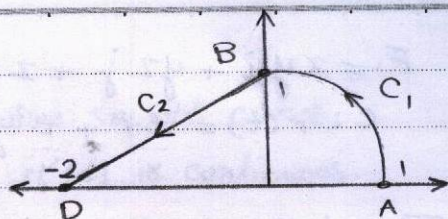
• $\int_{\underbrace{-C}} F \cdot dr = - \int_C F \cdot dr$ نفس المسار لكن بعكس الاتجاه

Ex

$$F = x^2 i - xy j$$

$$C = C_1 + C_2$$

(property #3)



$$\int_C F \cdot dr = \underbrace{\int_{C_1} F \cdot dr}_{\textcircled{1}} + \underbrace{\int_{C_2} F \cdot dr}_{\textcircled{2}}$$

① Calculated in previous Example.

$$x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq \frac{\pi}{2}$$

② This one has different x , y , and t .

C_2 is the line segment $r(t) = r_0 + (r_1 - r_0)t$
from $B[0, 1]$ to $D[-2, 0]$

and Remember: For line segment: $0 \leq t \leq 1$

$$\begin{aligned} \text{Now } r(t) &= [0, 1] + [-2, -1]t \\ &= [-2t, 1-t]. \end{aligned}$$

~> Note that for C_1 and C_2 , there are two different separated Integrals, no one affects the other.

Continue: $r'(t) = [-2, -1]$

$$F(r(t)) = [4t^2, 2t(1-t)]$$

$$\int_{C_2} (-8t^2 - 2t(1-t)) dt = \dots$$

$$\text{Finally: } \int_C F \cdot dr = \textcircled{1} + \textcircled{2}$$

APPLICATION

The work done by a force F in the displacement along a curve C is

$$W = \int_C F \cdot dr$$

Ex $F = [z, x, y]$ is a force.

find the work along curve C .

$$C: r(t) = [\cos t, \sin t, 2t] \quad 0 \leq t \leq 2\pi$$

$$[1] \quad F(r(t)) = [2t, \cos t, \sin t]$$

$$[2] \quad r'(t) = [-\sin t, \cos t, 2]$$

$$\int_0^{2\pi} -2t \sin t + \cos^2 t + 2 \sin t \, dt$$

$$\begin{array}{l} t \\ 1 \\ 0 \end{array} \begin{array}{l} \swarrow \sin t \\ \rightarrow -\cos t \\ \searrow -\sin t \end{array}$$

$$= -2(-t \cos t + \sin t) + \frac{1}{2}t + \frac{1}{4} \sin 2t - 2 \cos t \Big|_0^{2\pi}$$

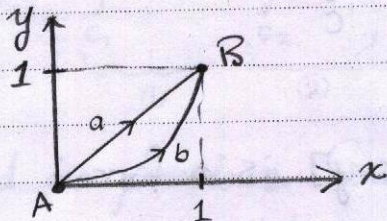
$$= -2(-2\pi + 0) + \pi + 0 - 2 - (-2) = \underline{5\pi}$$

PATH DEPENDENT

Ex Evaluate $\int_C F \cdot dr$, where $F = [0, xy]$ along.

a) C is a line segment from A to B .

b) C is a parabola $y = x^2$ (From A to B).



Sol

a) $\int_A^B F \cdot dr = \int_{C_a} F(r(t)) \cdot r'(t) \cdot dt$

$y = x$, $y = t$, $x = t$

$\rightarrow r(t) = [t, t]$

$r'(t) = [1, 1]$

$0 \leq t \leq 1$
(من 0 إلى 1)

$$\int_0^1 [0, t^2] \cdot [1, 1] dt = \int_0^1 t^2 dt = \left[\frac{1}{3} \right]$$

b) $r(t) = [t, t^2]$

$r'(t) = [1, 2t]$

$0 \leq t \leq 1$

$F(t) = [0, t^3]$

$$\int_0^1 2t^4 dt = \left[\frac{2}{5} \right]$$

اختلاف كل حسب الطريقة التي انتمى
من A إلى B.

(مسار معتمد path dependent)

LINE INTEGRAL FOR THE SCALAR FIELD.

$$f(x, y, z)$$

$$C: r(t) = [x(t), y(t), z(t)] \quad a \leq t \leq b$$

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) ds$$

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

يعني بصيغة
المعادلة:

$$\int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

Ex Evaluate $\int_C xy^4 ds$, where C is the right half of the circle $x^2 + y^2 = 16$ in ccw direction.

Sol:

$$C: r(t) = [4\cos t, 4\sin t] \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

$$r'(t) = [-4\sin t, 4\cos t]$$

$$f = 4\cos t (4\sin t)^4$$

$$ds = \sqrt{(-4\sin t)^2 + (4\cos t)^2} dt = 4 dt$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t) ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^6 \cos t \sin^4 t dt = 4^6 \frac{\sin^5 t}{5} \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{8192}{5}$$

Ex $\int_C y \sin z \, ds$

$C: x = \cos t, \quad y = \sin t, \quad z = t$
 $0 \leq t \leq 2\pi$

$$\int_0^{2\pi} \sin t \cdot \sin t \cdot \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} \, dt$$

$$= \int_0^{2\pi} \sin^2 t \sqrt{2} \, dt$$

* If we want to Integrate over a line,
 with Respect to x, y, z .

• w.r.t x : $\int_C f \, dx = \int_a^b f(x(t), y(t), z(t)) x'(t) \, dt$

• $\int_C f(x, y, z) \, dy = \int_a^b f(x(t), y(t), z(t)) y'(t) \, dt$

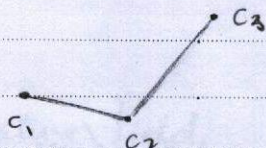
• $\int_C f(x, y, z) \, dz = \int_a^b f(x(t), y(t), z(t)) z'(t) \, dt$

Ex evaluate $\int_C y dx + z dy + x dz$

where C consists of 2 line segments
 from $C_1(2, 0, 0)$ to $C_2(3, 4, 5)$
 and from $C_2(3, 4, 5)$ to $C_3(3, 4, 0)$.

Sol

$$C = C_1C_2 + C_2C_3$$



$$r = r_0 + (r_1 - r_0)t \quad 0 \leq t \leq 1$$

$$\rightarrow C_1C_2: r = [2+t, 4t, 5t]$$

$$\rightarrow C_2C_3: r = [3, 4, 5-5t]$$

$$\begin{aligned} \int_C &= \int_{C_1C_2} 4t(dt) + 5t(4dt) + (2+t)(5 \cdot dt) \\ &+ \int_{C_2C_3} 4 \cdot 0 + (5-5t) \cdot 0 + 3(-5dt) \end{aligned}$$

$$= 2t^2 + 10t^2 + 5t^2 \Big|_0^1 + 15t \Big|_0^1$$

$$= 17 + 15 = 32$$

10.2 PATH INDEPENDENCE OF LINE INTEGRAL

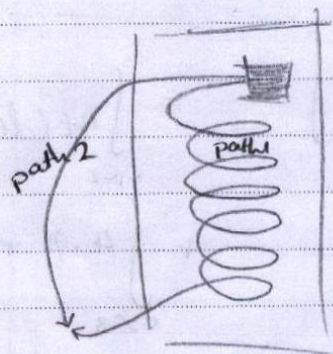
$$\int_C \mathbf{F} \cdot d\mathbf{r}, \quad \mathbf{F} = [F_1, F_2, F_3]$$

F_i are continuous function in some open domain.

< not $\leq$
(not [

- line integral is path independent if it has the same value for all paths in D
(same amount of work, for example, taking different paths to same point)

ISEW!



Th. 1 A line integral $\int F \cdot dr$ is path-independent if and only if F is the gradient of some function f in D . ($F = \nabla f$)

* If $F = \nabla f$, then F is conservative field, and f is potential.

OR $\int F \cdot dr$ is path indep. $\Leftrightarrow F$ is conservative field

* A line integral of a conservative field:

$$\int_C F \cdot dr = f(B) - f(A)$$

Ex Show that $\int F \cdot dr$, where $F = [2x, 2y, 4z]$,

is path independent. and Evaluate

$$\int_{(0,0,0)}^{(2,2,2)} F \cdot dr.$$

Sol if $F = \nabla f \Rightarrow$ path independent (conserv.)

$$F = \underbrace{\frac{\partial f}{\partial x}}_{2x} i + \underbrace{\frac{\partial f}{\partial y}}_{2y} j + \underbrace{\frac{\partial f}{\partial z}}_{4z} k \quad \text{ans given}$$

Now;

$$[1] \quad f(x, y, z) = \int 2x \, dx + g(y, z) = x^2 + g(y, z)$$

$$[2] \quad \frac{\partial f}{\partial y} = 2y \rightarrow 0 + \frac{\partial g}{\partial y} = 2y \rightarrow g = y^2 + l(z)$$

$$[3] \quad \frac{\partial f}{\partial z} = 4z \rightarrow 0 + 0 + l'(z) = 4z \rightarrow l(z) = 2z^2$$

$$\text{finally; } f(x, y, z) = x^2 + y^2 + 2z^2$$

$$\leadsto \text{Checking: } \nabla f = [2x, 2y, 4z] = F \quad \checkmark$$

$$\begin{aligned} & \bullet \int_{(0,0,0)}^{(2,2,2)} F \, dr = f(2,2,2) - f(0,0,0) \\ & = 4 + 4 + 8 - 0 = \underline{\underline{16}} \end{aligned}$$

that's it!

* Consider $A=B$, then

$$\int F \cdot dr = f(B) - f(A) = 0$$

Th. 2

The line integral $\int_C F \cdot dr$ is path independent

if and only if its value around every
over?

closed path in D is zero.

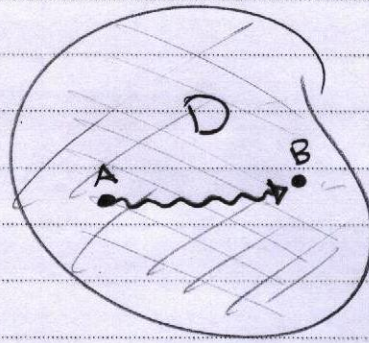
Ex evaluate $\int_C F \cdot dr$, where $F = [2x, 2y, 4z]$.

$$C: r(t) = [4 \cos t, 4 \sin t, 0].$$

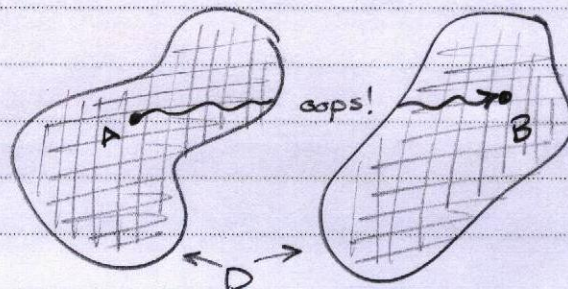
(circle, closed!)

$$\int_C F \cdot dr = 0$$

D is simply connected.



D is not connected



$\int_C F dr$ is path independent

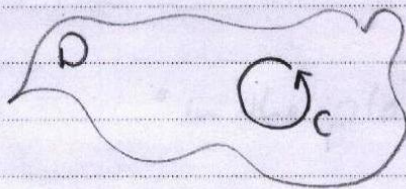
Th. 1 $F = \nabla f$ (F is conservative)

Th. 2 $\oint F dr = 0$

$$\int_A^B F dr = f(B) - f(A)$$

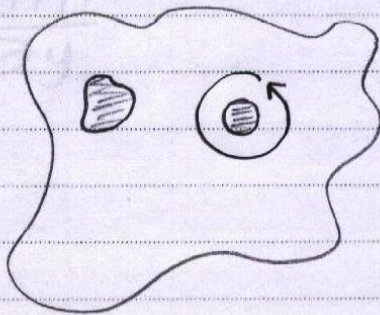
* D is simply connected.

"no holes
inside"



* D is multiply connected.

"some holes
inside"



→ Test for conservative fields

$$F = [F_1, F_2, F_3]$$

$$dr = [dx, dy, dz]$$

$$F \cdot dr = F_1 dx + F_2 dy + F_3 dz$$

$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$

(Exact differential form)

Th. 3

let F be continuous in D , then:

- (a) if F is conserv. field ($F = \nabla f$)
or $F \cdot dr$ is Exact.

$$\Rightarrow \text{curl } F = 0.$$

- (b) if $\text{curl } F = 0$ and D is simply-connected, then F is conserv.
($F \cdot dr$ is Exact)

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

* if $F = [F_1, F_2]$ "in the plane"

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Ex Determine whether the vector fields are conservative:

1 $F(x,y) = [y+x, y-x]$

$$\frac{\partial(y-x)}{\partial x} - \frac{\partial(y+x)}{\partial y} = -1 - 1 = -2 \quad \text{not } 0$$

not conserv.

2 $F(x,y) = [e^y, x e^y]$

$$e^y - e^y = 0 \rightarrow \text{conserv.} \checkmark$$

Ex Show that the form Under the Integral is Exact, then evaluate the Integral.

$$\int_{(5,0)}^{(0,5)} (y^2 e^{2x} dx + y e^{2x} dy)$$

$$\begin{array}{ccc} F_1 & & F_2 \\ \downarrow \partial y & & \downarrow \partial x \\ 2y e^{2x} & \checkmark & 2y e^{2x} \quad (\text{conserv.}) \end{array}$$

• "مسألة" $y \int_0^x e^{2x} dx + \int_0^y x e^{2x} dy$

Evaluating
the Integral next page. →

$$F = \nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

$$\bullet f(x, y) = \int y^2 e^{2x} dx = y^2 \frac{e^{2x}}{2} + g(y)$$

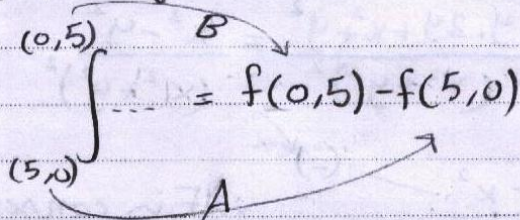
$$\bullet \frac{\partial f}{\partial y} = \frac{1}{2} y e^{2x} + g'(y) = y e^{2x}$$

$$g'(y) = 0$$

$$g(y) = k$$

$$f(x, y) = y^2 \frac{e^{2x}}{2} + k$$

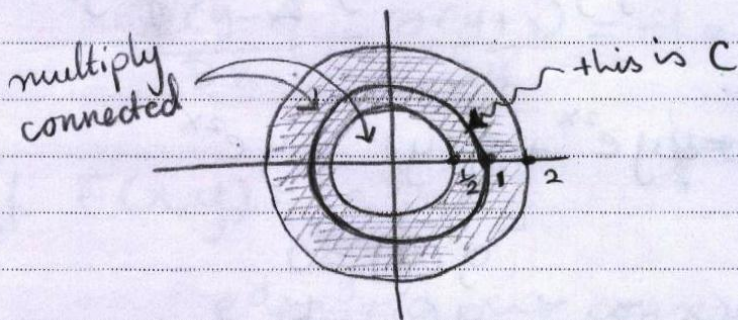
Now, finally!! :



$$\int_{(5,0)}^{(0,5)} = f(0,5) - f(5,0)$$

Ex Given that:

$$F = \left[\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right]$$



Evaluate: $\int_C F \cdot dr$ where $C: x^2+y^2=1$

"from theorem 2: the $\int$ must yield zero"

$$\bullet \quad \frac{\partial F_1}{\partial y} = \frac{-y \cdot 2y + x^2 + y^2}{(x^2+y^2)^2} = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$\bullet \quad \frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2} \xrightarrow{(-)} (=) \quad \therefore F \text{ is conservative}$$

$$\int_C F \cdot dr ; \quad \begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned} \quad 0 \leq t \leq 2\pi$$

in polar:

$$\int_0^{2\pi} \left[\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t} \right] \cdot [-\sin t, \cos t] dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t dt = \underline{\underline{2\pi}}$$

! $\rightarrow$ W @ W b

10.3 Double INTEGRAL

$$\iint_R f(x, y) dA$$

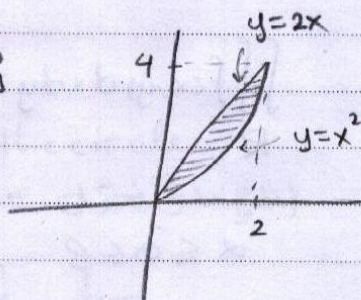
(R)

→ always constant limits
function غير ثابت ، يكون ثابت ، يكون ثابت

Ex

$$\iint_R x^2 + y^2 dA$$

constant
← function
2 2x



$$\textcircled{1} \int_{x^2}^{2x} \int_0^2 (x^2 + y^2) dy dx$$

$$= \int_0^2 (x^2 y + y^3/3) \Big|_{x^2}^{2x} dx = \int_0^2 (2x^3 + \frac{8}{3}x^3 - x^4 - \frac{x^6}{3}) dx = \dots$$

$$\textcircled{2} \int_{y/2}^{\sqrt{y}} \int_0^4 (x^2 + y^2) dx dy = \int_{y/2}^{\sqrt{y}} (\frac{x^3}{3} + y^2 x) \Big|_{y/2}^{\sqrt{y}} dy$$

$$= \int_0^4 (\frac{\sqrt{y}^3}{3} + y^2 \sqrt{y} - \frac{y^3}{24} - \frac{y^3}{2}) dy = \dots$$

Polar Coordinates.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\iint_R f(x, y) dx dy = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) \underline{r dr d\theta}$$

$$\alpha \leq \theta \leq \beta, \quad h_1(\theta) \leq r \leq h_2(\theta)$$

Ex $\iint_R (x^2 + y^2) dx dy$

(R)

↳ is circle $x^2 + y^2 = 4$

$$\int_0^{2\pi} \int_0^2 r^2 \cdot r dr d\theta = \int_0^{2\pi} \left. \frac{r^4}{4} \right|_0^2 d\theta = \frac{2^4}{4} (2\pi)$$

* $\int_0^1 \sqrt{1-x^2} dx$ is half a circle.

1) • $\iint_R 1 dA = \text{area of the region } R$

2) • $\iint_R 1 dx dy = \text{area of the circle}$
(polar description)
 $R \rightarrow x^2 + y^2 = 9$

$A = \pi R^2$ here 9π .

$\rightarrow$ not 81!

if $x^2 + y^2 = 9^2 \rightarrow 81 \checkmark$

• $\iint_R f(x, y) dx dy$

no specific application.

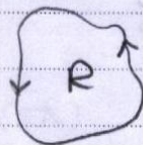
10.4 GREEN'S THEOREM

let R be a closed bounded Region in the xy -plane, whose boundary C is a positive oriented, piecewise smooth closed curve.

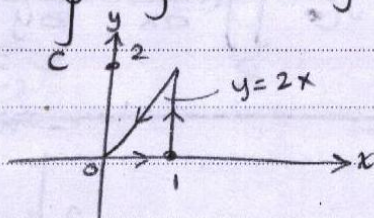
If $F_1(x, y)$ & $F_2(x, y)$ have continuous partial derivatives on an open region containing R , then:

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy = \oint_C F_1 dx + F_2 dy$$

positive orientation : CCW



Ex Evaluate $\oint_C xy dx + x^2 y^3 dy$



$$\frac{\partial F_1}{\partial y} = x$$

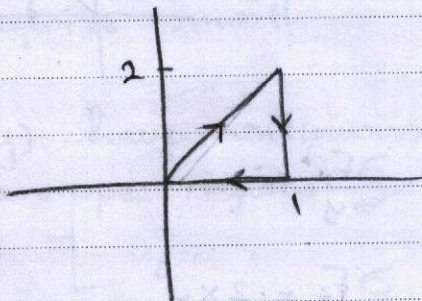
$$\frac{\partial F_2}{\partial x} = 2xy^3$$

تختلف

$$\oint = \int_0^1 \int_0^{2x} (2xy^3 - x) dy dx = \dots = \frac{2}{3}$$

$$\oint F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

Σx



$$\oint_C (xy dx + x^2 y^3 dy)$$

$$= - \iint 2xy^3 - x = -\frac{2}{3}$$

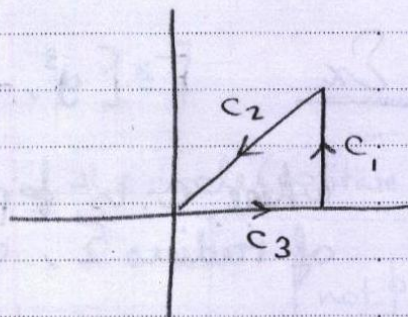
تقريباً في بي بي

Green's thm.
(positive orie) يتقابل مع دوران كس هاد كمل

عنا في هاد مبره

$$\sum_x \int_{C_1 \cup C_2} xy dx + x^2 y^3 dy$$

closed curve



$$\oint_{C_1 \cup C_2 \cup C_3} - \int_{C_3}$$

closed loop $\Rightarrow$ Green's line

$\Rightarrow$ from previous Example.

$$= \frac{2}{3} - \int_{C_3} xy dx + x^2 y^3 dy$$

$$C_3: r = (0,0) \rightarrow (1,0)$$

$$r(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}$$

$\begin{matrix} x \\ y \end{matrix}$

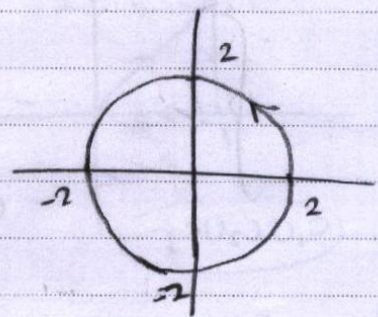
$$= \int_0^1 0 dx + 0 dy = 0$$

$$\text{total} = \frac{2}{3}$$

Ex $F = [y^3, -x^3]$ Evaluate $\oint_C F \cdot dr$

where C is a positively oriented circle of radius 2, centered at the origin

$$= \iint_R (-3x^2 - 3y^2) dx dy$$



$$= -3 \iint x^2 + y^2 dx dy$$

circle : into polar : $x = r \sin \theta$, $0 \leq \theta \leq 2\pi$
 $y = r \cos \theta$, $0 \leq r \leq 2$

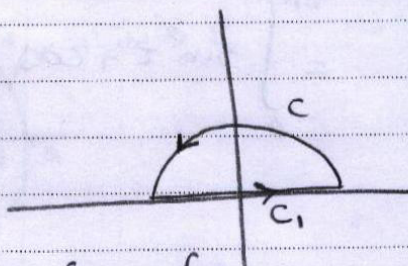
$$= -3 \int_0^{2\pi} \int_0^2 r^2 \cdot r dr d\theta = -3 \cdot 2\pi \cdot \frac{2^4}{4} = -24\pi$$

Ex $F = [e^x + 4y, \frac{\sin y}{y^2 + 1} + 5x]$

C is the upper half of the circle $x^2 + y^2 = 4$
(positive oriented)

Evaluate: $\int F dr$

By Definition $\Rightarrow$



* because it is not closed ($\int$ not $\oint$),
we create C_1 .

$$\int_C = \oint_{C \cup C_1} - \int_{C_1}$$

$$\textcircled{1} = \iint_R 5 - 4 \, dx \, dy = \text{area of } \frac{1}{2} \text{ circle} = \frac{\frac{1}{2} \pi \cdot 2^2}{2} = \underline{\underline{2\pi}}$$

$$\textcircled{2} \, C_1: r(t) = [-2 + 4t, 0] \leadsto r'(t) = [4, 0]$$

$$\int_{C_1} F dr = \int [e^{-2+4t}, 5(-2+4t)] \cdot [4, 0] \, dt$$

$$= 4 \int_0^1 e^{-2} e^{4t} \, dt = \frac{4}{e^{2t}} \cdot \frac{e^{4t}}{4} \Big|_0^1 = \frac{e^4 - 1}{2}$$

$$\text{total} = 2\pi - \frac{e^4 - 1}{2}$$

APPLICATIONS OF GREEN'S THEOREM

$$\iint_R 1 dx dy = \text{area of } R$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$$

- If $F_2 = x$, $F_1 = 0$

$$\oint_C x dy = \iint_R 1 dx dy$$

- If $F_2 = 0$, $F_1 = -y$

$$\oint_C -y dx = \iint_R 1 dx dy$$

→ Summed together:

$$\oint_C x dy - y dx = 2 \iint_R dx dy$$

$$\boxed{\text{AREA OF } R = \frac{1}{2} \oint_C x dy - y dx}$$

Ex Find the area of ellips $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$x = a \cos t, \quad y = b \sin t \quad 0 \leq t \leq 2\pi$$

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (a \cos t)(b \cos t) dt + (b \sin t)(-a \sin t) dt \\ &= \frac{ab}{2} \int_0^{2\pi} 1 dt = ab\pi \end{aligned}$$

قد بآدم green's ؟ لا توافقان دفت!

