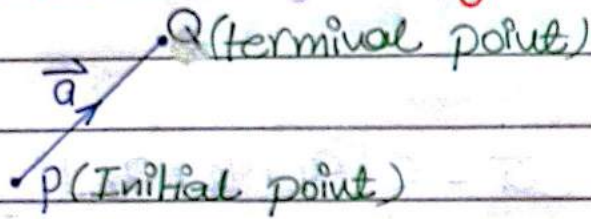


Scalars and Vectors:

(number)

→ magnitude + Direction (norm, Length).



** $\vec{a} = \overrightarrow{PQ}$, $|\vec{a}| \equiv$ is the magnitude.

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2) \Rightarrow$

** $\vec{a} = \overrightarrow{PQ} = [a_1, a_2, a_3] = [(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)]$

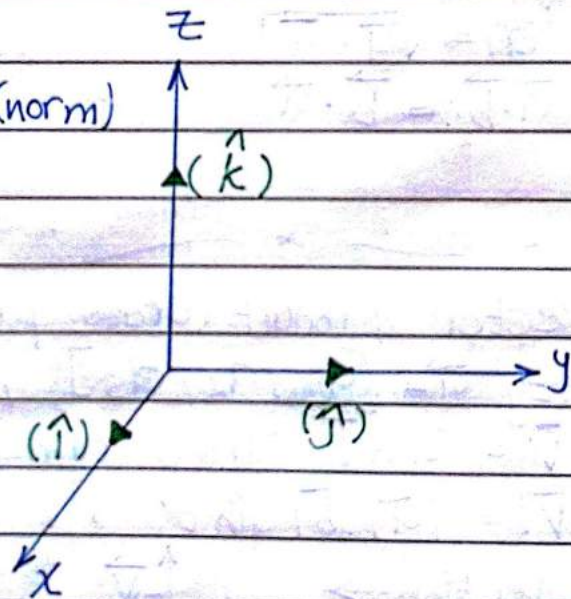
→ $\langle a_1, a_2, a_3 \rangle \rightarrow$ Component Form.

** $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

→ $(\hat{i}, \hat{j}, \hat{k})$ are Basis in $\mathbb{R}^3$

→ $\hat{i} = [1, 0, 0]$, $\hat{j} = [0, 1, 0]$, $\hat{k} = [0, 0, 1]$

$|\vec{a}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ (norm)



Let $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

** $\vec{a} \pm \vec{b} = [a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3]$

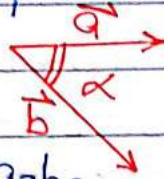
$\vec{0} = [0, 0, 0] \rightarrow$ Zero vector in any direction.

Scalar Multiplication:

If (C) is a scalar $\rightarrow C\vec{a} = C[a_1, a_2, a_3]$
 $= [Ca_1, Ca_2, Ca_3]$

Scalar Product "Dot product":

** $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \alpha$



** $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

If $\vec{a}$ & $\vec{b}$ are perpendicular ($\vec{a} \perp \vec{b}$):

$\therefore \vec{a} \cdot \vec{b} = 0$, since "Cos 90° = 0".

** $\vec{a} \cdot \vec{a} = \sqrt{a_1^2 + a_2^2 + a_3^2} = (a^2)$

** $|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$

** $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

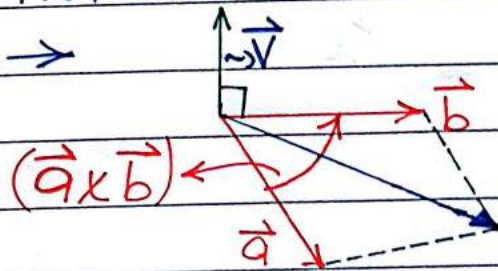
Vector Product (Cross product):

Let $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

** $\vec{V} = \vec{a} \times \vec{b}$ "cross product".

** $|\vec{V}| = |\vec{a}| |\vec{b}| \sin \alpha$ **

- Direction $\rightarrow$



* By the Right hand Rule *

** $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

** If the vectors ($\vec{a}$ & $\vec{b}$) are parallel $\rightarrow$
 $\vec{a} \times \vec{b} = 0$, Since $\sin 0 = 0$ //

$$\# \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$(-)$ $(+)$ $(-)$ $(+)$

* Ex: If $\vec{a} = [1, 0, -2]$ & $\vec{b} = [3, 1, 0]$, Find $\vec{a} \times \vec{b}$?

$$\# \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2 \\ 3 & 1 & 0 \end{vmatrix}$$

$$= \hat{i}(0+2) - \hat{j}(0+6) + \hat{k}(1-0)$$

$$= 2\hat{i} - 6\hat{j} + \hat{k}$$

9.4 # Vector and scalar functions:

** $y = f(x)$, $y = x^2$ (scalar function).

$$f: \mathbb{R}^1 \rightarrow \mathbb{R}^1$$

** Distance from $P(x, y, z)$ to $P_0(x_0, y_0, z_0)$

$$d = \sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}$$

Vector Function:

$$\vec{r}(t) = [x(t), y(t), z(t)]$$

$$\vec{r}(t) = [t, t^2, t^3]$$

$$\Rightarrow \vec{r}'(t) = [1, 2t, 3t^2]$$

$$** \vec{F}(x, y, z) = F_1(x, y, z)\hat{i} + F_2(x, y, z)\hat{j} + F_3(x, y, z)\hat{k}$$

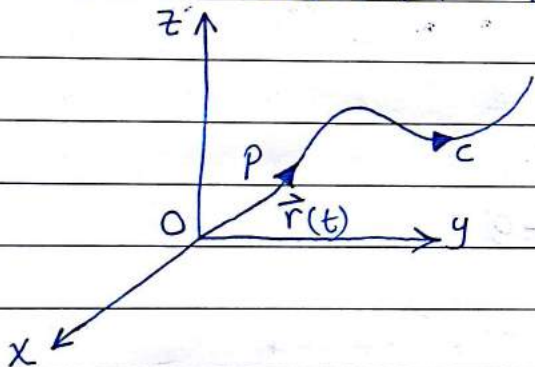
$$\# \text{ Ex: } F(x, y, z) = [x+y^2, \ln y, x^2yz]$$

$$** \frac{df}{dx} = [1, 0, 2xy]$$

$$** \frac{df}{dy} = [2y, \frac{1}{y}, x^2z]$$

$$** \frac{df}{dz} = [0, 0, x^2y]$$

* 9.5 : Curves (ex 1-7, 11-19, 24-28)

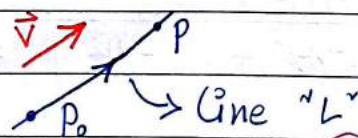


** Parametric representations of the Curve C is $\vec{r}(t)$,
 $\vec{r}(t) = [x(t), y(t), z(t)]$, to each point corresponds
 a position vector $\vec{r}(t_0)$; t is parameter.

1 * Lines!

$$P_0 = (x_0, y_0, z_0)$$

$$\vec{v} = [a, b, c]$$



** Find an equation of $P(x, y, z)$

$$\vec{P_0P} = t\vec{v}$$

$$[x-x_0, y-y_0, z-z_0] = [ta, tb, tc]$$

خطی (Line)
 و سائل Vector

$$\rightarrow x - x_0 = ta$$

$$x = x_0 + ta$$

$$\rightarrow y - y_0 = tb$$



$$y = y_0 + tb$$

$$\rightarrow z - z_0 = tc$$

$$z = z_0 + tc$$

* Parametric eqns of the line *

* Vector equation of the line:

$$\begin{aligned}\vec{r}(t) &= [x_0 + ta, y_0 + tb, z_0 + tc] \\ &= [x_0, y_0, z_0] + t[a, b, c]\end{aligned}$$

$$* x(t) = x_0 + ta$$

$$* y(t) = y_0 + tb$$

$$* z(t) = z_0 + tc$$

* Note: vector equation (parametric equations) is not unique:

- Initial point P .
- Vector $\vec{v} \parallel$ to the line.
- Parameter " t ".

* Ex 1: Find a parametric representation of the line through $P_0(3, 1, 2)$ in the direction of $(\hat{i} + 4\hat{k})$.

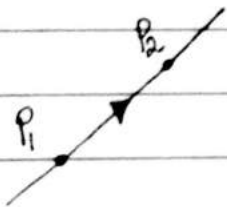
$$\star \vec{v} = \hat{i} + 4\hat{k} = [1, 0, 4]$$

$$\rightarrow x = 3 + t \quad y = 1 \quad z = 2 + 4t$$

$$\rightarrow \vec{r}(t) = [3, 1, 2] + t[1, 0, 4]; \text{ "vector eqn."}$$

* Ex 2: Find parametric equations of line (L) passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$

$\vec{r}_0$ is the position vector of P_1

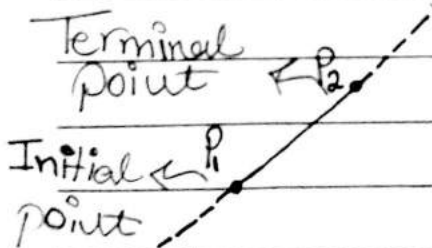


$$\star \vec{v} = \vec{P_1P_2} = [3, -4, 8]$$

$$\star \vec{r}(t) = [2, 4, -1] + t[3, -4, 8]$$

$$= [2 + 3t, 4 - 4t, -1 + 8t], \quad -\infty < t < \infty$$

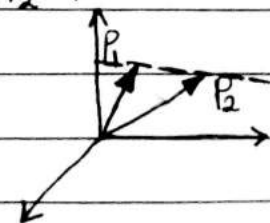
$\rightarrow$ Line Segment:



* * let P_1 be an initial point $\vec{v} = \vec{P_1P_2}$
choose $t, (0 < t < 1) \rightarrow$ Vector eqn.
of the line segment P_1P_2 .

$$\star \vec{r}(t) = \vec{r}_0 + (\vec{r}_1 - \vec{r}_0)t$$

$\rightarrow$ Vector representation.



* Ex 3: Find parametric equations of the line segment joining the points $P_1(2, 4, -1)$ & $P_2(1, 0, 5)$

$$* \vec{r}_0 = [2, 4, -1], \quad \vec{r}_1 = [1, 0, 5]$$

$$* \vec{r}(t) = [2, 4, -1] + ([1, 0, 5] - [2, 4, -1])t$$

$$= [2-t, 4-4t, -1+6t]$$

$$* x = 2-t$$

$$y = 4-4t$$

$$z = -1+6t$$

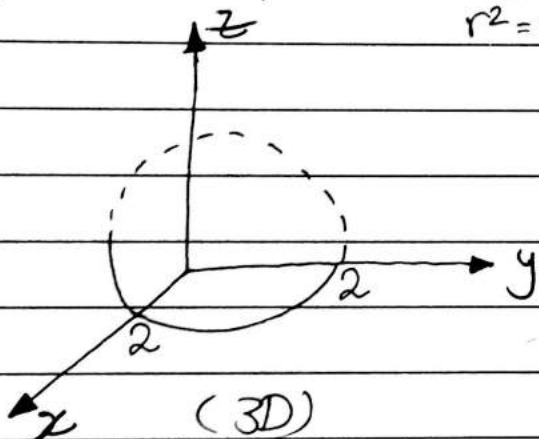
2 # Circles $x^2 + y^2 = 4, z=0$

$$r^2 = 4 \rightarrow r = 2.$$

$$\rightarrow x = r \cos t$$

$$\rightarrow y = r \sin t$$

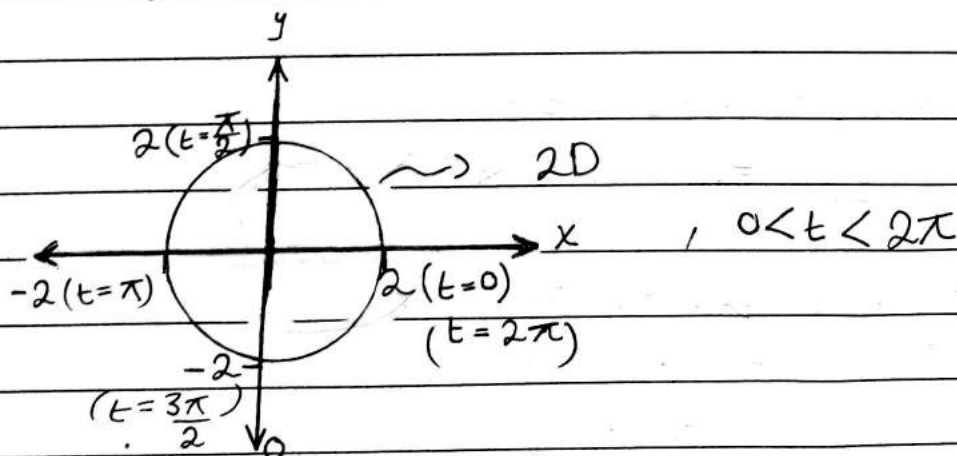
$$\rightarrow z = 0, \text{ where } r \text{ is the radius.}$$



** Parametric equations of the Circle:

$$x = 2 \cos t, \quad y = 2 \sin t, \quad z = 0$$

$$\rightarrow \vec{r}(t) = [2 \cos t, 2 \sin t, 0]$$



$$** x^2 + z^2 = 9, y=4$$

$$\rightarrow \vec{r}(t) = [3 \cos t, 4, 3 \sin t]$$

$$x^2 + z^2 = 9, y = -1$$

$$\rightarrow x = 3 \cos t, y = -1, z = 3 \sin t, 0 \leq t \leq 2\pi$$

$$y^2 + 4y + z^2 = 5, x = 3$$

$$\rightarrow (y^2 + 4y + 4) - 4 + z^2 = 5$$

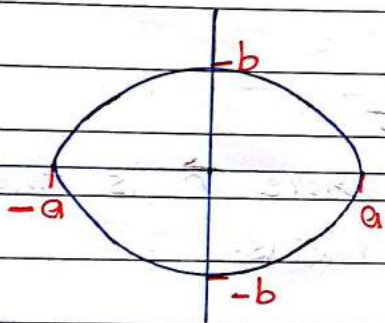
$$\text{إكمال مربع} \quad (y+2)^2 + z^2 = 9, x = 3$$

$$y = -2 + 3 \cos t$$

$$z = 3 \sin t$$

3 * Ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$$



$$\rightarrow x = a \cos t$$

$$\rightarrow y = b \sin t$$

$$\rightarrow z = 0$$

$$0 \leq t \leq 2\pi$$

$$* \frac{y^2}{4} + \frac{z^2}{5} = 1, x = 3$$

$$\rightarrow \vec{r}(t) = [3, 2 \cos t, \sqrt{5} \sin t], 0 \leq t \leq 2\pi$$

$$* (x-2)^2 + 9(y+1)^2 = 36, z = 4$$

$$\rightarrow \frac{(x-2)^2}{36} + \frac{(y+1)^2}{4} = 1, 0 \leq t \leq 2\pi$$

$$x - 2 = 6 \cos t \leadsto x = 2 + 6 \cos t$$

$$y + 1 = 2 \sin t \leadsto y = 2 \sin t - 1$$

$$z = 4 \leadsto z = 4$$

$$* * \vec{r}(t) = [2 + 6 \cos t, -1 + 2 \sin t, 4]$$

4 # Helix:

* Is the twisted Curve represented by a vector function:-

$$\vec{r}(t) = [a \cos t, a \sin t, ct]$$

→ If $c=0 \rightarrow \vec{r}(t)$ is a circle.

→ If $\vec{r}(t) = [a \cos t, b \sin t, ct]$ elliptical Helix.

Ex: Find parametric equations of the curve results from the intersection of $x^2 + y^2 = 2$, $x+z=3$.

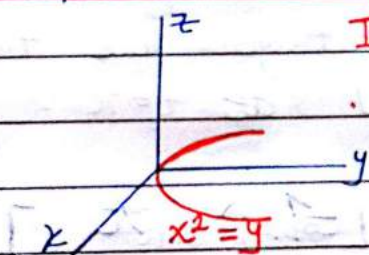
* $x = \sqrt{2} \cos t$

* $y = \sqrt{2} \sin t$

* $z = 3 - \sqrt{2} \cos t$

→ Circular Helix.

5 # Plane Curves:



Independent var. y is the independent variable. x and z are dependent variables. $\vec{r}(t) = [x(t), y(t), z(t)]$ is the vector function.

* $y = x^2$, $z = 0$

→ $x = t$, $y = t^2$, $z = 0$

$$\vec{r}(t) = [t, t^2, 0]$$

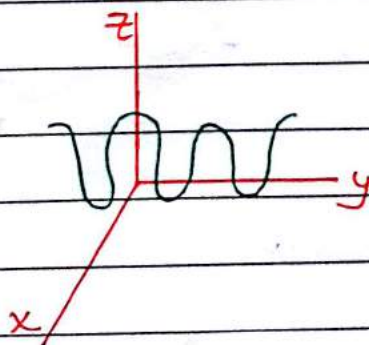
* Parametric:

$$x = 0, z = \cos y$$

→ $x = 0$

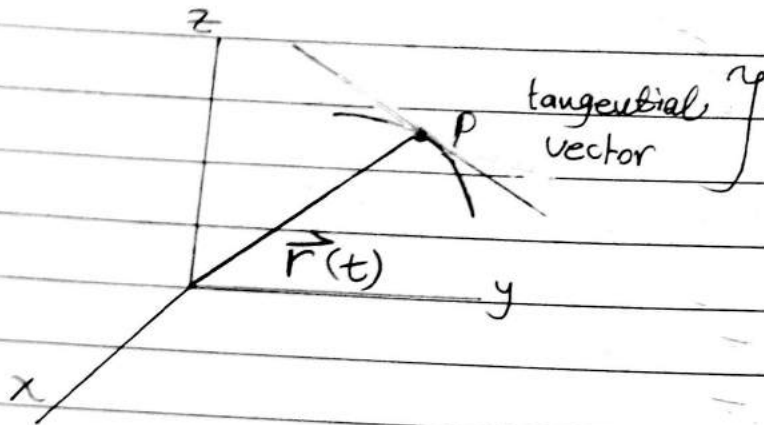
→ $y = t$

→ $z = \cos t$



Equation of the tangent line to a curve $\underline{C}$ at a point $C: \vec{r}(t) = [x(t), y(t), z(t)]$

** $\vec{r}'(t)$ is called a tangent vector to the graph of $\underline{C}$.



** $P(x_0, y_0, z_0)$

$\vec{q}(t) = \vec{r}_0 + \vec{r}'(t_0)t \leadsto$ The equation of the tangent line.

* Ex: Find parametric equations of tangent line to the Helix $\vec{r}(t) = [\cos t, \sin t, t]$, at $t = \pi$.

** $t = \pi \rightarrow P(-1, 0, \pi) \rightarrow \vec{r}_0 = [-1, 0, \pi]$

$\vec{r}'(t) = [-\sin t, \cos t, 1]$

$\vec{r}'(\pi) = [0, -1, 1]$

$\vec{q}(t) = [-1, 0, \pi] + t[0, -1, 1] = \begin{bmatrix} -1 \\ -t \\ \pi+t \end{bmatrix}$

$\underline{\bar{x}} \quad \underline{\bar{y}} \quad \underline{\bar{z}}$

Param. equations : $x = -1$

$y = -t$

$z = \pi + t$

(1)

9.7 : Gradient of a scalar Field (ex 1-6, 11-15, 30-35)

* Scalar Function = Scalar Field

* Vector Function = Vector Field

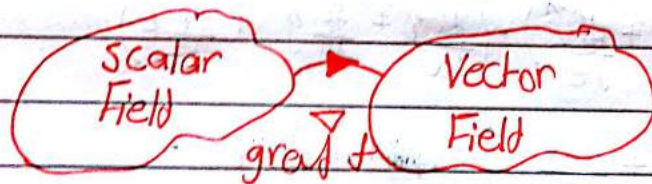
Def: Given a "scalar Function" $f(x, y, z)$ that is defined and differentiable in a domain in $\mathbb{R}^3$.

The gradient of $f(x, y, z)$ is defined as the vector functions,

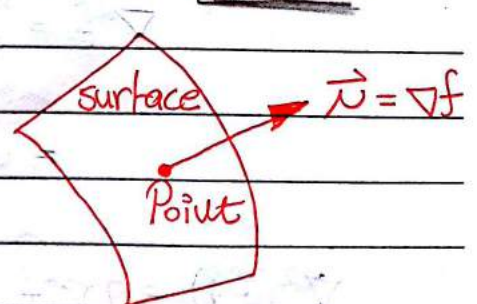
$$\text{grad } f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

xx $\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$, del operator. xx

*** $f(x, y, z) = 3y^2 + 5xz - 3x$
→ $\nabla f = (5z - 3) \hat{i} + 6y \hat{j} + 5x \hat{k}$



note: → The Gradient " ∇f " at any point P is the normal vector to the surface at the given point.



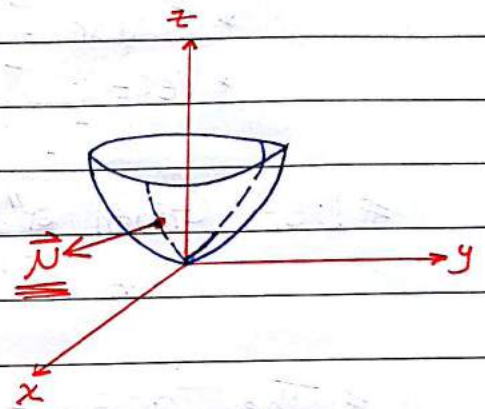
$f(x, y, z) = xy^2 + \sin xz$
→ $\nabla f = (y^2 + z \cos xz) \hat{i} + (2xy) \hat{j} + (x \cos xz) \hat{k}$
 $\nabla f(1, 2, 0) = 4 \hat{i} + 4 \hat{j} + \hat{k}$
x y z

- Given a paraboloid $z = 4x^2 + y^2$, Find a normal vector at the point $P(1, 0, 4)$.

$$z - 4x^2 - y^2 = 0$$

$$\vec{N} = \nabla f = [-8x, -2y, +1]$$

$$\nabla f(1, 0, 4) = [-8, 0, +1]$$



*** OR !

$$4x^2 + y^2 - z = 0$$

$$\nabla f = [8x, 2y, -1]$$

$$\nabla f(1, 0, 4) = [8, 0, -1]$$

(2) * Laplacian:

$$\nabla^2 = \nabla \cdot \nabla$$

Scalar Field

(∇^2)

Scalar Field

$$\nabla^2 f = \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right) \cdot \left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \left\{ \text{Laplacian} \right.$$

Ex: $f(x, y, z) = xy^3 + z^4$

$$\nabla^2 f = \frac{\partial^2}{\partial x^2} (xy^3 + z^4) + \frac{\partial^2}{\partial y^2} (xy^3 + z^4) + \frac{\partial^2}{\partial z^2} (xy^3 + z^4)$$

$$= 6xy + 12z^2$$

$$\nabla^2(fg) = g \nabla^2 f + 2 \nabla f \cdot \nabla g + f \nabla^2 g$$

** $\nabla^2 u = 0$ is Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

properties: $(f, g) \rightarrow$ are scalar functions.

1. $\nabla(f^n) = n f^{(n-1)} \cdot \nabla f$

2. $\nabla(fg) = f \nabla g + g \nabla f$

3. $\nabla\left(\frac{f}{g}\right) = \frac{1}{g^2} \cdot (g \nabla f - f \nabla g)$

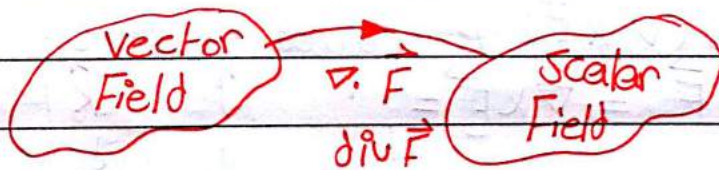
* (3) *
9.8 : Divergence of a vector field (ex 1-6, 9, 15-20)

** Let $\vec{F}(x, y, z) = [F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)]$;
Be a vector functions.

$\therefore$ then $\nabla \cdot \vec{F}$ is called a divergence.

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot [F_1, F_2, F_3]$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$



Ex: $\vec{F} = x\hat{i} - xy\hat{j} + z^2\hat{k}$, Find $\text{div } \vec{F}$?

$$\text{div } \vec{F} = 1 - x + 2z$$

$$\text{div } \vec{F}(1, 1, 1) = 1 - 1 + 2 = 2$$

Properties:

1. $\text{div}(\text{grad } f) = \nabla^2 f$

$$\text{div}(\text{grad } f) = \text{div}\left(\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

ex: $f = xy^3 + z^4$

$$\nabla f = [y^3, 3xy^2, 4z^3]$$

$$\text{div}(\nabla f) = 0 + 6xy + 12z^2 = (\nabla^2 f)$$

2. $\text{div}(k\vec{u}) = k \cdot \text{div}(\vec{u})$ / k is constant.

9.9: Curl of a vector field (ex 4-8, 15-20)

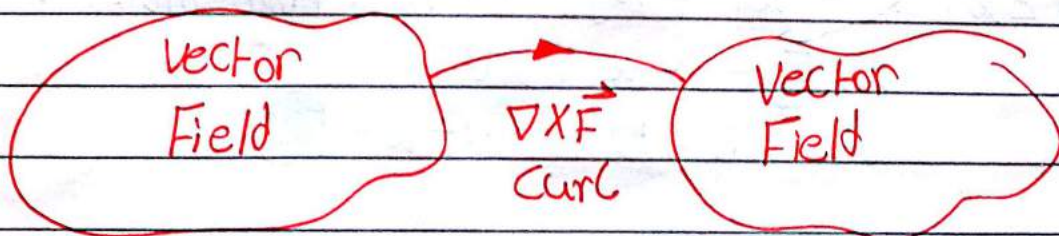
(Review P. 410: ex 32-37, 40)

Let $\vec{F}(x, y, z) = [F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)]$,

Be a vector function.

$\therefore$ then $(\nabla \times \vec{F})$ is called a curl.

