** Diff. **

29 - Sep. - 2015
Tuesday.

$* \; y' + 2y = 0 \rightsquigarrow$ DE "1st order".

$y'' + 2y + x = 0 \rightsquigarrow$ DE "2nd order". $\rightsquigarrow$ ODE ordinary DE

$*$ Solve the DE $\xrightarrow{\text{نجد}}$ $y' = ?!$

$\frac{\partial u^2}{\partial^2 x} + \frac{\partial u^2}{\partial^2 y} = 0 \rightsquigarrow$ Partial DE $\rightarrow$ PDE

$* \; y' + xy = 0 \;, y(0) = 1 \rightarrow$ Initial value Problem. "IVP"

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* CH 1 :- (First order)

1. Separable /

$y' = F(x,y)$

$M(x,y)\,\partial x + N(x,y)\,\partial y =$ zero

* Ex: Solve: $(y+1)\,\partial x + (x-1)\,\partial y = 0$

* divide by $(y+1)(x-1)$

$$\frac{\partial x}{(x-1)} + \frac{\partial y}{(y+1)} = 0$$

$\rightsquigarrow$ Integral. $\frac{\partial x}{x-1} = \frac{-\partial y}{y+1}$

$$\ln|y-1| = -\ln|y+1| + c$$

$$\ln|y+1| = c - \ln|x-1|$$

$$y+1 = e^{c-\ln|x-1|}$$

$$y = \frac{c^*}{x-1} - 1$$

* Ex: solve $(xy - 2x)\,\partial x - (xy + y)\,\partial y = 0 \;, y(0) = 3 \;, \overset{\text{"IVP"}}{x = 0}$
$y = 3$

$$x(y-2)\,\partial x - y(x+1)\,\partial y = 0$$

$$\frac{x}{x+1}\,\partial x = \frac{y}{y-2}\,\partial y$$

$$\left(1 - \frac{1}{x+1}\right)\partial x = \left(1 + \frac{2}{y-2}\right)\partial y$$

$x - \ln|x+1| = y + 2\ln|y-2| + c$

* $x = 0$

$y = 3$

$0 = 3 + 0 + c$

$\boxed{C = -3}$

$\therefore\ x - \ln|x+1| = y + 2\ln|y-2| - 3$ ~> Implicit solution.

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* Ex: Solve $e^{x-y^3}\,dx + \frac{3y^2}{x}\,dy = 0$

$e^{x} \cdot e^{-y^3}\,dx + \frac{3y^2}{x}\,dy = 0$

$xe^{x}\,dx + 3y^2 e^{y^3}\,dy = 0$

* $\underbrace{xe^{x}\,dx}_{\text{by parts.}} = \underbrace{-3y^2 e^{y^3}\,dy}_{\text{sub.}}$ $u = y^3$

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※ Ex: Solve $y' + 4xy = 0$

$$\frac{dy}{dx} + 4xy = 0$$

$$dy = -4xy\,dx$$

$$\frac{dy}{y} = -4x\,dx$$

$$\ln y = -2x^2 + c$$

$$y = c^{*} e^{-2x^2}$$

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* Ex: Solve:

$(xy+3x+2y+6)\,dx + (x^2y^2+4y^2+4x^2+16)\,dy = 0$

$(y+3)(x+2)\,dx + (y^2+4)(x^2+4)\,dy = 0$

$\frac{(x+2)}{(x^2+4)}\,dx = -\frac{(y^2+4)}{(y+3)}\,dy$

$\frac{x}{(x^2+4)}\,dx + \frac{2}{x^2+4}\,dx + \;----$

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* Reduction of separable:

* Ex: Solve $2xyy' = y^2 - x^2$ (Not Sep.)

$2xy\,dy = (y^2 - x^2)\,dx \rightsquigarrow$ Reduce to sep.

$\boxed{\begin{array}{l} * y = ux \\ * x = uy \end{array}} \rightsquigarrow$ $y = ux$

$y' = u'x + u$ (الاشتقاق بالنسبة لـ x)

** $2x(ux)(u+u'x) = u^2x^2 - x^2$

$2u(u + xu') = u^2 - 1$

$2xuu' = -(u^2+1)$

$2xu\,du = -(u^2+1)\,dx$

$\frac{2u}{u^2+1}\,du = \frac{-dx}{x}$

$\rightsquigarrow \ln|u^2+1| = -\ln|x| + c$

$u^2+1 = e^{-\ln|x|+c}$

$u^2+1 = \frac{c^*}{x}$

$(\frac{y}{x})^2 + 1 = ---$

If $f(kx,ky) = f(x,y)$ then the ODE is reduced to separable

* If $M(x,y)$ and $N(x,y)$ homogenous function of same degree $\rightsquigarrow$ the ODE is reduced to sep. *

*$y' = f(x,y)$

If $y' = h(\frac{y}{x})$ then reduced to sep. *

$y' = \frac{y^2 - x^2 / x^2}{2xy / x^2}$

$= \frac{(\frac{y}{x})^2 - 1}{2(\frac{y}{x})}$

$*$ Ex: Solve $y\,dx + (y-x)\,dy = 0$ ~> Reduced to sep.

$* \; x = uy \rightarrow \boxed{dx = u\,dy + y\,du}$

$y(u\,dy + y\,du) + (y - uy)\,dy = 0$

$uy\,dy + y^2\,du + (y - uy)\,dy = 0$

$u\,dy + y\,du + (1-u)\,dy = 0$

$y\,du = -dy$

$du = \frac{-dy}{y}$

$u = -\ln|y| + c$

$\frac{x}{y} = -\ln|y| + c$

$x = -y\ln|y| + c$

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$*$ Ex: Solve $(2xy + 3y^2)dx - (2xy + x^2)dy = 0$ ~> Reduced to sep

$y = ux$

$dy = x\,du + u\,dx$

$*(2u + 3u^2)\,dx - (2u+1)(x\,du + u\,dx) = 0$

$(2u + 3u^2)\,dx - [2ux\,du + 2u^2\,dx + x\,du + u\,dx] = 0$

$(2u + 3u^2)\,dx - 2ux\,du - 2u^2\,dx - x\,du - u\,dx = 0$

$(u + u^2)\,dx = x(2u+1)\,du.$

$\frac{dx}{x} = \frac{2u+1}{u+u^2}\,du$

$\ln|x| + = \ln|u+u^2|$

$u^2 + u = c^*\,x$

$\frac{y^2}{x^2} + \frac{y}{x} = c^*\,x$

$y^2 + yx = c^*\,x^3$

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* Exact ODE :

$M(x,y)\,\partial x + N(x,y)\,\partial y = 0$

→ IF $M_y = N_x$

∴ ODE is exact.

- $* \; z = F(x,y) = c$
- $* \; \partial F = \partial z = F_x \partial x + F_y \partial y$
- $* \; \partial F = M(x,y)\partial x + N(x,y)\partial y$.

* the solution of this ODE is given by $F(x,y) = c$

where $F_x = M$, $F_y = N$.

* Ex : Solve $\overset{M}{(y^2 + 3yx^2)}\,\partial x + \overset{N}{(x^3 + 2xy)}\,\partial y = 0$

$$\left.\begin{array}{l} * \; M_y = 2y + 3x^2 \\ * \; N_x = 3x^2 + 2y \end{array}\right\} \rightsquigarrow \text{(exact)},\; F(x,y) = ?!$$

$* \; F_x = M = y^2 + 3x^2 y$

$F(x,y) = \int (y^2 + 3yx^2)\,\partial x \Rightarrow F(x,y) = y^2 x + yx^3 + g(y)$

$F(y) = 2yx + x^3 + g'(y)$

$= x^3 + 2xy$

$* \; g'(y) = 0 \rightsquigarrow g(y) = c_1$

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* The solution of ODE :

$F(x,y) = y^2 x + yx^3 + \underline{g(y)} \sim x0)$

$y^2 x + yx^3 + \underset{(0)}{c_1} = c_2$

$$\boxed{y^2 x + yx^3 = c}$$

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#Ex! Solve: $\overset{N}{(ye^{xy} - 2y^3)}\, dx + \overset{M}{(xe^{xy} - 6xy^2 - 2y)}\, dy = 0$

* $M_y = e^{xy} + xye^{xy} - 6y^2$
* $N_x = e^{xy} + xye^{xy} - 6y^2$ $\Big\} \rightsquigarrow$ Exact

#$F_x = M = ye^{xy} - 2y^3$

#$F(x,y) = \int (ye^{xy} - 2y^3) .dx = e^{xy} - 2xy^3 + g(y)$

#$F_y = N \Rightarrow e^{xy} - 6xy^2 + g'(0) = xe^{xy} - 6xy^2 - 2y$

$g'(y) = -2y \rightsquigarrow \boxed{g(y) = -y^2 + c_1}$

* The solution is: $\boxed{e^{xy} - 2xy^3 - y^2 = c}$

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*Ex! Solve: $y' = \dfrac{\sin(y) + y\sin(x)}{\cos(x) - x(\cos(y))}$

* $\underbrace{(\cos(x) - x\cos(y))\, \partial y}_{(N)} - \underbrace{(\sin(y) + y\sin(x))\, \partial x}_{(M)} = 0$

#$M_y = -\cos(x) - \sin(x)$ $\Big\}$ Exact

#$N_x = -\sin(x) - \cos(x)$

#$F_y = N = \cos x - x\cos y$

#$F(x,y) = \int (\cos x - x\cos y) dy$

$= y\cos x - x\sin y + g(x)$

#$F_x = -y\sin(x) - \sin(y) + g'(x) = -\sin(y) - y\sin(x)$

$g'(x) = 0 \rightsquigarrow g(x) = c_1$

* The solution is: $y\cos(x) - x\sin(y) = c$.

Reduction to exact :

$M(x,y)\,dx + N(x,y)\,dy = 0$

* If $\dfrac{N_x - M_y}{M} = h(y) \rightarrow$ Reduced to exact.

or $\dfrac{M_y - N_x}{N} = h(x) \rightarrow$ Reduced to exact.

$*\,u(y) = e^{\int h(y)\,dy}$, $*\,u(x) = e^{\int h(x)\,dx}$ [Integrating Factor]

Ex: Solve

$y(y^2\cos x + 1)dx + (y^2\sin x - x + y)dy = 0$

$*\ M_y = 3y^2\cos x + 1$, $*\ N_x = y^2\cos x - 1$

$*\ \dfrac{N_x - M_y}{M} = \dfrac{(y^2\cos x - 1) - (3y^2\cos x + 1)}{y(y^2\cos x + 1)}$

$= \dfrac{-2(y^2\cos x + 1)}{y(y^2\cos x + 1)} = \boxed{\dfrac{-2}{y}} \rightsquigarrow$ Reduced to exact.

$*\,u(y) = e^{-2\ln y} = \dfrac{1}{y^2}$ (نضرب المعادلة الأصلية فيها)

$(y\cos x + \frac{1}{y})dx + (\sin x - \frac{x}{y^2} + \frac{1}{y})dy = 0$

$*\ M_y = \cos(x) - \frac{1}{y^2}$

$*\ N_x = \cos(x) - \frac{1}{y^2}$

$*\ \partial u = M\,dx$

$\int \partial u = \int (y\cos(x) + \frac{1}{y})\partial x$

$u = y\sin(x) + \frac{x}{y} + g(y)$

$u_y = \sin(x) - \frac{x}{y^2} + g'(y) = N = \sin(x) - \frac{x}{y^2} + \frac{1}{y}$

$\int g'(y) = \int \frac{1}{y}\,dy$

$g(y) = \ln|y| + c$

$\therefore u = y\sin(x) + \frac{x}{y} + \ln|y| + c = 0$

* Ex: Solve:

$$y(2x^2 - xy + 1)\,\partial x + (x - y)\,\partial y = 0$$

$$*\, M_y = 2x^2 - 2xy + 1 \quad , *\, N_x = 1 \longrightarrow \text{Not exact}$$

$$\#\ \frac{M_y - N_x}{N} = \frac{2x * (x-y)}{(x-y)} = 2x = h(x).$$

"Reduced to exact"

$$*\, u(x) = e^{\int 2x \cdot \partial x} = e^{x^2}$$ " Integration Factor":

$$\left(y e^{x^2} 2x^2 - x e^{x^2} y^2 + y e^{x^2}\right)\partial x + \left(x e^{x^2} - y e^{x^2}\right)\partial y = 0$$

$$*\, M_y = y\left(e^{x^2} \cdot 2x \cdot 2x^2 + e^{x^2} \cdot 4x\right) - y^2\left(e^{x^2} + x \cdot 2x \cdot e^{x^2}\right) + y x 2 e^{x^2}$$

$$*\, N_x = e^{x^2} + x \cdot 2x \cdot e^{x^2} - y \cdot 2x e^{x^2}.$$

$$\#\ F(x,y) = ? \quad *\, f_y = N^* = x e^{x^2} - y e^{x^2}$$

$$\therefore f(x,y) = \int x e^{x^2} - y e^{x^2} \cdot \partial y$$

$$= x e^{x^2} y - \frac{y^2 e^{x^2}}{2} + g(x)$$

$$*\, f_x = y\left(e^{x^2} + x \cdot 2x e^{x^2}\right) - x y^2 e^{x^2} + g'(x)$$

$$= 2y x^2 e^{x} - x^2 y^2 e^{x^2} + y e^{x^2}$$

$$\therefore g'(x) = 0 \rightsquigarrow g(x) = c$$

$$\therefore F(x,y) = x e^{x^2} y - \frac{y^2 e^{x^2}}{2} = c$$

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\# Proof:

$$u M\,\partial x + u N\,\partial y = 0$$

$$(uM)_y = (uN)_x$$

$$u_y M + u M_y = u_x N + u N_x$$

$$*\ \text{If } u = u(x)$$

$$u M_y = u_x N + u N_x$$

$$u(M_y - N_x) = u_x N$$

$$\frac{u_x}{u} = \frac{M_y - N_x}{N} \rightsquigarrow \frac{\partial u}{u} = \frac{M_y - N_x}{N}\,\partial x$$

$$\ln|u| = \int \frac{M_y - N_x}{N}\,\partial x \rightsquigarrow \boxed{u = e^{\int \square\, \partial x}}$$

* Ex: Solve:

$$(x^2\sqrt{x^2+1}+y^2)\partial x+(2xy\ln x)\partial y=0$$

$* M_y=2y \quad , \quad * N_x=y(2\ln x+2x\cdot\frac{1}{x})=2y\ln x+2y$

$$* \frac{M_y-N_x}{N}=\frac{-2y\ln x}{2xy\ln x}=\frac{-1}{x}$$

$$* u(x)=e^{\int\frac{-1}{x}\cdot\partial x}=e^{-\ln x}=\boxed{\frac{1}{x}}$$

$$*\overset{(M)}{\underbrace{(x\sqrt{x^2+1}+\frac{y^2}{x})}}\partial x+\overset{(N)}{\underbrace{(2y\ln x)}}\partial y=0$$

$* M_y=\frac{2y}{x} \quad , \quad * N_x=\frac{2y}{x}$ ⇝ Exact.

$$\frac{\partial u}{\partial y}=N \rightsquigarrow \int\partial u=\int N\partial y$$

$$u=\int(2y\ln x)\partial y \rightsquigarrow u=y^2\ln x+F(x)$$

$$u_x=\frac{y^2}{x}+F'(x)=M=x\sqrt{x^2+1}+\frac{y^2}{x}$$

$$\int F'(x)=\int x\sqrt{x^2+1}\,\partial x \text{ (Sub.)}$$

$$F(x)=\frac{1}{3}(x^2+1)^{3/2}+c$$

$$\therefore u=y^2\ln x+\frac{1}{3}(x^2+1)+c=0$$

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* Ex: Solve:

$$\underbrace{(x+x^2\cos t)}_{M}\cdot\partial t+\underbrace{(2t+3x\sin t)}_{N}\partial x=0$$

$* M_x=1+2x\cos t \quad , \quad * N_t=2+3x\cos t$ ⇝ Not Exact

* Integrating Factor:

$$\frac{N_t-M_x}{M}=\frac{2+3x\cos t-1-2x\cos t}{x(1+\cos t)}=\frac{1}{x}$$

$$=e^{\int\frac{1}{x}\partial x}=e^{\ln x}=\boxed{x}$$

$$* \underbrace{(x^2+x^3\cos t)}_{M}\partial t+\underbrace{(2xt+3x^2\sin t)}_{N}\partial x=0$$

$* M_x=2x+3x^2\cos t \quad , \quad * N_t=2x+3x^2\cos t$ ⇝ Exact

$$* \int\partial u=\int M\partial t \rightsquigarrow u=\int x^2+x^3\cos t\,\partial t=x^2t+x^3\sin t+g(x)$$

$$u_x=2xt+3x^2\sin t+g'(x)=N=2xt+3x^2\sin t$$

$$g'(x)=0 \rightsquigarrow g(x)=c$$

$$\therefore u=x^2t+x^3\sin t+c=0$$

5* Linear first order ODE :

$y' + P(x)y = Q(x) \rightsquigarrow$ linear in (y)

$x' + P(y)x = Q(y) \rightsquigarrow$ linear in (x)

* Ex: $y' + 3xy^2 = x^3 \rightsquigarrow$ Non - linear

$x^3 y' + y = \sin x \rightsquigarrow$ linear

* $u(x) = e^{\int P(x)\,\partial x}$ (Integrating Factor)

$$e^{\int P(x)\partial x} y' + P(x) y e^{\int P(x)\partial x} = Q(x) e^{\int P(x)\partial x}$$

$$\left(y e^{\int P(x)\partial x}\right)' = Q(x) e^{\int P(x)\partial x}$$

$$\boxed{y = \frac{1}{u(x)} \int Q(x)\, u(x) \cdot \partial x}$$

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* Ex: Solve : $xy' + 3y = 6x^3$

$$y' + \frac{3}{x} y = 6x^2$$

$$* u(x) = e^{\int \frac{3}{x} \cdot \partial x} = x^3$$

$$y = \frac{1}{x^3} \int 6x^2 \cdot x^3 \cdot \partial x$$

$$y = \frac{1}{x^3} (x^6 + c)$$

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#Ex: Solve

$$(1+y^2)\,\partial x-(xy+y+y^3)\,\partial y=0$$

#$x'-\dfrac{xy+y+y^3}{y^2+1}=0$

$$x'-\frac{y}{y^2+1}x=\frac{y+y^3}{y^2+1}$$

$$x'-\frac{y}{y^2+1}x=y$$

#$u(y)=e^{\int\frac{-y}{y^2+1}\cdot\partial y}=\dfrac{1}{\sqrt{y^2+1}}$

$$x=\sqrt{y^2+1}\int\frac{y}{\sqrt{y^2+1}}\cdot\partial y$$

$$=\sqrt{y^2+1}\left(2\cdot\frac{1}{2}\sqrt{y^2+1}+c\right)$$

$$\therefore\ x=y^2+1+c\sqrt{y^2+1}$$

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$$y'+P(x)y=\varphi(x)$$

$$e^{\int P(x)}y'+P(x)ye^{\int P(x)}=\varphi(x)e^{\int P(x)}$$

$$\underbrace{\left(e^{\int P(x)}\right)}_{N_x}\partial y+\underbrace{e^{\int P(x)}\left(P(x)y-\varphi(x)\right)}_{M_y}\partial x=0$$

$$N_x\longleftrightarrow M_y$$

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#Ex: Solve

$$(1+y^2)\partial x - (xy + y + y^3)\partial y = 0$$

#$x' - \frac{xy+y+y^3}{y^2+1} = 0$

$$x' - \frac{y}{y^2+1}x = \frac{y+y^3}{y^2+1}$$

$$x' - \frac{y}{y^2+1}x = y$$

#$u(y) = e^{\int \frac{-y}{y^2+1} \cdot \partial y} = \frac{1}{\sqrt{y^2+1}}$

$$x = \sqrt{y^2+1}\int \frac{y}{\sqrt{y^2+1}} \cdot \partial y$$

$$= \sqrt{y^2+1}\left(2 \cdot \frac{1}{2}\sqrt{y^2+1} + c\right)$$

$$\therefore x = y^2+1 + c\sqrt{y^2+1}$$

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$$y' + P(x)y = \varphi(x)$$

$$e^{\int P(x)} y' + P(x) y e^{\int P(x)} = \varphi(x) e^{\int P(x)}$$

$$\underbrace{\left(e^{\int P(x)}\right)\partial y}_{} + \underbrace{e^{\int P(x)}\left(P(x)y - \varphi(x)\right)\partial x}_{} = 0$$

$$N_x \longleftrightarrow M_y$$

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*Ex! Solve

$\sin y\, \partial x + 2(x - 3\sin y)\cos y\, \partial y = 0$

*$x' + \left(\frac{2x}{\sin y} - 6\right)\cos y = 0$

$x' + \frac{2x\cos y}{\sin y} = 6\cos y$

$x' + 2\cot y\, x = 6\cos y$

∴ linear in x:

