

Partial Fractions

Matthew M. Peet
Illinois Institute of Technology

The Partial Fraction Expansion

Introduction

In this Lecture, you will learn: The Inverse Laplace Transform

- Simple Forms

The Partial Fraction Expansion

- How poles relate to dominant modes
- Expansion using single poles
- Repeated Poles
- Complex Pairs of Poles
 - ▶ Inverse Laplace

Recall: The Inverse Laplace Transform of a Signal

To go from a frequency domain signal, $\hat{u}(s)$, to the time-domain signal, $u(t)$, we use the **Inverse Laplace Transform**.

Definition 1.

The **Inverse Laplace Transform** of a signal $\hat{u}(s)$ is denoted $u(t) = \Lambda^{-1}\hat{u}$.

$$u(t) = \Lambda^{-1}\hat{u} = \int_0^{\infty} e^{i\omega t} \hat{u}(i\omega) d\omega$$

- Like Λ , the inverse Laplace Transform Λ^{-1} is also a Linear system.
- **Identity:** $\Lambda^{-1}\Lambda u = u$.
- Calculating the Inverse Laplace Transform can be tricky. e.g.

$$\hat{y} = \frac{s^3 + s^2 + 2s - 1}{s^4 + 3s^3 - 2s^2 + s + 1}$$

Poles and Rational Functions

Definition 2.

A **Rational Function** is the ratio of two polynomials:

$$\hat{u}(s) = \frac{n(s)}{d(s)}$$

Most transfer functions are rational.

Definition 3.

The point s_p is a **Pole** of the rational function $\hat{u}(s) = \frac{n(s)}{d(s)}$ if $d(s_p) = 0$.

- It is convenient to write a rational function using its poles

$$\frac{n(s)}{d(s)} = \frac{n(s)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$

- The Inverse Laplace Transform of an isolated pole is easy:

$$\hat{u}(s) = \frac{1}{s + p} \quad \text{means} \quad u(t) = e^{-pt}$$

Partial Fraction Expansion

Definition 4.

The **Degree** of a polynomial $n(s)$, is the highest power of s with a nonzero coefficient.

Example: The degree of $n(s)$ is 4

$$n(s) = s^4 + .5s^2 + 1$$

Definition 5.

A rational function $\hat{u}(s) = \frac{n(s)}{d(s)}$ is **Strictly Proper** if the degree of $n(s)$ is less than the degree of $d(s)$.

- We assume that $n(s)$ has lower degree than $d(s)$
- Otherwise, perform long division until we have a **strictly proper** remainder

$$\frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5} = s + 1 + \frac{2}{s^2 + s + 5}$$

Poles and Inverse Laplace Transforms

A Strictly Proper Rational Function is The Sum of Poles: We can *usually* find coefficients r_i such that

$$\frac{n(s)}{(s - p_1)(s - p_2) \cdots (s - p_n)} = \frac{r_1}{s - p_1} + \cdots + \frac{r_n}{s - p_n}$$

- Except in the case of repeated poles.

Poles Dominate the Motion: Because a signal is the sum of poles, The inverse Laplace has the form

$$u(t) = r_1 e^{p_1 t} + \cdots + r_n e^{p_n t}$$

- p_i may be complex.
 - ▶ If p_i are complex, r_i may be complex.
- Doesn't hold for repeated poles.

Examples

Simple State-Space: Step Response

$$\hat{y}(s) = \frac{s-1}{(s+1)s} = \frac{2}{s+1} - \frac{1}{s}$$

$$y(t) = e^{-t} - \frac{1}{2}\mathbf{1}(t)$$

Suspension System: Impulse Response

$$\hat{y}(s) = \frac{1}{J} \frac{1}{s^2 - \frac{Mgl}{2J}} = \frac{1}{J} \sqrt{\frac{2J}{Mgl}} \left(\frac{1}{s - \sqrt{\frac{Mgl}{2J}}} - \frac{1}{s + \sqrt{\frac{Mgl}{2J}}} \right)$$

$$y(t) = \frac{1}{J} \sqrt{\frac{2J}{Mgl}} \left(e^{\sqrt{\frac{Mgl}{2J}}t} - e^{-\sqrt{\frac{Mgl}{2J}}t} \right)$$

