

MAESTRO

DIFFERENTIOL EQUATION

إعداد: أنس أبو زهرة



Anas Abu-Zahra
Anas Abu Zahra
0795476962

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Differential equation "DE"

← هي معادلة تحتوي على متغيرين او اكثر وتحتوي على مشتقة لاحد المتغيرات

→ Examples:

1. $3x^2+y'+2yx=2$
2. $2y''+y'+2y=0$
3. $2x\,dy - 2y\sin x\,dx = 0$
4. $x^2y\frac{d^2z}{dxdy} - \frac{d^2z}{dx^2} + \frac{dz}{dy} = \frac{1}{x^2}$

→ Remark:

1- $y=f(x) \rightarrow \frac{dy}{dx} = f'(x)$ (ordinary derivative تعتبر مشتقة عادية وتسمى)

2- $z=f(x,y) \rightarrow \frac{dz}{dx}, \frac{dz}{dy}$ (Partial derivativ تعتبر مشتقة جزئية وتسمى)

← في مادة المعادلات التفاضلية سنتعامل فقط مع ordinary derivative ليصبح اسم المعادلة "ODE"

→ Classification of DE:

1- Type → ODE (ordinary)
→ PDE (partial) → يوجد مادة كاملة لها

2- order هي اعلى مشتقة موجودة في السؤال

3- linearity

← حتى تسمى المعادلة linear يجب ان تكون قوة y, y', y'' تساوي 1 ويجب الا يحتوي السؤال على $y^2, \sqrt{y}, e^y, \sin y$ ولا يحتوي على عملية ضرب بين y و y'

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→ Examples:

1- $y' + y = x$ linear

2- $y' + y^2 = \sin x$ not linear → بسبب y^2

3- $yy' = \cos x + x^3$ not linear → بسبب عملية الضرب بين yy'

4- $y'x - x^2 + y'y'' = 0$ not linear → بسبب عملية الضرب بين y', y''

4- homogeneous or non-homo

→ case 1 standard form اذا كانت المعادلة على شكل

(standard form for first order) $y' + P(x)y = g(x)$ $\begin{cases} g(x) = 0 \rightarrow (\text{homo}) \\ g(x) \neq 0 \rightarrow (\text{non-homo}) \end{cases}$

→ case 2 standard form اذا لم تكن المعادلة على شكل

← نفوض tx مكان كل x و ty مكان كل y

اذا بقيت المعادلة نفسها تكون hom

اذا لم تعد المعادلة كما كانت تكون non-homo

Examples on case 2

Q1- $xy' = y + xe^{y/x}$

$txy' = ty + tx\, e^{ty/tx}$

$\rightarrow txy' = t(y + x\, e^{y/x}) \Rightarrow xy' = y + xe^{y/x}$ → so homo

Q2- $y' = \frac{x+y}{x-y}$

$y' = \frac{tx+ty}{tx-ty} = \frac{t(x+y)}{t(x-y)} = \frac{x+y}{x-y}$ → so homo

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Q_3 One of the following is homo-DE

a- $y(9x-2y)dx - (6x-y^3)dy = 0$

b- $2\,dy - (2y^2 + \sqrt{x^2+y^2})\,dx = 0$

c- $2x\,dy - (2y + \sqrt{x^2+y^2})\,dx = 0$

d- $2\,dy - (2y + \sqrt{x^2+y^2})\,dx = 0$

e- none

ans is [c] $2tx\,dy - (2ty + \sqrt{4t^2x^2 + t^2y^2})\,dx = 0$

$$\frac{2tx\,dy - (2ty + t\sqrt{x^2+y^2})\,dx = 0}{t}$$

$$2x\,dy - (2y + \sqrt{x^2+y^2})\,dx = 0$$

→ Classify the following ODE

1- $y'' - \sin x\, y' - y = \sin x$

2nd order, linear, non homo

2- $y''' - 2x(y'')^3 = 0$

3rd order, non-linear, homo

3- $\frac{(x-1)y'}{x-1} + \frac{y}{x-1} = \frac{0}{x-1}$

$y' + \left(\frac{1}{x-1}\right)y = 0$

1st order, linear, homo

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4- $y'' + 3y = 2$

2nd order, linear, non-homo

5- $y' + e^x y = x^2$

1st order, linear, non-homo

6- $(x + xy)dx + 2y\,dy = 0$

dx (قسم)

$\frac{x}{2y} + \frac{xy}{2y} + \frac{2yy'}{2y} = 0$

1st order, non-linear, non-homo

7- $y'' - 4y' + 4y = 0$

2nd order, linear, homo

8- $y'' - 4y' + 4 = 0$

2nd order, linear, non-homo

← الهدف دائماً من اسئلة المعادلات التفاضلية هو ايجاد حل للمعادلة ويوجد نوعين من الحلول

→ General solution (G.S) $\int x^3 dx = \frac{x^4}{4} + c$

هذا الحل يسمى G.S لانه يحتوي على c

→ Particular solution

→ if $f'(x) = x^3$, $f(1) = 0$, find $f(x)$

by Integration

$f(x) = \frac{x^4}{4} + c$

we can find c → $f(1) = \frac{1}{4} + c = 0$

$c = -\frac{1}{2}$

← هو الحل الذي يمكننا من خلاله ايجاد قيمة الثابت c من خلال معلومة مساعدة تسمى initial value

← يسمى السؤال الذي يحتوي على معلومة مساعدة بـ initial value Problem (IVP)

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Q_1 $y'' - 4y' + 3y = 0$, which of the following functions below is solution for DE:

a- $y = e^{3x}$

b- $y = \sin x$

c- $y = x^2$

d- $y = 3e^x$

Solution:

test $y = e^{3x}$

↳ $y' = 3e^{3x}$

↳ $y'' = 9e^{3x}$

now substitute in $y'' - 4y' + 3y = 0$

$9e^{3x} - 12e^{3x} + 3e^{3x} \stackrel{?}{=} 0$

yes so $y = e^{3x}$ is solution

Q_2 show that $y = 10 - ce^{-x}$ is solution to $y' + y = 10$

$y = 10 - ce^{-x}$

$y' = ce^{-x}$ → $10 - ce^{-x} + ce^{-x} \stackrel{?}{=} 10$

$10 = 10$ so its solution

Q_3 show that $y = ce^{2x}$ is solution of IVP $y' = 2y$, $y(0) = 1$

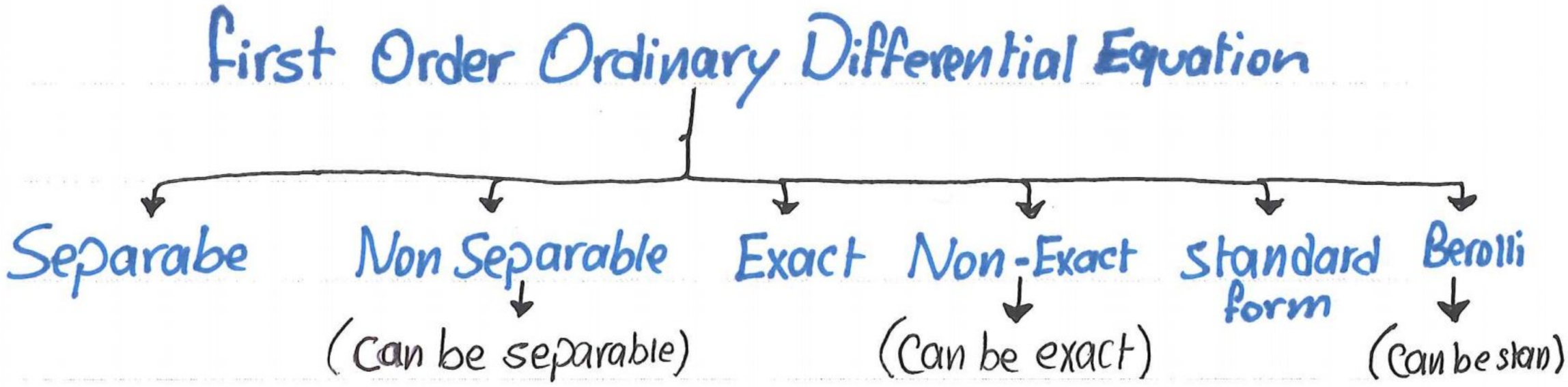
$1 = ce^0$ → so $\boxed{c = 1}$

$y = e^{2x}$

$y' = 2e^{2x}$ → $2e^{2x} \stackrel{?}{=} 2e^{2x}$

yes, so its solution

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1. Separable DE

تستخدم هذه الطريقة عندما نتمكن من فصل الاقترانات التي تكون بدلالة x في جهة والاقترانات التي تكون بدلالة y في جهة اخرى . وعادة ما تصادف الاقترانات التي تكون على شكل $\frac{dy}{dx} = f(x)\,f(y)$

Q_1 Solve the DE $y' = y^2 \cos x$

$$\rightarrow \frac{dy}{dx} = y^2 \cos x$$

$$\rightarrow \frac{dy}{y^2} = \cos x \, dx$$

Note !!

يجب ان يكونوا في البسط دائما dx,dy

$$\rightarrow \int \frac{1}{y^2} dy = \int \cos x \, dx$$

$$\rightarrow \frac{y^{-1}}{-1} = \sin x \qquad \rightarrow y = \frac{-1}{\sin x} + C$$

Q_2 solve $y' = x e^{x+y}$

$$\rightarrow \frac{dy}{dx} = x e^x . e^y$$

$$\rightarrow \int e^{-y} dy = \int x e^x dx \quad \text{(by Parts)}$$

$$\rightarrow \frac{e^{-y}}{-1} = x e^x - e^x + C$$

$$e^{-y} = e^x - x e^x + C$$

$$-y = \ln(e^x - x e^x + C)$$

$$y = -\ln(e^x - x e^x + C)$$

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Q3 Solve $y' - xy^2 = x$

$$\frac{dy}{dx} = x + xy^2$$

$$\frac{dy}{dx} = x(1+y^2)$$

$$\int \frac{1}{1+y^2}\,dy = \int x\,dx$$

$$\tan^{-1} y = \frac{x^2}{2} + c$$

$$y = \tan\left(\frac{x^2}{2} + c\right)$$

Q4 Solve $e^{x+y}dx = e^{x-2y}dy$

$$\cancel{e^x} e^y dx = \cancel{e^x} \cdot e^{-2y} dy$$

$$dx = \frac{e^{-2y}}{e^y}\,dy$$

$$\int dx = \int e^{-3y}\,dy$$

$$x = \frac{e^{-3y}}{-3}$$

$$e^{-3y} = -3x + c$$

$$-3y = \ln(-3x+c) \rightarrow y = \frac{-1}{3}\ln(-3x+c)$$

Q5 Solve $y' - x^2 - y^2 = x^2y^2 + 1$, $y(0) = 2$

$$\frac{dy}{dx} = x^2y^2 + x^2 + y^2 + 1$$

x^2 مشترك عامل $\frac{dy}{dx} = x^2(y^2+1) + (y^2+1)$

(y^2+1) مشترك عامل $\frac{dy}{dx} = (y^2+1)(x^2+1)$

$$\int \frac{dy}{(y^2+1)} = \int (x^2+1)\,dx$$

$$\tan^{-1} y = \frac{x^3}{3} + x + c$$

$$y = \tan\left(\frac{x^3}{3} + x + c\right)$$

$y(0) = 2$

$2 = \tan c$

$c = \tan^{-1} 2$

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Q6 $dy - xydx = (4y+3x+12)dx$

$$dy = xydx + (4y+3x+12)dx$$

$$dy = dx(xy+4y+3x+12)$$

$$dy = dx(y(x+4)+3(x+4))$$

$$dy = dx(x+4)(y+3)$$

$$\int \frac{dy}{(y+3)} = \int (x+4)dx$$

$$\ln(y+3) = \frac{x^2}{2} + 4x + C$$

Q7 $y' = \sqrt{x^2+y^2x^2}$, $y(0) = 1$

$$\frac{dy}{dx} = \sqrt{x^2(1+y^2)}$$

$$\frac{dy}{dx} = x\sqrt{1+y^2}$$

$$dy \cdot \frac{1}{\sqrt{1+y^2}} = x\,dx$$

$$\ln\left(\sqrt{1+y^2}+y\right) = \frac{x^2}{2} + C$$

$y(0)=1 \rightarrow \ln(\sqrt{2}+1) = C$

homework

Q8 solve $y' + xy^2 + x = 0$

Q9 solve $y' = \frac{x\sin x}{y e^{y^2}}$

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2. Non Separabe DE that can be made separable

تستخدم هذه الطريقة للمسائل التي يمكن تحويلها الى Separable عن طريق تعويض مناسب

حيث يوجد نوعان من التعويض

$U = \frac{y}{x}$

$U = ax + by$

Q1 Solve $2xyy' = y^2 - x^2$

$$y' = \frac{y^2 - x^2}{2xy}$$

← نلاحظ انه لا يمكن فصل $x+y$

$$y' = \frac{y^2}{2xy} - \frac{x^2}{2xy}$$

$$y' = \frac{y}{2x} - \frac{x}{2y}$$

← نلاحظ وجود $\frac{y}{x}$ بشكل متكرر اذا

يمكن تحويل هذه المسألة الى sep.

assume $U = \frac{y}{x}$

$y = xU$

$y' = U + xU'$

$$\rightarrow U + xU' = \frac{U}{2} - \frac{1}{2U}$$

$$xU' = \frac{U}{2} - \frac{1}{2U} - U$$

$$xU' = -\frac{U}{2} - \frac{1}{2U} \quad \text{(توحيد المقامات)}$$

$$xU' = -\frac{U^2 - 1}{2U} = -\frac{(U^2+1)}{2U}$$

$$x\frac{dU}{dx} = -\frac{(U^2+1)}{2U}$$

$$\int \frac{2U}{U^2+1}\,dU = \int \frac{-1}{x}\,dx$$

$$\ln(U^2+1) = -\ln x + C$$

$$U^2 + 1 = e^{(\ln(\frac{1}{x}) + C)} = \frac{1}{x} + e^C$$

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$Q_2: xy' = y + xe^{y/x}$

$$y' = \frac{y}{x} + e^{y/x}$$

$\rightarrow v = \frac{y}{x}$

$y = vx$

$y' = v + xv'$

$\rightarrow xv' + v = v + e^v$

$x\frac{dv}{dx} = e^v$

$e^{-v}dv = \frac{1}{x}dx$

$-e^{-v} = \ln x + c$

$e^{-v} = -\ln x + c$

$e^{-y/x} = -\ln x + c$

$Q_3 \quad y' = \frac{x+y}{x-y}$

نقسم كل المعادلة على x

$y' = \frac{1+y/x}{1-y/x}$

$v = \frac{y}{x}$

$y' = v + xv'$

$\rightarrow xv' + v = \frac{1+v}{1-v}$

$xv' = \frac{1+v}{1-v} - v \cdot \frac{(1-v)}{(1-v)}$ ← توحيد المقامات

$xv' = \frac{1+v-v+v^2}{1-v}$

$x\frac{dv}{dx} = \frac{1+v^2}{1-v}$

$\int \frac{1-v}{1+v^2}dv = \int \frac{1}{x}dx$

$\rightarrow \int \frac{1}{1+v^2} - \frac{1}{2}\int \frac{2v}{1+v^2} = \int \frac{1}{x}dx$

$\tan^{-1}v - \frac{1}{2}\ln(1+v^2) = \ln x$

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homework

Q_4 solve $(x^3+y^2)\,dx + 2xy\,dy = 0$

Q_5 solve $xy' = y + 2x^3 \sin^2\frac{y}{x}$

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Q6 $y' = (x+y)^2$

assume $v = (x+y)$ → $v' - 1 = v^2$

$y = v - x$ $\quad v' = v^2 + 1$

$y' = v' - 1$ $\quad \frac{dv}{dx} = v^2 + 1$

$$\int \frac{1}{v^2+1}\, dv = \int dx$$

$$\tan v = x + c$$

$$v = \tan(x+c)$$

$$x + y = \tan(x+c)$$

$$y = \tan(x+c) + -x$$

homework

Q7 Solve $y' = (x+3y)^2$

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3. Exact 1st order ODE

$$dU = \frac{dU}{dx}dx + \frac{dU}{dy}dy$$

$$\rightarrow dU = U_x dx + U_y dy$$

Exact form $\overset{\rightarrow U_x}{M}dx + \overset{\rightarrow U_y}{N}dy = 0$

Example $(x^3y + \overset{\rightarrow M}{y^2})dx + (x^2y + \overset{\rightarrow N}{1})dy = 0$

Rule $M_y = N_x$ (exact)

$M_y \neq N_x$ (not exact but can be exact by factor)

Steps of Solution:

step1 check if exact or not (if not make it exact)

step2 from $M = U_x$ or $N = U_y$ find U by Integration

step3 Derivate U to find constant

Note if $\int dU = \int y x^3 dx$

then $U = \frac{y x^4}{4} + C$ → constant / f(y)

$$U = \frac{y x^4}{4} + f(y)$$

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Q1 solve $(x\sin y + y^2)dx + (\frac{1}{2}x^2\cos y + 2xy)dy = 0$

M N

step1 $My = x\cos y + 2y$

$Nx = x\cos y + 2y$

$> My = Nx$ so its exact

step2 $M = x\sin y + y^2$

$$\frac{du}{dx} = x\sin y + y^2$$

$$\int du = \int (x\sin y + y^2)dx$$

$$U = \frac{x^2}{2}\sin y + y^2 x + f(y)$$

step3 to find $f(y)$ $\frac{du}{dy} = \frac{x^2}{2}\cos y + 2xy + f'(y)$ ($\frac{du}{dy}$ → N)

$$\frac{1}{2}x^2\cos y + 2xy = \frac{x^2}{2}\cos y + 2xy + f'(y)$$

$$f'(y) = 0$$

$$f(y) = \text{constant}(k)$$

G.S $U = \frac{x^2}{2}\sin y + y^2 x + K = c$

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Q12 solve $(1-\sin x\tan y)\,dx + (\cos x\sec^2 y)\,dy = 0$

M N

step 1 $M_y = -\sin x\sec^2 y$, $N_x = -\sin x\sec^2 y$ $\Rightarrow M_y = N_x$ so exact

step 2 $M = 1-\sin x\tan y$

$$\frac{du}{dx} = 1-\sin x\tan y$$

$$\int du = \int (1-\sin x\tan y)\,dx$$

$$U = x + \cos x\tan y + f(y)$$

step 3 N

$$\frac{du}{dy} = 0 + \cos x\sec^2 y + f'(y)$$

$$\cos x\sec^2 y = 0 + \cos x\sec^2 y + f'(y)$$

$$f'(y) = 0$$

$$f(y) = k$$

$$U = x + \cos x\tan y + k = c$$

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Q_3 solve $(y e^{xy} - 2y^3)dx + (x e^{xy} - 6xy^2 - 2y)dy = 0$