*$u(y) = e^{\int 2\cos y/\sin y \,.\, \partial y} = e^{2\ln|\sin y|} = \boxed{\sin^2 y}$

*$x = \csc^2(y)\int 6\underbrace{\sin^2 y}_{\text{subs.}}\cos y\, \partial y$

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*Ex! Solve

$(2x - y^2)y' = 2y$

$(2x - y^2)\frac{\partial y}{\partial x} = 2y$

$2y\,\partial x = (2x - y^2)\partial y$

*$x' = \frac{2x - y^2}{2y}$

$x' - \frac{1}{y}x = \frac{-1}{2}y$

*$u(y) = e^{\int \frac{-1}{y}\partial y} = e^{-\ln y} = \boxed{\frac{1}{y}}$

$x = y\int \frac{1}{y} \cdot \frac{-1}{2}y\, \partial y$

$x = y\left(\frac{-y}{2} + c\right)$

$x = \frac{-y^2}{2} + yc$

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\# Ex: Solve

$$y' = \frac{y}{2y\ln y + y - x}$$

$$\# \frac{\partial y}{\partial x} = \frac{y}{2y\ln y + y - x} \rightsquigarrow y\,\partial x = (2y\ln y + y - x)\,\partial y$$

$$x' - \left(2\ln y + 1 - \frac{x}{y}\right) = 0$$

$$x' + \frac{1}{y}x = 2\ln y + 1$$

$$\# \ u(y) = e^{\int \frac{1}{y} dy} = y$$

$$x = \frac{1}{y}\int (\underbrace{2y\ln y}_{\text{By parts}} + \underbrace{y}_{\frac{y^2}{2}})\, dy$$

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* Bernoulli's ODE: "Reduced to linear".

$$*\left(y' + P(x)y = Q(x)y^n\right) / y^n$$

$$z = y^{1-n}$$

$$z' = (1-n)\,y^{-n}y'$$

$$y^{-n}y' + P(x)y^{(1-n)} = Q(x)$$

$$\frac{z'}{(1-n)} + P(x)z = Q(x)$$

$$z' + (1-n)P(x)\,z = (1-n)Q(x) \rightsquigarrow \text{"linear"}$$

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\# Ex: Solve: $y' + y = y^3$ "sep."

$$z = y^{-2}$$

$z' = -2y^{-3}y'$ (الاشتقاق بالنسبة لـ x)

$$-2y^{-3}y' - 2y^{-2} = -2$$

$$z' - 2z = -2 \rightsquigarrow \text{linear}$$

* $u(x) = e^{\int -2\,\partial x} = e^{-2x}$ Integrating Factor.

$$\# \ z = e^{2x}\int -2e^{-2x}\,\partial x.$$

$$z = e^{2x}(e^{-2x} + c) \rightsquigarrow y^{-2} = 1 + ce^{2x}$$

*Ex: Solve

$$(2xy - x^2y^2)\,\partial x + (1+x^2)\,\partial y = 0$$

* $\frac{2x}{1+x^2}\, y - \frac{x^2}{1+x^2}\, y^2 + y' = 0$

$$y' + \frac{2x}{1+x^2}\, y = \frac{x^2}{1+x^2}\, y^2$$

* $z = y^{-1}$

$$z' = -y^{-2} y'$$

$$z' - \frac{2x}{x^2+1}\, z = \frac{-x^2}{x^2+1} \rightsquigarrow \text{linear}$$

* $u(x) = e^{-\int (2x/x^2+1)\partial x} = \frac{1}{x^2+1}$

* $y = (x^2+1) \cdot \int \frac{1}{x^2+1} \cdot \frac{-x^2}{x^2+1}\, \partial x = \int \frac{-x^2}{x^2+1}\, \partial x$

$$\int \frac{-x^2+1-1}{x^2+1}\, \partial x = \int -\frac{(x^2+1)}{(x^2+1)} + \left(\frac{1}{x^2+1}\right) \partial x = -x + \tan^{-1} x + c$$

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*Ex: Solve

$$y' = \frac{y(1+x-6y^2)}{2x}$$

$$y' - \frac{(1+x)}{2x}\, y = \frac{-3}{x}\, y^3$$

* $z = y^{-2}$

$$z' = -2y^{-3} y'$$

$$z' + \frac{(1+x)}{x}\, z = \frac{6}{x}$$

$$z' + \left(1 + \frac{1}{x}\right) z = \frac{6}{x} \rightsquigarrow \text{linear}$$

* $u(x) = e^{\int 1+\frac{1}{x}} = xe^{x}$

* Ex :

1. $xy' - 2y = 12x^3 \sqrt{y}$

$y' - \frac{2}{x} y = 12x^2 y^{1/2}$

* $z = y^{1/2}$

$z' = \frac{1}{2} y^{-1/2} y'$

$\frac{1}{2} y^{(-1/2)} y' - \frac{2}{x} \cdot \frac{1}{2} \cdot y^{(-1/2)} \cdot y = 12x^2 \cdot \frac{1}{2} y^{(-1/2)} \cdot y^{(1/2)}$

$z' - \frac{1}{x} z = 6x^2$

2. $y \partial x - x(1 + xy \sin y) \partial y = 0$

$x' - \frac{x}{y} - x^2 \sin y = 0$

$x' - \frac{1}{y} x = \sin y (x^2)$

* $z = x^{-1} \rightsquigarrow z' = -x^{-2} x'$

$-x^{-2} x' + \frac{1}{y} x^{-2} x = -\sin y \, x^{-2} x^2$

$z' + \frac{1}{y} z = -\sin y$

*

* Ex: Solve

1) $(x^2-y^2)\partial x + 2xy\partial y = 0$

* Reduced to sep.
* Reduced to exact.
* Bernolli in (y).

2) $2x\tan^{-1}(y)\,\partial x + \frac{x^2}{1+y^2}\,\partial y = 0$

$$\frac{2x}{x^2}\,\partial x = \frac{-1}{\tan^{-1}(y)\,(1+y^2)}\,dy$$

* Sep.
* Exact.

3- $y\,dx + (3x+y)\,dy = 0$

* Reduced to sep.

* Linear $x' + \frac{3x}{y} + 1 = 0$

* Reduced to exact $x' + \frac{3}{y}(x) = -1$

*** $x = uy \rightsquigarrow x' = u'y + u$

$\# \, yx' + (3x+y) = 0$

$y(u'y+u) + (3(uy) + y) = 0$

$y^2u' + yu + 3yu + y = 0$

$y^2u' = -4yu - y$

$y^2u' = -y(4u+1)$

$-yu' = 4u+1$

$\frac{-dy}{y} = \frac{du}{4u+1} \rightsquigarrow -\ln|y| + c = \frac{1}{4}\ln|4u+1|$

$\left[u = \frac{x}{y}\right]$

4- $\frac{x}{y^2}y' + \frac{1}{y} = x^2 \quad , y(1) = 4$

* Bernolli in y

* $y' + \frac{y}{x} = xy^2$

*** $z = y^{-1}$

$z' = -1y^{-2}y_2$

$z' - \frac{1}{x}z = -x$

* $u(x) = e^{\int \frac{-1}{x}dx} = e^{-\ln|x|} = \frac{1}{x}$

$z = x \cdot \int \frac{1}{x} \cdot -x \, dx$

$= x\left[-x + c\right] = -x^2 + cx$

$z = -x^2 + cx$

$\frac{1}{y} = -x^2 + cx \rightsquigarrow y = \frac{-1}{x^2} + \frac{1}{cx}$

5- $xy' \ln x \ln y + 1 = 0$

* Seperable

* Exact

$\underbrace{\ln y \cdot dy}_{\text{By parts}} = \frac{-1}{x\ln x} dx$

$y\ln|y| - y =$

6- $y' + \frac{2y}{x} = \sqrt{y}$

* $(z = y^{1/2})$ ~> Bernolli in y

$z' = \frac{1}{2} y^{\frac{-1}{2}} y'$

$z' + \frac{1}{x} z = \frac{1}{2}$

$u(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = x$

$z = \frac{1}{x} \cdot \int \frac{1}{2} x \, dx$

$= \frac{1}{x} \cdot \left[\frac{x^2}{4} + c\right] = \frac{x}{4} + \frac{c}{x}$

$z = \frac{x}{4} + \frac{c}{x}$

$y^{\frac{1}{2}} = \frac{x}{4} +$

$y = \left(\frac{x}{4} + \frac{c}{x}\right)^2$

7- $xy' = y\ln x - y\ln y$

*Reduced to seperable

$$y' = \frac{y(\ln x - \ln y)}{x}$$

$$y' = \frac{y}{x}\ln\left(\frac{x}{y}\right)$$

*** $y = ux$

$y' = u'x + u$

$u'x + u = u\ln\left(\frac{1}{u}\right)$

$$u'x = u\ln\left(\frac{1}{u}\right) - u$$

$$\frac{dx}{x} = \frac{du}{u\ln\left(\frac{1}{u}\right) - u}.$$

[$*u = \frac{y}{x}$]

8- $xy' = y + \sqrt{y^2 - x^2}$

*Reduced to seperable

*** $y = ux$

$y' = u'x + u$

* $x(u'x + u) = ux + \sqrt{u^2x^2 - x^2}$

$u'x^2 + \cancel{ux} = \cancel{ux} + x\sqrt{u^2 - 1}$

$u'x^2 = x\sqrt{u^2 - 1}$

$u'x = \sqrt{u^2 - 1}$

$$\frac{dx}{x} = \frac{du}{\sqrt{u^2 - 1}}$$

$(\ln|x| + c)$ Trigonometric Sub.

' $u = \sec\theta$ '

9- $(2r\sin\theta + \cos\theta)\partial r - (r\sin\theta - r^2\cos\theta)\partial\theta = 0$

$* \; N_\theta = 2r\cos\theta - \sin\theta$

$* \; M_r = 2r\cos\theta - \sin\theta$ $\Big| \rightsquigarrow$ Exact

$$\int f_r \partial r = \int (2r\sin\theta + \cos\theta)\partial r$$

$$= r^2\sin\theta + r\cos\theta + f_\theta$$

$\# \; r^2\cos\theta - r\sin\theta + f'_\theta = -r\sin\theta + r^2\cos\theta$

$f'_\theta = 0$

$f_\theta = c$

$\# \; f(r,\theta) = r^2\sin\theta + r\cos\theta = c$

#CH.2

Second order ODE

$y'' + p(x)y' + q(x)y = r(x) \rightsquigarrow$ linear "S.O" ODE

* If $r(x) = 0 \rightsquigarrow$ homogenous

* The solution of

$y'' + P(x)y' + q(x)y = 0$; is given by,

$y = c_1 \underline{\underline{y_1}} + c_2 \underline{\underline{y_2}}$

. (Basis of the Function)

"S.O" ODE can be reduced to first order:

1. y is missing:

\# Ex: Solve

$$x^2 y'' + (y')^2 = 0$$

* $z = y'$

* $z' = y''$

$x^2 z' + z^2 = 0 \rightsquigarrow$ seperable.

$$\int \frac{dz}{z^2} = \int -\frac{dx}{x^2}$$

$$\frac{-1}{z} = \frac{1}{x} + c = \frac{c_1 x + 1}{x}$$

$$y' = z = \frac{-x}{c_1 x + 1}$$

$$\begin{array}{r|l} & \frac{1}{c_1} \\ \hline c_1 x + 1 & \cancel{x} \\ & \cancel{-x} \mp \frac{1}{c_1} \\ \hline & \frac{-1}{c_1} \end{array}$$

$$\frac{dy}{dx} = \frac{-x}{c_1 x + 1} \rightsquigarrow dy = \frac{-x}{c_1 x + 1} dx$$

$$\int dy = \int \left(\frac{-1}{c_1} + \frac{\left(\frac{1}{c_1}\right)}{c_1 x + 1} \right) dx$$

$$y = \frac{-1}{c_1} x + \frac{1}{c_1^2} \ln|c_1^2 x + 1| + c_2$$

$*$ Ex: Solve

$$xy'' = y' + (y')^3$$

$$* z - y'$$

$$* z' = y''$$

$$xz' = z + z^3$$

$$z' = \frac{(z+z^3)}{x}$$

$$\frac{\partial z}{z(z^2+1)} = \frac{\partial x}{x}$$

"Partial Fraction".

$$\frac{A}{z} + \frac{Bz+c}{z^2+1} = 1$$

$$A(z^2+1) + (Bz+C)(z) = 1$$

$$* z=0 \rightsquigarrow \boxed{A=1}$$

$$* z=1 \rightsquigarrow 2+B+c=1$$

$$* z=-1 \rightsquigarrow 2+B-c=1$$

$$\boxed{B=-1} \quad \boxed{C=0}$$

$$\left(\frac{1}{z} - \frac{z}{z^2+1}\right)\partial z = \frac{\partial x}{x}$$

$$\ln|z| - \frac{1}{2}\ln|z^2+1| = \ln x + c_1 \qquad * \underline{2}$$

$$\ln\left|\frac{z^2}{z^2+1}\right| = \ln x^2 + c_1^* \qquad * e$$

$$\frac{z^2}{z^2+1} = c_1^{**} x^2$$

$$z^2 = c_1^{**} x^2 z^2 + c_1^{**} x^2$$

$$z^2 = (1 - c_1^{**} x^2) = c_1^{**} x^2$$

$$z^2 = \frac{c_1^{**} x^2}{1 - c_1^{**} x^2}$$

$$y' = z = \mp \frac{c_2 x}{\sqrt{1 - c_2^2 x^2}}$$

⋮

*Ex: Solve

1. $xy'' = 2y' + x^4$
2. $y'' = y'(1+y')$
3. $xy' + y'(3xy'-2) = 0$ *

2. X missing: can be reduced to "f.o" First order.

*Ex: Solve

$4y(y')^2 y'' = (y')^4 + 1$

$z = y' = \frac{\partial y}{\partial x}$

$y'' = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$

$y'' = z\frac{\partial z}{\partial y}$

$4yz^2\, z.\frac{\partial z}{\partial y} = z^4+1$

$\frac{4z^3}{z^4+1} = \frac{\partial y}{y}$

$\ln|z^4+1| = \ln|y| + c_1$

$z^4+1 = c_1^* y$

$\frac{\partial y}{\partial x} = z = (c_1^* y - 1)^{(1/4)}$

$(c_1^* y - 1)^{-1/4} \partial y = \partial x$

⋮

* Ex: Solve

$yy'' - (y')^2 = y^2 y'$

* $y' = z$

* $y'' = z \frac{\partial z}{\partial y} = zz'$

$yzz' - z^2 = y^2 z \quad / (yz) \rightsquigarrow$ linear in $\underline{\underline{z}}$

$z' - (\frac{1}{y}) z = y$

* $u(y) = e^{\int \frac{-1}{y} dy} = \frac{1}{y}$

$z = y \int dy$

$= y \, (y + c_1)$

$y' = y^2 + c_1 y$

$\frac{dy}{y(y+c_1)} = dx \rightsquigarrow$ sep.

* * *

* Ex: Solve:

1. $yy'' = y' + (y')^2$
2. $y^3 y'' + 1 = 0$, $y(1) = 1$ $y'(1) = 0$
3. $yy''(y')^2 = y^3 y'$

One Solution Given :

$y'' + P(x)y' + q(x)y = r(x)$

* y_1 is Solution for C.H.ODE

Corrosponding homogenous ordinary diff. equation

* The general solution :

$y = y_1 v$

$y' = y_1' v + y_1 v'$

$y'' = y_1'' v + 2y_1' v' + y_1 v''$

$\therefore (y_1'' v + 2y_1' v' + y_1' v') + P(x)(y_1' v + y_1 v') + q(x) y_1 v = r(x)$

$= v\,(y_1'' + P(x) y_1' + q(x) y_1) + y_1 v'' + (2y' + \quad v' = r(x)$

$$v'' + \left(\frac{2y_1'}{y_1} + P(x)\right) v' = \frac{r(x)}{y_1}$$

$z = v'$

* * *

Ex: Solve $x^2 y'' - 2y = x^2$

* $y_1 = \frac{1}{x}$, is Solution for C.H.ODE.
* $y = y_1 v = \frac{v}{x}$

$y_1 = \frac{1}{x}$; $y_1' = \frac{-1}{x^2}$; $P(x) = 0$; $r(x) = 1$

$$v'' - \frac{2}{x} v' = x$$

* $z = v' \rightsquigarrow z' = v''$

$z' - \frac{2}{x} z = x \rightsquigarrow$ (Linear in z)

$u(x) = e^{\int \frac{-2}{x} dx} = \frac{1}{x^2}$

$z = x^2 \int \frac{1}{x^2} \cdot x \, dx$

$z = x^2 (\ln x + c_1)$

$z = x^2 \ln x + x^2 c_1$

**

$\int x^2 \ln x + c_1 x^2 \, dx$

$\ln x - \frac{x^3}{9} + \frac{c_1 x^3}{3} + c_2$

$n x - \frac{x^2}{9} + c_1 x^2) + \frac{c_2}{x}$

*   *

$' - 7xy' + 16y = x^5$

solution for C.H. ODE.