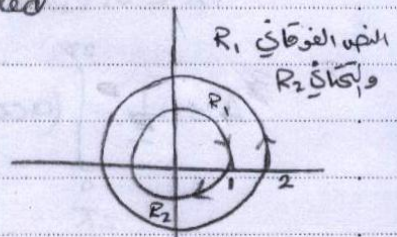
قالنا آرينا!

Ex Evaluate $\oint_C y^3 dx - x^3 dy$

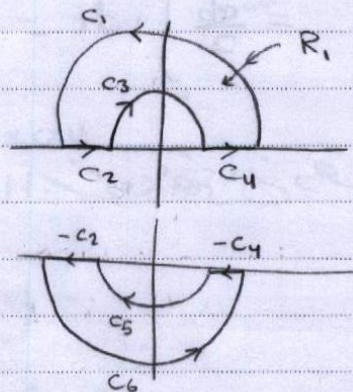
C_i are two circles of radius 1 and 2

* region R is not simply connected

$$\iint_R = \iint_{R_1} + \iint_{R_2}$$



$$= \oint_{C_1 \cup C_2 \cup C_3 \cup C_4} + \oint_{C_5 \cup C_2 \cup C_6 \cup -C_4}$$



$$= \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} + \int_{C_5} - \int_{C_2} + \int_{C_6} - \int_{C_4}$$

$$= \int_{C_1} + \int_{C_6} + \int_{C_3} + \int_{C_5}$$

الدائرة الكبيرة، الدائرة الصغيرة

$$\oint_C = \iint_R (-3x^2 - 3y^2) dx dy$$

$$= -3 \int_0^{2\pi} \int_1^2 r^2 r dr d\theta$$

10.5 SURFACES

(ex: 1-8, 14-19)

$$z = f(x, y) \quad \text{or} \quad g(x, y, z) = 0$$

$$z = \sqrt{a^2 - x^2 - y^2} \leadsto z^2 + x^2 + y^2 - a^2 = 0$$

$$* \mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$$

$\leadsto$ Normal vector $\mathbf{N}$ of Surface S at the point p .

(Normal to tangent surface)

$$\boxed{1} \quad \mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$$

$$\boxed{2} \quad \mathbf{N} = \nabla g(x, y, z)$$

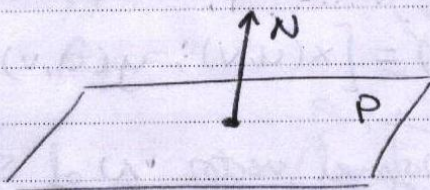
$\leadsto$ Unit Normal vector

$$\mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|}$$

1 PLANES

$$ax + by + cz + d = 0$$

to determine a plane $P_0(x_0, y_0, z_0)$
and normal vector $N = ai + bj + ck$



Ex find the Equation of the plane through
the point $P(x_0, y_0, z_0)$ and with $N = [a, b, c]$

$$* * \left| a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \right|$$

$$2(x - 2) + 3(y - 4) + 4(z + 1) = 0$$

$$2x + 3y + 4z = 12 \quad \Rightarrow$$

Ex Find the equation of the plane that passes through points

$$P(1,3,2), Q(3,-1,6), R(5,2,0)$$

$$\overrightarrow{PQ} = [2, -4, 4]$$

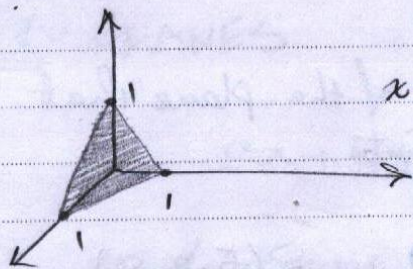
$$\overrightarrow{PR} = [4, -1, -2]$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}$$

$$12(x-1) + 20(y-3) + 14(z-2) = 0$$

$$12x + 20y + 14z = 100$$

نفس الشيء $\rightarrow 100 - 12x - 20y - 14z = 0$



$$x + y + z = 1$$

Sew!

Ex write parametrical Equations of
 [1] $6x + 5y - z = 4$

$$x = u, \quad y = v$$

$$z = 6u + 5v - 4$$

$$r(u, v) = u\mathbf{i} + v\mathbf{j} + (6u + 5v - 4)\mathbf{k}$$

while

$$r(x, y) = x\mathbf{i} + y\mathbf{j} + (6x + 5y - 4)\mathbf{k}$$

[2] $4x - 2y + 10z = 16$

$$r(u, v) = \left[u, v, \frac{16 - 4u + 2v}{10} \right]$$

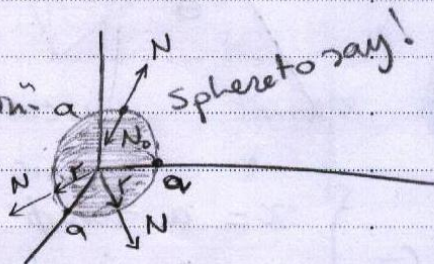
2. SPHERE

$$x^2 + y^2 + z^2 = a^2$$

$$\begin{aligned} N &= \nabla(x^2 + y^2 + z^2 - a^2) \\ &= [2x, 2y, 2z] \end{aligned}$$

Normal will be the same direction as $\vec{r}$ "position"

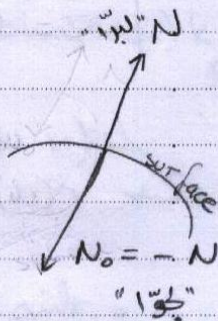
i.e. ~~out~~ direction
outward $\rightarrow$ (positive direction)



في اتجاه "خارجي" $\vec{N}$

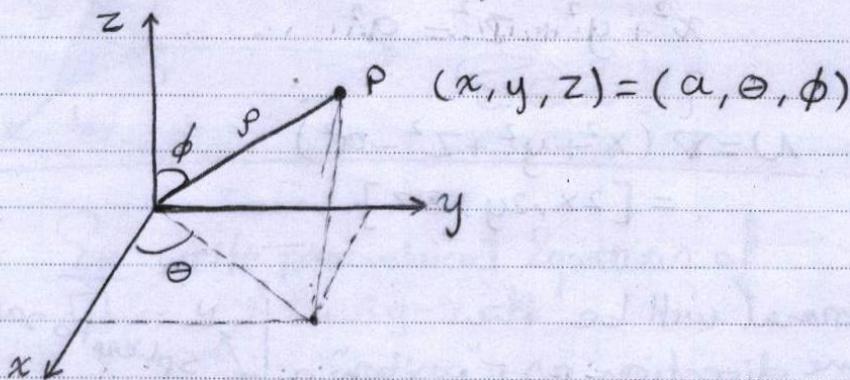
$$N_o = -N = \nabla(a^2 - x^2 - y^2 - z^2)$$

"Inward direction"



PARAMETRIZATION

→ Use spherical coordinates.



$$C = \begin{cases} x = a \sin \phi \cos \theta \\ y = a \sin \phi \sin \theta \\ z = a \cos \phi \end{cases}$$

let $u = \phi$, $v = \theta$

$$r(u, v) = (a \cos v \cdot \sin u) \mathbf{i} + (a \cdot \sin v \cdot \sin u) \mathbf{j} + (a \cos u) \mathbf{k}$$

- this is the "outward sphere"
- if we swap $u \leftrightarrow v$, this would be "inward"

$$0 \geq u \geq \pi$$

$$0 \geq v \geq 2\pi$$

$$r_u = a \cos v \cos u i + a \sin v \cos u j - a \sin u k$$

$$r_v = -a \sin v \sin u i + a \cos v \sin u j + 0 k$$

$$N = r_u \times r_v$$

$$= \begin{vmatrix} i & j & k \\ a \cos v \cos u & \sin v \cos u & -\sin u \\ -a \sin v \sin u & \cos v \sin u & 0 \end{vmatrix}$$

$$= (a^2 \cos v \sin^2 u) \hat{i} + (a^2 \sin^2 u \sin v) \hat{j} + (a^2 \cos^2 v \sin u \cos u + a^2 \sin^2 v \sin u \cos u) k$$

$$= a \sin u (a \cos v \sin u i + a \sin v \sin u j + a \cos u k)$$

$$= \underline{a \sin u} (x i + y j + z k)$$

(positive)

outward sphere in the direction of the position vector.

3. SYLINDER

$$x^2 + y^2 = a^2 \quad -1 \leq z \leq 1$$

"circular cylinder"

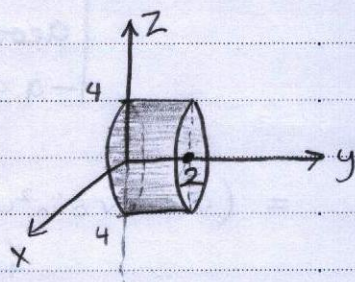
$$r(u, v) = [a \cos u, a \sin u, v]$$

$$0 \leq u \leq 2\pi, \quad -1 \leq v \leq 1$$

$$(*) \quad \frac{x^2}{4} + \frac{z^2}{16} = 1, \quad 0 \leq y \leq 2$$

$$r(u, v) = [2 \cos u, \underline{v}, 4 \sin u]$$

$$r_u = -2 \sin u \mathbf{i} + 4 \cos u \mathbf{j} + 0 \mathbf{k}$$



4. CONE

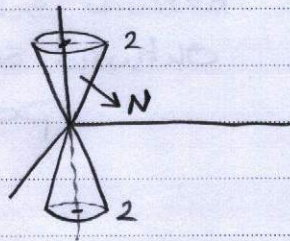
$$z^2 = x^2 + y^2; \quad -2 \leq z \leq 2$$

$$r(u, v) = [u \cos v, u \sin v, u] \quad \begin{matrix} -2 \leq u \leq 2 \\ 0 \leq v \leq 2\pi \end{matrix}$$

$$N = \text{grad}(x^2 + y^2 - z^2)$$

$$= 2x \mathbf{i} + 2y \mathbf{j} - 2z \mathbf{k}$$

"outward"



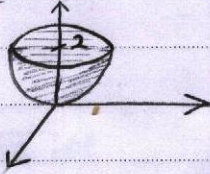
$$-N = \text{grad}(z^2 - x^2 - y^2) \quad \text{"inward"}$$

5. PARABOLOID

حل $z = x^2 + y^2, \quad 0 \leq z \leq 2$

طريقة أخرى
لحلها

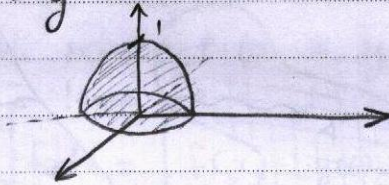
$$r(u, \theta) = [u \cos \theta, u \sin \theta, u^2]$$



طريقة أخرى
 $r(x, y) = [x, y, x^2 + y^2]$

حل

$$z = 1 - x^2 - y^2$$



10.6 SURFACE INTEGRAL (ex. 1-10, 12-16)

S is given by parametrical Equations.

$$r(u, v) = [x(u, v), y(u, v), z(u, v)]$$

let S be a piecewise smooth, so that S has a normal vector N .

Unit Norm vector: $n = \frac{N}{|N|}$

- Surface Integral Over S :

$$\iint_S F n \, dA$$

flux Integral.

$$\therefore \iint_S F n \, dA = \iint_R F(r(u, v)) \cdot N \, du \, dv$$

↘ double integral

*** R is the projection of S onto uv -plane.

$$\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N} du dv$$

$S: \mathbf{r}(u,v)$

R is the projection of S on uv -plane.

Ex Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n} dA$, where:

$\mathbf{F} = [x^2, 0, 3y^2]$, S is the portion of the plane $x+y+z=1$ in the first octant (upward oriented)

S : Parametrical Equations

$$x = u$$

$$y = v$$

$$z = 1 - u - v$$

$$\mathbf{F} = [u^2, 0, 3v^2]$$

N.B. [1] $\mathbf{r}_u \times \mathbf{r}_v = \dots$

[2] ∇ "grad"

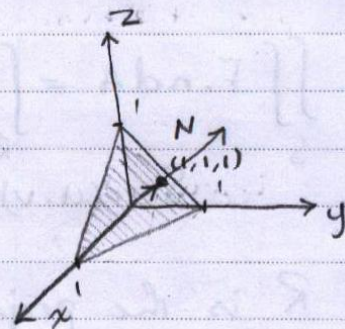
$$x + y + z = 1 \Rightarrow 0$$

$$\nabla = [1, 1, 1]$$

outer

$$\text{or } 1 - x - y - z = 0$$

$$\nabla = [-1, -1, -1]$$

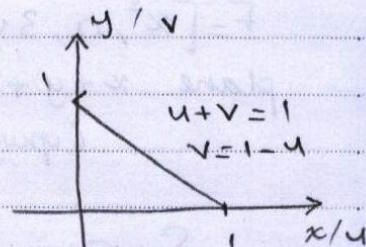


$$N = [1, 1, 1]$$

$$F \cdot N = [u^2, 0, 3v^2] \cdot [1, 1, 1] = u^2 + 3v^2$$

$$\int_0^1 \int_0^{1-u} (u^2 + 3v^2) dv du$$

$$= \int_0^1 (u^2 v + v^3) \Big|_0^{1-u} du = \dots$$



but let x, y be parameters

$$\int_0^1 \int_0^{1-x} x^2 + 3y^2 dy dx$$

Ex Find the flux of $F = zk = [0, 0, z]$
across the outward sphere $x^2 + y^2 + z^2 = a^2$

Sol parametrical Equation of the outward sphere.
 $r(u, v) = [a \cos v \sin u, a \sin v \sin u, a \cos u]$

$$0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$$

$$N = r_u \times r_v = \dots$$
$$= [a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \sin u \cos u]$$

$$F \cdot N = a^2 \sin u \cos u$$

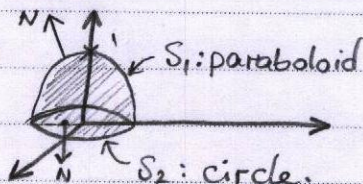
$$\int_0^{2\pi} \int_0^{\pi} a^2 \sin u \cos^2 u \, du \, dv = \frac{4}{3} \pi a^3$$

عَنْ دَنَا حَيْثُ دَنَا مِنْهُ السُّرُورُ
وَأَعْلَى قَدْرًا تَسْأَلُ كَالْبُورِ

Ex Evaluate $\iint_S F \cdot n \, dA$, where:

$$F = [y, x, z]$$

S is the boundary of the solid enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$, outward oriented.



$$\iint_S = \iint_{S_1} + \iint_{S_2} \quad \text{zero}$$

$\rightarrow F = [x, y, 0], N = \nabla z = -k = [0, 0, -1]$
 $F \cdot N = 0$

$\rightarrow S_1: z = 1 - x^2 - y^2$
 in Parametrical form:

$$r(x, y) = [y, x, 1 - x^2 - y^2]$$

$$N = \text{grad}(z + x^2 + y^2 - 1) = [2x, 2y, 1]$$

$$\iint_{S_1} F \cdot N \, dA = \iint [y, x, 1 - x^2 - y^2] \cdot [2x, 2y, 1] \, dx \, dy$$

$(R_1) \rightarrow \text{circle: } x = r \cos \theta \quad 0 \leq \theta \leq 2\pi$
 $y = r \sin \theta \quad 0 \leq r \leq 1$

$$\iint_R \underbrace{4xy}_{2xy + 2xy} + 1 - x^2 - y^2 \, dx \, dy \quad \pi!$$

$$= \iint_{\theta=0}^{2\pi} \int_{r=0}^1 (4r^2 \cos \theta \sin \theta + 1 - r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[r^2 \cos \theta \sin \theta + \frac{r^2}{2} - \frac{r^4}{4} \right] \Big|_0^1 d\theta = \frac{\theta}{4} + \left(\frac{\sin^2 \theta}{2} \right) \Big|_0^{2\pi} = \left| \frac{\pi}{2} \right|$$

$\hookrightarrow 0$

/ /

$$0 \leq u \leq 2\pi$$

05251

$$N = r_u \times r_v = \begin{vmatrix} 1 & \dot{y} & k \\ -v \sin u & v \cos u & 0 \\ \cos u & \sin u & -2v \end{vmatrix}$$

!cylq̃.. oḡōinūip hup .. quānūik *
(inward) ləḡwē - z

$$\iint_{S_1} \vec{F} \cdot \vec{n} \, dS = \iiint_V \vec{F} \cdot \vec{n} \, dV = \iiint_V 4v^3 \cos u \sin u + v(1-v^2) \, dV$$

SURFACE INTEGRAL WITH SCALAR FIELD

$$\iint_S G(r) dA = \iint_R G(r(u,v)) \underbrace{|N(u,v)|}_{\text{"norm of"}} du dv$$

G : Scalar function

S : $r(u,v)$

R : Projection of S on the uv plane.

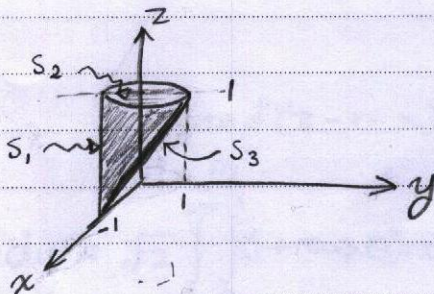
Ex Evaluate $\iint_S z dA$

S_1 : the cylinder $x^2 + y^2 = 1$

S_2 : the disk $x^2 + y^2 \leq 1$ in plane $z = 1$

S_3 : the plane $z = 1 + x$

$$S = S_1 + S_2 + S_3$$



Solution in
next page.

$S_1: r(u, v) = [\cos u, \sin u, v]$
 $0 \leq u \leq 2\pi$
 $0 \leq v \leq 1 + \cos u \quad (1+x = 1+\cos u)$

$$\iint_{S_1} = \iint_R G(r(u, v)) |r_u \times r_v| du dv$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = [\cos u, \sin u]$$

$$|r_u \times r_v| = \sqrt{\cos^2 u + \sin^2 u} = 1$$

$$G(r(u, v)) = v \quad (z = v)$$

$$= \int_0^{2\pi} \int_0^{1+\cos u} v \cdot 1 dv du = \frac{3}{2} \pi$$

$$\iint_{S_2} = 0$$

$S_3: z = 1+x, \quad N = \text{grad}(z - 1 - x) = [-1, 0, 1]$
 $|N| = \sqrt{2}$

$$\iint_R (1+x) \sqrt{2} dx dy = \sqrt{2} \int_0^{2\pi} \int_0^1 (1+\cos \theta) r dr d\theta = \sqrt{2}$$

(R) circle.

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = \left| \frac{3}{2}\pi + \sqrt{2} \right|$$

* Area of the surface S .

$$A = \iint_S dA$$

Ex Find the area of the surface: $x^2 + y^2 + z^2 = a^2$

Sol. Use $\iint_S dA$

$$r(u, v) = a \sin u \cos v \mathbf{i} + a \sin u \sin v \mathbf{j} + a \cos u \mathbf{k}$$

$$0 \leq u \leq \pi$$

$$0 \leq v \leq 2\pi$$

$$|r_u \times r_v| = a^2 \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \sin^2 u \cos^2 u}$$

$$= a^2 \sin u \sqrt{\sin^2 u (\cos^2 v + \sin^2 v) + \cos^2 u}$$

$$= a^2 \sin u$$

now:

$$\int_0^{2\pi} \int_0^{\pi} a^2 \sin u \, du \, dv = 4\pi a^2$$

Student Name:

أسامة عمر صالح

Student number:

0110445

Lecture time:

9-11:30 صباحاً

Q1) Let $f(x, y, z) = e^{yx}z$, $\vec{F}(x, y, z) = xyi + yzj + zxk$ and

$\vec{r} = xi + yj + zk$. State whether each expression is meaningful, then find:

- a) $\nabla \cdot (\underbrace{f \nabla f}_{\text{vector}})$;
- b) $(\vec{F} \cdot \nabla) \underbrace{f}_{\text{vector}}$;
- c) $\nabla f \times (\nabla \cdot f)$;
- d) $\text{div}(\nabla \times \underbrace{\nabla f}_{\text{vector}})$;
- e) $\nabla(\vec{r} \cdot \vec{F})$.

(5 points)

$$\begin{aligned} \text{a) } \nabla f &= \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \\ &= yze^{yx} \hat{i} + xze^{yx} \hat{j} + e^{yx} \hat{k} \\ &= [ye^{yx}, xze^{yx}, e^{yx}] \end{aligned}$$

$$\vec{G} = f \nabla f = [yze^{2yx}, xz^2e^{2yx}, ze^{2yx}]$$

$$\begin{aligned} \nabla \cdot \vec{G} &= \left(\frac{\partial}{\partial x} (yze^{2yx}) + \frac{\partial}{\partial y} (xz^2e^{2yx}) + \frac{\partial}{\partial z} (ze^{2yx}) \right) \\ &= (2y^2ze^{2yx} + 2x^2z^2e^{2yx} + e^{2yx}) \end{aligned}$$

$$\nabla \cdot (f \nabla f) = e^{2yx} (2y^2z^2 + 2x^2z^2 + 1) \checkmark$$

$$\text{b) } \vec{F} \cdot \nabla = [xy, yz, zx] \cdot \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]$$

$$= xy \frac{\partial}{\partial x} + yz \frac{\partial}{\partial y} + zx \frac{\partial}{\partial z}$$

$$(\vec{F} \cdot \nabla) f = xy \frac{\partial f}{\partial x} + yz \frac{\partial f}{\partial y} + zx \frac{\partial f}{\partial z}$$

$$= xy(yze^{yx}) + yz(xze^{yx}) + zx(e^{yx})$$

$$(\vec{F} \cdot \nabla) f = xy^2ze^{yx} + xyz^2e^{yx} + zxe^{yx} \checkmark$$

Continue
in the next
page.

$$c) \nabla f \times (\nabla \cdot f)$$

$$d) \underline{\text{div}} (\nabla \times \nabla f)$$

$$e) \nabla(\vec{r} \cdot \vec{F})$$

$$f = e^{yx} z$$

$$\vec{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

c) ^{it} is meaningless because divergence ~~is~~ can't be applied to scalars $\nabla \cdot f$
vector scalar

$$d) \nabla f = [yze^{yx}, xze^{yx}, e^{yx}]$$

$$\nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yze^{yx} & xze^{yx} & e^{yx} \end{vmatrix} = \hat{i} (xe^{yx} - xe^{yx}) - \hat{j} (ye^{yx} - ye^{yx}) + \hat{k} (ze^{yx} + xze^{yx} - xze^{yx} - yze^{yx})$$

$$= \underline{\underline{\vec{0}}} \text{ (zero vector)}$$

$$e) \vec{r} \cdot \vec{F} = [x, y, z] \cdot [xy, yz, zx]$$

$$= x^2y + y^2z + z^2x$$

$$\nabla(\vec{r} \cdot \vec{F}) = (2xy + z^2)\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 + 2zx)\hat{k}$$

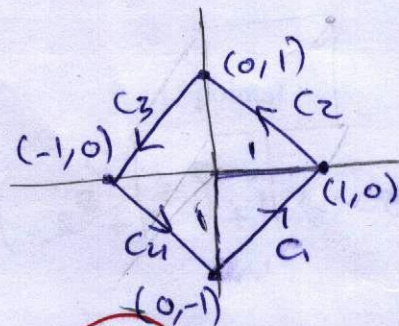
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

Q2) Evaluate $\oint_C (y \sin xy - 3y) dx + (2x + x \sin xy) dy$, where C is the square with vertices (1,0), (0,1), (-1,0), (0,-1). (4 points)

$$\oint_C (y \sin xy - 3y) dx + (2x + x \sin xy) dy$$

By applying Green's Thm.

closed & +ve oriented curve c



4

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

$$= \iint_R ((2 + yx \cos xy + \sin xy) - (xy \cos xy + \sin xy - 3)) dx dy$$

$$= \iint_R (2 + 3) dx dy$$

$$= 5 \iint_R dx dy$$

Area of the square

$$= 5 (\text{length}^2)$$

$$= 5 ((\sqrt{1^2 + 1^2})^2)$$

$$= 5 \times 2 (\sqrt{2})^2$$

$$\oint_C \vec{F} \cdot d\vec{r} = 10$$

Q3) Find the value of $\int_C \vec{F} \cdot d\vec{r}$ from the point (0, 0, 0) to the point (1, 1, 1),

where $C: \vec{r}(t) = ti + t^2j + t^3k$

$$\vec{F}(x, y, z) = (2 - e^z)i + (2y - 1)j + (2 - xe^z)k$$

(5 points)

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2 - e^z) & (2y - 1) & (2 - xe^z) \end{vmatrix} = \hat{i}(0 - 0) - \hat{j}(-e^z + e^z) + \hat{k}(0 - 0) = 0 \checkmark$$

Since $\text{curl } \vec{F} = 0$ then $\vec{F}$ is a conservative field

and $\vec{F} = \nabla f$

$$\vec{F} = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} = (2 - e^z)\hat{i} + (2y - 1)\hat{j} + (2 - xe^z)\hat{k}$$

$$f = \int (2 - e^z) dx + g(y, z) = 2x - e^z x + g(y, z)$$

$$\frac{\partial f}{\partial y} = 2y - 1$$

$$\int \frac{\partial g}{\partial y} dy = \int (2y - 1) dy$$

$$g = y^2 - y + l(z)$$

$$\frac{\partial f}{\partial z} = -xe^z + \frac{\partial l}{\partial z} = 2 - xe^z$$

$$\frac{\partial l}{\partial z} = 2 \checkmark$$

$$l = 2z + k$$

$$\therefore f = 2x - e^z x + y^2 - y + 2z + k$$

$$\int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

$$= f(1, 1, 1) - f(0, 0, 0) = 2 - e + 1 - 1 = 4 - e$$

$$= (4 - e) - (0) = 4 - e$$

(5)

4 - e

Q4) Evaluate $\iint_S \vec{F} \cdot d\vec{A}$, where $\vec{F}(x, y, z) = (y^2 i + x^2 j + 5z k)$,

S is the boundary of the solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$, oriented outward.