M N

step1 $M_y = y e^{xy} x + e^{xy} - 6y^2$

$N_x = x e^{xy} \cdot y + e^{xy} - 6y^2$

$\Rightarrow M_y = N_x$ so exact

step2 $M = y e^{xy} - 2y^3$

$\int dU = \int (y e^{xy} - 2y^3)\,dx$

$U = \frac{y e^{xy}}{y} - 2y^3 x + f(y)$

step3 $\frac{dU}{dy} = e^{xy} \cdot x - 6y^2 x + f'(y)$

$x e^{xy} - 6xy^2 - 2y = e^{xy} x - 6y^2 x + f'(y)$

$f'(y) = -2y \rightarrow f(y) = -y^2$

$U = e^{xy} - 2y^3 x + (-y^2) = c$

Q_4 solve $y' = \frac{\sin y + y \sin x}{\cos x - x \cos y}$, ans $U = -y\cos x + x \sin y = c$

Q_5 solve $y' = \frac{3x^2 y - y^3}{3xy^2 - x^3}$, ans $U = xy^3 - x^3 y + k = c$

Q_6 solve $(2x e^{x^2} y)dx + (e^{x^2})dy = 0$, ans $y = c e^{-x^2}$

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Q7 Solve the IVP

$$2e^{2x}\cos y\,dx - e^{2x}\sin y\,dy = 0, \quad y(0)=0$$

Method 1

$$2e^{2x}\cos y\,dx = e^{2x}\sin y\,dy$$

$$\int 2\,dx = \int \tan y\,dy$$

$$2x = -\ln|\cos y| + c$$

$$0 = 0 + c$$

$$c = 0$$

$$-e^{2x} = \ln\cos y$$

$$y = \cos e^{-2x}$$

Method 2

$$My = -2e^{2x}\sin y$$
$$Nx = -2e^{2x}\sin y \quad > \text{exact}$$

Note: $2e^{2x}\cos y\,dx - e^{2x}\sin y\,dy = 0 \rightarrow$ exact

but $2\cos y\,dx - \sin y\,dy = 0 \rightarrow$ not exact

نلاحظ ان نفس الاقتران عند وجود e^{2x} يصبح exact وعند عدم وجودها يصبح not exact

وفي هذه الحالة يسمى الاقتران e^{2x} ← Integration factor والذي سنتعلم طريقة ايجاده في الصفحة التالية

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4- Non-exact DE that can be made exact

← اذا كانت $My \neq N_x$ اي كانت المعادلة not exact فيلمكانا تحويلها

الى exact عن طريق ضربها بـ Integration factor

Integration factor (If) ←

- $e^{\int \frac{My - N_x}{N} dx}$ ← يجب ان يكون الناتج بدلالة x فقط
- $e^{\int \frac{N_x - My}{M} dy}$ ← يجب ان يكون الناتج بدلالة y فقط

* نستخدم القانون الذي يحقق الشرط

Q1 $(\overset{M}{y})dx + (2x - \overset{N}{ye^y})dy = 0$

step1 $My = 1$, $N_x = 2$ > $My \neq N_x$ → not exact → we need (IF)

step2 (If) بـ $R = \frac{My - N_x}{N} = \frac{1-2}{2x - ye^y} = \frac{-1}{2x - ye^y}$ (x, y يحتوي على)

$R = \frac{N_x - My}{M} = \frac{2-1}{y} = \frac{1}{y}$

$IF = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$

step3 $y^2 dx + (2xy - y^2 e^y) dy = 0$

IF بضرب

$My = 2y$, $N_x = 2y$ > $My = N_x$ so it's exact now

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Q2 $(3x^2y+2xy+y^3)dx+(x^2+y^2)dy=0$

step1 $My = 3x^2+2x+3y^2$

$Nx = 2x$ > not exact

step2 $R = \frac{My-Nx}{N} = \frac{3x^2+2x+3y^2-2x}{x^2+y^2} = 3$

$IF = e^{\int 3dx} = e^{3x}$

step3 $(3x^2ye^{3x}+2xye^{3x}+y^3e^{3x})dx + (x^2e^{3x}+y^2e^{3x})dy = 0$

(M → $3x^2ye^{3x}+2xye^{3x}+y^3e^{3x}$, N → $x^2e^{3x}+y^2e^{3x}$)

$My = Nx$ → so exact

step4 $N = x^2e^{3x}+y^2e^{3x}$

$\int dv = \int (x^2e^{3x}+y^2e^{3x})dy$

$U = x^2e^{3x}y + \frac{y^3}{3}e^{3x} + f(x)$

step5 $\frac{dv}{dx} =$

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Q_3 $y' = e^{2x} + y - 1$

$$\frac{dy}{dx} = e^{2x} + y - 1$$

$$dy = (e^{2x} + y - 1)\,dx$$

$$\overset{\rightarrow M}{-(e^{2x} + y - 1)}\,dx + \overset{\rightarrow N}{(1)}\,dy = 0$$

Q_4 $(e^x)dx + (e^x \cot y + 2y \csc y)\,dy = 0$, solve

Q_5 $\frac{1}{x}\cosh y\,dx + \sinh y\,dy = 0$, find IF

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5- standard form for first order "linear" ODE

$$y' + P(x)y = g(x)$$

the General solution for standard form is

$$y = \frac{1}{M_x}\left[\int M_x\, g(x)\, dx + c\right]$$

where M_x is Integrating factor $= e^{\int P(x)dx}$

Q1 solve $xy' + 2y = \sin x$

$$y' + \underbrace{\frac{2}{x}}_{P(x)}y = \underbrace{\frac{\sin x}{x}}_{G(x)}$$

step 1 $M_x = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

step 2 $y = \frac{1}{x^2}\int x^2 \frac{\sin x}{x}\, dx + c$

$$y = \frac{1}{x^2}(-x\cos x + \sin x + c)$$

Q2 solve $\cos x\, y' + \sin x\, y = \cos x \sin 2x$, $0 < x < \frac{\pi}{2}$

$$y' + \tan x\, y = \sin 2x$$

step 1 $M_x = e^{\int \tan x\, dx} = e^{\ln|\sec x|} = \sec x$

step 2 $y = \frac{1}{\sec x}\int \frac{1}{\cos x} \cdot 2\sin x \cos x + c$

$$y = \cos x\,(-2\cos x + c)$$

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Q_3 $x y' + 4y = 8x^4$, $y(1) = 2$

Q_4 $y' + \sin x \cdot y = e^{\cos x}$

Q_5 $(1+x^2)\,dy + 4xy\,dx = (1+x^2)^{-2}\,dx$

$(1+x^2)\,y' + 4xy = (1+x^2)^{-2}$

$y' + \underbrace{\frac{4x}{1+x^2}}_{P} y = \underbrace{(1+x^2)^{-3}}_{G}$

Q_6 $y' = \frac{1}{6e^y - 2x}$

$\frac{dy}{dx} = \frac{1}{6e^y - 2x}$

$\frac{dx}{dy} = 6e^y - 2x \rightarrow x' = 6e^y - 2x$

$x' + \underbrace{2}_{P} x = \underbrace{6e^y}_{g}$

step1 $M_y = e^{\int 2\,dy} = e^{2y}$

step2 $y = \frac{1}{e^{2y}} \left(\int e^{2y} \cdot 6e^y + c \right)$

$= \frac{1}{e^{2y}} \left(2e^{3y} + c \right)$

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6- Non-linear 1st order that can be made linear

Bernoulli DE

$$y' + P(x)\,y = g(x)\,y^n \quad , \quad n \neq 0, 1$$

if 0 then its linear

if 1 then its separable

→ steps of solutions:

step 1 let $U = y^{1-n}$

step 2 $U' = \underline{(1-n)\,y^{-n}\,y'}$

step 3 وسوف ← y' اضرب المعادلة بعامل

تصبح المعادلة linear

Q1 solve $y' + y = y^3$

step 1 $U = y^{1-3} = y^{-2}$

step 2 $U' = -2\,y^{-3}\,y'$

step 3 $-2\,y^{-3}\,y' + -2(y^{-3}\,y) = -2\,y^{-3}\,y^3$ (y^{-2})

$$U' - 2U = -2$$

step 4 $M_x = e^{\int -2\,dx} = e^{-2x}$

step 5 $U = \frac{1}{e^{-2x}} \int e^{-2x} \cdot (-2)\,dx$

$$U = e^{2x}\left(e^{-2x} + C\right)$$

$$y^{-2} = e^{2x}\left(e^{-2x} + C\right)$$

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Q_2 $y' + xy = x^3 y^{-1}$

Q_3 $y' + y = xy^2$

Q_4 $y' = \frac{1}{6e^y x^3 - 2x}$

Q_5 $2xy\,y' + (x-1)y^2 = x^2 e^x$

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Example : solve the following ODE's :

1] $y' = 3y$

Sol :

(Seperable) $\frac{dy}{dx} = 3y \rightarrow \int \frac{dy}{3y} = \int dx$

Intgrate $\rightarrow \frac{1}{3} \ln|y| = x + C$

$y = e^{3(x+C)}$

So the general solution is :

$$\boxed{y = e^{3(x+C)}}$$

2] $y' = 1 + y^2$

Sol :

(Seperable) $\frac{dy}{dx} = 1 + y^2 \rightarrow \int \frac{dy}{1+y^2} = \int dx$

Integrate $\rightarrow \tan^{-1}(y) = x + C$

$y = \tan(x+C)$

So the general solution is :

$$\boxed{y = \tan(x+C)}$$

3] $y' = xy^2 + x$

Sol :

(Seperable) $\frac{dy}{dx} = x(y^2+1) \rightarrow \int \frac{dy}{y^2+1} = \int x\,dx$

Integrate $\rightarrow \tan^{-1}(y) = \frac{x^2}{2} + C$

So the general solution is :

$$\boxed{y = \tan\left(\frac{x^2}{2} + C\right)}$$

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4] $yy' = e^{x+y}$

Sol:

(seprable) $\frac{dy}{dx} = e^x \frac{e^y}{y} \rightarrow \int \frac{y}{e^y} dy = \int e^x dx$

$$\underbrace{\int e^{-y} y\, dy}_{\text{by parts}} = \int e^x dy$$

$$\boxed{-y e^{-y} - e^{-y} = e^x + C} \rightarrow \text{Implicit solution}$$

$u = y$ $\quad dv = e^{-y}$ (×)

$du = dy$ (−∫) $\quad v = -e^{-y}$

$$\int e^{-y} y\, dy = -y e^{-y} + \int e^{-y} dy$$

$$= -y e^{-y} - e^{-y}$$

5] $y' = \frac{x^2 + 3y^2}{2xy}$

Sol: $/x^2$

(Non seprable)

$$y' = \frac{1 + 3(\frac{y}{x})^2}{2(\frac{y}{x})}$$

$\boxed{u = \frac{y}{x}, \quad y' = u + xu'}$, Then $u = xu' = \frac{1+3u^2}{2u}$

$$xu' = \frac{1+3u^2}{2u} - \frac{u}{u} + u$$

$$x\frac{du}{dx} = \frac{1+u^2}{2u}$$

$$\int \frac{2u}{1+u^2} du = \int \frac{dx}{x}$$

$$\ln(1+u^2) = Ln(x) + c_1$$

$$\ln(1+u^2) = Ln(x) + Ln c_1$$

$$\ln(1+u^2) = Ln(c_1 x)$$

$$1+u^2 = c_1 x$$

$$1 + \frac{y^2}{x^2} + ax$$

$\rightarrow \boxed{x^2 + y^2 = ax^3}$

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6] $(2x+y)y' = -4x-3y$

Sol:

(Non Seprable) $\quad y' = \dfrac{-4x-3y}{2x+y}$

$[/x] \rightarrow y' = \dfrac{-4-3\frac{y}{x}}{2+\frac{y}{x}}$

$u = \dfrac{y}{x}\ ,\ y' = u'x+u$

$u'x + y - u = \dfrac{-4-3u}{2+u} - u$

$\dfrac{du}{dx}\ u'x = \dfrac{-4-3u-2u-u^2}{2+u}$

$\dfrac{-(2+u)}{+4+5u+u^2}\,du = \dfrac{dx}{x}$

$+4+5u+u^2 \rightarrow (u+4)(u+1)$

$\dfrac{2+u}{u^2+5u+4} = \dfrac{A}{u+4} + \dfrac{B}{u+1}$

$2+u = A(u+1) + B(u+4)$

if $u=-1 \rightarrow B = \frac{1}{3}$

if $u=-4 \rightarrow A = \frac{2}{3}$

$+\int \dfrac{\frac{2}{3}}{u+4} + \dfrac{\frac{1}{3}}{u+1}\,dy = -\int \dfrac{dx}{x}$

$\frac{2}{3}\ln(u+4) + \frac{1}{3}\ln(u+1) = -\ln(x) + \ln|C|$

$\ln|(u+4)^{\frac{2}{3}} \cdot (u+1)^{\frac{1}{3}}| = \ln\left(\frac{C}{x}\right)$

e

$(u+4)^{\frac{2}{3}} \cdot (u+1)^{\frac{1}{3}} = \dfrac{C}{x}$

$\boxed{\left(\frac{y}{x}+4\right)^{\frac{2}{3}} \cdot \left(\frac{y}{x}+1\right)^{\frac{1}{3}} = \frac{C}{x}}$

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7] $(x^2 e^{2\frac{y}{x}} + xy)\,dx = x^2\,dy$

Sol:

$$\frac{dy}{dx} = e^{2\frac{y}{x}} + \frac{y}{x}$$

(Non seprable) $u = \frac{y}{x}$, $u'x + u = y'$

$$u'x + u = e^{2u} + u$$

$$x\frac{du}{dx} = e^{2u} \longrightarrow \int \frac{du}{e^{2u}} = \int \frac{dx}{x}$$

$$-\frac{1}{2} e^{-2u} = \ln|x| + C$$

$$\boxed{-\frac{1}{2} e^{-2\frac{y}{x}} = \ln|x| + C}$$

8] $\underbrace{\cos(x+y)}_{M}\,dx + \underbrace{(3y^2 + 2y + \cos(x+y))}_{N}\,dy = 0$

Sol

(exact)

$My = -\sin(x+y)$

$Nx = -\sin(x+y)$

$\boxed{My = Nx}$ → exact

$Ux = M = \cos(x+y)$

$U = \sin(x+y) + F(y)$

$Uy = \cos(x+y) + F'(y)$

$3y + 2y + \cos(x+y) = \cos(x+y) + F'(y)$

$F'(y) = 3y^2 + 2y$

$\boxed{F(y) = y^3 + y^2 + C}$

G.S → $\sin(x+y) + y^3 + y^2 = C$

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9] $\underbrace{(e^x \sin y - 2y \sin x)}_{M} dx + \underbrace{(e^x \cos y + 2\cos x)}_{N} dy = 0$

Sol: (exact)

$My = e^x \cos y - 2\sin x$

$Nx = e^x \cos y - 2\sin x$

$\boxed{My = Nx} \rightarrow$ exact

$Ux - M = e^x \sin y - 2y \sin x$, $Uy = N = e^x \cos y + 2\cos x$

$\rightarrow U = e^x \sin y + 2y \cos x + F(y)$

$\rightarrow Uy = e^x \cos y + 2\cos x + F'(y)$

$e^x \cos y + 2\cos x + F'(y) = e^x \cos y + 2\cos x$

$F'(y) = 0$

$\boxed{F(y) = C}$

G.S $\rightarrow$ $e^x \sin y + 2y \cos x = C$

10] $\underbrace{(3x^2y + 2xy + y^3)dx}_{M} + \underbrace{(x^2 + y^2)dy}_{N} = 0$

$My = 3x^2 + 2x + 3y^2$, $Nx = 2x$

$\boxed{My \neq Nx} \rightarrow$ Non exact

$h(x) = \frac{My - Nx}{N} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2}$

$\rightarrow \boxed{h(x) = 3}$

$F(x) = e^{\int h(x) dx} = e^{\int 3 dx} = e^{3x}$

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$$\rightarrow e^{3x}\underbrace{[3x^2y+2xy+y^3]}_{M_1}dx+e^{3x}\underbrace{[x^2+y^2]}_{N_1}dy=C$$

$M_{1y}=N_{1x}\rightarrow$ exact

$u_x=M_1=(3x^2y+2xy+y^3)e^{3x}$, $u_y=N_1=e^{3x}(x^2+y^2)$

$\rightarrow u=e^{3x}(x^2y+\frac{y^3}{3})+F(x)$

$\rightarrow u_x=??$

$u_x=e^{3x}(2xy)+3(x^2y+\frac{y^3}{3})e^{3x}+F'(x)$

$=3e^{3x}x^2y+2xye^{3x}+y^3e^{3x}$

$e^{3x}(2xy)+3x^2ye^{3x}+y^3e^{3x}+F'(x)=3e^{3x}x^2y+2xye^{3x}+y^3e^{3x}$

$F'(x)=0\rightarrow F(x)=C$

G.S $\rightarrow$ $\boxed{e^{3x}(x^2y+\frac{y^3}{3})=C}$

II $(y+1)dx+(x-1)dy=0$

(seperable) $\frac{dy}{y+1}=\frac{dx}{x-1}\rightarrow +\ln|y+1|=-\ln|x-1|+C$

$y+1=e^{C-\ln|x-1|}$

$y+1=e^C\cdot\frac{1}{e^{\ln|x-1|}}$

$y+1-1=\frac{C^x}{x-1}-1$

G.Sis $\rightarrow$ $\boxed{y=\frac{C^x}{x-1}-1}$

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12 $y' + xy^2 = x$

$y' + xy = \frac{x}{y}$

$y' + xy = xy^{-1}$ → Bernouli Equ (Non Homol.