$= 4x^3 \quad , r(x) = x^3 \quad , P(x) = \frac{-7}{x}$

$- \frac{16}{x^2} y = x^3$

${}^4 u$

$\frac{7}{x}) u' = \frac{1}{x}$

$= \frac{1}{x}$

$z' = u''$

$= \frac{1}{x}$

$$\# \; y'' + P(x)y' + q(x)y = r(x)$$

$$y = y_1 u$$

$$u'' + \left(\frac{2y_1'}{y_1} - P(x)u'\right) = 0$$

$$z' + \left(\frac{2y_1'}{y_1} - P(x)z\right) = 0$$

$$\frac{\partial z}{z} = -\left(\frac{2y_1'}{y_1} + P(x)\right)\partial x$$

$$\ln|z| = -2\ln|y_1| - \int P(x)\,\partial x$$

$$z = c_1^* \frac{1}{y_1^2} e^{-\int P(x)\,\partial x}$$

$$z = \frac{c_1^* e^{-\int P(x)\partial x}}{y_1^2}$$

$$\frac{\partial u}{\partial x} = c_1^* \frac{e^{-\int P(x)\partial x}}{y_1^2}$$

$$\# \; u = c_1^* \int \frac{e^{-\int P(x)\partial x}}{y_1^2}\,\partial x + c_2$$

$$y = c_1^* y_1 \left(\int \frac{e^{-\int P(x)\partial x}}{y_1^2}\right) + \underline{\underline{c_2 y_2}}$$

* If the ODE is homogenous

$r(x) = 0$, then :

$$y_2 = y_1 \int \frac{e^{-\int P(x)\partial x}}{y_1^2}\,\partial x \, .$$

→ General solution.

↳ $y = c_1 y_1 + c_2 y_2$

* * *

* Ex: Solve :

$$xy'' - (x+2)y' + 2y = 0$$

$y_1 = e^x$ is solution for C.H. ODE

$$* \; y_2 = ?! \, , \quad P(x) = \frac{-(x+2)}{x} = -1 - \frac{2}{x}$$

$$* \; e^{-\int P(x)\partial x} = e^{\int (1+\frac{2}{x})\partial x} = e^{x + \ln x^2} = \boxed{x^2 e^x}$$

$$y_2 = e^x \int \frac{x^2 e^x}{e^{2x}}\,\partial x = e^x \int x^2 e^{-x}\,\partial x$$

$$= e^x(-x^2 e^{-x} - 2xe^{-x} - 2e^{-x})$$

$$* \; y_2 = -(x^2 + 2x + 2)$$

$$y = c_1 e^x + c_2(x^2 + 2x + 2)$$

* Tabular Int.

$$\begin{array}{ccc} x^2 & & e^{-x} \\ & \searrow + & \\ 2x & & -e^{-x} \\ & \searrow - & \\ 2 & & e^{-x} \\ & \searrow + & \\ 0 & & -e^{-x} \end{array}$$

*Ex : Solve $xy'' + 2y' + xy = 0$

* $y_1 = \frac{\sin x}{x}$

* $y_2 = ?!$

$P(x) = \frac{2}{x}$

$$e^{-\int P(x)dx} = e^{-\int \frac{2}{x} dx} = \frac{1}{x^2}$$

* $$y_2 = \frac{\sin x}{x} \cdot \int \frac{\left(\frac{1}{\cancel{x^2}}\right)}{\left(\frac{\sin^2 x}{\cancel{x^2}}\right)} dx$$

$$= \frac{\sin x}{x} \int \csc^2 x \, dx$$

$$= \frac{\cancel{\sin x}}{x} \cdot - \frac{\cos x}{\cancel{\sin x}} = - \frac{\cos x}{x}$$

* General Solution :

$$y = c_1 \frac{\sin x}{x} + c_2 \frac{\cos x}{x}$$

* * *

*2.2 : Second order linear homogenous ODE with constant coefficients:

* $y'' + ay' + by = 0$

* General Solution

$y = c_1 e^{\lambda x} + c_2 e^{\lambda x}$

\# القانون العام *

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

* Ex : Solve $y'' - y' - 12y = 0$

$$\lambda^2 - \lambda - 12 = 0$$

$$(\lambda - 4)(\lambda + 3) = 0$$

$$\lambda_1 = 4, \; \lambda_2 = -3$$

* The general solution:

$$y = c_1 e^{4x} + c_2 e^{-3x}$$

Ex: Solve $y'' + 3y' = 0$, $y(0) = 2$, $y'(0) = 6$ (IVP)

$\lambda^2 + 3\lambda = 0$

$\lambda(\lambda + 3) = 0$

$\lambda = 0, \lambda = -3$

$* \; y = c_1 + c_2 e^{-3x} \rightsquigarrow y = \underline{\underline{4}} - \underline{\underline{2}}e^{-3x}$

$y' = -3c_2 e^{-3x}$

$y'(0) = 6 \rightsquigarrow -3c_2 = 6 \rightsquigarrow \boxed{c_2 = -2}$

$y(0) = 2 \rightsquigarrow 2 = c_1 - 2 \rightsquigarrow \boxed{c_1 = 4}$

* * *

* Ex: Solve $y'' - 10y' + 25y = 0$

$\lambda^2 - 10\lambda + 25 = 0$

$(\lambda - 5)^2 = 0$

$\lambda = 5, 5$

$* \; y = c_1 e^{5x} + c_2 x e^{5x}$

λ is repeated.

$y = c_1 e^{\lambda x} + c_2 e^{\lambda x} x$

* * *

* Ex: Solve $4y'' - 4y' + y = 0$

$* \; y'' - y' + \frac{y}{4} = 0$

$\lambda^2 - \lambda + \frac{1}{4} = 0$

$(\lambda - \frac{1}{2})^2 = 0$

$\lambda = \frac{1}{2}, \frac{1}{2}$

$* \; y = c_1 e^{\frac{x}{2}} + c_2 x e^{\frac{x}{2}}$.

※ Ex: Solve $y'' - 6y' + 10y = 0$

$$\lambda^2 - 6\lambda + 10 = 0$$

$$\lambda = \frac{-b \pm \sqrt{\Delta}}{2a} \quad , \Delta = 36 - 40 = -4$$

$$\lambda = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm i$$

$$* \, y = e^{3x} (c_1 \cos x + c_2 \sin x).$$

* ——— * ——— * ———

*Note, 1** If $\lambda = a \pm bi$ ~ General Solutions.

$$\hookrightarrow y = e^{ax} (c_1 \cos bx + c_2 \sin bx).$$

2** If $\lambda = \lambda_1 , \lambda_1$ (نفس القيمة)

$$\hookrightarrow y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

* Ex: Solve

$y'' + 4y = 0 \quad , \; y(0) = 0 \quad , y'(0) = 2$

* $\lambda^2 + 4 = 0$

$\lambda = \mp 2i$

$y = c_1 \cos 2x + c_2 \sin 2x$

$y'_1 = -2 c_1 \sin 2x + 2 c_2 \cos 2x$

* $y'(0) = \underline{2 = 2 c_2} \rightsquigarrow \boxed{c_2 = 1}$

* $y(0) = \underline{0 = \; c_1} + 0$

$y = \sin 2x$

~~~~ * ~~~~ * ~~~~ *

*Q : Find S.O ODE whose solution is given by:

1) $y = (c_1 e^{5x} + c_2 e^{x})$.

2) $y = e^{2x} (c_1 + c_2 x)$.

3) $y = e^{3x} (c_1 \cos 2x + c_2 \sin 2x)$.

1) $\lambda = 1, 5$

$(\lambda - 1)(\lambda - 5) = 0$

$\lambda^2 - 6\lambda + 5 = 0$

$y'' - 6y' + 5y = 0$

2) $\lambda = 2, 2$

$(\lambda - 2)^2 = \lambda^2 - 4\lambda + 4 = 0$

$y'' - 4y' + 4y = 0$

3) $\lambda = 3 \mp 2i$

$(\lambda - (3 + 2i))(\lambda - (3 - 2i)) = 0$

$\lambda^2 - (3 - 2i)\lambda - (3 + 2i)\lambda + (3 - 2i)(3 + 2i) = 0$

$\lambda^2 - 6\lambda + 13 = 0$

$y'' - 6y' + 13y = 0$
~~~~

*2.5 : Euler - Cauchy ODE :

* $x^2y'' + axy' + by = 0$

$y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{(m-2)}$

* $m(m-1)x^m + amx^m + bx^m = 0$

$x^m(m^2 + (a-1)m + b) = 0$

$\Rightarrow m^2 + (a-1)m + b = 0$

.. IF ..

1. $m = m_1, m_2 \rightsquigarrow y = C_1x^{m_1} + C_2x^{m_2}$
2. $m = m, m \rightsquigarrow y = C_1x^m + C_2x^m \cdot \underline{\ln x}$
3. $m = C \mp di \rightsquigarrow y = x^c(C_1\cos(d\ln x) + C_2\sin(d\ln x))$.

* $x^2y'' + axy' + by = 0$

$x = e^t \rightsquigarrow \frac{\partial x}{\partial t} = e^t$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$y'_x = e^{-t}y'_t = \frac{1}{x}y'_t$$

$$y''_x = \frac{\partial}{\partial x}(y'_x) = \frac{\partial}{\partial x}\left(e^{-t}\frac{\partial y}{\partial t}\right).$$

$$= \frac{\partial}{\partial t}\left(e^{-t}\frac{\partial y}{\partial t}\right)\cdot\frac{\partial t}{\partial x}$$

$$= \left(-e^{-t}\frac{\partial y}{\partial t} + e^{-t}\frac{\partial^2 y}{\partial t^2}\right)e^{-t}$$

$$= \frac{1}{x^2}(y''_t - y'_t)$$

$$= (y''_t - y'_t) + ay'_t + by_t = 0$$

$$= y''_t + (a-1)y'_t + by_t = 0$$

* $\lambda^2 + (a-1)\lambda + b = 0$

→

1. $\lambda = \lambda_1, \lambda_2$

$y = c_1 e^{\lambda t} + c_2 e^{\lambda_2 t}$

$= c_1 (e^t)^{\lambda_1}$

2. $\lambda = \lambda, \lambda$

$y = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$

* ~ * ~ * ~ *

* Ex: Solve:

$x^2 y'' - xy' - 3y = 0$

$* y = x^m$

$m^2 + (a-1)m + b = 0$

$m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m = 3, -1$

$* y = c_1 x^3 + c_2 x^{-1}$

* ~ * ~ * ~ *

* Q: Find S. O ODE whose solution is:

$y = c_1 x^2 + c_2 x^5$

$* m = 2, 5$

$(m-2)(m-5) = 0$

$m^2 - 7m + 10 = 0$

$b = 10$

$-7 = a - 1 \rightsquigarrow a = -6$

$* \quad x^2 y'' - 6xy' + 10y = 0 \quad *$

※ Ex: Solve:

$2x^2y'' + 5xy' - 2y = 0$

* $y = x^m$

* $2m(m-1) + 5m - 2 = 0$

$2m^2 + 3m - 2 = 0$

$(2m - 1)(m + 2) = 0$

$m = \frac{1}{2}, -2.$

$y = c_1\sqrt{x} + \frac{c_2}{x^2}$

* * *

※ Ex: Solve:

$x^2y'' - xy' + y = 0$

* $y = x^m$

* $m^2 - 2m + 1 = 0$

$(m - 1)^2 = 0$

$m = 1, 1$

$y = c_1 x + c_2 x \ln x$

* * *

※ Ex: Solve:

$x^2y'' + 7xy' + 10y = 0$

$m^2 + 6m + 10 = 0$

※ $m = \frac{-6 \pm \sqrt{-4}}{2} = -3 \mp i$

$y = x^{-3}(c_1 \cos(\ln x) + c_2 \sin(\ln x)).$

*Q : Find SO ODE whose solution :

$*y = x\left(c_1 \cos(2\ln x) + c_2 \sin(2\ln x)\right).$

* Solution:

$m = 1 \mp 2i$

$\left(m - (1+2i)\right)\left(m - (1-2i)\right) = 0$

$m^2 - 2m + 5 = 0$

$* \; x^2y'' - xy' + 5y = 0$

* * *

\# Sec. (2.7) → Non-homogenous ODE:

$y'' + P(x)y' + q(x)y = r(x)$

→ General Solution:

\# $y = \underline{y_h} + \underline{y_p}$

* y_h ⇒ solution for corrosponding homogenous.

→ $y_h = c_1 y_1 + c_2 y_2$

* y_p ⇒ solution for Non-homogenous.

* Undetermined Coefficients method:

	$r(x)$	y_p
1	$Ke^{\lambda x}$	$Ce^{\lambda x}$
2	Kx^n	$K_n x^n + K_{(n-1)} x^{(n-1)} + --- + K_0$
3	$K\cos\omega x$ $K\sin\omega x$	$(k_1 \cos\omega x + k_2 \sin \omega x)$
4	$Ke^{ax}\cos\omega x$ $Ke^{ax}\sin\omega x$	$e^{ax}(k_1 \cos\omega x + k_2 \sin\omega x)$.

* * *

\# Ex: Solve:

$y'' + 3y' + 2y = x^2$

* $y = y_h + y_p$

* $y_h = ?!$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+2)(\lambda+1) = 0$

$\lambda = -2, -1$

$y_h = c_1 e^{-2x} + c_2 e^{-x}$

* $y_p = ?$

* $y_p = k_2x^2 + k_1x + k_0$ (From the table) ⇝ "polynomial"

* $y_p' = 2k_2x + k_1$

* $y_p'' = 2k_2$

$(2k_2) + 3(2k_2x + k_1) + 2(k_2x^2 + k_1x + k_0) = x^2$

* $2k_2 + 3k_1 + 2k_0 = 0$ --- (1) ⇝ "k_2, k_1, k_0" الثوابت

* $6k_2 + 2k_1 = 0$ --- (2) ⇝ معاملات $\underline{x}$

$2k_2 = 1 \rightsquigarrow \boxed{k_2 = \frac{1}{2}}$, $\boxed{k_1 = \frac{-3}{2}}$