(6 points)

$$z = x^2 + y^2 \text{ \& } z = 1$$

$$\iint_S \vec{F} \cdot d\vec{A} = \iint_{S_1} + \iint_{S_2} \quad \checkmark$$

$$S_2: [x, y, 1]$$

$$\vec{n} =$$

~~$$\iint_{S_1} \vec{F} \cdot d\vec{A} =$$~~

~~$$\iint_{S_1} [u^2 \sin^2 v, u^2 \cos^2 v, 5u^2] \cdot [2u \cos v, 2u \sin v, -1] du dv$$~~

~~$$\iint_{S_1} 2u^3 \cos v \sin^2 v + 2u^3 \sin v \cos^2 v - 5u^3 du dv$$~~

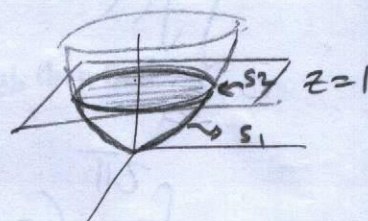
$$r(x, y) = [x, y, x^2 + y^2]$$

$$\vec{F}(r(x, y)) = [y^2, x^2, 5x^2 + 5y^2]$$

$$\vec{n} = [2x, 2y, -1]$$

$$\vec{F} \cdot \vec{n} = 2xy^2 + 2x^2y - 5x^2 - 5y^2$$

$$= 2xy(y+x) - 5(x^2 + y^2)$$



4.5

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$S_1: x^2 + y^2 = z$$

$$r(u, v) = [u \cos v, u \sin v, u^2]$$

$$\begin{cases} 0 \leq v \leq 2\pi \\ 0 \leq u \leq 1 \end{cases}$$

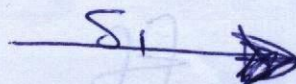
$$r(x, y) = [x, y, x^2 + y^2]$$

$$\vec{n} = \nabla f(x, y, z)$$

$$= \nabla (x^2 + y^2 - z)$$

$$\vec{n} = [2x, 2y, -1] \quad \checkmark$$

~~$$\vec{n} = [2u \cos v, 2u \sin v, -1]$$~~



Q4 (3)

$$S_2: r(x, y) = [x, y, 1]$$

$$\vec{n} = \nabla (z-1)$$

$$= \hat{k} \checkmark$$

$$\vec{F}(r(x, y)) = [y^2, x^2, 5]$$

$$\vec{F} \cdot \vec{n} = 5 \checkmark$$

$$\iint_{S_2} 5 \, r \, dr \, d\theta$$

$$\int_0^{2\pi} \int_0^1 5r \, dr \, d\theta$$

$$= 5 \left[\frac{r^2}{2} \right]_0^1 \int_0^{2\pi} d\theta$$

$$= \frac{5}{2} \times 2\pi = 5\pi \checkmark$$

$$x^2 + y^2 = z$$

at $z=1$

circle with

← radius = 1

$$\therefore \iint_S \vec{F} \cdot n \, dA = 5\pi - \frac{10\pi}{3}$$

① $2xy(y+x) - 5(x^2+y^2) = \vec{F} \cdot \vec{n}$ $x = u \cos v$
 $y = u \sin v$
 $z = u^2$

~~$\int_0^{2\pi} \int_0^1 (2u^3 \cos^2 v \sin v + 2u^3 \cos v \sin^2 v) - 5u^2 \, du \, dv$~~

$\int_0^{2\pi} \left(\frac{2u^4}{4} \cos^2 v \sin v + \frac{2u^4}{4} \cos v \sin^2 v - \frac{5u^3}{3} \right) \Big|_0^1 \, dv$

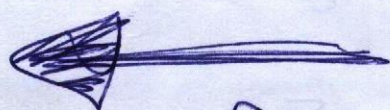
$= \int_0^{2\pi} \left(\frac{1}{2} \cos^2 v \sin v + \frac{1}{2} \cos v \sin^2 v - \frac{5}{3} \right) \, dv$
 $\begin{matrix} z = \cos v \\ dz = -\sin v \, dv \end{matrix} \quad \begin{matrix} L = \sin v \\ L = \sin v \end{matrix}$

$= \left[\frac{1}{2} \left(-\frac{z^3}{3} \right) + \frac{1}{2} \left(\frac{L^3}{3} \right) - \frac{5}{3} v \right]_{v=0}^{v=2\pi}$

$= \frac{1}{2} \left(-\frac{\cos^3 v}{3} \right) + \frac{1}{2} \left(\frac{\sin^3 v}{3} \right) - \frac{5}{3} v \Big|_0^{2\pi}$

$= -\frac{1}{6} \times 1 + \frac{1}{6} \times 0 - \frac{5}{3} \times 2\pi - \left(-\frac{1}{6} + 0 - 0 \right)$

$= -\frac{1}{6} - \frac{10\pi}{3} + \frac{1}{6} = \boxed{-\frac{10\pi}{3}}$



$\frac{F \cdot n}{S_2}$

10.7 DIVERGENCE THEOREM OF GAUSS (ex. 9-18)

- let T be a closed bounded region in space whose boundary is a piecewise smooth surface S with positive (outward) orientation.
- let $F(x, y, z)$ be continuous and has continuous first-order partial derivatives in some domain containing T .

• Then:

$$\iiint_T \operatorname{div} F \, dV = \iint_S F \cdot n \, dA$$

بشكل متساوي

فهو متساوي بالمتجه

* ملاحظة:

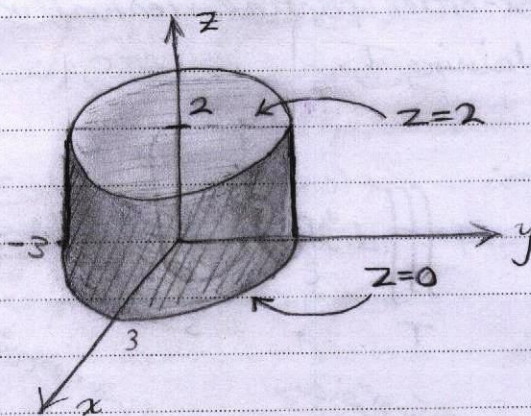
كل $\nabla \cdot F$ يعرف حجة .. إذا كان $\nabla \cdot F$ ثابتاً ..
النقطة

- * Note that this theorem takes region in space bounded by surfaces.
- * while in 10.6 region in plane bounded by curves.

Ex Evaluate $\iint_S F \cdot n dA$, where

$$F(x, y, z) = x^3 i + y^3 j + z^2 k$$

S : is the region enclosed by the circular cylinder $x^2 + y^2 = 9$, and the planes $z=0$ & $z=2$.



$$\text{div } F = 3x^2 + 3y^2 + 2z$$

$$\iint_S F \cdot n dA = \iiint_T (3x^2 + 3y^2 + 2z) dv$$

for the cir. cylinder:

cylindrical coordinates:

$$0 \leq r \leq 3$$

$$x = r \cos \theta$$

$$0 \leq \theta \leq 2\pi$$

$$y = r \sin \theta$$

$$0 \leq z \leq 2$$

$$z = z$$

$$dv = r dr d\theta dz$$

$$= \int_0^{2\pi} \int_0^3 \int_0^2 (3r^2 + 2z) r \, dz \, dr \, d\theta$$

$(\cos^2 + \sin^2) = 1$
 $\downarrow$

نفس العمل، ولكن قبل ما ندمج...
 $\rightarrow$ $\int_0^2 (3r^2 + 2z) r \, dz = \left[3r^2 z + z^2 \right]_0^2 = 6r^2 + 4$

$$= \int_0^{2\pi} \int_0^3 (6r^2 + 4) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{6}{4} (3)^4 + 2(3)^2 \right] d\theta = \underline{\underline{279\pi}}$$

(= سواد)

Ex Same as the previous but:

region is not closed from above

(no $z=2$, but $0 \leq z \leq 2$) بفتح إيشال قبل إكاسية

"closed" كاسية المنطقة من أسفله، لكن غير مغلقة من فوق
 من مغلقة

الآن... بقية أسفله، لكن غير مغلقة (مع الأسفل) وأفتح فوق
 as surface Integ. التي هي غير مغلقة أسفله

(مثل ما كنا فعلنا بـ Green's Theorem)

$$\iint_S F \cdot n \, dA = \iint_{S \cup S_3} - \iint_{S_3}$$

$\uparrow$ div thm. $\uparrow$ surface integral.

$$\iint_{S_3} F \cdot n \, dA$$

$\rightarrow z - 2 = 0$

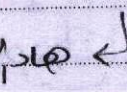
$$S_3: z = 2 \rightarrow r(x, y) = [x, y, 2]$$

$$F = [x^3, y^3, 4]$$

$$N = \text{grad}(z - 2) = k = [0, 0, 1]$$

$$F \cdot N = 4$$

$$= \iint_{R: x^2 + y^2 = 3^2} 4 \, dx \, dy = 4 \iint_R dx \, dy = \text{Area} \times 4 = 9\pi \times 4 = 36\pi$$

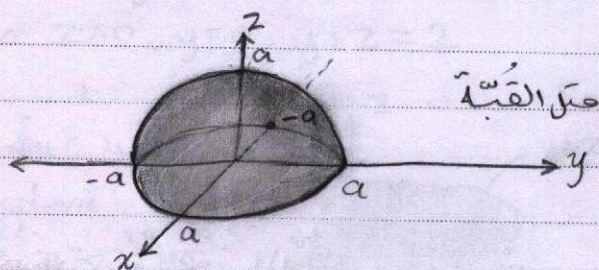
Area of R:  ←

$$\text{total} = (279 - 36)\pi = \underline{\underline{243\pi}}$$

Ex Evaluate Integral

$$\iint_S F \cdot n \, dA, \text{ where } F = [x^3, y^3, z^3]$$

S is the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$,
and $z = 0$



$$\text{div } F = 3x^2 + 3y^2 + 3z^2$$

$$\iint_S F \cdot n \, dA = 3 \iiint_T x^2 + y^2 + z^2 \, dV$$

always: for spheres: Use spherical coordinates

$$x = \rho \sin \phi \cos \theta$$

$$0 \leq \rho \leq a$$

$$y = \rho \sin \phi \sin \theta$$

$$0 \leq \phi \leq \frac{\pi}{2}$$

$$z = \rho \cos \phi$$

$$0 \leq \theta \leq 2\pi$$

$$| \, dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \, |$$

$$= 3 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a \rho^2 \cdot \underbrace{\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta}_{dV}$$

sphere above

$$= \frac{6\pi a^5}{5}$$

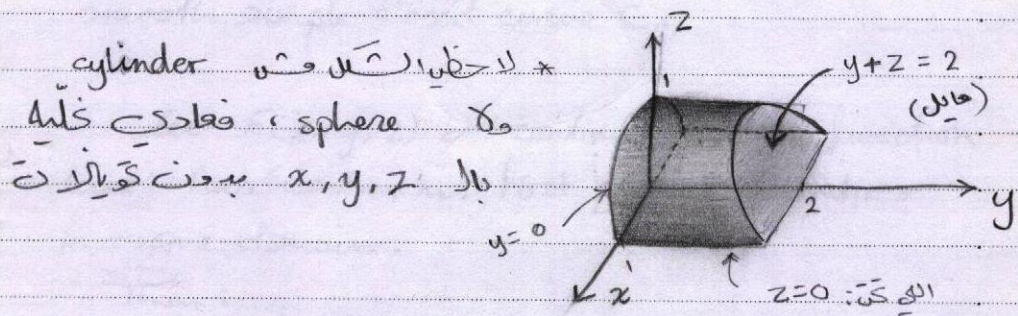
كانت الحسابات بهذه السهولة

END OF LEC. 1

Ex Evaluate $\iint_S F \cdot n dA$, where $F(x, y, z) =$

$$= (xy)i + (y^2 + e^{xz^3})j + (\sin xy^2)k$$

S is the Surface of the region bounded by parabolic cylinder $z = 1 - x^2$, and the planes $z = 0$, $y = 0$, $y + z = 2$

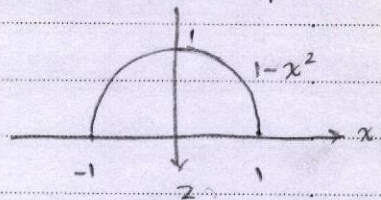


$$\text{div } F = y + 2y + 0 = 3y$$

$\times$ the region F is not a sphere, but a cylinder, so $\text{div } F$ is not zero.

$$\begin{aligned} & \int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} 3y \, dy \, dz \, dx \\ &= \int_{-1}^1 \int_0^{1-x^2} \frac{3}{2} (2-z)^2 \, dz \, dx = \dots \end{aligned}$$

projection on xz plane



$\times$ the region F is not a sphere, but a cylinder, so $\text{div } F$ is not zero.

10.9

STOKES'S THEOREM

(ex. 1-10, 13-20)

* It's all about converting from
 "line Integral" to "Surface Integral".

Th.

let S be a piecewise smooth oriented surface in space.

and let the boundary of S be a piecewise smooth simple closed curve C .

let $F(x, y, z)$ be continuous vector function that has continuous first partial derivatives in some domain.

Then:

$$\iint_S \text{curl}(F) \cdot n \, dA = \oint_C F \cdot dr$$

where, as you know;

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If $F = F(x, y)$ only two dimensions

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Oh! it's just like
Green's thm.

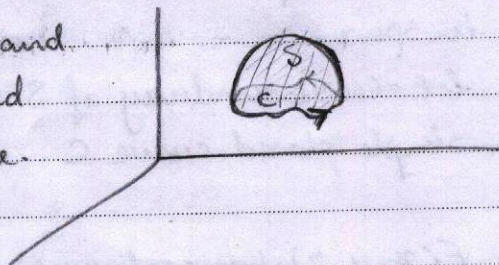
* Note that if F is conservative, that is

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

we say $\oint F \cdot dr = 0$

(conserv. $\nabla \cdot F = 0$) $\Rightarrow \oint F \cdot dr = 0$ *
 (conserv. $\nabla \cdot F = 0$) $\Rightarrow \oint F \cdot dr = 0$

S is the surface, and
 C is a curve that
 bound the surface.



RIGHT-HAND RULE

امشي بأصابعك مع C "إلى اليمين"
 فتكون إصبعك هو N (outward vect.)

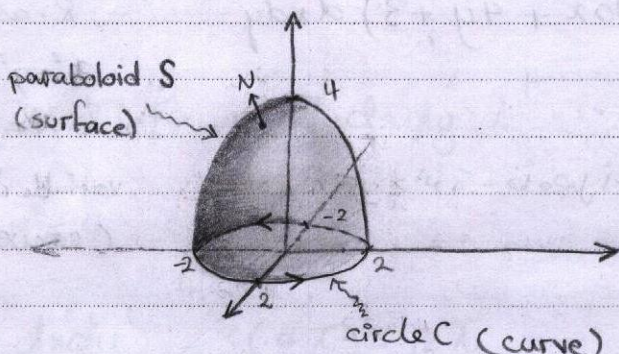
* تعريف: $\nabla \times F$ هو المجال الناتج عن F
 إذا الواحد عش مع C بنفس الاتجاه
 يكون المجال $\nabla \times F$ هو N ، N هو N !!

Ex Verify Stoke's theorem for:

$$F(x, y, z) = 2z\mathbf{i} + 3xy\mathbf{j} + 5y\mathbf{k}$$

S is the portion of the paraboloid $z = 4 - x^2 - y^2$ for $z \geq 0$ with upward orientation.

and C is the positively oriented circle $x^2 + y^2 = 4$ that forms the boundary of S in the xy -plane.



$$\textcircled{1} \oint \vec{F} \cdot d\vec{r} = \int \vec{F}(x(t), y(t)) \cdot \vec{r}'(t) dt$$

$$C: \vec{r}(t) = [2 \cos t, 2 \sin t, 0]$$

$$\vec{r}'(t) = [-2 \sin t, 2 \cos t, 0]$$

$$\vec{F}(\vec{r}(t)) = [0, 6 \cos t, 10 \sin t]$$

$$\oint = \int_0^{2\pi} 12 \cos^2 t dt = \frac{12}{2} \int_0^{2\pi} 1 + \cancel{\cos 2t} dt$$

$$= \underline{12\pi}$$

END OF LEC. 2

$$(2) \iint_S (\text{curl } F) \cdot n \, dA$$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5i + 2j + 3k$$

$$N = \text{grad}(x^2 + y^2 + z - 4) = [2x, 2y, 1]$$

$$= \iint_R (10x + 4y + 3) \, dx \, dy$$

, R = area of projection.

$$x^2 + y^2 = 4$$

$$\rightarrow x = r \cos \theta, y = r \sin \theta$$

variable r , not a number (not 2)

(converting not parametrization)

polar $\rightarrow$ دائرة - فافعل اي شئ

$$\int_0^{2\pi} \int_0^2 (10r \cos \theta + 4r \sin \theta + 3) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{10(8) \cos \theta}{3} + \frac{4 \cdot 8 \sin \theta}{3} + \frac{3 \cdot 4}{2} d\theta \right] = 6 \times 2\pi = 12\pi$$

zero zero

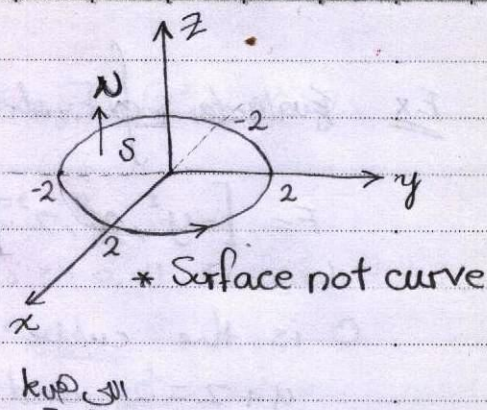
$$(1) = (2)$$

$$\textcircled{3} \quad \oint_C \mathbf{F} \cdot d\mathbf{r} =$$

$$C: x^2 + y^2 = 4$$

S is the disk, where $z=0$

$$\mathbf{N} = [0, 0, 1]$$



$$= \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dA$$

$$= \iint_R [5, 2, 3] \cdot [0, 0, 1] dx dy$$

$$R: x^2 + y^2 = 4$$

$$= 3 \underbrace{\left\{ \iint dx dy \right\}}_{\text{area}} = 3(4\pi) = 12\pi$$

$$\textcircled{1} = \textcircled{2} = \textcircled{3}$$

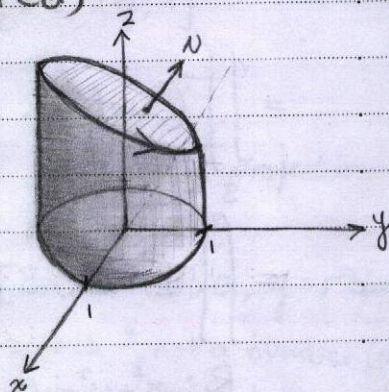
Ex Evaluate $\oint_C F \cdot dr$

حساب الجيب

$$F = [-y^2, x, z^2]$$

C is the curve of intersection of the plane $y+z=2$ and the circular cylinder $x^2+y^2=1$ (positive oriented)

$$\begin{aligned} \text{curl } F &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} \\ &= [0, 0, 1+2y] \end{aligned}$$



$$S: y+z=2$$

$$N = \text{grad}(y+z-2) = [0, 1, 1]$$

$$\oint_C F \cdot dr = \iint_S (\text{curl } F) \cdot n \, dA = \iint_R (\text{curl } F) \cdot N \, dx \, dy$$

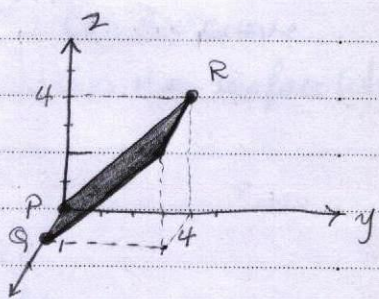
$$\iint_{R: x^2+y^2 \leq 1} (1+2y) \, dx \, dy = \int_0^{2\pi} \int_0^1 (1+2r \sin \theta) r \, dr \, d\theta = \pi$$

END OF LEC. 3

Ex Evaluate $\oint_C F \cdot dr$, where $F(x,y,z) = [z^2, -x^2, 0]$

C is Rectangle with vertices $(0,0,0)$, $(1,0,0)$, $(0,4,4)$ and $(1,4,4)$

$$\oint_C F \cdot dr = \iint_S (\text{curl } F) \cdot n \, dA$$



* $\text{curl } F = 2z \, j - 2x \, k$

* $N = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 0 & 4 & 4 \\ 1 & 0 & 0 \end{vmatrix} = [0, -4, 4]$

* S is the plane $|z=y|$ parametrization $\rightarrow r(x,y) = [x, y, y]$
 (Note: The original image has some scribbles and corrections here, including "choice of normal" and "dist. no. 4.5.")

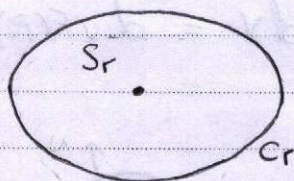
$$\iint_R [0, 2y, -2x] \cdot [0, -1, 1] \, dx \, dy$$

(R) $\rightarrow$ projection on xy.

$$= \int_0^4 \int_0^1 -2y - 2x \, dx \, dy$$

PHYSICAL INTERPRETATION OF CURL

- $P_0(x_0, y_0, z_0)$ is any point in the fluid.
- C_r is a small circle with radius r , centered at P_0 .