$y' + P(x)y = Q(x)y^n$

$z = y^{1-n} = y^{1-(-1)} = y^2$

→ $z' = 2yy'$ → $y' + xy = xy^{-1}$

$2yy' + 2xy^2 = 2x$

$z' + \frac{2x}{P(x)} z = \frac{2x}{Q(x)}$

$F(x) = e^{\int Q(x)dx} = e^{\int 2x\,dx} = e^{x^2}$

$z = \frac{\int F(x)P(x)dx + C}{e^{x^2}}$

$= \frac{\int e^{x^2} \cdot 2x\,dx + C}{ex^2}$

$z = 1 + Ce^{-x^2}$

$\boxed{y^2 = 1 + ce^{-2x}}$

13 $(1+x^2)y' + 4xy = \frac{1}{(1+x^2)^2}$

Sol:

$y' + \frac{4x}{1+x^2} y = \frac{1}{(1+x^2)^2}$ → Bernouli Equ (Homo)

$F(x) = e^{\int p(x)\,dx} = e^{\int \frac{4}{1+x^2} dx} = e^{2\ln(1+x^2)} = (1+x^2)^2$

$y = \frac{\int (1+x^2)^2 \cdot \frac{1}{(1+x^2)^2} dx + C}{(1+x^2)^2} = \boxed{\frac{\tan^{-1}(x) + C}{(1+x^2)^2}}$

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14] $y' - \frac{2}{x} y = \frac{x^2 e^x}{Q(x)}$

$\frac{1}{P(x)}$

Sol: $F(x) = e^{\int P(x)dx} = e^{\int \frac{-2}{x} dx} = e^{-2\ln(x)} = e^{\ln(x^{-2})}$

$= x^{-2}$

$$y = \frac{\int F(x)\, Q(x)\, dx + C}{F(x)}$$

$$= \frac{\int x^{-2} \cdot x^2 e^x dx + C}{x^{-2}} = \frac{e^x + C}{x^{-2}}$$

$$\boxed{y = \frac{e^x + C}{x^{-2}}}$$

15] $(1+y^2)dx - (xy + y + y^3)dy = 0$

Sol: $x' - \frac{xy + y + y^3}{1+y^2} = 0$

$$x' - \frac{y}{1+y^2} x = \frac{y+y^3}{1+y^2} = \frac{y(1+y^2)}{1+y^2}$$

$\rightarrow x' \underbrace{\boxed{-\frac{y}{1+y^2}}}_{P(y)} x = \boxed{y}\, Q(y)$

$F(y) = e^{\int P(y)dy}$

$= e^{\int \frac{-y}{1+y^2} dy} = e^{-\frac{1}{2}\ln|1+y^2|} = \frac{1}{\sqrt{1+y^2}}$

$$x = \frac{\int F(y)\, Q(y)\, dy + C}{F(y)} = \frac{\int \frac{1}{\sqrt{1+y^2}} \cdot y\, dy + C}{\frac{1}{\sqrt{1+y^2}}}$$

$$\boxed{x = \sqrt{1+y^2}\ \left[\sqrt{y^2+1} + C\right]}$$

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16 $(2x - y^2)\, y' = 2y$

$(2x - y^2)\frac{dy}{dx} = 2x$

$2y\,dx = (2x - y^2)\,dy$

$x' = \frac{2x - y^2}{2y}$

$x' - \underbrace{\frac{1}{y}}_{P(y)} x = \frac{-y}{2}\ Q(y)$

$F(y) = e^{\int P(y)\,dy} = e^{\int -\frac{1}{y}\,dy} = e^{-\ln|y|} = \frac{1}{y}$

$x = \frac{\int F(y)\,Q(y)\,dy + c}{F(y)}$

$= \left(\int \frac{1}{y} \cdot \frac{-y}{2}\,dy + c\right) / \frac{1}{y} = y\left[-\frac{1}{2}y + c\right]$

$$\boxed{x = \frac{-y^2}{2} + cy}$$

17 $x y^{-3} y' + y^{-2} = x$

Sol $y' + \frac{y}{x} = y^3$ .... Bern Equ

$u = y^{1-n} = y^{1-3} = y^{-2}$

$u' = -2y^{-3}y'$

Sol $-2y^{-3}y' - \frac{2}{x}y^{-2} = -2$

$u' - \frac{2}{x}u = -2 \rightarrow$ linear

$F(x) = e^{\int \frac{-2}{x}dx} = \frac{1}{x^2}$

$u = \frac{\int F(x)\,Q(x)\,dx + c}{F(x)}$

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$$u = \int \frac{\frac{1}{x^2} - 2\,dx + c}{\frac{1}{x^2}}$$

$$u = x^2 [2x^{-1} + c]$$

$$u = 2x + cx^2$$

$$y^{-2} = 2x + cx^2$$

$$y^2 = 1/(2x + cx^2)$$

$$\boxed{y = \sqrt{\frac{1}{2x + cx^2}}}$$

18 $(2xy + 3y^2)\,dx - (2xy + x^2)\,dy = 0$

(NoN seperable) $u = \frac{y}{x}$, $y' = u'x = u$

$$(1/x^2) \quad \left(2\frac{y}{x} + 3\left(\frac{y}{x}\right)^2\right)dx - \left(2\frac{y}{x} + 1\right)dy = c$$

$$\frac{\left(\frac{2y}{x} + 3\left(\frac{y}{x}\right)^2\right)}{\frac{2y}{x} + 1} = y'$$

$$\frac{2u + 3u^2}{2u + 1} - u = u'x + u - u$$

$$u'x = \frac{du}{dx}x = \frac{2u + 3u^2 - 2u^2 - u}{2u + 1} = \frac{u + u^2}{2u + 1}$$

$$\int \frac{2u+1}{u+u^2}\,du = \int \frac{dx}{x} \longrightarrow \ln|u^2+u| = \ln|x| + \ln|c|$$

$$e^{\ln|u^2+u|} = e^{\ln(cx)}$$

$$\longrightarrow cx = u^2 + u \longrightarrow cx = \frac{y^2}{x^2} + \frac{y}{x}$$

$$\boxed{cx^3 = y^2 + yx}$$

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19 $(\underbrace{y^2 + 3yx^2}_{M})dx + (\underbrace{x^3 + 2xy}_{N})dy = 0$

Sol:

$My = 2y + 3x^2$
$Nx = 3x^2 + 2y$
$\Big\}$ $My = Nx$ → exact

$Ux = M = y^2 + 3yx^2$, $Uy = N = x^3 + Nxy$

$U = y^2x + yx^3 + F(y)$

$Uy = 2yx + x^3 + F'(y) = x^3 + 2xy$

$F'(y) = 0$ → $F(y) = C$

→ So the G.S is :-

$y^2x + yx^3 = C$

20 $(x + x^2\cos t)dt + (2t + 3x\sin t)dx = 0$

$Mx = 1 + 2x\cos t$
$Nt = 2 + 3x\cos t$
$\Big\}$ $Mx \neq Nt$ → Non exact.

$\frac{Nt - Mx}{M} = h(x) = \frac{2 + 3x\cos t - 1 - 2x\cos t}{x + x^2\cos t}$

$= \frac{1 + x\cos t}{x(1 + x\cos t)} = \frac{1}{x} = h(x)$

$F(x) = e^{\int h(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$

$(\underbrace{x^2 + x^3\cos t}_{M})dt - (\underbrace{2tx + 3x^2\sin t}_{N})dx = c$

$Mx = Nx$ → exact.

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$U_t = M = x^2 + x^3 \cos t$, $U_x = 2tx + 3x^2 \sin t$

$u = x^2 t + x^3 \sin t + F(x)$

$u_x = 2xt + 3x^2 \sin t + F'(x) = 2tx + 3x^2 \sin t$

$F'(x) = 0 \longrightarrow \boxed{F(x) = C}$

So the G.S is :-

$$\boxed{x^2 t + x^3 \sin t = C}$$

21 $xy' - 2y = x^3 e^x$

Sol:

$$y' \underbrace{-\frac{2}{x}}_{P(x)} y = \underbrace{x^2 e^x}_{Q(x)}$$

$$u(x) = e^{\int P(x)\,dx} = e^{\int \frac{-2}{x} dx} = e^{-2\ln x} = x^{-2}$$

$$y = \frac{\int x^{-2} \cdot x^2 e^x \, dx + C}{x^{-2}}$$

$$= \frac{e^x + C}{x^{-2}} = x^2 (e^x + C)$$

So the G.S is :

$$\boxed{y = x^2 (e^x + C)}$$

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QUESTIONS



Calculus first Exam

sum 2019/2020

By Anas Abu Zahra

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Q_1 The solution of $y' = \sqrt{3x+y} - 3$

a- $2\sqrt{3x+y} = x + c$

b- $3\sqrt{3x+y} = x + c$

c- $2\sqrt{3x+y} = c$

d- $3x+y = c$

e- none

solution $U = 3x+y$

$y = U - 3x \rightarrow U' - 3 = \sqrt{U} - 3$

$y' = U' - 3$

$\frac{dU}{dx} = \sqrt{U}$

$dU \cdot \frac{1}{\sqrt{U}} = dx$

$2U^{1/2} = x + c$

$2\sqrt{3x+y} = x + c$ [a]

Anas Abu Zahra

best wishes ☺

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Q_2 if $y''y'-y=x$ is linear, then the set of all Possible valves of n is

a- $\{1\}$ → $y'y-y=x$ (non-linear) بسبب الضرب

(b-) $\{0\}$ → $y'-y=x$ ✓

c- $\{0,1\}$

d- $\{-1\}$ → $\frac{y'}{y}-y=x \rightarrow y'-\frac{y^2}{x^2}$ → non linear
نضرب المعادلة بـ y

e- none

Q_3 The D.E $\frac{dy}{dx}=\frac{y^2-x^2}{xy}$ can be classified as

a- first order linear

b- separable

c- homogeneous and linear

d- homogeneous and Bernoulli

e- none

solution: → check homogeneous

$$y'=\frac{t^2y^2-t^2x^2}{(tx)(ty)}=\frac{t^2(y^2-x^2)}{t^2xy}$$ so homo

$$\rightarrow y'=\frac{y^2}{xy}-\frac{x^2}{xy}=y.\frac{1}{x}-xy^{-1}$$

$$y'-\frac{1}{x}y=-xy^{-1}\ \text{(Bernoulli)}$$

so ans is [D]

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Q4 if $(x^2y+y^n+x^2+1)\,dx+(y+x^m)\,dy=0$ is separable, then

a- $n=0, m=0$

b- $n=0, m=1$

c- $n=1, m=0$

d- $n=2, m=0$

e- none

solution $x^2y+y^n+x^2+1+(y+x^m)\frac{dy}{dx}=0$

check $n=1$, $m=0$ $\quad x^2y+y+x^2+1+(y+1)\frac{dy}{dx}=0$

نأخذ y المشتركة ← $y(x^2+1)+(x^2+1)+(y+1)\frac{dy}{dx}=0$

نأخذ القوس المشترك ← $(x^2+1)(y+1)+(y+1)\frac{dy}{dx}=0$

$\cancel{(y+1)}\frac{dy}{dx}=-(x^2+1)\cancel{(y+1)}$

$dy=-(x^2+1)\,dx$

so ans is C

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Q_5 if $(e^x + 4xy^2)dx + (4x^2y^n + 4y^3)dy = 0$ is exact then $n =$

a- 3

b- 1

c- 2

d- 0

e- none

solution exact means $My = Nx$

$$8xy = 8xy^n$$

$$y = y^n$$

$$n = 1 \quad \boxed{b}$$

Q_6 The solvtion of $y' + \frac{1}{x}y = 2$, $y(1) = 2$ is

a- $y = \frac{x^2+1}{x}$

b- $y = x^2 + 1$

c- $y = \frac{x^2+2}{x}$

d- $xy = 2$

e- none

solution $M_x = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$$y = \frac{1}{x}\left[\int 2x\,dx + c\right] = \frac{1}{x}(x^2 + c)$$

$$2 = \frac{1}{1}(1 + c) \rightarrow y = \frac{x^2+1}{x} \quad \boxed{a}$$

$$c = 1$$

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Q7 The D.E $M(x,y)dx + N(x,y)dy = 0$ is Not exact and $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = 2$, the the integrating factor is $M =$

a- $2x$

b- $2y$

c- e^x

d- e^{2y}

e- e^{2x}

$M = e^{\int \frac{My - Nx}{N} dx} = e^{\int 2dx} = e^{2x}$ [e]

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QUESTIONS



Calculus first Exam

First sem. 2020/2021

By Anas Abu Zahra

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If y_1 and y_2 are two linearly independent solutions of the ODE $xy'' + y' - y/x = 0$
and the Wronskian $W(y_1, y_2)(4) = -1/2$, then
$W(y_1, y_2)(10) =$

A) $-1/5$
B) $-1/2$
C) $-1/6$
D) $-1/3$
E) $-1/4$

A suitable integrating factor $\mu(x)$ to convert the ordindary differential equation

$$xy' - y = x$$

to

$$(\mu(x)y)' = 1/x$$

is $\mu(x) =$
A) x^2
B) $1/x$
C) x
D) e^{-x}
E) e^x

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Consider the ODE

$$y' + xy = \frac{1}{3y^2}.$$

Using $u = y^3$, we get

A) $u' - 3xu = 9$

B) $u' - 3xu = 3$

C) $u' + 3xu = 9$

D) $u' + 3xu = 3$

E) $u' + 3xu = 1$

The solution of the ODE

$$\frac{dy}{dx} = \frac{y \sin x}{y - 3},$$

with initial condition $y(0) = 1$ is

A) $y + 3\ln|y| = 2 - \cos x$

B) $y - 3\ln|y| = 2 - \cos x$

C) $-y + 3\ln|y| = -\cos x$

D) $y + 3\ln|y| = \cos x$

E) $y + 3\ln|y| = 2 + \cos x$

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The solution of the ODE

$(y + e^x)dx + (x + 3y^2)dy = 0$

is

A) $xy + e^x - y^3 = c$

B) $xy + e^x + y^3 = c$

C) $xy + e^x - 2y^3 = c$

D) $xy - e^x + 2y^3 = c$

E) $xy - e^x - y^3 = c$

A suitable integrating factor to convert the ODE

$(x^2 + y/x)dx + (x \sin y - 1)dy = 0$

to exact is

A) x^2

B) x

C) $1/x^2$

D) $1/x$

E) x^3

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GINEERING MATHEMATICS (1)/جميع الشعب

My Courses ENGINEERING MATHEMATICS (1)/جميع الشعب General Eng Math I Ist Exam

If the Wronskian $W(f,g) = 4x$, then $W(3f, f-g) =$

A) $-9x$
B) $-6x$
C) $6x$
D) $12x$
E) $-12x$

Using $u = \frac{y}{x}$ the ODE $\frac{dy}{dx} = \frac{y^2}{xy+2x^2}$ becomes

A) $\frac{du}{dx} = \frac{-2u}{x(u+2)}$
B) $\frac{du}{dx} = \frac{2u}{x(u-2)}$
C) $\frac{du}{dx} = \frac{u}{x(u+1)}$
D) $\frac{du}{dx} = \frac{-u}{x(u-2)}$
E) $\frac{du}{dx} = \frac{-u}{x(u+1)}$

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ch2: Second order ODE

→ Standard form for linear 2nd order ODE

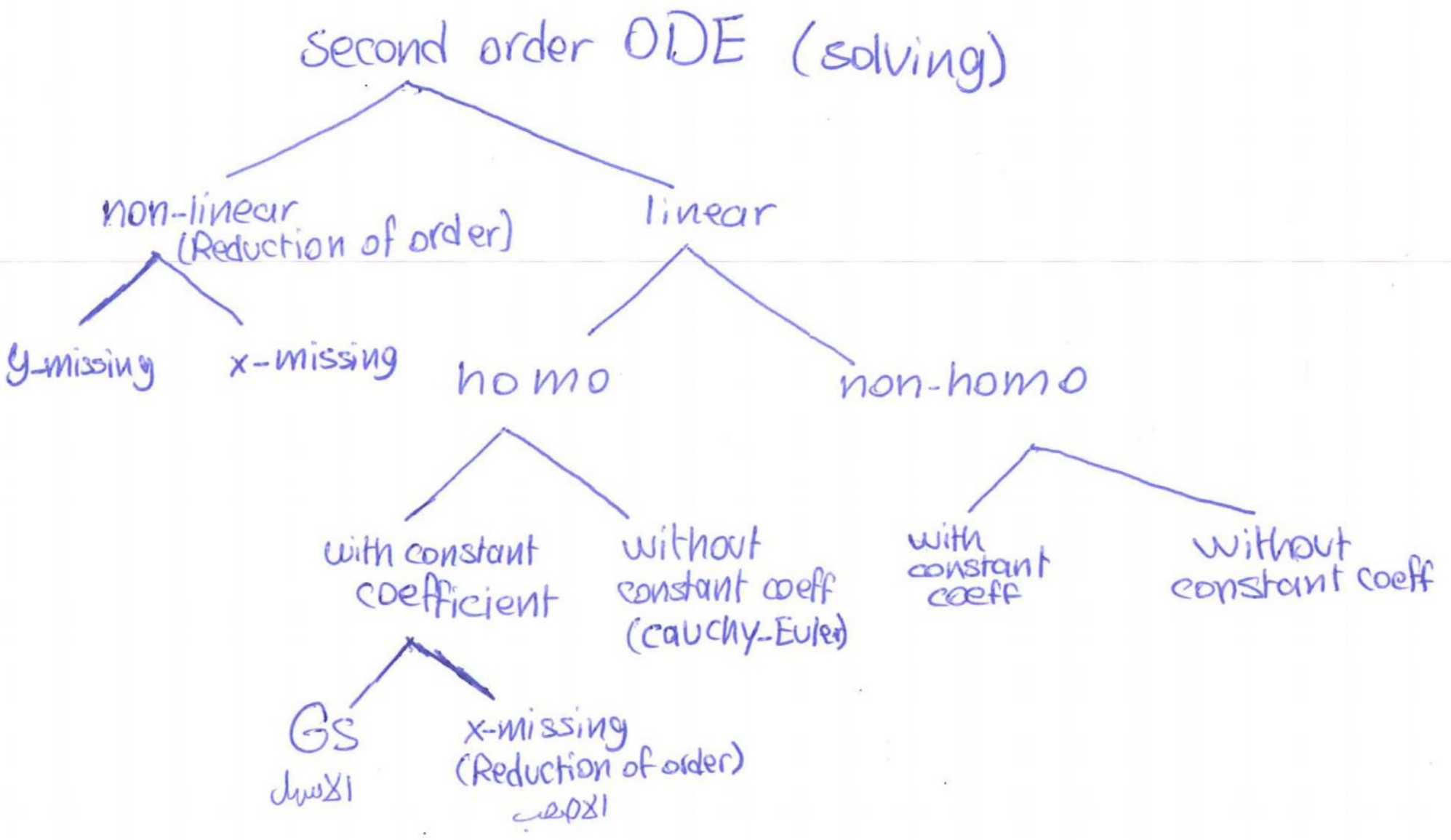
$$y'' + P(x)y' + Q(x)y = g(x)$$

- if $g(x) = 0$ then homo
- if $g(x) \neq 0$ then non-hom

the solution of homo 2nd order ODE is:

$$y = C_1 y_1 + C_2 y_2$$ where y_1 : first solution

y_2: second solution



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→ Second order DDE that can be reduced to 1st order