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$



* Ex: $\vec{F} = [x, -xy, z^2]$, Find curl?

$$* \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -xy & z^2 \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial z^2}{\partial y} + \frac{\partial xy}{\partial z} \right)$$

$$- \hat{j} \left(\frac{\partial z^2}{\partial x} - \frac{\partial x}{\partial z} \right)$$

$$+ \hat{k} \left(\frac{\partial (-xy)}{\partial x} - \frac{\partial y}{\partial z} \right)$$

$$= -y\hat{k} = [0, 0, -y] \checkmark$$

* ~~~~~ *

$$\# \vec{F} = [F_1, F_2, F_3]$$

$$\nabla \times \vec{F} = \text{curl } \vec{F}$$

$$\nabla \cdot \vec{F} = \text{div } \vec{F}$$

Properties:

$$1. \text{div}(\text{curl } \vec{F}) = 0$$

$$2. \text{curl}(\text{grad } f) = \vec{0}$$

→ Proof of 2.

$$** \text{grad } f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$$

$$** \text{curl}(\text{grad } f) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) - \hat{j} \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \right) + \hat{k} \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

$$= \vec{0} \text{ (all terms are zeros)}$$

* Ex 1: let f be a scalar function, $\vec{F}$ is a vector function; state whether each expression is meaningful.

1. $\text{Curl } f$ $\times$
2. $\text{grad } f$ $\checkmark$
3. $\text{div } \vec{F}$ $\checkmark$
4. $\text{grad}(\text{div } \vec{F})$ $\checkmark \rightarrow \nabla(\nabla \cdot \vec{F})$
5. $\text{div}(\text{grad } f)$ $\checkmark \rightarrow \nabla \cdot (\nabla f)$
6. $\text{Curl}(\text{Curl } \vec{F})$ $\checkmark$
7. $\text{div}(\text{div } \vec{F})$ $\times$
8. $\nabla \cdot (\nabla \times (\nabla f))$ $\checkmark$

* Ex 2: Given $f = x^2y - z$ & $\vec{F} = [x, -xy, z^2]$, Find:

a) $\nabla f - \nabla \times \vec{F} = [2xy, x^2, -1] - [0, 0, -y]$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -xy & z^2 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(-y)$$

$$\nabla f = [2xy, x^2, -1]$$

$$\rightarrow \nabla f - \nabla \times \vec{F} = [2xy, x^2, -1] - [0, 0, -y]$$

* Ex 3: let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, verify:

a) $\nabla \cdot \vec{r} = 3$

b) $\nabla \cdot (|\vec{r}| \vec{r}) = 4|\vec{r}|$

c) $\nabla^2 |\vec{r}|^3 = 12\vec{r}$

→ a) $\nabla \cdot \vec{r} = \text{div } \vec{r} = 1+1+1 = 3$

** let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a}$ is a constant vector
 $\vec{a} = [a_1, a_2, a_3]$; show that $\nabla \cdot (\vec{a} \times \vec{r}) = 0$

$$\nabla \times (\vec{a} \times \vec{r}) = 2\vec{a}$$

$$\nabla (\vec{a} \cdot \vec{r}) = \vec{a}$$

→ $\nabla (\vec{a} \cdot \vec{r})$

$$= \nabla (a_1 x + a_2 y + a_3 z)$$

$$= [a_1, a_2, a_3] = \vec{a}$$

Ch. 9:

9.5 → 1-7, 11-19, 24-28

9.7 → 1-6, 11-15, 30-35

9.8 → 1-6, 9, 15-20

9.9 → 4-8, 15-20

Review → 32-37, 40
P. (4/10)

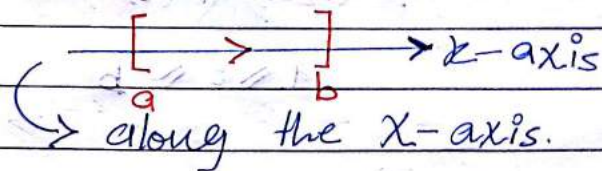
1. $\text{div}(\nabla f) = \nabla^2 f$

2. $\text{curl}(\nabla f) = \vec{0}$

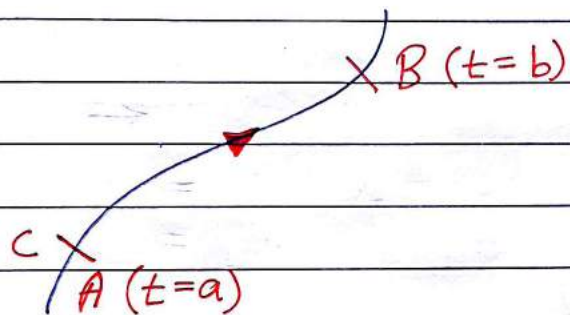
3. $\text{div}(\text{curl } \vec{v}) = 0$

CH 10: Vector Integral Calculus.

10.1: Line integrals (ex 2-11)
definite integral $\int_a^b f(x) \cdot dx$

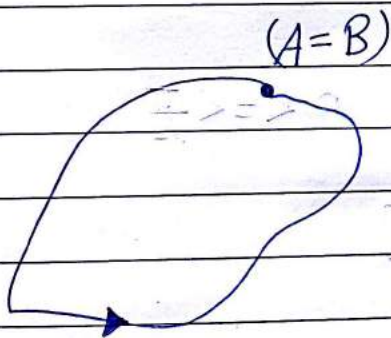


In a line integral we integrate along a curve "C" in space.



* $\vec{r}(t) = [x(t), y(t), z(t)]$
t is a parameter.

* The direction from A to B in which t increases is called positive direction.



→ * C is a closed Curve.

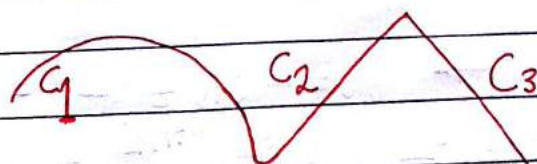
→ Assumption:

$C | \vec{r}(t)$ is a piecewise

Smooth Curve.

→ $\vec{r}'(t)$ is continuous.

It consists of finite number of curves.



Def: A line integral of a vector function $\vec{F}$ over a curve $C: \vec{r}(t)$ is defined by

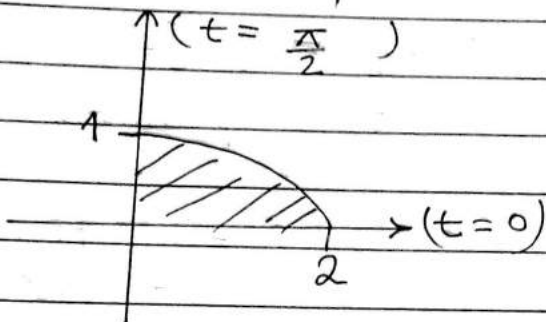
$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

$$a \leq t \leq b$$

Ex: Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where

$$\vec{F} = [x^2, -xy], \quad C \text{ is the quarter Ellipse}$$

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$



1. Parameterize:

$$C: \vec{r}(t) = [\underbrace{2\cos t}_x, \underbrace{\sin t}_y], \quad 0 \leq t \leq \frac{\pi}{2}$$

$$2. \vec{F} = [4\cos^2 t, -2\cos t \sin t]$$

$$\vec{r}'(t) = [-2\sin t, \cos t]$$

$$\rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} [4\cos^2 t, -2\cos t \sin t] \cdot [-2\sin t, \cos t] dt$$

$$= \int_0^{\pi/2} (-8\cos^2 t \sin t - 2\cos^2 t \sin t) dt$$

$$= -10 \int_0^{\pi/2} \cos^2 t \sin t dt \quad \text{"Substitution"}$$

$$V = \cos t \quad \rightarrow \quad dV = -\sin t dt = -\int V^2 dV$$

$$= -\frac{V^3}{3} = -10 \left(-\frac{1}{3}\right) \cos^3 t \Big|_0^{\pi/2} = -\frac{10}{3} \quad \checkmark$$

Ex 2: Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = [xy, yz, zx]$

$C: \vec{r}(t) = \left[\underset{x}{t}, \underset{y}{t^2}, \underset{z}{t^3} \right]; 0 \leq t \leq 1$!

* $\vec{F}(\vec{r}(t)) = [t^3, t^5, t^4]$

* $\vec{r}'(t) = [1, 2t, 3t^2]$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 [t^3, t^5, t^4] \cdot [1, 2t, 3t^2] dt$$

$$= \int_0^1 (t^3 + 2t^6 + 3t^6) dt$$

$$= \left. \frac{t^4}{4} + \frac{5t^7}{7} \right|_0^1 = \frac{27}{28}$$

Connection between line integral of a vector function and line integral of a scalar function.

$\rightarrow \vec{F} = [F_1, F_2, F_3]; \vec{r}(t) = [x(t), y(t), z(t)]$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

$$= \int_a^b [F_1, F_2, F_3] \cdot [x'(t), y'(t), z'(t)] dt$$

$$= \int_a^b \underbrace{F_1 x'(t)}_{dx} + \underbrace{F_2 y'(t)}_{dy} + \underbrace{F_3 z'(t)}_{dz} dt$$

$$= \int_C F_1 dx + F_2 dy + F_3 dz$$

$x'(t) dt = dx$

* Evaluate $\int_C \underbrace{xy}_{F_1} dx + \underbrace{yz}_{F_2} dy + \underbrace{zx}_{F_3} dz$, $\rightarrow$ Scalar Form

$$C: \vec{r}(t) = [t, t^2, t^3] ; 0 \leq t \leq 1$$

$$\rightarrow \int_0^1 (t)(t^2)dt + (t^2)(t^3)(2t)dt + t^4(3t^2)dt$$

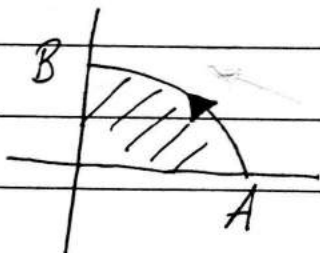
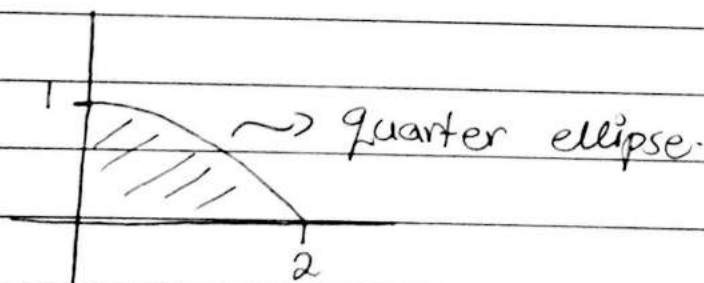
$$= \int_0^1 (t^3 + 2t^6 + 3t^6)dt$$

$$\left. \frac{t^4}{4} + \frac{5t^7}{7} \right|_0^1 = \frac{27}{28}$$

* Ex 1: $\vec{F} = [x^2, -xy]$, $\vec{r}(t) = [2\cos t, \sin t]$; $\int_C \vec{F} \cdot d\vec{r}$

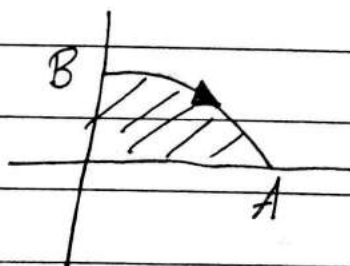
* Evaluate $\int x^2 dx - xy dy$, $\vec{r}(t)$
 Scalar Form

$$\rightarrow \int_0^{\pi/2} 4\cos^2 t (-2\sin t)dt - 2\cos t \sin t (\cos t dt)$$



(A $\rightarrow$ B)

$$-\int_0^{\pi/2} = \int_{\pi/2}^0$$



(B $\rightarrow$ A)

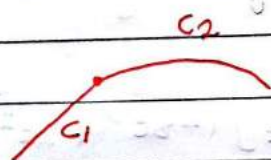
properties:

$$1. \int_C k (\vec{F} \cdot d\vec{r}) = k \int_C \vec{F} \cdot d\vec{r}, \text{ where } k \text{ is const.}$$

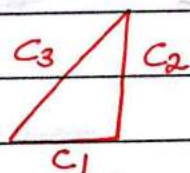
$$2. \int_C (\vec{F} + \vec{G}) \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_C \vec{G} \cdot d\vec{r}$$

$$3. \int_{-C} \vec{F} \cdot d\vec{r} = - \int_C \vec{F} \cdot d\vec{r}$$

$$4. \text{ If } C = C_1 + C_2$$



$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$



Ex 3: Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = [x^2, -xy]$

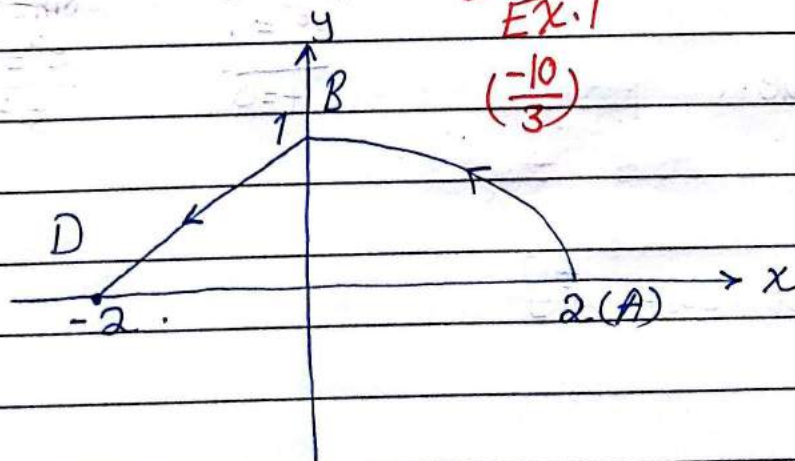
$C = C_1 + C_2$; C_1 is the quarter ellipse from (Ex.1)

$$(\vec{r}(t)) = [2\cos t, \sin t], \quad 0 \leq t \leq \frac{\pi}{2}$$

C_2 is a line segment (BD)

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

EX.1
 $(-\frac{10}{3})$



$$\rightarrow c_2: B(0,1) \rightarrow D(-2,0)$$

$$** \vec{r}(t) = \vec{r}_0 + (\vec{r}_1 - \vec{r}_0)t, \quad 0 \leq t \leq 1$$

$$** \vec{r}(t) = [0,1] + ([-2,0] - [0,1])t$$

$$= [-2t, 1-t]$$

$x, \quad y=1-t$

$$\rightarrow \int_{c_2} \vec{F} \cdot d\vec{r} = \int_0^1 [4t^2, 2t(1-t)] \cdot [-2, -1] dt$$

$$= \int_0^1 (-8t^2 + 2t^2 - 2t) dt = -\frac{6t^3}{3} - t^2$$

$$= -2 - 1 = -3$$

* line segment (1 ← 0) $\rightarrow$ $\int_0^1$ (CH. 9)

The Work W done by a variable force $\vec{F}$ in the displacement along curve C is given by $\left(\int_C \vec{F} \cdot d\vec{r} \right)$

Ex. 4: let $\vec{F} = [x^2, y^2, z^2]$ be a force.

Find the work done in the disp. along a curve $C: \vec{r}(t) = [\cos t, \sin t, e^t]$ from $A(1,0,1)$ to $B(1,0,e^{2\pi})$.

exponent. Helix.

$$\begin{array}{l} \cos t = 1 \\ \sin t = 0 \\ e^t = 1 \\ \hline t = 0 \end{array}$$

$$\begin{array}{l} \cos t = 1 \\ \sin t = 0 \\ e^t = e^{2\pi} \\ \hline t = 2\pi \end{array}$$

$\rightarrow$ Find t value.
(0 $\rightarrow$ 2π)

$$\rightarrow \vec{r}'(t) = [-\sin t, \cos t, e^t]$$

$$** W = \int_C \vec{F} \cdot d\vec{r}$$

$$W = \int_0^{2\pi} [\cos^2 t, \sin^2 t, e^{2t}] \cdot [-\sin t, \cos t, e^t] dt$$

$$= \int_0^{2\pi} (-\cos^2 t \sin t + \sin^2 t \cos t + e^{3t}) dt \rightarrow \text{Subst.}$$

$$= \left. \frac{\cos^3 t}{3} \right|_0^{2\pi} + \left. \frac{\sin^3 t}{3} \right|_0^{2\pi} + \left. \frac{e^{3t}}{3} \right|_0^{2\pi} = \frac{1}{3} (e^{6\pi} - 1).$$

* Ex: Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = [0, x, y]$ and

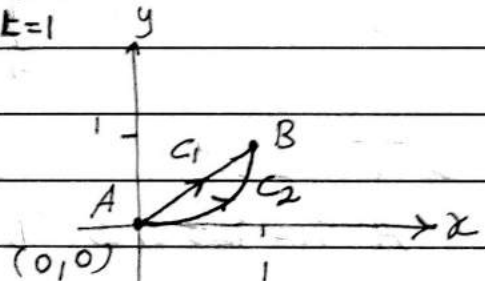
C: a) along the line segment $C_1: \vec{r}(t) = [t, t]$

b) along the parabola $C_2: \vec{r}(t) = [t, t^2]$

from $A(0,0)$ to $B(1,1)$.
 $t=0 = \quad = t=1$

$\rightarrow$ a) $0 \leq t \leq 1$

$$\begin{aligned} \int_{C_1} \vec{F} \cdot d\vec{r} &= \int_0^1 [0, t^2] \cdot [1, 1] dt \\ &= \int_0^1 t^2 dt = \boxed{\frac{1}{3}} \end{aligned}$$



$$\begin{aligned} \rightarrow \text{b) } \int_{C_2} \vec{F} \cdot d\vec{r} &= \int_0^1 [0, t^3] \cdot [1, 2t] dt \\ &= \int_0^1 2t^4 dt = \boxed{\frac{2}{5}} \end{aligned}$$

** The line integral depends on:

1. $\vec{F}$

2. endpoints

3. path of integral (curve C).

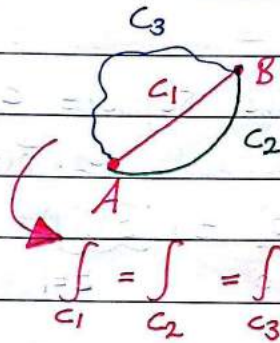
#10.2: Path independence of line integral (ex 3-9, 13-19)

* * $\vec{F}(x,y,z) = [F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)]$ is a vector function.

* * A line integral $\int_C \vec{F} \cdot d\vec{r}$

$$= \int_C F_1 dx + F_2 dy + F_3 dz \dots\dots\dots (1)$$

$\rightarrow$ is called path independent if it has the same value for all paths (from A to B)



Theorem 1:

A line integral (1) with continuous F_1, F_2, F_3 in some open domain (D) in space is path independent (only if) $\vec{F}$ is the gradient of some function f in (D).

$$\rightarrow \vec{F} = \text{grad } f \Rightarrow F_1 = f_x, F_2 = f_y, F_3 = f_z$$

* * f is called a "potential" (function).
 $\vec{F}$ is called "conservative".

* * The value of integral is:

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A) \dots\dots\dots (2)$$

Ex 1: Evaluate $\int_{(0,0,0)}^{(2,2,2)} 2x dx + 2y dy + 4z dz = \int_C \vec{F} \cdot d\vec{r}$

$$\vec{F} = [2x, 2y, 4z], \quad \vec{F} = \text{grad } f$$

$$\begin{aligned} \Rightarrow F_1 &= f_x & f_x &= 2x \\ \Rightarrow F_2 &= f_y & f_y &= 2y \\ \Rightarrow F_3 &= f_z & f_z &= 4z \end{aligned}$$

$$\rightarrow f_x = 2x \Rightarrow f(x, y, z) = \int 2x dx = x^2 + h(y, z)$$

$$\rightarrow f_y = 2y \Rightarrow f_y = h_y(y, z) \rightarrow h_y = 2y$$

$$h(y, z) = \int 2y dy = y^2 + g(z)$$

$$f(x, y, z) = x^2 + y^2 + g(z)$$

$$\rightarrow f_z = F_3 = 4z \Rightarrow f(z) = g'(z) \rightarrow g(z) = \int 4z dz = 2z^2 + k$$

$\therefore f(x, y, z) = x^2 + y^2 + 2z^2 + k$ Potential, $\vec{F}$ is conservative field.

* Verify!

$$f_x = 2x$$

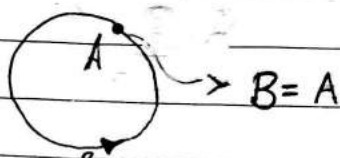
$$f_y = 2y$$

$$f_z = 4z$$

(✓).

$$f(2, 2, 2) - f(0, 0, 0) = 4 + 4 + 8 + k - k = 16$$

** If C is a closed curve:



ex.1: $\int_A^B 2x dx + 2y dy + 4z dz = 0$ / since " $A=B$ "

Initial point =
terminal point

Theorem.2:

The line integral (1) is path independent in (D) (if & only if) its value around every closed curve in (D) is (zero).

$$\int_C \vec{F} \cdot d\vec{r} \text{ is path independent} \Leftrightarrow \vec{F} = \text{grad } f \Leftrightarrow \oint \vec{F} \cdot d\vec{r} = 0$$

(if and only if)

$$\vec{F} = [F_1, F_2, F_3]$$

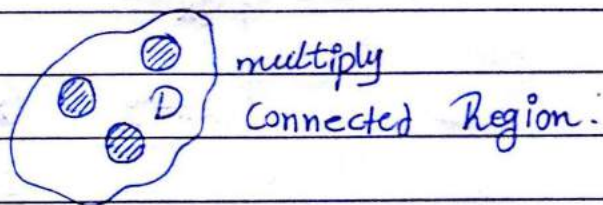
*** A line integral $\int_C \vec{F} \cdot d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz$ ---- (1)
is path independent $\Leftrightarrow$

1. $\vec{F} = \nabla f$ ($\vec{F}$ is a conservative field)

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

2. $\oint_C \vec{F} \cdot d\vec{r} = 0$

*** Test for conservative vector field:



*** Theorem (3):

$\rightarrow$ If $\vec{F}$ is conservative, then $\text{Curl } \vec{F} = 0$, provided D is a simply connected region.

3. $\text{Curl } \vec{F} = \vec{0}$ (D is simply connected region).

*** $\vec{F} = \nabla f$, $F_1 = f_x$, $F_2 = f_y$, $F_3 = f_z$

$\rightarrow \vec{F} \cdot d\vec{r} = F_1 dx + F_2 dy + F_3 dz$
 $= f_x dx + f_y dy + f_z dz = df$
exact differential.

4. $\vec{F} \cdot d\vec{r}$ is exact differential.

* Ex:

- a - Show that the differential form under the integral sign is exact.

$$\int (2xz^2y)dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz$$

$\xrightarrow{\quad} \begin{matrix} f_x & f_y & f_z \end{matrix}$

[Show that $\vec{F} = [2xz^2y, \dots]$ is the conser. Field].

- b - Evaluate the integral from $A(0,0,1)$ to $B(1, \pi/4, 2)$

* Solution:

(a) Test $\text{Curl } \vec{F} =$

+	-	+
$\hat{i}$	$\hat{j}$	$\hat{k}$
$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial y}$	$\frac{\partial}{\partial z}$
$(2xz^2y)$	$(x^2z^2 + z\cos yz)$	$(2x^2yz + y\cos yz)$

$$= \hat{i}(2xz^2 + \cos yz - yz \sin yz - 2x^2z - \cos yz + zy \sin yz) - \hat{j}(4xyz - 4xyz) + \hat{k}(2x^2z^2 - 2x^2z^2) = \vec{0}$$

$\therefore \vec{F}$ is conservative field

$\vec{F} \cdot d\vec{r}$ is exact diff.

(b) $f(x,y,z) = \int 2xz^2y dx = x^2z^2y + h(y,z)$

$f_y = x^2z^2 + h_y(y,z) = x^2z^2 + z\cos yz$

$h(y,z) = \int z\cos yz dy = \frac{z}{z} \sin yz + g(z)$

$f(x,y,z) = x^2z^2y + \sin yz + g(z)$

$f_z = 2x^2zy + y\cos yz + g'(z) = 2x^2yz + y\cos yz$

$g'(z) = 0 \rightarrow g(z) = \text{const} = k$

$$\rightarrow f(x,y,z) = x^2 y z^2 + \sin y z + k \rightarrow \text{"potential"} \vec{F} = \nabla f$$

$$\int_{(0,0,1)}^{(1,\pi/4,2)} \vec{F} \cdot d\vec{r} = f(1, \frac{\pi}{4}, 2) - f(0,0,1) = \frac{\pi}{4} * 4 + \sin \frac{\pi}{2} + k - k = (\pi + 1)$$

* In the (plane):

$$\vec{F}(x,y) = [F_1, F_2]$$

$$\rightarrow \text{Test } \text{Curl } \vec{F} = \vec{0} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1(x,y) & F_2(x,y) & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = \vec{0} = [0, 0, 0]$$

$$*** \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} *** \checkmark$$

$$* \text{ Ex: } \vec{F}(x,y) = \underbrace{e^y}_{F_1} \hat{i} + \underbrace{x e^y}_{F_2} \hat{j}$$

- a - verify that $\vec{F}$ is conservative field

$$\text{- b - Evaluate } \int_{(1,0)}^{(-1,0)} \vec{F} \cdot d\vec{r}$$

$$a) \frac{\partial F_1}{\partial y} = e^y = \frac{\partial F_2}{\partial x} = e^y \Rightarrow \vec{F} = \nabla f$$

$$b) F_1 = f_x = e^y \\ F_2 = f_y = x e^y$$

$$f(x, y) = \int e^y dx = x e^y + h(y)$$

↓

$$f_y = x e^y + h'(y) = x e^y$$

$$h'(y) = 0 \Rightarrow h(y) = k$$

$$f(x, y) = x e^y + k$$

$$\int_{(1,0)}^{(-1,0)} \vec{F} \cdot d\vec{r} = f(-1,0) - f(1,0) = -1 + k - k - 1 = (-2)$$

* Evaluate: $\oint_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \left[\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right]$

$$C: x^2 + y^2 = 1, \quad C \in D: \frac{1}{2} < \sqrt{x^2 + y^2} < \frac{3}{2}$$

** $\vec{F}(x, y) \Rightarrow$ discontinuous at $(0, 0)$

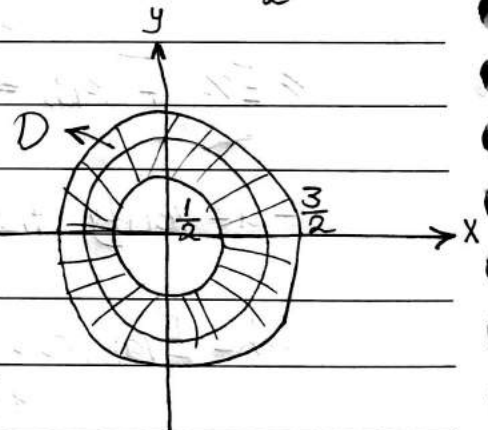
Test for conservative field:

$$\rightarrow \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

$$\frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial F_1}{\partial y}$$

$\rightarrow \vec{F}$ is conservative field in any region in the xy-plane that does not contain the origin $(0, 0)$

In this case $\oint_C \vec{F} \cdot d\vec{r} = 0$



$\left(D \right)$ is multiply connected region.

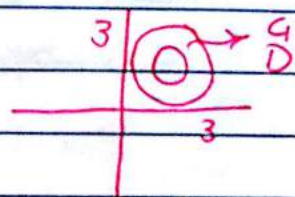
→ But the region D is multiply connected $\Rightarrow$ evaluate the integral
By def.

$$\oint_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot (\vec{r}'(t)) dt$$

→ $C: x^2 + y^2 = 1$, $\vec{r}(t) = [\underbrace{\cos t}_x, \underbrace{\sin t}_y]$, $0 \leq t \leq 2\pi$

$$= \int_0^{2\pi} \left[\frac{-\sin t}{1}, \frac{\cos t}{1} \right] \cdot [-\sin t, \cos t] dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 2\pi$$



$$\oint \vec{F} \cdot d\vec{r} = 0$$

Shifted Circle

$\int_C \vec{F} \cdot d\vec{r}$ is path indep. $\int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A)$

→ $\vec{F} = \nabla f$ is conservative field

$$\Leftrightarrow \oint \vec{F} \cdot d\vec{r} = 0$$

$\Leftrightarrow$ If $\text{curl } \vec{F} = \vec{0}$ and D is simply connected. $\left(\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \right)$

$\Leftrightarrow F_1 dx + F_2 dy + F_3 dz$ is exact.

10.3: Double Integrals (ex 1-9)

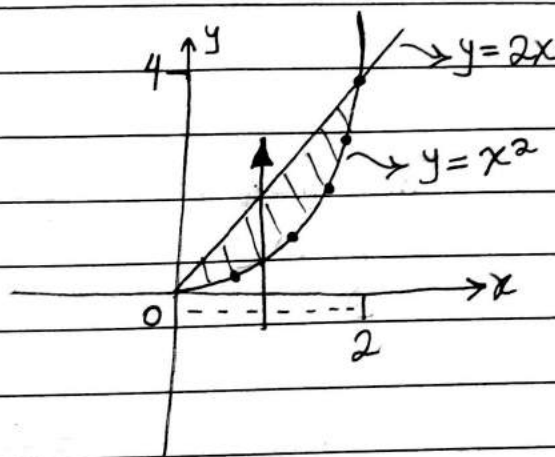
A double integral of $f(x,y)$ over the region D is

$$\iint_D f(x,y) dA$$

$\swarrow \searrow$
 $dx dy \quad dy dx$

Ex 1: Find $\iint_D (x^2 + y^2) dA$

$$\int_0^2 \int_{x^2}^{2x} (x^2 + y^2) dy dx$$



$$= \int_0^2 \left(x^2 y + \frac{y^3}{3} \right) \Big|_{x^2}^{2x} dx = \int_0^2 \left(2x^3 + \frac{8x^3}{3} - x^4 - \frac{1}{3}x^6 \right) dx$$

$$= \left(\frac{2x^4}{4} + \frac{8}{3} \frac{x^4}{4} - \frac{x^5}{5} - \frac{1}{3} \frac{x^7}{7} \right) \Big|_0^2 = \checkmark$$

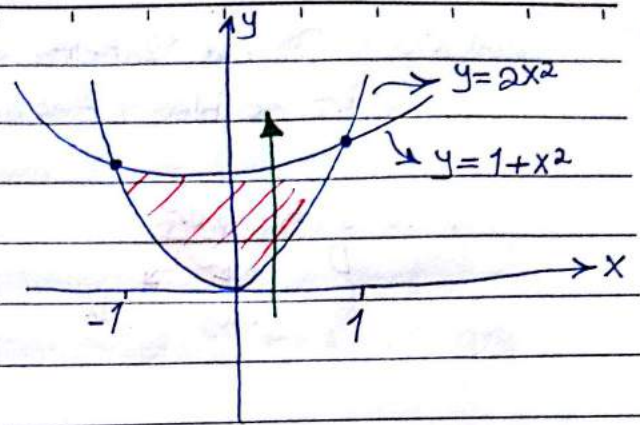
→ OR

$$\iint_D (x^2 + y^2) dA = \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (x^2 + y^2) dx dy$$

$$= \int_0^4 \left(\frac{x^3}{3} + y^2 x \right) \Big|_{\frac{y}{2}}^{\sqrt{y}} dy = \checkmark$$

* Ex: $\iint_D (x+2y) dA$?!

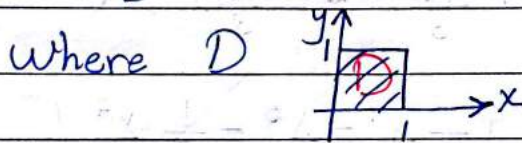
$$\rightarrow \int_{-1}^1 \int_{2x^2}^{1+x^2} (x+2y) dy dx.$$



* The Area of a region D is:

$$A = \iint_D dx dy$$

* Ex 3: Find $\iint_D dx dy \equiv$ the area of $D = (1 \times 1 = 1)$



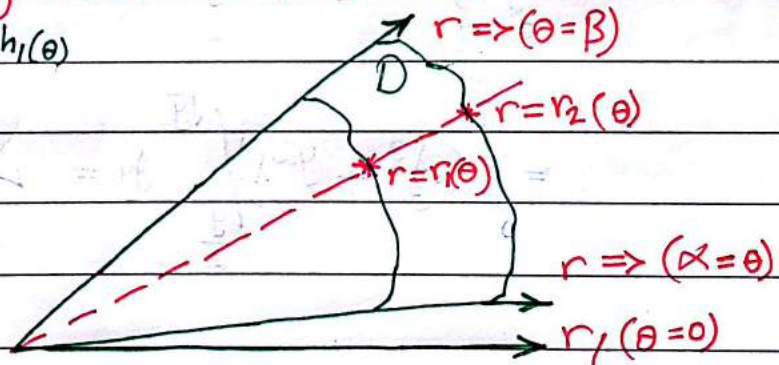
→ Polar Coordinates:

$$\iint_D f(x,y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$dA = r dr d\theta$$

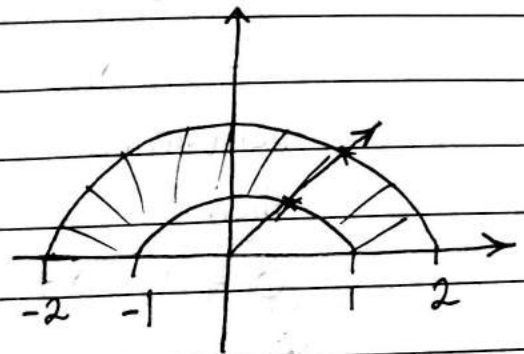


$$\# \text{ Ex. 4: } \iint_R (3x + 4y^2)$$

$$\Rightarrow \int_0^{\pi} \int_1^2 (3r \cos \theta + 4r^2 \sin^2 \theta) r dr d\theta$$

$$x^2 + y^2 = 1 \rightarrow r = 1$$

$$x^2 + y^2 = 4 \rightarrow r = 2$$

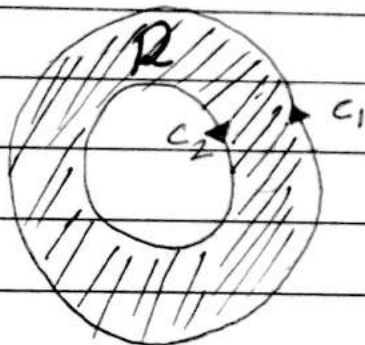
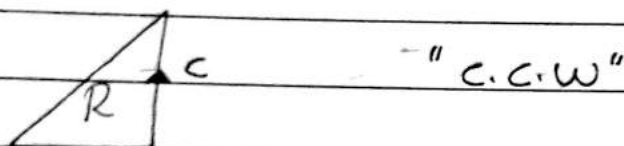
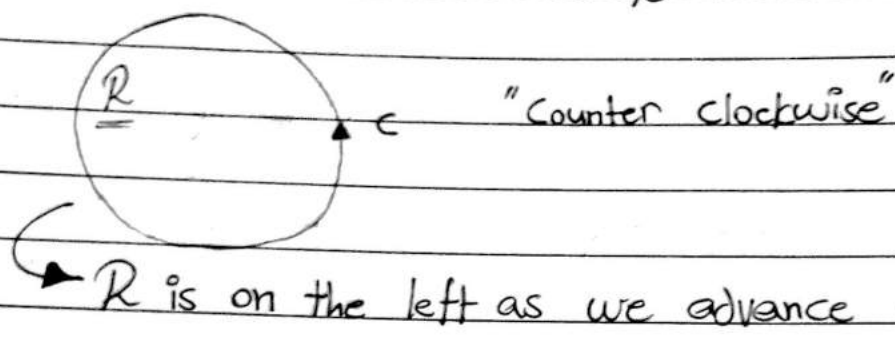


* 10.4 : [Green's theorem] in the plane (ex 1-10)

Let R be a closed bounded region in the xy -plane whose boundary is a positively oriented piecewise smooth curve C .

If $F_1(x,y)$ and $F_2(x,y)$ have continuous partial derivatives on an open region containing R , then:

$$\oint_C F_1(x,y)dx + F_2(x,y)dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy.$$



* Ex 1: Evaluate $\oint_C xy dx + x^2 y^3 dy$, where C is the

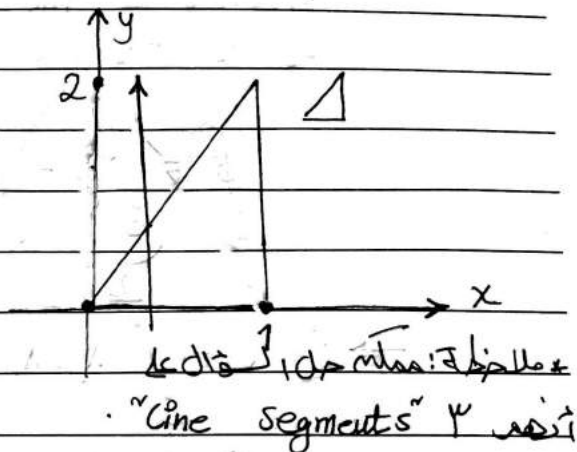
triangle with vertices $(0,0)$, $(1,0)$ and $(1,2)$
with counter clockwise direction.

$$** \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2xy^3 - x$$

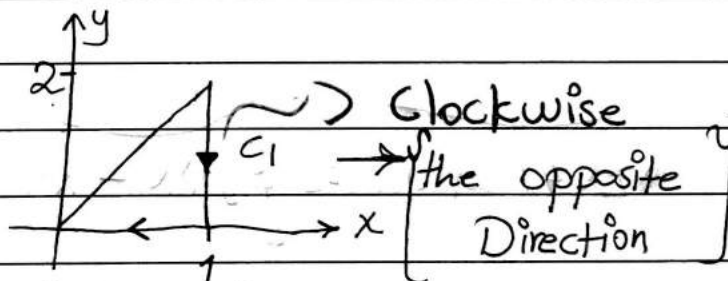
$$\oint = \int_0^1 \int_0^{2x} (2xy^3 - x) dy dx$$

$$= \int_0^1 \left(\frac{xy^4}{2} - (xy) \right) \Big|_0^{2x} dx$$

$$= \int_0^1 \left(\frac{8x^5}{2} - 2x^2 \right) dx = \left(\frac{4x^6}{3} - \frac{2x^3}{3} \right) \Big|_0^1 = \frac{2}{3}$$



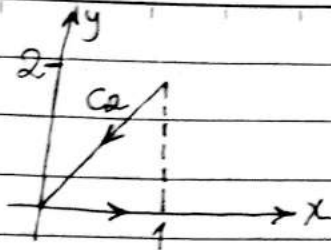
* Ex 2: = Ex 1, But



$$(1) \oint = \int_0^1 \int_0^{2x} (2xy^3 - x) dy dx = \frac{2}{3}$$

$$** \oint_{C_1} = - \oint_C = -\frac{2}{3}$$

* Ex 3: = Ex. 1



→ Introduce the Curve $C_2 \Rightarrow C = C_2 + C_3$,
 C_3 is the line segment from $(1,0)$ to $(1,2)$

$$** \oint_{C_2} = \oint_C - \oint_{C_3}$$

$\frac{2}{3}$ "Ex 1"

→ By def. $\int_{C_3} (xy dx) + (x^2 y^3 dy) =$

$$C_3: \vec{r}(t) = \vec{r}_0 + (\vec{r}_1 - \vec{r}_0)t, \quad 0 \leq t \leq 1$$

$$(1,0): \vec{r}_0 = [1, 0]$$

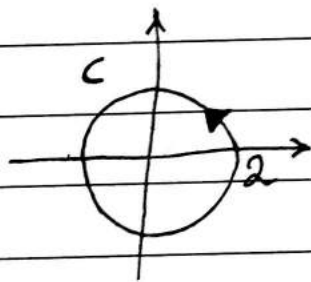
$$(1,2): \vec{r}_1 = [1, 2]$$

$$** \vec{r}(t) = [1, 0] + ([1, 2] - [1, 0])t = [\underbrace{1}_x, \underbrace{2t}_y]$$

$$dy = (2t)' dt$$

* Ex 4: Evaluate $\oint_C y^3 dx - x^3 dy$, where C is a positively

oriented circle of radius "2" centered in the Origin.



$$"x^2 + y^2 = 4" \leadsto \text{Circle eqn.}$$

$$** \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -3x^2 - 3y^2$$

$$\oint_C = -3 \iint_R (x^2 + y^2) dx dy$$

$R : (x^2 + y^2) = 4$, Using Polar coordinates.

$$* x = r \cos \theta \quad , \quad 0 \leq \theta \leq 2\pi$$

$$* y = r \sin \theta$$

$$* x^2 + y^2 = 4 \rightarrow r = 2$$

$$= -3 \int_0^{2\pi} \int_0^2 (r^2 \cos^2 \theta + r^2 \sin^2 \theta) r dr d\theta$$

$$= -3 \int_0^{2\pi} \int_0^2 r^3 dr d\theta = -3 \int_0^{2\pi} d\theta \int_0^2 r^3 dr = -3 \cdot 2\pi \cdot \frac{r^4}{4} \Big|_0^2 = \frac{-3 \cdot 2^4 \pi}{2}$$

$$= -24\pi$$

$$\oint_C F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\text{curl } \vec{F}) \cdot \vec{k} \, dx dy$$

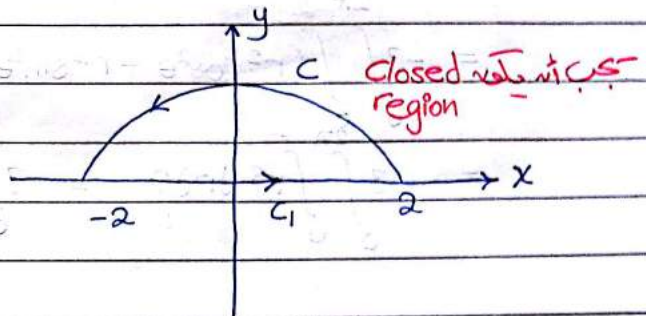
* Ex: Evaluate $\int_C (e^x + 4y) dx + (5xy + \sqrt{y^4 + 1} + 7x) dy$, where

C is the upper half of the circle $x^2 + y^2 = 4$

** $C_1: (-2, 0) \rightarrow (2, 0)$

$$\int_C = \oint_{C \cup C_1} - \int_{C_1}$$

use Green th.



$$\oint_{C \cup C_1} = \iint_R (7 - 4) dx dy = 3 \iint_R dx dy = 3 \left(\frac{1}{2} \pi (4) \right) = 6\pi$$

$$3 \int_0^{2\pi} \int_0^2 r dr d\theta = 3\pi \frac{r^2}{2} \Big|_0^2 = 6\pi$$

** Area of R
 $\iint_R 1 dx dy$

→ C_1 is the line segment $\vec{r}(t) = \vec{r}_0 + (\vec{r}_1 - \vec{r}_0)t$, $0 \leq t \leq 1$

$$\begin{aligned} \vec{r}(t) &= [-2, 0] + ([2, 0] - [-2, 0])t \\ &= [-2 + 4t, 0] \end{aligned}$$

$$\begin{aligned} \int_{C_1} &= \int_0^1 (e^{-2+4t})(4 dt) + (-----)0 \\ &= 4e^{-2} \int_0^1 e^{4t} dt = e^{-2} e^{4t} \Big|_0^1 = e^{-2} (e^4 - 1) \end{aligned}$$

$$\begin{aligned} dx &= x'(t) dt \\ dy &= 0 \end{aligned}$$

* Application of the Green theorem : (Computing areas)
→ Area of R is $A = \iint_R 1 dx dy$

$$\therefore \oint_C F_1 dx + F_2 dy = \iint_R \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)}_{(1)} dx dy$$

$$\text{Let } \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$$

** Choose F_2 and F_1 ($\vec{F} = [F_1, F_2]$)

If $F_2 = x$ and $F_1 = 0 \Rightarrow$

$$\boxed{\iint_R 1 dx dy} = \oint_C x dy$$

If $F_2 = 0$ and $F_1 = -y \Rightarrow$

$$\boxed{\iint_R 1 dx dy} = \oint_C -y dx$$

$$2 \iint_R 1 dx dy = \oint_C x dy - y dx$$

$$\iint_R 1 dx dy = \frac{1}{2} \oint_C x dy - y dx$$

* Ex: Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$C: \vec{r}(t) = [a \cos t, b \sin t] \quad , \quad 0 \leq t \leq 2\pi$$

$$\rightarrow A = \frac{1}{2} \int_0^{2\pi} (a \cos t)(b \cos t) dt - (b \sin t)(-a \sin t) dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab (\cos^2 t + \sin^2 t) dt = \frac{1}{2} (a \cdot b) 2\pi = ab\pi$$