C_r : the curve

S_r : the surface (disk)

Stokes' :

$$\oint_{C_r} \vec{F} \cdot d\vec{r} = \iint_{S_r} (\text{curl } \vec{F}) \cdot \vec{n} dA$$

$$\text{Curl } \vec{F}(P) \approx \text{Curl } \vec{F}(P_0)$$

$$P \in S_r$$

$$\oint_{C_r} \vec{F} \cdot d\vec{r} \approx \iint_{S_r} \text{Curl } \vec{F}(P_0) \cdot \vec{n}(P_0) dA$$

$$= \text{Curl } \vec{F}(P_0) \cdot \vec{n}(P_0) \left\{ \iint_{S_r} dA \right\}$$

area

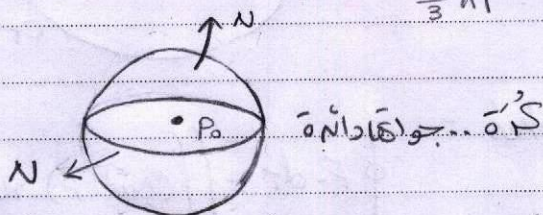
$$= \text{Curl } \vec{F}(P_0) \cdot \vec{n}(P_0) \underbrace{A_r}_{\pi r^2}$$

$$\text{Curl } \vec{F}(P_0) \cdot \vec{n}(P_0) = \lim_{r \rightarrow 0} \frac{1}{A_r} \oint_{C_r} \vec{F} \cdot d\vec{r}$$

INTERPRETATION OF DIVERGENCE

$$\iint_{S_r} F \cdot n dA = \iiint_{T_r} \operatorname{div} F \cdot dV \approx \iiint_{T_r} \operatorname{div} F(P_0) dV$$

$$= \operatorname{div} F(P_0) \iiint_{T_r} dV = \operatorname{div} F(P_0) \underbrace{V_r}_{\substack{\rightarrow \text{Volume} \\ \frac{4}{3}\pi r^3}}$$



S_r is a small sphere with radius r , centered at P_0 .

$T_r = S_r + \text{interior}$

$$\operatorname{div} F(P) \approx \operatorname{div} F(P_0)$$

$$\operatorname{div} F(P_0) = \lim_{r \rightarrow 0} \frac{1}{V_r} \left(\iint_{S_r} F \cdot n dA \right)$$

flux

COROLLARIES

نتائج

f and g are scalar functions
 F is a vector function.

$$\boxed{1} \quad \iint_S f \cdot F \cdot n dA = \iiint_T F \cdot \nabla f + f(\nabla \cdot F) dV$$

$$\boxed{2} \quad F = f \bar{C} \quad \bar{C} \text{ is a constant vector.}$$

$$\iint_S f \cdot n dA = \iiint_T \nabla f dV$$

GREEN'S IDENTITY

$$\boxed{1} \quad \iint_S f(\nabla g \cdot n) dA = \iiint_T f(\nabla^2 g + \nabla g) dV$$

$$\boxed{2} \quad \iint_S (f \nabla g - g \nabla f) \cdot n dA = \iiint_T (f \nabla^2 g - g \nabla^2 f) dV$$

END OF LEC. 4

CH. 1111.1 & 11.2

- $1 + 2 + 3 + \dots + n$ (doesn't have sum)
- $1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}}$ (has sum)
- $e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$
 $\rightarrow$ sum of a series = function
 (for some series)

Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad (*)$$

$\hookrightarrow$ when $n=0$: $f(x) = a_0 \cos 0 + b_0 \sin 0 = \underline{a_0}$.

- $f(x)$: periodic function.
- period of $\sin$ and $\cos$ is 2π .
- period of the expression above = $2L$.

$$\{ \Rightarrow f(x+2L) = f(x)$$

$$L = \frac{\text{Period}}{2}$$

(11.1)

• The set of functions:

$$1, \cos \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \dots, \sin \frac{\pi x}{L}, \sin \frac{2\pi x}{L}, \dots$$

is orthogonal.

$$\boxed{1} \quad \int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \begin{cases} 0, & n \neq m \\ L, & n = m \end{cases}$$

$$\boxed{2} \quad \int_{-L}^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} 0, & n \neq m \\ L, & n = m \end{cases}$$

$$\boxed{3} \quad \int_{-L}^L \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0 \quad \text{for all } n \& m.$$

.. in Fourier functions , clear is

~> we can prove these three Integrations , proof of [2] is in the next page.

[1] and [3] can be proved in similar way.

Proof $\int_{-L}^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx$

identity:

$$\sin A * \sin B = \frac{1}{2} (\cos(A-B) - \cos(A+B))$$

$$\begin{aligned} \text{جواب:} &= \int_{-L}^L \frac{1}{2} \left(\cos \frac{(n-m)\pi x}{L} - \cos \frac{(n+m)\pi x}{L} \right) dx \\ &= \frac{1}{2} \frac{L}{(n-m)\pi} \left. \sin \frac{(n-m)\pi x}{L} \right|_{-L}^L - \frac{1}{2} \frac{L}{(n+m)\pi} \left. \sin \frac{(n+m)\pi x}{L} \right|_{-L}^L \\ &\quad \downarrow \quad \quad \quad \downarrow \\ &\quad \quad \quad = 0 \quad \quad \quad = 0 \end{aligned}$$

$= 0$ (for $n \neq m$)

→ But for $n = m$:

$$\int_{-L}^L \left(\sin \frac{n\pi x}{L} \right)^2 dx = \frac{1}{2} \int_{-L}^L \left\{ 1 - \cos \frac{2n\pi x}{L} \right\} dx$$

$\int \cos = 0$

$$= \frac{1}{2} x \Big|_{-L}^L = L \quad (\text{for } n = m)$$

* Find a_0, a_n, b_n .

(مثلاً إذا كان $f(x)$ زوجاً)

إذاً إذا ضربنا المعادلة (*) بـ $\cos \frac{n\pi x}{L}$

$$\int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx =$$

$$= \int_{-L}^L a_0 \cos \frac{n\pi x}{L} dx + \sum_{n=1}^{\infty} \left(\int_{-L}^L a_n \cos \frac{n\pi x}{L} \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \cos \frac{n\pi x}{L} \right) dx$$

(مثلاً إذا كان $f(x)$ زوجاً) $a_m = \frac{2}{L} \int_0^L f(x) \cos \frac{m\pi x}{L} dx$ $n = m$ $a_n = 0$ $b_n = 0$ $n \neq m$

$$= a_m \int_{-L}^L \left(\cos \frac{n\pi x}{L} \right)^2 dx = a_m L$$

$$a_m = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx$$

أيضاً إذا كان $f(x)$ زوجاً

a_n, b_n

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

• to find a_0 :

$$f(x) = a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

for $n=0$:

$$\begin{aligned} f(x) &= a_0 * (1) + b_0 * (0) \\ &= a_0 \end{aligned}$$

$$\int_{-L}^L f(x) dx = \int_{-L}^L a_0 dx = a_0(L - (-L)) = a_0(2L)$$

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

فهو القانون للحقبة :

$$\boxed{1} \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$\boxed{2} \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

$$\boxed{3} \quad a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

END OF LEC. 5

Ex

Find the Fourier Series for:

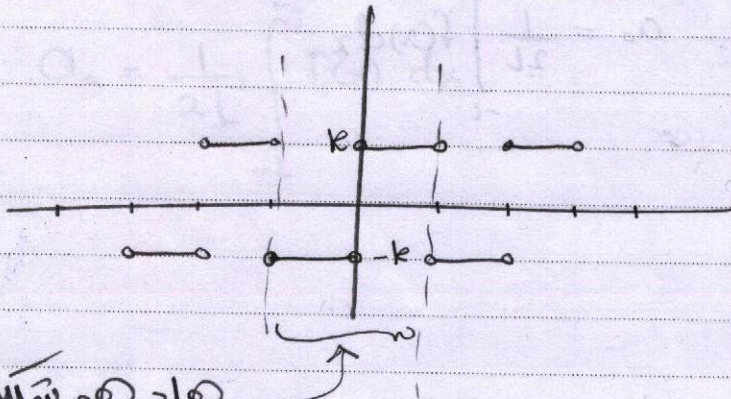
$$f(x) = \begin{cases} -k & , -1 < x < 0 \\ k & , 0 < x < 1 \end{cases}$$

$$\text{and } f(x+2) = f(x) \quad 2L$$

- k is constant.
- $f(x)$ must be periodic in order to have F.S.

- periodic $\Rightarrow$ $f(x)$.periodic $\Rightarrow$ $f(x)$ is periodic.periodic $\Rightarrow$ $f(x)$ is periodic.

(periodic)



periodic $\Rightarrow$ $f(x)$ is periodic.

periodic $\Rightarrow$ $f(x)$ is periodic.

$$2L = 2$$

$$L = 1$$

To find F.S: find a_0 , a_n , and b_n first.

$$\begin{aligned} \boxed{1} \quad a_0 &= \frac{1}{2L} \int_{-L}^L f(x) dx \\ &= \frac{1}{2} \left(\int_{-1}^0 (-k) dx + \int_0^1 (k) dx \right) = \frac{1}{2} (-k + k) = \underline{\underline{0}} \end{aligned}$$

$$\begin{aligned} \boxed{2} \quad a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \\ &= -k \int_{-1}^0 \cos \frac{n\pi x}{1} dx + k \int_0^1 \cos \frac{n\pi x}{1} dx \\ &= \frac{-k}{n\pi} \sin(n\pi x) \Big|_{-1}^0 + \frac{k}{n\pi} \sin(n\pi x) \Big|_0^1 = \underline{\underline{0}} \end{aligned}$$

$$\begin{aligned} \boxed{3} \quad b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \\ &= -k \int_{-1}^0 \sin n\pi x dx + k \int_0^1 \sin n\pi x dx \\ &= \frac{+k}{n\pi} \cos n\pi x \Big|_{-1}^0 - \frac{k}{n\pi} \cos n\pi x \Big|_0^1 \end{aligned}$$

$$(-1)^n = \cos n\pi$$

$$= \frac{k}{n\pi} (1 - \cos n\pi - \cos n\pi + 1) = \frac{2k}{n\pi} (1 - \cos n\pi)$$

$$= \frac{2k}{n\pi} (1 - (-1)^n) = \begin{cases} 0, & n \text{ even} \\ \frac{4k}{n\pi}, & n \text{ odd.} \end{cases}$$

Note that: $(1 - (-1)^n) = \begin{cases} 0, & n \text{ is even. "2n"} \\ 2, & n \text{ is odd. "2n-1"} \end{cases}$

Now:

$$f(x) \equiv \overset{\text{zero}}{a_0} + \sum_{n=1}^{\infty} (\overset{\text{zero}}{a_n} \cos \frac{n\pi x}{L} + \underbrace{b_n \sin \frac{n\pi x}{L}}_{\text{displacement}})$$

$\swarrow$
 b_n

$$f(x) \equiv \sum_{n=1}^{\infty} \frac{4k}{(2n-1)\pi} \sin (2n-1)\pi x$$

n is odd $\rightarrow$ $2n-1$ is odd

as a series:

$$f(x) = \frac{4k}{\pi} \left(\sin \pi x + \frac{1}{3} \sin 3\pi x + \frac{1}{5} \sin 5\pi x + \dots \right)$$



Fourier Series ... الجواب ...

CONVERGENCE OF FOURIER SERIES.

Th.

let $f(x)$, $f'(x)$ be piecewise, continuous on $(-L, L)$; $f(x+2L) = f(x)$ periodic

Then $f(x)$ has the Fourier Series Representation

The F.S converges to $f(x)$ at points where $f(x)$ is continuous, and to $\frac{f(x^+) + f(x^-)}{2}$ at points where $f(x)$ is discontin.

$f(x^+)$: right-hand limit = $\overline{f(x)}$

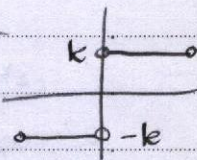
$f(x^-)$: left-hand limit = $\underline{f(x)}$

مثال: $f(x) = \begin{cases} k & x > 0 \\ -k & x < 0 \end{cases}$

* At $x=0$, the F.S converges to

Sol: $f(x^+) = f(0^+) = k$

$$f(x^-) = f(0^-) = -k$$



converges to $\frac{k - k}{2} = 0$

no for the same question.

$$\text{At } x = \frac{1}{2} \rightarrow f\left(\frac{1}{2}\right) = k$$

$$\text{At } x = -\frac{1}{3} \rightarrow f\left(-\frac{1}{3}\right) = -k$$

↓
where $f(x)$ is cont. (Theorem $\epsilon-\delta$)

- At $x = 10$ $10-$

$$f(10) = f(5 \times \text{period}) = f(0) = 0$$

- At $x = 21$

$$f(21) = f(21 - 10 \times \text{period}) = f(1) = \frac{f(1^+) + f(1^-)}{2} = \frac{k - k}{2} = 0$$

- At $x = 21\frac{1}{2}$

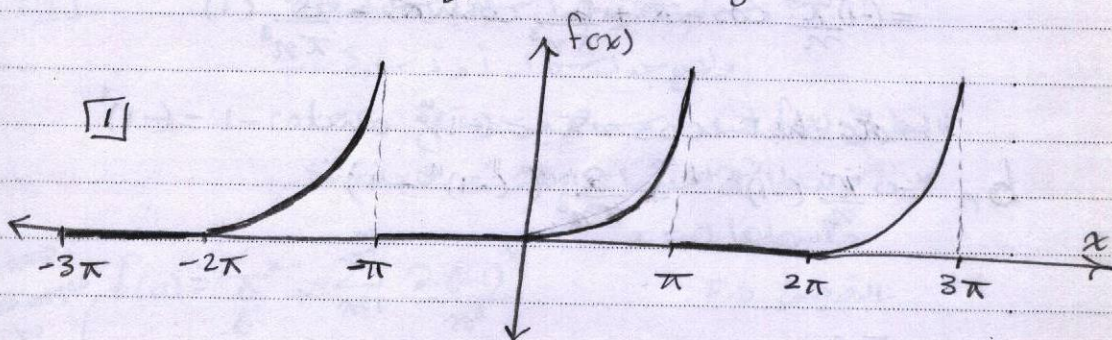
$$f(21\frac{1}{2}) = f(21\frac{1}{2} - 11 \times \text{period}) = f(-\frac{1}{2}) = -k$$

END OF LEC. 6

سؤال آخر

$$\text{Ex } f(x) = \begin{cases} 0 & , -\pi < x < 0 \\ x^2 & , 0 < x < \pi \end{cases} , f(x+2\pi) = f(x)$$

- ① Sketch the graph of $f(x)$.
- ② Find the F.S.
- ③ Convergence.
- ④ Show that $\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \dots$



2 $L = \pi$

$$a_0 = \frac{1}{2\pi} \int_0^{\pi} x^2 dx = \frac{\pi^2}{6} \quad (\text{if } -\pi < x < 0, \text{ then } f(x) = 0)$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} x^2 \cos \frac{n\pi x}{\pi} dx$$

by parts:

x^2	$\cos nx$
$\downarrow$	$\downarrow$
$2x$	$\frac{1}{n} \sin nx$
$\downarrow$	$\downarrow$
2	$\frac{1}{n^2} \cos nx$
$\downarrow$	$\downarrow$
0	$-\frac{1}{n^3} \sin nx$

$$= \frac{1}{\pi} \left(\underbrace{\frac{x^2}{n} \sin nx}_{\text{zero}} + \underbrace{\frac{2x}{n^2} \cos nx}_{\text{zero}} - \frac{2}{n^3} \sin nx \right) \bigg|_0^{\pi}$$

$$= \frac{2\pi}{\pi n^2} \cos n\pi = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} x^2 \sin nx \, dx$$

$$\begin{array}{rcl} x^2 & \cdot & \sin nx \\ \downarrow + & & \\ 2x & \cdot & -\frac{1}{n} \cos nx \\ \downarrow + & & \\ 2 & \cdot & -\frac{1}{n^2} \sin nx \\ \downarrow + & & \\ & \cdot & \frac{1}{n^3} \cos nx \end{array}$$

$$= \frac{1}{\pi} \left(\frac{-x^2}{n} \cos nx + \frac{2x}{n^2} \sin nx + \frac{2}{n^3} \cos nx \right) \Big|_0^{\pi}$$

$$= (-1)^n \frac{\pi}{n} \cos n\pi + \frac{2}{\pi n^3} \cos n\pi - \frac{2}{\pi n^3} (1)$$

note that $\cos n\pi = (-1)^n$ and $-1 = (-1)^1$

$$b_n = \frac{\pi}{n} (-1)^{n+1} + \frac{2}{\pi n^3} ((-1)^n - 1)$$

Now; F.S is:

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left(\frac{2(-1)^n}{n^2} \cos nx + \left(\frac{\pi}{n} (-1)^{n+1} + \frac{2}{\pi n^3} ((-1)^n - 1) \right) \sin nx \right)$$

.. الفس، 1 ip no 21, 6 ne) 2 by 2 am F.S uo oia

[3] At $x=1$, the F.S converges to $f(1)=1$
 At $x=2$ $f(2)=2^2$
 $x=\pi$ $f(\pi)=\frac{\pi^2-0}{2}$
 $x=11\pi$ not like $x=11\pi-(2\pi)*5=\pi$
 the F.S converges to

[4] $\sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}$ متسلسلة يوجد 2 و 1 فقط
 ولا يوجد 3 أو 4
 بالنظر إلى $f(x)$ في $x=0$ موجودة مع $\cos \pi x$
 لأن $\sin \pi x = 0$ و $\cos \pi x = 1$ حين $x=0$ و $x=\pi$

at $x=0$ f converges to 0

$$f(0) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}$$

F.S converges

$$0 = \frac{\pi^2}{6} + 2\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots\right)$$

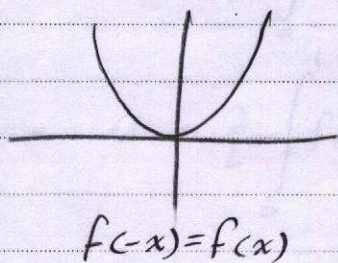
$$\frac{\pi^2}{6 \times 2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

90% there is a question in the Exam
 like branch [4] i.e: Show that

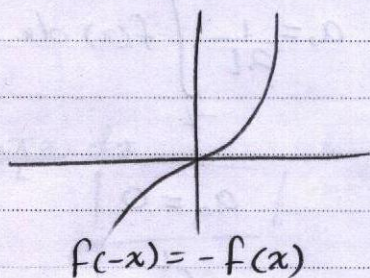
- * أول شي تنفيذاً إلى شي المطلوب في السؤال Show أن متسلسلة $\sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}$ بال F.S
- * بعد جيباً قيمة x إلى تنفيذاً إلى فيه إلى شي مشابه وتبشيل كدور لثانية
- * آخر شي رتبنا الشكل حتى يسهل فهمه قبل السؤال

EVEN AND ODD FUNCTIONS

EVEN (E.f)



ODD (O.f)



PROPERTIES (Important)

[1] $E.f \pm E.f = E.f$

$E.f \times E.f = E.f$

[2] $O.f \pm O.f = O.f$

$O.f \times O.f = E.f$

[3] $E.f \pm O.f = \text{neither Even nor odd.}$

$E.f \times O.f = O.f$

[4] If $f(x)$ is even $\Rightarrow \int_{-L}^L f(x) dx = 2 \int_0^L f(x) dx$

[5] If $f(x)$ is odd $\Rightarrow \int_{-L}^L f(x) dx = \underline{\underline{0}}$

Finding F.S: Cases:

Case #1 if $f(x)$ is odd, then:

$$\bullet a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = 0$$

$$\boxed{a_0 = 0}$$

$$\bullet a_n = \frac{1}{L} \int_{-L}^L \underbrace{f(x)}_{\substack{\text{O.F} \\ \downarrow \\ \text{odd}}} \underbrace{\cos \frac{n\pi x}{L}}_{\substack{\text{E.F} \\ \downarrow \\ \text{even}}} dx = 0$$

$$\boxed{a_n = 0}$$

$$\bullet b_n = \frac{1}{L} \int_{-L}^L \underbrace{f(x)}_{\substack{\text{O.F} \\ \downarrow \\ \text{odd}}} \underbrace{\sin \frac{n\pi x}{L}}_{\substack{\text{O.F} \\ \downarrow \\ \text{odd}}} dx = \frac{2}{L} \int_0^L \dots$$

$$\boxed{b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx}$$

$$\text{Hence } f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad \text{Fourier Sine Series.}$$

(For O.F)

case #2

If $f(x)$ is even, and periodic

properties, a_0, a_n, b_n is

$$a_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L \underbrace{f(x)}_{\text{Even}} \underbrace{\cos \frac{n\pi x}{L}}_{\text{Even}} dx$$

$$b_n = 0 \quad \text{Even} \times \text{Even} = \text{Even} \quad \int_{-L}^L \underbrace{f(x)}_{\text{Even}} \underbrace{\sin \frac{n\pi x}{L}}_{\text{Odd}} dx = 0$$

$$f(x) \equiv a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

Fourier Cosine Series.

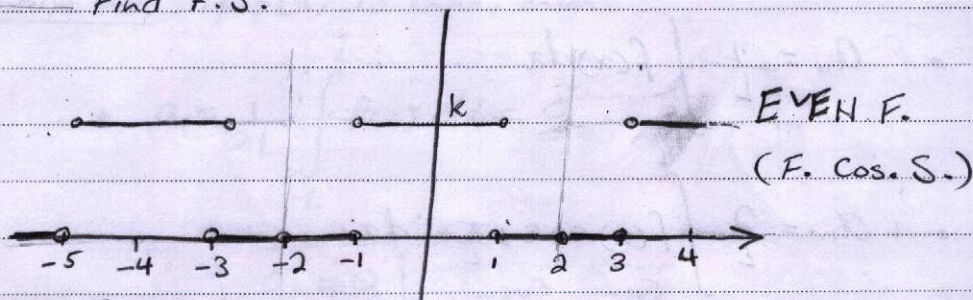
(for E.f)

السؤال جانباً ونستكمل F. Cos. S. المطلوب هو إيجاد F.S. Even

Ex

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases} \quad \begin{matrix} f(x+4) = f(x) \\ L=2 \end{matrix}$$

Find F.S.



المطلوب هو إيجاد F.S. Even
المطلوب هو إيجاد F.S. Even

• $a_0 = \frac{1}{2} \int_0^1 k dx = \frac{k}{2}$

• $a_n = \frac{2}{L} \int_0^1 k \cos \frac{n\pi x}{2} dx = \frac{2k}{n\pi} \left[\sin \frac{n\pi x}{2} \right]_0^1$
 $= \frac{2k}{n\pi} \sin \frac{n\pi}{2}$

$$f(x) \approx \frac{k}{2} + \sum_{n=1}^{\infty} \left(\frac{2k}{n\pi} \sin \frac{n\pi}{2} \right) \cos \frac{n\pi x}{2}$$

EHD OF LEC. 7

- Fourier Cosine Series, for Even fun.

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$= \frac{1}{L} \int_0^L f(x) dx$$

$$= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

- Fourier Sine Series, for Odd fun.

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

* في دالة فردية

(small letter) $f(x)$ دالة فردية

(capital) $F(x)$ دالة زوجية

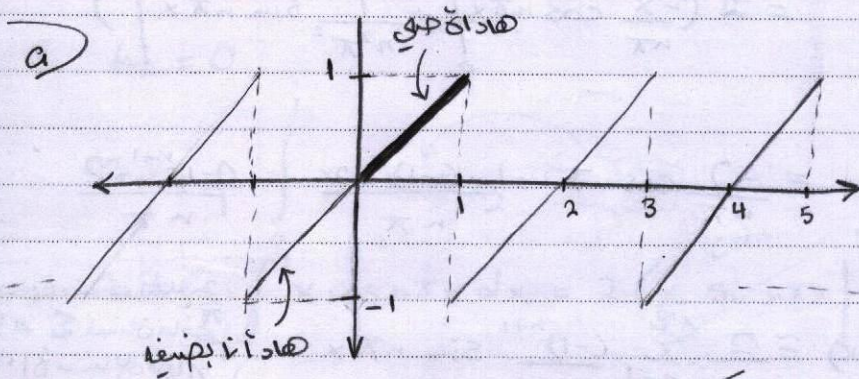
... انظر الى واعرف

HALF-RANGE EXPANSION.

↳ $f(x)$ is defined on $(0, L)$

Ex Given $f(x) = x$, $0 < x < 1$ ($\therefore L = 1$)

Find: a) the Fourier Sine Series. هذه (فالس) مشرقة.
b) the Fourier Cosine Series. هذه (فالس) مشرقة.