[linear or non-linear] → Reduction of order

Reduction of order

y-missing

$U = y'$

$\frac{du}{dx} \nwarrow U' = y''$

x-missing

$U = y'$

$U\frac{du}{dy} = y''$

الاثبات

$$\frac{du}{dx} = \frac{du}{dy}\cdot\frac{dy}{dx}$$

$$= \frac{du}{dy}\cdot y' = \frac{du}{dy}\cdot U$$

* فكرة هذه الطريقة هي تحويل 2nd-order الى 1st-order

* في حال لم يحتوي السؤال على x,y الافضل استخدام طريقة y-missing

Q1 solve $y'' + y' = 0$

x, y-missing _ solve by y-missing

$U = y'$, $U' = y''$

$$U' + U = 0$$

$$\frac{du}{dx} = -U$$

$$\int\frac{du}{-U} = \int dx$$

$$-\ln|U| = x + C$$

$$-\ln|y'| = x + C$$

$$\frac{1}{y'} = e^{x+C}$$

$$y' = \frac{1}{C}e^{-x}$$

$$\int dy = \int\frac{1}{C}e^{-x}dx$$

$$y = (\tfrac{1}{-1})\frac{-1}{C_1}e^{-x} + C_2$$

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Q2 $y'' = 1+(y')^2$

x and y-missing
solve by y-missing

$U = y'$

$U' = y''$

$U' = 1+U^2$

$\frac{dU}{dx} = 1+U^2$

$\tan^{-1} U = x + C$

$U = \tan(x+C)$

$y' = \tan(x+C)$

$\int dy = \int \tan(x+c)\, dx$

$y = \ln|\sec(x+c)| + C_2$

Q3 $y'' = 2x(y')^2$

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Q4 solve $yy'' = 3(y')^2$

X-missing

$v = y'$

$v\frac{dv}{dy} = y''$

$\rightarrow y\,v\frac{dv}{dy} = 3v^2$

$\int\frac{dv}{v} = \int\frac{3}{y}dy$

$\ln|v| = 3\ln|y| + c$

$v = y^3 . e^c = c_1y^3$

$y' = c_1y^3$

$\frac{dy}{dx} = cy^3$

$\frac{y^{-2}}{-2} = \pm c_1x + c_2$

Q5 $y'' + (y')^3\cos y = 0$

x-missing $\quad v = y'$

$v\frac{dv}{dy} = y''$

$v\frac{dv}{dy} + v^3\cos y = 0$

$\int\frac{v\,dv}{v^3} = \int -\frac{v^3}{v^3}\cos y\,dy$

$\frac{v^{-1}}{-1} = -\sin y + c$

$v = \frac{1}{\sin y - c}$

$\frac{dy}{dx} = \frac{1}{\sin y - c}$

$\int(\sin y - c)\,dy = \int dx$

$-\cos y - cy = x + c_2$

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Q6 $y^{11}+(y^1)^2 = y(y^1)^3$

x-missing

$v = y^1$

$v\frac{dv}{dy} = y^{11}$

$v\frac{dv}{dy} + v^2 = y v^3$

$v\frac{dv}{dy} = yv^3 - v^2$

$\frac{dv}{dy} = yv^2 - v$

$\frac{dv}{dy} + v = yv^2$ (bernolli in y)

نحل السؤال الكم جنب بالصور الضبيحة بعدين نعيد كل شي لاصله

$\frac{dy}{dx} + y = xy^2$

$v = y^{-1}$

$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$

$-y^2y^1 + y.(-y^{-2}) = -x$

$\frac{dv}{dx} - v = -x$

$v^1 - v = -x$

$Mv = e^{\int -1 dx} = e^{-x}$

$y = \frac{1}{e^{-x}} \int e^{-x} . -x$

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Homogenous 2nd order linear ODE

→ $y'' + P(x)y' + Q(x)y = g(x)$ → $g(x) = 0$ for homo

* if → $\frac{f}{g} = c$, then f and g called propotional or linearly dependent
  → $\frac{f}{g} \neq c$, then f and g are not propotional (linearly independent)

* Wronskian $W(f,g) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x)$

if → $W = 0$ then f, g linearly dependent
   → $W \neq 0$ then f, g linearly independent

G.S $y = C_1 y_1 + C_2 y_2$

→ If y_1 given then wa can find y_2 using

$$y_2 = y_1 . v$$

where $v = \int \frac{1}{y_1^2} . e^{-\int p(x) dx}$.

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Wronskian

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'$$

→ if $W(f,g) = 0$ then f,g linearly dependent

→ if $W(f,g) \neq 0$ then f,g linearly independent

Q_1 find $W(\cos x, \sin x)$

$$\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Q_2 find $W(f,g)(1)$ where $f(x) = x^2$, $g(x) = x$

$$W(f,g)(x) \begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix} = x^2 - 2x^2 = -x^2$$

$$W(1) = -1$$

$$W(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$$

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Q3 $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$, find $W(y_1, y_2, y_3)$

$$\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x(12x^2 - 6x^2) - x^2(6x - 0) + x^3(2)$$

$$= 6x^3 - 6x^3 + 2x^3 = 2x^3$$

Q4 le $W(y_1, y_2)(x) = c\, e^{-\int P(x)\,dx}$

where $y'' + P(x)\,y' + q(x)\,y = 0$

Q4 if y_1, y_2 are linearly independent solutions of $xy'' + 2y' + xe^x y = 0$ and $W(y_1, y_2)(2) = 3$, find $W(y_1, y_2)(5)$

$$y'' + \left(\frac{2}{x}\right)y' + e^x y = 0 \qquad (P = \tfrac{2}{x})$$

$$\rightarrow W(y_1, y_2) = c\, e^{-\int \frac{2}{x}dx} = c\, e^{-2\ln x} = c\, e^{\ln \frac{1}{x^2}} = \frac{c}{x^2}$$

$$\rightarrow W(2) = \frac{c}{4} = 3 \quad \boxed{c = 12}$$

$$\rightarrow W(5) = \frac{12}{25}$$

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Q5 if $W(f,g)=2$, find $W(f, f+2g)$

$$\begin{vmatrix} f & f+2g \\ f' & f'+2g' \end{vmatrix} = \cancel{ff'} + 2fg' - \cancel{ff'} - 2f'g$$

$$= 2fg' - 2f'g$$

$$= 2(fg' - f'g)$$

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

$$= fg' - gf' = 2 \rightarrow = 2(2) = 4$$

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Q6 if $W(f,g) = 3e^{4x}$ and $f(x) = e^{2x}$, find $g(x)$

$$\begin{vmatrix} e^{2x} & g \\ 2e^{2x} & g' \end{vmatrix} = e^{2x}g' - 2e^{2x}g = 3e^{4x}$$

*لايجاد الاقتران g يجب حل solve للمعادلة الخلاه حسب التشابر السابق

$$\rightarrow e^{-2x}g' - 2e^{-2x}g = 3$$

$$\int (e^{-2x}g)' = \int 3$$

$$e^{-2x}g = 3x + c$$

hint!

$\frac{d}{dx} e^{-2x} \cdot g =$

$e^{-2x}g' - 2e^{-2x}g$

$$g = \frac{3x+c}{e^{-2x}}g$$

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Q1 $2y'' - 8y' + 8y = 0$. Given $y_1 = e^{2x}$. Find y_2

$y'' - 4y' + 4y = 0$ (P(x) = −4, Q(x) = 4)

$$U = \int \frac{1}{e^{4x}} \cdot e^{-\int -4\,dx}\,dx$$

$$U = \int \frac{e^{4x}}{e^{4x}}\,dx = x$$

$y_2 = x e^{2x}$ G.S $y = c_1 y_1 + c_2 y_2$

Q2 If $y_1 = x^2$ is a solution of $x^2y'' + xy' - 4y = 0$

Find the General solution

$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$ (P = $\frac{1}{x}$, Q = $-\frac{4}{x^2}$)

$$U = \int \frac{1}{x^4} \cdot e^{-\int \frac{1}{x}}\,dx$$

$e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$

$$U = \int \frac{1}{x^5}\,dx = \frac{x^{-4}}{-4}$$

$$y_2 = \frac{-1}{4x^4} \cdot x^2 = \frac{-1}{4x^2}$$

G.S $y = c_1 x^2 + c_2\left(\frac{-1}{4x^2}\right)$

Q3 If $y_1 = x$ is solution of $(x^2 - x)y'' + xy' + y = 0$

Find the General solution

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→ 2nd Order linear homogenous with constant coefficiant

$a y'' + b y' + c y = 0$ where a,b,c are constant

G.S $y = e^{rx}$

→ charachterestic equation $ar^2 + br + c = 0$

we use this equation to find r



Q_1 solve $y'' + 3y' + 2y = 0$

step 1 $r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r = -2$ $r = -1$ (case 1)

$y = c_1 e^{-2x} + c_2 e^{-x}$

Q_2 solve $y'' - 4y = 0$

$r^2 - 4 = 0$

$(r-2)(r+2) = 0$

$r = 2$ $r = -2$

$y = c_1 e^{2x} + c_2 e^{-2x}$

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Q7 solve $y'' + y' + y = 0$

$$r^2 + 1 + 1 = 0$$

$D = 1 - 4 = -3$ (case 3)

$r = \frac{-1 \pm \sqrt{-3}}{2}$ $\quad r = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$

$\alpha = \frac{-1}{2}$, $\beta = \frac{\sqrt{3}}{2}$

$$y = c_1 e^{\frac{-1}{2}x} \cos\frac{\sqrt{3}}{2}x + c_2 e^{\frac{-1}{2}x} \sin\frac{\sqrt{3}}{2}x$$

Q8 solve $2y'' + 3y' + 2y = 0$

$$2r^2 + 3r + 2 = 0$$

$D = -7$ (case 3)

$r = \frac{-3 \pm \sqrt{7}\, i}{4}$ $\quad \alpha = \frac{-3}{4}$

$\beta = \frac{\sqrt{7}}{4}$

$$y = c_1 e^{\frac{-3}{4}x} \cos\frac{\sqrt{7}}{4}x + c_2 e^{-3/4x} \sin\frac{\sqrt{7}}{4}x$$

Q9 solve $y'' + 4y = 0$

$$r^2 + 4 = 0$$

$r^2 = -4$

$r = \pm 2i$ $\quad \alpha = 0$, $\beta = 2$

$$y = c_1 \cos 2x + c_2 \sin 2x$$

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Q3 Solve $3y'' + 4y' - y = 0$

$$3r^2 + 4r - 1 = 0$$

$$D = 16 - 4(3)(-1) = 28 > 0 \quad \text{(case 1)}$$

$$r = \frac{-b \pm \sqrt{D}}{2a} = \frac{-4 \pm \sqrt{28}}{6}$$

$$r_1 = \frac{-4 + \sqrt{28}}{6} \qquad r_2 = \frac{-4 - \sqrt{28}}{6}$$

Q4 solve $y'' - 6y + 9 = 0$

$$r^2 - 6r + 9 = 0$$

$$(r-3)(r-3) = 0$$

$$r = 3 \quad \text{(case 2)}$$

$$y = c_1 e^{3x} + c_2 x e^{3x}$$

Q5 $y'' - 3y' = 0$

$$r^2 - 3r = 0$$

$$r = 0, 3 \quad \text{(case 1)}$$

$$y = c_1 + c_2 e^{3x}$$

Q6 $2y'' - 12y' + 18y = 0$

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Q_{10} Find the ODE where the General solution is

$$y = c_1 e^{x} + c_2 e^{-2x}$$

solution $r_1 = 1$

$r_2 = -2$

$(r+2)(r-1) = 0$

$r^2 + r - 2 = 0$

$y'' + y' - 2y = 0$

Q_{11} Find the ODE where G.S $y = c_1 e^{2x} + c_2 x e^{2x}$

$r_1 = r_2 = 2$

$(r-2)^2 = 0$

$r^2 - 4r + 4 = 0$

$y'' - 4y' + 4y = 0$

Q_{12} find the ODE where G.S $y = c_1 e^{2x}\cos\sqrt{3}x + c_2 e^{2x}\sin\sqrt{3}x$

$\alpha = 2 \;, \beta = \sqrt{3}$

$r = 2 \pm \sqrt{3}i$

$(r - (2+\sqrt{3}i))(r - (2-\sqrt{3}i)) = 0$

$((r-2) - \sqrt{3}i)((r-2) + \sqrt{3}i) = 0$

$(r-2)^2 - (\sqrt{3}i)^2 = 0$

$r^2 - 4r + 4 + 3 = 0 \rightarrow r^2 - 4r + 7 = 0$

$y'' - 4y' + 7y = 0$

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Operations on complex number i

① multiplication with constant

$$z = 3+2i$$

$$4z = 12+8i$$

② Addition

$$(1-4i)+(5+7i) =$$

$$=6+3i$$

③ multiplication

Note $i^2 = -1$

$$(2+3i).(4+5i)$$

$$= 8 + 10i + 12i + 15(i)^2$$

$$= 8 + 22i - 15 = -7+22i$$

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Q_{13} Find 2nd ODE which has G.S $y = C_1\cos 2x + C_2 \sin 2x$

$\alpha = 0 \quad , \beta = 2$

$r = \pm 2i$

$(r - 2i)(r + 2i) = 0$

$r^2 + 2ri - 2ri + 4 = 0$

$r^2 + 4 = 0$

$y'' + 4y = 0$

2nd Order homogenous linear ODE with non-constant coefficient (cauchy-Euler) equation CEE

$\rightarrow \quad a x^2 y'' + bxy' + cy = 0$

Charachterestic equation $a(r)(r-1) + br + c = 0$

case 1
$D > 0 \ (r_1, r_2)$

$y = C_1 x^{r_1} + C_2 x^{r_2}$

case 2
$D = 0 \ (r_1 = r_2)$

$y = C_1 x^{r_1} + C_2 x^{r_2} (\ln x)$

case 3
$D < 0$

$y = C_1 x^{\alpha} \cos(\beta \ln x) + C_2 x^{\alpha} \sin(\beta \ln x)$

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Q1 solve $x^2y'' - 2xy' + 2y = 0$

$r(r-1) - 2r + 2 = 0$

$(r-1)(r-2) = 0$

$r = 1, 2$ (case 1)

$y = C_1 x + C_2 x^2$

Q2 solve $xy'' + 2y' + \frac{3y}{x} = 0$

step 1 ضرب $\times x$

$x^2y'' + 2xy' + 3y = 0$

$r(r-1) + 2r + 3 = 0$

$r^2 + r + 3 = 0$

$D = -11$ (case 3)

$r = \frac{-1 \pm \sqrt{11}\, i}{2}$ $\quad \alpha = \frac{-1}{2}$, $\beta = \frac{\sqrt{11}}{2}$

$y = C_1 x^{-\frac{1}{2}} \cos\frac{\sqrt{11}}{2}\ln x + C_2 x^{-\frac{1}{2}} \sin\left(\frac{\sqrt{11}}{2}\ln x\right)$

Q3 $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0$

$x^2y'' - xy' + y = 0$

$r(r-1) - r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$ (case 2)

$y = C_1 x + C_2 x \ln x$

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Q4 $2x^2y'' + 3xy' - y = 0$

Q5 $x^2y'' + 7xy' + 13y = 0$

Q6 $xy'' + 2y' = 0$ → y-missing → CEE

$x^2y'' + 2xy' = 0$

$r(r-1) + 2r = 0$

$r = 0, -1$

$y = c_1 + c_2 x^{-1}$

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Q7 Find the ODE which GS is $y = c_1x^3 + c_2x^2$

$r = 3, 2$

$(r-3)(r-2) = 0$

$r^2 - 5r + 6 = 0$ (not char eq)

$r^2 - r + r - 5r + 6 = 0$

$r(r-1) - 4r + 6 = 0$

$x^2y'' - 4xy' + 6y = 0$

Q8 Find the ODE which GS is $y = c_1\cos 2\ln x + c_2 \sin 2\ln x$

$\alpha = 0$, $\beta = 2$

$r = \pm 2i$

$(r - 2i)(r + 2i)$

$r^2 + 4 = 0$

$r^2 - r + r + 4 = 0$

$r(r-1) + r + 4 = 0$

$x^2y'' + xy' + 4y = 0$

Q9 Find the second order CEE which char. eq is $2r^2 + 4r + 3 = 0$

$2r^2 - 2r + 2r + 4r + 3 = 0$

$2r(r-1) + 6r + 3 = 0$

$2x^2y'' + 6xy' + 3y = 0$

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Second order ODE

- non-linear
  - x-missing
  - y-missing
  - Reduction of order
- linear
  - homogeneous
    - with constant Coefficient
      - → homo with constant coefficient
        - r_1, r_2: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
        - r: $y = c_1 e^{rx} + c_2 x e^{rx}$
        - $D<0$: $y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$
    - without constant coefficient (cauchy-Euler)
  - non-homogeneous
    - with constant coefficient
    - without constant coefficient

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Part II : Non-homogeneous 2nd order linear ODE :

→ Case 1 with constant coefficients

General formula $ay'' + by' + cy = g(x)$ ← $g(x) \neq 0$

The G.S has the form $y = y_h + y_p$

→ y_h : the general solution of $ay'' + by' + cy = 0$

← اي هو حل المعادلة عل اعتبار انها homo اي الطرف الثاني يساوي 0

→ y_p : the particular solution

← وهو الحل المتعلق بقيمة الاقتران $g(x)$

There are two methods to find y_p

1. Undetermined coefficients
2. Variation of parameters

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Method (1) Undetermined Coefficients

$$ay'' + by' + cy = g(x) \neq 0$$

General solution $y = y_h + y_p$

where → $a\,y_h'' + b\,y_h' + c\,y_h = 0$

and

→ $a\,y_p'' + b\,y_p' + c\,y_p = g(x)$

Remarks :

تستخدم هذه الطريقة في حال كان $g(x)$ أحد الافترانات التالية فقط

1. Exponential (e^x)
2. Polynomial
3. sin or cos
4. اقتران يحتوي عملية ضرب بين الاقترانات الاولاه
5. اقتران يحتوي عملية جمع بين الاقترانات الاولاه

الهدف من هذه الطريقة هو ايجاد y_p والتي تعتمد على قيمة $g(x)$ حيث سنقوم بفرض y_p بالاعتماد على نوع $g(x)$ وحسب الجدول في الصفحة التالية

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$G(x)$	y_p	example
- جميع المعاملات يجب ان تكون معلومة في السؤال	المعاملات غير معلومة وعلينا ايجادها	
1. Ae^{rx} (عدد معطى في السؤال)	$y_p = Ae^{rx}$ (ثابت علينا ايجاد قيمته، نفس معامل القوة في السؤال)	$g(x) = \frac{1}{3}e^{-2x}$? ↳ $y_p = Ae^{-2x}$
2. $a_nx^n+\dots+a_1x+a_0$	$y_p = A_nx^n+\dots A_1x+A_0$ (نفس معامل الزاوية في السؤال)	$g(x) = 3x^2+2$? ↳ $y_p = Ax^2+Bx+C$
		$g(x) = x$? ↳ $y_p = Ax+B$
		$g(x) = x^3$? ↳ $y_p = Ax^3+Bx^2+Cx+d$
3. $a\sin\beta x$, $b\cos\beta x$, $a\sin\beta x+b\cos\beta x$	$y_p = A\cos\beta x+B\sin\beta x$	$g(x) = -3\cos 2x$? ↳ $y_p = A\cos 2x+B\sin 2x$
		$g(x) = \sin x+4\cos x$? ↳ $y_p = A\cos x+B\sin x$

← في حال كان $g(x)$ حاصل ضرب اقترانين فان y_p تكون وفقا حاصل ضرب y_p للاقتران الاول و y_p للاقتران الثاني

$g(x) = 2x\cos(\pi x)$?

