MidTerm Exam - Solution

Problem I (3 Marks)

A steel [E = 200 GPa] rod with a circular cross section is 7.5-m long. Determine the minimum diameter required if the rod must transmit a tensile force of 50 kN without exceeding an allowable stress of 180 MPa or stretching more than 5 mm.

Solution

If the normal stress in the rod cannot exceed 180 MPa, the cross-sectional area must equal or exceed

$$A \geq \frac{P}{\sigma} = \frac{(50 \text{ kN})(1{,}000 \text{ N/kN})}{180 \text{ N/mm}^2} = 277.7778 \text{ mm}^2$$

If the elongation must not exceed 5 mm, the cross-sectional area must equal or exceed

$$A \geq \frac{PL}{E\delta} = \frac{(50 \text{ kN})(1{,}000 \text{ N/kN})(7{,}500 \text{ mm})}{(200{,}000 \text{ N/mm}^2)(5 \text{ mm})} = 375.0000 \text{ mm}^2$$

Therefore, the minimum cross-sectional area that may be used for the rod is $A_{min} = 375 \text{ mm}^2$. The corresponding rod diameter is

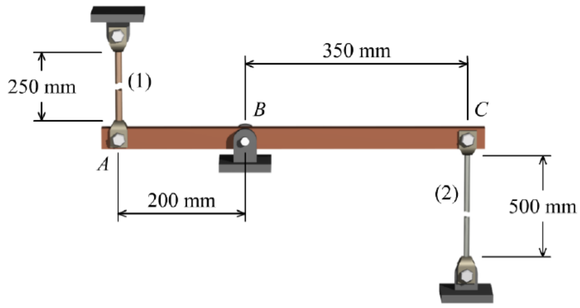
$$\frac{\pi}{4} d_{rod}^2 \geq 375 \text{ mm}^2 \qquad \therefore d_{rod} \geq 21.8510 \text{ mm} = \boxed{21.9 \text{ mm}}$$

Ans.

Problem II (8 Marks)

The pin-connected structure shown consists of a rigid bar *ABC*, a solid bronze [E = 100 GPa, α = 16.9 × 10^{-6}/°C] rod (1), and a solid aluminum alloy [E = 70 GPa, α = 22.5×10^{-6}/°C] rod (2). Bronze rod (1) has a diameter of 24 and aluminum rod (2) has a diameter of 16 mm. The bars are unstressed when the structure is assembled at 25°C.

After assembly, the temperature of rod (2) is decreased by 40°C while the temperature of rod (1) remains constant at 25°C. Determine the normal stresses in both rods for this condition.

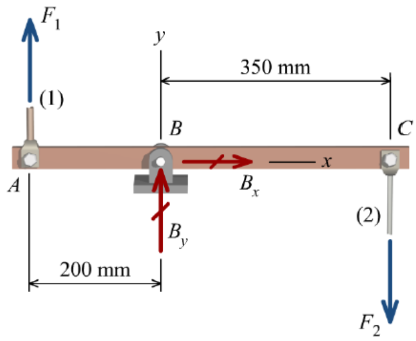


Solution

Equilibrium

Consider a FBD of the rigid bar. Assume tension forces in members (1) and (2). A moment equation about pin B gives the best information for this situation:

$$\Sigma M_B = -(200 \text{ mm})F_1 - (350 \text{ mm})F_2 = 0 \qquad \textbf{(a)}$$



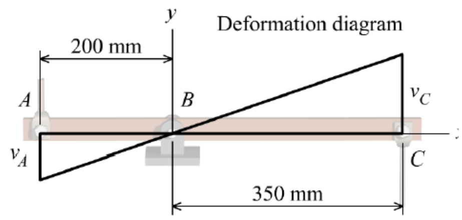
Geometry of Deformations Relationship

Draw a deformation diagram of the rigid bar. The deflections of the rigid bar are related by similar triangles:

$$\frac{v_A}{200 \text{ mm}} = \frac{v_C}{350 \text{ mm}} \qquad \text{(b)}$$

There are no gaps, clearances, or other misfits at pins A and C; therefore, Eq. (c) can be rewritten in terms of the member deformations as:

$$\frac{\delta_1}{200 \text{ mm}} = \frac{\delta_2}{350 \text{ mm}} \qquad \text{(c)}$$