~~~~~\*~~~~~\*~~~~~\*

## \* 10.5 : Surfaces (ex 1-8, 14-19)

\*\* Representation of a surface in space is

$$z = f(x, y), \quad g(x, y, z) = 0$$

$$z = x^2 + y^2$$

$$x^2 + y^2 + z^2 = 4$$

$$z = \sqrt{4 - x^2 - y^2}$$

"S" is two dimensional  $\Rightarrow$  we need two parameters  $u$  and  $v$ .

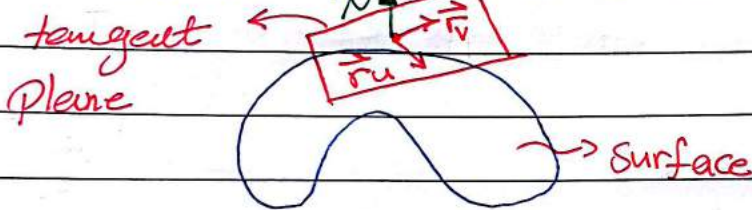
$\rightarrow$  A parametric representation of "S" in space is of the form:

$$\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$$

$$= x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$$

$$* (u, v \in \mathbb{R}) *$$

\* Normal vector  $\vec{N}$  of "S" at the point P



$$** \vec{N} = \vec{r}_u \times \vec{r}_v **$$

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u}, \quad \vec{r}_v = \frac{\partial \vec{r}}{\partial v}$$

$\rightarrow$  two diff. tangential vectors.  $\{$  Plane  $\{ \vec{e}_1, \vec{e}_2 \} \equiv \vec{e}'_u, \vec{e}'_v$  cross product  $\vec{e}_1 \times \vec{e}_2$   $\}$

$$\rightarrow \text{Unit normal vector } \hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

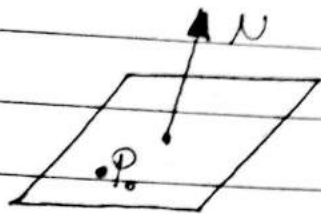
\*\* If "S" is represented by  $g(x, y, z) = 0$ ,

then  $\vec{N} = \underline{\underline{\text{grad } g(x, y, z)}}$



# planes:

$P_0(x_0, y_0, z_0)$  and Normal  $\vec{N} = [a, b, c]$



$$\leadsto a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

"plane equation"

\* Ex: Given  $P_0(2, 4, -1)$  and  $\vec{N} = [2, 3, 4]$ .

$$2(x-2) + 3(y-4) + 4(z+1) = 0$$
$$\rightarrow 2x + 3y + 4z - 12 = 0$$

## \* Revision \*

\* Planes:

\* Ex. 2: Find an equation of the plane that passes through  $P(1, 3, 2)$ ,  $Q(3, -1, 6)$  and  $R(5, 2, 0)$ .

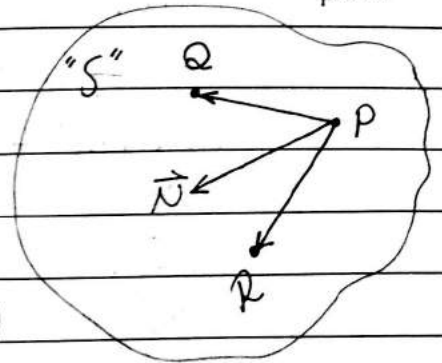
we choose it as an "initial" point

$$* \vec{PQ} = [2, -4, 4]$$

$$* \vec{PR} = [4, -1, -2]$$

$$* * \vec{N} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$= \hat{i}(12) - \hat{j}(-20) + \hat{k}(14)$$



$$\rightarrow 12(x-5) + 20(y-2) + 14z = 0$$

$$6(x-5) + 10(y-2) + 7z = 0$$

$$6x + 10y + 7z = 50$$

\* Ex: Find a parametric representation of:

$$6x + 5y - z = 4 \rightarrow z = 6x + 5y - 4$$

$$\left[ \begin{array}{l} u = x \\ v = y \\ z = 6u + 5v - 4 \end{array} \right] \vec{r}(u, v) = [u, v, 6u + 5v - 4]$$

\* OR

$$\left[ \begin{array}{l} x = x \\ y = y \\ z = 6x + 5y - 4 \end{array} \right] \vec{r}(x, y) = [x, y, 6x + 5y - 4]$$

\* Ex: Find a parametric representation of:  
 $4x - 2y + 10z = 16$

$$x = u$$

$$y = v$$

$$z = \frac{1}{10} (16 - 4u + 2v)$$

$$* \vec{r}(u, v) = \left[ u, v, \frac{1}{10} (16 - 4u + 2v) \right]$$

OR

$$x = x$$

$$y = y$$

$$z = \frac{1}{10} (16 - 4x + 2y)$$

$$* \vec{r}(x, y) = \left[ x, y, \frac{1}{10} (16 - 4x + 2y) \right]$$

\* ~~~~~ \*

# Cylinder:

$$x^2 + y^2 = a^2, \quad -2 \leq z \leq 2$$

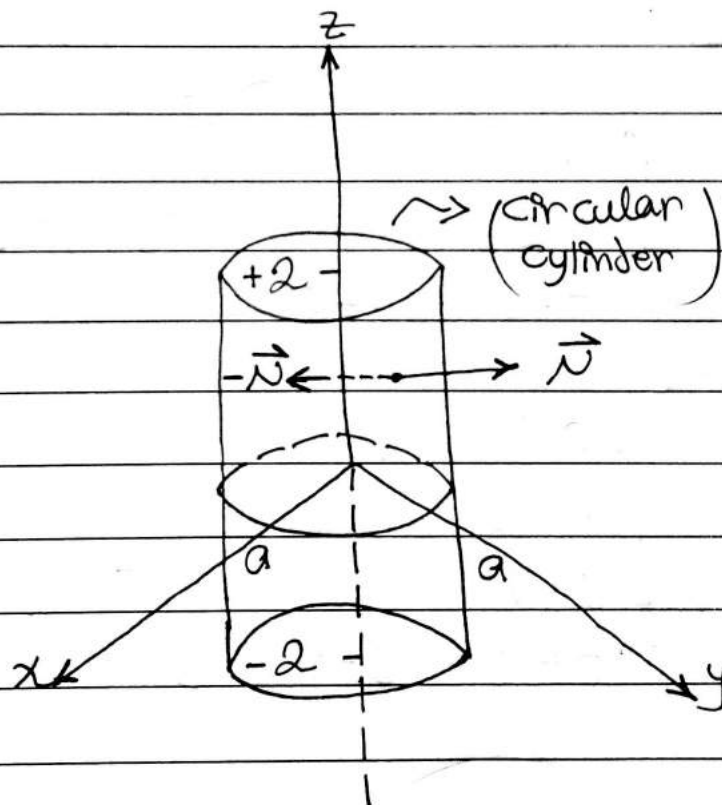
$$* x = a \cos u$$

$$* y = a \sin u$$

$$* z = v$$

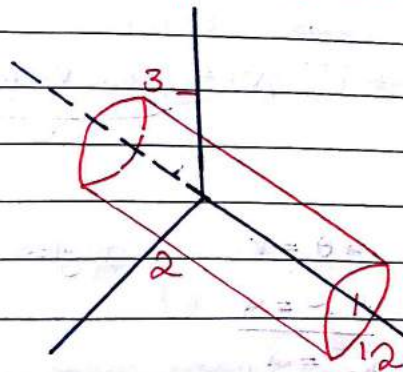
$$0 \leq u \leq 2\pi$$

$$-2 \leq v \leq 2$$





\* Ex:  $\frac{x^2}{4} + \frac{z^2}{9} = 1$ ,  $0 \leq y \leq 12$  "elliptic cylinder"



\* Ex:  $y = x^2$ ,  $0 \leq z \leq 1$  "Parabolic cylinder"

\*  $x = u$

\*  $y = u^2$

\*  $z = v$

or

\*  $x = x$

\*  $y = x^2$

\*  $z = z$

$\rightarrow \vec{r}(u,v) = [a \cos u, a \sin u, v]$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a \sin u & a \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \underbrace{(a \cos u)}_x \hat{i} + \underbrace{(a \sin u)}_y \hat{j}$

outward direction

\*\*  $(\hat{k} = 0) \leadsto z = \int \text{div}(\vec{N}) \, dA = \int \nabla \cdot \vec{N} \, dA$

$\rightarrow$  Inward direction:  $(-\vec{N}) = -(\vec{r}_u \times \vec{r}_v)$

\* Sphere: " $x^2 + y^2 + z^2 = a^2$ " ( $\rho = a$ )

Use spherical coordinates:

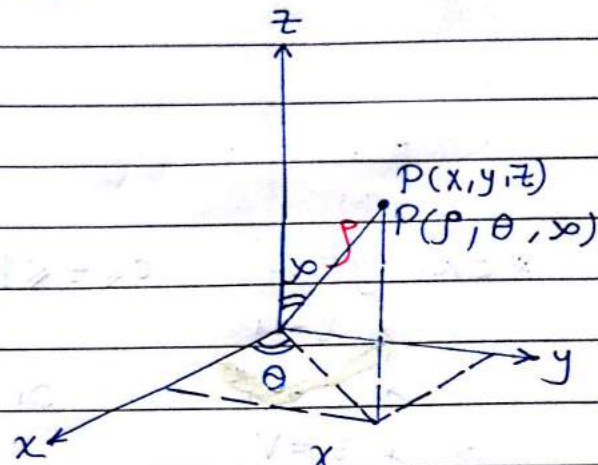
$$\vec{r}(u,v) = (\underbrace{a \cos v \cdot \sin u}_x) \hat{i} + (\underbrace{a \sin v \cdot \sin u}_y) \hat{j} + (\underbrace{a \cos v}_z) \hat{k}$$

$\theta = v$   
 $\phi = u$   
 $\rho = a$

angles

$$\sim 0 \leq v \leq 2\pi$$

$$\sim 0 \leq u \leq \pi$$

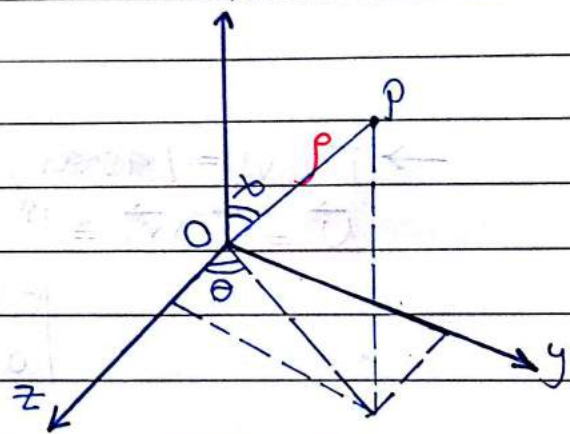


\* Spherical coordinates:

$$x = \rho \cos \theta \sin \phi$$

$$y = \rho \sin \theta \sin \phi$$

$$z = \rho \cos \phi$$



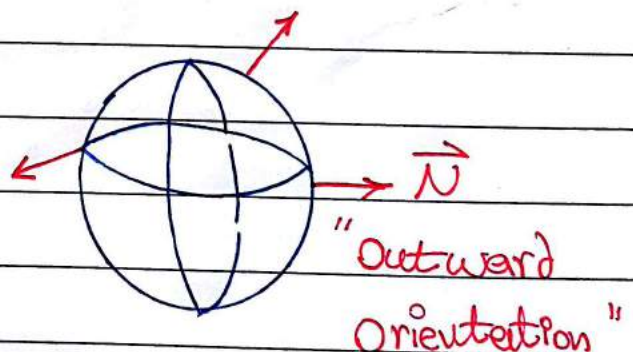
$$\vec{r}(u,v) = (a \cos u \cdot \sin v) \hat{i} + (a \sin u \cdot \sin v) \hat{j} + (a \cos v) \hat{k}$$

$$\sim 0 \leq u \leq 2\pi$$

$$\sim 0 \leq v \leq \pi$$

\* Note:-

A positive Orientation of " $s$ " is outward orientation (for closed surfaces).





$$\# x^2 + y^2 + z^2 = a^2$$

$$\vec{r}(u, v) = (a \cos v \sin u) \hat{i} + (a \sin v \sin u) \hat{j} + (a \cos u) \hat{k}, \quad 0 \leq u \leq \pi$$

$0 \leq v \leq 2\pi$

→ "outward sphere"