$1 - \frac{9}{2} + 2k_0 = 0$

$2k_0 = \frac{7}{2}$, $\boxed{k_0 = \frac{7}{4}}$

* $y_p = \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$

$$y = \underbrace{c_1e^{-2x} + c_2e^{-x}}_{y_h} + \underbrace{\frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}}_{y_p}$$

* ~~~~ * ~~~~ *

* $y = y_h + y_p$

*اختيار (y_p) يعتمد ع:

1. Basic rule.
2. Modification rule. $\rightsquigarrow$ إذا كان في تشابه بين ال y_p , ال y_h بنضرب ال y_p ب x .
3. Sum rule.

$(\sim\sim\sim) = \underline{r(x)}$, $y_h = c_1 e^{2x} + c_2 e^{3x}$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$y_{p_1} \leftarrow \underline{x^2} + \underline{e^{2x}} \rightarrow y_{p_2}$ $\qquad$ $r(x) = e^{2x}$

* * *

*Ex: Find the Form of y_p for the ODE:

$y'' - 3y' + 2y = 6e^{-x} + e^{x} + e^{2x}$

* $y_h = ?$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda = 2, \ \lambda = 1$

* $y_h = c_1 e^{2x} + c_2 e^{x}$

* $y_p = Ae^{-x} + Bxe^{x} + ke^{2x}x.$

* Ex: $y'' + 9y = 6\sin 3x$, Solve:

$*y_h = ?$

$\lambda^2 + 9 = 0$

$\lambda = \pm 3i$

$*y_h = c_1 \cos 3x + c_2 \sin 3x$

$*y_p = \underline{x}(A\cos 3x + B\sin 3x)$

$y_p' = (A\cos 3x + B\sin 3x) + x(-3A\sin 3x + 3B\cos 3x)$

$y_p'' = (-3A\sin 3x + 3B\cos 3x) + (-3A\sin 3x + 3B\cos 3x) + x(-9A\cos 3x - 9B\sin 3x)$

$$y'' + 9y = -6(A\sin 3x - B\cos 3x) - 9x(A\cos 3x + B\sin 3x) + 9x(A\cos 3x + B\sin 3x)$$

$* -6A = 6$ $\quad$ $* 6B = 0$

$\boxed{A = -1}$ $\quad$ $\boxed{B = 0}$

$\# y_p = -x\cos 3x$.

$\# y_h = c_1 \cos 3x + c_2 \sin 3x$

$*** y = c_1 \cos 3x + c_2 \sin 3x - x\cos 3x$.

* ~ * ~ *

* Ex : $y'' + 3y' + 4 = e^{2x}$

$*y'' + 3y' = e^{2x} - 4$

$\lambda^2 + 3\lambda = 0$

$\lambda(\lambda + 3) = 0$, $\lambda = 0$, $\lambda = -3$

$y_h = c_1 e^{0x} + c_2 e^{-3x} \Rightarrow y_h = c_1 + c_2 e^{-3x}$

$* y'' + 3y' = e^{2x} - 4$

$*y_p = \underline{\underline{k \cdot x + Ae^{2x}}}$

*Ex: Find the form of y_p for the ODE:

$y'' + y' + y = \cos x - 9x^2 e^x$

* $\lambda^2 + \lambda + 1 = 0$

$\lambda = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \mp \frac{\sqrt{3}}{2} i$

* $y_h = e^{\frac{-x}{2}} \left(C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right)$

* $y_p = k_1 \cos x + k_2 \sin x + (A_2 x^2 + A_1 x + A_0) e^x$

* * *

*Ex: Find S.O ODE whose solution is

$y = \underbrace{C_1 \cos 2x + C_2 \sin 2x}_{(y_h)} - \underbrace{\sin 3x}_{(y_p)}$

* $y_h = C_1 \cos 2x + C_2 \sin 2x$

$x = \pm 2i$

$(\lambda - 2i)(\lambda + 2i) = r(x)$

$\lambda^2 + 4 = 0$

$y'' + 4y = r(x)$

* $y_p = -\sin 3x$

$y'_p = -3\cos 3x$

$y''_p = 9 \sin 3x$

* $y''_p + 4y_p = r(x)$

$9 \sin 3x - 4 \sin 3x = r(x)$

$r(x) = 5 \sin 3x$

$\therefore y'' + 4y = 5 \sin 3x$

* Ex: $x^2y'' - xy' + y = e^x$, $y_h \to$ Euler

y_p

$(m(m-1) - m + 1) = 0$

$m^2 - 2m + 1 = 0$

$m = 1, 1$

** $y_h = c_1 x + c_2 x \ln(x)$

** $y_p = Ae^x$

$\boxed{A = 1}$

$y = y_h + y_p$

$= c_1 x + c_2 x \ln(x) + e^x$

* * *

wroskian:

$y'' + P(x)y' + q(x)y = 0$

$y = c_1 y_1 + c_2 y_2$

* $w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= y_1 y_2' - y_2 y_1'$

If it is "zero" linearly. dependent.

* Ex: $w(x, 2x) = \begin{vmatrix} x & 2x \\ 1 & 2 \end{vmatrix} = 0$

$\Rightarrow x, 2x,$ L.d.

* Ex: $w(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$

$\Rightarrow x, x^2$ L.I.

* Q: If $w(\sin x, f(x)) = \sin^2 x$, $0 < x < \frac{\pi}{2}$, Find $f(x)$ given that $f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{8}\pi$ (IVP)

$$* w(\sin x, f(x)) = \begin{vmatrix} \sin x & f(x) \\ \cos x & f'(x) \end{vmatrix}$$

$$= \sin x f'(x) - \cos x f(x) = \sin^2 x$$

→ First order ODE

$$* f'(x) - \frac{\cos x}{\sin x} f(x) = \sin x$$

$$u(x) = e^{\int -\frac{\cos x}{\sin x} dx} = e^{-\ln|\sin x|} = \frac{1}{\sin x}$$

$$f(x) = \sin x \cdot \int \frac{1}{\sin x} \cdot \sin(x)\, dx$$

$$f(x) = \sin x \, (x + c)$$

$$= x \sin x + c \sin x$$

* IVP:

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \cdot \frac{1}{\sqrt{2}} + \frac{c}{\sqrt{2}} = \frac{\sqrt{2}}{8}\pi$$

$$\frac{\pi}{4} + c = \frac{\pi}{4}$$

$$\rightarrow c = 0$$

$$\therefore f(x) = x \sin x$$

Variation of parameters method for solving non-homogenous:

$y'' + p(x)y' + q(x)y = r(x)$

* $y = y_h + y_p$

* $y_h = y_1 c_1 + y_2 c_2$

* $y_p = y_1 \int \frac{w_1}{w} + y_2 \int \frac{w_2}{w}$, where :

1. $w = w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

2. $w_1 = \begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}$

3. $w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & r(X) \end{vmatrix}$

* * *

* Ex: Solve : $y'' - 3y' + 2y = \underbrace{\cos(e^{-x})}$

↳ not exist in the previous table.

* $y_h = ?$

$y_h \Rightarrow \lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda = 2, \lambda = 1$

* $y_h = c_1 \underbrace{e^x}_{y_1} + c_2 \underbrace{e^{2x}}_{y_2}$

* $W = w(y_1, y_2) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$

* $w_1 = \begin{vmatrix} 0 & e^{2x} \\ \cos(e^{-x}) & 2e^{2x} \end{vmatrix} = -e^{2x} \cos(e^{-x})$

* $w_2 = \begin{vmatrix} e^x & 0 \\ e^x & \cos(e^{-x}) \end{vmatrix} = e^x \cos(e^{-x})$

1* $\int \frac{w_1}{w} dx = \int \frac{-e^{2x}\cos(e^{-x})}{e^{3x}} = \int \underbrace{-e^{-x}\cos(e^{-x})}_{\text{Sub.}} dx$

$= \sin(e^{-x})$

2* $\int \frac{\omega}{\omega} = \int \frac{e^{x}\cos(e^{-x})}{e^{3x}} = \int \underbrace{e^{-2x}\cos(e^{-x})}_{\text{1. sub} \quad \text{2. By parts.}} dx$

$* z = e^{-x}$

$* dz = -e^{-x}dx \rightarrow -\int \underbrace{z\cos z}_{\text{By parts.}} dz$

$= -(z\sin z + \cos z)$

$= -e^{-x}\sin(e^{-x}) - \cos(e^{-x})$

$* y_p = e^{x}\sin(e^{-x}) + e^{2x}(-e^{-x}\sin(e^{-x}) - \cos(e^{-x}))$

$= -e^{2x}\cos e^{-x}$

$\therefore y = c_1 e^{x} + c_2 e^{2x} - e^{2x}\cos(e^{-x}).$

* * *

another Formula

$$y_p = -y_1 \int \frac{y_2\, r(x)}{\omega} + y_2 \int \frac{y_1\, r(x)}{\omega}$$

* CH.3

* Higher Order ODE *

* Ex: Solve $y^{(4)} - 5y'' + 4y = 0$ "4th order". "homogenous with constant coefficients".

$\lambda^4 - 5\lambda^2 + 4 = 0$

$(\lambda^2 - 4)(\lambda^2 - 1) = 0$

$(\lambda-2)(\lambda+2)(\lambda-1)(\lambda+1) = 0$

$\lambda = 2, -2, 1, -1$

$\therefore y = c_1e^{x} + c_2e^{-x} + c_3e^{2x} + c_4e^{-2x}$

* If $\lambda = 1, 1, 2, -2$ "Repeated" → $\underline{\underline{x}}$ نضرب ب

→ $c_1e^{x} + c_2xe^{x} + c_3e^{2x} + c_4e^{-2x}$.

~ * ~ * ~ *

* Ex: Solve $x^3y''' - 3x^2y''' + 6xy' = 0$ "3rd order". "homogenous with non-Const. coefficients".

* $y = x^m$

$m(m-1)(m-2) - 3m(m-1) + 6m = 0$

$m^3 - 3m^2 + 2m - 3m^2 + 3m + 6m = 0$

$m^3 - 6m^2 + 11m = 0$

$m(m^2 - 6m + 11) = 0$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{6 \pm \sqrt{36-44}}{2} = \boxed{3 \pm \sqrt{2}\,i} , \boxed{m=0}$$

* $y = c_1 + x^3(c_2 \cos\sqrt{2}\ln x + c_3 \sin\sqrt{2}\ln x)$.

~ * ~ * ~ *

*Ex: Solve $y^{(5)} - 3y^{(4)} + 3y^{(3)} - y'' = 0$

$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$

$\lambda^2 (\underbrace{\lambda^3 - 3\lambda^2 + 3\lambda - 1}) = 0$

حَل عن طريق نظرية العوامل "بالتجريب".

* $\lambda = 1$, is a factor for $(\lambda^3 - 3\lambda^2 + 3\lambda - 1) = 0$

$\lambda^2 - 2\lambda + 1$

$(\lambda - 1) \overline{)(\lambda^3 - 3\lambda^2 + 3\lambda - 1)}$

$\mp\lambda^3 \pm \lambda^2$

$-2\lambda^2 + 3\lambda - 1$

$\pm\lambda^2 \mp 2\lambda$

$\lambda - 1$

$\lambda - 1$

0

* $\lambda^2 (\lambda - 1)(\lambda^2 - 2\lambda + 1) = 0$

$\lambda^2 (\lambda - 1)(\lambda - 1)(\lambda - 1) = 0$

$\lambda = \underbrace{0, 0, 1, 1, 1}$

* $y = c_1 + c_2 x + c_3 e^x + c_4 x e^x + c_5 x^2 e^x$

*Ex: Solve $y^{(4)} + 5y'' + 4y = 0$

$\lambda^4 + 5\lambda^2 + 4 = 0$

$(\lambda^2 + 4)(\lambda^2 + 1) = 0$

$\lambda^2 = -4 \rightarrow \lambda = \pm 2i$

$\lambda^2 = -1 \rightarrow \lambda = \pm i$

* $y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos x + c_4 \sin x$.

* Ex: Solve $y^{(4)} + 6y'' + 9y = 0$

$\lambda^4 + 6\lambda^2 + 9 = 0 \rightarrow (z = \lambda^2)$

$z^2 + 6z + 9 = 0$

$(z+3)(z+3) = 0$

$z = -3 \rightarrow \lambda^2 = -3 \rightsquigarrow \lambda = \pm\sqrt{3}i$

$z = -3 \rightarrow \lambda^2 = -3 \rightsquigarrow \lambda = \pm\sqrt{3}i$

$*y = c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x + c_3x\cos\sqrt{3}x + c_4x\sin\sqrt{3}x$

* * *

* Ex: Solve $y^{(6)} + 6y^{(4)} + 9y'' + 4y = 0$

$\lambda^6 + 6\lambda^4 + 9\lambda^2 + 4 = 0 \rightarrow (z = \lambda^2)$

$z^3 + 6z^2 + 9z + 4 = 0$

* try $z = -1$

-1

$-1 + 6 - 9 + 4 = 0$

$-4 + 4 = 0$ ✓

$$\begin{array}{r|l} & z^2 + 5z + 4 \\ \hline z+1 & z^3 + 6z^2 + 9z + 4 \\ & \mp z^3 \mp z^2 \\ \hline & 5z^2 + 9z + 4 \\ & \mp 5z^2 \mp 5z \\ \hline & 4z + 4 \\ & \mp 4z \mp 4 \\ \hline & 0 \end{array}$$

$*(z+1)(z^2 + 5z + 4) = 0$

$(z+1)(z+4)(z+1) = 0$

$z = -1, -1, -4 = \underline{\lambda^2}$

$\lambda^2 = -1 \rightsquigarrow \lambda = \pm i$

$\lambda^2 = -1 \rightsquigarrow \lambda = \pm i$

$\lambda^2 = -4 \rightsquigarrow \lambda = \pm 2i$

$*y = c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x + c_5\cos 2x + c_6\sin 2x.$

*** Non-homogenuous higher order ODE :

*Ex: Find the form of y_p for :

$$y''' + 3y'' + 3y' + y = 30e^{-x}$$

$* \; y = y_h + y_p$

$y_h = ?$

$y''' + 3y'' + 3y' + y = 0$

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

*try $\boxed{\lambda = -1}$ ∴ is a factor.

$-1 + 3 - 3 + 1 = 0 \Leftarrow$

$(\lambda + 1)^3 = 0$

$\lambda = -1, -1, -1$

$y_h = c_1 e^{-x} + c_2 x . e^{-x} + c_3 x^2 e^{-x}$

$* \; y_p = A e^{-x} \underline{x^3}$

$\therefore y = y_h + y_p$

$y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + A x^3 e^{-x}$

* * *

*Ex: Find the form of y_p for:

$$y''' + 4y' = 24x^2 + \sin x$$

$* \; y = y_h + y_p$

$\lambda^3 + 4\lambda = 0$

$\lambda(\lambda^2 + 4) = 0$

$\boxed{\lambda = 0}$, $\lambda^2 = -4 \rightarrow \boxed{\lambda = \pm 2i}$

$* \; y_h = \underline{c_1} + c_2 \cos 2x + c_3 \sin 2x$

$* \; y_p = A_2 x^2 + A_1 x + \underline{A_0} + k_1 \cos x + k_2 \sin x.$

نضرب بـ x بسبب التشابه .

$y_p = x(A_2 x^2 + A_1 x + A_0) + k_1 \cos x + k_2 \sin x.$

$\therefore y = c_1 + c_2 \cos 2x + c_3 \sin 2x + x(A_2 x^2 + A_1 x + A_0) + k_1 \cos x + k_2 \sin x.$

* * *

*Ex: Solve $x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 \ln x$

$x = e^t \rightsquigarrow \frac{\partial x}{\partial t} = e^t \rightsquigarrow \frac{\partial t}{\partial x} = e^{-t} = \frac{1}{x}$

* نحولهم $\frac{\partial y}{\partial x}$ ل $\frac{\partial y}{\partial t}$ *

$*\frac{\partial y}{\partial x} = \frac{\partial y}{\partial t} \cdot \frac{\partial t}{\partial x}$

$y'_x = y'_t \cdot e^{-t}$

$y'_x = y'_t \cdot \frac{1}{x}$

$y''_x = \frac{1}{x^2}(y''_t - y'_t)$

*ممكن حلّه Undetermined coefficients لكن بعد تحويل الطرف الأيمن r(x) إلى صيغة من الصيغ الموجودة في الجدول ($t^4 \cdot e^t$) وممكن حله Variation فوراً *

* * *

*Ex: Solve $x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 \cdot \ln x$

$* y = y_h + y_p$

$y_h = ? \quad , \quad y = x^m$

$m(m-1)(m-2) - 3(m)(m-1) + 6m - 6 = 0$

عامل مشترك $(m-1)\left[(m^2-2m) - 3m + 6\right] = 0$

$(m-1)(m^2 - 5m + 6) = 0$

$(m-1)(m-2)(m-3) = 0$

$m = 1, 2, 3$

$* y_h = c_1 \underbrace{x}_{y_1} + c_2 \underbrace{x^2}_{y_2} + c_3 \underbrace{x^3}_{y_3}$

$$* w(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$w(x, x^2, x^3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

*Ex: Find $\det(A) = |A|$, where

$$A = \begin{vmatrix} 2 & 4 & 1 \\ 3 & -2 & 5 \\ 4 & 2 & 7 \end{vmatrix}$$

1* using First row:

$$|A| = 2 * \begin{vmatrix} -2 & 5 \\ 2 & 7 \end{vmatrix} \ominus 4 \begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix} \oplus 1 \begin{vmatrix} 3 & -2 \\ 4 & 2 \end{vmatrix}$$

$= -48 - 4 + 14 = \underline{\underline{-38}}$

2* using First column:

$$|A| = 2 * \begin{vmatrix} -2 & 5 \\ 2 & 7 \end{vmatrix} \ominus 3 \begin{vmatrix} 4 & 1 \\ 2 & 7 \end{vmatrix} \oplus 4 \begin{vmatrix} 4 & 1 \\ -2 & 5 \end{vmatrix}$$

$= -48 - 78 + 88 = \underline{\underline{-38}}$

$$*W = x\begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} - 1\begin{vmatrix} x^2 & x^3 \\ 2 & 6x \end{vmatrix}$$

$$= 6x^3 - 4x^3 = \boxed{2x^3}$$

$$*W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ x\ln x & 2 & 6x \end{vmatrix} = x\ln x * x^4 = \boxed{x^5 \ln x}\ W_1$$

$$*W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & x\ln x & 6x \end{vmatrix}$$

$$= -x\ln x * 2x^3 = \boxed{-2x^4 \ln x}$$

$$*W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & x\ln x \end{vmatrix}$$

$$= \boxed{x^3 \ln x}$$

$$*** \; y_p = y_1\int\frac{W_1}{W} + y_2\int\frac{W_2}{W} + y_3\int\frac{W_3}{W} \; ***$$

$$= x\int\frac{x^2}{2}\ln x \cdot dx + x^2\int -x\cdot\ln x\cdot dx + \frac{x^3}{2}\int \ln x \cdot dx$$

*By parts:

* * *

#Ex: $y''' + 3y'' + 5y' = 0$

$* \; z = y'$

$z'' + 3z' + 5z = 0$

* CH. 7

* Matrices *

* Determine:

* Ex: let $A = \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix}$

$|A| = 8 - 15 = -7$

* Ex: $A = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 5 & 2 \\ 1 & 3 & 4 \end{bmatrix}$

* $|A| = 2\begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix} - 4\begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix} + 1\begin{bmatrix} -1 & 5 \\ 1 & 3 \end{bmatrix}$

$= 28 + 24 - 8 = 44$

* * *

*** Operations on matrices:

* Ex: let $A = \begin{bmatrix} 2 & 4 & 3 \\ 7 & 2 & -1 \end{bmatrix}$; $B = \begin{bmatrix} 5 & 1 & 2 \\ 9 & -1 & 4 \end{bmatrix}$

1. $A + B = \begin{bmatrix} 7 & 5 & 5 \\ 16 & 1 & 3 \end{bmatrix}$ *note: A,B must have the same order "size".
*$a_{12} = 4$ *$b_{22} = -1$ (calling by Index).

2. $5A = \begin{bmatrix} 10 & 20 & 15 \\ 35 & 10 & -5 \end{bmatrix}$ ~> Scalar Multiplication.

* * *

* Multiplication of matrices:

$A_{n\times \textcircled{r}} \ x. \ B_{\textcircled{r}\times m} = (AB)_{n\times m}$

يجب أن يكون عدد أعمدة المصفوفة الأولى مساوي عدد صفوف المصفوفة الثانية

*Ex: $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \\ 4 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 1 \\ 2 & 7 \end{bmatrix}$

*Find (if possible):

1. $A_{3\times 2} B_{2\times 2} \rightsquigarrow$ ✓
2. $B_{2\times 2} A_{3\times 2} \rightsquigarrow$ ✗

1 $AB = \begin{bmatrix} 2 & 5 \\ 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 16 & 37 \\ 9 & 22 \\ 16 & 18 \end{bmatrix} = \begin{bmatrix} AB_{11} & AB_{12} \\ AB_{21} & AB_{22} \\ AB_{31} & AB_{32} \end{bmatrix}$

صف أول عمود أول

$(2*3) + (5*2) = 16$

*AB_{11} → first row of A with first column of B

→ $A_{11} * B_{11} + A_{12} * B_{21}$

$(2*3) + (5*2) = 16$

*AB_{12} → $(2*1) + (5*7) = 37$

---

*(Inverse) :- (Just for square matrices) ⇝ (2×2) (3×3) (4×4)

* $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ → Identity مصفوفة عبارة (عنصر محايد لضرب المصفوفات)

$$\begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix} \searrow$$

(Diagonal) + $\underbrace{\text{على القطر}}_{(1)}$ والباقي أصفار +

→ The inverse of A is A^{-1}, such that : $AA^{-1} = A^{-1}A = I$

*Note: If $\det(A) \neq 0$, then (A^{-1}) exists.

* رقم * رقم = 1 عكس بعض

$7 * \frac{1}{7} = 1$

* If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then:

$$* \, A^{-1} = \frac{1}{ad-bc} * \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

→ the main diagonal. (↘) (نعكس أماكنها)

→ the other diagonal (↙) (نضرب بالسالب)

* * *

※ Ex: $A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$, Find A^{-1} (if exists), check the det(A), $|A| \neq 0$

$$A^{-1} = \frac{1}{-2} * \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & 1 \\ \frac{3}{2} & -2 \end{bmatrix}$$

*** CHECK: $\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} * \begin{bmatrix} \frac{-1}{2} & 1 \\ \frac{3}{2} & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

* * *

※ Ex: Find A^{-1}, where:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

* Use gauss elmination:

$$\left[\begin{array}{c|c} A & I_3 \end{array}\right]$$

$$\left[\begin{array}{c|c} I_3 & A^{-1} \end{array}\right]$$

$$**\ \left[\begin{array}{ccc|ccc} \overset{\text{leader}}{\boxed{1}} & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & \underset{\text{leader}}{\boxed{1}} & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & \underset{\text{leader}}{\boxed{1}} & 5 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$* A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

$$** Ex: \underset{3\times3\ \ 3\times\textcircled{2}}{AB} = \begin{bmatrix} 2 & 4 \\ 7 & 3 \\ 1 & 5 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & 1 & 2 \\ 5 & 1 & 2 \\ 7 & -1 & 3 \end{bmatrix} \rightsquigarrow A^{-1} = ?$$

2. Find (B)

$$A^{-1}AB = A^{-1}[\quad]$$

$$B = A^{-1}\begin{bmatrix} 2 & 4 \\ 7 & 3 \\ 1 & 5 \end{bmatrix}$$

$$B = [\quad]$$

*Ex: Solve the following linear system of equations

$$2x + 3y + z = 1$$
$$x - 2y + 2z = 3$$
$$x + 12y - 4z = \underline{-7}$$

↳ If it was (8)

*** $\underset{\text{Coeff.}}{A}\,\underset{\text{variables}}{x} = b$ ~> Solving the system using gauss elimination:

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 2 \\ 1 & 12 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & -2 & 2 & 3 \\ 1 & 12 & -4 & -7 \end{array}\right] \rightsquigarrow \text{"Augmented"}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & -2 & 2 & 3 \\ 2 & 3 & 1 & 1 \\ 1 & 12 & -4 & -7 \end{array}\right] \quad \begin{array}{l} -2\,\text{row}(1) + \text{row}(2) \\ -1\,\text{row}(1) + \text{row}(3) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 7 & -3 & -5 \\ 0 & 14 & -6 & -10 \end{array}\right] \rightsquigarrow (\text{row } \underline{2} \div 7)$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 1 & \frac{-3}{7} & \frac{-5}{7} \\ 0 & 14 & -6 & -10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 1 & \frac{-3}{7} & \frac{-5}{7} \\ 0 & 0 & \underline{0} & \underline{0} \end{array}\right]$$

(1) ↙ ↳ If it was (2) it means no solution.

It means **one** solution.

- $x-2y+2z=3$
- $y-\frac{3}{7}z=\frac{-6}{7}$

\# $z=t$, $t\in\mathbb{R}$

* any real no.

$y=\frac{3}{7}t-\frac{5}{7}$

$x=3+2\left(\frac{3}{7}t-\frac{5}{7}\right)-2t$

$x=\frac{11}{7}-\frac{8}{7}t$

... Note:

If the number of variables is greater than the number of equations , there is infinite solutions.

* The system may have :

1- One unique solution (2 variables with 3 equ.s).

2- No solution

3- Infinite number of solutions.

* Ex: Solve:

$$x_1 + 2x_2 + 3x_3 + x_4 = 1$$
$$2x_1 - x_2 + 5x_3 - x_4 = 0$$
$$2x_1 + 2x_3 - 5x_4 = 2$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 1 \\ 2 & -1 & 5 & -1 & 0 \\ 2 & 0 & 2 & -5 & 2 \end{array}\right]$$

* * *

* Ex: Solve:

$$2x - y + z = 1$$
$$x - 3y + 2z = 0$$
$$x + y - z = 2$$

$$*A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -3 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

$$*|A| = 2 - 3 + 4 = 3$$

$$*Ax = b$$
$$A^{-1}Ax = A^{-1}b$$
$$\boxed{x = A^{-1}b}$$

* Cramer's Rule : For unique solution (square matrix) "a, a" same no. of rows and columns.

*** $x = \frac{|A1|}{|A|}$, $y = \frac{|A2|}{|A|}$, $z = \frac{|A3|}{|A|}$

... where :

$$A1 = \begin{bmatrix} \underset{b}{1} & -1 & 1 \\ 0 & -3 & 2 \\ \underline{\underline{2}} & 1 & -1 \end{bmatrix} = 3$$

$$A2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & \underset{b}{\underline{\underline{2}}} & -1 \end{bmatrix} = -8$$

$$A3 = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -3 & 0 \\ 1 & 1 & \underset{b}{\underline{\underline{2}}} \end{bmatrix} = 12$$

* $x + y = 4$

* $2x + 3y = 1$

* $3x + 4y = 5$

This system has unique solution, but it's not square matrix.

*Ex: Find A^{-1}, where :

$$A=\begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

use adjoint matrix method:

$*\det(A) = |A| = 36 + 12 + 16$

$= 64$

*** $A^{-1} = \frac{1}{\det(A)} \cdot \text{adjoint}(A)$

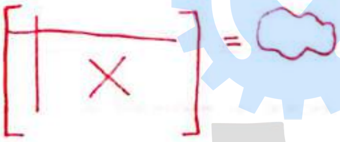
$C(A)$

↳ Cofactor matrix:

$* C_{11} = (-1)^{1+1} * 12 = 12$ $\quad * C_{21} = 4$ $\quad * C_{31} = (-1)^{4} * 12 = 12$

$* C_{12} = (-1)^{1+2} * -6 = 6$ $\quad * C_{22} = 2$ $\quad * C_{32} = (-1)^{5} * 10 = -10$

$* C_{13} = (-1)^{4} * -16 = -16$ $\quad * C_{23} = 16$ $\quad * C_{33} = (-1)^{6} * 16 = 16$

*** $(-1)^{r+n}$ *  ; r → Row no. , n → Col. no.

$$* C(A) = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

$$* \text{adj}(A) = (C(A))^{t} = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

, t or T ~ Transpose ~ تقلب المصفوفة

$$* A^{-1} = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & -16 \end{bmatrix}$$

* Ex : Find A^{-1}

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}$$

* $C(A)=\begin{bmatrix}40 & -13 & -5\\ -16 & 5 & 2\\ -9 & 3 & 1\end{bmatrix}$

* $\det(A)=(-1)$
* $A^{-1}=\begin{bmatrix}-40 & 16 & 9\\ 13 & -5 & -3\\ 5 & -2 & -1\end{bmatrix}$

** properties of Det. :

1) $\det(A) = \det(A^t)$

2) $\det(AB) = \det(A)\det(B)$ ~> (they must be square matrices) (2*2), (3*3) of the same order.

3) If A is invertable, then :

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

4) $\det(I_n) = \underline{1}$ ~> (determine for any Identity = 1)

5) $\det(AA^{-1}) = \det(A)\det(A^{-1}) = \underline{1} \rightarrow |A| * \frac{1}{|A|} = \underline{1}$

6) $\underset{A_{n*n}}{\det(kA)} = k^n . \det(A)$

* ——— *

* Ex: If A is 3X3 matrix and $\det(A) = 5$, Find $\det(2A)$

* $\det(2A) = 2^3 . \det(A) = 8 * 5 = 40$

* ——— *

7) If $A_{n \times n}$ is $\underbrace{\text{diagonal}}_{(1)}$, $\underbrace{\text{upper triangular}}_{(2)}$ or $\underbrace{\text{lower triangular}}_{(3)}$ matrix, then :

$$\det(A) = \prod_{i=1}^{n} a_{ii}$$

(Σ) Product (*)

* Ex: $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ ~> Diagonal.

* $|A| = 2 * 4 * 7 = 56$

* Ex: $A = \begin{bmatrix} 2 & 9 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ ~> upper. triangular matrix.

* ملاحظة : يجب أن يكون ما تحت القطر = 0

* $|A| = 2 * 4 * 1 = 8$

$※ Ex: A = \begin{bmatrix} 2 & 0 & 0 \\ 9 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ → lower triangular matrix.

$* |A| = 2 * 4 * 7 = 56$

ملاحظة ← يجب أن يكون ما خارج قطرة = 0 .

** 8)

$※ Ex: A = \begin{bmatrix} 2 & 4 \\ 7 & 10 \end{bmatrix}$, $|A| = \underline{-}\,8$

الإشارة

$A_1 = \begin{bmatrix} 7 & 10 \\ 2 & 4 \end{bmatrix}$, $|A| = 8$

+ ما ينطبق على الصف ينطبق على العمود

$A_2 = \begin{bmatrix} \overset{(2*3)}{\underline{6}} & \overset{(4*3)}{\underline{12}} \\ 7 & 10 \end{bmatrix}$, $|A| = -24$ $\quad [|A| * 3]$

$A_3 = \begin{bmatrix} 2 & 4 \\ \underset{(2*2+7)}{\underline{11}} & \underline{18} \end{bmatrix} = -8$ $\quad \to (4*2+10)$

(+ ضعف الصف الأول + الصف الثاني .)

* إذا جمعنا صف إلى صف (أو) عمود إلى عمود
نحصل على نفس (det) وإذا طرحنا
عكس الإشارة *

*

$※ Ex: A = \begin{bmatrix} 2 & 1 & -3 & 5 \\ 4 & 2 & 9 & 1 \\ 7 & 1 & 2 & 4 \\ -2 & -1 & 3 & -4 \end{bmatrix}$, $|A| = ?$

$B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 4 & 2 & 9 & 1 \\ 7 & 1 & 2 & 4 \\ -2 & -1 & 3 & -4 \end{bmatrix}$, $|B| =$ ←

* قمنا بجمع الصف الاخير
مع الصف الاول *

$* |B| = |A|$ $\quad$ * لأننا جمعنا *

* Eigan Value and eigan vectors for matrices:

* Ex: let $A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$,

Find the Eigan values and Vectors for A.

$$\underset{n\times n}{A}\,\underset{n\times 1}{x} = \overset{\text{Scalar}}{\lambda} x$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\lambda x - Ax = 0$$

$$\underbrace{(\lambda I - A)}_{B}\, x = 0$$

$$Bx = 0$$

$$* \; \lambda I - A = \begin{bmatrix} \lambda-1 & -4 \\ -1 & \lambda+2 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda+2)(\lambda-1) - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0$$

$\lambda = 2, -3$ → Eigan Values.

* for $\lambda = 2$:

$$\left[\begin{array}{cc|c} 1 & -4 & 0 \\ -1 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & \underline{\underline{0}} \end{array}\right] \overset{\text{row}}{(1+2)}$$ [illegible] +

→ Infinite no. of solutions.

$x_1 - 4x_2 = 0$

$x_2 = t \; , \; t \in \mathbb{R}$

$\therefore \; x_1 = 4t$

$x = \begin{bmatrix} 4t \\ t \end{bmatrix}$ * Eigan vector for $\lambda = 2$ is $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ * t مشترك عامل

* CHECK *

$$\underbrace{\begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\underline{\text{Vector}}} = \underset{\lambda}{\underline{2}} \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\underline{\text{Vector}}}$$

* For $\lambda = -3$

$$\left[\begin{array}{cc|c} -4 & -4 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & \underline{0} \end{array}\right]$$

↳ Infinite no. of solutions.

$x_1 + x_2 = 0$

$x_2 = t$

$x_1 = -t$

$x = \begin{bmatrix} -t \\ t \end{bmatrix}$. * Eigen vector for $\lambda = -3$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