به این بکتره عیسو شوال

This is Odd periodical extension of $f(x)$

↓
(قررتے آئے odd عینان اطلب sine فی سوال)

Period = 2

Continue →

For I want "F.S.S", then

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{1} \int_0^1 x \sin n\pi x \, dx$$

• by parts:

$$\begin{array}{l} x \quad \sin n\pi x \\ \downarrow \quad \downarrow \\ 1 \quad -\frac{1}{n\pi} \cos n\pi x \\ 0 \quad -\frac{1}{n^2\pi^2} \sin n\pi x \end{array}$$

$$= 2 \left(\frac{-x}{n\pi} \cos n\pi x \Big|_0^1 + \frac{1}{n^2\pi^2} \sin n\pi x \Big|_0^1 \right)$$

$$= \frac{-2}{n\pi} \cos n\pi = \frac{-(-1)^n \cdot 2}{n\pi} = \frac{(-1)^{n+1} \cdot 2}{n\pi}$$

Now;

$$F(x) \cong \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x$$

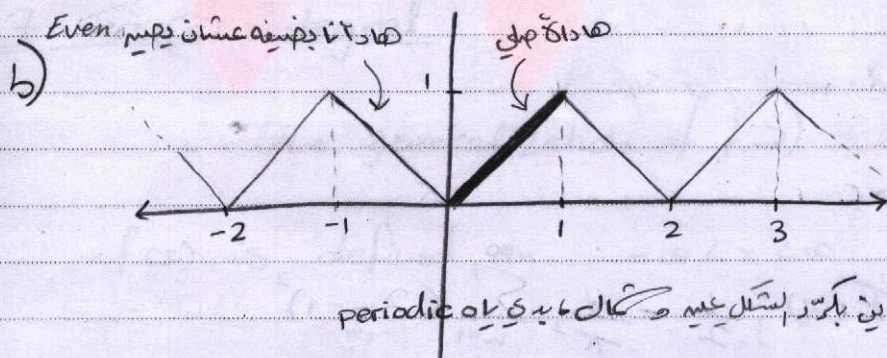
ملاحظة: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ متناهي
لأنه تقبل جوا $\sum$

This is the Fourier Sine Series for Odd periodic Extension of $f(x)$

→ Convergence: $f(-x) = -f(x)$
at $x = 7\frac{1}{2}$ $7\frac{1}{2} - (2 \times 4) = -\frac{1}{2}$

$$F(7\frac{1}{2}) = F(-\frac{1}{2}) = -F(\frac{1}{2}) = -\frac{1}{2}$$

F.S no $\rightarrow$ متناهي التقارب (Gibbs)
(Gibbs)



Even periodic extension of $f(x)$, period = 2

$$b_n = 0$$

$$a_0 = \frac{1}{1} \int_0^1 x dx = \frac{1}{2}$$

$$a_n = \frac{2}{1} \int_0^1 x \cos n\pi x dx = 2 \left(\frac{x}{2\pi} \sin n\pi x \Big|_0^1 + \frac{1}{n^2\pi^2} \cos n\pi x \Big|_0^1 \right)$$

zero

$$= \frac{2}{n^2\pi^2} ((-1)^n - 1) = \frac{4}{(2n-1)^2\pi^2}$$

التفسير: ← الحد $((-1)^n - 1)$ يساوي 0 إذا n زوجي (يعني زوجي)
 يساوي -2 إذا n فردي
 (بمعنى $\frac{2}{n^2\pi^2} (-2)$)
 لكن نحافظ للفردي لأن n زوجي $(2n-1)$
 that's it. =)

So;

$$F(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

— This is the F. Cos. S for Even periodic Extension of $f(x)$.

→ for (b): $f(-x) = f(x)$

$$f\left(-\frac{1}{2}\right) = f\left(\frac{1}{2}\right) = \frac{1}{2}$$

11.7 Fourier Integral

(the generalization of F.S)

$f(x)$ is defined on $-\infty < x < \infty$

$$(1) \quad f(x) \equiv \int_{-\infty}^{\infty} (A(\omega) \cos \omega x + B(\omega) \sin \omega x) d\omega$$

← وليه x

Sum of Integral
n dw

(كيف مثل كذا F.S. في $n=0$ و ∞)

ولييه $x=0$ و ∞

where:

$$[1] \quad A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v dv$$

$$[2] \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin(\omega v) dv$$

هناك القانون حفظ ..

Th.

Assume that f, f' are piecewise continuous for $-\infty < x < \infty$,

and $f(x)$ is absolutely integrable fun.

$$\left(\int_{-\infty}^{\infty} |f(x)| dt \text{ exists} \right)$$

then:

$f(x)$ has F.I Representation

at points where $f(x)$ is discontinuous.

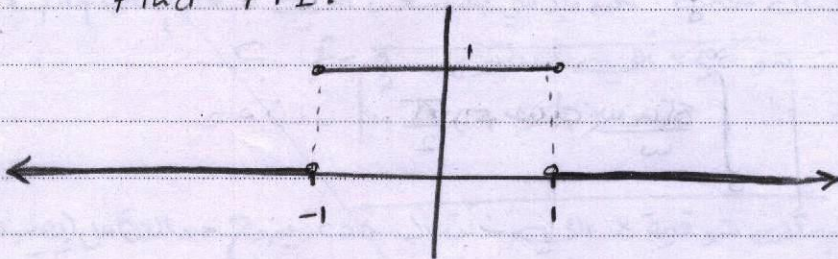
the F.I converges to $\frac{f(x^+) + f(x^-)}{2}$

EHD OF LEC. 8

Ex

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Find F.I.



$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v dv = \frac{1}{\pi} \int_{-1}^1 \cos \omega v dv$$

مع إشارة الـ ω من $(-\infty, \infty)$

أخذت به $(-1, 1)$ لأن باقيها صفر
فأزل أثره ..

$$= \frac{1}{\pi} \left. \frac{\sin \omega v}{\omega} \right|_{-1}^1 = \frac{2 \sin \omega}{\pi \omega}$$

$$B(\omega) = \frac{1}{\pi} \int_{-1}^1 1 \cdot \sin \omega v dv = 0 \quad (\text{odd})$$

هاد هو الجواب

F.I.:

$$f(x) \approx \frac{2}{\pi} \int_0^{\infty} \frac{\sin \omega \cos \omega x}{\omega} d\omega$$

بمبدأ التفاضل

$$\int_0^{\infty} \frac{\sin \omega}{\omega} \cos \omega x d\omega = f(x) \cdot \frac{\pi}{2} = \begin{cases} \frac{\pi}{2}, & |x| < 1 \\ \frac{1}{2} \left(\frac{\pi}{2} \right), & x = 1 \\ 0, & |x| > 1 \end{cases}$$

بمساعدة حق لو فوسن وقوة الاقتدار الـ ω

وتساوي النقط بين القوسين

بمبدأ التفاضل (conv. thm.)

$$\frac{1}{2} = \frac{0+1}{2}$$

أخذ x من $|x|$

لأننا نعلم أنه

في نفس السؤال

* Show that $\int_0^{\infty} \frac{\sin w}{w} dw = \frac{\pi}{2}$

الطريقة 1. بسأل جاك: هم الإثبات المشعب الذي يأخذ صيغة بديهية.
شوقية x التي أتفق $f = \frac{\pi}{2}$ ؟
الجواب ($1 > x$)

الطريقة 2. بسأل شوقية x التي خلت الـ $\cos$ يروح منه التكامل (بصرا) ؟
الجواب $x=0$

اذن بنحار $x=0$ وبفرض في F.I التي حسبته

$$f(0) = \int_0^{\infty} \cancel{\cos w(0)} \cdot \frac{\sin w}{w} dw = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{\sin w}{w} dw = \frac{\pi}{2}$$

} أسئلة Show that كثر

* If $x=3$: $\int_0^{\infty} \frac{\sin w}{w} \cos 3w dw = 0$

في الحقيقة أن المشعب الذي يأخذ صيغة بديهية

$$\bullet \int_0^{\infty} \frac{\sin x}{x} \cos \frac{x}{2} dx = ?$$

بَعْرِفْ نَفْسَ الْاِقْتِرَانِ بِسَمَّا الْمَتَغَيِّرِ كَانِ w حَقْنِ x ..
 حَافِظَةً ، تَغْيِيرُ اسْمٍ فَقَطَا (dummy)

$$\int_0^{\infty} \frac{\sin w}{w} \cos \frac{w}{2} dw = ? = \boxed{\frac{\pi}{2}}$$

$\xrightarrow{x = \frac{w}{2}}$

• FOURIER COSINE INTEGRAL

If $f(x)$ is Even, it can be represented as F.COS.I:

$$\left| f(x) \cong \int_0^{\infty} A(\omega) \cos \omega x d\omega \right| \quad \left| A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \cos \omega v dv \right|$$

(because in B there is $\int_{-\infty}^{\infty} \underbrace{\text{Even}}_f \times \underbrace{\text{Odd}}_{\sin} = \int_{-\infty}^{\infty} \text{odd} = 0$)

• FOURIER SINE INTEGRAL.

If $f(x)$ is Odd, it can be represented as F.SIN.I:

$$\left| f(x) \cong \int_0^{\infty} B(\omega) \sin \omega x d\omega \right| \quad \left| B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \sin \omega v dv \right|$$

(because in A there is $\int_{-\infty}^{\infty} \underbrace{\text{Odd}}_f \cdot \underbrace{\text{Even}}_{\cos} = \int_{-\infty}^{\infty} \text{odd} = 0$)

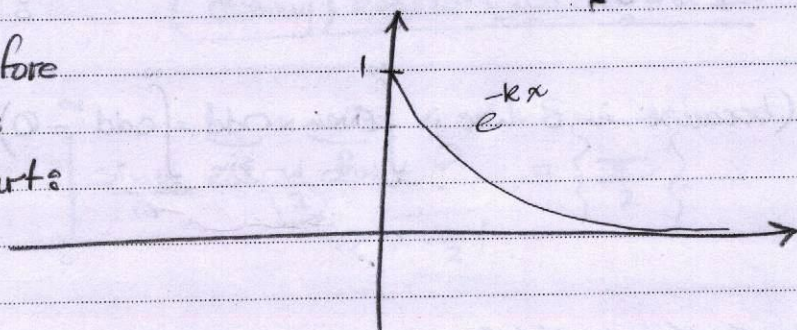
Ex $f(x) = e^{-kx}$, $x > 0$, $k > 0$.

FIND:

a) the F.S.I of $f(x)$.

b) the F.C.I of $f(x)$.

before
we
start:



from Calculus

$$\bullet \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$\bullet \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

a) F.S.I : $A(\omega) = 0$

عقانون:

$$a = -k$$

$$b = \omega$$

$$B(\omega) = \frac{2}{\pi} \int_0^{\infty} e^{-kv} \sin \omega v dv$$

$$B(\omega) = \frac{2}{\pi} \frac{e^{-kv}}{k^2 + \omega^2} (-k \sin \omega v - \omega \cos \omega v) \Big|_0^{\infty}$$

as $v \rightarrow \infty : e^{-kv} \rightarrow 0$ يعني نزل كل

$$= \frac{2}{\pi} \left(0 - \frac{-\omega(1)}{k^2 + \omega^2} \right) = \frac{2\omega}{\pi(k^2 + \omega^2)}$$

The F.S.I for $f(x)$ is:

$$e^{-kx} = \frac{2}{\pi} \int_0^{\infty} \frac{\omega}{k^2 + \omega^2} \sin \omega x d\omega$$

b) F.C.I : $B(\omega) = 0$

$$A(\omega) = \frac{2}{\pi} \int_0^{\infty} e^{-kv} \cos \omega v dv$$

$$A(\omega) = \frac{2}{\pi} \frac{e^{-kv}}{k^2 + \omega^2} (-k \cos \omega v + \omega \sin \omega v) \Big|_0^{\infty}$$

$$= \frac{2}{\pi} \left(0 - \frac{-k}{k^2 + \omega^2} \right) = \frac{2k}{\pi(k^2 + \omega^2)}$$

The F.C.I for $f(x) =$

$$e^{-kx} = \frac{2}{\pi} \int_0^{\infty} \frac{k}{k^2 + w^2} \cos wx \, dw$$

c) Show that $\int_0^{\infty} \frac{w \sin w}{4 + w^2} \, dw = \frac{\pi}{2} e^{-2}$ ↗ sin: so compare it with F.S.I.

Comparing this with the F.S.I,
we note that $k = +2$, and $x = 1$.

F.S.I $\rightarrow$ $\sin \leftarrow 1$

$$e^{-(2)(1)} = \frac{2}{\pi} \int_0^{\infty} \frac{w}{(2)^2 + w^2} \sin(w \cdot 1) \, dw$$

comparing
↙

$$\int_0^{\infty} \frac{w \sin w}{4 + w^2} \, dw = \frac{\pi}{2} e^{-2}$$

"))

/ /

$$\int_0^{\infty} \frac{x \sin 5x}{3+x^2} dx$$

• الخيار 5 : (بالعادة اسماء x ، بـ y ، استخدمه x بدل y)
 كما يصير اسماء x ، بعض بنقر كاتبة "الخيار" للعميم
 • $k^2 = 3$ $\leadsto$ $k = \sqrt{3}$

From F.S.I.:

this integration = $\frac{\pi}{2} e^{-k(\frac{\pi}{2})}$

$$= \frac{\pi}{2} e^{-5\sqrt{3}}$$

END OF LEC. 9

11.8 FOURIER COSINE AND SINE TRANSFORM.

• FOURIER SINE TRANSFORM

$$\underset{\text{Sine transform}}{F_s\{f\}} = \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x \, \underline{dx} \quad \downarrow \text{dx in transform.}$$

→ INVERSE

$$F_s^{-1}\{\hat{f}_s(\omega)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(\omega) \sin \omega x \, \underline{d\omega} \quad \downarrow \text{d\omega in inverse}$$

• FOURIER COSINE TRANSFORM

$$\underset{\text{cosine transform}}{F_c\{f\}} = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, \underline{dx}$$

→ INVERSE.

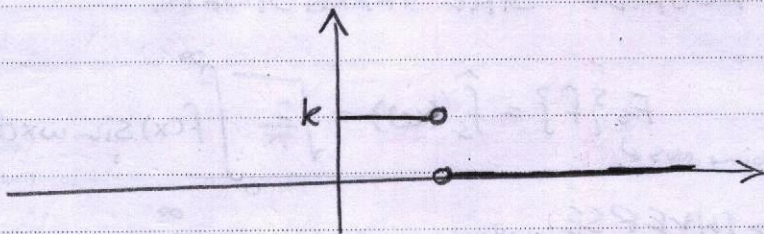
$$F_c^{-1}\{\hat{f}_c(\omega)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(\omega) \cos \omega x \, \underline{d\omega}$$

: ملاحظة

• حدود التكامل في F.C.T, F.S.T هي من 0 إلى ∞

أيضا في section القادم : حدود تكامل F.T (سواء sine أو cosine) هي من $-\infty$ إلى ∞

$$\underline{\underline{E \times}}$$



$$\bullet \quad F_c \{f\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f \cos wx dx = \sqrt{\frac{2}{\pi}} \int_0^1 k \cos wx dx$$

$$= \sqrt{\frac{2}{\pi}} \left. \frac{k}{w} \sin wx \right|_0^1 = \sqrt{\frac{2}{\pi}} \frac{k \sin w}{w}.$$

$$\bullet \quad F_s \{f\}^2 = \sqrt{\frac{2}{\pi}} \int_0^1 k \sin wx dx = -\sqrt{\frac{2}{\pi}} \frac{k}{w} \cos wx \Big|_0^1$$

$$= \sqrt{\frac{2}{\pi}} \frac{k}{w} (1 - \cos w)$$

[illegible]

$$f(x) = x, \quad f(x) = k \quad \text{für } x \geq 0$$

• PROPERTIES

[1] Linearity

$$F_s \{ a f(x) \pm b f(x) \} = a F_s \{ f(x) \} \pm b F_s \{ g(x) \}$$

F_c مجموع

[2] Transforms of Derivatives. (تحويل المشتقات)

$$F_c \{ f' \} = \omega F_s \{ f \} - \sqrt{\frac{2}{\pi}} f(0)$$

$$F_s \{ f' \} = -\omega F_c \{ f \}$$

→ Proof (property [2])

$f(x)$ is cont. and absol. integ.

$$\left(\int_{-\infty}^{\infty} |f(x)| dx < \infty \right)$$

$f'(x)$ is a piecewise cont. function,
and $f(x) \rightarrow 0$ as $x \rightarrow \infty$

Contin. →

/ /

integrate by parts

$$\begin{aligned}
 \bullet \quad F_c \{f'\} &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f'(x) \cos wx \, dx \\
 &= \sqrt{\frac{2}{\pi}} \left(f(x) \cos wx \Big|_0^{\infty} + w \int_0^{\infty} f(x) \sin wx \, dx \right) \\
 &= \sqrt{\frac{2}{\pi}} \left(\underset{\infty}{0} - \underset{0}{f(0)} \right) + w \underbrace{F_s \{f\}}_{\substack{\sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \cdot 2}} \\
 &= w F_s \{f\} - \sqrt{\frac{2}{\pi}} f(0)
 \end{aligned}$$

$$\begin{aligned}
 \bullet \quad F_s \{f'\} &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f'(x) \sin wx \, dx \\
 &= \sqrt{\frac{2}{\pi}} \left(f(x) \sin wx \Big|_0^{\infty} - \int_0^{\infty} f(x) \cos wx \, dx \right) \\
 &= \sqrt{\frac{2}{\pi}} \left(0 - 0 - w \int_0^{\infty} f(x) \cos wx \, dx \right) \\
 &= -w F_c \{f\}
 \end{aligned}$$

[3] for Second Derivative

$$\bullet F_c \{f''\} = -\omega^2 F_c \{f\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$\bullet F_s \{f''\} = -\omega^2 F_s \{f\} + \sqrt{\frac{2}{\pi}} \omega f(0)$$

Proof

$$F_s \{f''\} = F_s \{(f')'\} = -\omega F_c \{f'\}$$

$$= -\omega (\omega F_s \{f\} - \sqrt{\frac{2}{\pi}} f(0))$$

$$= -\omega^2 F_s \{f\} + \sqrt{\frac{2}{\pi}} \omega f(0)$$

$\overline{a_i} > 0$
 من الـ property 2
 property 2 no ... = = property 3

(= ...)

Ex Find the F. Cos. Tran of $f(x) = e^{-ax}$, $a > 0$

Use

$$F_c\{f''\} = -w^2 F_c\{f\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$f'(x) = -ae^{-ax}$$

$$\leadsto f'(0) = -a$$

$$f''(x) = a^2 e^{-ax} = a^2 f(x)$$

نطبق
 $F_c\{ \}$

$$\begin{aligned} \rightarrow a^2 F_c\{f(x)\} &= F_c\{f''(x)\} = -w^2 F_c\{f\} - \sqrt{\frac{2}{\pi}} f'(0) \\ &= -w^2 F_c\{f\} - \sqrt{\frac{2}{\pi}} f'(0) = -w^2 F_c\{f\} + \sqrt{\frac{2}{\pi}} a \end{aligned}$$

بعض الترتيب:
من هنا نطلب

$$(a^2 + w^2) F_c\{f\} = \sqrt{\frac{2}{\pi}} a$$

$$F_c\{f\} = \frac{a}{a^2 + w^2} \sqrt{\frac{2}{\pi}}$$

$$F_c\{e^{-ax}\} = \frac{a}{a^2 + w^2} \sqrt{\frac{2}{\pi}}$$

11.9 FOURIER TRANSFORM

$f(x)$ is absol. integ. and piecewise cont.
for $-\infty < x < \infty$

$$F\{f\} = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

→ Inverse:

i : imaginary number (j)

$$F^{-1}\{\hat{f}(\omega)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

Ex Find the Fourier Trans. of $f(x)$.

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

By def:

$$F\{f\} = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left. \frac{e^{-i\omega x}}{-i\omega} \right|_{-1}^1$$