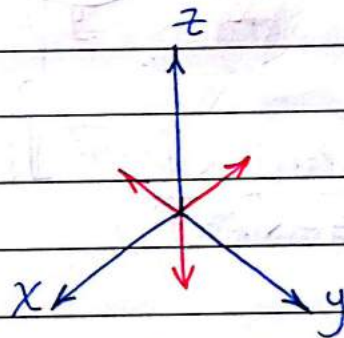
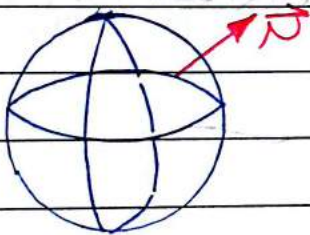
→ Find the normal:

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (a \cos v \cos u) & (a \sin v \cos u) & (-a \sin u) \\ (-a \sin v \sin u) & (a \cos v \sin u) & 0 \end{vmatrix}$$

$$= \hat{i} (a^2 \cos v \sin^2 u) + \hat{j} (a^2 \sin v \sin^2 u) + \hat{k} (a^2 \cos^2 v \cos u \sin u + a^2 \sin^2 v \sin u \cos u)$$

$$= a \sin u \left[ \underbrace{(a \cos v \sin u)}_x \hat{i} + \underbrace{(a \sin v \sin u)}_y \hat{j} + \underbrace{(a \cos u)}_z \hat{k} \right]$$

→  $\vec{N}$  has the same direction as a position vector  $[x, y, z]$ :  
Outward from the sphere.



$$** \vec{r}(u, v) = (a \cos u \sin v) \hat{i} + (a \sin u \sin v) \hat{j} + (a \cos v) \hat{k}$$

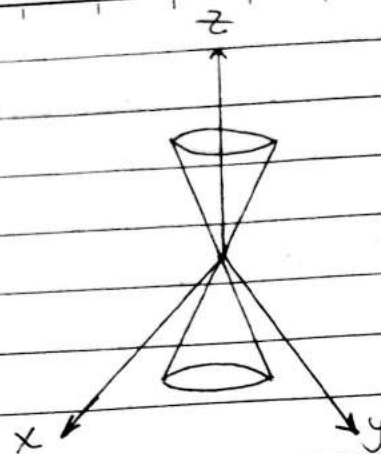
→ "Inward sphere"



# \* Revision \*

\* Cone \*

$$z^2 = x^2 + y^2$$

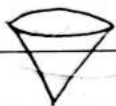


$$\rightarrow z = -\sqrt{x^2 + y^2}$$



"lower half"

$$\rightarrow z = \sqrt{x^2 + y^2}, \quad 0 \leq z \leq 2$$



"upper half"

$$** \vec{r}(u, v) = \left[ \underbrace{u \cos v}_x, \underbrace{u \sin v}_y, \underbrace{u}_z \right], \quad \begin{matrix} 0 \leq u \leq 2 \\ 0 \leq v \leq 2\pi \end{matrix}$$

$$** \vec{r}(x, y) = [x, y, \sqrt{x^2 + y^2}]$$

$$\vec{N} = \text{grad}(z - \sqrt{x^2 + y^2})$$

$$= \left[ \frac{-2x}{2\sqrt{x^2 + y^2}}, \frac{-y}{\sqrt{x^2 + y^2}}, 1 \right] \text{ ; upward "Inside cone"}$$

\* Paraboloid \*

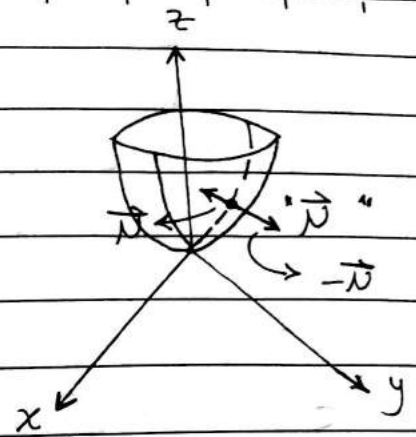
$$z = x^2 + y^2, \quad 0 \leq z \leq 5$$

"1"  $\vec{r}(u,v) = [u \cos v, u \sin v, u^2]$

"2"  $\vec{r}(x,y) = [x, y, (x^2 + y^2)]^T$

"1"  $\rightarrow \vec{N} = \vec{r}_u \times \vec{r}_v$

"2"  $\rightarrow \vec{N} = \text{grad}(z - (x^2 + y^2))$



\* Ellipsoid \*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\vec{r}(u,v) = [a \cos v \sin u, b \sin v \sin u, c \cos u]$$

### # 10.6 : "Surface Integral" (ex 1-10, 12-16)

\*\* A surface "S" is given by,

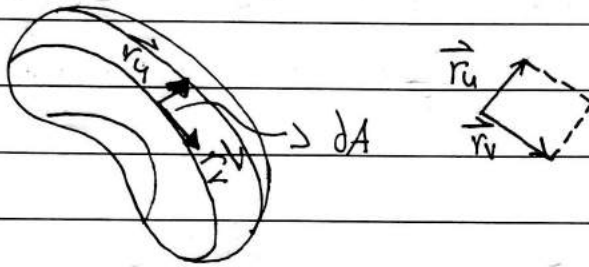
$$\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)],$$

and it is piecewise smooth.

\*\* "S" has a normal vector  $\vec{N} = \vec{r}_u \times \vec{r}_v$  and a unit normal vector  $\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

→ A surface integral over S:

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_R \vec{F}(\vec{r}(u,v)) \cdot \vec{N} du dv$$



$$\hat{n} dA = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \cdot |\vec{r}_u \times \vec{r}_v| du dv$$

→ R is projection of "S" onto the uv-plane:

$$\iint_S \vec{F} \cdot \hat{n} dA = \iint_R \vec{F}(\vec{r}(u,v)) \cdot \underbrace{(\vec{r}_u \times \vec{r}_v)}_{\vec{N}} du dv$$

Projection onto the uv-plane

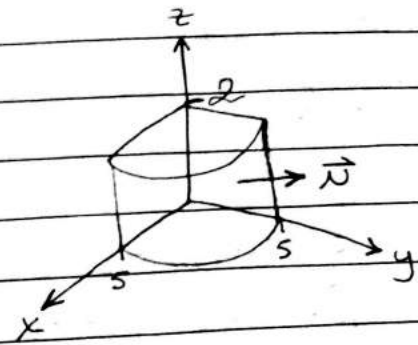


\* Ex. 1: Evaluate  $\iint_S \vec{F} \cdot \hat{n} dA$ , where  $\vec{F} = [y, z, e^x]$  and

$$S: x^2 + y^2 = 25$$

$$x \geq 0, y \geq 0$$

$$0 \leq z \leq 2$$



$$** S \Rightarrow \vec{r}(u, v) = [5\cos u, 5\sin u, v]$$

$$0 \leq u \leq \left(\frac{\pi}{2}\right) \quad x \quad y \quad z$$

$$0 \leq v \leq 2$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (-5\sin u) & (5\cos u) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \hat{i}(5\cos u) + \hat{j}(5\sin u)$$

$$\rightarrow \iint_S \vec{F} \cdot \hat{n} dA = \int_0^2 \int_0^{\pi/2} [5\sin u, v, e^{5\cos u}] \cdot [5\cos u, 5\sin u, 0] du dv$$

*du dv*

$$= \int_0^2 \int_0^{\pi/2} (25 \sin u \cos u + 5v \sin u) du dv$$

$$= \int_0^2 \left[ \left( \frac{25}{2} \cdot \frac{1}{2} \right) (-\cos 2u) + 5v (-\cos u) \right] \bigg|_0^{\pi/2} dv$$

$$= -\int_0^2 \left( \frac{25}{4} (-1) + 5v (-1) \right) dv = \int_0^2 \left( \frac{25}{4} + 5v \right) dv = \dots$$

$$= \frac{25}{2} v + \frac{5v^2}{2} = 25 + 10 = 35$$

# Ex. 2: Evaluate  $\iint_S \vec{F} \cdot \hat{n} dA$ , where  $\vec{F} = [x^2, 0, 3y^2]$

$\vec{n}$  also called

$\vec{n} = \frac{\nabla(x+y+z-1)}{|\nabla(x+y+z-1)|}$

$\rightarrow [1, 1, 1]$   
normal vector

and  $S$  is the portion of the plane  $x+y+z=1$  in the first octant "upward oriented".  $x \geq 0, y \geq 0, z \geq 0$

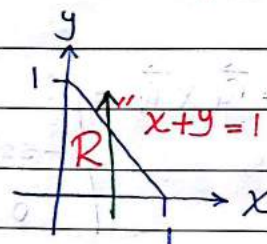
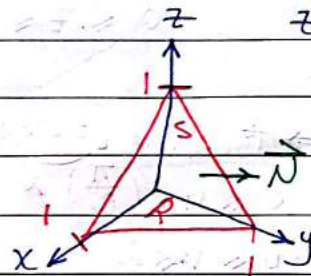
$S: \vec{r}(x,y) = [x, y, 1-x-y]$

$\vec{N} = \text{grad}(x+y+z-1) = [1, 1, 1]$

(or) " $1-x-y-z=0$ "

$\rightarrow \iint_S = \int_0^1 \int_0^{1-x} [x^2, 0, 3y^2] \cdot [1, 1, 1] dy dx$

$= \int_0^1 \int_0^{1-x} [x^2 + 3y^2] dy dx = \int_0^1 (x^2 y + y^3) \Big|_0^{1-x} dx$   
 $= \left(\frac{1}{3}\right)$



$0 \leq x \leq 1$

$0 \leq y \leq 1-x$

# Ex. 3: Evaluate  $\iint_S \vec{F} \cdot \hat{n} dA$ , where  $\vec{F}(x,y,z) = z\hat{k}$

$S$  is the outward sphere  $x^2+y^2+z^2=a^2$

$S: \vec{r}(u,v) = [a \cos v \sin u, a \sin v \sin u, a \cos u]$ ,  $0 \leq u \leq \pi$   
 $0 \leq v \leq 2\pi$

$\vec{N} = \vec{r}_u \times \vec{r}_v = [a^2 \cos v \sin^2 u, a^2 \sin v \sin^2 u, a^2 \cos u \sin u]$

$\vec{F} \cdot \vec{N} = a \cos u (a^2 \cos u \sin u)$

$\iint_S = \int_0^{2\pi} \int_0^\pi a^3 \cos^2 u \sin u du dv = a^3 \int_0^{2\pi} \left( \frac{-\cos^3 u}{3} \right) \Big|_0^\pi dv = \frac{2\pi}{3} a^3 \cdot 2$

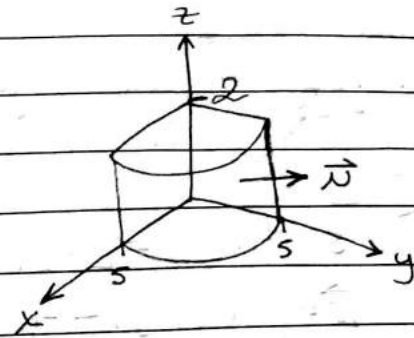
$\int \cos^2 u \sin u du = -\frac{\cos^3 u}{3}$ ,  $t = \cos u$   
 $dt = -\sin u du$

# Ex. 1: Evaluate  $\int_S \vec{F} \cdot \hat{n} dA$ , where  $\vec{F} = [y, z, e^x]$  and

$$S: x^2 + y^2 = 25$$

$$x \geq 0, y \geq 0$$

$$0 \leq z \leq 2$$



$$** S \Rightarrow \vec{r}(u, v) = [5\cos u, 5\sin u, v]$$

$$0 \leq u \leq \left(\frac{\pi}{2}\right) \quad x \quad y \quad z$$

$$0 \leq v \leq 2$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (-5\sin u) & (5\cos u) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \hat{i}(5\cos u) + \hat{j}(5\sin u)$$

$$\rightarrow \int_S \vec{F} \cdot \hat{n} dA = \int_0^2 \int_0^{\pi/2} [5\sin u, v, e^{5\cos u}] \cdot [5\cos u, 5\sin u, 0] du dv$$

$$= \int_0^2 \int_0^{\pi/2} (25 \sin u \cos u + 5v \sin u) du dv$$

$$= \int_0^2 \left( \frac{25}{2} \cdot \frac{1}{2} (-\cos 2u) + 5v (-\cos u) \right) \Big|_0^{\pi/2} dv$$

$$= \int_0^2 \left( \frac{25}{4} (-1) + 5v (-1) \right) dv = \int_0^2 \left( -\frac{25}{4} - 5v \right) dv = \dots$$

$$= \frac{25}{2} v + \frac{5v^2}{2} = 25 + 10 = 35$$



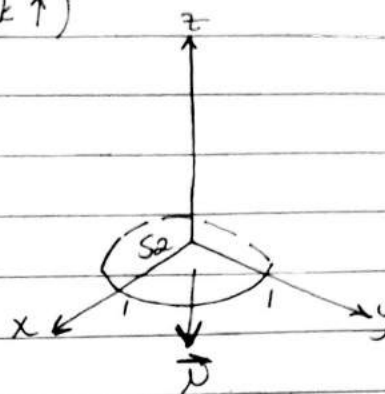
→  $S_2: x^2 + y^2 = 1$  in the  $z=0$  ( $\text{grad } z = \hat{k} \uparrow$ )

$$\vec{r}(x,y) = [x, y, 0]$$

$$\vec{n} = -\hat{k} = [0, 0, -1]$$

$$\vec{F} \cdot \vec{n} = [y, x, 0] \cdot [0, 0, -1]$$

$$\iint_{S_2} = 0$$



$$\therefore S = S_1 + S_2$$

$$= \frac{\pi}{2} + 0 = \left(\frac{\pi}{2}\right)$$

\* H.W. Solve in  $(u,v)$  variables. "Parameterization"

\* Surface integral without regard to orientation.

$$\int_S G(r) dA = \iint_R G(\vec{r}(u,v)) \cdot |\vec{r}_u \times \vec{r}_v| du dv$$

R "is the projection of S onto uv-plane"

\*\*  $G(r)$  is a scalar function.

# Ex: Evaluate  $\iint_S y^2 z^2 dA$  where  $S$  is the part of the cone only this part of the cone  
 $z = \sqrt{x^2 + y^2}$  that lies between the planes  $z=1$  and  $z=2$ .

$$\rightarrow S: \vec{r}(u, v) = [u \cos v, u \sin v, u]$$

$$1 \leq u \leq 2$$

$$0 \leq v \leq 2\pi$$

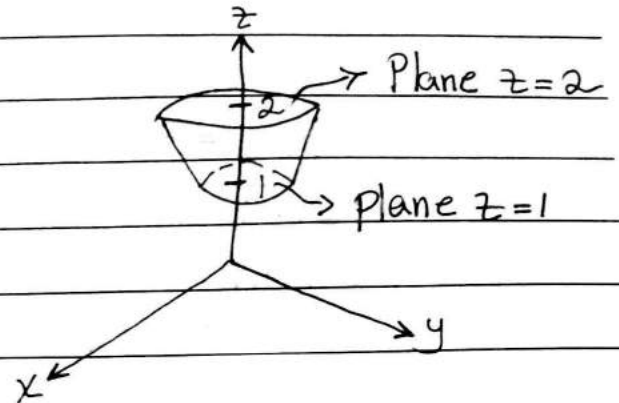
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix}$$

$$= \hat{i}(-u \cos v) - \hat{j}(u \sin v) + \hat{k}(u)$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2 \cos^2 v + u^2 \sin^2 v + u^2} = (u\sqrt{2})$$

$$\iint_S = \int_0^{2\pi} \int_1^2 (u^2 \sin^2 u) u^2 \cdot u\sqrt{2} du dv$$

$$= \sqrt{2} \int_0^{2\pi} \left. \frac{u^6}{6} \right|_1^2 dv = \frac{\sqrt{2}}{6} (2^6 - 1) 2\pi$$



\* If  $(G=1)$  then the surface integral gives that the Area of "S"

$$\rightarrow \text{Area of the Surface "S"} = \iint_S dA = \iint_R |\vec{r}_u \times \vec{r}_v| du dv$$

# Ex: Find the Area of  $x^2 + y^2 + z^2 = 1$

$$\vec{r}(u,v) = \dots$$

$$\vec{r}_u \times \vec{r}_v = [a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u]$$

$$\rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{a^4 \sin^4 u \cos^2 v + a^4 \sin^4 u \sin^2 v + a^4 \cos^2 u \sin^2 u}$$

$$= a^2 \sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u}$$

$$= a^2 \sin u$$

$$A = \iint_R a^2 \sin u du dv = a^2 \int_0^{2\pi} \int_0^{\pi} \sin u du dv = \dots = 4\pi a^2$$

ch. 10 :

$$10.1 \rightarrow (2-11)$$

$$10.2 \rightarrow (3-9) (13-19)$$

$$10.3 \rightarrow (1-9)$$

$$10.4 \rightarrow (1-10)$$

$$10.5 \rightarrow (1-8, 14-19)$$

$$10.6 \rightarrow (1-10, 12-18)$$

(9.)

$$\vec{F} = [0, \sinh z, \cosh z]$$

$$S: x^2 + z^2 = 4$$

$$0 \leq x \leq 1/\sqrt{2}$$

$$0 \leq y \leq 5$$

$$z \geq 0$$

Parameterize:

$$1) x^2 + (y+2.8)^2 +$$

$$(z-3.2)^2 = 2.25$$

Sphere

$$2) z = \sqrt{x^2 + 4y^2}$$

elliptic cone



# # 10.7: Divergence theorem of Gauss (ex 9-5)

## \*\* Theorem:

Let  $T$  be a closed boundary region in space whose boundary is a piecewise-smooth orientable surface " $s$ " (positive orientable).

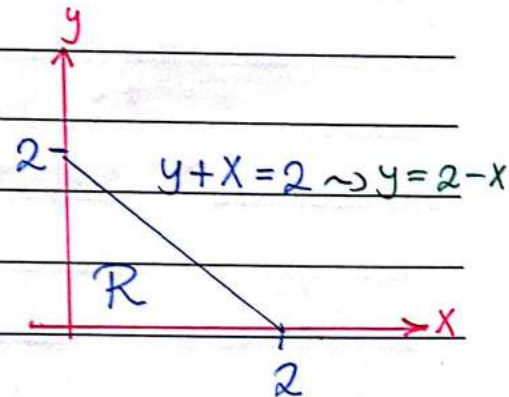
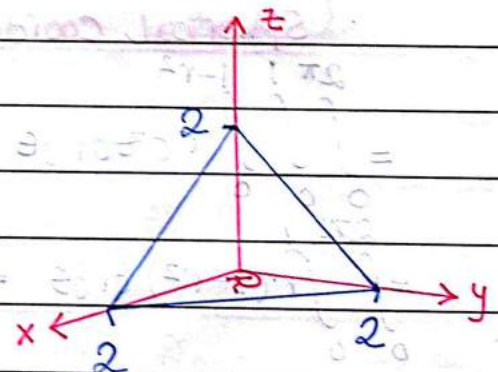
Let  $\vec{F}(x, y, z)$  be a vector function that is continuous partial derivatives in some region, then:

$$\iiint_S \vec{F} \cdot \hat{n} dA = \iiint_V \text{div} \vec{F} \cdot dV$$

\* Ex. 1: Evaluate  $\iiint_S \vec{F} \cdot \hat{n} dA$ , where  $\vec{F}(x, y, z) = [x^2 + 2y^2z, 2xz, xe^{-y}]$ .

$S \equiv$  The boundary of the region enclosed by the three coordinate planes and the plane  $x + y + z = 2$

$$\begin{aligned} \leadsto \text{div } \vec{F} &= 2x \\ \iiint_S &= \iiint_R (2-x-y) dz \\ &= \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2x dz dy dx = \int_0^2 \int_0^{2-x} 2x(2-x-y) dy dx \\ &= \int_0^2 \left[ 2x(2-x)y - xy^2 \right]_0^{2-x} dx \\ &= \int_0^2 (2x)(2-x)^2 - x(2-x)^2 dx \\ &= \int_0^2 x(2-x)^2 dx = \int_0^2 (4x - 4x^2 + x^3) dx = 2x^2 - \frac{4}{3}x^3 + \frac{x^4}{4} \Big|_0^2 \\ &= 8 - \frac{48}{3} + 4 \checkmark \end{aligned}$$

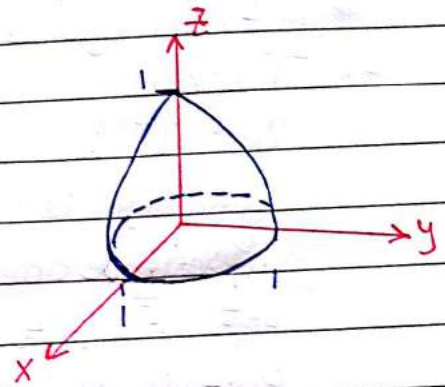


$$= 8 - \frac{48}{3} + 4 \checkmark$$

\* Ex. 2: From (10.6)  $\iint_S \vec{F} \cdot \hat{n} dA$

$$\vec{F} = [x, y, z], \quad S: z = 1 - x^2 - y^2$$

$z=0$



$$\leadsto \operatorname{div} \vec{F} = 3$$

$$\iint_S \vec{F} \cdot \hat{n} dA = 3 \iiint_R (1 - x^2 - y^2) dz dx dy$$

$$\leadsto R: x^2 + y^2 = 1 \rightarrow \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$dv = r dz dr d\theta$$

(cylindrical coord.)  $\leftarrow$  cylinder

Cone

paraboloid

(Spherical coord.)  $\leftarrow$  sphere

$$= \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r dz dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 r(1-r^2) dr d\theta = \int_0^{2\pi} \left( \frac{1}{2} - \frac{1}{4} \right) d\theta = \frac{\pi}{2}$$



\* Ex. 3: Evaluate  $\iint_S \vec{F} \cdot \hat{n} dA$ , where function  $\vec{F} = [x^3, y^3, z^2]$

S: the region that is enclosed by the cylinder  $x^2 + y^2 = 9$  and the planes  $z=0$ ,  $z=2$ .

(2, 0) in cylinder  $\parallel$   $(1, 0, 2)$

$$\leadsto \text{div } \vec{F} = 3x^2 + 3y^2 + 2z$$

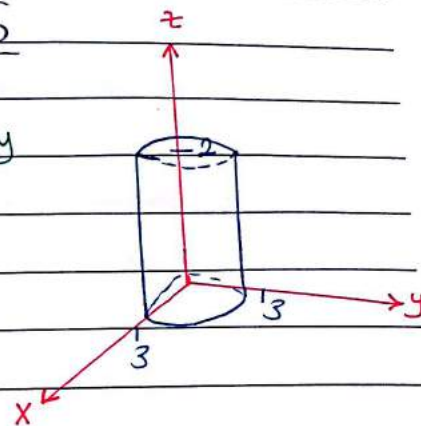
$$\iint_S \vec{F} \cdot \hat{n} dA = \iiint_R (3x^2 + 3y^2 + 2z) dz dx dy$$

$$= \int_0^{2\pi} \int_0^3 \int_0^2 (3r^2 + 2z) dz r dr d\theta$$

$$= \int_0^{2\pi} \int_0^3 (6r^3 + 4r) dr d\theta$$

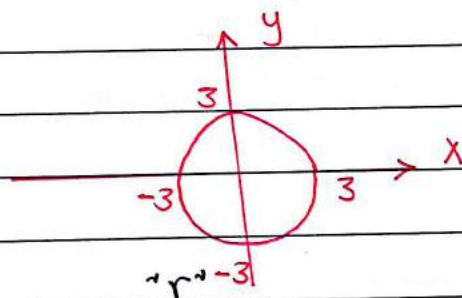
$$= 2\pi \left[ \frac{6r^4}{4} + 2r^2 \right]_0^3$$

$$= 279\pi$$



$\leadsto$  (Projection  $z=0$ )

$$R: x^2 + y^2 = 9$$



$$x = 3 \cos \theta, \quad z = z$$

$$y = 3 \sin \theta$$

$$0 \leq r \leq 3$$

$$0 \leq \theta \leq 2\pi$$



\* Ex:  $\iint_S \vec{F} \cdot \hat{n} dA$   $\vec{F} = [x^3, y^3, z^2]$

$S: x^2 + y^2 = 9$

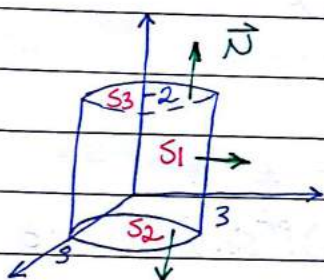
$z = 0$

$z = 2$

\* Ex: Evaluate  $\iint_S \vec{F} \cdot \hat{n} dA$ , where  $\vec{F} = [x^3, y^3, z^2]$

$S: x^2 + y^2 = 9, z = 0$

$(0 \leq z \leq 2)$



\*\*  $S = S_1 + S_2$  Introduce a surface " $S_3: z = 2$ "  $\rightarrow g(x, y, z) = 0 \therefore z - 2 = 0$   
( $x^2 + y^2 = 9$ )

\*\*  $S = S_1 + S_2 - S_3$

$\iint_S = \iint_{S_1 + S_2} - \iint_{S_3} = 279\pi$   
(See ex =  $279\pi$ )

$\Rightarrow$  Evaluate  $\iint_{S_3} \vec{F} \cdot \hat{n} dA$ ,  $S_3: \vec{r}(x, y) = [x, y, 2]$

\*\*  $\vec{N} = \text{grad}(z - 2) = [0, 0, 1]$  upward

$\iint_{S_3} = \iint_R \vec{F} \cdot \vec{N} dx dy = \iint_R [x^3, y^3, 4] \cdot [0, 0, 1] dx dy$   
"R =  $x^2 + y^2 = 9$ "

$= 4 \iint_R dx dy = 4 \cdot 9\pi$   
"area of R"



\* Evaluate:  $\iint_S \vec{F} \cdot \hat{n} dA$ , where  $\vec{F} = [x^3, y^3, z^3]$

"S" is the region that is enclosed by the hemisphere  $z = \sqrt{a^2 - x^2 - y^2}$  and the plane  $z = 0$ .

$$\rightarrow \text{div } \vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$\iint_S \vec{F} \cdot \hat{n} dA = \iiint_V 3\rho^2 (\cos^2\theta \sin^2\varphi + \sin^2\theta \sin^2\varphi + \cos^2\varphi) \rho^2 \sin\varphi d\rho d\varphi d\theta$$

Use spherical coordinates:  $x = \rho \cos\theta \sin\varphi$

$$y = \rho \sin\theta \sin\varphi$$

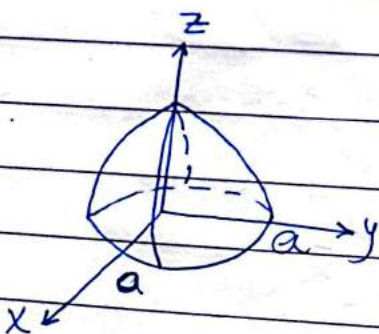
$$z = \rho \cos\varphi$$

$$\rightarrow dV = \rho^2 \sin\varphi \cdot d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a 3\rho^2 (\cos^2\theta \sin^2\varphi + \sin^2\theta \sin^2\varphi + \cos^2\varphi) \rho^2 \sin\varphi d\rho d\varphi d\theta$$

$$= 3 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^a \rho^4 \sin\varphi d\rho d\varphi d\theta = \frac{3}{5} a^5 \int_0^{2\pi} (-\cos\varphi) \Big|_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{3a^5 \cdot 2\pi}{5}$$



Sphere

Spherical coordinate:

$$x = \rho \cos\theta \sin\varphi$$

$$y = \rho \sin\theta \sin\varphi$$

$$z = \rho \cos\varphi$$

$$dV = \rho^2 \sin\varphi d\rho d\varphi d\theta$$

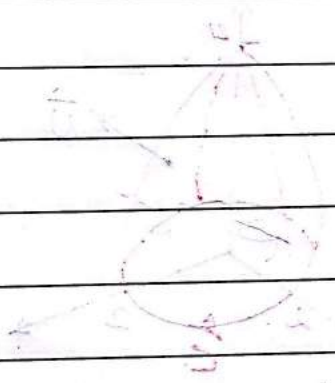
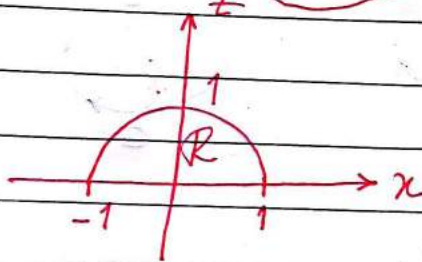
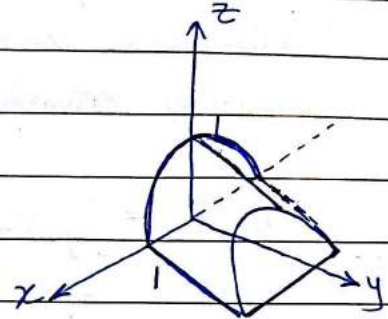
\* Ex: Evaluate  $\iint_S \vec{F} \cdot \hat{n} dA$ ,  $\vec{F}(x,y,z) = [xy, y^2 + e^{xz^2}, \sin xy]$ .