↳ $y_p = (Ax+B)\cos\pi x + (Cx+D)\sin\pi x$

$g(x) = e^{-3x}\sin(\frac{1}{2}x)$?

↳ $y_p = e^{-3x}(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x)$

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Examples : solve

1. $y'' - 7y' + 10y = 3xe^{3x}$

$$y = y_h + y_p$$

→ to find y_h $\quad y'' - 7y' + 10y = 0$

$$r^2 - 7r + 10 = 0$$

$$(r-5)(r-2) = 0$$

$r = 5 \quad r = 2$

$$y_h = c_1 e^{2x} + c_2 e^{5x}$$

→ to find y_p

$g(x) = 3xe^{3x}$

$y_p = (ax+b)e^{3x}$

لايجاد قيم الثوابت a, b نجد كل من y'_p ، y''_p ونعوضهم في المعادلة الاصلية

$$y'_p = (ax+b)e^{3x} \cdot 3 + e^{3x} \cdot a$$

$$y''_p = (ax+b)e^{3x} \cdot 9 + 3e^{3x} \cdot a + 3ae^{3x} = 9e^{3x}(ax+b) + 6ae^{3x}$$

→ $9e^{3x}(ax+b) + 6ae^{3x} - 21e^{3x}(ax+b) - 7ae^{3x} + 10e^{3x}(ax+b) = 3xe^{3x}$

نفكك الاقواس

$9axe^{3x} + 9be^{3x} + 6ae^{3x} - 21axe^{3x} - 21be^{3x} - 7ae^{3x} + 10axe^{3x} + 10be^{3x} = 3xe^{3x}$

$-2axe^{3x} - 2be^{3x} - ae^{3x} = 3xe^{3x}$

$-2b - a = 0$ $\quad -2a = 3$ ← أي كل

$-2b + \frac{3}{2} = 0$ $\quad \boxed{a = \frac{-3}{2}}$

$\boxed{b = \frac{3}{4}}$

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Modification Rules:

1. if you choose y_p according to the table and you notice that one of the terms of y_p is similar to one of the terms of y_h, then multiply y_p by x, if you face the same Problem multiply by x one again....
Keep multiplying by x until no terms in y_p appears in y_h

Q find a suitable form for y_p if the method of undetermined coefficient is to be used

1. $y'' - 3y' + 2y = 2e^x$

→ to find y_h

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r = 2 \quad r = 1$

$y_h = C_1 e^x + C_2 e^{2x}$

→ to find y_p

$g(x) = 2e^x$

$y_p = Ae^x$ بما انها نفس احد حلول y_h يجب ضربها بـ x

$\boxed{y_p = Axe^x}$

2. $y'' - 2y' + y = xe^x$

→ $r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$\boxed{y_h = C_1 e^x + C_2 xe^x}$

→ $g(x) = xe^x$ مكبد مكبد

$y_p = (ax+b)e^x = axe^x + be^x$

ضرب بـ x ← $y_p = ax^2e^x + bxe^x$ مكبد

ضرب بـ x ← $\boxed{y_p = ax^3e^x + bx^2e^x}$

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3. $y'' + 4y = xe^{3x}\sin 2x$

$\rightarrow r^2 + 4 = 0$

$r = \pm 2i$

$y_h = c_1 \cos 2x + c_2 \sin 2x$

$\rightarrow y(x) = x e^{3x}\sin 2x$

$y_p = e^{3x}(ax+b)\cos 2x + e^{3x}(cx+d)\sin 2x$

4. $y'' + 4y = x\sin 2x$

$\rightarrow y_h = c_1 \cos 2x + c_2 \sin 2x$

$\rightarrow y_p = (ax+b)\cos 2x + (cx+d)\sin 2x$

$= ax\cos 2x + b\cos 2x + cx\sin 2x + d\sin 2x$

(b cos 2x: نفس حل ; d sin 2x: نفس حل)

نضرب في x: $y_p = ax^2\cos 2x + bx\cos 2x + cx^2\sin 2x + dx\sin 2x$

homework

solve : $y'' + 9y = 4\sin 3x$?

ans: G.S $\Rightarrow$ $y = c_1\cos 3x + c_2\sin 3x - \frac{2}{3}x\cos 3x$

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2. if $ay''+by'+cy=g_1(x)+g_2(x)+g_3(x)$ then the Particular solution $Y_p = Y_{P_1} + Y_{P_2} + Y_{P_3}$

where Y_{P_1} is the particular solution for

$ay''+by'+cy=g_1(x)$

Examples: find the suitable form of Y_p

1. $y''+4y'+3y = xe^{-x}+\sin^2 x$

$$y''+4y'+3y = xe^{-x}+\frac{1}{2}-\frac{1}{2}\cos 2x$$

g_1 g_2 g_3

بسبب استخدام هذه الخطابقات هو انه لا يوجد في الجدول قيم Y_p لـ $\sin^2$ و $\cos^2$

hint

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

$\rightarrow Y_h = C_1 e^{-x} + C_2 e^{-3x}$

$\rightarrow Y_p$:

- $Y_{P_1} = (Ax+B)e^{-x}$ ← ضرب بـ x لانه مكرر → $\boxed{(Ax+B)xe^{-x}}$
- $Y_{P_2} = C$
- $Y_{P_3} = D\cos 2x + E\sin 2x$

$Y_p = Y_{P_1} + Y_{P_2} + Y_{P_3}$

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2. $y'' + 4y = xe^{-2x} + x\sin^2 x$

$$y'' + 4y = \underset{g_1}{xe^{-2x}} + \underset{g_2}{\frac{x}{2}} - \underset{g_3}{\frac{x}{2}\cos 2x}$$

$$\rightarrow y_h = c_1 \cos 2x + c_2 \sin 2x$$

$\rightarrow y_p$:

- $\rightarrow y_{p_1} = (ax+b)e^{-2x}$
- $\rightarrow y_{p_2} = cx + d$
- $\rightarrow y_{p_3} = (ex+f)\overset{\text{نفس}}{\cos 2x} + (gx+h)\overset{\text{نفس}}{\sin 2x}$

$$y_{p_3} = x(ex+f)\cos 2x + x(g(x+h)\sin 2x$$

$$y_p = y_{p_1} + y_{p_2} + y_{p_3}$$

to find y_p

$y_{p_1} \rightarrow \quad y''_{p_1} + 4y_{p_1} = x e^{-2x}$

$y_{p_2} \rightarrow \quad y''_{p_2} + 4y_{p_2} = \frac{x}{2}$

$y_{p_3} \rightarrow \quad y''_{p_3} + 4y_{p_3} = \frac{-x}{2}\cos 2x$

$\vdots$

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Method 2 : Variation of Parameters

let $ay''+by'+cy = g(x)$ $\rightarrow \neq 0$

$$C.S \Rightarrow y = y_h + y_p$$

assume $y_h = c_1y_1 + c_2y_2$

then $$Y_p = -y_1\int \frac{y_2\,g(x)}{w}dx + y_2\int \frac{y_1\,g(x)}{w}dx$$

where $w = w(y_1, y_2) \neq 0$

w ← wronskian

ديروا بالكم تنسوا !!

لايجاد $g(x)$ يجب ان يكون معامل y'' = 1+

Examples :

1. $y''+9y = \csc 3x$

* نلاحظ بداية ان هذا السؤال لايمكن حله بالطريقة الاولى لكون $\csc 3x$ لا يوجد لها مقابل بالجدول

$$\rightarrow y_h = c_1\cos 3x + c_2\sin 3x$$

$$w(y_1,y_2) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$$

$$\rightarrow Y_p = -\cos 3x\int \frac{\sin 3x\,\csc 3x}{3}dx + \sin 3x\int \frac{\cos 3x\,\csc 3x}{3}dx$$

$$\int \frac{\sin 3x}{3}\cdot\frac{1}{\sin 3x} = \frac{1}{3}x$$

$$\int \frac{\cos 3x}{3\sin 3x} = \frac{1}{3}\int \cot 3x = \frac{1}{3}\ln|\sin 3x| = \frac{\ln|\sin 3x|}{9}$$

$$\rightarrow Y_p = \frac{-1}{3}x\cos 3x + \frac{1}{9}\sin 3x\,\ln|\sin 3x|$$

$$Y = Y_h + Y_p$$

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2. $y'' + 6y' + 9y = \frac{16e^{-3x}}{x^2+1}$

$r^2 + 6r + 9 = 0$

$(r+3)(r+3) = 0$

$\rightarrow y_h = c_1 e^{-3x} + c_2 x e^{-3x}$

$\rightarrow W(y_1, y_2) = \begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & -3xe^{-3x} + e^{-3x} \end{vmatrix}$

$= -3xe^{-6x} + e^{-6x} + 3xe^{-6x} = e^{-6x}$

$\rightarrow Y_p = -e^{3x} \int \frac{xe^{-3x}}{e^{-6x}} \cdot \frac{16e^{-3x}}{x^2+1} dx + xe^{-3x} \int \frac{e^{-3x}}{e^{-6x}} \frac{16e^{-3x}}{x^2+1} dx$

$\int \frac{16x}{x^2+1} dx = 8 \ln|x^2+1|$

$\int \frac{16}{x^2+1} dx = 16 \tan^{-1} x$

$\rightarrow Y_p = -8e^{3x} \ln|x^2+1| + 16xe^{-3x} \tan^{-1} x$

$Y = Y_h + Y_p$

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Case 2: Non-homo with non-constant coefficient

→ to find y_p we use method II only

Example: Solve

1. $x^2y'' - 4xy' + 6y = x^3$. $x>0$

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Q Find the DE whose G.S is

1. $y = c_1 + c_2 e^{3x} + x^3 + 1$

$\rightarrow y_h = c_1 + c_2 e^{3x}$

$r_1 = 0, \; r_2 = 3$

$(r)(r-3) = 0$

$r^2 - 3r = 0$

$y'' - 3y' = 0$

$\rightarrow$ Now to find $g(x)$

$Y_p = x^3 + 1$

$Y_p' = 3x^2$

$Y_p'' = 6x$

$6x - 9x^2 = g(x)$

so the D.E is $Y'' - 3y' = 6x - 9x^2$

2. $y = c_1 x + c_2 x \ln x + x \cos x$

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chapter 3 : Higher order ODE

Standard form :-

$$P_n(x)\,y^{(n)} + \dots + P_1(x)\,y' + P_0(x)\,y = g(x)$$

Part I: homogeneous ($g(x)=0$)

→ case 1 : with constant coefficient

$$G.S : y = e^{rx}$$

!! تذكر
$y^4 \rightarrow$ 4 مشتقة
$y^{(4)} \rightarrow$ مشتقة الرابعة

Examples : Solve

1. $y^{(4)} - 5y'' + 4y = 0$

$$r^4 - 5r^2 + 4 = 0$$

* assume $z = r^2$

$$\rightarrow z^2 - 5z + 4 = 0$$
$$(z-4)(z-1) = 0$$
$$(r^2-4)(r^2-1) = 0$$
$$(r-2)(r+2)(r-1)(r+1) = 0$$
$$r=2 \quad r=-2 \quad r=1 \quad r=-1$$

$$G.S : y = C_1 e^{2x} + C_2 e^{-2x} + C_3 e^{x} + C_4 e^{-x}$$

* اذا كانت المعادلة من الدرجة الرابعة يجب ان يكون لها 4 حلول ، واذا كانت من الدرجة الثالثة يجب ان يكون لها ثلاثة حلول
* في حال تكرار احد الحلول نضربه بـ x ، واذا بقي متكررًا نضربه مرة ثانية بـ x كما كنا نتعامل مع الحلول في 2nd ordrer ODE

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2. $y''' - 3y'' + 3y' - y = 0$

$$r^3 - 3r^2 + 3r - 1 = 0$$

→ P.R.R = ±1

try r = 1

$(1)^3 - 3(1)^2 + 3(1) - 1 =$

$1 - 3 + 3 - 1 = 0$

so r = 1 is solution

القوس الاول $(r-1)$

$$\begin{array}{r|l} r-1 & r^3 - 3r^2 + 3r - 1 \\ & -r^3 + r^2 \\ \hline & -2r^2 + 3r - 1 \\ & +2r^2 - 2r \\ \hline & r - 1 \\ & r - 1 \\ \hline & 0 \end{array} \quad r^2 - 2r + 1$$

→ يجب ان يكون الباقي 0 والا فان الحل خطأ

$(r-1)(r^2 - 2r + 1) = 0$

$(r-1)(r-1)(r-1) = 0$

$r=1 \quad r=1 \quad r=1$

G.S $y = C_1 e^x + C_2 x e^x + C_3 x^2 e^x$

important Notes!!

1. لحل المعادلة التكعيبية علينا ايجاد الحل الاول بالتجريب ومن ثم اجراء قسمة طويلة عام الحل
2. في حال كان معامل r^3 يساوي 1 فان الحل الاول هو احد عوامل الحد الاخير!

P.R.R حيث تعني عوامل الحد الاخير

Possible Rational Roots

اي الحلول المحتملة للمعادلة

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3. $y^{(4)} + 13y'' + 36y = 0$

$r^4 + 13r^2 + 36 = 0$

$(r^2+4)(r^2+9) = 0$

$r^2 = -4$ $\quad$ $r^2 = -9$

$r = \pm 2i$ $\quad$ $r = \pm 3i$

$\alpha = 0$, $\beta = 2$ $\quad$ $\alpha = 0$, $\beta = 3$

G.S: $y = C_1\cos 2x + C_2 \sin 2x + C_3 \cos 3x + C_4 \sin 3x$

4. $y^{(5)} - 4y^{(4)} + 3y''' = 0$

$r^5 - 4r^4 + 3r^3 = 0$

$r^3(r^2 - 4r + 3) = 0$

$r^3(r-3)(r-1) = 0$

$r = 0$ $\quad$ $r = 3$ $\quad$ $r = 1$

انتبه! مكرر 3 مرات

G.S: $y = C_1 e^{0x} + C_2 x e^0 + C_3 x^2 e^0 + C_4 e^{3x} + C_5 e^x$

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5. $y^4 - 16y = 0$

$$r^4 - 16 = 0$$

$$(r^2 - 4)(r^2 + 4) = 0$$

$$(r-2)(r+2)(r^2+4) = 0$$

$r = 2$ $\quad r = -2$ $\quad r = \pm 2i$

$$G.S \quad y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$$

6. $y^{(4)} + 2y''' + 2y'' + 2y' + y = 0$

$$r^4 + 2r^3 + 2r^2 + 2r + 1 = 0$$

P.R.R $= \pm 1$ (معاملات الحد الأخير) → هنا ممكن نستخدم نفس الملاحظة عن معادلة غير كلية ؟ آه ممكن !!

→ try $r = -1$ ✓

$(r+1)$ first root

$$\rightarrow (r+1)(r^3 + r^2 + r + 1) = 0$$

P.R.R $= \pm 1$

$(r+1)$ second root

$$(r+1)(r+1)(r^2+1) = 0$$

$r = -1$ $\quad r = -1$ $\quad r = \pm i$

	1	2	2	2	1
-1		-1	-1	-1	-1
	1	1	1	1	0

$r^3 + r^2 + r + 1$

	1	1	1	1
-1		-1	0	-1
	1	0	1	0

$r^2 + 1 = 0$

$$G.S \quad y = c_1 e^{-x} + c_2 x e^{-x} + c_3 \cos x + c_4 \sin x$$

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7. $y^{(6)} + 3y^{(4)} + 3y^{(2)} + y = 0$

$r^6 + 3r^4 + 3r^2 + 1 = 0$

$z^3 + 3z^2 + 3z + 1 = 0$

P.R.R $= \pm 1$

$(z+1)(z^2+2z+1) = 0$

$(z+1)(z+1)(z+1) = 0$

$(r^2+1)(r^2+1)(r^2+1) = 0$

$r = \pm i$

$z = -1$

	1	3	3	1
		-1	-2	-1
	1	2	1	0

$y = C_1 \cos x + C_2 \sin x + C_3 x\cos x + C_4 x\sin x + C_5 x^2\cos x + C_6 x^2 \sin x$

→ case 2 : with non-constant coefficient (cauchy Euler's)

G.S : $y = x^r$

Examples : solve

1. $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$

$r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$

$r(r-1)(r-2) - 3r(r-1) + 6(r-1) = 0$

$(r-1)(r(r-2) - 3r + 6) = 0$

$(r-1)(r(r-2) - 3(r-2)) = 0$

$(r-1)(r-2)(r-3) = 0$

$r=1 \quad r=2 \quad r=3$

$y = C_1 x + C_2 x^2 + C_3 x^3$

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2. $2x^3y''' + x^2y'' + 2xy' - 2y = 0$

$$2r(r-1)(r-2) + r(r-1) + 2r - 2 = 0$$

$2r$ → $2(r-1)$

$$(r-1)(2r(r-2) + r + 2) = 0$$

$$(r-1)(2r^2 - 4r + r + 2) = 0$$

$$(r-1)(2r^2 - 3r + 2) = 0$$

$$D = 9 - (4)(2)(2) = -7$$

$$r = \frac{3 \pm \sqrt{-7}}{4}$$

$r = \frac{3}{4} + \frac{\sqrt{7}}{4}i$, $\alpha = \frac{3}{4}$

$r = \frac{3}{4} - \frac{\sqrt{7}}{4}i$, $\beta = \frac{\sqrt{7}}{4}$

G.S $y = C_1 x + C_2 x^{3/4}\cos\left(\frac{\sqrt{7}}{4}\ln x\right) + C_3 x^{3/4}\sin\left(\frac{\sqrt{7}}{4}\ln x\right)$

3. find the homogeneous ODE, given that e^x, e^{2x} and $e^x\cos x$ are 3 linearly independent solutions for the equation

$y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = e^x\cos x$ → $\alpha = 1, \beta = 1$

$r_1 = 1$, $r_2 = 2$, $r_3 = 1+i$, $r_4 = 1-i$

$$(r-1)(r-2)(r-1-i)(r-1+i) = 0$$

$$(r-1)(r-2)\left((r-1)^2 - (i)^2\right) = 0$$

$$(r^2 - 3r + 2)\left(r^2 - 2r + \frac{1+1}{2}\right)$$

$$r^4 - 2r^3 + 2r^2 - 3r^3 + 6r^2 - 6r + 2r^2 - 4r + 4 = 0$$

$$r^4 - 5r^3 + 10r^2 - 10r + 4 = 0$$

$$y^{(4)} - 5y^{(3)} + 10y^{(2)} - 10y' + 4y = 0$$

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Part 2 : Non-homogeneous higher order

