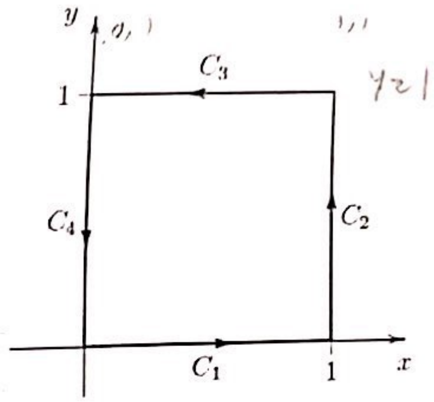
Q1 [7 pts]. Evaluate $\oint_C \overset{F_1}{xy}dx + \overset{F_2}{(x^2 - y^2)}dy$ where



(a) $C = C_3$.

$\vec{r}(t) = [0,1] + [1,0]t$

$= [t, 1]$

$\int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

$\int [t, t^2 - 1] \cdot [1, 0]$

$\int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$

-2

(b) $C = C_1 \cup C_2 \cup C_3 \cup C_4$. **(Hint**: use Green's Theorem)

$\iint \frac{\partial F_2}{\partial x} - \frac{\partial F}{\partial y}\,dy\,dx$

$\int_0^1\int_0^1 2x - x\,dy\,dx$

$\int_0^1 xy\Big|_0^1 dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

Q2 [8 pts]. Evaluate the surface integral $\iint_S \mathbf{F}.d\mathbf{S}$, where $\mathbf{F} = x\hat{i} + y\hat{j} + (2x+2y)\hat{k}$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the unit disk ($x^2 + y^2 \le 1$) with upward orientation. $\vec{r}(u,v) = [u\cos v, u\sin v, u^2]$

$\iint F \cdot ds = \iint \vec{F}(\vec{r}(u,v)) \cdot \vec{N}\, du\, dv$

$\vec{F}(\vec{r}(u,v)) = u\cos v\hat{i} + u\sin v\hat{i} + (2u\cos v + 2u\sin v)\hat{k}$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \nabla(x^2 + y^2 + z - 4 = 0)$

$= [2x, 2y, 1]$

$\iint \vec{F}(r(u,v)) \cdot \vec{N}\, du\, dv = \iint [u\cos v, u\sin v, 2u\cos v + 2u\sin v]$

$\cdot [2u\cos v, 2u\sin v, 1]\, du\, dv$

$\iint 2u^2\cos^2 v\hat{i} + 2u^2\sin^2 v\hat{j} + (2u\cos v + 2u\sin v)\hat{k}$

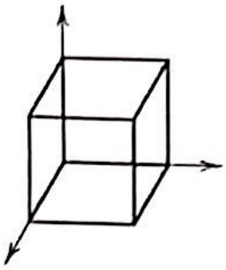
-5

$r_u = \cos v, \sin v, 2u$

$r_v = -u\sin v, u\cos v, 0$

$\begin{vmatrix} \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = -2u^2\cos v\hat{i} + 2u^2\sin v\hat{j} + (u\cos^2 v + u\sin^2 v)\hat{k}$

Q3 [8 pts]. Let E be the solid unit cube with opposing corners at the origin and (1, 1, 1) and faces parallel to the coordinate planes. Let S be the boundary surface of E, oriented with the outward-pointing normal. If $\mathbf{F} = [2xy, 3ye^z, x\sin z]$, then



(c) Compute $div(\mathbf{F})$.

$= 2y + 3e^z + x\cos z$

(d) Compute the flux integral $\iint_S \mathbf{F}.d\mathbf{S}$, using the divergence theorem.

$\iiint \frac{F_1}{\partial x} + \frac{F_2}{\partial y} + \frac{F_3}{\partial z}\, dA$

$= \int_0^1\int_0^1\int_0^1 2y + 3e^z + x\cos z \; dz\, dy\, dx$

$= \int_0^1\int_0^1 2yz + 3e^z + x\sin z \Big|_0^1 dy\, dx$

$= \iint 2y$

2

0

-6

(e) Show that $\iint_S curl(\mathbf{F}).d\mathbf{S} = 0$.

$curl\,(\nabla F) = \vec{0}$

Q4 [7 pts]. Suppose $\mathbf{F} = -y\hat{i} + x\hat{j} + z\hat{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 25$ below the plane $z = 4$, oriented with the outward-pointing normal (so that the normal at $(5, 0, 0)$ is $\hat{i}$).

(f) Compute $curl(\mathbf{F})$.

$$curl F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = 0i - 0j + (1+1)k = 2\hat{k}$$

(g) Compute the flux integral $\iint_S curl(\mathbf{F}).d\mathbf{S}$.

$$\iint [0,0,2] \cdot \vec{N}\, du\, dv$$

$$\vec{r}(u,v) = [\cos v \cos u, \cos v \sin u, \sin v]$$

$$\vec{r}_u = [-\cos v \sin u, \cos v \cos u, 0]$$

$$\vec{r}_v = [-\cos u \sin v, -\sin v \sin u, \cos v]$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v$$

$$= \iint 2\cos v \sin v \sin u^2 + 2\cos u^2 \cos v \sin v\, du\, dv$$

$-y$