"S" is the surface of the region bounded by  $z = 1 - x^2$  and  $z = 0$ ,  $y = 0$ ,  $y + z = 2$ .

$$\sim y = 2 - z$$

$$\rightarrow \text{div } \vec{F} = y + 2y = 3y$$

$$\iiint_S = \int_{-1}^1 \int_0^{2-z} \int_0^{1-x^2} 3y \, dy \, dz \, dx$$

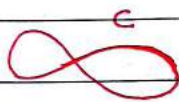
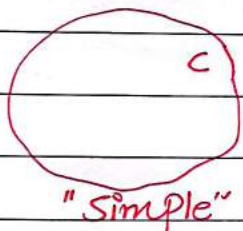




# \* 10.9: Stoke's Theorem (ex 1-10, 13-20).

\* Th. let "S" be a piecewise smooth oriented surface in space and let the boundary of "S" be a piecewise smooth simple closed curve C. Let  $\vec{F}(x,y,z)$  be a continuous vector function that has continuous first partial derivatives in some domain, then:

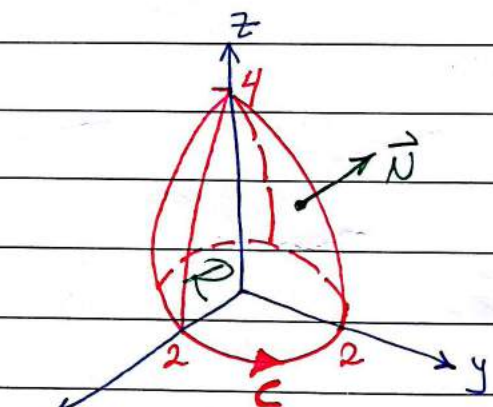
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{Curl}(\vec{F})) \cdot \hat{n} dA$$



\* Ex. 1: Evaluate  $\oint_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F} = 2z\hat{i} + 3x\hat{j} + 5y\hat{k}$ .

"S" is the intersection of the paraboloid  $z = 4 - x^2 - y^2$  and  $z = 0$ , It is counter clockwise oriented.

$$\begin{aligned} z=0 &\Rightarrow 0 = 4 - x^2 - y^2 \\ &\Rightarrow x^2 + y^2 = 4 \text{ "circle"} \end{aligned}$$



$$\begin{aligned} \text{Curl } \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} \\ &= 5\hat{i} + 2\hat{j} + 3\hat{k} \end{aligned}$$

$$= 5\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \hat{n} dA = \iint_R (10x + 4y + 3) dx dy$$

$R: x^2 + y^2 = 4$  = "Disk inside the circle"

$$S: \vec{r}(x, y) = [x, y, 4 - x^2 - y^2]$$

$$** \vec{N} = \text{grad}(g(x, y, z))$$

$$= \text{grad}(z - 4 + x^2 + y^2) = [2x, 2y, 1]$$

$$** (\text{curl } \vec{F}) \cdot \vec{N} = [5, 2, 3] \cdot [2x, 2y, 1]$$

$$= 10x + 4y + 3$$

→ " $x^2 + y^2 = 4$ ", Disk → using polar coordinates.

$$x = r \cos \theta, \quad 0 \leq r \leq 2$$

$$y = r \sin \theta, \quad 0 \leq \theta \leq 2\pi$$

$$= \int_0^{2\pi} \int_0^2 (10r \cos \theta + 4r \sin \theta + 3) r dr d\theta$$

$$= \int_0^{2\pi} \left( \frac{10}{3} \cdot \underbrace{8 \cos \theta}_0 + \frac{4}{3} \cdot \underbrace{8 \sin \theta}_0 + 6 \right) d\theta = (12\pi)$$

# Ex. 2: Evaluate  $\oint_C \vec{F} \cdot d\vec{r}$  (= ex 1),  $C$  is the boundary of  $S_1: x^2 + y^2 = 4, z = 0$  "disk".

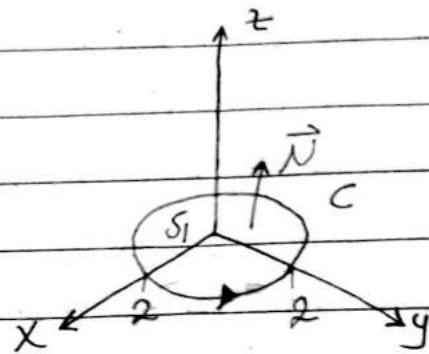
$$\therefore \vec{N} = [0, 0, 1] = \text{grad}(z)$$

$$\text{curl } \vec{F} = [5, 2, 3]$$

$$(\text{curl } \vec{F}) \cdot \vec{N} = 3$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R 3 dx dy = 3$$

$$\equiv (\text{Area of } R) = 3 \cdot 4\pi = (12\pi)$$



→ Note! The paraboloid can be replaced by the disk.

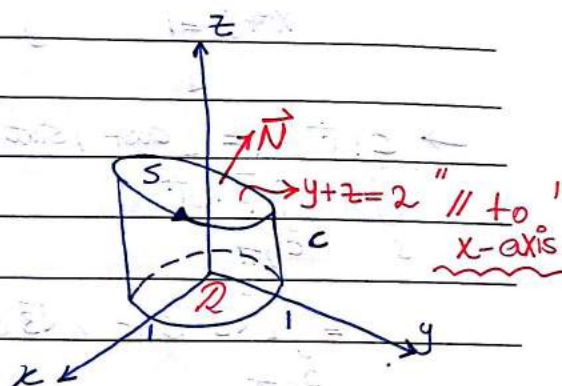


$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \hat{n} dA$$

\* Ex. 3: Evaluate  $\oint_C \vec{F} \cdot d\vec{r}$ , if  $\vec{F}(x,y,z) = [-y^2, x, z^2]$ ,  $C$  is the

curve of the intersection of the plane ( $y+z=2$ ) and the cylinder  $x^2+y^2=1$ ,  $C$  is oriented counter clockwise.

$$\begin{aligned} * \text{curl } \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} \\ &= \hat{i}(0) - \hat{j}(0) + \hat{k}(1+2y) \\ &= [0, 0, 1+2y] \end{aligned}$$



$$\rightarrow S: y+z=2; \vec{r}(x,y) = [x, y, 2-y]$$

$$\vec{N} = \text{grad}(y+z-2) = [0, 1, 1] \text{ upward.}$$

$$\oint = \iint_S = \iint_R [0, 0, 1+2y] \cdot [0, 1, 1] dx dy$$

$$= \iint_R (1+2y) dx dy$$

$R$ : disk  $x^2+y^2=1$ .

\*\* using polar coordinates;

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$= \int_0^{2\pi} \int_0^1 (1+2r \sin \theta) r dr d\theta = \int_0^{2\pi} \left( \frac{1}{2} + \frac{2}{3} \sin \theta \right) d\theta = 2\pi \cdot \left( \frac{1}{2} \right) = (\pi).$$

\* Ex. 4: Evaluate  $\iint_S (\text{curl } \vec{F}) \cdot \hat{n} dA$ ,  $\vec{F}(x, y, z) = [xz, yz, xy]$

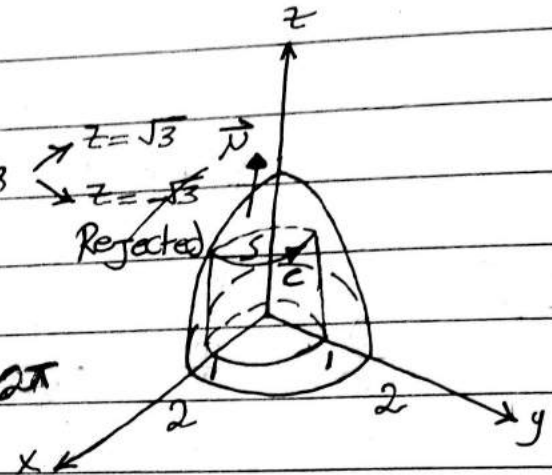
"S" is the part of the sphere  $x^2 + y^2 + z^2 = 4$  that lies inside the cylinder  $x^2 + y^2 = 1$ , and above the xy-plane.

\*\* C:  $x^2 + y^2 = 1, z = \sqrt{3}$

$x^2 + y^2 + z^2 = 4$

$x^2 + y^2 = 1$

$1 + z^2 = 4, z^2 = 3$



$\rightarrow C: \vec{r}(t) = [\cos t, \sin t, \sqrt{3}], 0 \leq t \leq 2\pi$

$\iint_S = \oint_C \vec{F} \cdot d\vec{r}$

$= \int_0^{2\pi} [\sqrt{3} \cos t, \sqrt{3} \sin t, \cos t \sin t] \cdot [-\sin t, \cos t, 0] dt$

$= \int_0^{2\pi} (-\sqrt{3} \cos t \sin t + \sqrt{3} \cos t \sin t) dt = 0$



\* \* \* \* \*



## CH. 11. Fourier Analysis :

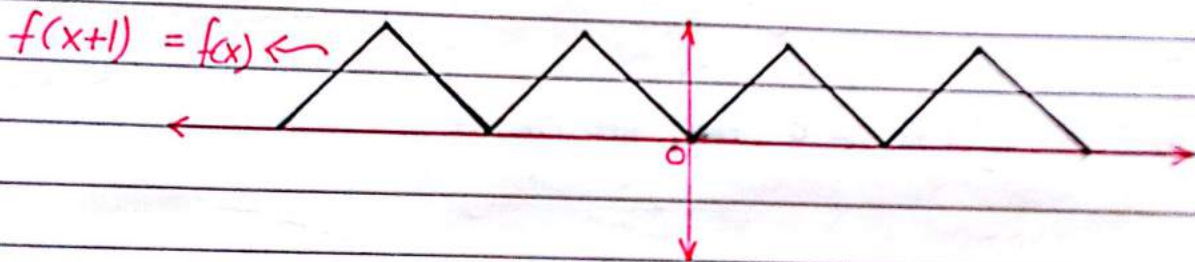
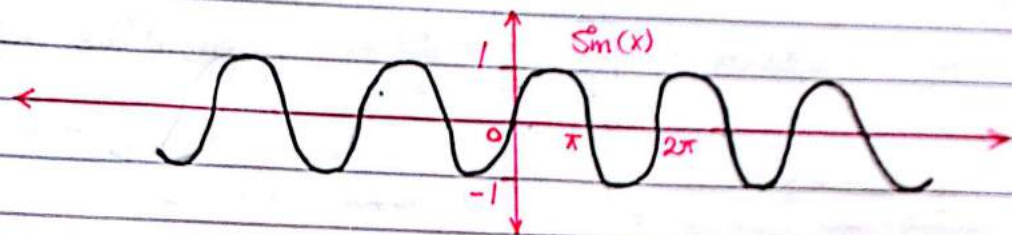
### \* 11.1 : Fourier Series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

\* A function  $f(x)$  is periodic function, if  $f(x+p) = f(x)$  where  $p$  is some positive number,  $p$  is called a period.

$$\rightarrow \sin(x+2\pi) = \sin x$$

$\cos x, \tan x, \dots$  Is the smaller period.



$$* \sin(x+4\pi)$$

(period)

$$\therefore \sin(x+n\pi)$$
$$x \rightarrow \int_{-L}^L \frac{\cos m\pi x}{L} \cdot \frac{\cos n\pi x}{L} dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$x \rightarrow \int_{-L}^L \frac{\sin m\pi x}{L} \cdot \frac{\sin n\pi x}{L} dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$x \rightarrow \int_{-L}^L \frac{\cos m\pi x}{L} \cdot \frac{\sin n\pi x}{L} dx = 0 \rightarrow \int \cos Ax \sin Bx dx$$



\* 11.1 + 11.2 : Fourier series (ex. 12-16), (8-17, 23-29).

\* Consider a system:

$$1. \cos \frac{\pi x}{L}, \sin \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \sin \frac{2\pi x}{L}, \dots, \cos \frac{n\pi x}{L}, \sin \frac{n\pi x}{L}, \dots$$

→ where  $L$  is some number.

all these functions are periodic with period  $2L$ .

This system is orthogonal, that is ---

$$* \int_{-L}^L \sin \frac{m\pi x}{L} \cdot \sin \frac{n\pi x}{L} dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \quad (m, n) \text{ are integers.}$$

$$* \int_{-L}^L \cos \frac{m\pi x}{L} \cdot \cos \frac{n\pi x}{L} dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

$$* \int_{-L}^L \sin \frac{m\pi x}{L} \cdot \cos \frac{n\pi x}{L} dx = 0 \text{ for all } (m, n)$$

$\sin u \sin v = \frac{1}{2} [\cos(u-v) - \cos(u+v)]$   
 $\cos u \cos v = \frac{1}{2} [\cos(u-v) + \cos(u+v)]$   
 $\sin u \cos v = \frac{1}{2} [\sin(u+v) + \sin(u-v)]$

\*\* proof: ( $m \neq n$ )

$$\int_{-L}^L \sin \frac{m\pi x}{L} \cdot \sin \frac{n\pi x}{L} dx =$$

→ Use  $\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$

$$= \frac{1}{2} \int_{-L}^L (\cos \frac{(m-n)\pi x}{L} - \cos \frac{(m+n)\pi x}{L}) dx$$

$$= \frac{1}{2} \left[ \frac{L}{(m-n)\pi} \sin \frac{(m-n)\pi x}{L} - \frac{L}{(m+n)\pi} \sin \frac{(m+n)\pi x}{L} \right] \Bigg|_{-L}^L$$



→ If 'm=n' ;

$$\int_{-L}^L \left( \frac{\sin \frac{n\pi x}{L}}{L} \right)^2 dx = \frac{1}{2} \int_{-L}^L (1 - \cos \frac{2n\pi x}{L}) dx$$

$$= \frac{1}{2} (2L) = L$$

\* Def. The series of the form:

$$\left[ a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right];$$

is called a Fourier series for a function  $f(x)$

"Periodic with period (2L)"

\*\* Find coefficients ( $a_0, a_n, b_n$ )

"Multiply the Fourier series by  $\cos \frac{m\pi x}{L}$  and integrate from (-L) to (L)"

$$\int_{-L}^L f(x) \cdot \cos \frac{m\pi x}{L} dx = \left( a_0 \int_{-L}^L \cos \frac{m\pi x}{L} dx + \sum_{n=1}^{\infty} a_n \int_{-L}^L \cos \frac{n\pi x}{L} \cdot \cos \frac{m\pi x}{L} dx + b_n \int_{-L}^L \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx \right)$$

$$** \int_{-L}^L f(x) \cdot \cos \frac{n\pi x}{L} dx = a_n \cdot L \Rightarrow \left( a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx \right) **$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \cdot \sin \frac{n\pi x}{L} dx$$

$$\left( a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx \right)$$

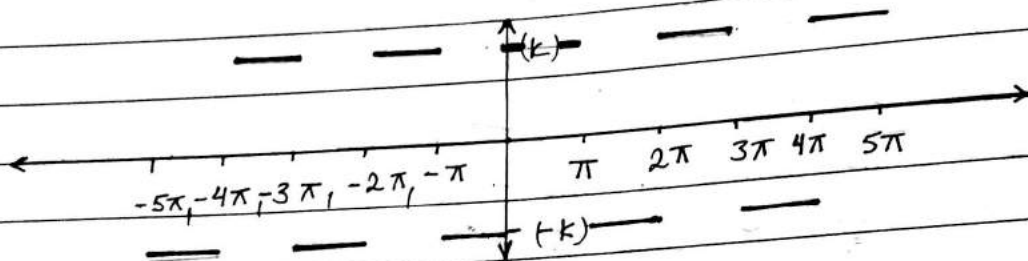
"Average"

\* Ex. 1: Find the Fourier series of the function:

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$$

and  $f(x+2\pi) = f(x)$   
(period is  $2\pi = 2L$ )  
 $L = \pi$

$k$  is some constant.



$$\begin{aligned} * a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \left[ \int_{-\pi}^0 (-k) dx + \int_0^{\pi} k dx \right] = \\ &= \frac{1}{2\pi} [-k(0+\pi) + k(\pi+0)] = 0 \end{aligned}$$

$$\begin{aligned} * a_n &= \frac{1}{\pi} \left[ \int_{-\pi}^0 (-k) \cos nx dx + \int_0^{\pi} k \cos nx dx \right] = \\ &= \frac{k}{\pi} \left[ -\frac{\sin nx}{n} \Big|_{-\pi}^0 + \frac{\sin nx}{n} \Big|_0^{\pi} \right] = 0 \end{aligned}$$

$$\begin{aligned} * b_n &= \frac{1}{\pi} \left[ \int_{-\pi}^0 (-k) \sin nx dx + \int_0^{\pi} k \sin nx dx \right] = \frac{k}{\pi} \left[ \frac{\cos nx}{n} \Big|_{-\pi}^0 - \left[ \frac{\cos nx}{n} \right] \Big|_0^{\pi} \right] \end{aligned}$$

$$= \frac{k}{\pi n} [1 - \cos n\pi - \cos n\pi + 1] = \frac{k}{\pi n} [2 - 2\cos n\pi]$$

$$= \frac{2k}{\pi n} [1 - (-1)^n] = \frac{2k}{\pi n} \begin{cases} 0, & \text{if } n \text{ is even "2n"} \\ 2, & \text{if } n \text{ is odd "2n-1"} \end{cases}$$

$$\therefore f(x) \sim \sum_{n=1}^{\infty} \frac{4k}{(2n-1)\pi} \sin (2n-1)x.$$

$$\leadsto (-1)^n = \cos n\pi$$



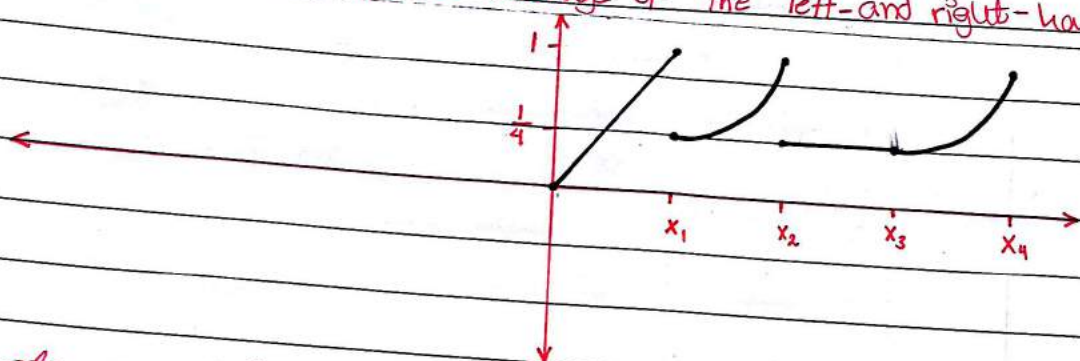
## # Convergence of Fourier series:

### \*\* Theorem:

Let  $f(x)$  be piecewise continuous in " $-L \leq x \leq L$ " and periodic with period " $2L$ ". Let  $f(x)$  have left hand and Right hand limits at each point of  $[-L, L]$ . Then the Fourier series for  $f(x)$  converges.

It sum is  $f(x)$ , except points where  $f(x)$  is discontinuous.

There the sum is the average of the left- and right-hand limits.



$\sim \rightarrow$  at  $x = x_1$  "F.S" conv.  $\left( \frac{1 + \frac{1}{4}}{2} \right)$

# Ex. 2: Given  $f(x) = \begin{cases} 0 & , -\pi < x < 0 \\ x^2 & , 0 \leq x \leq \pi \end{cases}$  ,  $f(x+2\pi) = f(x)$   
 $2\pi = 2L$   
 $\rightarrow " \pi = L "$

a) Find the Fourier series:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$\sim \rightarrow a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2\pi} \left[ \int_{-\pi}^0 0 dx + \int_0^{\pi} x^2 dx \right] = \frac{1}{2\pi} \cdot \frac{x^3}{3} \Big|_0^{\pi} = \frac{\pi^2}{6}$$

$$\sim \rightarrow a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} x^2 \cos nx dx$$

$$= \frac{1}{\pi} \left[ \frac{x^2}{n} \sin nx + \frac{2x}{n^2} \cos nx - \frac{2}{n^3} \sin nx \right] \Big|_0^{\pi}$$

$$= \frac{2\pi}{\pi n^2} \cos n\pi = \frac{2(-1)^n}{n^2}$$

\* Derivative \*

\* Integral \*

|       |   |                          |
|-------|---|--------------------------|
| $x^2$ | + | $\cos nx$                |
| $2x$  | - | $\frac{1}{n} \sin nx$    |
| $2$   | + | $\frac{-1}{n^2} \cos nx$ |
| $0$   | - | $\frac{-1}{n^3} \sin nx$ |

$$\leadsto b_n = \frac{1}{\pi} \int_0^{\pi} x^2 \sin nx \, dx = \frac{1}{\pi} \left[ \frac{-x^2}{n} \cos nx + \frac{2x}{n^2} \sin nx + \frac{2}{n^3} \cos nx \right] \Big|_0^{\pi}$$

(Tabular Integration)

\* Derivative \*

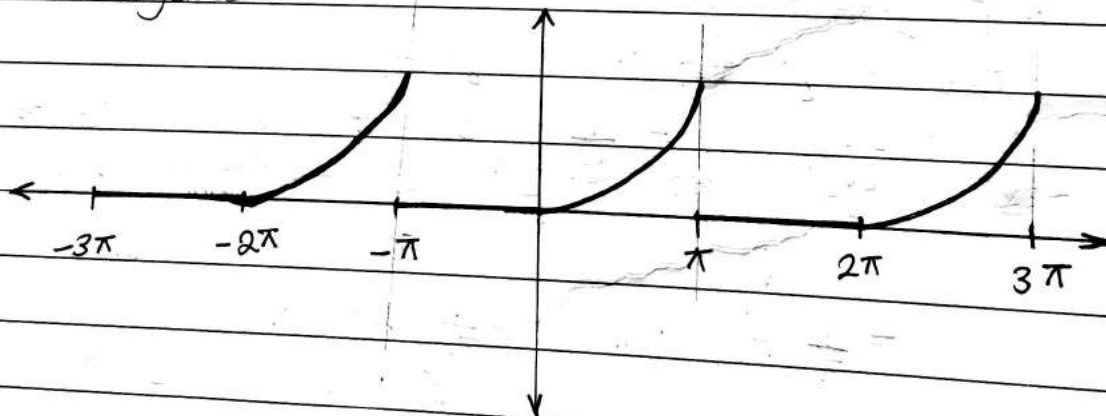
\* Integral \*

|       |   |                          |
|-------|---|--------------------------|
| $x^2$ | + | $\sin nx$                |
| $2x$  | - | $\frac{-1}{n} \cos nx$   |
| $2$   | + | $\frac{-1}{n^2} \sin nx$ |
| $0$   | - | $\frac{1}{n^3} \cos nx$  |

$$b_n = \frac{1}{\pi} \left[ -\frac{\pi^2}{n} \cos n\pi + \frac{2}{n^3} (\cos n\pi - 1) \right] = \frac{\pi}{n} (-1)^{n+1} + \frac{2}{\pi n^3} ((-1)^n - 1)$$

$$** f(x) \sim \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2}{n^2} \cos nx + \left( \frac{(-1)^{n+1} \pi}{n} + \frac{2}{\pi n^3} ((-1)^n - 1) \right) \sin nx$$

b) Convergence!



$\leadsto$  At  $(x=1)$  the F.S converges to  $f(1)=1$

$(x=-2)$

=

$$f(-2) = 0$$

$(x=\pi)$

=

$$\frac{f(\pi^-) + f(\pi^+)}{2}$$

2



$$x \quad \frac{f(\pi^-) + f(\pi^+)}{2} = \frac{\pi^2 + 0}{2} = \frac{\pi^2}{2}$$

$\leadsto f(\pi^-)$  is the left-hand limit.

$\leadsto f(\pi^+)$  is the right-hand limit.

$(x = -\pi)$  the F.S. converges to  $\frac{f(\pi^-) + f(\pi^+)}{2} = \frac{\pi^2 + 0}{2} = \frac{\pi^2}{2}$

c) Show that  $\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

$\leadsto \sin nx = 0$

$\leadsto \cos nx = 1$

At  $x=0$  the F.S. converges to "0"

$\sin nx = 0$

$\cos nx = 1$

$$0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}$$

$$-\frac{\pi^2}{6} = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{12} = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$



\* Given  $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x^2 & 0 \leq x \leq \pi \end{cases}$  (فئة جرد)

$\leadsto$  use  $f(x) \sim \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} \cos(nx) + \left[ \frac{\pi(-1)^{n+1}}{n} + \frac{2}{\pi n^2} ((-1)^n - 1) \sin(nx) \right]$   
to find the sum  $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sin(nx) = 0$  ,  $(x = \pi) \leadsto \sin(n\pi) = 0$

$(-1)^n \cos nx = 1$

$\cos(n\pi) = (-1)^n$

$(-1)^n * (-1)^n = (-1)^{2n} = (1)^n = 1$

$\leadsto$  At  $x = \pi$  ; the f.s Converges to  $\frac{f(\pi^-) + f(\pi^+)}{2} = \frac{\pi^2 + 0}{2} = \left(\frac{\pi^2}{2}\right)$

$\frac{\pi^2}{2} = \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$

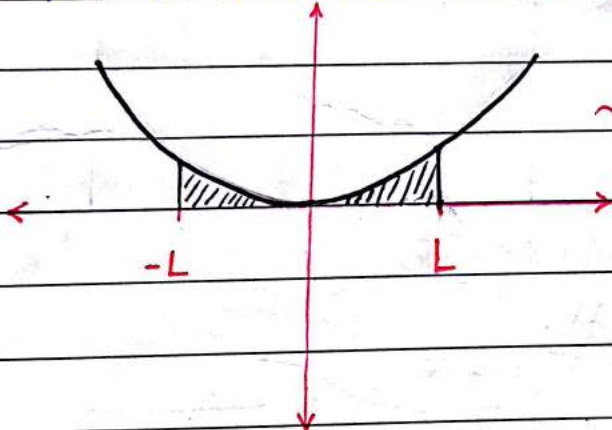
$\frac{\pi^2}{3} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

\* ~~~~~ \*

# Even and Odd functions:

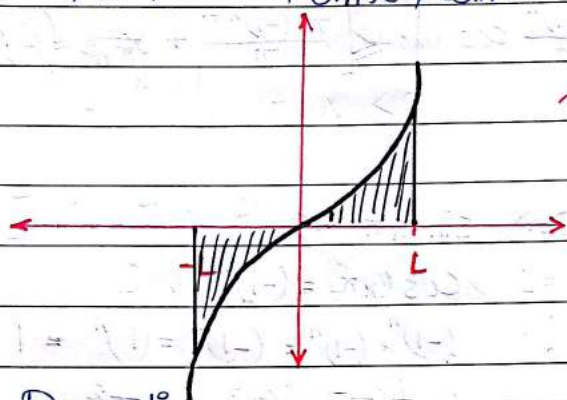
$\leadsto f(x)$  is even  $\Leftrightarrow f(-x) = f(x)$



$\leadsto$  (Symmetry with respect to the y-axis)

$5, x^2, x^4, \dots, \cos x, |x|$

$\leadsto f(x)$  is odd  $\Leftrightarrow f(-x) = -f(x)$   
 $x, x^3, \dots, \sin x, \tan x$



$\leadsto$  Symmetry with respect to the origin.