Simple State-Space: Sinusoid Response

$$\hat{y}(s) = \frac{s-1}{(s+1)(s^2+1)} = \left(\frac{s}{s^2+1} - \frac{1}{s+1} \right)$$

$$y(t) = \frac{1}{2} \cos t - \frac{1}{2} e^{-t}$$

Partial Fraction Expansion

Conclusion: If we can find coefficients r_i such that

$$\begin{aligned}\hat{u}(s) &= \frac{n(s)}{d(s)} = \frac{n(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \\ &= \frac{r_1}{s-p_1} + \cdots + \frac{r_n}{s-p_n},\end{aligned}$$

then this is the **PARTIAL FRACTION EXPANSION** of $\hat{u}(s)$.

We will address several cases of increasing complexity:

1. $d(s)$ has all real, non-repeating roots.
2. $d(s)$ has all real roots, some repeating.
3. $d(s)$ has complex, repeated roots.

Case 1: Real, Non-repeated Roots

This case is the easiest:

$$\hat{y}(s) = \frac{n(s)}{(s - p_1) \cdots (s - p_n)}$$

where p_n are all real and distinct.

Theorem 6.

For case 1, $\hat{y}(s)$ can always be written as

$$\hat{y}(s) = \frac{r_1}{s - p_1} + \cdots + \frac{r_n}{s - p_n}$$

where the r_i are all real constants

The trick is to solve for the r_i

- There are n unknowns - the r_i .
- To solve for the r_i , we evaluate the equation for values of s .
 - ▶ Potentially unlimited equations.
 - ▶ We only need n equations.
 - ▶ We will evaluate at the points $s = p_i$.

Case 1: Real, Non-repeated Roots

Solving

We want to solve

$$\hat{y}(s) = \frac{r_1}{s - p_1} + \cdots + \frac{r_n}{s - p_n} \quad \text{for } r_1, r_2, \dots$$

For each r_i , we **Multiply by $(s - p_i)$** .

$$\begin{aligned}\hat{y}(s)(s - p_i) &= r_1 \frac{s - p_i}{s - p_1} + \cdots + r_i \frac{s - p_i}{s - p_i} + \cdots + r_n \frac{s - p_i}{s - p_n} \\ &= r_1 \frac{s - p_i}{s - p_1} + \cdots + r_i + \cdots + r_n \frac{s - p_i}{s - p_n}\end{aligned}$$

Evaluate the Right Hand Side at $s = p_i$: We get

$$\begin{aligned}r_1 \frac{p_i - p_i}{p_i - p_1} + \cdots + r_i + \cdots + r_n \frac{p_i - p_i}{p_i - p_n} \\ = r_1 \frac{0}{p_i - p_1} + \cdots + r_i + \cdots + r_n \frac{0}{p_i - p_n} = r_i\end{aligned}$$

Setting $LHS = RHS$, we get a simple formula for r_i :

$$r_i = \hat{y}(s)(s - p_i)|_{s=p_i}$$

Case 1: Real, Non-repeated Roots

Calculating r_i

So we can find all the coefficients:

$$r_i = \hat{y}(s)(s - p_i)|_{s=p_i}$$

$$\begin{aligned}\hat{y}(s)(s - p_i) &= \frac{n(s)}{(s - p_1) \cdots (s - p_{i-1})(s - p_i)(s - p_{i+1}) \cdots (s - p_n)}(s - p_i) \\ &= \frac{n(s)}{(s - p_1) \cdots (s - p_{i-1})(s - p_{i+1}) \cdots (s - p_n)}\end{aligned}$$

- This is \hat{y} with the pole $s - p_i$ removed.