~~~~~~~ * ~~~~~~~ * ~~~~~~~ *

※ Ex: Find Eigen values and vectors for A:

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$* \lambda I - A = \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix}$$

$$|\lambda I - A| = \lambda(\lambda-2)(\lambda-3) + 2(\lambda-2) = 0$$

$$= (\lambda-2)(\lambda^2 - 3\lambda + 2) = 0$$

$$(\lambda-2)^2(\lambda-1) = 0$$

$$\lambda = \underline{2}, \underline{1}$$

[ ] [ ] [ ]
~~~~~~~

CH.4

System of ODE's

* Ex: Solve :

$y_1' = \underline{-3y_1 + y_2} \quad \underset{\sim\sim}{+(t^2)}$

$y_2' = y_1 - 3y_2 \quad \underset{\sim\sim}{+(3t)}$

$\boxed{y' = Ay}$ ↳ homo. ↳ non-homo.

$$y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

* $y' = Ay$ ⇝ homogenous.

* $y'(t) = Ay(t) + g(t)$ ⇝ non-homogenous.

* * *

*** $A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$, we have to find Eigan values and Eigan vectors to solve this problem.

$$A - \lambda I = \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix}$$

** يمكن أن تكتب (λI−A) تعطي نفس الجواب

$$|A - \lambda I| = (-3-\lambda)^2 - 1 = 0$$

$$\underset{\sim\sim}{(-3-\lambda)} = \pm 1$$

$\boxed{\lambda = -4, -2}$ ⇝ Eigan values.

* for $\lambda = -4$.

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \rightsquigarrow x_1 + x_2 = 0$$

* $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_2 = t, \quad x_1 = -t$

* $x = \begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ⇝ Eigan vector ⇒ $x^{(1)}$ by $\underline{\lambda_1}$

* for $\lambda = -2$:

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$-x_1 + x_2 = 0$

$x_1 = x_2$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow$ Eigen vector $\Rightarrow x^{(2)}$ by λ_2

*** The general solution of the system is given by:

$y = c_1 y^{(1)} + c_2 y^{(2)}$, where:

$* \, y_1 = (x)^{(1)} e^{\lambda_1 t}$, $* \, y_2 = (x)^2 e^{\lambda_2 t}$

(G.S) $** \; y = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$

الصف الاول ← $y_1 = -c_1 e^{-4t} + c_2 e^{-2t}$

الصف الثاني ← $y_2 = c_1 e^{-4t} + c_2 e^{-2t}$

* * *

* Ex: Solve:

$y_1' = y_1 + y_2$, IVP $y_1(0) = 1$ → To Find the Values of (c_1, c_2)

$y_2' = 4y_1 + y_2$ $\quad y_2(0) = 6$

$* \, A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$

$|A - \lambda I| = (1-\lambda)^2 = 4$

$1 - \lambda = \pm 2$

$** \; \lambda = 3, -1$

* for $\lambda_1 = 3$:

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right]$$ *نفس المعادلة بس مضروبة ب (-2)

$-2x_1 + x_2 = 0$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightsquigarrow$ Eigan Vector $\Rightarrow x^{(1)}$ by λ_1

* for $\lambda_2 = -1$:

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right]$$ *نفس المعادلة بس مضروبة ب (2)

$2x_1 + x_2 = 0$

$\begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightsquigarrow$ Eigan vector $\Rightarrow x^{(2)}$ by λ_2

$$** \, y = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} 1 + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} 1 \longrightarrow t = 0, \; \therefore e^{3(0)} = 1, \; e^{-1(0)} = 1$$

$1 = c_1 + c_2$

$6 = 2c_1 - 2c_2$

$\therefore c_1 = 2, \; c_2 = -1$

$$\boxed{* \, y = \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{3t} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t}}$$

* Ex : solve :

$\vec{y}' = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \vec{y}$; $\vec{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ the same meaning.

$y_1' = 3y_1$

$y_2' = 3y_2$

* $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix}$

$|A - \lambda I| = (3-\lambda)^2 = 0$

$\lambda = 3$, (Repeated)

** for $\lambda_1 = 3 = \lambda_2$:

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$, $x_1 = \underline{\underline{t}}$, $(s,t) \in \mathbb{R}$

$x_2 = \underline{\underline{s}}$

كل الأعداد

* $X = \begin{bmatrix} t \\ s \end{bmatrix}$, Eigan vectors : $\underline{\underline{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}}_{X^{(1)}}$, $\underline{\underline{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}}_{X^{(2)}}$

نسحب t عامل مشترك ←

→ نسحب s عامل مشترك

* G.S :

$y = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}$

$\left.\begin{array}{l} y_1 = c_1 e^{3t} \\ y_2 = c_2 e^{3t} \end{array}\right\}$ (لا يوجد بينهم علاقة).

* ~~~ * ~~~ * ~~~ *

** Ex: Solve :

$$y' = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix} y$$

$* A = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$, $A - \lambda I = \begin{bmatrix} 2-\lambda & 2 \\ -1 & -\lambda \end{bmatrix}$

$$|A - \lambda I| = -(2-\lambda)\lambda + 2 = 0$$
$$= \lambda^2 - 2\lambda + 2 = 0$$

$* \lambda = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i \rightsquigarrow$ complex no.

$*$ for $\lambda = 1+i$:

$$\left[\begin{array}{cc|c} 1-i & 2 & 0 \\ -1 & -1-i & 0 \end{array}\right]$$

$$\xrightarrow{-r_2} \left[\begin{array}{cc|c} 1 & 1+i & 0 \\ 1-i & 2 & 0 \end{array}\right]$$

$$\xrightarrow{-(1-i)r_1 + r_2} \left[\begin{array}{cc|c} 1 & 1+i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

gauss.

$* \; -(1-i)(1+i) = -2$

$(1-i)x_1 + 2x_2 = 0$

Eigan vector : $\begin{bmatrix} 1 \\ \frac{i-1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ i-1 \end{bmatrix} \Rightarrow x^{(1)}$

$*$ for $\lambda = 1-i$:

نفس الحاطة

$$\left[\begin{array}{cc|c} 1+i & 2 & 0 \\ -1 & i-1 & 0 \end{array}\right]$$

$(1+i)x_1 + 2x_2 = 0$

Eigan vector : $\begin{bmatrix} 2 \\ -(1+i) \end{bmatrix} \Rightarrow x^{(2)}$

*** G.S :

$$y = c_1 \begin{bmatrix} 2 \\ i-1 \end{bmatrix} e^{(1+i)t} + c_2 \begin{bmatrix} 2 \\ -(1+i) \end{bmatrix} e^{(1-i)t}$$

※ Ex : Solve:

$y_1' = 2y_1 - 2y_2$

$y_2' = 2y_1 + 2y_2$

$* A = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$, $A - \lambda I = \begin{bmatrix} 2-\lambda & -2 \\ 2 & 2-\lambda \end{bmatrix}$

$|A - \lambda I| = (2-\lambda)^2 + 4 = 0$

$\rightsquigarrow 2 - \lambda = \mp 2i$

$\lambda = 2 \mp 2i$

$* \lambda_1 = 2 - 2i$:

$\left[\begin{array}{cc|c} 2i & -2 & 0 \\ 2 & 2i & 0 \end{array}\right]$

$2i x_1 - 2x_2 = 0$

Eigan vector: $\begin{bmatrix} 1 \\ i \end{bmatrix}$, when $(x_1 = 1) = x^{(1)}$

$* \lambda_2 = 2 + 2i$:

$\left[\begin{array}{cc|c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right]$

$-2i x_1 - 2x_2 = 0$

Eigan vector $\begin{bmatrix} 1 \\ -i \end{bmatrix}$, when $(x_1 = 1) = x^{(2)}$

※ G.S:

$y = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(2-2i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(2+2i)t}$

$* y_1 = c_1 e^{(2-2i)t} + c_2 e^{(2+2i)t}$

$* y_2 = i c_1 e^{(2-2i)t} + (-i) c_2 e^{(2+2i)t}$.

※ Ex: Solve:

$$y' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} y$$

* $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $A - \lambda I = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$

$|A - \lambda I| = (2-\lambda)^2 = 0$

* $\lambda_1 = 2 = \lambda_2$:

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = t$

$x_2 = s$, $(t, s) \in \mathbb{R}$

$$x = \begin{bmatrix} t \\ s \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x^{(1)}} + s \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{x^{(2)}}$$

$$y = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}$$

* Repeated Eigan Values with diff. Eigan vectors.

* * *

※ Ex: Solve:

$$y' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} y$$

$A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$, $A - \lambda I = \begin{bmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix}$

$$|A-\lambda I| = (4-\lambda)(2-\lambda)+1=0$$

$$\lambda^2-6\lambda+9=0$$

$$(\lambda-3)^2=0$$

$$\boxed{\lambda=3}$$

$* \lambda_1 = 3 = \lambda_2$:

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right]$$

$$x_1 + x_2 = 0$$

Eigan vector: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$* \; y^{(1)} = x^{(1)} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$$

$$* \; y^{(2)} = x^{(1)} t e^{\lambda_1 t} + u e^{\lambda_1 t}, \text{ where:}$$

$$(A-\lambda I)u = x^{(1)}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & -1 \end{array}\right]$$

$$u_1 + u_2 = 1.$$

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$* \; y = \underbrace{c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}}_{y^{(1)}} + \underbrace{c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{3t}}_{y^{(2)}}$$

(another solution)

$$\rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

※ Repeated Eigan Values with One Eigan Vector.