$$= \frac{-1}{i\omega \sqrt{2\pi}} (e^{-i\omega} - e^{+i\omega})$$

~~~~~>



$$\left| \begin{array}{l} \text{But } e^{\alpha i} = \cos \alpha + i \sin \alpha \\ e^{-\alpha i} = \cos \alpha - i \sin \alpha \end{array} \right|$$

$$\text{الفعل للـ } F = \frac{-1}{i\omega \sqrt{2\pi}} (\cos \omega - i \sin \omega - (\cos \omega + i \sin \omega))$$

$$= \frac{-1}{i\omega \sqrt{2\pi}} - 2i \sin \omega = \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega}$$



Ex Find  $F\{f\}$ , where  $f = \begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0 \end{cases}$

$$F\{f\} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} e^{-i\omega x} dx$$

$$\left(\frac{1}{\sqrt{2\pi}}\right) \Rightarrow \int_0^{\infty} e^{-(a+i\omega)x} dx = \frac{-1}{(a+i\omega)\sqrt{2\pi}} e^{-(a+i\omega)x} \Big|_0^{\infty}$$

$$= \frac{-1}{(a+i\omega)\sqrt{2\pi}} \overset{x \text{ goes to } \infty}{\underset{\downarrow}{(0-1)}} \overset{x=0}{\uparrow}$$

$$= \frac{1}{(a+i\omega)\sqrt{2\pi}}$$



## Properties

[1] F.T is linear

$$F\{af(x) \pm bg(x)\} \quad a \text{ and } b \text{ are constants.}$$

$$= a F\{f(x)\} \pm b F\{g(x)\}$$

$$[2] F\{f'(x)\} = i\omega F\{f(x)\} \quad (\text{مشتق... بالاعتماد على...})$$

$$F\{f''\} = i\omega F\{f'\} = -\omega^2 F\{f\}$$

Ex Find the F.T of function  $x e^{-x^2}$

Use  $F\{f'\} = i\omega F\{f\}$  and

$$F\{e^{-x^2}\} = \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}}$$

$$f = x e^{-x^2}$$

$$f' = -2x e^{-x^2} \Rightarrow x e^{-x^2} = -\frac{1}{2} f'(x)$$

$$F\{x e^{-x^2}\} = -\frac{1}{2} F\{f'(x)\} = -\frac{1}{2} i\omega F\{f\}$$

$$= -\frac{1}{2} i\omega \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}}$$

END OF LEC. 10



Student Name:....

Student number:.....

Lecture time:.....

احمد العبد

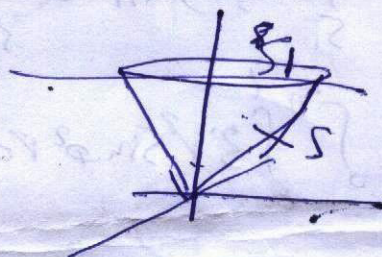
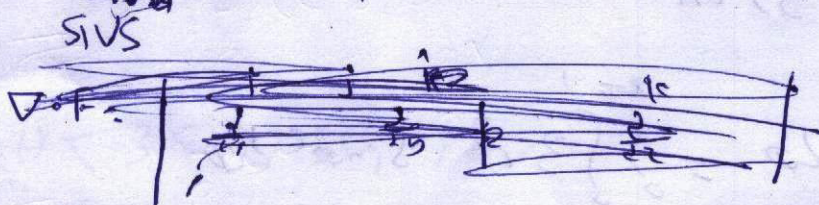
1. Use the Divergence Theorem to evaluate  $\iint_S \vec{F} \cdot \vec{n} dA$ , where

$$\vec{F}(x, y, z) = (xz \sin(yz) + x^3)\mathbf{i} + (\cos(yz))\mathbf{j} + (3zy^2)\mathbf{k} \text{ and}$$

S is the surface of the cone  $z = \sqrt{x^2 + y^2}$ ,  $0 \leq z \leq 1$ .

[8 points]

$$\iiint_V \vec{F} \cdot \vec{n} dA = \iiint_V \nabla \cdot \vec{F} dV$$

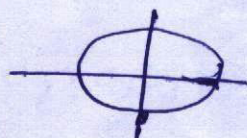


$$\nabla \cdot \vec{F} = z \sin(yz) + 3x^2 + \sin yz + (z) + 3y^2$$

$$= z \sin yz + 3x^2 + \sin yz + z + 3y^2$$

$$= 3x^2 + 3y^2$$

$$= \iiint_T 3(x^2 + y^2) dV = \int_{Rxy} \int_{V_{xyz}} 3(x^2 + y^2) dz dA$$



$$= \iint_{Rxy} 3(x^2 + y^2) (1 - \sqrt{x^2 + y^2}) dA$$

$$= \int_0^{2\pi} \int_0^1 3r^2(1-r) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 3r^3(1-r) dr d\theta$$

$$= \int_0^{2\pi} (3r^4 - 3r^5) dr \times 2\pi$$

$$= \left[ \frac{3}{5} r^5 - \frac{3}{6} r^6 \right]_0^1 \times 2\pi$$

$$= \left[ \frac{3}{5} (1-0) - \frac{3}{6} (1-0) \right] \times 2\pi$$

$$= \frac{15-12}{30} \times 2\pi = \frac{3}{10} \times 2\pi = \frac{3\pi}{5}$$

النتيجة



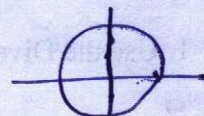
$$\iint_{S \cup S_1} F \cdot n \, dA = \iint_S F \cdot n \, dA + \iint_{S_1} F \cdot n \, dA$$

$$\iint_S F \cdot n \, dA = \iint_{S \cup S_1} F \cdot n \, dA - \iint_{S_1} F \cdot n \, dA$$

$$S_1: x^2 + y^2 = 1 \quad z = 1$$

$$n = \hat{k}$$

$$n \cdot F = 3zy^2$$



$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$\iint_{S_1} 3zy^2 \, dA = \iint_{S_1} 3y^2 \, dA$$

$$\int_0^{2\pi} \int_0^1 3r^2 \sin^2 \theta \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 3r^3 \sin^2 \theta \, dr \, d\theta$$

$$= \left[ \frac{3r^4}{4} \right]_0^1 \times \frac{1}{2} \int_0^{2\pi} (1 - \cos 2\theta) \, d\theta$$

$$= \frac{3}{4} (1-0) \times \frac{1}{2} \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi}$$

$$= \frac{3}{4} \times \frac{1}{2} (2\pi - \frac{1}{2} (0 - 0))$$

$$= \frac{3}{4} \times \pi = \frac{3\pi}{4}$$

$$\iint_S F \cdot n \, dA = \frac{2\pi}{10} - \frac{3\pi}{4} = \frac{12\pi - 30\pi}{40} = -\frac{18\pi}{40}$$



2. Use Stokes's theorem to evaluate  $\iint_S (\text{curl } \vec{F}) \cdot \vec{n} dA$ , where

$S$  is the surface of the hemisphere  $x^2 + y^2 + z^2 = 9$  with  $z \geq 0$ , oriented outward;

$$\vec{F}(x, y, z) = (2y \cos z)\vec{i} + (e^x \sin z)\vec{j} + (xe^y)\vec{k}$$

[7 points]

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y \cos z & e^x \sin z & xe^y \end{vmatrix} = (xe^y - e^x \cos z)\vec{i} - (e^y + 2y \sin z)\vec{j} + (e^x \sin z - 2 \cos z)\vec{k}$$

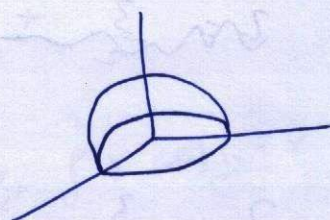
$$\vec{n} = [0, 0, 1]$$

$$(\text{curl } \vec{F}) \cdot \vec{n} = e^x \sin z - 2 \cos z$$

$$z = 0$$

$$(\text{curl } \vec{F}) \cdot \vec{n} = -2$$

7



$$\iint_S -2 dA = -2 (\pi) 9 = -18\pi$$



3. If  $f(x) = \begin{cases} x+2, & -2 \leq x < 0, \\ 2, & 0 \leq x < 2, \end{cases}$  ,  $f(x+4) = f(x)$  then

a) Find the Fourier series for  $f(x)$ ,

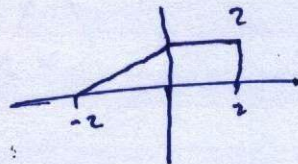
b) Find  $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ .

[8 points]

$$P = 2 \times 4$$

$$L = 2$$

$$c_0 = \frac{1}{4} \int_{-2}^2 f(x) dx = \frac{1}{4} \left( \int_{-2}^0 (x+2) dx + \int_0^2 2 dx \right)$$



$$= \frac{1}{4} (2 + 4) = \frac{6}{4} = \frac{3}{2}$$

$$a_n = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx = \frac{1}{2} \left( \int_{-2}^0 (x+2) \cos \frac{n\pi x}{2} dx + \int_0^2 2 \cos \frac{n\pi x}{2} dx \right)$$

$$= \frac{1}{2} \left( \frac{2(x+2) \sin(\frac{n\pi x}{2})}{n\pi} + \frac{4 \cos \frac{n\pi x}{2}}{n^2 \pi^2} \right) \Big|_{-2}^0$$

$$\begin{aligned} & (x+2) \cos \frac{n\pi x}{2} \\ & \Big|_0^2 \frac{2 \sin(\frac{n\pi x}{2})}{n\pi} \\ & - \frac{4 \cos(\frac{n\pi x}{2})}{n^2 \pi^2} \end{aligned}$$

8

$$+ \frac{(2) 2 \sin(\frac{n\pi x}{2})}{n\pi} \Big|_{-2}^0$$

$$= \frac{1}{2} \left( \frac{4 - 4 \cos(-n\pi)}{n^2 \pi^2} \right) = \frac{2(1 - (-1)^n)}{n^2 \pi^2}$$

$$b_n = \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx = \frac{1}{2} \left( \int_{-2}^0 (x+2) \sin \frac{n\pi x}{2} dx + \int_0^2 2 \sin \frac{n\pi x}{2} dx \right)$$

$$= \frac{1}{2} \left( \frac{2(x+2) \cos(\frac{n\pi x}{2})}{n\pi} + \frac{4 \sin(\frac{n\pi x}{2})}{n^2 \pi^2} \right) \Big|_{-2}^0$$

$$\begin{aligned} & x+2 \sin(\frac{n\pi x}{2}) \\ & \Big|_0^2 \frac{-2 \cos(\frac{n\pi x}{2})}{n\pi} \\ & - \frac{4 \sin(\frac{n\pi x}{2})}{n^2 \pi^2} \end{aligned}$$

$$= \frac{-1}{n\pi} (2 \cos(n\pi)) - 0$$

$$= \frac{-2(-1)^n}{n\pi}$$

في الصفحة التالية



$$\frac{2(1-(-1)^n)}{n^2 \pi^2}$$

n is odd  $\frac{4}{n^2 \pi^2}$

n is even zero

$$\sum_{n=1}^{\infty} \frac{-2(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

$$a) x(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n^2 \pi^2} \cos\left(\frac{n\pi x}{2}\right)$$

$$b) x(0) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n^2 \pi^2}$$

$$2 = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2 \pi^2}$$

$$2 = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2 \pi^2}$$

$$2 - \frac{3}{2} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\text{So } \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$



4. Find the Fourier transform of  $f(x) = e^{-c|x|}$ ,  $-\infty < x < \infty$ ,  $c > 0$

[7 points]

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx$$

$$f(x) = \begin{cases} e^{cx} & -\infty < x < 0 \\ e^{-cx} & 0 < x < \infty \end{cases}$$

$$= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^0 e^{cx} e^{-j\omega x} dx + \int_0^{\infty} e^{-cx} e^{-j\omega x} dx \right) \checkmark$$

$$= \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^0 e^{(c-j\omega)x} dx + \int_0^{\infty} e^{(-c-j\omega)x} dx \right) \checkmark$$

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{e^{(c-j\omega)x}}{c-j\omega} \Big|_{-\infty}^0 + \frac{e^{(-c-j\omega)x}}{-c-j\omega} \Big|_0^{\infty} \right) \checkmark$$

$$= \frac{1}{\sqrt{2\pi}} \left( \left( \frac{1}{c-j\omega} - 0 \right) + \left( 0 - \frac{1}{-c-j\omega} \right) \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{c-j\omega} + \frac{1}{c+j\omega} \right)$$

7

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{c+j\omega + c-j\omega}{c^2 + \omega^2} \right) = \frac{1}{\sqrt{2\pi}} \frac{2c}{c^2 + \omega^2} \checkmark$$

$$= \frac{2c}{\sqrt{2\pi} (c^2 + \omega^2)}$$



# CHAPTER 12

## PARTIAL DIFFERENTIAL EQUATION

### 12.1 INTRO

ODE:  $y'' + 2y' - 3xy = e^x$   
(Diff. الـ  $y$  فقط)

PDE: is an equation involving partial derivatives of unknown function.

✓ in  $\begin{cases} u = u(x, y) \\ \text{or } u = u(t, y) \\ u = u(x_1, x_2, \dots, x_n) \end{cases}$

### Examples

$$u_x + u_y = 1$$

1st order / nonhomo

$$u_{tt} = c^2 u_{xx}$$

2nd = / homo

$$u_t = a^2 u_{xx}$$

2nd = / homo

$$u_{tt} - c^2 u_{xx} = t$$

" " / nonhomo

\* An order of a PDE is the order of the highest derivative.

\* Linearity.

(All of the above Equations are linear).



Examples for non linear PDE:

$$(u_x)^2 + u_y = 1$$

$$u_x + \underline{u} \underline{u}_y = 1$$

NOTE that

$$u_x + xy u_y = 1 \text{ is } \underline{\text{linear}}$$

(dependent) Funct.  $u$  is  $\downarrow$   
linearity  $\swarrow$  functions  $x, y$   $\swarrow$   
(independent)

Remember: we need some initial  
Conditions to solve the D.E.

- without initial Conditions we can get the general Solution, but not the exact one.



Ex Solve  $u_{xx} - u = 0$  ( $u = u(x, y)$ )

→ one kind of deriv. (w.r.t.  $x$ ), I can write it as:

$$u'' - u = 0$$

$$k^2 - 1 = 0 \rightarrow k = \pm 1 \rightarrow \begin{matrix} e^x \\ e^{-x} \end{matrix}$$

$$u(x, y) = A(y) e^x + B(y) e^{-x}$$

← عشان  $u$  فيها  $x$  و  $y$

فانا بد اوجدت اللي فيه  $x$

فنبقى ايجو فيه  $y$

Ex  $u_{yy} - 4u_y + 4u = 0$   
(w.r.t  $y$  only)

$$u'' - 4u' + 4u = 0$$

$$k^2 - 4k + 4 = 0 \quad k_1 = k_2 = 2$$

$$u(x, y) = \underline{A(x)} e^{2y} + \underline{B(x)} y e^{2y}$$

عشان طلعنا به الجزء اللي فيه  $y$

فبقي كل وحدة فيه  $x$  function of  $x$



Ex (complex numbers)

$$u_{xx} + 4u = 0$$

$$u'' + 4u = 0$$

$$\rightarrow k^2 + 4 = 0$$

$$k_1 = 2i$$

$$k_2 = -2i$$

$$u(x, y) = A(y) \cos 2x + B(y) \sin 2x$$

لذا الآن بدد حلينا وانو  $u$  تكون حصة  
بالنسبة للاحتر واحد بد  $(x$  او  $y)$ .

وكان حالة جوال  $O.D.E$  حالة

$$u'' \rightsquigarrow u_{xx}$$

$$u' \rightsquigarrow u_y$$

و صلا ..



Ex Solve  $u_{xy} = -u_x$

$p = u_x$  (if  $p$  is a function of  $y$  only)  
 $p_y = u_{xy} = -u_x = -p \rightarrow \underline{p_y = -p}$

Separable:

$$\frac{dp}{dy} = -p$$

$$\int \frac{dp}{p} = - \int dy$$
$$\ln p = -y + \underbrace{A(x)}_{\text{constant}}$$

$$p = e^{(-y+A(x))} = A_1(x) e^{-y} \quad (A_1 = e^{A(x)})$$
$$= u_x$$

$$u(x, y) = e^{-y} \underbrace{\int A_1(x) dx}_{A_2(x)} + B(y)$$

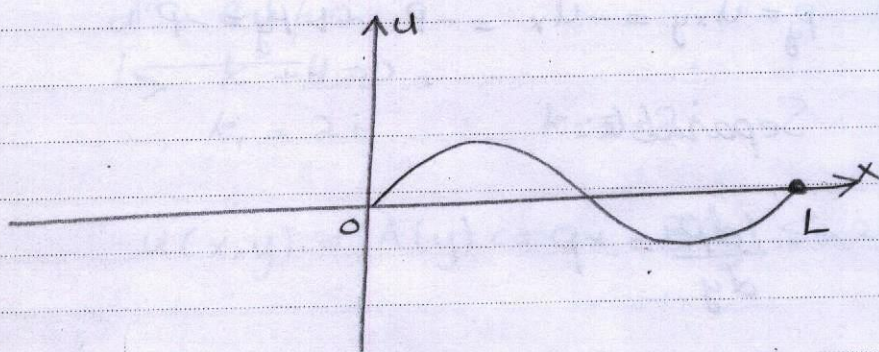
$$u(x, y) = A_2(x) e^{-y} + B(y)$$

general Sol.



### 12.3 WAVE EQUATION

Consider a string over interval  $L$



$u(x,t)$  = height of the string at  
point  $x$  at time  $t$ .

↳ position fun.  
or displacement fun.

$$\left| \begin{array}{l} u_{tt} = c^2 u_{xx} \\ c \text{ is constant} \end{array} \right| \quad \text{wave equation (one dimension)}$$



## \* Boundary Conditions (for $x$ )

[1] Dirichlet Cond. "ديريكلية"

$$u(0, t) = 0 \quad \text{Fixed Ends (مثبتة بالمحصلة و 0)} \quad L$$

$$u(L, t) = 0 \quad (\text{homogeneous cond.})$$

nonhomog.  $u(x, t) = f(x)$   
(غير متجانسة،  $f(x)$ )

[2] Neumann B.C.

$$u_x(0, t) = 0 \quad \text{Free Ends}$$

$$u_x(L, t) = 0 \quad (\text{كأنه حرة / مثبتة / مربوطة بحرة})$$

[3]

$$\begin{cases} \bullet u(0, t) = 0 \\ \bullet u_x(l, t) = 0 \end{cases}$$

OR

$$\begin{cases} \bullet u_x(0, t) = 0 \\ \bullet u(l, t) = 0 \end{cases}$$



Solve  $u_{tt} = C^2 u_{xx} \dots (1) \quad 0 < x < L, \quad 0 < t$

→ Boundary Conditions (B.C.)

$$u(0, t) = 0$$

$$u(L, t) = 0$$

$$t \geq 0 \dots (2)$$

→ Initial Conditions (3)

- $u(x, 0) = f(x)$  (يعني بس في  $x$ )

initial displacement

- $u_t(x, 0) = g(x)$  (تجيب في  $t$  يعني displacement  $u$  و velocity  $u_t$ )

initial velocity

Solution Steps:

#1 Separation of variables.

PDE becomes → pair of ODE's

#2 Solve ODEs (using B.C.)

#3 Use Fourier Series.





## Step #1

let  $u(x, t) = F(x) G(t)$  (4)

(for Example:  $e^{-t} \sin 2x$ )

فرضیه: let the solution be a product of a function of  $x$  and a function of  $t$ .

- Substitute (4) into (1):

$$u_t = F(x) G'(t) \rightarrow u_{tt} = F(x) G''(t)$$

$$u_x = F'(x) G(t) \rightarrow u_{xx} = F''(x) G(t)$$

$$F(x) G''(t) = C^2 F''(x) G(t)$$

ب. س. ت. ع. ح. ط. و. خ. ع. ج. هـ.

$$\frac{F''(x)}{F(x)} = \frac{G''(t)}{C^2 G(t)} = -k$$

أنا بي أفكرهم قمر .. وخطبة البعشان لا أتبه  
المعادلة مان شوي ع أفكره ع جهة أتبه وبعده موبه

$k$  is a separation constant that must be found.

$$\frac{F''}{F} = -K \quad \text{AND} \quad \frac{G''}{C^2 G} = -K$$

$$\textcircled{5} \quad \underline{F'' + kF = 0} \quad \underline{G'' + kC^2 G = 0} \quad \textcircled{6}$$

ODE  ODE

Step #1: فصل نَسَبًا



1 / 1  
 1. خطوة: قم بتفكيك PDE إلى زوج من ODEs.  
 2. وفقاً لـ Step #1 وبوقف.

Step #2: Solve (5) with B.C

$$\text{B.C(1): } u(0, t) = \underline{F(0)} \underline{G(t)} = 0$$

one of them = 0

if  $G(t) = 0$ :

$$u(x, t) = F(x) \cdot 0 = 0 \quad (\text{trivial Solution})$$

! ok  $\forall x, t$

So:  $\boxed{F(0) = 0}$

$$\text{B.C(2): } u(L, t) = \underline{F(L)} \underline{G(t)} = 0$$

$$\boxed{F(L) = 0}$$

• Solve  $F'' + kF = 0$ ,  $F(L) = F(0) = 0$

(we have 3 cases for  $k$ :

1) if  $k = 0$

2) if  $k < 0$

3) if  $k > 0$ )

~



### Case [1]

$$k=0 \rightarrow F''=0 \rightarrow F(x)=Ax+B$$

$$\text{But } F(0)=0 \rightarrow B=0$$

$$\text{And } F(L)=0 \rightarrow AL=0 \rightarrow A=0$$

$$F(x)=0 \text{ then.}$$

$$u(x,t)=0 \cdot G(t)=0 \quad (\text{trivial! ignore it})$$

### Case [2]

$$k < 0 \rightarrow k = -\lambda^2 \quad (\text{التردد سالب... موجب... سالب... موجب})$$

$$F'' - \lambda^2 F = 0$$

$$\text{char. eq. : } m^2 - \lambda^2 = 0 \rightarrow m_1 = \lambda, m_2 = -\lambda$$

$$F(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

$$(F(0)=F(L)=0 \rightarrow \text{نقطة } C_2, C_1 \text{ مألوفة})$$

$$\bullet F(0) = C_1 + C_2 = 0 \rightarrow C_1 = -C_2$$

$$\bullet F(L) = C_1 e^{\lambda L} + C_2 e^{-\lambda L} = 0$$

$$C_2 (e^{-\lambda L} - e^{\lambda L}) = 0$$

$$\text{و } \rightarrow e^{-\lambda L} - e^{\lambda L} = 0 \rightarrow \lambda = 0 \rightarrow k = 0 \quad \leftarrow \begin{array}{l} \text{مفردة في } \lambda=0 \\ k < 0 \end{array}$$

$$\text{و } \rightarrow C_2 = 0 \rightarrow C_1 = 0 \rightarrow F(x) = 0 \quad (\text{trivial!})$$

### Case [3]

$$k > 0 \rightarrow k = \lambda^2 \quad (\text{التردد موجب... موجب})$$

$$F'' + \lambda^2 F = 0$$

$$m^2 + \lambda^2 = 0 \rightarrow m = \pm \lambda i$$

$$F(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$$



$$F(0) = 0 \rightarrow \underline{C_1 = 0}$$

$$F(L) = C_2 \sin \lambda L$$

! لا بد أن يكون  $\sin \lambda L = 0$  لأن  $C_2 \neq 0$

$$\text{إذن: } \sin \lambda L = 0$$

$$\rightarrow \lambda L = n\pi$$

$$n = 1, 2, 3, \dots$$

$$\boxed{\lambda_n = \frac{n\pi}{L}}$$

$$k = \lambda^2 = \rightarrow \boxed{k = \frac{n^2 \pi^2}{L^2}} \quad \text{eigen values} \quad \text{--- (7)}$$

(القيم الذاتية)

$$F_n(x) = C_2 \sin \frac{n\pi x}{L}$$

--- (8)

eigen functions  
(infinitely many func.)

$\rightarrow$  we still in step 2, we want to find  $G(t)$  from (6)

$$G''(t) + C^2 k G(t) = 0$$

no need to test the cases of  $k$  again  
we've just done that with  $F(x)$ ...

$$\text{Just take } k = \frac{n^2 \pi^2}{L^2} \dots$$

$$G'' + \frac{C^2 n^2 \pi^2}{L^2} G = 0 \rightarrow m^2 = -\frac{C^2 n^2 \pi^2}{L^2}$$

$$m_1 = \frac{Cn\pi}{L} i, \quad m_2 = -\frac{Cn\pi}{L} i$$

$\rightarrow$



$$\text{So, } G_n(t) = A_1 \cos \frac{cn\pi}{L} t + A_2 \sin \frac{cn\pi}{L} t$$

Step #3 "F.S"

$$u_n(x, t) = F_n(x) \cdot G_n(t)$$

$$u_n(x, t) = C_2 \sin \frac{n\pi x}{L} \left( A_1 \cos \frac{cn\pi}{L} t + A_2 \sin \frac{cn\pi}{L} t \right)$$

General Solution

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( \underbrace{P_n}_{C_2 A_1} \cos \frac{cn\pi}{L} t + \underbrace{R_n}_{C_2 A_2} \sin \frac{cn\pi}{L} t \right)$$

→ we have to find  $P_n$  and  $R_n$ .

(Using Initial Condition)

→ I.C. 1

$$u(x, 0) = f(x)$$

Series }  $\sum_{n=1}^{\infty}$   $\left\{ \begin{array}{l} \text{فوق} \\ \text{سلسلة} \end{array} \right\}$

$$\sum_{n=1}^{\infty} P_n \sin \frac{n\pi x}{L} = f(x)$$

$f(x)$   $\left\{ \begin{array}{l} \text{فوق} \\ \text{دالة} \end{array} \right\}$   
 $3x-1$   $\left\{ \begin{array}{l} \text{فوق} \\ \text{دالة} \end{array} \right\}$

This is a F.S.S. So

$$P_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$P_n$   $\left\{ \begin{array}{l} \text{فوق} \\ \text{دالة} \end{array} \right\}$



## I.C. 2

$$u_t(x, 0) = g(x) \quad \text{بعضها يكون له قيمة، مثلاً 4 أو x}$$

عشان نتوصل.. لازم بالاول نشق 4 النسبة لـ t

$$u_t(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( -P_n \frac{cn\pi}{L} \sin \frac{n\pi t}{L} + P_n \frac{cn\pi}{L} \cos \frac{n\pi t}{L} \right)$$

$$\downarrow u_t(x, 0) = \sum_{n=1}^{\infty} R_n \frac{cn\pi}{L} \sin \frac{n\pi x}{L} = g(x) \quad \text{F.S.S}$$

$$R_n \frac{cn\pi}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

$P_n, R_n$  صحيحين  $R_n$  .. صحيحين  $P_n$  .. صحيحين  $R_n$

وبعضها في التوقيت .. General Solution

كيف بيحس السؤال بالافغان ؟!

$$u_{tt} = 1 u_{xx} \quad \therefore c = 1$$

$$0 < x < 2 \quad \therefore L = 2$$

$$u(2, t) = u(0, t) = 0 \quad \text{B.C.}$$

$$u(x, 0) = 3x \quad \text{I.C.}$$

$$u_t(x, 0) = 0 \quad \text{I.C.}$$

وبحل بنفس الطريقة .. بد أرقام بلك الرفع.

$$R_n = 0$$

$$P_n = \frac{12(-1)^{n+1}}{n\pi}$$

من بعده لقيم أعطاة: ديكوه كل:



# LEC. #3 (final)

1 1

Solve

$$u_{tt} = c^2 u_{xx} \quad 0 < x < L, \quad t > 0$$

$$\left. \begin{aligned} u(0, t) &= 0 \\ u(L, t) &= 0 \end{aligned} \right\} \text{B.C.}$$

$$\left. \begin{aligned} u(x, 0) &= f(x) \\ u_t(x, 0) &= g(x) \end{aligned} \right\} \text{I.C.}$$

عادة في الامتحان واحد منهم صفر وبالتالي "مثلا"  $5x+3$ ,  $2x$

Solution

Step #1 Separation of variables.

Copy, paste from the previous Example!

$$\rightarrow \text{we got: } F'' + kF = 0$$

$$\text{and } G'' + c^2 k G = 0$$

Step #2 Solve  $F'' + kF = 0$  + B.C.

$$u(x, t) = F(x) \cdot G(t)$$

$$u_x(x, t) = F'(x) \cdot G(t)$$

$$\text{B.C.1} \rightarrow F'(0) = 0$$

$$\text{B.C.2} \rightarrow F'(L) = 0$$

$G(t) = 0$  is trivial  
we ignore it.

for k:

$$\text{case 1 } k=0 \quad F''=0 \rightarrow F = Ax + B \rightarrow F' = A$$

$$F'(0) = F'(L) = 0 \rightarrow \boxed{A=0} \text{ and } B \text{ is a number.}$$

we say that  $k=0$  is an eigen value.

and then  $F_0 = \text{Constant}$  is eigen function

$$\text{let } F_0 = 1 \quad (\cos(0) = 1, \sin(0) = 0)$$



case 2  $k < 0$  (no nontrivial solution)

Do it as a homework.

case 3  $k > 0$  ( $k = \lambda^2, \lambda > 0$ )

$$F'' + \lambda^2 F = 0$$

$$F(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$$

$$F'(x) = -C_2 \lambda \cos \lambda x - C_1 \lambda \sin \lambda x$$

$$F'(0) = \lambda C_2 \rightarrow \lambda > 0 \neq 0$$

$$\rightarrow |C_2 = 0| \quad C_2 \text{ is not zero}$$

$$F'(l) = -C_1 \lambda \sin \lambda l$$

$$\neq 0 \neq 0 \rightarrow = 0 \rightarrow \lambda l = n\pi$$

$$\lambda = \frac{n\pi}{L}, n = 1, 2, \dots$$

$$k = \frac{n^2 \pi^2}{L^2} \text{ is eig. val.}$$

$$F' \text{ is not } F \text{ is not } \sin$$

$$\uparrow F_n(x) = C_1 \cos \frac{n\pi}{L} x \text{ is eigen. function}$$

$\rightarrow$  Now, we want to find  $G(t)$

$$G'' + C^2 k G = 0$$

• we have two eigen values:  $k=0$ ,  $k = \frac{n^2 \pi^2}{L^2}$

$$\rightarrow k=0 \rightarrow G''(t) = 0 \rightarrow G(t) = At + B$$

$$G_0(t) = At + B$$

( $F_0(x)$  was a constant)



$$\rightarrow k = \frac{n^2 \pi^2}{L^2} \rightarrow G_n(t) = A_1 \cos \frac{n\pi c}{L} t + A_2 \sin \frac{n\pi c}{L} t$$

Step #3

$$u_0(x, t) = F_0 \cdot G_0 = N_0 t + M_0$$

$$u_n(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi x}{L} \left( A_1 \cos \frac{n\pi c}{L} t + A_2 \sin \frac{n\pi c}{L} t \right)$$

$\sum \text{below } u_n \text{ is } \leftarrow$

$\rightarrow$  The General Solution

$$u(x, t) = M_0 + N_0 t + \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \left( P_n \cos \frac{n\pi c}{L} t + R_n \sin \frac{n\pi c}{L} t \right)$$

\* Use I.C.'s to determine  $P_n$  &  $R_n$

$$\boxed{1} \quad u(x, 0) = M_0 + \sum_{n=1}^{\infty} P_n \cos \frac{n\pi x}{L} = f(x)$$

F.C.S

$$M_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$\text{and } P_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

$$\boxed{2} \quad u_t = N_0 + \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \left( -P_n \frac{n\pi c}{L} \sin \frac{n\pi c}{L} t + R_n \frac{n\pi c}{L} \cos \frac{n\pi c}{L} t \right)$$

$$u_t(x, 0) = N_0 + \sum_{n=1}^{\infty} R_n \frac{n\pi c}{L} \cos \frac{n\pi x}{L} = g(x)$$

F.C.S

$$\text{Q.S } \rightarrow \frac{2}{L} \int_0^L g(x) \cos \frac{n\pi x}{L} dx$$

$$\text{and } N_0 = \frac{1}{L} \int_0^L g(x) dx$$

خاتمة السوال



مثل هاد السؤال في الامتحان، لكن مع قيم:

$$u_{tt} = u_{xx} \quad 0 < x < 1$$

$\frac{1}{c} = 1$                        $\frac{1}{L} = 1$

B.C       $u(0, t) = u(1, t) = 0$

I.C       $u(x, 0) = x - 1$       "f(x)"  
 $u_x(x, 0) = 0$       "g(x)"

بحل بنفس الطريقة مع تهريره القيم

So, any wave equation has B.C.

أو       $u(0, t) = u(l, t) = 0$       يمكن أن يكون  
 أو       $u_x(0, t) = u_x(l, t) = 0$       يمكن أن يكون  
 أو       $u_x(0, t) = u(l, t) = 0$  }      ما حليناهم...  
 أو       $u(0, t) = u_x(l, t) = 0$  }      بنفس  
                                          الطريقة وخطوات  
                                          (Home work)

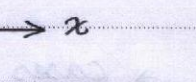
سأه سؤال بالفانيك: أحل الاحتمالات الآتية التي فوقه  
 مع wave equation  
 Solve it! و

END OF WAVE  
 EQUATION



8-1096

- length  $l$ .



$\langle e, t \rangle$

درجہ کاغذ

(2) الامتحان

les .....

四

$$= 0 \quad \boxed{3}$$



## Step #2 Solving the two ODE's (from step 1)

→ [2] with B.C. to solve  $F$

$$u(0,t) = F(0) \quad \underbrace{G(t)=0}_{t \neq 0} \rightarrow \boxed{F(0)=0}$$

$$u(l,t) = 0 \rightarrow \boxed{F(l)=0}$$

with

$$\boxed{F'' + kF = 0}$$

\* cases for value of  $k$ : (see wave equation, Exactly the same)

• for  $k=0$  and  $k < 0 \rightarrow$  no solution.

• for  $k > 0$

$$\hookrightarrow k = \frac{n^2 \pi^2}{L^2} \quad \text{eigen values} \quad (n=1,2,\dots)$$

$$\left\{ F_n(x) = C_n \sin \frac{n\pi}{L} x \right\} \text{ eigen functions.}$$

→ [3] to solve  $G$ .

$$G' + \frac{c^2 n^2 \pi^2}{L^2} G = 0 \rightarrow \frac{G'}{G} = -\left(\frac{cn\pi}{L}\right)^2$$

مبدأ جدلية:  $\ln G = -\left(\frac{cn\pi}{L}\right)^2 t + \text{const.}$

$$G_n(t) = e^{(\text{const.} - (\frac{cn\pi}{L})^2 t)}$$

$$G_n(t) = A_n e^{-\left(\frac{c^2 \pi^2 n^2}{L^2} t\right)}$$

Now, the general Sol. →



$$u_n(x,t) = \underbrace{C_n}_{F_n} \sin \frac{n\pi x}{L} \cdot \underbrace{A_n}_{G_n} e^{-\frac{c^2 n^2 \pi^2}{L^2} t}$$

let  $C_n A_n = B_n$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\dots}$$

→ Find  $B_n$  from I.C.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x)$$

F.S.S  $\rightarrow = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

وَكُنْ كَالطَّيْرِ.. خَفَّاقَ الْجَنَاحِ..



Ex Solve  $u_t = c^2 u_{xx}$   $0 < x < l, t > 0$

$$u_x(0,t) = u_x(l,t) = 0$$

$$u(x,0) = f(x)$$

Step #1 Same as previous Example

Step #2 Solve ODE's

→ for  $F$ :  $k=0$  is an eigen value

$F_0 = \text{const.}$  eigen function

$k < 0$  no solution

$k > 0$   $k = \frac{n^2 \pi^2}{L^2}$  eigen values.

$$\left\{ F_n(x) = A_n \cos \frac{n\pi x}{L} \right\} \text{ eigen functions.}$$

→ for  $G$ : when  $k=0$   $G'(t)=0$

$G_0 = \text{const.}$

when  $k = \frac{n^2 \pi^2}{L^2}$ :

$$\left\{ G_n = B_n e^{-\frac{c^2 n^2 \pi^2}{L^2} t} \right\}$$

Step #3

→  $u_0 = F_0 \cdot G_0 = \text{constant, say: } C_0$

→  $u_n = F_n \cdot G_n$