Force-Temperature-Deformation Relationships

$$\delta_1 = \frac{F_1 L_1}{A_1 E_1} + \alpha_1 \Delta T_1 L_1 \qquad \delta_2 = \frac{F_2 L_2}{A_2 E_2} + \alpha_2 \Delta T_2 L_2 \qquad \text{(d)}$$

Compatibility Equation

$$\frac{1}{200 \text{ mm}}\left[\frac{F_1 L_1}{A_1 E_1} + \alpha_1 \Delta T_1 L_1\right] = \frac{1}{350 \text{ mm}}\left[\frac{F_2 L_2}{A_2 E_2} + \alpha_2 \Delta T_2 L_2\right] \qquad \text{(e)}$$

Solve the Equations

Solve Eq. (a) for F_1:

$$F_1 = -\frac{350 \text{ mm}}{200 \text{ mm}} F_2 = -1.75 F_2 \qquad \text{(f)}$$

Substitute this result into Eq. (e):

$$\frac{1}{200 \text{ mm}}\left[\frac{(-1.75F_2)L_1}{A_1 E_1} + \alpha_1 \Delta T_1 L_1\right] = \frac{1}{350 \text{ mm}}\left[\frac{F_2 L_2}{A_2 E_2} + \alpha_2 \Delta T_2 L_2\right]$$

Note that $\Delta T_1 = 0°C$; therefore, the equation simplifies to:

$$-\frac{1.75F_2L_1}{A_1E_1} = \frac{200 \text{ mm}}{350 \text{ mm}}\left[\frac{F_2L_2}{A_2E_2} + \alpha_2\Delta T_2 L_2\right]$$

Solve this equation for F_2:

$$-\frac{1.75F_2L_1}{A_1E_1} = \frac{200 \text{ mm}}{350 \text{ mm}}\left[\frac{F_2L_2}{A_2E_2} + \alpha_2\Delta T_2 L_2\right]$$

$$F_2\left[\frac{1.75L_1}{A_1E_1} + \frac{200}{350}\frac{L_2}{A_2E_2}\right] = -\frac{200}{350}\alpha_2\Delta T_2 L_2$$

$$F_2 = -\frac{\frac{200}{350}\alpha_2\Delta T_2 L_2}{\frac{1.75L_1}{A_1E_1} + \frac{200}{350}\frac{L_2}{A_2E_2}} \qquad \text{(g)}$$

For this structure, the lengths, areas, coefficients of thermal expansion, and elastic moduli are given below.

$L_1 = 250 \text{ mm}$

$A_1 = \frac{\pi}{4}(24 \text{ mm})^2 = 452.3893 \text{ mm}^2$

$E_1 = 100{,}000 \text{ MPa}$

$\alpha_1 = 16.9 \times 10^{-6} / °C$

$L_2 = 500 \text{ mm}$

$A_2 = \frac{\pi}{4}(16 \text{ mm})^2 = 201.0619 \text{ mm}^2$

$E_2 = 70{,}000 \text{ MPa}$

$\alpha_2 = 22.5 \times 10^{-6} / °C$

Substitute these values along with $\Delta T_2 = -40°C$ into Eq. (g) and calculate $F_2 = -8{,}579.65$ N. From Eq. (f), calculate $F_1 = -15{,}014.39$ N.

Normal Stresses

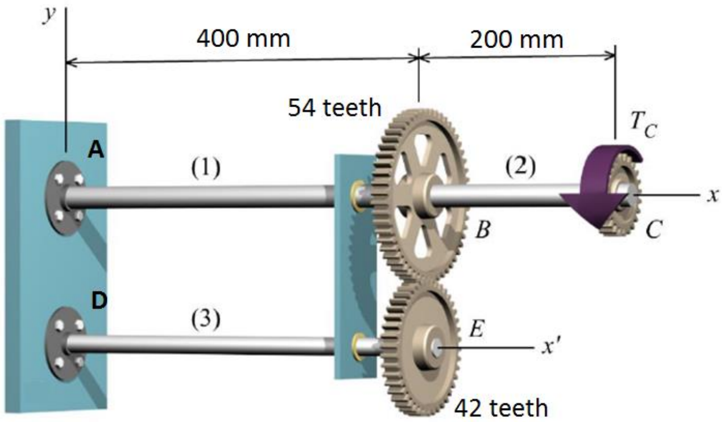
The normal stresses in each axial member can now be calculated:

$$\sigma_1 = \frac{F_1}{A_1} = \frac{-15{,}014.39 \text{ N}}{452.3893 \text{ mm}^2} = \boxed{33.2 \text{ MPa (C)}}$$ **Ans.**

$$\sigma_2 = \frac{F_2}{A_2} = \frac{8{,}579.65 \text{ N}}{201.0619 \text{ mm}^2} = \boxed{42.7 \text{ MPa (T)}}$$ **Ans.**

Problem III (10 Marks)

A torque of T_C = 460 N-m acts on gear C of the assembly shown. Shafts (1) and (2) are solid 35-mm-diameter aluminum shafts and shaft (3) is a solid 25-mm-diameter aluminum shaft. G = 28 GPa. Determine:
(a) the maximum shear stress magnitude in shaft (1).
(b) the maximum shear stress magnitude in shaft segment (3).
(c) the rotation angle of gear E.
(d) the rotation angle of gear C.



Solution

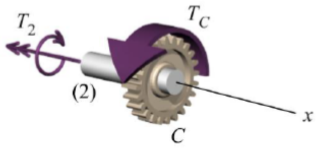
Section Properties: The polar moments of inertia for the shafts are:

$$J_1 = J_2 = \frac{\pi}{32}(35\text{ mm})^4 = 147{,}323.5\text{ mm}^4$$

$$J_3 = \frac{\pi}{32}(25\text{ mm})^4 = 38{,}349.5\text{ mm}^4$$

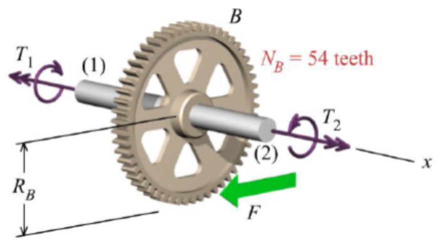
Equilibrium: Consider a free-body diagram cut through shaft (2) and around gear C:

$$\Sigma M_x = -T_2 + T_C = 0 \qquad \therefore T_2 = T_C = 460\text{ N-m} \quad \text{(a)}$$



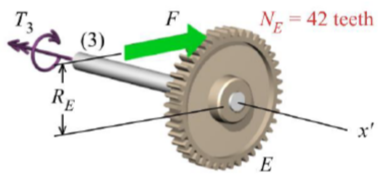
Next, consider a FBD cut around gear B through shafts (1) and (2). The teeth of gear E exert a force F on the teeth of gear B. This force F opposes the rotation of gear B. The radius of gear B will be denoted by R_B for now (even though the gear radius is not given explicitly).

$$\Sigma M_x = T_2 - T_1 - F \cdot R_B = 0 \quad \text{(b)}$$



Finally, consider a FBD cut around gear E through shaft (3). The teeth of gear B exert an equal magnitude force F on the teeth of gear C, acting opposite to the direction assumed in the previous FBD. The radius of gear E will be denoted by R_E for now.

$$\Sigma M_{x'} = -T_3 - F \cdot R_E = 0 \qquad \therefore F = -\frac{T_3}{R_E} \quad \text{(c)}$$



The results of Eqs. (a) and (c) can be substituted into Eq. (b) to give

$$T_1 = 460 \text{ N-m} + T_3 \frac{R_B}{R_E}$$

The ratio R_B/R_E is simply the gear ratio between gears B and E, which can also be expressed in terms of gear teeth N_B and N_E:

$$T_1 = 460 \text{ N-m} + T_3 \frac{N_B}{N_E} = 460 \text{ N-m} + T_3 \left(\frac{54 \text{ teeth}}{42 \text{ teeth}} \right) = 460 \text{ N-m} + 1.285714 T_3 \qquad \text{(d)}$$

Equation (d) summarizes the results of the equilibrium considerations, but there are still two unknowns in this equation: T_1 and T_3. Consequently, this problem is statically indeterminate. To solve the problem, an additional equation must be developed. This second equation will be derived from the relationship between the angles of twist in shafts (1) and (3).