→ case 1: with constant coefficient

G.S $y = y_h + y_p$

To find y_p:

Method 1 : undetermined coefficient

The same table and modification rules as in second order

Examples : solve

1. $y''' + 3y'' + 3y' + y = 9e^{-x}$

solution $y''' + 3y'' + 3y' + y = 0$

$r^3 + 3r^2 + 3r + 1 = 0$

$(r+1)(r+1)(r+1) = 0$

$\rightarrow y_h = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$

$\rightarrow y_p = a e^{-x}$ → مكررة

↳ $y_p = a x e^{-x}$ → مكررة

↳ $y_p = a x^3 e^{-x}$

$y'_p = -a x^3 e^{-x} + 3a x^2 a^{-x} = (3a x^2 - a x^3) e^{-x}$

$y''_p = -(3a x^2 - a x^3) e^{-x} + e^{-x}(6a x - 3a x^2)$

$= e^{-x}(6a x - 3a x^2 - 3a x^2 + a x^3)$

$= e^{-x}(6a x - 6a x^2 + a x^3)$

$y'''_p = e^{-x}(6a - 18a x + 9a x^2 + a x^3)$ ← هذه الاشتقاق والاختصارات

← هذه التعويض في المعادلة ينتج $\boxed{a = \frac{3}{2}}$

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2. find the suitable form for y_p for

$$y^{(5)} - 16y' = xe^{2x}\sin 2x + \cos 2x$$

$$y^{(5)} - 16y' = 0$$

$$r^5 - 16r = 0$$

$$r(r^4 - 16) = 0$$

$$r(r^2 - 4)(r^2 + 4) = 0$$

$$r = 0, 2, -2, 2i, -2i$$

$$y_h = C_1 + C_2 e^{2x} + C_3 e^{-2x} + C_4 \cos 2x + C_5 \sin 2x$$

$y_p \rightarrow \quad y^{(5)} - 16y' = xe^{2x}\sin 2x + \cos 2x$

$g_1 \qquad g_2$

$$y_{p_1} = (ax+b)e^{2x}\cos 2x + (cx+d)e^{2x}\sin 2x$$

$$y_{p_2} = e\cos 2x + f\sin 2x \quad !! \text{ no}$$

$$\hookrightarrow = ex\cos 2x + fx\sin 2x$$

$$y_p = y_{p_1} + y_{p_2}$$

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howework:

Solve $y^{(4)} - y^{(3)} - 2y'' = -12x$

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Method 2: Variation of parameters

$$y_p = y_1 \int \frac{W_1\, g(x)}{W}\,dx + y_2 \int \frac{W_2 g(x)}{W}\,dx + y_3 \int \frac{W_3 g(x)}{W}\,dx + \cdots$$

→ Where $W = W(y_1, y_2, y_3)$ هي الـ W الطبيعية كما في الشابتر السابق

example find $W(x, \cos x, \sin x)$

$$\begin{vmatrix} x & \cos x & \sin x \\ 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= x(\sin^2 x + \cos^2 x) - \cos x(-\sin x \quad) + \sin x(-\cos x)$$

$$= x$$

→ W_1 is $W(x)$ but replace 1st column by $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

→ W_2 is $W(x)$ but replace 2nd column by $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

⋮

$$W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 1 & -\cos x & -\sin x \end{vmatrix} = 0 - \cos x(0 - \cos x) + \sin x(0 + \sin x)$$

$$= \cos^2 x + \sin^2 x = 1$$

$$W_2 = \begin{vmatrix} x & 0 & \sin x \\ 1 & 0 & \cos x \\ 0 & 1 & -\sin x \end{vmatrix} = x(0 - \cos x) + 0 + \sin x(1)$$

$$= -x\cos x + \sin x$$

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Example : Solve

1. $y''' - 6y'' + 12y' - 12y = \sqrt{x}\, e^x$

2. $x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 \ln x$

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Matrices

ما هي Matrix ؟

هي عبارة عن صفوف واحدة من الاعداد او المقدرات وتوضع داخل اقواس [ ] , ( )

وتسمى عادة بـ Capital letters مثل A,B,C

$$A = \begin{bmatrix} 1 & 3 \\ 5 & -2 \end{bmatrix} , B = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 0 & -1 \\ 1 & 5 & 4 \end{bmatrix}$$

In general we will write matrices as $A = (a_{ij})_{n \times m}$

$n \times m$: size of the matrix

n : num of rows

m : num of columns

a_{ij} : the entery in the i^{th} row and j^{th} column

Example : A $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ find

1. Size of the Matrix

   3x3

2. $a_{12} = 2$

3. $a_{21} = 4$

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Important facts and Definitions

1. $n \neq m$, A is called rectangular

2. $n = m$, A is called square

3. $n = 1$, A is called a row matrix (row vector)

ex: $A = (1\ 2\ 3\ 4)$

4. $m = 1$, A is called a column matrix (column vector)

ex: $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

5. **Zero Matrix** A is zero Matrix if $a_{ij} = 0$ for all i, j

ex. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

6. **Diagonal** $a_{nn} : a_{11}, a_{22}, a_{33}$

ex: $A \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ → diagonal

7. **Identity matrix** (I_n) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. **The transpose of A** $(A^T) = a_{ij(m \times n)}$

ex $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

$\rightarrow A^T = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B^T = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

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Operations on Matrices

1. scalar multiplication

$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ then $3A = \begin{bmatrix} 6 & 3 \\ 9 & 12 \end{bmatrix}$

2. Addition

* حتى نتمكن من جمع two matrices يجب ان تكونا من نفس الحجم

Example: $A \begin{bmatrix} 3 & -1 & 5 \\ 6 & -2 & 0 \end{bmatrix}$ $B \begin{bmatrix} 1 & 12 \\ 2 & 33 \end{bmatrix}$ $C \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

find 1. A+B $\begin{bmatrix} 4 & 0 & 7 \\ 8 & 1 & 3 \end{bmatrix}$

2. A+C undefined (different sizes)

3. Matrix multiplication

If $A = a_{ij}(n \times r)$, $B = b_{ij}(r \times m)$ then

$A.B = C_{ij}(n \times m)$

← اي ان ناتج ضرب two matrices دائما عبارة عن عدد صفوف الاولى وعدد اعمدة الثانية

$A \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ $B \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$ if $C = A.B$ then

$C \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ where

c_{11} (1st row in A).(1st column in b)

c_{12} (1st row in A).(2nd column in b)

c_{21} (2nd row in A).(1st column in b)

c_{22} (2nd row in A).(2nd column in b)

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Example $A\begin{bmatrix}1 & 2\\ 10 & 1\end{bmatrix}$ $B\begin{bmatrix}2\\ 1\end{bmatrix}$ find AB

$$AB = C = \begin{bmatrix}C_{11}\\ C_{21}\end{bmatrix} = \begin{bmatrix}4\\ 21\end{bmatrix}$$

Note: $AB \neq BA$

4. Inverse of Matrix

let $A\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ then $A^{-1} = \frac{1}{\det A}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

تبدیل اشارہ

تبدیل اماکن

Example $A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ find A^{-1}

$$\det(A) = 4 - 6 = -2$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix} = \begin{bmatrix}-2 & 1\\ \frac{3}{2} & \frac{-1}{2}\end{bmatrix}$$

Notes

1. if $\det(A) \neq 0$ then A has an inverse
2. if $\det(A) = 0$ then A has no inverse
3. $AA^{-1} = A^{-1}A = In = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

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Chapter 4 System of 1st order linear ODE

$$\left.\begin{aligned} a\,y_1 + b\,y_2 &= g_1 \\ c\,y_1 + d\,y_2 &= g_2 \end{aligned}\right\} \text{linear system}$$

This system can be written as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$\text{or} \quad A\ y = g$$

→ if $g = 0$ (zero matrix) then system is called homogeneous

Eigenvalues (λ) and Eigenvectors x_λ

Identity matrix

Rule 1 $\vec{x}_\lambda (A - \lambda \vec{I}) = 0$ →

Rule 2 $\det(A - \lambda I) = 0$ →

تستخدم لايجاد قيم $\vec{x}_\lambda$

تستخدم لايجاد قيم λ

In this chapter we will deal with 2x2 matrices only

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rightarrow \quad \lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\rightarrow \quad \vec{x}_\lambda = \begin{pmatrix} a \\ b \end{pmatrix}$$

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Examples : find eigen values and eigen vectors

1. $A=\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}$

→ to find λ: step 1 find $A-\lambda I$

$$\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}-\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}=\begin{pmatrix}2-\lambda & 1\\ 3 & 4-\lambda\end{pmatrix}$$

Step 2 $\det(A-\lambda I)=0$

$$\begin{vmatrix}2-\lambda & 1\\ 3 & 4-\lambda\end{vmatrix}=(2-\lambda)(4-\lambda)-3=0$$

$$=\lambda^2-2\lambda-4\lambda+\lambda^2-3=0$$

$$=\lambda^2-6\lambda+5=0$$

$$(\lambda-5)(\lambda-1)=0$$

$\boxed{\lambda=5}$ $\boxed{\lambda=1}$ eigen values

→ to find $\vec{x}_\lambda$

* إذا احتوى السؤال على λ_1, λ_2 فالسؤال يحتوي على $\vec{x}_{\lambda_1}, \vec{x}_{\lambda_2}$

step 1 $(A-\lambda_1 I)x_{\lambda_1}=0$

تعويض λ_1 في معادلة $A-\lambda I$

$$\begin{pmatrix}-3 & 1\\ 3 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$-3a+b=0$$
$$3a-b=0$$

$$b=3a$$

$$X_{\lambda_1}=\begin{pmatrix}a\\ 3a\end{pmatrix}=a\begin{pmatrix}1\\ 3\end{pmatrix}$$

if $a=1 \rightarrow X_{\lambda_1}=\begin{pmatrix}1\\ 3\end{pmatrix}$

يمكن تعويض اي قيمة كان a غير الصفر

step 2 $(A-\lambda_2 I)X_{\lambda_2}=0$

$$\begin{pmatrix}1 & 1\\ 3 & 3\end{pmatrix}\begin{pmatrix}c\\ d\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$c+d=0$$
$$d=-c$$

$$\vec{X}_{\lambda_2}=\begin{pmatrix}c\\ -c\end{pmatrix}=c\begin{pmatrix}1\\ -1\end{pmatrix}$$

if $c=1 \rightarrow X_{\lambda_2}=\begin{pmatrix}1\\ -1\end{pmatrix}$

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2. $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$

Step1 $A - \lambda I = \begin{pmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{pmatrix}$

step2 $\begin{vmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$(2-\lambda)(3-\lambda) - 0 = 0$

$6 - 2\lambda - 3\lambda + \lambda^2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\boxed{\lambda_1 = 2}$ $\boxed{\lambda_2 = 3}$

→ for $\lambda_1 = 2$ $\vec{x}_{\lambda_1} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$a + b = 0$ $\begin{pmatrix} a \\ -a \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$b = -a$

→ for $\lambda_2 = 3$

$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-c + 0 = 0$ $\begin{pmatrix} 0 \\ d \end{pmatrix} = d \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$c = 0$

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3. $A=\begin{pmatrix}2 & -13\\ 1 & -2\end{pmatrix}$

step 1 $A-\lambda I=\begin{pmatrix}2-\lambda & -13\\ 1 & -2-\lambda\end{pmatrix}$

step 2 $\begin{vmatrix}2-\lambda & -13\\ 1 & -2-\lambda\end{vmatrix}=0$

$(2-\lambda)(-2-\lambda)+13=0$

$-4-2\lambda+2\lambda+\lambda^2+13=0$

$\lambda^2+9=0$

$\boxed{\lambda_1=3i}$, $\boxed{\lambda_2=-3i}$

→ for $\lambda_1=3i$, $\vec{x}_{\lambda_1}=\begin{pmatrix}a\\ b\end{pmatrix}$

$\begin{pmatrix}2-3i & -13\\ 1 & -2-3i\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$(2-3i)a-13b=0$

$b=\frac{(2-3i)a}{13}$ → $\begin{pmatrix}a\\ \frac{(2-3i)}{13}a\end{pmatrix}=a\begin{pmatrix}1\\ \frac{2-3i}{13}\end{pmatrix}$

if $a=13$ → $x_{\lambda_1}=\begin{pmatrix}13\\ 2-3i\end{pmatrix}=\begin{pmatrix}13+0i\\ 2-3i\end{pmatrix}$

→ for $\lambda_2=-3i$

Rule: if $\lambda_1=\alpha+\beta i$
$\lambda_2=\alpha-\beta i$
and $x_{\lambda_1}=a+bi$
then $x_{\lambda_2}=a-ib$

$X_{\lambda_1}=\begin{pmatrix}13\\ 2\end{pmatrix}+i\begin{pmatrix}0\\ -3\end{pmatrix}$

→ $X_{\lambda_2}=\begin{pmatrix}13\\ 2\end{pmatrix}-i\begin{pmatrix}0\\ -3\end{pmatrix}$

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Repeated eigen values

$$\lambda_1 = \lambda_2 = \lambda$$

To find $X_{\lambda_1}, X_{\lambda_2}$ where $X_{\lambda_1}, X_{\lambda_2}$ are linearly independent

1. solve $(A - \lambda_1 I) X_{\lambda_1} = 0$ من خلالها نجد X_{λ_1} كما في الأسئلة السابقة

2. solve $(A - \lambda_2 I) X_{\lambda_2} = X_{\lambda_1}$

4. $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$

step1 $(A - \lambda I) = \begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix}$

step2 $\begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0$

$(2-\lambda)(4-\lambda) + 1 = 0$

$8 - 2\lambda - 4\lambda + \lambda^2 + 1 = 0$

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)(\lambda - 3) = 0$

$\lambda_1 = 3 \quad \lambda_2 = 3$

step3 For $\lambda_1 = 3 \quad \vec{X}_{\lambda_1} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-a + b = 0 \quad X_{\lambda_1} = \begin{pmatrix} a \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$a = b$

for $a = 1 \rightarrow X_{\lambda_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

step4 $X_{\lambda_2} = \begin{pmatrix} c \\ d \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$-c + d = 1$

$\boxed{d = 1 + c}$

$X_{\lambda_2} = \begin{pmatrix} c \\ 1+c \end{pmatrix}$

for $c = 1 \rightarrow X_{\lambda_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

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System of 1st order linear ODE (2x2)

consider the system

$$y_1'(t) = a\,y_1 + b\,y_2 + g_1(t)$$

$$y_2'(t) = c\,y_1 + d\,y_2 + g_2(t)$$

$$\rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

$$\rightarrow \vec{Y}(t) = A\,\vec{y}(t) + \vec{g}(t)$$

In this chapter we will study only systems with constant coefficients

1. Homogeneous system of 1st ODE

standard form:

$$\vec{Y}(t) = A\,y(t) \quad , \quad \vec{g}(t) = 0$$

G.S

- Case 1: $\lambda_1 \neq \lambda_2$ → $Y(t) = C_1 X_{\lambda_1} e^{\lambda_1 t} + C_2 X_{\lambda_2} e^{\lambda_2 t}$
- case 2: $\lambda_1 = \lambda_2$ → $y(t) = C_1 X_{\lambda_1} e^{\lambda_1 t} + C_2 (t X_{\lambda} e^{\lambda t} + \cancel{X_{\lambda_2}} e^{\lambda t})$
- case 3: $\lambda = \alpha \pm Bi$

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Examples

1. find G.S of $y_1' = 4y_1 - 3y_2$

$y_2' = 8y_1 - 6y_2$

step1 $A = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$

$(A - \lambda I) = \begin{pmatrix} 4-\lambda & -3 \\ 8 & -6-\lambda \end{pmatrix}$

step2 $\begin{vmatrix} 4-\lambda & -3 \\ 8 & -6-\lambda \end{vmatrix} = (4-\lambda)(-6-\lambda) + 24 = 0$

$\Rightarrow -24 - 4\lambda + 6\lambda + \lambda^2 + 24 = 0$

$\lambda^2 + 2\lambda = 0$

$\lambda_1 = 0, \lambda_2 = -2$

step3 $X\lambda_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$4a - 3b = 0 \quad b = \frac{4}{3}a$

$X_{\lambda_1} = \begin{pmatrix} a \\ \frac{4a}{3} \end{pmatrix} = a\begin{pmatrix} 1 \\ 4/3 \end{pmatrix}$

let $a = 3 \rightarrow X_{\lambda_1} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

step4 $X_{\lambda_2} = \begin{pmatrix} c \\ d \end{pmatrix}$

$\begin{pmatrix} 6 & -3 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$6c - 3d = 0 \quad X_{\lambda_2} = \begin{pmatrix} c \\ 2c \end{pmatrix} = c\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$d = 2c$ let $c = 1 \rightarrow X_{\lambda_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

step5 $Y(t) = C_1\begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{0t} + C_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$

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2. $y_1'(t) = y_1 - y_2$

$y_2'(t) = y_1 + 3y_2$

step1 $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$

$A - \lambda I = \begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix}$

step2 $\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = 0$

$= 3 - \lambda - 3\lambda + \lambda^2 + 1 = 0$

$= \lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)(\lambda - 2) = 0$

$\lambda_1 = 2 \quad \lambda_2 = 2$

step3 $X_{\lambda_1} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-a - b = 0 \quad b = -a \rightarrow X_{\lambda_1} = \begin{pmatrix} a \\ -a \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

let $a = 1 \rightarrow X_{\lambda_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

step4 $X_{\lambda_2} = \begin{pmatrix} c \\ d \end{pmatrix}$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$c + d = -1 \quad d = -1 - c \rightarrow X_{\lambda_2} = \begin{pmatrix} c \\ -1-c \end{pmatrix}$

let $c = 0 \Rightarrow X_{\lambda_2} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

G.S $\Rightarrow y = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + c_2\left(t\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}e^{2t}\right)$

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Case3 $\lambda = \alpha \pm \beta i$

Remember $X_{\lambda_1} = \vec{a} + i\vec{b}$

$X_{\lambda_2} = \vec{a} - i\vec{b}$

G.S $y = c_1 y_1 + c_2 y_2$

$\rightarrow y_1 = e^{\alpha t}(\vec{a}\cos\beta t - \vec{b}\sin\beta t)$

$\rightarrow y_2 = e^{\alpha t}(\vec{b}\cos\beta t + \vec{a}\sin\beta t)$

3. $y_1' = 3y_1 - 2y_2$

$y_2' = 4y_1 - 2y_2$

step1 $A - \lambda I = \begin{vmatrix} 3-\lambda & -2 \\ 4 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) + 8 = 0$

$\lambda^2 - 2\lambda + 5 = 0$

$D = 4 - 20 = -16$

$\lambda = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$\lambda_1 = 1 + 2i$

$\lambda_2 = 1 - 2i$

$\alpha = 1$, $\beta = 2$

step2 $X_{\lambda_1} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(2-2i)a - 2b = 0$