~~~~~


General Solution

$$u(x, t) = C_0 + \sum_{n=1}^{\infty} C_n \cos \frac{n\pi x}{L} e^{-\frac{n^2 \pi^2 c^2}{L^2} t}$$

Use I.C. to Find C_0 and C_n .

$$\begin{aligned} u(x, 0) &= C_0 + \sum_{n=1}^{\infty} C_n \cos \frac{n\pi x}{L} = f(x) \quad \text{F.C.S.} \\ &\quad \swarrow \quad \searrow \\ &= \frac{1}{L} \int_0^L f(x) dx \quad = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx \end{aligned}$$

في الاقطاب نفس الشيء (طبعا مع قيم l, c)
لكن الفرق في B.C. يكونوا احدى التايمة التالية:

$$\boxed{1} \quad u(x, 0) = u(x, l) = 0$$

$$\boxed{2} \quad u(x, 0) = u_x(x, l) = 0$$

$$\boxed{3} \quad u_x(x, 0) = u(x, l) = 0$$

$$\boxed{4} \quad u_x(x, 0) = u_x(x, l) = 0$$

END OF HEAT EQUATION

LEC. #5 (final)

1 1

Ex Solve $u_t = 4u_{xx}$ $0 < x < 3$, $t > 0$

1. $u(0, t) = 0$

2. $u(3, t) = 0$

3. $u(x, 0) = -5 \sin \frac{2\pi x}{3} + 4 \sin \frac{4\pi x}{3}$

Notes: This is heat Equation, with values for L and C given,

Conditions 1 and 2 are B.C. $(x \in 0, 3, 0)$

Condition 3 is the I.C. $(t \in 0)$

Solution

let $u(x, t) = F(x) \cdot G(t)$

{ through same steps } - - -

General Solution will be:

$$u(x, t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{3} e^{-\frac{4\pi^2 n^2}{9} t}$$

$\rightarrow$ we find C_n from the I.C.

$$u(x, 0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{3} = \underbrace{-5 \sin \frac{2\pi x}{3} + 4 \sin \frac{4\pi x}{3}}_{f(x)}$$

This is F.S.S, but with complicated $f(x)$,
finding C_n by integration is not
easy and straightforward,
So what to do?

• we write the terms of the Series:

$$C_1 \sin \frac{\pi x}{3} + C_2 \sin \frac{2\pi x}{3} + C_3 \sin \pi x + C_4 \sin \frac{4\pi x}{3} + C_5 \sin \frac{5\pi x}{3} + \dots$$
$$= -5 \sin \frac{2\pi x}{3} + 4 \sin \frac{4\pi x}{3}$$

$$\rightarrow C_2 = -5$$

$$C_4 = 4$$

$$C_1 = C_3 = C_5 = C_6 = C_7 = \dots = 0$$

$$u(x, t) = -5 \sin \frac{2\pi x}{3} e^{\frac{-4(4)\pi^2 t}{9}} + 4 \sin \frac{4\pi x}{3} e^{\frac{-4(16)\pi^2 t}{9}}$$

(TRUE) !!
END OF HEAT
EQUATION

LAPLACE EQUATION.

first like the heat Equation, but in 2D: x and y

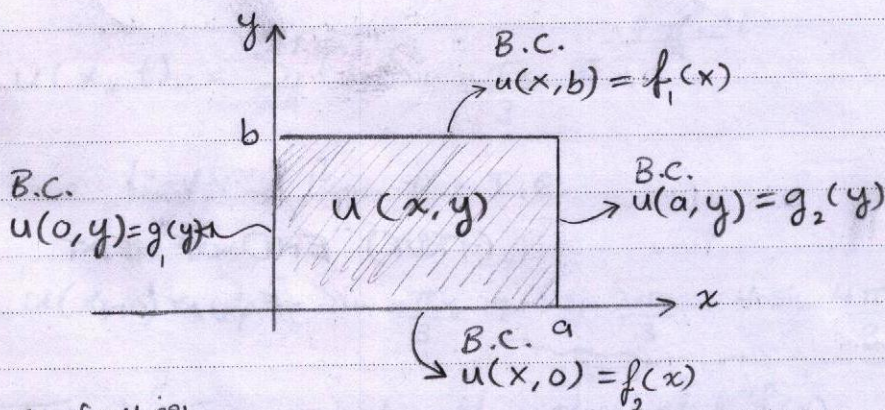
$$u_t = C^2(u_{xx} + u_{yy})$$

and if function u is time independent, i.e. " $u(x, y)$ "
then $u_t = 0$, and the equation becomes:

$$\boxed{u_{xx} + u_{yy} = 0} \quad \text{laplace Equation.}$$

→ Since there is no t in the equation,
then there is no I.C. in laplace eqn.

→ Since there is 2 variables (x and y),
then there are 4 B.C.s in laplace eqn.



- Above is Dirichlet Problem in a Rectangle.
- if they were $u_x(,) = \dots$: Neumann.

4 B.C's : one of them = function.
the other three = 0

Ex Solve $u_{xx} + u_{yy} = 0$ $0 \leq x \leq a$, $0 \leq y \leq b$

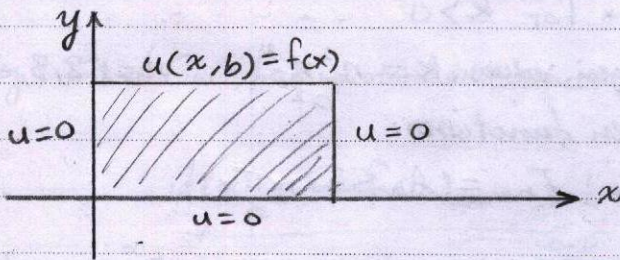
Nonhomog $\begin{cases} u(x, 0) = 0 \\ u(x, b) = f(x) \end{cases}$ 4 B.C. $\sin \frac{\pi x}{2}, \sin \frac{3\pi x}{2}, \dots$
• function = $\sum S_n$

Homog. $\begin{cases} u(0, y) = 0 \\ u(a, y) = 0 \end{cases}$

* ملاحظة هامة: بالنظر الى B.C. أجد أن B.C. تكون Homo

عندما x هو المتغير الذي لا قيمة له (مثلاً: 0 و 1)

لذلك: سأبذل بال ODE التي فيها $F(x)$ وليس $G(y)$ (في step 2)



Step #1 Separation of variables

let $u(x, y) = F(x) G(y)$

from lap. Equ.: $F''G = -FG''$

$$\frac{F''}{F} = -\frac{G''}{G} = -k$$

بما! في F ... بخطية حرة الوحدانية في معادلة F
حرجية .. يعرّف بخطية من ساحة ..

معادلة تفاضلية

$$F'' + kF = 0$$

$$G'' - kG = 0$$

Step #2 Solving the ODEs from Step 1.

Start with $F(x)$ and $G(y)$

$$\boxed{1} \quad F'' + kF = 0$$

with the homo. B.C.s:

$$\bullet \quad u(0, y) = F(0) \quad G(y) = 0 \rightarrow \boxed{F(0) = 0}$$

$$\bullet \quad u(a, y) = F(a) \quad G(y) = 0 \rightarrow \boxed{F(a) = 0}$$

Cases of k :

* $k = 0$ } no nontrivial Solution
* $k < 0$ } (See wave Equ.)

* for $k > 0$

$$\bullet \text{ eigen values: } k = \frac{n^2 \pi^2}{a^2}, \quad n = 1, 2, 3, \dots$$

$\bullet$ eigen functions:

$$\boxed{F_n = A_n \sin \frac{n\pi x}{a}}$$

$$\boxed{2} \quad G'' - \frac{n^2 \pi^2}{a^2} G = 0$$

$$\text{char. equ: } m^2 - \frac{n^2 \pi^2}{a^2} = 0 \quad \begin{cases} m_1 = \frac{n\pi}{a} \\ m_2 = -\frac{n\pi}{a} \end{cases}$$

$$G(y) = B_1 e^{\frac{n\pi y}{a}} + B_2 e^{-\frac{n\pi y}{a}}$$

Find B_1 and B_2 from B.C.:

- $u(x,0) = F(x) \quad G(0)=0 \rightarrow G(0)=0$

$$G(0) = B_1 + B_2 = 0 \rightarrow \underline{B_2 = -B_1}$$

$$G_n(y) = 2B_1 \left(\frac{e^{+\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}}{2} \right) \quad (\sinh \text{ definition } \frac{e^x - e^{-x}}{2})$$

$$\boxed{G_n = B_n^* \sinh \frac{n\pi y}{a}} \quad (B_n^* = 2B_1)$$

General Solution

$$u_n(x,y) = F_n(x) G_n(y)$$

$$u(x,y) = \sum_{n=1}^{\infty} \underbrace{C_n}_{A_n B_n^*} \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

Finally, Find C_n from nonhomogeneous B.C.

$$u(x,b) = f(x)$$

$$u(x,b) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a} = f(x)$$

F.S.S.

$$C_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

• $f(x)$ لا يتغير C_n فيه

LAPLACE EQUATION IN CIRCLE

(Dirichlet Problem)

$$u_{xx} + u_{yy} = 0 \quad \text{for a circle.}$$

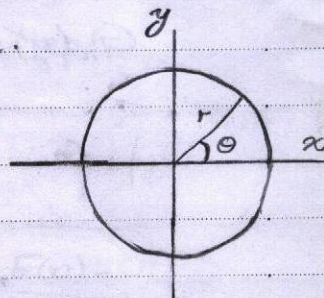
$$x = r \cos \theta$$

$$y = r \sin \theta$$

المساحة الداخلية

$$0 \leq r < a$$

$$-\pi < \theta < \pi$$



$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{معادلة لابلاس}$$

$$u(a, \theta) = f(\theta) \quad \text{non-homo. B.C.}$$

يوجد النوع من المعادلات يوجد فقط 1 B.C.

General Sol. فقط بأفراط عشان نوجد

Step #1 let $u(r, \theta) = R(r)F(\theta)$

$$R''F + \frac{1}{r} R'F + \frac{1}{r^2} RF'' = 0$$

$$(r^2 R'' + rR')F + RF'' = 0$$

$$\frac{r^2 R'' + rR'}{R} = \frac{-F''}{F} = k \quad (\text{نفس القيمة في الطرفين})$$

$$\textcircled{1} \quad r^2 R'' + rR' - kR = 0$$

$$\textcircled{2} \quad F'' + Fk = 0$$

Because $-\pi < \theta < \pi$ and $r < a$

$u(r, \theta) \therefore$ • periodic function for θ : period = 2π $F(\theta)$

• bounded function for r : $0 \leq r \leq a$ $R(r)$

* Cases for k :

II $k=0$ $r^2 R'' + r R' = 0$

first

$$F'' = 0 \leadsto F(\theta) = B\theta + A$$

But, as we said above, F must be periodic,
and linear function is not periodic,
unless it was a constant. ($B=0$)

So, $F(\theta) = \text{constant}$

i.e. $\boxed{F_0(\theta) = A_0}$

Second $r^2 R'' + r R' = 0$ Euler Equation

char. Equ: $m(m-1) + m = 0$

$$m^2 = 0, m_1 = m_2 = 0$$

$$R = B_1 r^0 + B_2 r^0 \ln r$$

Euler Equ. is (Solved)

$$= B_1 + B_2 \ln r$$

(Diff 1)

But $\ln r$ is not bounded! it must be so!

So $B_2 = 0$, and

$$R = B_1 = \text{constant}$$

$\boxed{R_0(r) = B_0}$

[2] $k < 0$ (let $k = -\lambda^2, \lambda > 0$)

my ODE's: $r^2 R'' + rR' + \lambda^2 R = 0$

$\hookrightarrow F'' - \lambda^2 F = 0$

char. equ. $\rightarrow m^2 - \lambda^2 = 0$

$m_1 = \lambda, m_2 = -\lambda$

$F(\theta) = C_1 e^{\lambda\theta} + C_2 e^{-\lambda\theta}$ (must be periodic)

to make it periodic, C_1 & C_2 must equal zero.

$C_1 = C_2 = 0$

$F(\theta) = 0$ trivial Solution

case Ignored. $\Rightarrow$ If R is periodic, F is trivial F if $k > 0$ or $k = 0$

[3] $k > 0$ (let $k = \lambda^2, \lambda > 0$)

my ODE's: $r^2 R'' + rR' - \lambda^2 R = 0$

$F'' + \lambda^2 F = 0$

first

$F(\theta) = A_{1n} \cos \lambda\theta + A_{2n} \sin \lambda\theta$

values of λ that make F periodic: integers only

$\lambda = n$ where $n = 1, 2, 3, \dots$

$F_n(\theta) = A_{1n} \cos n\theta + A_{2n} \sin n\theta$

Second

$$r^2 R'' + r R' - \lambda^2 R = 0$$

Euler Equation

$$m(m-1) + m - n^2 = 0$$

($\lambda = n$)

$$m^2 - n^2 = 0$$

$$m_1 = n, \quad m_2 = -n$$

$$R(r) = B_1 r^n + B_2 \frac{1}{r^n} \rightarrow \text{when } r=0 \rightarrow \text{it goes to } \infty$$

So, $B_2 = 0$ (unacceptable)

$$R_n = B_n r^n$$

For the General Solution

$$u_0(r, \theta) = A_0 B_0 = C_0$$

$$u_n(r, \theta) = \underbrace{A_n B_n}_{C_n} r^n \cos n\theta + \underbrace{A_{2n} B_n}_{P_n} r^n \sin n\theta$$

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} C_n r^n \cos n\theta + P_n r^n \sin n\theta$$

Find C_0, C_n, P_n Using the B.C.

$$u(a, \theta) = C_0 + \sum_{n=1}^{\infty} C_n a^n \cos n\theta + P_n a^n \sin n\theta = f(\theta)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

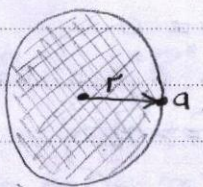
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$$

$-\pi < \theta < \pi$ and f is periodic in $L = \pi$: $\theta \rightarrow \theta + \pi$

في $C_n a^n \sin$ و $C_n a^n \cos$: $\theta \rightarrow \theta + \pi$

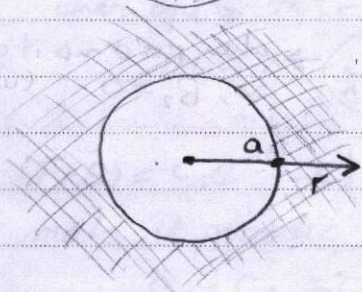
و $P_n a^n$: $\theta \rightarrow \theta + \pi$



Interior Problem

(جوانا الألة $r < a$)

التي أخذنا كل الألة



Exterior Problem

(جوانا الألة $r > a$)

التي أخذنا كل الألة

Exterior Dirichlet Problem.

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$$r > a, -\pi < \theta < \pi$$

with B.C. $u(a, \theta) = f(\theta)$

Step #1 Separation

$$\text{let } u(r, \theta) = R(r) F(\theta)$$

$$\textcircled{1} F'' + KF = 0$$

$$\textcircled{2} r^2 R'' + rR' - KR = 0$$

Step #2

$$\square \text{ for } k=0$$

$$F_0(\theta) = A_0$$

$$R_0(r) = B_0$$

نفس التي كان

Interior

[2] for $k > 0$ (let $k = \lambda^2, \lambda > 0$)

$$F_n(\theta) = A_n \cos n\theta + A_n \sin n\theta$$

$$R(r) = B_1 r^n + B_2 \frac{1}{r^n}$$

when $r \rightarrow \infty$, this term $\rightarrow \infty$
(unacceptable)

$$\text{So, } R_n(r) = B_n \frac{1}{r^n}$$

[3] for $k < 0$: case ignored (see Interior Problem)

General Solution

$$u(r, \theta) = C_0 + \sum_{n=1}^{\infty} \frac{A_n B_n}{r^n} \cos n\theta + \frac{A_n B_n}{r^n} \sin n\theta$$

$$\text{let } A_n B_n = C_n, \quad A_n B_n = K_n$$

and using the B.C.

$$f(\theta) = C_0 + \sum_{n=1}^{\infty} \frac{C_n}{a^n} \cos n\theta + \frac{K_n}{a^n} \sin n\theta$$

$$C_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$C_n = \frac{a^n}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta$$

$$K_n = \frac{a^n}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$$

END OF LAPLACE EQUATION.

SEMI-INFINITE DOMAIN

$$0 < x < \infty$$

Infinite: $(-\infty, \infty)$ يعني ∞ في الطرفين

FOURIER TRANSFORM will be used.

F.S.T. and F.C.T.