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix}$$

* * *

※Q → Find system of ODE's whose solution is given:

$$y = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

※ $y' = Ay$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

* $\lambda_1 = -2, \ x^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

* $\lambda_2 = -4, \ x^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} a+2 & b & 0 \\ c & d+2 & 0 \end{array}\right]$$

$(a+2)x_1 + bx_2 = 0$ --- (1)

$cx_1 + (d+2)x_2 = 0$ --- (2)

$$\left[\begin{array}{cc|c} a+4 & b & 0 \\ c & d+4 & 0 \end{array}\right]$$

$(a+4)x_1 + bx_2 = 0$ --- (3)

$cx_1 + (d+4)x_2 = 0$ --- (4)

$$x^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \begin{cases} a+2+b=0 \\ c+d+2=0 \end{cases}$$

$$x^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightsquigarrow \begin{cases} a+4-b=0 \\ c-(d+4)=0 \end{cases}$$

$a = -3$, $c = 1$

$b = 1$, $d = -3$

* $y_1' = -3y_1 + y_2$
* $y_2' = y_1 - 3y_2.$

* 4.6 – nonhomogenous systems and ODE's *

$$y' = Ay + g(t)$$

$$y = \underbrace{y^{(h)}}_{\text{Solve for homogenous}} + \underbrace{y^{(P)}}_{\text{Particular solution}}$$

** Undetermined coefficients method:

* Ex: Solve:

$$y_1' = y_2 + t$$

$$y_2' = y_1 - 3t$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} y + \begin{bmatrix} 1 \\ -3 \end{bmatrix} t$$

$$*\, y^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

$$*\, y^{(P)} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b1 \\ b2 \end{bmatrix}$$

$$y'(P) = \begin{bmatrix} a1 \\ a2 \end{bmatrix} + 0$$

$$y_p' = y'$$

$$y^{(P)\prime} = Ay^{(P)} + g(t).$$

$$\begin{bmatrix} a1 \\ a2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} a1 \\ a2 \end{bmatrix} t + \begin{bmatrix} b1 \\ b2 \end{bmatrix} \right) + \begin{bmatrix} 1 \\ -3 \end{bmatrix} t$$

$$\begin{bmatrix} a1 \\ a2 \end{bmatrix} = \begin{bmatrix} a2 \\ a1 \end{bmatrix} t + \begin{bmatrix} b2 \\ b1 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \end{bmatrix} t$$

$$0 = \underbrace{a2 + 1}_{}$$ مع (t)

$$(\underbrace{a1 = b2}_{})$$ بدون (t)

→ (صف أول) بدون (t)

$$0 = \underbrace{a1 - 3}_{}$$ مع (t)

$$(\underbrace{a2 = b1}_{})$$ بدون (t)

→ (صف ثاني)

$$\boxed{\begin{aligned} a2 &= -1 \\ a1 &= 3 \\ b2 &= 3 \\ b1 &= -1 \end{aligned}}$$

$$*y^{(P)} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 3 \end{bmatrix} \rightsquigarrow$$ لا يوجد تشابه مع $\underline{\underline{y^{(h)}}}$

$y_1' = y_2 + t^2$

$y_2' = y_1 - 3t$

$$*g(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ -3 \end{bmatrix} t$$

$$*y^{(P)} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t^2 + \begin{bmatrix} b1 \\ b2 \end{bmatrix} t + \begin{bmatrix} c1 \\ c2 \end{bmatrix}$$

* * *

***Ex: Solve:**

$y_1' = y_1 + y_2 + 5\cos t$.

$y_2' = 3y_1 - y_2 \underbrace{- 5\sin t}$.

[If it was $(-5\sin(2t))$, it will yields $\underline{4}$ solutions]

$$*y^{(h)} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$$

$$*g(t) = \underbrace{\begin{bmatrix} 5 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ -5 \end{bmatrix} \sin t}_{\begin{bmatrix} 5\cos t \\ -5\sin t \end{bmatrix}}$$

$$*y^{(P)} = \begin{bmatrix} a1 \\ a2 \end{bmatrix} \cos t + \begin{bmatrix} b1 \\ b2 \end{bmatrix} \sin t$$

$$y'^{(P)} = -\begin{bmatrix} a1 \\ a2 \end{bmatrix} \sin t + \begin{bmatrix} b1 \\ b2 \end{bmatrix} \cos t$$

$$y'p = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \left(\begin{bmatrix} a1 \\ a2 \end{bmatrix} \cos t + \begin{bmatrix} b1 \\ b2 \end{bmatrix} \sin t \right) + \begin{bmatrix} 5 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ -5 \end{bmatrix} \sin t .$$

$$-\begin{bmatrix}a_1\\a_2\end{bmatrix}\sin t+\begin{bmatrix}b_1\\b_2\end{bmatrix}\cos t=\begin{bmatrix}a_1+a_2\\3a_1+(-a_2)\end{bmatrix}\cos t+$$

$$\begin{bmatrix}b_1+b_2\\3b_1-b_2\end{bmatrix}\sin t+\begin{bmatrix}5\\0\end{bmatrix}\cos t+\begin{bmatrix}0\\-5\end{bmatrix}\sin t.$$

$\underline{\cos t}$ +حد أول ح

$b_1=a_1+a_2+5$

$\sin t$ ح+

$-a_1=b_1+b_2$

$\rightsquigarrow$ <u>حد أول</u>

$\underline{\cos t}$ ح+

$b_2=3a_1-a_2$

$-a_2=3b_1-b_2-5$

$\rightsquigarrow$ <u>حد ثاني</u>

$-a_1=4a_1+5 \rightsquigarrow \boxed{a_1=-1}$

$-a_2=4a_2+10 \rightsquigarrow \boxed{a_2=-10}$

$\boxed{\begin{matrix}b_1=2\\b_2=-1\end{matrix}}$

$$*y^{(P)}=\begin{bmatrix}-1\\-2\end{bmatrix}\cos t+\begin{bmatrix}2\\-1\end{bmatrix}\sin t.$$

※ Ex : write the Form of $y^{(P)} = ?!$

$$y_1' = 4y_2 + 5e^{t}$$
$$y_2' = -y_1 - 20e^{-t}$$

$$* \; y^{(h)} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 2 \\ -i \end{bmatrix} e^{-2it}$$

$$* \; g(t) = \begin{bmatrix} 5 \\ 0 \end{bmatrix} e^{t} + \begin{bmatrix} 0 \\ -20 \end{bmatrix} e^{-t}$$

$$** \; y^{(P)} = \begin{bmatrix} a1 \\ a2 \end{bmatrix} e^{t} + \begin{bmatrix} b1 \\ b2 \end{bmatrix} e^{-t}$$

* * *

* H.W: write the Form of $y^{(P)}$

$$y' = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix} y + \begin{bmatrix} 11t + 15 \\ 3e^{-t} - 15t^{2} - 20 \end{bmatrix}$$

$* \; y^{(h)} = ?!$

$$A - \lambda I_2 = \begin{bmatrix} -3-\lambda & -4 \\ 5 & 6-\lambda \end{bmatrix}$$

$$|A - \lambda I_2| = (-3-\lambda)(6-\lambda) + 20 = 0$$
$$\lambda^2 - 3\lambda + 20 = 0$$
$$(\lambda - 1)(\lambda - 2) = 0 \leadsto \lambda = 1, \lambda = 2$$

$$* \; y^{(P)} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{-t} + \begin{bmatrix} b1 \\ b2 \end{bmatrix} t^{2} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} t + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

*لا يوجد تشابه ، لا نستخدم
ال modification
e^{1t}, e^{2t}

$$* \; g(t) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ -15 \end{bmatrix} t^{2} + \begin{bmatrix} 11 \\ 0 \end{bmatrix} t + \begin{bmatrix} 15 \\ -20 \end{bmatrix}$$

※Ex: Solve:

$$y' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$* y^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$* y^{(P)} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} te^{-2t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{-2t}$$

$$* y^{(P)'} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{-2t} - 2 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} te^{-2t} - 2 \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{-2t}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{-2t} - 2 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} te^{-2t} - 2 \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{-2t} = ----$$

$$= \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} te^{-2t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{-2t} \right) + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$= \begin{bmatrix} -3a_1 + a_2 \\ a_1 - 3a_2 \end{bmatrix} te^{-2t} + \begin{bmatrix} -3b_1 + b_2 \\ b_1 - 3b_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$* \; -2a_1 = -3a_1 + a_2$ --- (1) $\rightarrow a_1 = a_2$

$* \; a_1 - 2b_1 = -3b_1 + b_2 - 6$ --- (2)

$* \; -2a_2 = a_1 - 3a_2$ --- (3) $\rightarrow a_1 = a_2$

$* \; a_2 - 2b_2 = b_1 - 3b_2 + 2$ --- (4)

$$\left.\begin{aligned} a_1 &= b_2 - b_1 - 6 \\ a_1 &= b_1 - b_2 + 2 \end{aligned}\right\} 2a_1 = -4 \rightsquigarrow \boxed{a_1 = a_2 = -2}$$

$b_1 = 2$

$b_2 = 6$

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

*- عوض بقيم b أي قيمة تحقق المعادلة (!) .

$$\left.\begin{aligned} & * \; b_2 - b_1 = 4 \\ & * \; b_1 - b_2 = -4 \\ & b_1 = 0, \; b_2 = 4 \end{aligned}\right\} * y^{(P)} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} te^{-2t} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} e^{-2t}$$

※ Ex: If $y = c_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} \cos t + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \sin t$, is a solution for $y' = Ay$, Find the form of the general solution of $y' = Ay + \begin{bmatrix} 2\cos t \\ 5 \sin t \end{bmatrix}$

$$* y^{(P)} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \cos t + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} \sin t + t\left(\begin{bmatrix} k_1' \\ k_2' \end{bmatrix} \cos t + \begin{bmatrix} k_3' \\ k_4' \end{bmatrix} \sin t \right)$$

$* \text{If it was } (\underline{\underline{\sin 3t}}) \rightsquigarrow \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 5 \end{bmatrix} \sin t$.

$$(\text{Same sol.}) + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos 3t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sin 3t \,.$$

* * *

※ Ex: $y' = Ay$

$$* y^{(h)} = c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{-2t} ,$$

Find the form of general solution of $y' = Ay + \begin{bmatrix} t^2 + e^{2t} \\ t - e^{-2t} \end{bmatrix}$

$$* g(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{2t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{-2t}$$

$$* y^{(P)} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t^2 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{2t} + \begin{bmatrix} k_1' \\ k_2' \end{bmatrix} t e^{2t} + \cdots$$

يوجد مشابه ←

(t) نضربه ب

$$+ \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} e^{-2t} + \begin{bmatrix} k_3' \\ k_4' \end{bmatrix} t e^{-2t} .$$

※ Ex: $y' = Ay$, The system is

$$y = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it} ,$$

Find the form of general solution , for:

$$y' = Ay + \begin{bmatrix} \sin 2t \\ \cos t \end{bmatrix}$$

$$* \; g(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(2t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t$$

$$y^{(P)} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos 2t + \begin{bmatrix} b' \\ b_2 \end{bmatrix} \sin 2t + \left(t * \begin{bmatrix} a_1' \\ a_2' \end{bmatrix} \cos 2t + \right.$$

$$\left. ---- + t * \begin{bmatrix} b_1' \\ b_2' \end{bmatrix} \sin 2t \right) + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \cos t + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} \sin t .$$

* ~ * ~ * ~

※ Variation of parameters:

* new method for solving system of ODE's:

※ Ex: solve:

$$y' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$* \; y^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$* \; y(t) = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix}$$

$$* \; y^{(P)} = y(t) \; u(t)$$

where:

$$u'(t) = y^{-1}(t) . g(t)$$

$$\rightarrow y^{-1}(t) = \frac{1}{-2e^{-6t}} * \begin{bmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{bmatrix} \begin{bmatrix} -6 e^{-2t} \\ 2 e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} -3+1 \\ -3e^{2t} - e^{2t} \end{bmatrix} = \begin{bmatrix} -2 \\ -4e^{2t} \end{bmatrix}$$

$$\rightarrow u(t) = \int_0^t \begin{bmatrix} -2 \\ -4e^{2\tau} \end{bmatrix} \partial\tau = \begin{bmatrix} -2\tau \\ -2e^{2\tau} \end{bmatrix} \Bigg|_0^t = \begin{bmatrix} -2t \\ -2e^{2t} + 2 \end{bmatrix}$$

$$* \; y^{(P)} = \begin{bmatrix} 3e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{bmatrix} -2t \\ 2 - 2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} -2te^{-2t} + 2e^{-4t} - 2e^{-2t} \\ -2t e^{-2t} - 2 e^{-4t} + 2e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} -2t - 2 \\ 2 - 2t \end{bmatrix} e^{-2t} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-4t}$$

$$* \, y^{(P)} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} t e^{-2t} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-4t}$$

** Conversion of an (n^{th}) order linear ODE; to a linear system:

* Rule *

$$\boxed{y_n = y^{(n-1)}}$$

※Ex: $y'' + 3y' + 2y = 0$

*let $y_1 = y$

$y_2 = y'$

$y'_2 = y''$

$\therefore y'_2 = -3y_1 - 2y_1$

※Ex: $y'''' - 2y''' + 5y'' + 3y' + 7y = 0$ ← CH.2 مهمة حلها كلها *

$y_n = y^{(n-1)}$

*let $y'''' = 2y''' - 5y'' - 3y' - 7y$

$y_1 = y$

$y_2 = y'$

$y_3 = y''$

$y_4 = y'''$

$y'_4 = y''''$

$\therefore y'_4 = 2y_4 - 5y_3 - 3y_2 - 7y_1$

*Ex: Solve (v. of. P)

$$y_1' = y_2 + t$$
$$y_2' = y_1 - 3t$$

$$*\, y^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$$

$$*\, y^{(\,)} = \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix}$$

$$*\, y^{-1}(t) = \frac{-1}{2} \begin{bmatrix} -e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{-t} & e^{-t} \\ e^{t} & -e^{t} \end{bmatrix}$$

$$*\, y^{-1} \cdot g(t) = \frac{1}{2} \begin{bmatrix} e^{-t} & e^{-t} \\ e^{t} & -e^{t} \end{bmatrix} \begin{bmatrix} t \\ -3t \end{bmatrix}$$

$$*\, u' = \frac{1}{2} \begin{bmatrix} te^{-t} - 3te^{-t} \\ te^{t} + 3te^{t} \end{bmatrix} = \begin{bmatrix} -te^{-t} \\ 2te^{t} \end{bmatrix}$$

$$u = \int_0^t \begin{bmatrix} -\tau e^{-\tau} \\ 2\tau e^{\tau} \end{bmatrix} \partial\tau = \begin{bmatrix} \tau e^{-\tau} + e^{-\tau} \\ 2(\tau e^{\tau} - e^{\tau}) \end{bmatrix} \Bigg|_0^t$$

$u = \tau ; \partial v = e^{-\tau}$

$\partial u = 1 ; v = -e^{-\tau}$

$-\tau e^{-\tau} - e^{-\tau}$

$u = \tau ; \partial v = e^{\tau}$

$\partial \tau , e^{\tau}$

$\tau e^{\tau} - e^{\tau}$

$$*\, \bar{u} = \begin{bmatrix} te^{-t} + e^{-t} - 1 \\ 2te^{t} - 2e^{t} + 2 \end{bmatrix}$$

* ناتج تعويض لصفر *

$$*\, y^{(P)} = \begin{bmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{bmatrix} \begin{bmatrix} te^{-t} + e^{-t} - 1 \\ 2te^{t} - 2e^{t} + 2 \end{bmatrix}$$

$$= \begin{bmatrix} t + 1 - e^{t} + 2t - 2 + 2e^{-t} \\ t + 1 - e^{t} - 2t + 2 - 2e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} 3t - 1 \\ -t + 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \end{bmatrix} e^{-t} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{t}$$

$$*\, y^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

* بدمج المتشابهين بـ $y^{(P)}$ مع $y^{(h)}$

CH.5: Power series.

1- Dec. - 2015

* $\sum_{k=0}^{\infty} a_k x^k$ → Power series around $\boxed{x=0}$.

* $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ → Power series around $\boxed{x=x_0}$.

- The Maclurin series (M.S) for f(x) is given by:

$$** f(x) = \sum_{k=0}^{\infty} \frac{f(0)^{(k)} x^k}{k!}$$

- The Taylor series for f(x) at $[x=x_0]$ is given by:

$$** f(x) = \sum_{k=0}^{\infty} \frac{f(x_0)^{(k)} * (x-x_0)^k}{k!}$$

f(x)	- M.S - -- (حفظ)
1) e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!} \rightarrow 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
) $\sin(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{(2k+1)}}{(2k+1)!}$
) $\cos(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k}}{2k!}$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k \; ; \; \lvert x \rvert < 1$

\# Consider the ODE:

$A(x)y'' + B(x)y' + C(x)y = 0$

↳ no common factor between (A, B, C) *(واذا موجود بلغيه)*

→ The ordinary points for ODE is all points at which $\boxed{A(x) \neq 0}$

→ The singular points for ODE are all points at which $\boxed{A(x) = 0}$

*Ex: Find the singular points for the ODE:

$$\underbrace{x(x^2-1)(x^2+4)}_{A(x)}y'' + 5xy' - 2y = 0$$

** $x(x^2-1)(x^2+4) = 0$

$x = 0 \, , \, \pm 1 \, , \, \pm 2i \rightarrow$ S.P

* * *

\# Ex: Find singular points for:

$x(x^2-1)y'' + x^2(x^2-1)y' + (x-1)y = 0$

→ There is a common factor $(x-1)$ نختصر

$x(x+1)y'' + x^2(x+1)y' + y = 0$

$x = 0, -1 \rightarrow$ S.P

** note: if he ask about the ordinary points:

ans: $\mathbb{R} - \{0, -1\} \rightarrow$ O.P

all real numbers.

* * *

\# Theorem:

If: $A(x)y'' + B(x)y' + C(x)y = 0$, have $(x=0)$ as an ordinary point, then the general (solution of the ODE around $(x=0)$ can be written as):

$$** \quad y = \sum_{n=0}^{\infty} a_n x^n = a_0 y_1 + a_1 y_2 \quad **$$

*Ex: Solve the ODE; using the series solution around $x=0$ ($y'' + y = 0$)

**$A(0) = 1 \neq 0$, the ($x=0$) → ordinary point.

→ the solution is given by:

* $y = \sum_{n=0}^{\infty} a_n x^n$

* $y' = \sum_{n=1}^{\infty} n a_n x^{(n-1)}$

* $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{(n-2)}$

(Shifting) $y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$

**$y''+y = \sum_{n=0}^{\infty} (n+2)(n+1)a_{(n+2)}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

$= x^n\left((n+2)(n+1)a_{n+2} + a_n\right) = 0$

$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$ → Recusion Relation.