Geometry of Deformation Relationship:
The rotation of gear B is equal to the angle of twist in shaft (1):

$$\phi_B = \phi_1$$

and the rotation of gear E is equal to the angle of twist in shaft (3):

$$\phi_E = \phi_3$$

However, since the gear teeth mesh, the rotation angles for gears B and E are not independent. The arclengths associated with the respective rotations must be equal, but the gears turn in opposite directions. The relationship between gear rotations can be stated as:

$$R_B \phi_B = -R_E \phi_E$$

where R_B and R_E are the radii of gears B and E, respectively. Since the gear rotation angles are related to the shaft angles of twist, this relationship can be expressed as:

$$R_B \phi_1 = -R_E \phi_3 \qquad \text{(e)}$$

Torque-Twist Relationships:

$$\phi_1 = \frac{T_1 L_1}{J_1 G_1} \qquad \phi_3 = \frac{T_3 L_3}{J_3 G_3} \qquad \text{(f)}$$

Compatibility Equation:
Substitute the torque-twist relationships [Eqs. (f)] into the geometry of deformation relationship [Eq. (e)] to obtain:

$$R_B \frac{T_1 L_1}{J_1 G_1} = -R_E \frac{T_3 L_3}{J_3 G_3}$$

which can be rearranged and expressed in terms of the gear ratio N_B/N_E:

$$\frac{N_B}{N_E} \frac{T_1 L_1}{J_1 G_1} = -\frac{T_3 L_3}{J_3 G_3} \qquad \text{(g)}$$

Note that the compatibility equation has two unknowns: T_1 and T_3. This equation can be solved simultaneously with Eq. (d) to calculate the internal torques in shafts (1) and (3).

Solve the Equations: Solve for internal torque T_3 in Eq. (g):

$$T_3 = -T_1 \left(\frac{N_B}{N_E} \right) \left(\frac{L_1}{L_3} \right) \left(\frac{J_3}{J_1} \right) \left(\frac{G_3}{G_1} \right)$$

and substitute this result into Eq. (d):

$$\begin{aligned} T_1 &= 460 \text{ N-m} + 1.285714 T_3 \\ &= 460 \text{ N-m} + 1.285714 \left[-T_1 \left(\frac{N_B}{N_E} \right) \left(\frac{L_1}{L_3} \right) \left(\frac{J_3}{J_1} \right) \left(\frac{G_3}{G_1} \right) \right] \\ &= 460 \text{ N-m} - 1.285714 \left[T_1 \left(\frac{54 \text{ teeth}}{42 \text{ teeth}} \right) \left(\frac{38,349.5 \text{ mm}^4}{147,323.5 \text{ mm}^4} \right) \right] \\ &= 460 \text{ N-m} - 0.430305 T_1 \end{aligned}$$

Group the T_1 terms to obtain:

$$T_1 = \frac{460 \text{ N-m}}{1.430305} = 321.610 \text{ N-m}$$

Backsubstitute this result into Eq. (d) to find the internal torque in shaft (3):

$$T_1 = 460 \text{ N-m} + 1.285714 T_3$$

$$\therefore T_3 = \frac{T_1 - 460 \text{ N-m}}{1.285714} = \frac{321.610 \text{ N-m} - 460 \text{ N-m}}{1.285714} = -107.637 \text{ N-m}$$

(a) Maximum Shear Stress Magnitude in Shaft (1):

$$\tau_1 = \frac{T_1 c_1}{J_1} = \frac{(321.610 \text{ N-m})(35 \text{ mm}/2)(1{,}000 \text{ mm/m})}{147{,}323.5 \text{ mm}^4} = \boxed{38.2 \text{ MPa}}$$ **Ans.**

(b) Maximum Shear Stress Magnitude in Shaft (3):

$$\tau_3 = \frac{T_3 c_3}{J_3} = \frac{(107.637 \text{ N-m})(25 \text{ mm}/2)(1{,}000 \text{ mm/m})}{38{,}349.5 \text{ mm}^4} = \boxed{35.1 \text{ MPa}}$$ **Ans.**