### \*\* Properties \*\*

1.  $O.f \pm O.f = O.f$

$O.f(x, \div) O.f = E.f \leadsto (x \cdot x^3 = x^4), \left(\frac{x^3}{x} = x^2\right)$

2.  $E.f \pm E.f = E.f$

$E.f(x, \div) E.f = E.f \leadsto (x^2 \cdot x^4 = x^6), \left(-\frac{x^4}{x^2} = -x^2\right)$

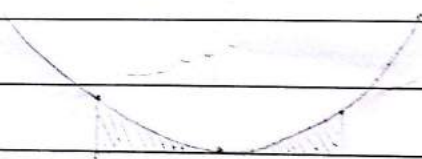
3.  $O.f \pm E.f = \text{"neither even nor odd"}$

$O.f(x, \div) E.f = O.f$

4.  $\int_{-L}^L f(x) \cdot dx$ , where  $f(x)$  is "even"

$\rightarrow = 2 \int_0^L f(x) \cdot dx$

5.  $\int_{-L}^L f(x) \cdot dx = 0$ , where  $f(x)$  is odd.





\*\* let  $f(x)$  be piecewise continuous and even periodic function on  $(-L \leq x \leq L)$ , Find the (F.S)

$$* a_0 = \frac{1}{2L} \int_{-L}^L \underbrace{f(x)}_{(E.f)} dx = \frac{1}{L} \int_0^L f(x) dx$$

$$* a_n = \frac{1}{L} \int_{-L}^L \underbrace{f(x)}_{(E.f)} \cdot \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{(E.f)} dx = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$* b_n = \frac{1}{L} \int_{-L}^L \underbrace{f(x)}_{(E.f)} \cdot \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{(O.f)} dx = 0$$

$$\leadsto f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) \quad \text{Fourier cosine series}$$

# let  $f(x)$  be piecewise continuous and odd periodic function on  $(-L \leq x \leq L)$

$$* a_0 = 0$$

$$* a_n = 0$$

$$* b_n = \frac{1}{L} \int_{-L}^L \underbrace{f(x)}_{(O.f)} \cdot \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{(O.f)} dx = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

(E.f)

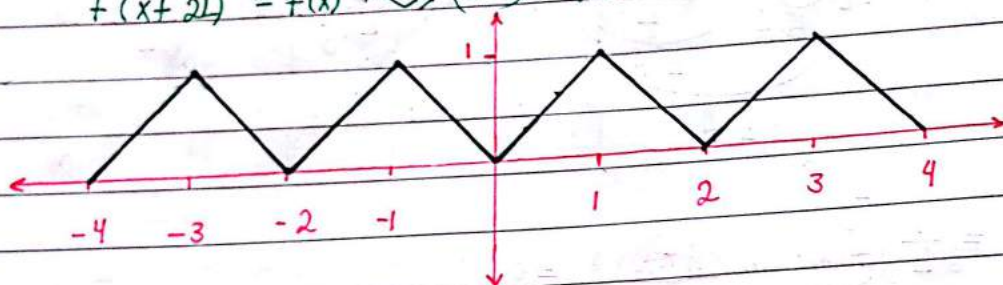


- \*  $n \sim 2n$  if "Even"
  - \*  $n \sim (2n-1)$  if "odd"
- (Replacement)

\*  $f(x) \sim \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right)$  Fourier Sine Series

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

# Ex. 1: Find the Fourier series for  $f(x) = |x|$ ,  $(f(x+2) = f(x))$ .  
 $f(x+2) = f(x) \sim (L=1)$  "Period".



$f(x) = |x|$  is even  $\sim$  use the Fourier cosine series.

\*  $a_0 = \frac{1}{1} \int_0^1 x dx = \frac{1}{2}$

\*  $a_n = \frac{2}{1} \int_0^1 x \cos(n\pi x) dx = 2 \left[ \frac{x}{n\pi} \sin(n\pi x) + \frac{1}{(n\pi)^2} \cos(n\pi x) \right]$

(\*  $b_n = 0$ )  $= \frac{2}{n^2\pi^2} [\cos n\pi - 1]$

$$\begin{pmatrix} x & + & \cos(n\pi x) \\ 1 & - & \frac{1}{n\pi} \sin(n\pi x) \\ 0 & & \frac{-1}{(n\pi)^2} \cos(n\pi x) \end{pmatrix}$$

$$= \frac{2}{n^2\pi^2} [(-1)^n - 1] = \frac{2}{n^2\pi^2} \begin{cases} -2, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} = \frac{-4}{(2n-1)^2\pi^2}$$

$$= \frac{f(1^-) + f(1^+)}{2} = \frac{1+1}{2} = 1, \quad f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right)$$

$\rightarrow$  at  $x=1$  the F.C.S. conv. to  $\uparrow$

at  $x=0$  the F.C.S. conv. to  $f(0) = 0$

at  $x=\frac{1}{2}$  the F.C.S. conv. to  $f(\frac{1}{2}) = \frac{1}{2}$

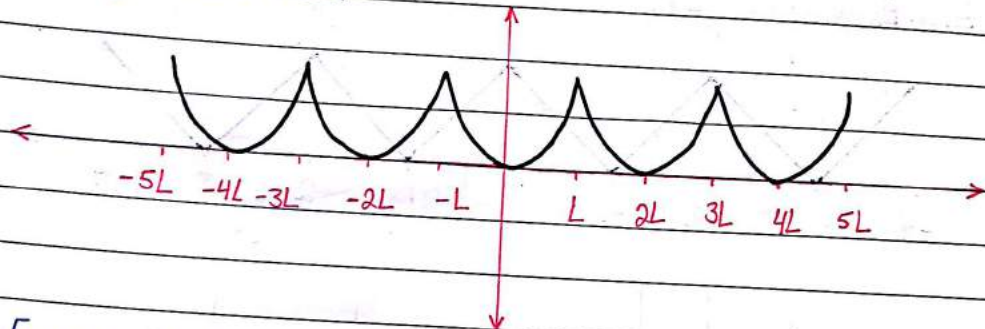
#  $f(x) \sim \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)\pi x)$



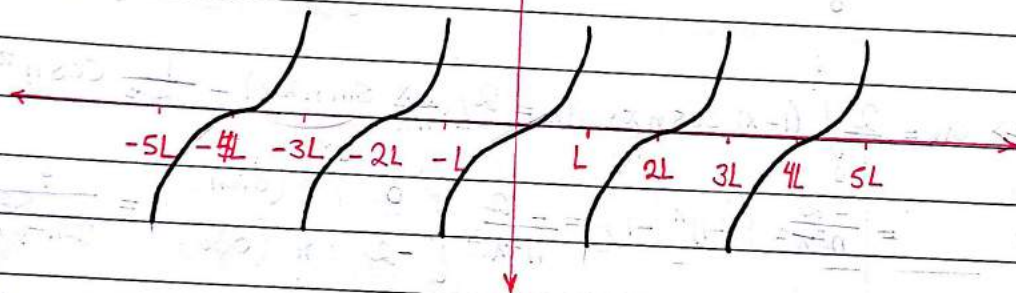
- \* If  $f(x)$  is odd periodic function  $\leadsto$  Fourier sine series.
- \* If  $f(x)$  is even periodic function  $\leadsto$  Fourier cosine series.

\* Half-range expansion:

If  $f(x)$  given on  $0 \leq x \leq L$



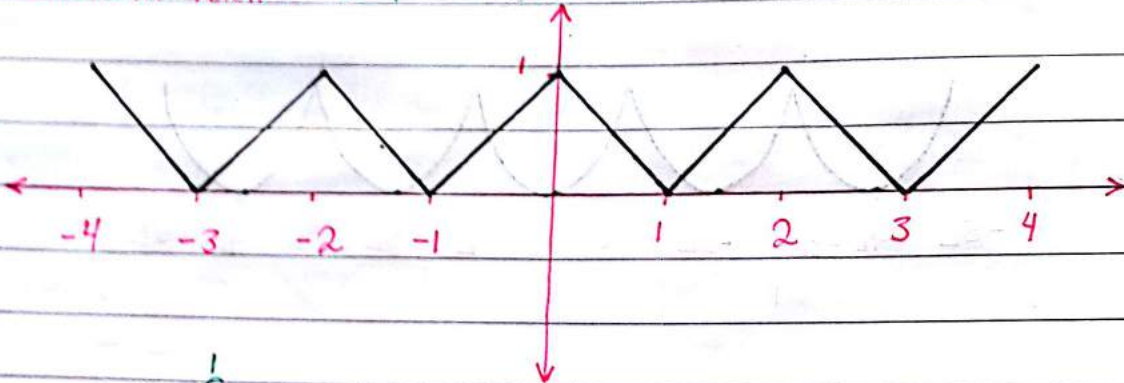
\*\* Use Fourier cosine series "period  $2L$ ", "even periodic extension of  $f(x)$ "



\*\* odd periodic extension of  $f(x) \leadsto$  use Fourier sine series "period  $2L$ "

Ex: Let  $f(x) = 1-x$ ,  $0 < x \leq 1$  ; sketch the graphs of the even and odd periodic extensions and find the Fourier series:

1. Even Extension:  $y = 1-x$ ,  $L=1$ ,  $2L=2$ .



$$\leadsto a_0 = \frac{1}{1} \int_0^1 (1-x) dx = x - \frac{x^2}{2} \Big|_0^1 = \left(\frac{1}{2}\right)$$

$$\begin{aligned} \leadsto a_n &= \frac{2}{1} \int_0^1 (1-x) \cos n\pi x dx = 2 \left[ \left( \frac{1-x}{n\pi} \sin n\pi x \right) - \frac{1}{n^2\pi^2} \cos n\pi x \right] \Big|_0^1 \\ &= \frac{-2}{n^2\pi^2} \left( (-1)^n - 1 \right) = \frac{-2}{n^2\pi^2} \begin{cases} 0 & n \text{ (even)} \\ -2 & n \text{ (odd)} \end{cases} = \frac{4}{(2n-1)^2} \end{aligned}$$

\* Derivative \*

$$\begin{array}{r} 1-x \\ -1 \\ 0 \end{array} \quad \begin{array}{c} + \\ - \end{array}$$

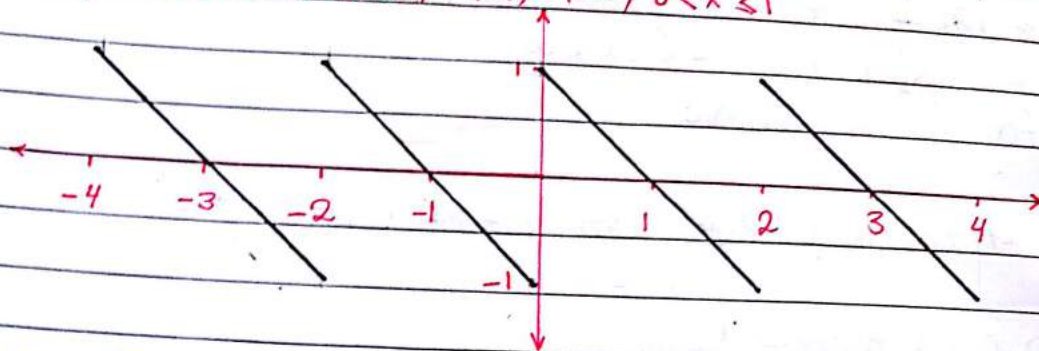
\* Integral \*

$$\begin{array}{l} \cos n\pi x \\ \frac{1}{n\pi} \cdot \sin n\pi x \\ -\frac{1}{n^2\pi^2} \cdot \cos n\pi x \end{array}$$

$$** f(x) \sim \frac{1}{2} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos (2n-1)\pi x.$$



2. Odd periodic Extension:  $f(x) = 1-x, 0 < x \leq 1$



# Use the Fourier sine series:

$$b_n = \frac{2}{1} \int_0^1 (1-x) \sin n\pi x = 2 \left[ -\frac{(x-1)}{n\pi} \cos n\pi x - \frac{1}{(n\pi)^2} \sin n\pi x \right] \Big|_0^1 = \frac{-2}{n\pi}$$

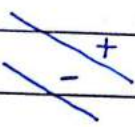
\* Derivative \*

\* Integral \*

$$1-x$$

$$-1$$

$$0$$



$$\sin n\pi x$$

$$\frac{-1}{n\pi} \cos n\pi x$$

$$\frac{-1}{(n\pi)^2} \cdot \sin n\pi x$$

$$\leadsto f(x) \sim \sum_{n=1}^{\infty} \frac{-2}{n\pi} \cdot \sin n\pi x \quad \text{"odd periodic extension"}$$

\* at  $x=0.5$  the Fourier sine series to  $(\frac{1}{2})$

\* at  $x=0$  the Fourier sine series to  $\frac{(-1+1)}{2} = (0)$

\* at  $x=2$  " " " " to  $(0)$

### # 11.7: Fourier Integrals: (ex 1-12, 16-20)

\*\*  $f(x)$  given for  $-\infty < x < \infty$

$f(x)$  is not periodic

~> If  $f(x)$  is periodic with  $p = 2L$ , let " $L \rightarrow \infty$ "

# Def.:  $f$  representation:

$$f(x) \sim \int_0^{\infty} (A(w) \cos wx + B(w) \sin wx) dw \quad \text{--- (1)}$$

Where:

$$* A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cdot \cos(wv) dv \quad \text{--- (2)}$$

$$* B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cdot \sin(wv) dv \quad \text{--- (3)}$$

is called a Fourier integral.

# Theorem:

\*\* If  $f(x)$  is piecewise continuous in every finite interval and has a right-hand derivative and left-hand derivative at every point, and if  $\int_{-\infty}^{\infty} |f(x)| dx$  exists

(absolutely integrable) then  $f(x)$  can be represented by the Fourier integral.

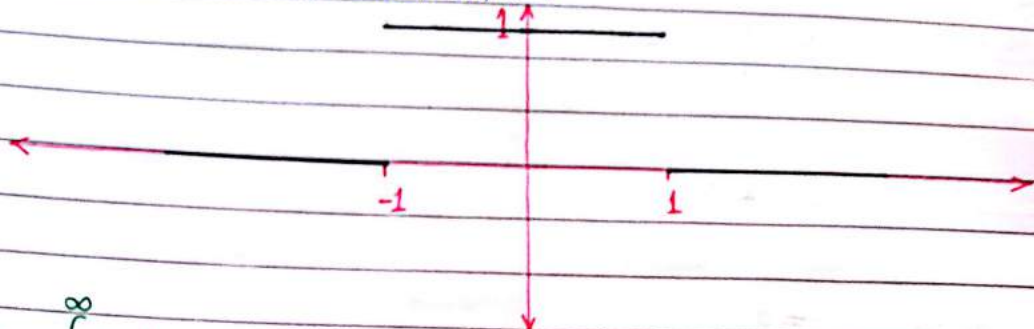
~> If  $f(x)$  is continuous at  $x = x_0$ , the F.S.I =  $f(x_0)$

~> If  $f(x)$  is discontinuous at  $x = x_0$ , the F.I =  $\frac{f(x_0^-) + f(x_0^+)}{2}$ .



\* Ex. 1: Find the Fourier Integral representation for function:  
 $f(x) = \int_1^1$ , when  $|x| < 1 \rightsquigarrow (-1 < x < 1)$

$$f(x) = \begin{cases} 1, & \text{when } |x| < 1 \rightarrow (-1 < x < 1) \\ 0, & \text{when } |x| > 1 \end{cases}$$



$$f(x) \sim \int_{-\infty}^{\infty} A(\omega) \cdot \cos \omega x + B(\omega) \sin \omega x \cdot d\omega$$

$$* A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos \omega v \, dv = \frac{1}{\pi} \int_{-1}^1 \cos \omega v \, dv$$

$$= \frac{2}{\pi} \int_0^1 \cos wv \, dv = \frac{2}{\pi} \left. \frac{\sin wv}{w} \right|_0^1 = \frac{2}{\pi} \frac{\sin w}{w}$$

$$* B(\omega) = \frac{1}{\pi} \int_{-1}^0 1 \sin \omega v dv = 0$$

$$\Rightarrow f(x) \sim \frac{2}{\pi} \int_0^{\infty} \frac{\sin w}{w} \cdot \cos wx \, dw \Rightarrow \frac{2}{\pi} \int_0^{\infty} \frac{\sin w}{w} \cdot \cos wx \, dw = \begin{cases} 1, & \text{if } |x| < 1 \\ \frac{1}{2}, & \text{if } x = \pm 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

\*\* If  $x=1$  then the F.I. converges to  $\frac{f(1^-) + f(1^+)}{2} = \frac{1+0}{2} = \frac{1}{2}$   
 If  $x=-1$

\* If  $x = -1$  // // // //  $= \frac{1}{2}$

\*\*\* If  $x=10$  " " " "  $= \frac{2}{f(10)} = 0$

$\therefore \text{If } x=0 \quad f(0) = f(0) = 1$

$$\# \int_0^{\infty} \frac{\sin w \cdot \cos wx}{w} \cdot dw = \begin{cases} \frac{\pi}{2} & , \text{ if } |x| < 1 \\ \frac{\pi}{4} & , \text{ if } x = \pm 1 \\ 0 & , \text{ if } |x| > 1 \end{cases}$$



\* Evaluate  $\int_0^{\infty} \frac{\sin w}{w} dw = \frac{\pi}{2}$

$$\int_0^{\infty} \frac{\sin w \cos w}{w} dw = \frac{\pi}{4} \quad (x=1)$$

$$\int_0^{\infty} \frac{\sin w \cdot \cos 2w}{w} dw = 0$$

$$\int_0^{\infty} \frac{\sin x \cdot \cos 3x}{x} dx = 0$$

\* If  $f(x)$  is odd, then we use the Fourier series sine integral  
 $f(x) \sim \int_0^{\infty} B(w) \sin wx \, dw$  / where

$$B(w) = \frac{2}{\pi} \int_0^{\infty} f(v) \cdot \sin wv \, dv$$

\* If  $f(x)$  is even, then the Fourier cosine integral.

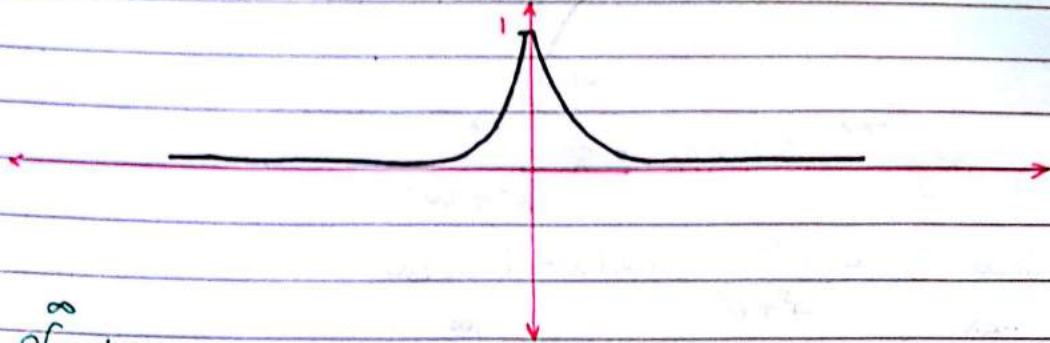
$$f(x) \sim \int_0^{\infty} A(w) \cos wx \, dw$$

$$A(w) = \frac{2}{\pi} \int_0^{\infty} f(v) \cos wv \, dv$$

$\leadsto$  If  $f(x)$  is given on  $(0, +\infty)$   $\nearrow$  F.C.I  
 $\searrow$  F.S.I

\* Ex. 2: Given  $f(x) = e^{-kx}$ ,  $0 < x < \infty$ ,  $k > 0$  is a constant.  
Find: a) the Fourier cosine integral.  
b) the Fourier sine integral.

\* a \*



$$A(\omega) = \frac{2}{\pi} \int_0^{\infty} e^{-kx} \cos \omega x \, dx$$

\* Use  $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$

$$\rightarrow a = -k, b = \omega$$

$$= \frac{2}{\pi} \left[ \frac{e^{-kx}}{k^2 + \omega^2} (-k \cos \omega x + \omega \sin \omega x) \right] \Big|_0^{\infty}$$

\*\* If  $x \rightarrow \infty \rightarrow (e^{-\infty} \rightarrow 0)$ ;  $\cos(\infty)$ ,  $\sin(\infty)$  are bounded functions.  
 $\therefore (0 \times \text{bounded}) = 0$

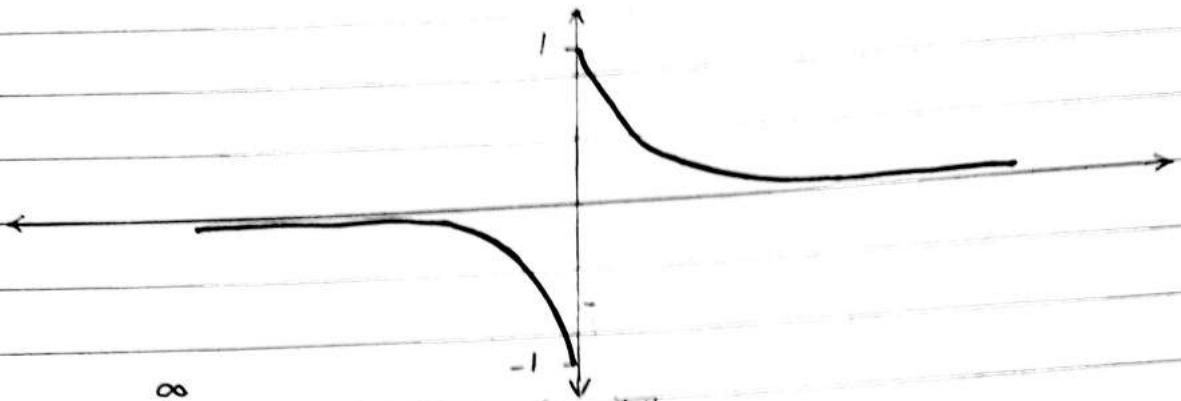
$$= \frac{2}{\pi} \cdot \frac{1}{k^2 + \omega^2} [0 - 1(-k)] = \frac{2k}{\pi(k^2 + \omega^2)}$$

$$f(x) \sim \int_0^{\infty} A(\omega) \cos \omega x \, d\omega = \frac{2k}{\pi} \int_0^{\infty} \frac{\cos \omega x}{k^2 + \omega^2} d\omega$$

$$\int_0^{\infty} \frac{\cos \omega x}{k^2 + \omega^2} d\omega = \frac{\pi}{2k} e^{-kx}$$

\* Evaluate  $\int_0^{\infty} \frac{1}{1 + \omega^2} d\omega = \frac{\pi}{2}$  ( $x=0$ ,  $k=1$ )

\*b\*



$$*B(\omega) = \frac{2}{\pi} \int_0^{\infty} e^{-kx} \sin \omega x \, dx = \frac{2}{\pi} \frac{e^{-kx}}{k^2 + \omega^2}$$

$$= \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$= \frac{2}{\pi} \frac{e^{-kx}}{k^2 + \omega^2} (-k \sin \omega x - \omega \cos \omega x) \Big|_0^{\infty} = \frac{2}{\pi} \cdot \frac{1}{k^2 + \omega^2} (-(-\omega))$$

$$= \frac{2\omega}{\pi (k^2 + \omega^2)}$$

$$\leadsto f(x) \sim \int_0^{\infty} \frac{2\omega}{\pi (k^2 + \omega^2)} \cdot \sin \omega x \, d\omega$$

$$= \int_0^{\infty} \frac{\omega}{k^2 + \omega^2} \cdot \sin \omega x \, d\omega = \frac{\pi}{2} e^{-kx}, \quad x > 0$$



Question 5:

Show that the integral represents the function:

$$\int_0^{\infty} \frac{\sin w - w \cos w}{w^2} \cdot \sin wx \, dw = \begin{cases} \frac{\pi}{2}x & \text{if } 0 < x < 1 \\ \frac{\pi}{4} & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$f(x) \sim \int_0^{\infty} \underbrace{A(w)}_{\text{"0"}} \cos wx + \underbrace{B(w)}_{\text{"B"}} \sin wx \, dw$$

→ Fourier sine integral:

$$f(x) = \begin{cases} \frac{\pi}{2}x & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$

→ Find the Fourier sine integral:

$$B(w) = \frac{2}{\pi} \int_0^{\infty} f(x) \cdot \sin wx \, dx = \frac{2}{\pi} \int_0^1 \frac{\pi}{2} x \sin wx \, dx$$

\* Derivative \*      \* Integral \*

|   |   |                          |
|---|---|--------------------------|
| x | + | $\sin wx$                |
| 1 | - | $\frac{1}{w} \cos wx$    |
| 0 |   | $\frac{-1}{w^2} \sin wx$ |

$$= -\frac{x}{w} \cos wx + \frac{1}{w^2} \sin wx \Big|_0^1 = -\frac{\cos w}{w} + \frac{1}{w^2} \sin w$$

$$** f(x) \sim \int_0^{\infty} \frac{\sin w - w \cos w}{w^2} \cdot \sin wx \, dw$$

→ At  $x=1$  the F. sine integral converges to  $\frac{f(1^-) + f(1^+)}{2} = \frac{\frac{\pi}{2} + 0}{2}$

$$\sim \int_0^{\infty} \frac{\sin w - w \cos w}{w^2} \cdot \sin wx \, dw = \begin{cases} \frac{\pi}{2}x & \text{if } 0 < x < 1 \\ \frac{\pi}{4} & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases} = \left(\frac{\pi}{4}\right)$$

\* Question 1: Show that the integral represents the function:

$$\int_0^{\infty} \frac{\cos wx + w \sin wx}{1+w^2} dw = \begin{cases} 0 & \text{if } x < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \\ \frac{\pi}{2e^x} & \text{if } x > 0 \end{cases}$$

$$f(x) \sim \int_0^{\infty} A(w) \cos wx + B(w) \sin wx \cdot dw$$

$$f(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{\pi e^{-x}}, & x > 0 \end{cases}, \text{ Find the Fourier integral of } f(x).$$

\* 11.8: Fourier cosine and sine transforms: ex (1-5, 9-13).

$$\begin{array}{c} f(x) \xrightarrow{\quad} \hat{f}_c(w) \\ \quad \quad \quad \hat{f}_s(w) \end{array}$$

$$f(x) = \int_0^{\infty} B(w) \sin wx \, dw \quad \text{and} \quad B(w) = \frac{2}{\pi} \int_0^{\infty} f(x) \sin wx \, dx$$

$\left( \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \right)$

\* The Fourier sine transform:

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cdot \sin wx \cdot dx$$

$$\begin{array}{c} f(x) \xrightarrow{\quad F_s \quad} \hat{f}_s(w) \\ \hat{f}_s(w) \xrightarrow{\quad F_s^{-1} \quad} f(x) \end{array}, \quad F_s[f] = \hat{f}_s(w)$$

\* The inverse Fourier sine transform:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(w) \sin wx \, dw.$$



\* The Fourier cosine transform:

$$\hat{f}_c(\omega) = F_c[f] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx$$

\* The inverse Fourier transform:

$$f(x) = F_c^{-1}[\hat{f}_c] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(\omega) \cos \omega x \, d\omega$$

\* Note:

If  $f(x)$  is a piecewise continuous function on  $(0, \infty)$  and  $f(x)$  goes to zero as  $x \rightarrow \infty$  then the Fourier transform exists.