**$a_2 = \frac{-a_0}{2}$, $a_4 = \frac{-a_2}{4*3} = \frac{a_0}{(2)(3)(4)} = \frac{a_0}{4!}$

$a_6 = \frac{-a_4}{6*5}$, $a_8 = \frac{-a_6}{8*7} = \frac{a_0}{8!}$

#$a_{2k} = \frac{a_0(-1)^k}{(2k)!}$ ---- [even]

**$a_3 = \frac{-a_1}{3!}$, $a_5 = \frac{-a_3}{5*4} = \frac{a_1}{5!}$

$a_7 = \frac{-a_1}{7!}$

#$a_{(2k+1)} = \frac{(-1)^k + a_1}{(2k+1)!}$ ---- [odd]

$$+\ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ----$$

$$= \left[\begin{array}{l} a_0 \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + ---\right) + \\ a_1 \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + ----\right) \end{array}\right]$$

<u>G.S</u> :- $$y = \underbrace{a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}}_{y_1} + \underbrace{a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)}}{(2k+1)!}}_{y_2}$$

$$y = a_0 \cos(x) + a_1 \sin(x) .$$

* *

\# Ex: Solve $(x^2+1)y'' + 6xy' + 6y = 0$, around $x=0$.

$x=0$ is ordinary point.

** The general solution as the form :

$$y = \sum_{n=0}^{\infty} a_n x^n .$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{(n-1)} \quad , y'' = \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2}$$

$$\rightarrow (x^2+1) \sum_{n=2}^{\infty} (n)(n-1) a_n x^{n-2} + 6x \sum_{n=1}^{\infty} n a_n x^{n-1} + 6 \sum_{n=0}^{\infty} a_n x$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=2}^{\infty} n(n+1) a_n x^{n-2} + 6 \sum_{n=1}^{\infty} n a_n x$$

$$6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$** \ x^n \left(n(n-1)a_n + (n+2)(n+1)a_{n+2} + 6n a_n + 6a_n\right) = 0$$

$$\underbrace{(n^2+5n+6)}_{(n+3)\cancel{(n+2)}} a_n + \cancel{(n+2)}(n+1)\, a_{n+2} = 0$$

$$\rightarrow a_{n+2} = \frac{-(n+3)a_n}{(n+1)}$$

$$**\ a_2 = \frac{-3a_0}{1} \quad , \quad a_4 = \frac{-5a_2}{3} = 5a_0$$

$$a_6 = \frac{-7a_4}{5} = -7a_0$$

$$\#\ a_{2k} = (-1)^k\ (2k+1)a_n \quad ---- [\text{even}]$$

$$**\ a_3 = -2a_1 \quad , \quad a_5 = \frac{-6}{4}a_3 = 3a_1$$

$$a_7 = \frac{-8}{6}a_5 = -4a_1$$

$$\#\ a_{(2k+1)} = (-1)^k\ (k+1)a_1 \quad ---- [\text{odd}]$$

$$\rightarrow y = a_0 + a_1x + a_2x^2 + a_3x^3 + ---$$
$$= a_0(1 + (-3x^2) + 5x^4 - 7x^6 + 9x^8 ---) +$$
$$a_1(x + (-2x^3) + 3x^5 - 4x^7 + 5x^9 ---) .$$

$$*\ y = a_0\sum_{k=0}^{\infty}(-1)^k\ (2k+1)x^{2k} + a_1\sum_{k=0}^{\infty}(-1)^k\ (k+1)x^{2k+1}$$

$\hookrightarrow$ not a known function.

* وإذا ما في اقتران معروف بنوقف ع هون *

* * *

\# Q ; solve $(1-x^2)y'' - 2xy' + 2y = 0$, around $x = 0$.

*Def. : consider the ODE:

$$A(x)y'' + B(x)y' + C(x)y = 0$$

let $x = x_0$ be singular point; if

1. $\lim (x-x_0)\cdot\frac{B(x)}{A(x)} = B_0$ (Finite), and

2. $\lim (x-x_0)^2\cdot\frac{C(x)}{A(x)} = C_0$ (Finite), then [∞ أي غير]

$(x=x_0)$ is called regular singular point.

* * *

*Ex: Find Regular singular point (r.s.p) for

$$(x^2-9)y'' + (x+3)y' + x(x-3)y = 0$$

$$(x^2-9) = (x-3)(x+3) = 0$$

$(x=3,\ x=-3) \rightarrow$ singular

*** $x = 3$.

$$\rightarrow \lim_{x\to 3} \frac{(x-3)(x+3)}{(x^2-9)} = 1$$

$$\rightarrow \lim_{x\to 3} (x-3)^2\cdot\frac{x(x-3)}{(x^2-9)} = 0$$

Finite $\therefore \boxed{x=3}$ is r.s

*** $x = -3$.

$$\rightarrow \lim_{x\to -3} \frac{(x+3)(x+3)}{(x^2-9)} = 0$$

$$\rightarrow \lim_{x\to -3} (x+3)^2\cdot\frac{x(x-3)}{(x^2-9)} = 0$$

Finite $\therefore \boxed{x=-3}$ is r.s.p

* * *

*Ex: Find r.s.p for:

$x^2(1-x^2)y'' + y' + xy = 0$ ⤳ no common Factor.

↳ $x^2(1-x^2) = 0$

$x = 0, 1, -1$ → (singular).

*** $X=0$

→ $\lim_{x \to 0} x \cdot \frac{1}{x^2(1-x^2)}$ infinite → $x=0$ irregular (not r.s.p).

*** $X=1$

→ $\lim_{x \to 1} (x-1) \frac{1}{x^2(1-x^2)} = \lim_{x \to 1} \frac{1}{-x^2(1+x^2)} = \frac{-1}{2}$ (finite).

→ $\lim_{x \to 1} (x-1)^2 \frac{x}{x^2(1-x^2)} = 0$ (finite)

∴ $\boxed{X=1}$ ⤳ (r.s.p)

*** $X=-1$

→ $\lim_{x \to -1} (x+1) \cdot \frac{1}{x^2(1-x^2)} = \lim_{x \to -1} \frac{1}{x(1-x)} = \frac{-1}{2}$

→ $\lim_{x \to -1} (x+1)^2 \frac{x}{x^2(1-x^2)} = 0$

(r.s.p)

∴ $\boxed{X=-1}$ ⤳

\# Theorem : [Frobenious]

If $x=0$ is (r.s.p) of $A(x)y'' + B(x)y' + C(x)y = 0$, then there exist at least one solution for the ODE of the form :

$$*** \quad y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad ***$$

where r is the row of indicial equation

$$r(r-1) + \underline{\underline{B_0}}r + \underline{\underline{C_0}} = 0$$

(limit من أوجدهم)

---

* Ex: Solve :

$$4xy'' + 2y' + y = 0 \quad , \text{ around } \underline{\underline{x=0}}$$

↳ singular point

$$\rightarrow \lim_{x\to 0} x \cdot \frac{2}{4x} = \frac{1}{2} \; (B_0)$$

$$\rightarrow \lim_{x\to 0} x^2 \cdot \frac{1}{4x} = 0 \; (C_0)$$

* Indicial eqn. :

$$r(r-1) + \frac{1}{2}r = 0$$

$$2r^2 - r + \frac{1}{2}r = 0$$

$$r(2r-1) = 0$$

$$r = 0 \quad , \quad r = \frac{1}{2}$$

$$\rightarrow y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

---

* Ex: Find the indicial equation for:

$x(x-1)y'' + (3x-1)y' + y = 0$, around $\underline{x=0}$ ↳ s.p

$\rightarrow \lim_{x\to 0} x \cdot \frac{(3x-1)}{x(x-1)} = 1 \quad (B_0)$

$\rightarrow \lim_{x\to 0} x^2 \cdot \frac{1}{x(x-1)} = 0 \quad (C_0)$

* Indicial equation:

$r(r-1) + r = 0$

$r^2 = 0$

$r = 0 \; , \; r = 0$

---

* Ex: Find r.s.p for:

$3x^2y'' + xy' - (1-x)y = 0$, around $\underline{x=0}$ ↳ s.p

$\rightarrow \lim_{x\to 0} x \cdot \frac{x}{3x^2} = \frac{1}{3} \quad (B_0)$

$\rightarrow \lim_{x\to 0} x^2 \cdot \frac{-(1+x)}{3x^2} = \frac{-1}{3} \quad (C_0)$

* In[illegible]uation:

[illegible] $= 0$

$r($[illegible]

[illegible] $r-1)$ [illegible]

[illegible] $r-1=$ [illegible]

[illegible]

* Theorem: (Frobenious).

If $x=0$ is (r.s.p) of $A(x)y'' + B(x)y' + C(x)y = 0$ and let $\underline{r_1}$ and $\underline{r_2}$ be roots for the indicial e

1. If $r_1 - r_2 \neq$ integer, then the two solutions of the ODE are:

* $y_1 = \sum_{n=0}^{\infty} a_n x^{n+n}$, * $y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}$

2. If $r_1 = r_2 = r$, then:

$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r}$, * $y_2 = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r}$

3. If $r_1 \neq r_2$ and $r_1 - r_2 =$ integer, then:

* $y_1 = \sum_{n=0}^{\infty} a_n \cdot x^{n+r_1}$, * $y_2 = k y_1 \ln x + \sum_{n=0}^{\infty} b_n \cdot x^{n+r_2}$

// Ex : $4xy'' + 2y' + y = 0$, around $x=0$

$(x=0) \rightsquigarrow (r.s.p)$

$[r_1 = 0 , r_2 = \frac{1}{2}]$ the first condition.

$$\cdot\ y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n\, x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{n+r-2}$$

$$** \quad 4\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} + 2\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r}}_{\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}} = 0$$

$$// \quad (4r(r-1) + 2r)\, a_0 x^{r-1} + \left[\sum_{n=1}^{\infty} (4(n+r)(n+r-1) + 2(n+r) a_n + \cdots + a_{n-1}\right] x^{n+r-1} = 0$$

$$(4r^2 - 2r)\, a_0 x^{r-1} = 0 \quad \& \quad a_n = \frac{-a_{n-1}}{2(n+r)(2n+2r-1)}$$

** To Find y_1 , y_2 $\Rightarrow a_n = \frac{-a_{n-1}}{2n(2n-1)}$ at $r=0$

$$a_1 = \frac{-a_0}{2n(2n-1) = 2} \quad , \quad a_2 = \frac{-a_1}{(4)(3)}$$

$$\rightarrow \quad a_n = \frac{(-1)^n a_0}{(2n)!}$$

$$a_3 = \frac{-a_2}{(6)(5)}$$

$$\therefore y_1 = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$ → not a common function.

$* \; r_2 = \frac{1}{2}$

$* \; b_n = \frac{-b_{n-1}}{2(n+\frac{1}{2})(2n)} = \frac{-b_{n-1}}{(2n)(2n+1)}$

$b_1 = \frac{-b_0}{(4)(5)} \;,\; b_2 = \frac{-b_1}{(4)(5)} = \frac{b_0}{5!} \;,\; b_3 = \frac{-b_2}{(6)(7)} = \frac{-b_0}{7!}$

$b_n = \frac{(-1)^n b_0}{(2n+1)!}$

$\therefore y_2 = b_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+\frac{1}{2}}}{(2n+1)!}$

The general solution :

$y = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} + b_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+\frac{1}{2}}}{(2n+1)!}$

*Ex: Solve $x(x-1)y'' + (3x-1)y' + y = 0$ around $\underset{\sim}{x=0}$.

إذاماحدد' كل انت $(x=0)$.

$\rightarrow \lim_{x\to 0} x \cdot \frac{(3x-1)}{x(x-1)} = 1$

$\rightarrow \lim_{x\to 0} x^2 \cdot \frac{1}{x(x-1)}$

, $x=0$ (r.s.p), then the indicial equation:

$r(r-1) + r = 0$

$r^2 = 0$ (2nd - case) $\boxed{r_1 = 0 = r_2}$

$$** \; y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$* \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + \text{-----}$$

$$+ 3\sum_{n=0}^{\infty} (n+r) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$*(-r(r-1) - r) a_0 x^{r-1} = 0$ and,

$-(n+r)(n+r-1) a_n + \text{----}$

$*(n+r-1)(n+r-2) a_{n-1} - (n+r) a_n + a_{n-1} = 0$

$+ 3(n+r-1) a_{n-1}$

$** \; -r^2 = 0 \rightarrow r = 0 = r_1 = r_2$.

$$a_n = (-1)\left[\frac{(n-1)(n-2) + 3(n-1) + 1}{-n(n-1) - n}\right] a_{n-1} \rightsquigarrow a_n = \frac{n^2}{-n^2} a_{n-1}$$

$$a_n = \frac{n^2 - 3n + 2 + 3n - 3 + 1}{-n^2 + n - n}$$

$$a_n = \underline{\underline{(-1)}} a_{n-1}$$

لازم انقلها ع الطرف الثاني

$* \ a_0 = a_1 = a_2 = ----- = a_n$

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$= a_0 \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

[One solution is given] ← $\underline{\underline{\text{CH.2}}}$ لإيجاد $\underline{\underline{y_2}}$ عن طريق *

$*$ Ex : solve $(x^2 - x)y'' - xy' + y = 0$, around $x = 0$

$$\rightarrow \lim_{x \to 0} x \cdot \frac{(-x)}{x(x-1)} = 0$$

$$\rightarrow \lim_{x \to 0} x^2 \cdot \frac{1}{x(x-1)} = 0$$

$**$ The indicial equation :

$r(r-1) = 0$

$r = 0 , \ r = 1$ (3rd case)

$$\# \ y = \sum_{n=0}^{\infty} a_n x^{n+r_1}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1} + - - - -$$

$$- \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$\rightarrow r(r-1) a_0 x^{r-1} = 0$, and

$(n+r-1)(n+r-2) a_{n-1} - (n+r)(n+r-1) a_n - -----$

$\boxed{r = 0, 1}$

$$\# \; a_n = \left(\frac{(n+r-1)(n+r-2) - (n+r-1)+1}{(n+r)(n+r-1)} \right) a_{n-1}$$

→ $r=1$

$$a_n = \frac{n(n-1) - n+1}{n(n+1)} \cdot a_{n-1}$$

$$a_n = \frac{n^2 - 2n + 1}{n(n+1)} \cdot a_{n-1} = \frac{(n-1)^2}{n(n+1)} a_{n-1}$$

$\therefore a_1 = 0 = a_2 = a_3 = ----$

$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r} = a_0 x^1 + a_1 x^{1+r} + a_2 x^{2+r} -----$

$= a_0 x + 0 + 0 ----$

$\therefore y_1 = x$

$y_2 =$ [one solution is given - CH.2].

→ $r=0$

$$a_n = \frac{(n-1)(n-2) - (n-1)+1}{n(n-1)} \cdot a_{n-1}$$

$$\# \; a_n = \frac{(n-2)^2}{n(n-1)} \text{, when } r-1$$

$n(n-1) a_n = (n-2) a_{n-1}$

$a_0 = 0$, $\underbrace{a_1 = \text{free}}_{\text{the same}}$, $a_2 = 0$, $a_3 = 0$ ----

/ integer[illegible]

CH.6

※ 6.1 : Laplace Transforms:

* Def: If f(t) is defined for $t \geq 0$, then:

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} * f(t) \partial t = F(s)$$

※ Ex: Find $\mathcal{L}(1)$:

$$\mathcal{L}(1) = \int_0^\infty e^{-st} \partial t = \lim_{b\to\infty} \int_0^b e^{-st} \partial t$$ "Improper Integral".

$$\lim_{b\to\infty} \frac{-e^{-st}}{s}\Bigg|_0^b = \lim_{b\to\infty}\left(\frac{-e^{-st}}{s} + \frac{1}{s}\right) , \underbrace{s > 0}$$

يجب وضع شرط حتى يكون الناتج اعتيادي
بدلالة (S) وليس (∞) ولا (0)

$$= \frac{1}{s} \Rightarrow \mathcal{L}(1) = \frac{1}{s}.$$

* * *

※ Rules:

1. $\mathcal{L}(1) = \frac{1}{s}$
2. $\mathcal{L}(t) = \frac{1}{s^2}$
3. $\mathcal{L}(t^n) = \frac{n!}{s^{(n+1)}}$
4. $\mathcal{L}(e^{at}) = \frac{1}{s-a}$
5. $\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$
6. $\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$

* Ex: show that $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} \, .dt = \int_0^\infty e^{(a-s)t} \, dt$$

$$\lim_{b\to\infty} \int_0^b e^{(a-s)t} \, dt = \lim_{b\to\infty} \left(\frac{e^{(a-s)b}}{(a-s)} - \frac{1}{(a-s)} \right) , \quad s > a$$

$$= \frac{1}{s-a}$$

* * *

* $\mathcal{L}(f(t)) = F(s)$

$\mathcal{L}^{-1}(F(s)) = f(t)$

* Properties of La-place:

1. $\mathcal{L}(af(t) + bg(t))$

$= a\,\mathcal{L}(f(t)) + b\,\mathcal{L}(g(t))$

$= aF(s) + bG(s)$

2. $\mathcal{L}^{-1}(aF(s) + bG(s))$

$= a\,\mathcal{L}^{-1}(F(s)) + b\,\mathcal{L}^{-1}(G(s))$

$= af(t) + bg(t)$.

* * *

* Ex: Find $\mathcal{L}(t^5 - 3t^2 + 2)$.

** $\mathcal{L}(t^5) - 3\mathcal{L}(t^2) + 2\mathcal{L}(1)$

$= \frac{5!}{s^6} - \frac{3 * 2!}{s^3} + \frac{3}{s}$ **

* Q: Find $\mathcal{L}(\sinh \omega t)$

$$= \mathcal{L}\left(\frac{e^{\omega t} - e^{-\omega t}}{2}\right) = \frac{1}{2}\left(\mathcal{L}(e^{\omega t}) - \mathcal{L}(e^{-\omega t})\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-\omega} - \frac{1}{s+\omega}\right) = \frac{1}{2}\left(\frac{2\omega}{s^2-\omega^2}\right)$$

$$= \frac{\omega}{s^2-\omega^2}$$

$$\left.\begin{array}{l} * \ \mathcal{L}(\sinh \omega t) = \dfrac{\omega}{s^2-\omega^2} \\ * \ \mathcal{L}(\cosh \omega t) = \dfrac{s}{s^2-\omega^2} \end{array}\right\} \text{* حفظ *}$$

* Ex: $\mathcal{L}(\cos^2(3t))$?!

$$\mathcal{L}\left(\frac{1+\cos 6t}{2}\right) = \frac{1}{2}\left(\mathcal{L}(1) + \mathcal{L}(\cos 6t)\right)$$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+36}\right)$$

* Ex: $\mathcal{L}(\cos(2t+\theta))$.

$$\mathcal{L}(\underbrace{\cos\theta}_{} \cos 2t - \underbrace{\sin\theta}_{} \sin 2t)$$

Constants

$$* \ \sin^2(u) = \frac{1-\cos(2u)}{2} \qquad * \ \cos^2(u) = \frac{1+\cos(2u)}{2}$$

$* \ \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$

$* \ \sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$

$* \ \sin(2u) = 2 \sin u \cos u$

$* \ \cos(2u) = \cos^2 u - \sin^2 u$

※Ex: Find $\mathcal{L}^{-1}\left(\frac{2s+5}{s^2-9}\right)$

$$* \; \mathcal{L}^{-1}\left(\frac{2s}{s^2-9} + \frac{5}{s^2-9}\right) \longrightarrow \frac{5}{3} * \frac{3}{s^2-9}$$

$$= 2\cosh 3t + \frac{5}{3} * \sinh 3t.$$

* * *

** Theorem: (s-shift)

If $\mathcal{L}(F(t)) = F(s)$, then:

$\mathcal{L}(e^{at} * F(t)) = F(s-a)$

※Ex: Find $\mathcal{L}(e^{2t} * \cos(5t))$.

$$* \; \mathcal{L}(\cos(5t)) = \frac{s}{s^2+25} \rightsquigarrow \mathcal{L}(e^{2t}\cos(5t)) = \frac{(s-2)}{(s-2)^2+25}$$

※Ex: Find $\mathcal{L}^{-1}\left(\frac{s+1}{s^2+6s-3}\right)$

$$* \; \frac{(s+1)}{s^2+6s-3} = \frac{(s+1)}{(s+3)^2-12} = \frac{s+3-2}{(s+3)^2-12}$$

$$= \frac{s+3}{(s+3)^2-12} + \frac{-2}{(s+3)^2-12}$$

$$* \; \mathcal{L}^{-1}\left(\frac{(s+1)}{s^2+6s-12}\right) = \mathcal{L}^{-1}\left(\frac{s+3}{(s+3)-12} - \frac{2}{(s+3)^2-12}\right)$$

$$= e^{-3t}\left(\cosh(\sqrt{12}\,t) + \frac{(-2)}{\sqrt{12}}\sinh(\sqrt{12}\,t)\right).$$

** $\mathcal{L}^{-1}(F(s-a)) = e^{at} * F(t)$ **

*Ex:

-1- $\mathcal{L}^{-1}\left(\frac{\pi}{s^2+10\pi s+24\pi}\right)$

-2- $\mathcal{L}^{-1}\left(\frac{10}{2s+\sqrt{2}}\right) = 5e^{(-t/\sqrt{2})}$.