$b = (1-i)a$

$X_{\lambda_1} = a\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

let $a=1 \rightarrow X_{\lambda_1} = \begin{pmatrix} 1+0i \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$X_{\lambda_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$y_1 = e^t\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin 2t\right)$

$y_2 = e^t\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin 2t\right)$

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homeworks

1. $y_1' = y_1 - y_2$

   $y_2' = 5y_1 - 3y_2$

2. $y' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} y$

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2. Non homogeneous system

$$\vec{Y}' = A\vec{Y} + \vec{g}(t) \quad (g(t) \neq 0)$$

G.S $Y_h + Y_p$

Method 1 Undetermined coefficient

The same table and modification rules in 2nd order with replacing constants by constant vectors

Note: if one of the terms of Y_p appears in Y_h then multiply by t and add a constant vector

Example if $g(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$ and $Y_h = c_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{t}$

$\rightarrow Y_p = \vec{a} e^{2t}$ (,,c_0)

$\hookrightarrow Y_p = (t\vec{a} + \vec{b}) e^{2t}$

Examples

1. find a suitable form for Y_p

$$Y_1' = -3Y_1 + Y_2 - 6e^{3t}$$
$$Y_2' = Y_1 - 3Y_2 + 2e^{3t}$$

$$Y_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$$

$\rightarrow$ for Y_p $\quad \vec{g}(t) = \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{3t}$

$$Y_p = \vec{v} e^{3t} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{3t}$$

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2. $Y' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} Y + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$

find a suitable form for Y_p

$$Y_h = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$g(t) = \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$Y_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \vec{e^{-2t}} \quad (\text{مش صفر})$$

$$\hookrightarrow Y_p = \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) e^{-2t}$$

$$= (\vec{a}t + \vec{b}) e^{-2t}$$

3. Find a suitable form for Y_p

$$Y_1' = 5y_1 + 3y_2 - 2e^{-t} + 1$$

$$Y_2' = -y_1 + y_2 + e^{-t} - 5t + 7$$

$$Y_n = c_1 x_{\lambda_1} e^{2t} + c_2 x_{\lambda_2} e^{4t}$$

$$Y' = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} Y + \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ -5 \end{bmatrix} t + \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$Y_p = \vec{a} e^{-t} + bt + \vec{c}$$

$$= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{-t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

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Method 2 Variation of Parameters

Steps of solution

step 1 find the fundamental matrix $F(t)$ or Φ

if $Y_n = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t}$

$$= c_1\begin{pmatrix}e^{-2t}\\e^{-2t}\end{pmatrix} + c_2\begin{pmatrix}e^{-4t}\\-e^{-4t}\end{pmatrix}$$

then $F(t) = \begin{bmatrix}e^{-2t} & e^{-4t}\\ e^{-2t} & -e^{-4t}\end{bmatrix}$

step 2 find $F^{-}(t)$

step 3 find $F^{-}(t)_{(2\times2)} \cdot g(t)_{(2\times1)} = (2\times1)$

step 4 evaluate $\int F^{-}(t)\, g(t)$

step 5 $Y_p = F(t)\int F^{-}(t)\, g(t)$

Example solve the system

$$Y' = \begin{bmatrix}0 & 2\\ -1 & 3\end{bmatrix}y + \begin{bmatrix}1\\-1\end{bmatrix}e^{t}$$

$$y_n = c_1\begin{bmatrix}2\\1\end{bmatrix}e^{t} + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{2t}$$

step 1 $F(t) = \begin{bmatrix}2e^{t} & e^{2t}\\ e^{t} & e^{2t}\end{bmatrix}$

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Step 2 $\det(F(t)) = 2e^{3t} - e^{3t} = e^{3t}$

$$F^{-}(t) = \frac{1}{\det F}\begin{bmatrix} e^{2t} & -e^{2t} \\ -e^{t} & 2e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}$$

Step 3 $F^{-}g = \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}\begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} = \begin{bmatrix} 2 \\ -3e^{-t} \end{bmatrix}$

Step 4 $\int F^{-}g = \int \begin{bmatrix} 2 \\ -3e^{t} \end{bmatrix} dt = \begin{bmatrix} 2t \\ +3e^{-t} \end{bmatrix}$

Step 5 $Y_p = F(t) \int F^{-}g$

$$= \begin{bmatrix} 2e^{t} & e^{2t} \\ e^{t} & e^{2t} \end{bmatrix}\begin{bmatrix} 2t \\ +3e^{-t} \end{bmatrix} = \begin{bmatrix} 4te^{t} + 3e^{t} \\ 2te^{t} + 3e^{t} \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix} te^{t} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{t}$$

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QUESTIONS

Calculus Sec. Exam

First sem. 2020/2021

By Anas Abu Zahra

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A particular solution of the non-homogeneous equation:

$$y''' + 4y'' = 1/x$$

is $y_p(x) =$

a) $\int \frac{-4x-1}{16x}dx + x\int \frac{1}{4x}dx + e^{4x}\int \frac{1}{16xe^{4x}}dx$

b) $\int \frac{-4x-1}{16x}dx + x\int \frac{1}{4x}dx + e^{-4x}\int \frac{1}{16xe^{-4x}}dx$

c) $\int \frac{4x+1}{16x}dx + x\int \frac{1}{4x^2}dx + e^{-4x}\int \frac{1}{16xe^{-4x}}dx$

d) $\int \frac{4x+1}{16x}dx + x\int \frac{1}{4x}dx + e^{4x}\int \frac{1}{16xe^{4x}}dx$

e) $\int \frac{-4x+1}{16x}dx + x\int \frac{1}{4x}dx + e^{-4x}\int \frac{1}{16xe^{4x}}dx$

○ a

○ b

○ c

A particular solution of the non-homogeneous ODE:

$$xy'' + y' - \frac{4}{x}y = \tan x$$

is $y_p(x) =$

a) $\frac{x^2}{4}\int \frac{1}{x^2}\tan x\,dx - \frac{1}{4x^2}\int x^2 \tan x\,dx$

b) $\frac{x^2}{4}\int \frac{1}{x}\tan x\,dx - \frac{1}{4x^2}\int x^3 \tan x\,dx$

c) $-\frac{1}{4x^2}\int \frac{1}{x^2}\tan x\,dx + \frac{x^2}{4}\int x^2 \tan x\,dx$

d) $-\frac{x^2}{4}\int \frac{1}{x}\tan x\,dx + \frac{1}{4x^2}\int x^3 \tan x\,dx$

e) $-\frac{x^2}{4}\int \frac{1}{x^2}\tan x\,dx + \frac{1}{x^2}\int x^2 \tan x\,dx$

○ a

○ b

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7

d

out of

n

The eigenvalues of

$$A = \begin{bmatrix} -5 & 2 \\ -2 & 0 \end{bmatrix}$$

are $\lambda =$

a) 2, -3
b) -2, 3
c) 1, 4
d) -1, -4
e) 2, 3

○ a

○ b

A suitable form of the particular solution of the non-homogeneous ODE:

$$y'' - 6y' + 9y = xe^{3x} + \cos x$$

is $y_p(x) =$

a) $(Ax + B)e^{3x} + C\cos x + D\sin x$
b) $(Ax + B)e^{3x} + C\cos x$
c) $(Ax^3 + Bx^2)e^{3x} + C\cos x$
d) $(Ax^3 + Bx^2)e^{3x} + C\cos x + D\sin x$
e) $(Ax^2 + Bx)e^{3x} + C\cos x + D\sin x$

○ a

○ b

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Given $y_1(x) = x$ is a solution of

$$x^4y'' - x(x+4)y' + (x+4)y = 0, x > 0.$$

A second linearly independent solution of this ODE is $y_2(x) =$

a) $x\int \frac{1}{x^2} e^{\int x(x+4)dx} dx$

b) $x\int \frac{1}{x} e^{\int \frac{x+4}{x^3} dx} dx$

c) $x\int \frac{1}{x^2} e^{\int \frac{x+4}{x^3} dx} dx$

d) $x\int \frac{1}{x^2} e^{-\int x(x+4)dx} dx$

e) $x\int \frac{1}{x^2} e^{-\int \frac{x+4}{x^3} dx} dx$

- a
- b
- c

The general solution of the homogeneous system $y_1' = -3y_1 + y_2$, $y_2' = y_1 - 3y_2$ is $\vec{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$.

In such a case a suitable form of the particular solution of

$y_1' = -3y_1 + y_2 - 5e^{-2t}$, $y_2' = y_1 - 3y_2 - 3t$ is $\vec{y}^{(p)} =$

a) $\vec{A}t^2 + \vec{B}t + \vec{C}te^{-2t}$

b) $\vec{A}t + \vec{B} + \vec{C}te^{-2t}$

c) $\vec{A}t^2 + \vec{B}t + \vec{C}e^{-2t} + \vec{D}te^{-2t}$

d) $\vec{A}t + \vec{B} + \vec{C}e^{-2t}$

e) $\vec{A}t + \vec{B} + \vec{C}e^{-2t} + \vec{D}te^{-2t}$

- a
- b
- c

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Question 4

Not yet answered

Marked out of 1.00

Flag question

The general solution of the homogeneous system $\begin{matrix} y_1' = y_1 + y_2 \\ y_2' = 3y_1 - y_2 \end{matrix}$

is $\vec{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$.

In such a case the particular solution of $\begin{matrix} y_1' = y_1 + y_2 + 3 \\ y_2' = 3y_1 - y_2 - 2t \end{matrix}$ is

a) $\frac{1}{4}\begin{bmatrix} e^{2t} & -e^{2t} \\ 3e^{-2t} & e^{-2t} \end{bmatrix} \int \begin{bmatrix} e^{-2t} & e^{2t} \\ -3e^{-2t} & e^{2t} \end{bmatrix} \begin{bmatrix} 3 \\ -2t \end{bmatrix} dt$

b) $\begin{bmatrix} e^{-2t} & e^{2t} \\ -3e^{-2t} & e^{2t} \end{bmatrix} \int \frac{1}{4}\begin{bmatrix} e^{2t} & -e^{2t} \\ 3e^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 3 \\ -2t \end{bmatrix} dt$

c) $\frac{1}{4}\begin{bmatrix} e^{2t} & e^{2t} \\ -3e^{-2t} & e^{-2t} \end{bmatrix} \int \begin{bmatrix} e^{-2t} & e^{2t} \\ -3e^{-2t} & e^{2t} \end{bmatrix} \begin{bmatrix} 3 \\ -2t \end{bmatrix} dt$

d) $\begin{bmatrix} e^{-2t} & e^{2t} \\ -3e^{-2t} & e^{2t} \end{bmatrix} \int \frac{1}{4}\begin{bmatrix} e^{-2t} & -e^{2t} \\ 3e^{-2t} & e^{2t} \end{bmatrix} \begin{bmatrix} 3 \\ -2t \end{bmatrix} dt$

e) $\begin{bmatrix} e^{-2t} & e^{2t} \\ -3e^{-2t} & e^{2t} \end{bmatrix} \int \begin{bmatrix} e^{2t} & e^{2t} \\ -3e^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 3 \\ -2t \end{bmatrix} dt$

○ a

○ b

○ c

5

d

out of

A suitable form of the particular solution of the non-homogeneous ODE:

$$y^{(6)} + 2y^{(4)} + y'' = 3 + \cos x$$

is $y_p(x) =$

a) $Ax^2 + Bx^2 \cos x + Cx^2 \sin x$

b) $A + B\cos x + C\sin x$

c) $Ax^2 + Bx\cos x + Cx\sin x$

d) $Ax + Bx^2 \cos x + Cx^2 \sin x$

e) $A + Cx\cos x + Dx\sin x$

○ a

○ b

○ c

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The functions $y_1(x) = x^2$ and $y_2(x) = x^2 \ln x$ are solutions of the ODE:

a) $x^2y'' + 3xy' + 4y = 0, x > 0.$
b) $x^2y'' + 4xy' + 4y = 0, x > 0.$
c) $x^2y'' - 3xy' + 4y = 0, x > 0.$
d) $x^2y'' - 5xy' + 4y = 0, x > 0.$
e) $x^2y'' - 4xy' + 4y = 0, x > 0.$

○ a

○ b

The solution of the initial value problem:

$$y'' + 9y = 0, \quad y(0) = 0, y'(0) = -3$$

is $y(x) =$

a) $-\cos 3x$
b) $-3 \sin 3x$
c) $-\sin 3x$
d) $\sin 3x$
e) $3 \sin x$

○ a

○ b

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Ch2+3+4 Pastpapers and suggested

Done by : Anas AbuZahra

Q1 The second ODE whose general solution is $y = c_1x^2 + c_2x^2\ln x$

A. $x^2y'' - 3xy' + 4y = 0$

B. $x^2y'' - 5xy' + 9y = 0$

C. $x^2y'' - 7xy' + 16y = 0$

d. $y'' - 4y' + 4y = 0$

e. none

$r_1 = 2, r_2 = 2$

Solution $(r-2)^2 = 0$

$r^2 - 4r + 4 = 0$

$r^2 - r + r - 4r + 4 = 0$

$r(r-1) - 3r + 4 = 0$

$x^2y'' - 3xy' + 4y = 0$

[a]

Q2 The form of the particular solution y_p for the ODE

$y''' - y'' + 4y' - 4y = 2e^x + 2\cos 2x$

a. $y_p = Ae^x + (B\sin 2x + D\cos 2x)x$

b. $y_p = Axe^x + B\sin 2x + D\cos 2x$

c. $y_p = Axe^x + (B\sin 2x + D\cos 2x)x$

d. $y_p = Ae^{2x} + (B\sin 2x + D\cos 2x)x$

e. none

solution: $r^3 - r^2 + 4r - 4 = 0$

$(r-1)(r^2+4) = 0$

$r = 1, r = \pm 2i$

	1	-1	4	-4
r=1		1	0	4
	1	0	4	0

$y_h = c_1e^x + c_2\cos 2x + c_3\sin 2x$

$y_{p_1} = Ae^x$ (مكرر) $\rightarrow y_{p_1} = Axe^x$

$y_{p_2} = B\cos 2x + C\sin 2x$ (مكرر) $\rightarrow y_{p_2} = x(B\cos 2x + C\sin 2x)$

$y_p = y_{p_1} + y_{p_2}$ [C]

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Q_3 the general solution of $xy^{(4)} + 6y^{(3)} = 0$ is

a. $y = c_1 + c_2 x + c_3 x^2 + c_4 x^{-3}$

b. $y = c_1 + c_2 e^x + c_3 e^{2x} + c_4 e^{-3x}$

c. $y = c_1 x + c_2 e^x + c_3 e^{2x} + c_4 e^{-3x}$

d. $y = c_1 e^{3x} + c_2 e^x + c_3 e^{2x} + c_4 e^{-3x}$

e. None

Solution:
بما ان السؤال non constant coeff. اذا يجب ان يكون جميع الاجابات على شكل x^r وليس e^{rx} وعليه فان الاجابة الصحيحة [a] للتأكد؟

$$x^4 y^{(4)} + 6x^3 y^{(3)} = 0$$

$$r(r-1)(r-2)(r-3) + 6r(r-1)(r-2) = 0$$

$$r(r-1)(r-2)(r-3+6) = 0$$

$r=0 \quad r=1 \quad r=2 \quad r=-3$

$y_1 = 1 \quad y_2 = x \quad y_3 = x^2 \quad y_4 = x^{-3}$

Q_4 if $A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$ then the eigen valves of A are

a. ± 2

b. 1

c. 5

d. 1 and 5

e. none

Note: نستثني اي خيار يحتوي قيمة واحدة الا اذا كانت مكررة

solution $\begin{vmatrix} 3-\lambda & 4 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 4 = 0$

$9 - 6\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 6\lambda + 5 = 0$

$(\lambda - 5)(\lambda - 1) = 0$

$\lambda = 5 \quad \lambda = 1$

[d]

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Q5 consider $y'''+y'=\sec x$, $y_n=c_1+c_2\cos x+c_3\sin x$ and $W[1,\cos x,\sin x]=1$, then $W_1=$

a. 1

b. $-\sin x$

c. $2\cos x$

d. $-2\cos x$

e. none

$$W_1=\begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 1 & -\cos x & -\sin x \end{vmatrix}=0+\cos^2x+\sin^2x=1$$

[a]

Q6 $y'=\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}y$, if repeated eigenvalue $\lambda_1=3$, then G.S is

a. $y=c_1\begin{bmatrix}1\\-1\end{bmatrix}e^{3t}+c_2e^{3t}\begin{bmatrix}1\\-1\end{bmatrix}t$

b. $y=c_1\begin{bmatrix}1\\-1\end{bmatrix}e^{3t}+c_2e^{3t}\left(\begin{bmatrix}1\\-1\end{bmatrix}t+\begin{bmatrix}0\\1\end{bmatrix}\right)$

c. $y=c_1\begin{bmatrix}1\\-1\end{bmatrix}e^{3t}+c_2e^{3t}\begin{bmatrix}0\\1\end{bmatrix}$

d. $y=c_1\begin{bmatrix}1\\-1\end{bmatrix}e^{3t}+c_2e^{3t}\begin{bmatrix}0\\1\end{bmatrix}t$

نلاحظ انه لا داعي لايجاد قيم X_h حيث انها مشتركة في جميع الاجابات وبالتالي نبحث فقط عن الاجابة الصحيحة بالشكل الصحيح وهي [b]

Q7 the general solution of $y''+4y'+y=0$ is $y(x)=$

a. $c_1e^{-2x}\cos\sqrt{3}x+c_2e^{-2x}\sin\sqrt{3}x$

b. $c_1e^{(-2-\sqrt{3})x}+c_2e^{(-2+\sqrt{3})x}$

c. $c_1e^{-2x}+c_2e^{2x}$

d. $c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x$

solution $r^2+4r+1=0$

$D=16-4=12$

$r=\frac{-4\pm\sqrt{12}}{2}$

$r=\frac{-4\pm\sqrt{4.3}}{2}=\frac{-4\pm2\sqrt{3}}{2}$

$=-2\pm\sqrt{3}$

[b]

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Q8 if $y_1 = x, y_2 = x^2$ are two linearly independent solutions of $x^2y'' - 2xy' + 2y = 0$ then Particular solution $y_p =$

@f $x^2y'' - 2xy' + 2y = r(x)$

a. $-x\int \frac{r(x)}{x^2}dx + x^2\int \frac{r(x)}{x^3}dx$

b. $x^2\int \frac{r(x)}{x^2} + -x\int \frac{r(x)}{x^3}dx$

c- $\int \frac{r(x)}{x^2}dx + \int \frac{r(x)}{x^3}dx$

d- $-x\int r(x)dx + x^2\int \frac{r(x)}{x}dx$

لازم تقسمها على x^2 عشان $r(x)$ تتشال

Solution $W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$

$y_p = -y_1\int \frac{y_2\, r}{W}\cdot x^2 + y_2\int \frac{y_1\, r}{W}\cdot x^2$

$= -x\int \frac{x^2 r}{x^2 \cdot x^2} + x^2\int \frac{x\, r}{x^2 \cdot x^2}$

[d]