(c) Rotation Angle of Gear *E*:

$$\phi_3 = \frac{T_3 L_3}{J_3 G_3} = \frac{(-107.637 \text{ N-m})(2)(200 \text{ mm})(1{,}000 \text{ mm/m})}{(38{,}349.5 \text{ mm}^4)(28{,}000 \text{ N/mm}^2)} = -0.040096 \text{ rad}$$

$$\therefore \phi_E = \phi_3 = \boxed{-0.0401 \text{ rad}}$$ **Ans.**

(d) Rotation Angle of Gear *C*:

$$\phi_1 = \frac{T_1 L_1}{J_1 G_1} = \frac{(321.610 \text{ N-m})(2)(200 \text{ mm})(1{,}000 \text{ mm/m})}{(147{,}323.5 \text{ mm}^4)(28{,}000 \text{ N/mm}^2)} = 0.031186 \text{ rad}$$

$$\phi_2 = \frac{T_2 L_2}{J_2 G_2} = \frac{(460 \text{ N-m})(200 \text{ mm})(1{,}000 \text{ mm/m})}{(147{,}323.5 \text{ mm}^4)(28{,}000 \text{ N/mm}^2)} = 0.022303 \text{ rad}$$

$$\therefore \phi_C = \phi_B + \phi_2 = \phi_1 + \phi_2 = 0.031186 \text{ rad} + 0.022303 \text{ rad} = \boxed{0.0535 \text{ rad}}$$ **Ans.**

Problem IV (9 Marks)

A solid stainless steel [G = 86 GPa] shaft that is 2.0 m long will be subjected to a pure torque of T = 75 N-m. Determine the minimum diameter required if the shear stress must not exceed 50 MPa and the angle of twist must not exceed 4°. Report both the maximum shear stress τ and the angle of twist ϕ at this minimum diameter.

Solution

Consider shear stress:

The polar moment of inertia for a solid shaft can be expressed as

$$J = \frac{\pi}{32} d^4$$

The elastic torsion formula can be rearranged to gather terms with d:

$$\frac{\pi}{32} \frac{d^4}{(d/2)} = \frac{\pi d^3}{16} = \frac{T}{\tau}$$

From this equation, the unknown diameter of the solid shaft can be expressed as

$$d = \sqrt[3]{\frac{16T}{\pi \tau}}$$

For the solid stainless steel shaft, the minimum diameter that will satisfy the allowable shear stress is:

$$d \geq \sqrt[3]{\frac{16(75 \text{ N-m})(1{,}000 \text{ mm/m})}{\pi(50 \text{ N/mm}^2)}} = 19.70 \text{ mm}$$

Consider angle of twist:

Rearrange the angle of twist equation:

$$\phi = \frac{TL}{JG} \qquad \therefore J = \frac{\pi}{32} d^4 \geq \frac{TL}{\phi G}$$

and solve for the minimum diameter that will satisfy the angle of twist limitation:

$$d \geq \sqrt[4]{\frac{32TL}{\pi \phi G}} = \sqrt[4]{\frac{32(75 \text{ N-m})(2{,}000 \text{ mm})(1{,}000 \text{ mm/m})}{\pi(0.069813 \text{ rad})(86{,}000 \text{ N/mm}^2)}} = 22.46 \text{ mm}$$

Therefore, the minimum diameter that could be used for the shaft is

$$d_{min} = \boxed{22.5 \text{ mm}}$$ **Ans.**

The angle of twist for this shaft is ϕ = 0.069813 rad. To compute the shear stress in a 22.46-mm-diameter shaft, first compute the polar moment of inertia:

$$J = \frac{\pi}{32}(22.46 \text{ mm})^4 = 24{,}983.625 \text{ mm}^4$$

The shear stress in the shaft is thus:

$$\tau_{max} = \frac{(75 \text{ N-m})(22.46 \text{ mm}/2)(1{,}000 \text{ mm/m})}{24{,}983.625 \text{ mm}^4} = \boxed{33.7 \text{ MPa}}$$ **Ans.**