Definition 7.

The **Residue** of \hat{y} at $s = p_i$ is the value of $\hat{y}(p_i)$ with the pole $s - p_i$ removed.

Thus we have

$$r_i = \hat{y}(s)(s - p_i)|_{s=p_i} = \frac{n(p_i)}{(p_i - p_1) \cdots (p_i - p_{i-1})(p_i - p_{i+1}) \cdots (p_i - p_n)}$$

Example: Case 1

$$\hat{y}(s) = \frac{2s + 6}{(s + 1)(s + 2)}$$

We first separate \hat{y} into parts:

$$\hat{y}(s) = \frac{2s + 6}{(s + 1)(s + 2)} = \frac{r_1}{s + 1} + \frac{r_2}{s + 2}$$

Residue 1: we calculate:

$$r_1 = \frac{2s + 6}{(s + 1)(s + 2)}(s + 1)|_{s=-1} = \frac{2s + 6}{s + 2}|_{s=-1} = \frac{-2 + 6}{-1 + 2} = \frac{4}{1} = 4$$

Residue 2: we calculate:

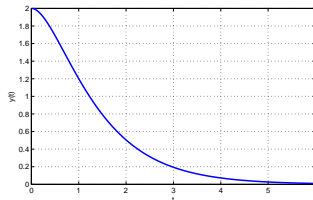
$$r_2 = \frac{2s + 6}{(s + 1)(s + 2)}(s + 2)|_{s=-2} = \frac{2s + 6}{s + 1}|_{s=-2} = \frac{-4 + 6}{-2 + 1} = \frac{2}{-1} = -2$$

Thus

$$\hat{y}(s) = \frac{4}{s + 1} + \frac{-2}{s + 2}$$

Concluding,

$$y(t) = 4e^{-t} - 2e^{-2t}$$



Case 2: Real, Repeated Roots

Sometimes \hat{y} has a repeated pole:

$$\hat{y}(s) = \frac{n(s)}{(s - p_1)^q (s - p_2) \cdots (s - p_n)}$$

In this case, we **CAN NOT** use simple expansion (As in case 1). Instead we have

$$\begin{aligned}\hat{y}(s) &= \frac{n(s)}{(s - p_1)^q (s - p_2) \cdots (s - p_n)} \\ &= \left(\frac{r_{11}}{(s - p_1)} + \frac{r_{12}}{(s - p_1)^2} + \cdots + \frac{r_{1q}}{(s - p_1)^q} \right) + \frac{r_2}{s - p_2} + \cdots + \frac{r_n}{s - p_n}\end{aligned}$$

- The r_{ij} are still real-valued.

Example: Find the r_{ij}

$$\frac{(s + 3)^2}{(s + 2)^3 (s + 1)^2} = \frac{r_{11}}{s + 2} + \frac{r_{12}}{(s + 2)^2} + \frac{r_{13}}{(s + 2)^3} + \frac{r_{21}}{s + 1} + \frac{r_{22}}{(s + 1)^2}$$

Case 2: Real, Repeated Roots

Problem: We have more coefficients to find.

- q coefficients for each repeated root.

First Step: Solve for r_2, \dots, r_n as before:

If p_i is not a repeated root, then

$$r_i = \frac{n(p_i)}{(p_i - p_1)^q \cdots (p_i - p_{i-1})(p_i - p_{i+1}) \cdots (p_i - p_n)}$$

Case 2: Real, Repeated Roots

New Step: Multiply by $(s - p_1)^q$ to get the coefficient r_{1q} .