-3- $\mathcal{L}^{-1}\left(\frac{7}{(s-1)^3}\right) = \underbrace{7e^t}_{\text{shift}} * \underbrace{\frac{1}{2}t^2}_{\mathcal{L}^{-1}\left(\frac{1}{s^3}\right)}$

※ Section 6.2:

** Transforms of Derivatives and Integral* *

→ $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$

$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$

$\mathcal{L}(f^n) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) ---- f^{n-2}(0)$

→ $\mathcal{L}\left(\int_0^t f(n)\cdot dn\right) = \frac{1}{s} * F(s)$;

* when: $\mathcal{L}(f(t)) = F(s)$.

$\mathcal{L}^{-1}\left(\frac{1}{s}F(s)\right) = \int_0^t \mathcal{L}^{-1}(F(s))\,dt$

* * *

※ Ex: $\mathcal{L}^{-1}\left(\frac{1}{s^3+25s}\right)$

* Partial Fraction *

$\mathcal{L}\left(\frac{1}{s}\cdot\frac{1}{s^2+25}\right)$

(Integral)

$= \int_0^t \mathcal{L}^{-1}\left(\frac{1}{s^2+25}\right)\cdot dt$

$= \frac{1}{5}\int_0^t \sin(5u)\cdot du = \frac{-1}{25}(\cos 5u)\Big|_0^t$

$= \frac{-1}{25}(\cos 5t - 1)$.

** $\mathcal{L}^{-1}\left(\frac{1}{s^4+25s^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\cdot\frac{1}{s^3+25s}\right)$

(Integral)

$= \int_0^t \frac{-1}{25}(\cos 5u - 1)\cdot du$.

* Ex: Find $y(t)$.

$$(y'' - 4y) = 5e^{3t} \quad , \; y(0) = 1 \quad y'(0) = 7$$

* $\mathcal{L}\, y''(t) - 4\,\mathcal{L}\,(y(t)) = 5\,\mathcal{L}(e^{3t})$

$$(s^2 y(s) - s y(0) - y'(0)) - 4(y(s)) = \frac{5}{s-3}$$

$$Y(s^2-4) - s - 7 = \frac{5}{s-3}$$

$$Y(s) = \frac{5}{s-3} + s + 7 = \frac{5 + (s-3)(s+7)}{(s-3)}$$

$$(s^2-4)\,Y(s) = \frac{s^2 + 4s - 16}{(s-3)}$$

* $Y(s) = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s+2}$.

* Finding the values of (A, B, C).

$[A = 1 \;,\; B = 1 \;,\; C = -1]$

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{1}{s-3} + \frac{1}{s-2} - \frac{1}{s+2}\right)$$

$$Y(t) = e^{3t} + e^{2t} - e^{-2t}.$$

* * *

※ Ex: solve:

$$y'' + y' + y = \int_0^t y(u)\,\partial u \quad , \quad y'(0) = 1 \quad y(0) = 0$$

→ $s^2 y(s) - sy(0) - y'(0) + sY(s) - y(0) - Y(s) = \frac{1}{s} y(s)$

(Integral)

$$Y(s) \left(s^2 + s - 1 - \frac{1}{s}\right) = 1$$

$$Y(s) \left(\frac{s^3 + s^2 - s - 1}{s}\right) = 1$$

$$Y(s) = \frac{s}{s^3 + s^2 - s - 1} = \frac{s}{(s-1)(s^2+2s+1)}$$

$$\begin{array}{r} s^2 + 2s + 1 \\ (s-1)\overline{\big)(s^3 + s^2 - s - 1)} \end{array}$$

→ $\frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$

$$\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2}\right) = 2e^{-t} \cdot t$$

* تحليل أو إكمال مربع *

* * *

Q: solve:

$$y'' + 2y' + 5y = 10 \quad , \quad y'(0) = 0 \quad , \quad y(0) = 1$$

→ $s^2 y(s) - sy(0) - y'(0) + 2sy(s) - 2y(0) + 5y(s) = \frac{10}{s}$

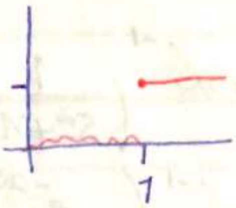
$$y(s) \cdot (s^2 + 2s + 5) = \frac{10}{s} + s + 2.$$

$$y(s) = \frac{(s^2 + 2s + 10)}{s(s^2 + 2s + 5)}$$

** section 6.3:

※ unit step function.

$$u(t-a) = \begin{cases} 0 & , \ t < a \\ 1 & , \ t > a \end{cases} \quad ; u_a(t)$$



$$** \{u(t-a)\} = \int_0^\infty e^{-st} * u(t-a)\, dt$$

$$= \int_a^\infty e^{-st}\, dt = \lim_{b \to \infty} \int_a^b e^{-st}\, dt$$

$$= \lim_{b\to\infty} \left(\frac{-e^{-st}}{s} \right) \Bigg|_a^b$$

$$= \lim_{b\to\infty} \left(\frac{-e^{-sb}}{s} + \frac{e^{-as}}{s} \right), \quad s > 0$$

حتى يطلع Finite

$$= \frac{e^{-as}}{s}$$

$$** \mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s} \ **$$

※ Rules:-

-1- If $\mathcal{L}(f(t)) = F(s)$, then:

$$\mathcal{L}(f(t-a) * u(t-a)) = e^{-as} F(s)$$

-2- $\mathcal{L}^{-1}(e^{-as} F(s)) = f(t-a)\, u(t-a)$

* Ex: Find $\mathcal{L}^{-1}\left(\frac{\boxed{e^{-2s}}}{s^2+\pi^2}\right)$!?

لأننا لانجد الكل:
١. نجعل الـ shift ع الـ Fun.
٢. تضرب بـ $u(t-9)$

* $\mathcal{L}^{-1}\left(\frac{1}{s^2+\pi^2}\right) = \frac{1}{\pi}\cdot \sin \pi t.$

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+\pi^2}\right) = \frac{1}{\pi} \sin \pi (t-2)\, u(t-2)$$

$$= \begin{cases} 0 & , t<2 \\ \frac{1}{\pi}\sin(\pi t - 2\pi) & , t>2 \end{cases}$$

* * *

* Ex: Find $\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2-4s+13}\right)$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-4s+13}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-2)^2+9}\right)$$

$$= \frac{1}{3}e^{2t}\cdot \sin 3t.$$

$$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s^2-4s+13}\right) = \frac{1}{3}e^{2(t-3)}\cdot \sin 3(t-3)\cdot u(t-3).$$

* * *

* Q: let $f(t) = \begin{cases} 2 & , t<1 \\ t & , 1<t<3 \\ \boxed{\sin t \quad , t>3} \end{cases}$;

(suppose) ←

write $f(t)$ using unit step function

$$* f(t) = 2\left(\underline{1} - u(t-1)\right) + t\left(u(t-1) - u(t-3)\right) + \sin t\left(u(t-3) - 0\right).$$

$u(t-0)=1$
نفرض بداية الفترة ($\underline{0}$)

$$* u(t-0) = \begin{cases} 0 & , t<0 \\ 1 & , t>0 \end{cases} \quad ; \quad * u(t-\infty) = \begin{cases} 0 & , t<\infty \\ \cancel{1 \quad , t>\infty} \end{cases}$$

لا يمكن أن تكون $\underline{t}$ أكبر من $\underline{\infty}$

* Ex! Let $f(t) = \begin{cases} 0 & , t < \pi \\ \cos t & , t > \pi \end{cases}$, Find $\mathcal{L}(f(t))$.

$* f(t) = \cos t . u(t-\pi)$.

$\mathcal{L}(f(t)) = \mathcal{L}(\cos t \cdot u(t-\pi))$.

$\hookrightarrow \cos t = \cos(t - \pi + \pi) = -\cos(t-\pi)$.

$= \mathcal{L}(-\cos(t-\pi).u(t-\pi))$.

$$= \frac{-e^{-\pi s}}{s^2+1}$$

* * *

* Ex: Solve $y'' + y = r(t)$, $y(0) = y'(0) = 0$
where $r(t) = t$ if $0<t<1$ and 0 otherwise.

$$* r(t) = \begin{cases} t & , 0<t<1 \\ 0 & , t>1 \end{cases} = t(1-u(t-1)) = t - t * u(t-1)$$

$$r(t) - t - (t-1+1)u(t-1) = t - (t-1)u(t-1) - u(t-1)$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} .$$

$$\rightarrow \mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(r(t))$$

$$s^2 y(s) - sy(0) - y'(0) + sy(s) - y(0) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$$* y(s) = \frac{1}{s^2(s^2+1)} - \frac{e^{-s}}{s^2(s^2+1)} - \frac{e^{-s}}{s(s^2+1)}$$

* 6.4 : Dirac's Delta Function :

$$\delta(t-a) = \begin{cases} \infty & , t=a \\ 0 & , \text{otherwise} \end{cases}$$

* $\mathcal{L}(\delta(t-a)) = e^{-as}$

* * * *

* Ex : solve $y'' + 2y' + 2y = e^{-t} + 5\,\delta(t-2)$, $y(0)=0$ $y'(0)=1$

* $s^2y(s) - sy(0) - y'(0) + 2sy(s) - 2y(0) + 2y(2) = \frac{1}{s+1} + 5e^{-2s}$.

$y(s)\,(s^2+2s+2) = 1 + \frac{1}{s+1} + 5e^{-2s}$

$y(s) = \frac{1}{s^2+2s+2} + \frac{1}{(s+1)(s^2+2s+2)} + \frac{5e^{-2s}}{(s^2+2s+2)}$.

* * * *

* Ex : Solve $y'' + 3y' + 2y = u(t-1) + \delta(t-2)$, $y(0) = y'(0) = 0$

* Ex : $y'' + 3y' + 2y = u(t-1) + u(t-2)$.

* Ex : Solve $y'' + 5y' + 6y = u(t-\pi)\cos t$, $y(0) = y'(0) = 0$

$s^2y(s) + 5\,sy(s) + 6y(s) = \frac{-se^{-\pi s}}{s^2+1}$

$y(s) = \frac{-se^{-\pi s}}{(s^2+1)(s+2)(s+3)}$

$\frac{s}{(s^2+1)(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$

$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)(s+2)(s+3)}\right) = Ae^{-2t} + Be^{-3t} + C\cos t + D\sin t$.

$y(t) = \mathcal{L}^{-1}(y(s)) = -(Ae^{-2(t-\pi)} + Be^{-3(t-\pi)} + C\cos(t-\pi) + D\sin(t-\pi$

$u(t-\pi))$.

* 6.6 : Diff and Integration

- $\mathcal{L}(t f(t)) = -F'(s) = -(\mathcal{L}(f(t)))'$
- $\mathcal{L}^{-1}(F'(s)) = -t f(t) = -t\,\mathcal{L}^{-1}(F(s))$.
- $\mathcal{L}(t^n f(t)) = (-1)^n F(s)^n$

proof * $F(s) = \int_0^\infty e^{-st} f(t) dt$

$$F'(s) = \int_0^\infty -t e^{-st} f(t) dt = -\int_0^\infty e^{-st} \underbrace{(t f(t))}_{g(t)} dt$$

* * *

*** Ex : Find $\mathcal{L}^{-1}\left(\ln\left(1+\frac{9}{s^2}\right)\right)$**

$$F(s) = \ln\left(1+\frac{9}{s^2}\right) = \ln\left(\frac{s^2+9}{s^2}\right) = \ln(s^2+9) - \ln(s^2).$$

$$F'(s) = \frac{2s}{s^2+9} - \frac{2}{s}$$

$$\mathcal{L}^{-1}(F'(s)) = \frac{2}{\cancel{2}}\cos 3t - 2 = -t\,\mathcal{L}^{-1}(F(s))$$

$$\rightarrow f(t) = \frac{2}{t}(1-\cos 3t).$$

* * *

*** Ex : Find $\mathcal{L}^{-1}(\tan^{-1}(s))$**

$$F(s) = \tan^{-1}(s) \;,\; F'(s) = \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}(F'(s)) = \sin t = -t f(t)$$

$$f(t) = \frac{-\sin t}{t}$$

* Ex: Find $\mathcal{L}^{-1}\left(\frac{2s}{(s^2+4)^2}\right)$

$$F'(s) = \frac{+2s}{(s^2+4)^2}$$

$$F(s) = \frac{-1}{s^2+4}$$

$$\mathcal{L}^{-1}\left(\frac{2s}{(s^2+4)^2}\right) = -t\,\mathcal{L}^{-1}\left(\frac{-1}{s^2+4}\right)$$

$$= \frac{1}{2}t.\sin 2t.$$

* ~~~~~~ * ~~~~~~ * ~~~~~~ *

* Ex: Solve $ty'' + (1-t)y' + y = 0$, $y(0) = y'(0) = 0$

$$\mathcal{L}(ty'') + \mathcal{L}(y') - \mathcal{L}(ty') + \mathcal{L}(y) = 0$$

$$-(s^2y(s) - \cancel{sy(0)} - \cancel{y'(0)})' + sy(s) - \cancel{y(0)} + (sy(s) - \cancel{y(0)})' + y(s) = 0$$

Reduced from 2nd order to 1st order

$$-(2sy(s) + s^2y'(s)) + sy(s) + y(s) + sy'(s) + y(s) = 0$$

$$y'(s - s^2) = (s-2)y(s)$$

$$\frac{\partial y}{\partial s} = \left(\frac{(s-2)}{(s-s^2)}\right)y$$

$$\frac{\partial y}{y} = \underbrace{\frac{(s-2)}{s(1-s)}}_{\text{(Partial Fraction)}}\partial s = \frac{-2}{s} - \frac{1}{(1-s)}$$

$$Y(s) = ?!$$

* Ex: Solve (ماضي)

$$y_1' = y_1 + y_2 \quad , y_1(0) = 0 \quad y_2(0) = 0$$

$$y_2' = 2y_1 - 3y_2$$

$$sY_1(s) - Y_1(0) = Y_1(s) + Y_2(s)$$

$$sY_2(s) - Y_2(0) = 2Y_1(s) - 3Y_2(s)$$

$$* \ \mathcal{L}\left(\frac{f(t)}{t}\right) = \int_{s}^{\infty} \mathcal{L}(f(t))\, du$$

$$* \ Ex : \text{Find } \mathcal{L}\left(\frac{\sin(t)}{t}\right).$$

$$\cdot \ \mathcal{L}\left(\frac{\cos(t)}{t}\right) \text{ not exist.}$$

$$= \int_{s}^{\infty} \underbrace{\frac{1}{1+u^2}}_{\mathcal{L}(\sin(t))} du \quad = \lim_{b \to \infty} \left(\tan^{-1} b - \tan^{-1} s\right)$$

$$= \frac{\pi}{2} - \tan^{-1}(s).$$

* * *

$$* \ \text{Find } \int_{s}^{\infty} \frac{\sin(t)}{t}\, dt = \frac{\pi}{2} \quad \text{since} \quad s=0$$

$$* \ \mathcal{L}\left(\frac{\sin(t)}{t}\right)$$

$$* \int_{0}^{\infty} \underset{\hookrightarrow s=0}{\underline{\underline{e^{-st}}}} \frac{\sin(t)}{t}\, dt = F(s)$$

* * *

$$1) \ \mathcal{L}^{-1}\left(\frac{s^2}{(s^2+4)^2}\right) = ?$$

$$2) \ \mathcal{L}^{-1}\left(\frac{1}{(s^2+4)^2}\right) = ?$$

$$(1) * \ \frac{s^2}{(s^2+4)^2} = \frac{1}{2}\left(\underbrace{\frac{s^2-4}{(s^2+4)^2}}_{\downarrow} + \frac{1}{s^2+4}\right)$$

$$-\left(\frac{s}{s^2+4}\right)'$$

$$\hookrightarrow \frac{s^2+4-2s^2}{(s^2+4)^2} = \frac{4-s^2}{(s^2+4)^2}$$

(2) * $\frac{1}{(s^2+4)^2} = \frac{-1}{8}\left(\frac{s^2-4}{(s^2+4)^2} - \frac{1}{(s^2+4)^2}\right)$

* * *

*** Ex: solve** [not Included in final Exam]

$y_1' = -y_1 - y_2 + \boxed{t}$ $\quad y_1(0)=0 \quad , y_2(0)=1$

$y_2' = y_1 - y_2 + \boxed{e^t}$

↳ non-homogeneous $\mathcal{L}(t)$, $\mathcal{L}(e^t)$

* Solution:

$$\left.\begin{aligned} sY_1(s) - y_1(0) &= -Y_1(s) - Y_2(s) \\ sY_2(s) - y_2(0) &= Y_1(s) - Y_2(s) \end{aligned}\right\}$$ حذف أوتعويض

$Y_2(s) = -(s+1)\,Y_1(s)$

$-s(s+1)\,Y_1(s) - 1 = Y_1(s) + (s+1)\,Y_1(s)$

$Y_1(s) = \frac{-1}{s^2+2s+2} = \frac{-1}{(s+1)^2+1}$

$\boxed{** \; y_1(t) = -e^{-t} \cdot \sin t}$

* $Y_2(s) = \frac{s+1}{(s+1)^2+1}$

$\boxed{** \; y_2(t) = e^{-t} \cos t}$