- Fourier Transforms for a function of one variable $f(x)$.

$$F_s \{ f(x) \} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx \, dx = \hat{f}_s(w)$$

$$F_c \{ f(x) \} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx = \hat{f}_c(w)$$

- Fourier Transform for a function of two variables $u(x, t)$, with respect to x .

$$\hat{u}_s(w, t) = F_s \{ u(x, t) \} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x, t) \sin wx \, dx$$

$$\hat{u}_c(w, t) = F_c \{ u(x, t) \} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x, t) \cos wx \, dx$$

من هنا: لأن ال Domain هو $(0, \infty)$ فستكون F.S.T. or F.C.T.

بأنها صيغة $(-\infty, \infty)$ في F.T. فستكون $\int_{-\infty}^{\infty}$

* for derivatives w.r.t t :

- first derivative.

$$F_c \{u_t\} = \frac{d}{dt} \hat{u}_c(\omega, t)$$

قناة $\hat{u}_c$

$$F_s \{u_t\} = \frac{d}{dt} \hat{u}_s(\omega, t)$$

- Second derivatives

$$F_s \{u_{tt}\} = \frac{d^2}{dt^2} \hat{u}_s(\omega, t)$$

قناة $\hat{u}_s$

$$F_c \{u_{tt}\} = \frac{d^2}{dt^2} \hat{u}_c(\omega, t)$$

Ex Solve $u_t = 4u_{xx}$, $0 < x < \infty$, $t > 0$

لإيجاد u نستخدم F.S.T أو F.C.T

لأننا نعلم أن $0 < x < \infty$ فنأخذ الشروط الابتدائية بال steps التي قبل.

$$\text{B.C. } u(0, t) = u_0 \quad (\text{constant})$$

$$\text{I.C. } u(x, 0) = 0$$

* بالنظر إلى B.C. نلاحظ أن u وليست u_x ، إذن نستخدم F.S.T.
(الملاحظة الأخيرة الصفة السابقة)

So, take F.S.T for the two sides:

$$F_s \{u_t\} = 4 F_s \{u_{xx}\}$$

الآن نستخدم المتطابقات التي كتبناها في صفحة ١١

$$\frac{d}{dt} \hat{u}_s = 4(-\omega^2 \hat{u}_s + \sqrt{\frac{2}{\pi}} \omega \cdot \overbrace{u(0,t)})$$

→ from B.C. $u(0,t) = u_0$

$$\boxed{\frac{d}{dt} \hat{u}_s + 4\omega^2 \hat{u}_s = 4\sqrt{\frac{2}{\pi}} \omega u_0}$$

That is a linear O.D.E !

"It is our goal to convert a PDE in x domain, to an ODE in ω domain Using Fourier transform, because ODE's are easier to solve,

then convert the solution back to x domain, using Fourier Inverse Transform"

It's Very Important to understand this!

→ Now, Solve the O.D.E.

Recall: For the form $y' + p(x)y = g(x)$

Integration Factor (I.F.) is $(e^{\int p(x) dx})$

The solution is $y \cdot (\text{I.F.}) = \int g(x) \cdot (\text{I.F.}) dx$

→ for this question (the O.D.E. inside the box)

$$\text{I.F.} = e^{\int 4\omega^2 dt} = e^{4\omega^2 t}$$

($\hat{u}_s$ is the function, t is the variable)

$$\hat{u}_s e^{4\omega^2 t} = \int \overset{\text{constant}}{4\sqrt{\frac{2}{\pi}}} u_0 \omega \overset{\text{by parts}}{e^{4\omega^2 t}} dt$$

$$= 4\sqrt{\frac{2}{\pi}} u_0 \left(\frac{\omega e^{4\omega^2 t}}{4\omega^2} - \frac{e^{4\omega^2 t}}{4^2 \omega^4} + C \right)$$

$$\hat{u}_s e^{4\omega^2 t} = \sqrt{\frac{2}{\pi}} u_0 \left(\frac{1}{\omega} - \frac{1}{4\omega^4} \right) e^{4\omega^2 t} + A$$

$$\hat{u}_s = \sqrt{\frac{2}{\pi}} u_0 \left(\frac{4\omega^3 - 1}{4\omega^4} \right) + A e^{-4\omega^2 t}$$

I have to find A, will use $\hat{u}_s(\omega, 0)$ with the previous Equation and with the definition of $F_s\{u\}$.

$$\textcircled{1} \hat{u}_s(\omega, 0) = \sqrt{\frac{2}{\pi}} u_0 \left(\frac{4\omega^3 - 1}{4\omega^4} \right) + A$$

$$\textcircled{2} \hat{u}_s(\omega, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{u(x, 0)}_{\Rightarrow 0 \text{ (from I.C.)}} \sin \omega x dx = 0$$

by definition

$$\text{So: } \sqrt{\frac{2}{\pi}} u_0 \left(\frac{4\omega^3 - 1}{4\omega^4} \right) + A = 0$$

$$A = -\sqrt{\frac{2}{\pi}} u_0 \left(\frac{4\omega^3 - 1}{4\omega^4} \right)$$

$$\hat{u}_s = \sqrt{\frac{2}{\pi}} u_0 \left(\frac{4\omega^3 - 1}{4\omega^4} \right) (1 - e^{-4\omega^2 t})$$

- This is the Solution in ω domain,
The only thing left is F_s^{-1} it!

$$u(x,t) = F_s^{-1} \{ \hat{u}_s(\omega, t) \}$$

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} u_0 \left(\frac{4\omega^3 - 1}{4\omega^4} \right) \overset{(\sin \omega x)}{\downarrow} (1 - e^{-4\omega^2 t}) d\omega$$

$$u(x,t) = \frac{2}{\pi} u_0 \int_0^{\infty} \left(\frac{4\omega^3 - 1}{4\omega^4} \right) (\sin \omega x) \left(\frac{4\omega^3 - 1}{4\omega^4} \right) d\omega$$

This is the Solution
leave it as it is (Integral form)

WAVE EQUATION OVER SEMI-INFINITE DOMAINEx Solve $u_{tt} = a^2 u_{xx}$, $0 < x < \infty$, $t > 0$

F.C.T. و F.S.T. لأن

B.C. $u(0, t) = 0$ (F.S.T. لأن سقيم u)I.C. $u(x, 0) = f(x)$ $u_t(x, 0) = 0$ Solution taking F_s for both sides:

$$F_s \{ u_{tt} \} = a^2 F_s \{ u_{xx} \}$$

$$\frac{d^2}{dt^2} \hat{u}_s = a^2 \left(-\omega^2 \hat{u}_s + \sqrt{\frac{2}{\pi}} \omega u(0, t) \right)$$

$\hookrightarrow = 0$ (from B.C.)

$$\frac{d^2}{dt^2} \hat{u}_s + a^2 \omega^2 \hat{u}_s = 0 \quad (\text{like } y'' + (a\omega)^2 y = 0)$$

$\hookrightarrow y = C_1 \cos a\omega x + C_2 \sin a\omega x$

$$\hat{u}_s(\omega, t) = A(\omega) \cos a\omega t + B(\omega) \sin a\omega t \quad \text{--- (A)}$$

($\frac{d^2}{dt^2} \hat{u}_s - \omega^2$). t لأن طرقت الجواب إلى $t=0$ $A(\omega), B(\omega) \rightarrow *$

Find $A(\omega), B(\omega)$ from I.C.'s and (A)

$$\text{[1]} \quad u(x, 0) = f(x)$$

by def: $\hat{u}_s(\omega, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{u(x, 0)}{f(x)} \sin \omega x \, dx$

from (A): $u_s(\omega, 0) = A(\omega) =$

$$A(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx$$

$$\boxed{2} \quad u_t(x, 0) = 0$$

$$\left(\frac{d}{dt} \hat{u}_s = A(w)(-aw \sin awt) + B(w)(aw \cos awt) \right)$$

$$\downarrow \frac{d}{dt} \hat{u}_s(w, 0) = aw B(w) \quad \text{"from } \Delta \text{"}$$

$$\text{by def: } F_s \{u_t(x, 0)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \underbrace{u_t(x, 0)}_{=0} \sin wx dx = 0$$

$$aw B(w) = 0 \Rightarrow \underline{B(w) = 0}$$

General

$$u_s(w, t) = A(w) \cos(awt)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx dx (\cos awt)$$

$$u(x, t) = F_s^{-1} \{u_s\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx dx \cos awt \sin wx dw$$

صلى كل إنشائي

HOMEWORK

Solve $u_t = 2u_{xx}$, $0 < x < \infty$, $t > 0$.

B.C. $u_x(0, t) = -4$

I.C. $u(x, 0) = 0$

Solution

Use F.C.T. because the B.C. is u_t .

$$F_c \{ u_t \} = 2 F_c \{ u_{xx} \}$$

$$\frac{d}{dt} \hat{u}_c = 2(-\omega^2 \hat{u}_c - \sqrt{\frac{2}{\pi}} u_x(0, t))$$

$\hookrightarrow -4$ (from B.C.)

$$\frac{d}{dt} \hat{u}_c + 2\omega^2 \hat{u}_c = 8\sqrt{\frac{2}{\pi}}$$

"Separable."

$$(-2\omega^2) dt = \frac{(-2\omega^2) d\hat{u}_c}{-2\omega^2 \hat{u}_c + 8\sqrt{\frac{2}{\pi}}}$$

$-2\omega^2$ is a constant

$$-2\omega^2 t + C = \ln(-2\omega^2 \hat{u}_c + 8\sqrt{\frac{2}{\pi}})$$

$$-2\omega^2 \hat{u}_c + 8\sqrt{\frac{2}{\pi}} = e^{-2\omega^2 t} A \quad (A = e^C)$$

$$\hat{u}_c = \frac{8\sqrt{\frac{2}{\pi}} - e^{-2\omega^2 t} A}{2\omega^2}, \text{ to Find } A:$$

[1] from previous Equation

$$\hat{u}_c(\omega, 0) = \frac{8\sqrt{\frac{2}{\pi}} - A}{2\omega^2}$$

[2] by definition

$$\hat{u}_c(\omega, 0) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x, 0) \cos \omega x dx = 0$$

$\hookrightarrow 0$ (I.C.)

$$A = 8\sqrt{\frac{2}{\pi}}$$

So;

$$\hat{u}_c(\omega, t) = 8 \sqrt{\frac{2}{\pi}} \frac{(1 - e^{-2\omega^2})}{2\omega^2}$$

Finally:

$$u(x, t) = F_c^{-1} \{ \hat{u}_c(\omega, t) \}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} 8 \sqrt{\frac{2}{\pi}} \frac{(1 - e^{-2\omega^2})}{2\omega^2} \cos \omega x \, d\omega$$

$$\left(u(x, t) = \frac{16}{\pi} \int_0^{\infty} \frac{(1 - e^{-2\omega^2})}{2\omega^2} \cos \omega x \, d\omega \right)$$

LAPLACE EQUATION OVER SEMI-INFINITE DOMAIN

Because this Equation is in 2D (x, y),
we must be very careful which one
we will take the F.T. with respect to it.
(according to the question)

Ex $u_{xx} + u_{yy} = 0$, $0 < x < \pi$, $y > 0$
 No I.C.'s (No t) (So, w.r.t y. $(0 < y < \infty)$)

B.C. • $u(0, y) = 0$
 • $u(\pi, y) = e^{-y}$
 • $u_y(x, 0) = 0$ (dy w.r.t F.C.T.)

$$F_c \{ u_{yy} \} = -\omega^2 \hat{u}_c - \sqrt{\frac{2}{\pi}} \omega u_y(x, 0) \quad \omega=0 \text{ (B.C.)}$$

$$F_c \{ u_{xx} \} = \frac{d^2 \hat{u}_c}{dx^2}$$

ملاحظة: إذا أخذنا متطابقة أطوار (مجموعتين متطابقتين) ..

$$\Rightarrow F_c \{ u_{xx} \} + F_c \{ u_{yy} \} = 0$$

$$\frac{d^2 \hat{u}_c}{dx^2} - \omega^2 \hat{u}_c = 0 \quad (\text{like } y'' - \omega^2 y = 0)$$

$$\hookrightarrow \hat{u}_c = A(\omega) e^{\omega x} + B(\omega) e^{-\omega x}$$

we put A(ω) & B(ω) because we solved
the O.D.E w.r.t x only.

Finding $A(\omega)$ & $B(\omega)$:

for each B.C. : use [1] previous Equation. [2] definition
 $x = \pi$ و $x = 0$ يعني

when $x = 0$

$$[1] \hat{u}_c(0, \omega) = A(\omega) + B(\omega)$$

$$[2] \hat{u}_c(0, \omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \underbrace{u(0, y)}_{\hookrightarrow 0 \text{ (B.C.)}} \cos \omega y dy = 0$$

$$[1] = [2] \leadsto A(\omega) + B(\omega) = 0$$

$$| \quad B(\omega) = -A(\omega) \quad |$$

when $x = \pi$

$$[1] \hat{u}_c(\pi, \omega) = A(\omega) e^{\pi \omega} + B(\omega) e^{-\pi \omega}$$

$$[2] \hat{u}_c(\pi, \omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \underbrace{u(\pi, y)}_{\hookrightarrow e^{-y} \text{ (B.C.)}} \cos \omega y dy = \sqrt{\frac{2}{\pi}} \frac{1}{1 + \omega^2}$$

الكمية قالت لو ارجع سؤال حتى صلي بالاعتقاد بتكون معكنا ال formula لهاد سائل

$$[1] = [2]$$

$$A(\omega) e^{\pi \omega} + B(\omega) e^{-\pi \omega} = \sqrt{\frac{2}{\pi}} \frac{1}{1 + \omega^2}$$

$$A(\omega) (e^{\pi \omega} - e^{-\pi \omega}) = \sqrt{\frac{2}{\pi}} \frac{1}{1 + \omega^2}$$

$$A(\omega) = \sqrt{\frac{2}{\pi}} \frac{1}{1 + \omega^2} \cdot \frac{1}{2 \sinh \omega \pi}$$

$$\leadsto \hat{u}_c = A(\omega) (e^{\omega x} - e^{-\omega x}) \quad (\text{because } B = -A)$$

$$= A(\omega) (2 \sinh \omega x)$$

$$\hat{u}_c(x, w) = \sqrt{\frac{2}{\pi}} \frac{\sinh wx}{(1+w^2) \sinh w\pi}$$

* Finally (Don't forget this step)

$$u(x, y) = F_c^{-1} \{ \hat{u}_c(x, w) \}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sinh wx}{(1+w^2) \sinh w\pi} \cdot \cos wy \, dw$$

wave, heat, laplace Equations on semi-infinite domain can be solved by F.T.'s.

(one of them will be a question in final Exam)

USING Fourier Transform TO SOLVE P.D.E.↳ from $-\infty$ to ∞ Ex Solve.

$$u_t = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$

No B.C.'s F.T. w.r.t. x

$$\text{I.C. } u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Remember:

$$F\{u_{xx}\} = -\omega^2 \hat{u}(\omega, t)$$

$$F\{u_t\} = \frac{d\hat{u}}{dt}$$

$$F\{u_t\} = c^2 F\{u_{xx}\}$$

$$\frac{d\hat{u}}{dt} = -c^2 \omega^2 \hat{u} \quad \text{Separable.}$$

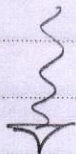
$$\frac{d\hat{u}}{dt} = -c^2 \omega^2 \hat{u} \Rightarrow \frac{d\hat{u}}{\hat{u}} = -c^2 \omega^2 dt$$

$$\hat{u}(\omega, t) = A(\omega) e^{-c^2 \omega^2 t}$$

Find $A(\omega)$ from I.C. (with the definition)
and previous Equation.

$$\boxed{1} \quad \hat{u}(\omega, 0) = A(\omega)$$

$$\boxed{2} \quad \text{by definition}$$



$$\hat{u}(\omega, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, 0) e^{-i\omega x} dx$$

from I.C.

$$= \frac{2}{\sqrt{2\pi}} \int_{-1}^1 e^{-i\omega x} dx = \frac{2}{\sqrt{2\pi} (-i\omega)} e^{-i\omega x} \Big|_{-1}^1$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{i\omega} (e^{i\omega} - e^{-i\omega})$$

$\sin \omega = \frac{e^{i\omega} - e^{-i\omega}}{2i}$

$$= 2 \sqrt{\frac{2}{\pi}} \frac{1}{\omega} \sin \omega$$

$$\boxed{1} = \boxed{2} \Rightarrow A(\omega) = 2 \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega}$$

$$\Rightarrow \hat{u}(\omega, t) = 2 \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega} e^{-c^2 \omega^2 t}$$

$$\Rightarrow u(x, t) = F^{-1} \{ \hat{u}(\omega, t) \}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot 2 \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin \omega}{\omega} e^{-c^2 \omega^2 t} e^{i\omega x} d\omega$$

(المسألة قاله خذ حد حده ما في داي للتبسيط)

THE END =)

Q1] Let $\vec{F} = [x + g(y), y]$. You are given that $\int_C \vec{F} \cdot d\vec{r}$ is independent of the path of integration and that $\int_{(0,0)}^{(1,1)} \vec{F} \cdot d\vec{r} = 10$. Find $g(y) :-$ (3 marks)

Q2] (4 points) Let $\vec{F}$ and $\vec{G}$ be any vector valued functions. Is it true that $\text{curl}(\vec{F} + \vec{G}) = \text{curl} \vec{F} + \text{curl} \vec{G}$? prove that:

Q3] Let $\vec{F} = [x^2 + y, y^3 + x] = (x^2 + y)\hat{i} + (y^3 + x)\hat{j}$.

a) (2 marks) Does there exist a scalar function f such that $\nabla f = \vec{F}$? Justify your answer:-

b) (3 marks) Evaluate $\int_{(0,0)}^{(1,1)} \vec{F} \cdot d\vec{r}$ along the line $y=x$:-

Q4] Let $f(x) = \begin{cases} e^x & ; 0 < x < 1 \\ 0 & ; \text{otherwise} \end{cases}$:-

a) (3 marks) Show that $\hat{f}(\omega) = \frac{e^{1-j\omega} - 1}{\sqrt{2\pi}(1-j\omega)} \therefore j = \sqrt{-1}$

b) (3 marks) Using the identity: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{j\omega x} d\omega = \frac{f(x^+) + f(x^-)}{2}$

Find the value of the integral:

$$\int_{-\infty}^{\infty} \frac{e^{1-j\omega} - 1}{(1-j\omega)} e^{j\omega} d\omega :-$$

Q5] (5 marks) Evaluate: $\int_{(0,2)}^{(1,2)} \nabla(x^2 y^3 + y) \cdot d\vec{r} :-$

Q 6 Consider the following Boundary value Problem (BVP):

$$u_{tt} - u_{xx} = 0$$

$$u_x(0, t) = u(L, t) = 0 \quad ; \quad t \geq 0$$

$$u(x, 0) = f(x) \quad ; \quad u_t(x, 0) = 0 \quad ; \quad 0 \leq x \leq L$$

- a) (4 marks) Using the method of separation of variables; show that this PDE reduces to the ODEs: $F'' - kF = 0$ & $G'' - kG = 0$:-
- b) (2 marks) Explain why we must have $F'(0) = 0$ & $F(L) = 0$, using the given boundary conditions:
- c) (4 marks) When solving the ODE: $F'' - kF = 0$; we have to take into account the all three cases of the k ; Explain why the cases $k = 0$ & $k > 0$ are rejected :-
- d) Obtain a ^{general} solution of the ODE: $F'' - kF = 0$; where $k < 0$; $k = -p^2$:- (2 marks)
- e) (3 marks) Use the given boundary conditions to find the particular solution of $F'' - kF = 0$:-
- f) (2 marks) Find the General solution of the 2nd ODE: $G'' - kG = 0$
- g) (1 mark) Having found both F & G , write the general form of the solution $u(x, t)$; this will take the form of an infinite sum:-
- h) (3 marks) Your answer of part (g) must take the form:

$$u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{(2n-1)\pi}{2L} t\right) + B_n \sin\left(\frac{(2n-1)\pi}{2L} t\right) \right] \cos\left(\frac{(2n-1)\pi}{2L} x\right)$$

Now use this information and the boundary condition $u(x, 0) = f(x)$, to find a formula for A_n :-

- i) (3 marks) Now use the condition $u_t(x, 0) = 0$ to find B_n :-

Q 7 (5 marks) Solve $u_{yx} = x^2 y$, given $u_y(0, y) = y^2$ & $u(x, 1) = \cos x$:-

Q8 (6 marks) For the vector field $\vec{F} = (2x^3y^2 + x)\hat{i} + (x^4y + y)\hat{j}$

a) Find the curl ($\vec{F}$) :-

b) Find a potential function $f(x, y)$ such that $\nabla f = \vec{F}[x, y]$:-

c) Evaluate $\int_C \vec{F} \cdot d\vec{r}$; where C is given by: $\vec{r}(t) = (t \cos(\pi t) - 1)\hat{i} + \hat{j} \sin(\frac{\pi t}{2})$
 $0 \leq t \leq 1$

Q9 (2 points) Verify that $u = x^4 + y^4$ satisfies: $u_{xx} + u_{yy} = 12(x^2 + y^2)$

Q10 (4 marks) Show that the given Integral represents the indicated function:

$$\int_0^\infty \frac{\sin w - w \cos w}{w^2} \sin x w dw = \begin{cases} \frac{\pi x}{2} & \text{if } 0 < x < 1 \\ \frac{\pi}{4} & \text{if } x = 0 \\ 0 & \text{if } x > 1 \end{cases}$$

Q11 (4 marks) Suppose that $f(x)$ is continuous on the x -axis & $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$; prove that $\mathcal{F}\{f'(x)\} = j\omega \mathcal{F}\{f(x)\}$

Q12 (3 points) Use separation of variables to transform:

$$U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta} = 0 \quad \text{into two ordinary DEs :-}$$

Q13 (5 marks) Given the following function $f(x) = e^{-x}$; which is continuous & absolutely integrable on the x -axis & $\lim_{x \rightarrow \infty} f(x) = 0$

Use the theorem $\mathcal{F}\{f'(x)\} = w \mathcal{F}\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$, to find $\mathcal{F}\{e^{-x}\}$:

Q14 Let $f_{xx} + f_{yy} + f_{zz} = 3$ & S be the surface $x^2 + y^2 + z^2 = 1$. Evaluate $\iint_S \nabla f \cdot \hat{n} \, dA$:
(5 points)

Q15 (4 marks) You are given that:

$$\mathcal{F}_c \{ e^{-ax^2} \} = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}}, \text{ for } a > 0$$

Use this together with the formula $\mathcal{F}_s \{ f'(x) \} = -w \mathcal{F}_c \{ f(x) \}$ to evaluate $\mathcal{F}_s \{ x e^{-\frac{x^2}{2}} \}$:

Q16 Find the Fourier transform of:

(4 marks)

$$f(x) = \begin{cases} k & \text{if } 0 < x < b \\ 0 & \text{otherwise} \end{cases}$$

Q17 Use Green's theorem to evaluate $\oint_C (xy \, dx + x^2 y^3 \, dy)$
(4 marks) where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation:-

Q18 (3 marks) let $f(x) \sim \frac{2}{\pi} \int_0^\infty \frac{\sin(2w) \cos(wx)}{w} \, dw$ be the Fourier integral representation for $f(x) = \begin{cases} 1 & ; |x| < 2 \\ 0 & ; |x| > 2 \end{cases}$

Evaluate $\int_0^\infty \frac{\sin(2w)}{w} \, dw$:-

Q 19 Consider the PDE : $u_t = -ux$; $u(x,0) = e^x$. We would like to solve this PDE using the method of separation of variables. so we let $u(x,t) = F(x) G(t)$:-

a) (3 marks) When the method of separation of variables is implemented, we get new Ordinary differential equations. Write down those Ordinary differential Equations:-

b) (2 marks) Suppose that your answer in (a) is : $\{F' + kF = 0$ & $G' - kG = 0\}$. Solve these differential equations:-

c) (2 marks) Suppose that your answer in (b) is $\{F = Ae^{kx}$ & $G = Be^{-kt}\}$. Hence $u = Ce^{kx}e^{-kt}$. Apply the given boundary condition to find k & C :-

Q 20 (3 marks) Let $f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)$ is the Fourier cosine series for $f(x) = x$, $x \in (0, \pi)$. Find $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$:-

Q 21 Represent $f(x)$ in the form of the Fourier Cosine Integral for $f(x) = \begin{cases} x^2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

Q 22 (5 marks) Solve $u_{yy} - 4u_y + 3u = 0$ to obtain a general solution $u(x,y)$:-