$$\begin{aligned}\hat{y}(s)(s - p_1)^q &= \left(r_{11} \frac{(s - p_1)^q}{(s - p_1)} + r_{12} \frac{(s - p_1)^q}{(s - p_1)^2} + \cdots + r_{1q} \frac{(s - p_1)^q}{(s - p_1)^q} \right) \\ &\quad + r_2 \frac{(s - p_1)^q}{s - p_2} + \cdots + r_n \frac{(s - p_1)^q}{s - p_n} \\ &= (r_{11}(s - p_1)^{q-1} + r_{12}(s - p_1)^{q-2} + \cdots + r_{1q}) \\ &\quad + r_2 \frac{(s - p_1)^q}{s - p_2} + \cdots + r_n \frac{(s - p_1)^q}{s - p_n}\end{aligned}$$

and evaluate at the point $s = p_i$ to get .

$$r_{1q} = \hat{y}(s)(s - p_1)^q|_{s=p_1}$$

- r_{1q} is $\hat{y}(p_1)$ with the repeated pole removed.

Case 2: Real, Repeated Roots

To find the remaining coefficients, we **Differentiate**:

$$\hat{y}(s)(s - p_1)^q = (r_{11}(s - p_1)^{q-1} + r_{12}(s - p_1)^{q-2} + \dots + r_{1q}) \\ + r_2 \frac{(s - p_1)^q}{s - p_2} + \dots + r_n \frac{(s - p_1)^q}{s - p_n}$$

to get

$$\frac{d}{ds} (\hat{y}(s)(s - p_1)^q) \\ = ((q - 1)r_{11}(s - p_1)^{q-2} + (q - 2)r_{12}(s - p_1)^{q-3} + \dots + r_{1,(q-1)}) \\ + \frac{d}{ds} (s - p_1)^q \left(\frac{r_2}{s - p_2} + \dots + \frac{r_n}{s - p_n} \right)$$

Evaluating at the point $s = p_1$.

$$\frac{d}{ds} (\hat{y}(s)(s - p_1)^q) \big|_{s=p_1} = r_{1,(q-1)}$$

Case 2: Real, Repeated Roots

If we differentiate again, we get

$$\begin{aligned} \frac{d^2}{ds^2} (\hat{y}(s)(s - p_1)^q) \\ = ((q - 1)(q - 2)r_{11}(s - p_1)^{q-3} + \dots + 2r_{1,(q-2)}) \\ + \frac{d^2}{ds^2}(s - p_1)^q \left(\frac{r_2}{s - p_2} + \dots + \frac{r_n}{s - p_n} \right) \end{aligned}$$

Evaluating at $s = p_1$, we get

$$r_{1,(q-2)} = \frac{1}{2} \frac{d^2}{ds^2} (\hat{y}(s)(s - p_1)^q) \big|_{s=p_1}$$

Extending this indefinitely

$$r_{1,j} = \frac{1}{(q - j - 1)!} \frac{d^{q-j-1}}{ds^{q-j-1}} (\hat{y}(s)(s - p_1)^q) \big|_{s=p_1}$$

Of course, calculating this derivative is often difficult.

Example: Real, Repeated Roots

Expand a simple example

$$\hat{y}(s) = \frac{s+3}{(s+2)^2(s+1)} = \frac{r_{11}}{s+2} + \frac{r_{12}}{(s+2)^2} + \frac{r_2}{s+1}$$

First Step: Calculate r_2

$$r_2 = \frac{s+3}{(s+2)^2} \Big|_{s=-1} = \frac{-1+3}{(-1+2)^2} = \frac{2}{1} = 2$$

Second Step: Calculate r_{12}

$$r_{12} = \frac{s+3}{s+1} \Big|_{s=-2} = \frac{-2+3}{-2+1} = \frac{1}{-1} = -1$$

The Difficult Step: Calculate r_{11} :

$$\begin{aligned} r_{11} &= \frac{d}{ds} \left(\frac{s+3}{s+1} \right) \Big|_{s=-2} \\ &= \left(\frac{1}{s+1} - \frac{s+3}{(s+1)^2} \right) \Big|_{s=-2} = \frac{1}{-2+1} - \frac{-2+3}{(-2+1)^2} = \frac{1}{-1} - \frac{1}{1} = -2 \end{aligned}$$