\* Ex. 1: Given  $f(x) = \begin{cases} x, & \text{if } 0 < x < 2 \\ 0, & \text{if } x > 2 \end{cases}$ , Find

a) the F. cosine transform.

b) the F. sine transform.

$$a) \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^2 x \cos \omega x \, dx$$

\* Derivative \*

\* Integral \*

|     |     |                                     |
|-----|-----|-------------------------------------|
| $x$ | $+$ | $\cos \omega x$                     |
| $1$ | $-$ | $\frac{1}{\omega} \sin \omega x$    |
| $0$ |     | $-\frac{1}{\omega^2} \cos \omega x$ |

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{x}{\omega} \sin \omega x + \frac{1}{\omega^2} \cos \omega x \right] \Big|_0^2$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{2}{\omega} \sin 2\omega + \frac{1}{\omega^2} (\cos 2\omega - 1) \right]$$



$$b) f_3(w) = F_3\{f\} = \sqrt{\frac{2}{\pi}} \int_0^2 x \sin wx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{-x}{w} \cos wx + \frac{1}{w^2} \sin wx \right] \Big|_0^2$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{-2}{w} \cos 2w + \frac{1}{w^2} \sin 2w \right]$$

\* Derivative \*

\* Integral \*

|   |   |                          |
|---|---|--------------------------|
| x | + | $\sin wx$                |
| 1 | - | $\frac{-1}{w} \cos wx$   |
| 0 |   | $\frac{-1}{w^2} \sin wx$ |



$$* \hat{f}_c(\omega) = \int_{-\infty}^{\infty} f(x) \cdot \cos \omega x \cdot dx$$

$$* \hat{f}_s(\omega) = \int_{-\infty}^{\infty} f(x) \cdot \sin \omega x \cdot dx$$

\* If  $f(x)$  is a piecewise continuous and absolutely integrable, then the Fourier transform exists!

\* Properties:

1. Fourier transforms are linear:

$$\leadsto \hat{F}_c[af(x) + bg(x)] = a\hat{F}_c[f(x)] + b\hat{F}_c[g(x)] \text{ , where } a, b \text{ are constants.}$$

$$\leadsto \hat{F}_s[af(x) + bg(x)] = a\hat{F}_s[f(x)] + b\hat{F}_s[g(x)]$$

$$\rightarrow f(x) = \begin{cases} 3x & , 0 < x < 2 \\ 0 & , x > 2 \end{cases}$$

2. Transforms of derivatives:

$$\leadsto \hat{F}_c[f'] = \int_{-\infty}^{\infty} f'(x) \cos \omega x \cdot dx = \left[ f(x) \cos \omega x \right]_{-\infty}^{\infty} + \omega \int_{-\infty}^{\infty} f(x) \sin \omega x \cdot dx$$

$$\rightarrow u = \cos \omega x$$

$$, dv = f'(x) \cdot dx$$

$$du = -\omega \sin \omega x$$

$$, v = f(x)$$

$$f(\infty) \cos \infty$$

$$\downarrow$$

bounded

$$f(x) \rightarrow 0 \text{ , as } x \rightarrow \infty$$

$$= \left[ \frac{2}{\pi} (0 - f(0)) + \omega \hat{F}_s[f] \right] , \hat{F}_c[f'(x)] = \omega \hat{F}_s[f] - \sqrt{\frac{2}{\pi}} f(0)$$



$$** \mathcal{F}_S \{f'\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f'(x) \sin \omega x dx = \sqrt{\frac{2}{\pi}} f(x) \sin \omega x \Big|_0^{\infty} - \omega \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x dx$$

$$\sim \rightarrow u = \sin \omega x \quad | \quad dv = f'(x) dx$$

$$du = \omega \cos \omega x \quad | \quad v = f(x)$$

$$f(\infty) \sin \infty \sim \rightarrow \sqrt{\frac{2}{\pi}} (0-0) - \omega \mathcal{F}_C \{f\}$$

$$\mathcal{F}_S \{f'\} = -\omega \mathcal{F}_C \{f\}$$

$$** \mathcal{F}_C \{f''\} = \mathcal{F}_C \{(f')'\} = \omega \mathcal{F}_S \{f'\} - \sqrt{\frac{2}{\pi}} f'(0) = -\omega^2 \mathcal{F}_C \{f\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$** \mathcal{F}_S \{f''\} = \mathcal{F}_C \{(f')'\} = -\omega \mathcal{F}_C \{f'\} = -\omega^2 \mathcal{F}_S \{f\} + \sqrt{\frac{2}{\pi}} \omega f(0)$$

\* Ex: Given  $f(x) = e^{-ax}$ , where  $a > 0$ , is a constant  
Find  $\mathcal{F}_C \{f(x)\} = ?!$

$$\sim \rightarrow f'(x) = -a e^{-ax} = -a f(x) \rightarrow f'(0) = -a$$

$$\sim \rightarrow f''(x) = a^2 e^{-ax} = a^2 f(x)$$

$$\mathcal{F}_C \{f''\} = \mathcal{F}_C \{a^2 f(x)\} = -\omega^2 \mathcal{F}_C \{f\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$a^2 \mathcal{F}_C \{f(x)\} = -\omega^2 \mathcal{F}_C \{f(x)\} + \sqrt{\frac{2}{\pi}} a$$

$$\mathcal{F}_C \{f\} (a^2 + \omega^2) = \sqrt{\frac{2}{\pi}} \quad , \quad \mathcal{F}_C \{e^{-ax}\} = \left( \frac{\sqrt{2/\pi} \cdot a}{a^2 + \omega^2} \right)$$



## # 11.9 : Fourier transform

→ The Fourier transform: , ( $i = -1$ )

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

→ The Inverse Fourier transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

$$\hat{f}(\omega) = F[\hat{f}] , f(x) = F^{-1}[\hat{f}]$$

### # Theorem:

If  $f(x)$  is absolutely integrable for all  $(x)$  and piecewise continuous of every finite interval, then the Fourier transform exists.

\* Ex-1: Given  $f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \text{ (otherwise)} \end{cases}$ , Find  $F[f(x)]$

$$** F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{i\omega} e^{-i\omega x} \Big|_{-1}^1 = \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{i\omega} (e^{-i\omega} - e^{i\omega})$$

$$\leadsto e^{ia} = \cos a + i \sin a$$

$$\leadsto e^{-ia} = \cos a - i \sin a$$

$$= -\frac{1}{\sqrt{2\pi} i\omega} (-2i \sin \omega) = \frac{2}{\sqrt{2\pi}} \cdot \frac{\sin \omega}{\omega}$$

\* Ex: Given  $f(x) = \begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$ , Find  $F[f(x)]$

$$** F[f(x)] = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} \cdot e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(a+i\omega)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{(a+i\omega)} e^{-(a+i\omega)x} \Big|_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left( -\frac{1}{(a+i\omega)} (0 - 1) \right) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(a+i\omega)}$$

# Properties:

1. The Fourier transform is linear. If  $a, b$  are constants, then  
$$F[af(x) + bg(x)] = aF[f] + bF[g]$$

2. Find  $F[f']$ .

$$F[f'] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} f(x) e^{-i\omega x} \Big|_{-\infty}^{\infty} + \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx =$$

$$\leadsto u = e^{-i\omega x}$$

$$1. \partial u = f'(x) dx$$

$$\leadsto du = -i\omega e^{-i\omega x} dx$$

$$1. v = f(x)$$

$$* f(x) \rightarrow 0$$

$$* \text{ as } x \rightarrow \pm \infty$$

$$* F[f'] = i\omega F[f]$$

$$3. F[f''] = F[(f')'] = i\omega F[f'] = i^2 \omega^2 F[f] = -\omega^2 F[f]$$



\* Ex. 3: Find  $F[xe^{-x^2}]$ , use  $F[f'] = i\omega \cdot F[f]$ , and

$$F[e^{-ax^2}] = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a}, \quad (a > 0)$$

$$\underbrace{xe^{-x^2}}_{f(x)} \quad \underbrace{e^{-x^2}}_{f'(x)}$$

$$f(x) = e^{-x^2} \Rightarrow f'(x) = -2xe^{-x^2}$$

$$\leadsto xe^{-x^2} = -\frac{1}{2} f'(x)$$

$$\therefore F[xe^{-x^2}] = -\frac{1}{2} F[f'(x)] = \frac{1}{2} i\omega \cdot F[f(x)]$$

$$= \frac{1}{2} i\omega \frac{1}{\sqrt{2}} e^{-\omega^2/4} = \frac{-i\omega}{2\sqrt{2}} e^{-\omega^2/4}$$

## Ch. 12 : Partial Differential Equations. "PDE's"

\* 12.1 : Basic concepts : (ex 16-23)

$$\cos x = 1, \quad y' + 2xy = e^x \quad \text{"Ordinary D.E., ODE"}$$

$$2x + 1 = 0, \quad y = y(x)$$

$$y' = \frac{dy}{dx}$$

$$\leadsto u = u(x, y)$$

$$\frac{\partial u}{\partial x} = u_x$$

$$\frac{\partial u}{\partial y} = u_y$$

$$\leadsto u_x + u_y = 1 \quad \text{"First-order D.E."}$$

\* Def. : The order of the highest derivative is called the order of a PDE.

$$** \quad x^2 y'' + 2xy' - 3y = \cos(2 \ln x)$$

# Examples:

$$1. \quad u_t = a^2 u_{xx} \quad , \quad \text{Second order, linear Heat eq-n}$$

$$2. \quad u_{tt} = c^2 u_{xx} \quad , \quad \text{Second order, linear wave eq-n}$$

$$3. \quad u_x + u u_y = 1 \quad , \quad \text{First order, nonlinear.}$$

$$** \quad u_{xx} + u_{yy} = 0 \quad , \quad \text{Second-order, linear homogeneous PDE.}$$

$$** \quad u_{xx} + u_{yy} = f(x, y) \quad , \quad \text{non-homogeneous PDE.}$$

$\leadsto$  A solution of a PDE in some region "R" is a function that has all partial derivatives appearing in the PDE. and satisfies this equation.



\* Ex. 1: Verify that  $u(x,y) = e^x \sin y$  is a solution of the Laplace's equation  $(u_{xx} + u_{yy} = 0)$

$$u_x = e^x \sin y$$

$$u_{xx} = e^x \sin y$$

$$u_y = e^x \cos y$$

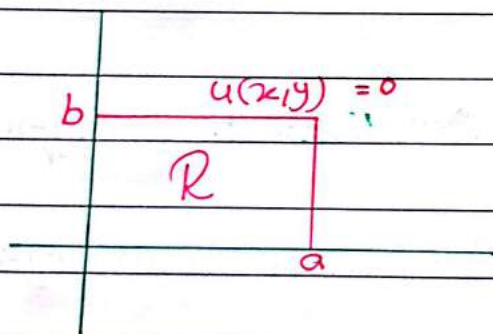
$$u_{yy} = -e^x \sin y$$

$$\leadsto e^x \sin y - e^x \sin y = 0$$

\* To find the unique solution we need additional conditions:

1. Boundary conditions:

If  $u(x,y)$  given on the boundary of the region  $(R)$



$$\leadsto u_{xx} + u_{yy} = 0$$

$$u = u(x,y)$$

$$0 \leq x \leq a$$

$$0 \leq y \leq b$$

2. Initial conditions (for  $t$ )

$$\leadsto u_t = 2u_{xx}$$

$$\leadsto u(0,x) = \sin x$$



**\*\*Solving like an ODE\*\***

\*Ex. 1: Solve  $u_{xx} - 4u_x + 3u = 0$ , ( $u = u(x, y)$ )

$$u'' - 4u' + 3u = 0$$

$$r^2 - 4r + 3 = 0$$

$$\rightarrow r_1 = 3, \quad r_2 = 1 \text{ "are real and distinct"}$$

$$\leadsto u(x, y) = c_1(y)e^{3x} + c_2(y)e^x, \text{ "is the general solution"}$$

\*Ex. 2: Solve  $u_{yy} - 10u_y + 25u = 0$ , (let  $u = u(x, y)$ )

$$u'' - 10u' + 25u = 0$$

$$r^2 - 10r + 25 = 0$$

$$\rightarrow (r-5)^2 = 0 \leadsto (r_1 = r_2 = 5)$$

$$\leadsto u(x, y) = c_1(x)e^{5y} + c_2(x)e^{5y} \cdot y \text{ "G.S."}$$

$$\rightarrow \text{If } r_1 = r_2 = r_3 = 1, \quad [c_1 e^y + c_2 e^y \cdot y + c_3 e^y \cdot y^2]$$

\*Ex. 3: Solve  $u_{xx} + 4u_x + 6u = 0$

$$u'' + 4u' + 6u = 0$$

$$r^2 + 4r + 6 = 0$$

$$\rightarrow D = 16 - 24 = -8 = (8i^2)$$

$$r_{1,2} = \frac{-4 \pm \sqrt{8i^2}}{2}$$

$$r_1 = -2 + \sqrt{2}i, \quad r_2 = -2 - \sqrt{2}i$$

$$\leadsto u(x, y) = e^{-2x} (c_1(y) \cos \sqrt{2}x + c_2(y) \sin \sqrt{2}x) \text{ "G.S."}$$

\* Ex. 4: Solve  $u_{xy} = -u_x$

Let  $u_x = P$

$P_y = -P$ , is ODE separable.

$$\frac{\partial P}{\partial y} = -P \leadsto \int \frac{\partial P}{P} = - \int dy$$

$$\left( \ln P = -y + C_1(x) \right) * e$$

$$\rightarrow P = e^{-y+C_1(x)} = e^{C_1(x)} e^{-y} = C_2(x) e^{-y}$$

$$\begin{aligned} u_x &= C_2(x) e^{-y} \\ \rightarrow u(x,y) &= \int C_2(x) e^{-y} \cdot dx = e^{-y} \underbrace{\int C_2(x) dx}_{C_4(x)} + C_3(y) \end{aligned}$$

$$\leadsto u(x,y) = C_4(x) e^{-y} + C_3(y)$$

\* Ex. 5: Solve  $x u_{xy} + 3 u_y = x$

$$\rightarrow u_{xy} = u_{yx} = (u_y)_x$$

Let  $q = u_y$

$$x q_x + 3q = x$$

$$x q' + 3q = x, \text{ "Linear, First Order"}$$

$$\rightarrow q' + \frac{3}{x} q = 1$$

$$\rightarrow \text{Integrating factor} \leadsto e^{\int \frac{3}{x} dx} = e^{3 \ln x} = (x^3)$$

$$\rightarrow \frac{d}{dx} (q x^3) = x^3$$

$$q x^3 = \int x^3 dx = \frac{x^4}{4} + C_1(y)$$

$$q = \left( \frac{x}{4} \right) + C_1(y) x^{-3}$$

$$\rightarrow u_y = \frac{x}{4} + C_1(y) x^{-3} \leadsto u(x,y) = \int \left( \frac{x}{4} + C_1(y) x^{-3} \right) dy$$

$$= \frac{xy}{4} + x^{-3} \int C_1(y) dy + C_2(x)$$



$$\leadsto y' + p(x)y = q(x)$$

$$h(x) = \int p(x) dx$$

$$y = e^{-\int p(x) dx} \left( \int e^{p(x)} q(x) dx + C \right)$$

\* Separation of variables:

$\leadsto$  Given a PDE:

Set a solution  $u(x,y) = F(x) \cdot G(y)$ , "Product solution" to obtain PDEs.

\* Ex. 1: Separate  $u_x + u_y = 0$  into a pair of PDE's.

$$u = u(x,y)$$

$\hookrightarrow u = F(x) \cdot G(y)$ , substitute  $u$  into PDE.

$\leadsto$  substitute  $u$  into PDE

$$u_x = F'(x) \cdot G(y) \quad , \quad u_y = F(x) \cdot G'(y)$$

$$\leadsto F'(x) G(y) + F(x) G'(y) = 0 \quad \div F(x) G(y)$$

$$\frac{F'}{F} + \frac{G'}{G} = 0$$

$$\frac{F'}{F} = -\frac{G'}{G} = \lambda$$

(function of  $x$ ) / (function of  $y$ )

$\leadsto \lambda$  is a separating constant, It must be found.

$$\frac{F'}{F} = \lambda \quad , \quad -\frac{G'}{G} = \lambda$$

$$F'(x) - \lambda F(x) = 0$$

$$G'(y) + \lambda G(y) = 0$$

# Ex. 2: Separate  $u_{xx} - u_y + u = 0$  into a pair of ODE's.

$$\begin{aligned} u(x,y) &= F(x) \cdot G(y) \\ \rightarrow u_x &= F'G, \quad u_{xx} = F''G \\ u_y &= FG' \end{aligned}$$

$$\frac{F''G}{FG} - \frac{FG'}{FG} + \frac{FG}{FG} = 0$$

$$\begin{aligned} \frac{F''}{F} - \frac{G'}{G} + 1 &= 0 \\ \rightarrow \frac{F''}{F} + 1 &= \frac{G'}{G} = \lambda \end{aligned}$$

$$\frac{F''}{F} + 1 = \lambda, \quad \frac{G'}{G} = \lambda$$

$$* F'' + F - \lambda F = 0 \leadsto F'' + (1-\lambda)F = 0$$

$$* G' - \lambda G = 0$$

# Sturm-Liouville Boundary value problems (SLP).

→ Boundary value problems:

$$\begin{aligned} y'' + \lambda y &= 0, \quad 0 < x < L \\ \rightarrow \alpha_1 y(0) + \alpha_2 y'(0) &= 0 \\ \rightarrow \beta_1 y(L) + \beta_2 y'(L) &= 0 \end{aligned}$$

\*  $\alpha_1, \alpha_2, \beta_1, \beta_2$  are constants\*  
Boundary conditions

$y \equiv 0$  is a solution "trivial solution".  
→ we want to find non-trivial solution (eigen functions), that correspond to certain  $\lambda$  (eigen values).



\* Ex. 1:  $y'' + \lambda y = 0$ ,  $[y(0) = 0, y(L) = 0]$

→ Case 1: If  $\lambda = 0 \Rightarrow y'' = 0$

$$\begin{aligned} y(x) &= Ax + B \\ \rightarrow y(0) &= 0, y(0) = B = 0 \sim y(x) = Ax \\ \rightarrow y(L) &= 0, y(L) = AL = 0 \Rightarrow A = 0 \end{aligned}$$

$\therefore y \equiv 0$  trivial solution

$\lambda = 0$  is not eigenvalue.

→ Case 2: If  $\lambda < 0$  (let  $\lambda = -\alpha^2, \alpha > 0$ ).

$$\begin{aligned} y'' - \alpha^2 y &= 0 \\ \rightarrow r^2 - \alpha^2 &= 0 \sim (r_1 = \alpha, r_2 = -\alpha) \end{aligned}$$

$$\begin{aligned} y(x) &= c_1 e^{\alpha x} + c_2 e^{-\alpha x} \\ \rightarrow y(0) &= c_1 + c_2 = 0 \\ \rightarrow y(L) &= c_1 e^{\alpha L} + c_2 e^{-\alpha L} = 0 \end{aligned}$$

$$c_1 = -c_2 \rightarrow -c_2 e^{\alpha L} + c_2 e^{-\alpha L} = 0$$

$$\begin{aligned} c_2 (e^{-\alpha L} - e^{\alpha L}) &= 0 \\ \rightarrow c_2 &= 0 \end{aligned}$$

$$\rightarrow c_1 = 0$$

→  $y \equiv 0$  trivial solution (no negative eigenvalues).

→ Case 3: If  $\lambda > 0$  (let  $\lambda = \alpha^2, \alpha > 0$ )  $\Rightarrow$

$$y'' + \alpha^2 y = 0$$

$$r^2 + \alpha^2 = 0 \leadsto (r_1 = \alpha i), (r_2 = -\alpha i)$$

$$\rightarrow y(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$y(0) = c_1 = 0 \Rightarrow y(x) = c_2 \sin \alpha x$$

$$y(L) = c_2 \sin \alpha L = 0$$

"trivial solution"

$$\leadsto \sin \alpha L = 0, \alpha L = n\pi, n=1, 2, \dots$$

$$\alpha_n = \frac{n\pi}{L} \leadsto \alpha_1 = \frac{\pi}{L}, \alpha_2 = \frac{2\pi}{L}$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \text{ eigenvalues.}$$

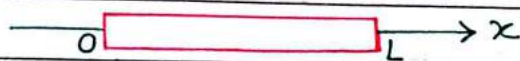
$$y_n = \sin \frac{n\pi x}{L}, \text{ eigenfunctions.}$$

$(0 < x < L)$



### \* Heat Equation:

Consider temperature in a metal rod of constant cross section and homogeneous material.



\*\* The Temperature in the rod describes by Heat eqn.

$$u_t = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \quad \dots (1)$$

$u = u(x, t)$ , "t" is time.

"c" is a constant (thermal diffusivity)

\*\* Boundary Condition: (t or x)

$$\rightarrow u(0, t) = 0$$

$$\rightarrow u(L, t) = 0 \quad \dots (2)$$

"the ends of the rod" are kept at zero temp.

x

\*\* Initial Condition: (f or t)

$$\rightarrow u(x, 0) = f(x) \quad \dots (3) \quad \text{at } t = 0$$

t

\* ~~~~~ \*

### \* Problem no. 1:

Solve  $u_t = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0$

B.C.  $u(0, t) = 0 \quad \dots (2)$

$$u(L, t) = 0$$

I.C.  $u(x, 0) = f(x) \quad \dots (3)$

1. Separation of variables:

$$u(x,t) = F(x) G(t)$$

$$u_t = F G' \quad , \quad u_{xx} = F'' G$$

$$F G' = c^2 F'' G \quad , \quad (\div FG)$$

$$\frac{G'}{G} = c^2 \frac{F''}{F} = -\lambda$$

$$\frac{G'}{G c^2} = -\lambda \quad , \quad \frac{F''}{F} = -\lambda$$

$$G'(t) + c^2 \lambda G(t) = 0 \quad \text{--- (5)}$$

$$F''(x) + \lambda F(x) = 0 \quad \text{--- (6)}$$

$$\rightarrow (\text{SLP}) \quad y'' + \lambda y = 0$$

$$y(x) = \text{---}$$

$$y(a) = \text{---}$$

$$2. u(0,t) = 0$$

$$u(0,t) = F(0) G(t) = 0 \Rightarrow F(0) = 0 \quad \text{--- (7)}$$

$$F(0) = 0 \quad \rightarrow \quad G(t) = 0$$

$$\text{If } G(t) = 0 \Rightarrow u(x,t) = F(x) \cdot 0 = 0 \quad (\text{trivial solution})$$

$$u(L,t) = \underset{0}{F(L)} \underset{\neq 0}{G(t)} = 0 \Rightarrow F(L) = 0 \quad \text{--- (8)}$$

3. Solve the problem (6) + B.C.

"7 + 8"

$$F'' + \lambda F = 0$$

"see ex. 1, SLP"

$$F(0) = 0 \quad , \quad F(L) = 0$$

$$y'' + \lambda y = 0$$

$$y(0) = 0$$

$$y(L) = 0$$



$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n=1, 2, 3, \dots \quad (9)$$

$$F_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \dots (10)$$

$$4. \text{ Solve (5) } , \quad G' + c^2 \underbrace{\frac{n^2 \pi^2}{L^2}}_{\lambda} G = 0.$$

$$\rightarrow \frac{\partial G}{\partial t} = -c^2 \frac{n^2 \pi^2}{L^2} \cdot G$$

$$\int \frac{\partial G}{G} = \int dt \cdot -c^2 \frac{n^2 \pi^2}{L^2}$$

$$\ln G = -c^2 \frac{n^2 \pi^2}{L^2} t + \ln A$$

$$\leadsto G_n(t) = A e^{-c^2 n^2 \pi^2 t / L^2}, \quad n=1, 2, \dots \quad (11)$$

5. Form the general solution.

$$U_n(x,t) = A \sin \frac{n\pi x}{L} e^{-n^2 \pi^2 c^2 t / L^2}, \quad n=1, 2, \dots \quad (12)$$

\* Superposition principle:

$$u(x,t) = \sum_{n=1}^{\infty} A_n U_n(x,t)$$

$\leadsto$  the general solution is!

$$U(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \cdot e^{-n^2 \pi^2 c^2 t / L^2} \quad \dots (13)$$

6. Find  $A_n$  from I.C (3)

$$t=0 \Rightarrow u(x,0) = \sum_{n=1}^{\infty} A_n \cdot \sin \frac{n\pi x}{L} = f(x)$$

$\hookrightarrow$  Fourier sine series for  $f(x)$

$$A_n = \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx \quad \text{--- (14)}$$



\* Heat equation, Rod with insulated ends.

\* Problem 2: (was)

$$u_t = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \quad \text{--- (1)}$$

$$\text{B.C. } \begin{cases} u_x(0, t) = 0, & t \geq 0 \quad \text{--- (2)} \\ u_x(L, t) = 0 \end{cases}$$

$$\text{I.C. } u(x, 0) = f(x), \quad 0 \leq x \leq L \quad \text{--- (3)}$$

1. Separation of (1):

$$u(x, t) = F(x) \cdot G(t) \quad \text{--- (4)}$$

$$F'' + \lambda F = 0 \quad \text{--- (5)}$$

$$G' + c^2 \lambda G = 0 \quad \text{--- (6)}$$

2. Boundary conditions:

$$\begin{aligned} u_x(0, t) = F'(0) G(t) = 0 & \leadsto F'(0) = 0 \\ u_x(L, t) = F'(L) G(t) = 0 & \leadsto F'(L) = 0 \end{aligned} \quad \text{--- (7)}$$

3. Solve (5) + (7)

$$F'' + \lambda F = 0$$

$$F'(0) = 0$$

$$F'(L) = 0$$

, Problem no. 2

$$\lambda \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases}$$

$$\times \text{ If } \lambda = 0 \Rightarrow F''(x) = 0 \Rightarrow F(x) = Ax + B$$

$$\leadsto \text{Find } (A, B); (F'(x) = A)$$

$$F'(0) = A = 0, \quad F'(L) = A = 0$$

$$F(x) = B, \quad F(x) = \text{const.}$$

Let  $F_0 = 1$ ,  $\lambda_0 = 0$  is eigenvalue.

\* If  $\lambda < 0$ , let  $\lambda = -\alpha^2$ ,  $\alpha > 0$

$$F'' - \alpha^2 F = 0$$

$$r^2 - \alpha^2 = 0, \quad (r_1 = \alpha, r_2 = -\alpha)$$

$$F(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$$

$$F'(x) = \alpha C_1 e^{\alpha x} - \alpha C_2 e^{-\alpha x}$$

$$F'(0) = \alpha C_1 - \alpha C_2 = 0 \Rightarrow C_1 = C_2$$

$$F'(L) = \alpha C_1 e^{\alpha L} - \alpha C_2 e^{-\alpha L} = 0$$

$$C_2 (e^{\alpha L} - e^{-\alpha L}) = 0, \quad C_2 = 0 \Rightarrow C_1 = 0$$

$\neq 0 \quad \neq 0$

$$F(x) = 0 \Rightarrow u(x, t) = 0, G(t) = 0 \text{ (Trivial solution)}$$

"No Nontriv. sol."