Example: Real, Repeated Roots

So now we have

$$\hat{y}(s) = \frac{s+3}{(s+2)^2(s+1)} = \frac{-2}{s+2} + \frac{-1}{(s+2)^2} + \frac{2}{s+1}$$

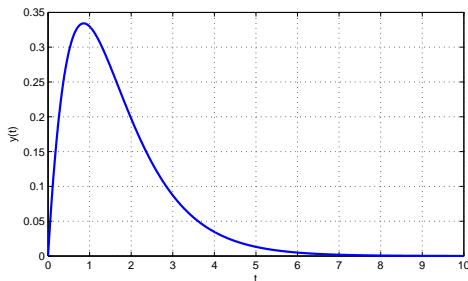
Question: what is the Inverse Fourier Transform of $\frac{1}{(s+2)^2}$?

Recall the Power Exponential:

$$\frac{1}{(s+a)^m} \rightarrow \frac{t^{m-1}e^{-at}}{(m-1)!}$$

Finally, we have:

$$y(t) = -2e^{-2t} - te^{-2t} + 2e^{-t}$$



Case 3: Complex Roots

Most signals have complex roots (More common than repeated roots).

$$\hat{y}(s) = \frac{3}{s(s^2 + 2s + 5)}$$

has roots at $s = 0$, $s = -1 - 2i$, and $s = -1 + 2i$.

Note that:

- Complex roots come in pairs.
- Simple partial fractions will work, but NOT RECOMMENDED
 - ▶ Coefficients will be complex.
 - ▶ Solutions will be complex exponentials.
 - ▶ Require conversion to real functions.

Best to Separate out the Complex pairs as:

$$\hat{y}(s) = \frac{n(s)}{(s^2 + as + b)(s - p_1) \cdots (s - p_n)}$$

Case 3: Complex Roots

Complex pairs have the expansion

$$\frac{n(s)}{(s^2 + as + b)(s - p_1) \cdots (s - p_n)} = \frac{k_1 s + k_2}{s^2 + as + b} + \frac{r_1}{s - p_1} + \cdots + \frac{r_n}{s - p_n}$$

Note that there are two coefficients for each pair: k_1 and k_2 .

NOTE: There are several different methods for finding k_1 and k_2 .

First Step: Solve for r_2, \dots, r_n as normal.

Second Step: Clear the denominator. Multiply equation by all poles.

$$n(s) = (s^2 + as + b)(s - p_1) \cdots (s - p_n) \left(\frac{k_1 s + k_2}{s^2 + as + b} + \frac{r_1}{s - p_1} + \cdots + \frac{r_n}{s - p_n} \right)$$

Third Step: Solve for k_1 and k_2 by examining the coefficients of powers of s .

Warning: May get complicated or impossible for multiple complex pairs.

Case 3 Example

Take the example

$$\hat{y}(s) = \frac{2(s+2)}{(s+1)(s^2+4)} = \frac{k_1s+k_2}{s^2+4} + \frac{r_1}{s+1}$$

First Step: Find the simple coefficient r_1 .

$$r_1 = \left. \frac{2(s+2)}{s^2+4} \right|_{s=-1} = 2 \frac{-1+2}{-1^2+4} = 2 \frac{1}{5} = \frac{2}{5}$$

Next Step: Multiply through by $(s^2+4)(s+1)$.

$$2(s+2) = (s+1)(k_1s+k_2) + r_1(s^2+4)$$

Expanding and using $r_1 = 2/5$ gives:

$$2s+4 = (k_1+2/5)s^2 + (k_2+k_1)s + k_2+8/5$$

Case 3 Example

Recall the equation

$$2s + 4 = (k_1 + 2/5)s^2 + (k_2 + k_1)s + k_2 + 8/5$$

Equating coefficients gives 3 equations:

- s^2 **term** - $0 = k_1 + 2/5$
- s **term** - $2 = k_2 + k_1$
- s^0 **term** - $4 = k_2 + 8/5$

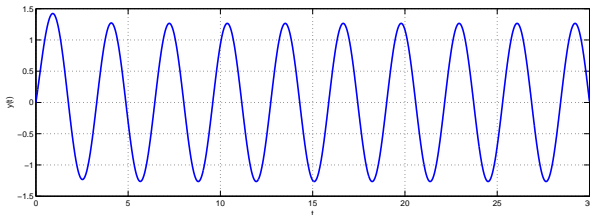
An over-determined system of equations (but consistent)

- First term gives $k_1 = -2/5$
- Second term gives $k_2 = 2 - k_1 = 10/5 + 2/5 = 12/5$
- Double-Check with last term: $4 = 12/5 + 8/5 = 20/5$

Case 3 Example

So we have

$$\begin{aligned}\hat{y}(s) &= \frac{2(s+2)}{(s+1)(s^2+4)} = \frac{2}{5} \left(\frac{1}{s+1} - \frac{s-6}{s^2+4} \right) \\ &= \frac{2}{5} \left(\frac{1}{s+1} - \frac{s}{s^2+4} + \frac{6}{s^2+4} \right) \\ y(t) &= \frac{2}{5} (e^{-t} - \cos(2t) + 3 \sin(2t))\end{aligned}$$



Case 3 Example

Now consider the solution to **A Different Numerical Example:** (Nise)

$$\frac{3}{s(s^2 + 2s + 5)} = \frac{3/5}{s} - \frac{3}{5} \frac{s + 2}{s^2 + 2s + 5}$$

What to do with term:

$$\frac{s + 2}{s^2 + 2s + 5}?$$

We can rewrite as the combination of a frequency shift and a sinusoid:

$$\begin{aligned} \frac{s + 2}{s^2 + 2s + 5} &= \frac{s + 2}{s^2 + 2s + 1 + 4} = \frac{(s + 1) + 1}{(s + 1)^2 + 4} \\ &= \frac{(s + 1)}{(s + 1)^2 + 4} + \frac{1}{2} \frac{2}{(s + 1)^2 + 4} \end{aligned}$$

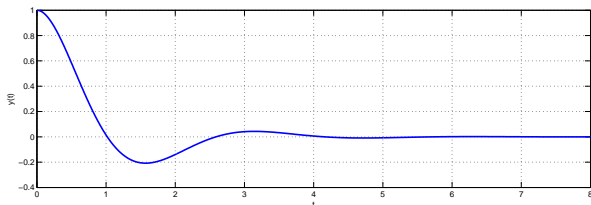
Case 3 Example

$$\frac{s+2}{s^2+2s+5} = \frac{(s+1)}{(s+1)^2+4} + \frac{1}{2} \frac{2}{(s+1)^2+4}$$

- $\frac{(s+1)}{(s+1)^2+4}$ is the sinusoid $\frac{s}{s^2+4}$ shifted by $s \rightarrow s+1$.
- $s \rightarrow s+1$ means multiplication by e^{-t} in the time-domain.
-

$$\Lambda^{-1} \left(\frac{(s+1)}{(s+1)^2+4} \right) = e^{-t} \Lambda^{-1} \left(\frac{s}{s^2+4} \right) = e^{-t} \cos 2t$$

- Likewise $\Lambda^{-1} \left(\frac{2}{(s+1)^2+4} \right) = e^{-t} \sin 2t$



$$\Lambda^{-1} \left(\frac{s+2}{s^2+2s+5} \right) = e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right)$$

Summary

What have we learned today?

In this Lecture, you will learn: The Inverse Laplace Transform

- Simple Forms

The Partial Fraction Expansion

- How poles relate to dominant modes
- Expansion using single poles
- Repeated Poles
- Complex Pairs of Poles
 - ▶ Inverse Laplace

Next Lecture: Important Properties of the Response