\* If  $\lambda > 0$ , let  $\lambda = \alpha^2$ ,  $\alpha > 0$

$$F'' + \alpha^2 F = 0$$

$$r^2 + \alpha^2 = 0, \quad (r_1 = \alpha i, r_2 = -\alpha i)$$

$$F(x) = C_1 \cos \alpha x + C_2 \sin \alpha x$$

$$F'(x) = -C_1 \alpha \sin \alpha x + C_2 \alpha \cos \alpha x$$

$$F'(0) = \alpha C_2 = 0 \Rightarrow C_2 = 0$$

$$F'(L) = -C_1 \alpha \sin \alpha L = 0$$

$0 \neq 0 \neq 0$

$$\sin \alpha L = 0, \quad \alpha L = n\pi, \quad (n = 1, 2, 3, \dots)$$

$$\alpha = \frac{n\pi}{L}, \quad \lambda = \alpha^2$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \dots \quad (9)$$

$$f_n(x) = \cos\left(\frac{n\pi x}{L}\right) \quad \dots \quad (10)$$



4. Solve  $G' + c^2 \lambda G = 0$

If  $\lambda = 0$ ,  $G' = 0 \Rightarrow G(t) = \text{const.}$ ; Let  $(G_0 = 1)$

$\lambda = \left(\frac{n\pi}{L}\right)^2 \Rightarrow G' + \frac{c^2 n^2 \pi^2}{L^2} G = 0$

$G_n(t) = e^{-c^2 n^2 \pi^2 t / L^2}$ ,  $n = 1, 2, 3, \dots$  (11)

5. The G.S is,

$u(x, t) = A_0 G_0 + \sum_{n=1}^{\infty} A_n F_n(x) G_n(t)$

$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2 \pi^2 c^2 t / L^2}$  ---- (12)

6. Find  $A_n$  from I.C. (13)

$u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} = f(x)$

"Fourier cosine series"

$A_0 = \frac{1}{L} \int_0^L f(x) dx$ ,  $A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$ .

# Problem 4:

$u_t = c^2 u_{xx}$ ,  $0 < x < L$ ,  $t > 0$

B.C  $\begin{cases} u(0, t) = 0 \\ u_x(L, t) = 0 \end{cases}$ ,  $t \geq 0$

I.C  $u(x, 0) = f(x)$ ,  $0 \leq x \leq L$

# Ex. 3:  $u_t = 100u_{xx}$ ,  $0 < x < 1$ ,  $t > 0$

B.C  $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}$ ,  $t \geq 0$

I.C  $u(x,0) = 2\sin 2\pi x - 3\sin 3\pi x$   
 $u(x,t) = F(x) G(t)$

→ The G.S:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 \pi^2 t} \cdot \sin n\pi x.$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x = A_1 \sin \pi x + A_2 \sin 2\pi x + A_3 \sin 3\pi x + \dots$$

$$= 2\sin 2\pi x - 5\sin 3\pi x$$

"n=2"

"n=3"

→  $A_1 = 0$ ,  $A_2 = 2$ ,  $A_3 = -5$   
 $A_4 = \dots$

$$u(x,t) = 2e^{-4\pi^2 t} \sin 2\pi x - 5e^{-9\pi^2 t} \sin 3\pi x.$$



# Wave equation:

Consider a string of length "L"

\* The ends of the string are fixed.

\* The Model of vibrating string consists of:

$u_{tt} = a^2 u_{xx}$ , is a wave equation ( $a$  is a constant.)

\* Boundary Conditions:

$$u(0, t) = 0$$

$$u(L, t) = 0 \quad \text{fixed ends.}$$

\* Initial Conditions:

$$u(x, 0) = f(x) \quad ; \text{ initial Displacement.}$$

$$u_t(x, 0) = g(x) \quad ; \text{ initial velocity,}$$



# Problem. 1:  $u_{tt} = a^2 u_{xx}$ ,  $0 < x < L$ ,  $t > 0$  --- (1)

$$\text{B.C. } u(0, t) = 0$$

$$u(L, t) = 0 \quad \text{--- (2)}$$

$$\text{I.C. } u(x, 0) = f(x)$$

$$u_t(x, 0) = g(x) \quad \text{--- (3)}$$

$$\text{"1"} \quad u(x, t) = F(x) G(t) \quad \text{--- (4)}$$

$$F G'' = a^2 F'' G$$

$$\frac{F''}{F} = \frac{G''}{a^2 G} = -\lambda$$

$$F \quad a^2 G$$

$$F'' + \lambda F = 0 \quad \dots (5)$$

$$G'' + a^2 \lambda G = 0 \quad \dots (6)$$

"2"  $u(0,t) = F(0) G(t) = 0 \leadsto F(0) = 0$   
 $u(L,t) = F(L) G(t) = 0 \leadsto F(L) = 0$

"3" Solve  $\begin{pmatrix} F'' + \lambda F = 0 \\ F(0) = 0 \\ F(L) = 0 \end{pmatrix}$ , See "ex. 1"  $\rightarrow$  (SLP)  
 or "Heat eqn. Prob. 1"

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \dots \quad (7)$$

$$F_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \dots (8)$$

"4" Solve  $G'' + \frac{a^2 n^2 \pi^2}{L^2} G = 0$

$$r^2 = -\frac{a^2 n^2 \pi^2}{L^2} \leadsto r_{1,2} = \pm i \left(\frac{an\pi}{L}\right)$$

$$G_n(t) = C_{1n} \cos \frac{an\pi}{L} t + C_{2n} \sin \frac{an\pi}{L} t \quad ; n = 1, 2, \dots \quad (9)$$

"5"  $u_n(x,t) = \left(\sin\left(\frac{n\pi x}{L}\right)\right) \left(\frac{A_n \cos \frac{an\pi}{L} t + B_n \sin \frac{an\pi}{L} t}{L}\right) \quad n = 1, 2, \dots$

$\leadsto$  The general solution:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left(\frac{A_n \cos \frac{an\pi}{L} t + B_n \sin \frac{an\pi}{L} t}{L}\right) \quad \dots (10)$$



"6" Find  $A_n$  and  $B_n$  from I.C. (3)

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = f(x)$$

F. sine ser. for  $f(x)$

$$* A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$\rightarrow u_{\pm}(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( -A_n \frac{a n \pi}{L} \sin \frac{a n \pi t}{L} + B_n \frac{a n \pi}{L} \cos \frac{a n \pi t}{L} \right)$$

$$\rightarrow u_{\pm}(x,0) = \sum_{n=1}^{\infty} B_n \frac{a n \pi}{L} \sin \frac{n\pi x}{L} = g(x) \text{ is the F.S.S.}$$

$$B_n \frac{a n \pi}{L} = \frac{2}{L} \int_0^L g(x) \sin \left( \frac{n\pi x}{L} \right) dx \quad \dots (12)$$

# Problem No. 2:

$$u_{tt} = a^2 u_{xx}$$

$$u_x(0,t) = 0$$

$$u_x(L,t) = 0$$

} Free ends

$$\text{I.C. } (u(x,0)) = f(x)$$

$$u_t(x,0) = g(x)$$

# Problem No. 3:

$$u_{tt} = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0$$

$$\text{B.C. } u(0,t) = 0$$

$$u_x(L,t) = 0$$

$$\text{I.C. } u(x,0) = f(x)$$

$$u_t(x,0) = 0$$

$$u(x,t) = F(x) G(t)$$

$$F'' + \lambda F = 0$$

$$G'' + a^2 \lambda G = 0$$

$$\rightarrow u(0,t) = F(0) G(t) = 0 \leadsto F(0) = 0$$

$$u_x(L,t) = F'(L) G(t) = 0 \leadsto F'(L) = 0$$

\*\* Solve:  $F'' + \lambda F = 0$

$$F(0) = 0$$

$$F'(L) = 0$$

$$\lambda \begin{matrix} \nearrow = 0 \\ \rightarrow < 0 \\ \searrow > 0 \end{matrix}$$

\*\* If  $\lambda = 0 \leadsto F'' = 0 \leadsto F(x) = Ax + B$

$$F(0) = B = 0$$

$$F'(x) = A, \quad F'(L) = A = 0$$

$$\begin{cases} F(x) = 0 \\ u(x,t) = 0 \cdot G = 0 \\ \text{trivial solution.} \end{cases}$$

\*\* If  $\lambda < 0$  (let  $\lambda = -\alpha^2, \alpha > 0$ )  $\leadsto (c_1 = -c_2)$

$$F'' - \alpha^2 F = 0$$

$$r^2 - \alpha^2 = 0, \quad (r_1 = \alpha, r_2 = -\alpha)$$

$$F(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

$$F(0) = c_1 + c_2 = 0$$

$$F'(L) = \alpha c_1 e^{\alpha L} - \alpha c_2 e^{-\alpha L} = 0$$

$$\leadsto -\alpha c_2 (e^{\alpha L} + e^{-\alpha L}) = 0$$

$$c_2 = 0 \leadsto c_1 = 0$$

$$F(x) = 0$$

$$u(x,t) = 0 \cdot G(t) = 0 \leadsto \text{Trivial Solution.}$$



$$* u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0$$

$$u(0, t) = 0$$

$$u_x(L, t) = 0$$

$$u_t(x, 0) = 0$$

$$u(x, 0) = f(x)$$

$$** u(x, t) = F(x) \cdot G(t)$$

$$F'' + \lambda F = 0$$

$$F(0) = 0$$

$$F'(L) = 0$$

$$\lambda \rightarrow \neq 0$$

$$\lambda \rightarrow < 0$$

~> If  $\lambda > 0$  (let  $\lambda = \alpha^2, \alpha > 0$ )

$$F'' + \alpha^2 F = 0, \quad F = 0$$

$$r^2 + \alpha^2 = 0, \quad r_1 = \alpha i, \quad r_2 = -\alpha i$$

$$** F(x) = C_1 \cos \alpha x + C_2 \sin \alpha x$$

$$F(0) = C_1 = 0$$

$$\rightarrow F'(x) = \alpha C_2 \cos \alpha x$$

$$F'(L) = \alpha C_2 \cos \alpha L = 0$$

$$\cos \alpha L = 0$$

$$\alpha L = \frac{(2n-1)\pi}{2}; \quad n = 1, 2, \dots$$

$$** \alpha = \frac{(2n-1)\pi}{2L}$$

$$\rightarrow \lambda_n = \frac{(2n-1)^2 \pi^2}{(2L)^2}, \quad n = 1, 2, \dots$$

$$\rightarrow F_n(x) = \sin \frac{(2n-1)\pi}{2L} x$$

$$\sim \rightarrow \text{Solve: } G'' + \frac{\alpha^2 (2n-1)^2 \pi^2}{4L^2} G = 0$$

$$x \times G_n(t) = G_n \cos \frac{a(2n-1)\pi}{2L} t + C_{2n} \sin \frac{a(2n-1)\pi}{2L} t \quad n=1, 2, \dots$$

~> The general solution:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{\sin \frac{(2n-1)\pi x}{2L}}{\sin \frac{(2n-1)\pi x}{2L}} \left( A_n \cos \frac{a(2n-1)\pi t}{2L} + B_n \sin \frac{a(2n-1)\pi t}{2L} \right)$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \frac{\sin \frac{(2n-1)\pi x}{2L}}{\sin \frac{(2n-1)\pi x}{2L}} = f(x)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{(2n-1)\pi x}{2L} dx$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{\sin \frac{(2n-1)\pi x}{2L}}{\sin \frac{(2n-1)\pi x}{2L}} \left( -A_n \dots \sin(\dots t) + B_n \dots \cos(\dots t) \right)$$

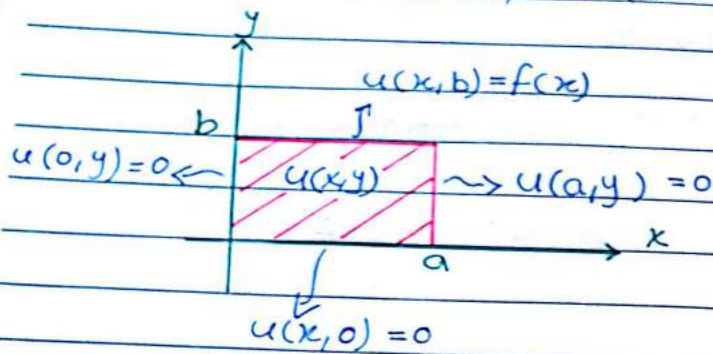
$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \dots \frac{\sin \frac{(2n-1)\pi x}{2L}}{\sin \frac{(2n-1)\pi x}{2L}} = \dots, B_n = 0$$

\* G.S \*

$$u(x,t) = \sum A_n \frac{\sin \frac{(2n-1)\pi x}{2L}}{\sin \frac{(2n-1)\pi x}{2L}} \cos \frac{a(2n-1)\pi t}{2L}$$



\* Laplace's equation: "in a rectangle"  
 $u_{xx} + u_{yy} = 0$ ,  $0 < x < a$ ,  $0 < y < b$



\* Boundary Conditions

$$\begin{aligned} \rightarrow u(0,y) = 0 \\ u(a,y) = 0 \end{aligned} \quad \left. \vphantom{\begin{aligned} \rightarrow u(0,y) = 0 \\ u(a,y) = 0 \end{aligned}} \right\} 0 \leq y \leq b$$

$$\begin{aligned} \rightarrow u(x,0) = 0 \\ u(x,b) = f(x) \end{aligned} \quad \left. \vphantom{\begin{aligned} \rightarrow u(x,0) = 0 \\ u(x,b) = f(x) \end{aligned}} \right\} 0 \leq x \leq a$$

\*\*  $u(x,y) = F(x) \cdot G(y)$

$F''G + FG'' = 0$

$\rightarrow \frac{F''}{F} = -\frac{G''}{G} = -\lambda$

$\leadsto F'' + \lambda F = 0$ ,  $G'' - \lambda G = 0$

$u(0,y) = F(0)G(y) = 0$ ,  $F(0) = 0$

$u(a,y) = F(a)G(y) = 0$ ,  $F(a) = 0$

$u(x,0) = F(x)G(0) = 0$ ,  $G(0) = 0$

\*\* Solve  $F'' + \lambda F = 0$ ,  $F(0) = 0$ ,  $F(a) = 0$  } see (ex.1), "SLP"  
 or heat, wave eqn.s.

$$** G_n(t) = G_n \cos \frac{a(2n-1)\pi}{2L} t + C_n \sin \frac{a(2n-1)\pi}{2L} t \quad n=1, 2, \dots$$

~> The general solution:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{\sin \frac{(2n-1)\pi x}{2L}}{\frac{2L}{2L}} \left( A_n \cos \frac{a(2n-1)\pi t}{2L} + B_n \sin \frac{a(2n-1)\pi t}{2L} \right)$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \frac{\sin \frac{(2n-1)\pi x}{2L}}{\frac{2L}{2L}} = f(x)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cdot \frac{\sin \frac{(2n-1)\pi x}{2L}}{\frac{2L}{2L}} dx$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{\sin \frac{(2n-1)\pi x}{2L}}{\frac{2L}{2L}} (-A_n \sin \frac{a(2n-1)\pi t}{2L} + B_n \cos \frac{a(2n-1)\pi t}{2L})$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{\sin \frac{(2n-1)\pi x}{2L}}{\frac{2L}{2L}} = 0, \quad B_n = 0$$

\* G.S \*

$$u(x,t) = \sum_{n=1}^{\infty} A_n \frac{\sin \frac{(2n-1)\pi x}{2L}}{\frac{2L}{2L}} \cos \frac{a(2n-1)\pi t}{2L}$$



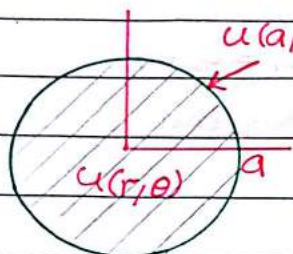
\* Laplace's equation & Problem for a circle

$$u_{xx} + u_{yy} = 0$$

~> Introduce polar coordinates:

i.e. in polar coordinates

$$(1) \quad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad | \quad "r < a" \quad , \quad "-\pi < \theta < \pi"$$



\* Boundary Condition:

$$u(a, \theta) = f(\theta) \quad \dots (2)$$

\* Interior Problem:

$$"r < a"$$

$$\sim u(r, \theta) = F(r) G(\theta)$$

$$F''G + \frac{1}{r} F'G + \frac{1}{r^2} F G'' = 0 \quad (\div FG)$$

$$\left( \frac{F''}{F} + \frac{1}{r} \frac{F'}{F} + \frac{1}{r^2} \frac{G''}{G} \right) = 0$$

$$\rightarrow r^2 \left( \frac{F''}{F} + \frac{1}{r} \frac{F'}{F} \right) = - \frac{G''}{G} = \lambda$$

$$\begin{cases} G'' + \lambda G = 0 \\ r^2 F'' + r F' - \lambda F = 0 \end{cases} \sim \text{Euler eq'n}$$

\*\* Notes: A function  $u(r, \theta)$  must be:

- periodic in  $\theta$  with period " $2\pi$ ".

- Bounded for  $r \leq a$ .

$$\lambda \begin{cases} \rightarrow 0 \\ \rightarrow < 0 \\ \rightarrow > 0 \end{cases}$$

1. If  $\lambda = 0 \Rightarrow G'' = 0 \Rightarrow$

$G(\theta) = c_1 + c_2 \theta$

$\rightarrow c_2 = 0, c_1 = \text{const.}, \text{ let } c_1 = 1$

$G_0 = 1$  is periodic with any period.

2. If  $\lambda < 0$ , (let  $\lambda = -\alpha^2, \alpha > 0$ )  $\Rightarrow$

$G(\theta) = c_1 e^{\alpha \theta} + c_2 e^{-\alpha \theta}$ ; is not periodic  $c_1 = c_2 = 0$   
"only trivial solution".

3. If  $\lambda > 0$  (let  $\lambda = \alpha^2, \alpha > 0$ )  $\Rightarrow$

$G'' + \alpha^2 G = 0$

$G(\theta) = c_1 \cos \alpha \theta + c_2 \sin \alpha \theta$

" $\alpha$  must be integer", ( $\alpha_n = n, n = 1, 2, \dots$ )

$G_n(\theta) = c_{1n} \cos n\theta + c_{2n} \sin n\theta$

" $\lambda_n = n^2$ "

non-trivial solutions.

\* Solve r-equations

1. If  $\lambda = 0 \Rightarrow$

$r^2 F'' + r F' = 0$

$k(k-1) + k = 0$

$k^2 = 0 \Rightarrow "k_1 = k_2 = 0"$

$\rightarrow F(r) = c_1 + c_2 \ln r$

If  $r \rightarrow 0 \Rightarrow \ln r \rightarrow -\infty \Rightarrow c_2 = 0$

$F_0 = 1$

$r^2 F'' + r F' - n^2 F = 0$

$k(k-1) + k - n^2 = 0$

$k^2 = n^2 \Rightarrow "k_1 = n, k_2 = -n"$

$\rightarrow F(r) = c_1 r^n + c_2 r^{-n}$

$F(r) = c_1$



$$r^{-n} = \frac{1}{r^n} \rightarrow \infty$$

(If  $r \rightarrow 0$ )  $r^n$

$$\rightarrow F_n(r) = r^n$$

\* The General Solution:

$$\sim u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) ;$$

\*\* Find  $A_n, B_n = ?$  from:

$$\text{B.C. } u(a, \theta) = f(\theta)$$

$$\rightarrow u(a, \theta) = A_0 + \sum_{n=1}^{\infty} a^n A_n \cos n\theta + a^n B_n \sin n\theta = f(\theta) \quad \text{+ Fourier Series +}$$

$$\sim A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$\sim a^n A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta$$

$$\sim a^n B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$$

\* Exterior Problem:

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$$u(a, \theta) = f(\theta) \quad , \quad r > a$$

If  $r \rightarrow \infty$

$$\rightarrow c_1 = 0$$

$$F(r) = c_2 r^{-n}$$

$$\sim u(r, \theta) = A_0^* + \sum_{n=1}^{\infty} r^{-n} (A_n^* \cos n\theta + B_n^* \sin n\theta)$$

$$\rightarrow A_0^* = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$\rightarrow a^{-n} A_n^* = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta$$

$$\rightarrow a^{-n} B_n^* = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta$$

\* Application of Fourier transforms:

$$\leadsto F_c \{ f(x) \} = -\omega^2 F_c \{ f \} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$\leadsto F_s \{ f(x) \} = -\omega^2 F_s \{ f \} + \sqrt{\frac{2}{\pi}} f(0) \omega$$

$$\rightarrow u_t = c^2 u_{xx}, \quad F_c \{ u_{xx} \} = -\omega^2 F_c \{ u \} - \sqrt{\frac{2}{\pi}} u_x(0,t)$$

$$u(x,t), \quad F_s \{ u_{xx} \} = -\omega^2 F_s \{ u \} + \sqrt{\frac{2}{\pi}} \omega u(0,t)$$

\* #Ex. 1! Solve  $u_t = c^2 u_{xx}$ ,  $0 < x < \infty$ ,  $t > 0$

$\leadsto$  B.C.  $u(0,t) = u_0 = \text{const.}$

$\leadsto$  I.C.  $u(x,0) = 0$

$\hookrightarrow$  use the F.S. transform:

$$F_s \{ u_t \} = c^2 F_s \{ u_{xx} \}$$

$$F_s \{ u_t \} = \frac{d}{dt} F_s \{ u \}, \quad \text{let } [F_s \{ u \} = u_s(\omega, t)]$$

$$\frac{\partial u_s}{\partial t} = c^2 u_s (-\omega^2) + \sqrt{\frac{2}{\pi}} \omega u(0,t)$$

$$= c^2 (-\omega^2 u_s + \sqrt{\frac{2}{\pi}} \omega u_0) \quad u_0 = 0$$

$\leadsto \frac{\partial u_s}{\partial t} + c^2 \omega^2 u_s = \sqrt{\frac{2}{\pi}} \omega^2 u_0$  "First order linear ODE"

$$\rightarrow u_s(\omega, t) = e^{-\omega^2 c^2 t} \left( \int e^{\omega^2 c^2 t} * \sqrt{\frac{2}{\pi}} c^2 \omega u_0 dt \right)$$

$$\rightarrow u_s(\omega, t) = e^{-\omega^2 c^2 t} \left( \sqrt{\frac{2}{\pi}} \frac{e^{\omega^2 c^2 t}}{\omega^2 c^2} e^{\omega^2 c^2 t} + A \right)$$

$$\hookrightarrow u_s(\omega, t) = \sqrt{\frac{2}{\pi}} \frac{u_0}{\omega} + A e^{-\omega^2 c^2 t}$$

\*\* Find A :-

$$u(x,0) = 0$$

$$u_s(\omega, 0) = \sqrt{\frac{2}{\pi}} \frac{u_0}{\omega} + A$$



~> by def.

$$U_s(\omega, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} u(x, t) \sin \omega x dx$$

$$\rightarrow U_s(\omega, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \underbrace{u(x, 0)}_{=0} \dots = 0$$

$$\sqrt{\frac{2}{\pi}} \frac{u_0}{\omega} + A = 0 \sim \rightarrow A = -\sqrt{\frac{2}{\pi}} \frac{u_0}{\omega}$$

$$\rightarrow U_s(\omega, t) = \sqrt{\frac{2}{\pi}} \frac{u_0}{\omega} - \sqrt{\frac{2}{\pi}} \frac{u_0}{\omega} e^{-\omega^2 c^2 t}$$

$$u(x, t) = F_s^{-1} \{ U_s \}$$

$$\rightarrow u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{u_0}{\omega} (1 - e^{-\omega^2 c^2 t}) \sin \omega x d\omega$$

# Ex. 2: Solve  $u_{tt} = a^2 u_{xx}$ ,  $0 \leq x < \infty$ ,  $t > 0$

B.C.  $u_x(0, t) = 0$ , use F.C.t

$$\text{I.C. } \begin{cases} u(x, 0) = \begin{cases} 2 & 0 \leq x < 1 \\ 0 & 1 \leq x < \infty \end{cases} \\ u_t(x, 0) = 0 \end{cases}$$

$$\sim \rightarrow F_c \{ u_{tt} \} = a^2 \underbrace{F_c \{ u_{xx} \}}_{\sim \rightarrow F_c}$$

$$\frac{\partial^2 U_c}{\partial t^2} = a^2 (-\omega^2 U_c - \underbrace{\sqrt{\frac{2}{\pi}} u_x(0, t)}_{=0})$$

$$\frac{\partial^2 U_c}{\partial t^2} + a^2 \omega^2 U_c = 0 \quad \text{"second-order with const. coefficients"}$$

$$U_c(\omega, t) = c_1(\omega) \cos a\omega t + c_2(\omega) \sin a\omega t$$

$$u(x, 0) = \begin{cases} 2 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$U_c(\omega, 0) = c_1(\omega)$$

$$c_1(\omega) = 2 \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega}$$

→ by def.

$$U_c(\omega, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} U(x, t) \cos \omega x \, dx$$

$$U_c(\omega, 0) = \sqrt{\frac{2}{\pi}} \int_0^1 2 \cos \omega x \, dx = 2 \sqrt{\frac{2}{\pi}} \frac{\sin \omega x}{\omega} \Big|_0^1$$

$$= 2 \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega}$$

$$\rightarrow \frac{\partial}{\partial t} (U_c)(\omega, t) = -c a \omega \sin \omega a t + c_2 a \omega \cos \omega a t$$

$$\frac{\partial}{\partial t} (U_c)(\omega, 0) = c_2 a \omega$$

$$\frac{\partial}{\partial t} (U_c)(\omega, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \underbrace{U_t(\omega, t)}_{=0} \cos \omega x \, dx$$

$$\frac{\partial U_c}{\partial t} = 0$$

$$\rightarrow U_c(\omega, t) = 2 \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega} \cos a \omega t.$$

$$U(x, t) = F^{-1} \left\{ U_c(\omega, t) \right\}.$$

$$U(x, t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \left( 2 \sqrt{\frac{2}{\pi}} \frac{\sin \omega}{\omega} \cos a \omega t \right) \cos \omega x \, d\omega.$$



\* Ex. 3: Solve  $u_t = u_{xx}$ ,  $-\infty < x < \infty$ ,  $t > 0$

I.C.  $u(x, 0) = \begin{cases} 3, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

$$F[f''] = -\omega^2 F[f], \text{ let } F(u) = u(\omega, t)$$

$$\leadsto F[u_t] = F[u_{xx}]$$

$$\frac{\partial u}{\partial t} = -\omega^2 u$$

$$\frac{\partial u}{u} = -\omega^2 dt$$

$$\rightarrow u(\omega, t) = A e^{-\omega^2 t}$$

$$u(\omega, 0) = A$$

$\leadsto$  by def. :

$$u(\omega, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-i\omega x} dx$$

$$u(\omega, 0) = \frac{1}{\sqrt{2\pi}} \int_0^1 3e^{-i\omega x} dx = \frac{3}{\sqrt{2\pi}} \cdot \frac{1}{(-i\omega)} e^{-i\omega x} \Big|_0^1$$

$$= \frac{3}{\sqrt{2\pi}} \cdot \frac{1}{(-i\omega)} (e^{-i\omega} - 1)$$

$$\leadsto A = \frac{3}{\sqrt{2\pi}} \left( \frac{1 - e^{-i\omega}}{i\omega} \right)$$

$$u(\omega, t) = \frac{3}{\sqrt{2\pi}} \left( \frac{1 - e^{-i\omega}}{i\omega} \right) e^{-\omega^2 t}$$

$$u(x, t) = F^{-1}\{u\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{3}{\sqrt{2\pi}} \left( \frac{1 - e^{-i\omega}}{i\omega} \right) e^{i\omega x} d\omega$$