Q9 $y^{(4)} + 4y'' = \sin^2 x$, then a suitable form of y_p is

a. $ax + x(b\cos 2x + c\sin 2x)$

b. $ax^2 + x(b\cos 2x + c\sin 2x)$

c- $ax^2 + b\cos 2x + c\sin 2x$

d- $ax + b\cos 2x + c\sin 2x$

solution

$r^4 + 4r^2 = 0$

$r^2(r^2 + 4) = 0$

$r = 0, r = \pm 2i$

$\rightarrow y_h = c_1 x^0 + c_2 x x^0 + c_3\cos 2x + c_4 \sin 2x$

$\rightarrow y'' + 4y'' = \frac{1}{2} - \frac{1}{2}\cos 2x$

$g_1 \quad g_2$

$y_{p_1} = a\,(\bar{a}, \bar{b}) \rightarrow y_{p_1} = ax\,(\bar{a}, \bar{b}) \rightarrow y_{p_1} = ax^2$

$y_{p_2} = b\cos 2x + c\sin 2x\,(\bar{a}, \bar{b}) \rightarrow y_{p_2} = x(b\cos 2x + c\sin 2x)$

$y_p = y_{p_1} + y_{p_2}$ [b]

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Q10 if $\{e^x, e^{-x}, e^{2x}\}$ is a basis of solutions of

$$y''' + ay'' + by' + cy = 0$$, then $a =$

a. 2

b. -2

c. -1

d. 1

e. none

$r_1 = 1, r_2 = -1, r_3 = 2$

$(r-1)(r+1)(r-2) = 0$

$(r^2-1)(r-2) = 0$

$r^3 - 2r^2 - r + 2 = 0$

$y''' - 2y'' - y' + 2y = 0$

$\boxed{a = -2}$ $\boxed{b}$

Q11 the solution of $y'' - 2y' - 3y = 0$, $y(0) = 1$, $y'(0) = -5$ is $y =$

a. $2e^x - e^{-3x}$

b. $e^{-x} - e^{3x}$

c. $2e^{-x} - e^{3x}$

d. $-2e^x + 3e^{-x}$

e. none

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = 3 \quad r = -1$

$y = c_1 e^{3x} + c_2 e^{-x}$

$\rightarrow y(0) = \boxed{c_1 + c_2 = 1} \rightarrow c_1 = 1 - c_2$

$y' = 3c_1 e^{3x} - c_2 e^{-x}$

$\rightarrow y'(0) = \boxed{3c_1 - c_2 = -5}$

$3(1 - c_2) - c_2 = -5$

$3 - 3c_2 - c_2 = -5$

$-4c_2 = -8$

$c_2 = 2$

$c_1 = -1$

ans is $\boxed{c}$

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Q_{12} if $y_p(t) = \sin 2t\, v_1 + \cos 2t\, v_2$ is particular solution of $y'' + 4y = \csc 2t$, find v_2 using variation of parameters

a. $\frac{1}{2}\ln|\cos 2t|$

b. $\frac{1}{2}\ln|\sin 2t|$

c. $\frac{-1}{2}t$

d. $\frac{-1}{2}\ln|\cos 2t|$

e. $\frac{1}{2}t$

$r^2 + 4 = 0$

$r = \pm 2i$

$y_1 = \cos 2t$

$y_2 = \sin 2t$

$$W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = 2$$

$$y_p = -\cos 2t \int \frac{\sin 2t}{2} \cdot \frac{1}{\sin 2t} + \sin 2t \int \frac{\cos 2t}{2} \cdot \frac{1}{\sin 2t}$$

$$= -\cos 2t \cdot \frac{1}{2}t$$

so $v_2 = \frac{-1}{2}t$ [c]

Q_{13} The linear system whose general solution $y = c_1 e^{2t}\begin{bmatrix} 4 \\ 0 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

solution $y' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} y$ نلاحظ ان لدينا 4 مجاهيل وبالتالي سنحتاج 4 معادلات

$\lambda_1 = 2$, $\lambda_2 = 3$, $X_{\lambda_1} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $X_{\lambda_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$X_{\lambda_1}(A - \lambda_1 I) = 0$

$$\begin{bmatrix} a-2 & b \\ c & d-2 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4(a-2) + b(0) = 0 \rightarrow$ eq 1

$a = 2$

$4c + (d-2)(0) = 0 \rightarrow$ eq 2

$c = 0$

$X_{\lambda_2}(A - \lambda_2 I) = 0$

$$\begin{bmatrix} a-3 & b \\ c & d-3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2(\overset{2}{\nearrow} a - 3) + b = 0 \rightarrow$ eq 3

$b = 2$

$2\overset{0}{\nearrow} c + d - 3 = 0 \rightarrow$ eq 4

$d = 3$

$$A = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$$

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Power Series

Power Series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

* if $c = 0$ then the power series called Maclaurin series

Remark:

1. $\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots\cdots$

2. تستخدم لتغيير العداد مع المحافظة على قوة القوس $(x-c)^n$

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + \sum_{n=1}^{\infty} a_n (x-c)^n$$
$$= a_0 + a_1 + \sum_{n=2}^{\infty} a_n (x-c)^n$$

3. تستخدم لتغيير العداد مع اجراء تعديل على قوة القوس $(x-c)^n$

في حالة القيام باي تعديل على العداد سواء بالطرح او الاضافة نقوم باجراء عكسه على n داخل السيريز

Example : $\sum_{n=3}^{\infty} n^2 a_{(n-1)} (x+4)^{n+2}$

$\sum_{n=0}^{\infty} (n+3)^2 a_{(n+2)} (x+4)^{n+5}$ ← نلاحظ ان القوة قد تغيرت

$\sum_{n=5}^{\infty} (n-2)^2 a_{(n-3)} (x+4)^n$

4. **To add/subtract two power series or more, they must have**

- the same power on (x) or $(x-c)$
- the same starting index

اي يجب ان يبدأوا من نفس الرقم في العداد

$$\sum_{n=0}^{\infty} a_n x^n \pm \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} x^n (a_n \pm b_n)$$

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Example: find $\sum_{n=1}^{\infty} 2a_n x^{n-1} + \sum_{n=1}^{\infty} a_n x^{n+1}$

Step 1 يجب تحقيق الشرط الاول وهو جعل القوة متساوية

$$\sum_{n=1}^{\infty} a_n x^{n+1} = \sum_{n=3}^{\infty} a_{(n-2)} x^{n-1}$$

Step 2 يجب تحقيق الشرط الثاني وهو جعل العداد يبدأ من نفس الرقم ونستخدم قاعدة رقم 2 حتى لا تتعدل القوى

$$\sum_{n=1}^{\infty} 2a_n x^{n-1} = 2a_1 x^0 + 2a_2 x + \sum_{n=3}^{\infty} 2a_n x^{n-1}$$

Step3 نقوم بعملية الجمع

$$2a_1 + 2a_2 x + \sum_{n=3}^{\infty} 2a_n x^{n-1} + \sum_{n=3}^{\infty} a_{(n-2)} x^{n-1} =$$

$$2a_1 + 2a_2 x + \sum_{n=3}^{\infty} x^{n-1} (2a_n + a_{(n-2)})$$

5. قيم الpower series لاهم الاقترانات

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

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let $y'' + p(x)y' + r(x)y = 0$

we can assume G.S $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

Types of x_0

Ordinary

$p(x), r(x)$ ↑

تسمى x_0 بهذا الاسم اذا كان الافتران متصلاً عندها

→ $\lim_{x \to x_0} p(x) =$ exist

→ $\lim_{x \to x_0} r(x) =$ exist

singular

$p(x), r(x)$ ↑

تسمى النقطة x_0 بهذا الاسم اذا لم يكن الافتران متصلاً عندها

→ $\lim_{x \to x_0} p(x) =$ doesn't exist

or → $\lim_{x \to x_0} r(x) =$ doesn't exist

اي يكفي ان تكون احدى النهايتين dne حتى تسمى singular

← نتوقع عادة $p(x), r(x)$ عندها مقام singular points

regular singular

if $\lim_{x \to x_0} (x - x_0) p(x)$ exist

and $\lim_{x \to x_0} (x - x_0)^2 r(x)$ exist

← المهم ان تكون النهايتين موجودات ولا يشترط ان تكون اجابتهم متساوية

irregular singular

اذا كانت احدى النهايتين doesn't exist او كلاهما

← تعتبر $\pm\infty$ احد اشكال DNE

Exaples: find all singular points of

1. $x(x-1)y'' + xy' + 4y = 0$

$$y'' + \frac{xy'}{(x)(x-1)} + \frac{4y}{(x-1)x} = 0$$

$p(x) = \frac{1}{x-1}$

$r(x) = \frac{4}{(x)(x-1)}$

> نتوقع ان singular عند اصفار المقام $x_0 = 0, 1$

وكنتحقق من جواب النهاية

check $\lim_{x \to x_0} p(x), r(x)$

$x_0 = 0$

$\lim_{x \to 0} \frac{1}{x-1} = -1$

$\lim_{x \to 0} \frac{4}{x(x-1)} = \pm\infty$ (dne)

so singular

$x_0 = 1$

$\lim_{x \to 1} \frac{1}{x-1} = \pm\infty$ (dne)

so singular

→ Ordinary points = $\mathbb{R} - \{0, 1\}$

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2. $xy'' + \frac{3x^2}{x-1}y' + \frac{1}{x-2}y = 0$

$$y'' + \frac{3x^2}{(x)(x-1)}y' + \frac{1}{x(x-2)}y = 0$$

$$P(x) = \frac{3x}{(x-1)}$$

$$r(x) = \frac{1}{(x)(x-2)}$$

check $\lim_{x \to x_0} P(x), r(x)$

$x_0 = 1$

$\lim_{x \to 1} \frac{3x}{x \to 1} = \pm\infty$

so singular

$x_0 = 0$

$\lim_{x \to 0} \frac{1}{x(x-2)} = \pm\infty$

so singular

$x_0 = 2$

$\lim_{x \to 2} \frac{1}{(x)(x-2)} = \pm\infty$

so singular

Singular Points $x_0 = 0, 1, 2$

Ordinary Points $x_0 = \mathbb{R} - \{0, 1, 2\}$

3. find singular points and classify them

$x^2(1-x)y'' + (x-2)y' - 3xy = 0$

$$P(x) = \frac{x-2}{x^2(1-x)}, \quad r(x) = \frac{-3x}{x^2(1-x)} = \frac{-3}{(x)(1-x)}$$

Singular points $x_0 = 0, 1$

→ for $x_0 = 0$

$$\lim_{x \to 0} (x - x_0)P(x) = \lim_{x \to 0} x\frac{x-2}{x^2(1-x)} = \pm\infty \text{ (dne)}$$

so $x_0 = 0$ is irregular singular point

→ for $x_0 = 1$

$$\lim_{x \to 1} (x - x_0)P(x) = \lim_{x \to 1} \frac{(x-1)^{-1}(x-2)}{x^2(1-x)} = 1 \text{ (exist)}$$

$$\lim_{x \to 1} (x - x_0)^2 r(x) = \lim_{x \to 1} (x-1)^2 \cdot \frac{-3}{x(1-x)} = 0 \text{ (exist)}$$

so $x_0 = 1$ is regular singular point

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4. $xy'' + \sin x\, y' + \frac{x^2}{1-x} y = 0$

$P(x) = \frac{\sin x}{x}$, $r(x) = \frac{x}{1-x}$

* check $x_0 = 0, 1$ if singular or not

$x_0 = 0$ → $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (exist)

→ $\lim_{x\to 0} \frac{x}{1-x} = 0$ (exist)

so $x_0 = 0$ is ordinary

!! بالرغم من كونها صفر مقام

$x_0 = 1$ → $\lim_{x\to 1} \frac{x}{1-x} = \pm\infty$

so $x_0 = 1$ is singular

* check $x_0 = 1$ is singular (regular or irregular)

→ $\lim_{x\to 1} (x-1)\frac{\sin x}{x} = 0$ (exist)

→ $\lim_{x\to 1} (x-1)^2 \frac{-x}{1-x} = 0$ (exist)

so $x_0 = 1$ is a regular singular point

homework

5. $x^2(1-x^2)y'' + \frac{2}{x} y' + 4y = 0$

6. $x\sin x\, y'' + 3y' + xy = 0$

solution for Q6 $x_0 = 0$ is irregular singular

$x_0 = n\pi$ are regular singular

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how to derivate Power Series ?

Case 1: $y = \sum_{n=0}^{\infty} a_n x^n$

$$\rightarrow y' = \sum_{n=1}^{\infty} a_n \, n \, x^{(n-1)}$$

$$\rightarrow y'' = \sum_{n=2}^{\infty} a_n \, n(n-1) \, x^{n-1}$$

انتبه!!
الاشتقاق دائما بدلالة x
اي ان x هو المتغير و n ثابت
ثابت لذلك a_n لا تتغير عند الاشتقاق

case 2: $y = \sum_{n=0}^{\infty} a_n \, x^{n+r}$

$$\rightarrow y' = \sum_{n=0}^{\infty} a_n \, (n+r) \, x^{n+r-1}$$

$$\rightarrow y'' = \sum_{n=0}^{\infty} a_n \, (n+r)(n+r-1) \, x^{n+r-2}$$

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Series solution near ordinary point

let $y'' + P(x)y' + r(x)y = 0$

the DE has a power series solution of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1 + a_1 y_2$$

where - x_0 is ordinary point

- a_0 and a_1 are constants
- y_1, y_2 are linearly independent solutions

Q1 find a series solution for $y'' + y = 0$ near $x_0 = 0$

* Since x_0 is ordinary point so $y = \sum_{n=0}^{\infty} a_n x^n$

$\rightarrow y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$

$\rightarrow y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$

* Now plug y, y'' into ODE

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{(n+2)} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left((n+2)(n+1) a_{(n+2)} + a_n \right) x^n = 0$$

$$(n+2)(n+1) a_{(n+2)} + a_n = 0$$

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}$$ → Recurrance relation

(داخل الكتب الكبير بدلالة الصغير)

Note: our aim is to find a general form of a_n

$\rightarrow$ at $n = 0$

$$a_2 = \frac{-a_0}{2.1} = \frac{-a_0}{2!}$$

$\rightarrow$ at $n = 1$

$$a_3 = \frac{-a_1}{3.2} = \frac{-a_1}{3!}$$

$\rightarrow$ at $n = 2$

$$a_4 = \frac{-a_2}{4.3} = \frac{a_0}{4.3.2!} = \frac{a_0}{4!}$$

$\rightarrow$ at $n = 3$

$$a_5 = \frac{-a_3}{5.4} = \frac{a_1}{5.4.3!} = \frac{a_1}{5!}$$

$\rightarrow$ at $n = 4$

$$a_6 = \frac{-a_4}{6.5} = \frac{-a_0}{6!}$$

$\rightarrow$ at $n = 5$

$$a_7 = \frac{-a_1}{7!}$$

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$$a_n = \begin{cases} \frac{(-1)^k a_0}{2k!}, & n = 2k \\ \frac{(-1)^k a_1}{(2k+1)!}, & n = 2k+1 \end{cases}$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k a_0}{2k!} x^n + \sum_{k=0}^{\infty} \frac{(-1)^k a_1}{(2k+1)!} x^n$$

$$= a_0 \cos x + a_1 \sin x$$

Q2 The recurrence relation of the power series solution of $y'' - xy' - y = 0$ near $x_0 = 0$ is

A. $a_{n+2} = \frac{a_n}{n+1}$

c. $a_{n+2} = \frac{-a_n}{n+2}$

d. $a_{n+2} = \frac{a_n}{n+2}$

b. $a_{n+2} = \frac{a_n}{(n+2)(n+1)}$

solution $x_0 = 0$ is ordinary point

so $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_{(n+1)} (n+1) x^n$

$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} a_{(n+2)} (n+2)(n+1) x^n$

$\sum_{n=0}^{\infty} a_{(n+2)} (n+2)(n+1) x^n - \sum_{n=0}^{\infty} a_{(n+1)} (n+1) x^{n+1} - \sum_{n=0}^{\infty} a_n x^n$

حطت الاس مشان x بس

$\sum_{n=0}^{\infty} a_{(n+2)} (n+2)(n+1) x^n - \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n$

$2a_2 + \sum_{n=1}^{\infty} a_{(n+2)} (n+2)(n+1) x^n - \sum_{n=1}^{\infty} n x^n a_n - a_0 - \sum_{n=1}^{\infty} a_n x^n$

$2a_2 - a_0 + \sum_{n=1}^{\infty} x^n \left((n+1)(n+2) a_{n+2} - n a_n - a_n \right)$ $\quad -a_n(n+1)$

$(n+1)(n+2) a_{n+2} = a_n(n+1)$

$\frac{(n+1)(n+2)}{(n+1)(n+2)} a_{n+2} = \frac{a_n(n+1)}{(n+1)(n+2)} = \frac{a_n}{(n+2)}$

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Q_3 if the power series is used to solve DE $y'' + (1+x^2)y = 0$ about $x_0 = 0$, then a_2 and a_3 are

A. $a_2 = \frac{-a_0}{3}$, $a_3 = \frac{-a_1}{3}$

b. $a_2 = \frac{-a_1}{3}$, $a_3 = \frac{-a_0}{2}$

c. $a_2 = \frac{-a_0}{2}$, $a_3 = \frac{-a_1}{6}$

d. $a_2 = \frac{-a_0}{2}$, $a_3 = \frac{-a_1}{2}$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n x^{n-1} a_n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) x^{n-2} a_n$$

$$\sum_{n=2}^{\infty} n(n-1) x^{n-2} a_n + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

(y → $\sum a_n x^n$; $x^2 y$ → $\sum a_n x^{n+2}$)

$$\sum_{n=0}^{\infty} (n+2)(n+1) x^n a_{(n+2)} + \sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_{(n-2)} x^n = 0$$

$$2a_2 + 6x a_3 + a_0 + a_1 x + \sum_{n=2}^{\infty} (n+2)(n+1) x^n a_{(n+2)} + \sum_{n=2}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_{(n-2)} x^n = 0$$

$$(2a_2 + a_0) + x(6a_3 + a_1) +$$

$$2a_2 + a_0 = 0 \qquad 6a_3 + a_1 = 0$$

$$\boxed{a_2 = -\frac{a_0}{2}} \qquad \boxed{a_3 = -\frac{a_1}{6}}$$

ans is c

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Q_4 find P.S solution for $y''-y=0$ near $x_0=0,1$

$x_0=0$ is ordinary point so $y=\sum_{n=0}^{\infty} a_n x^n$

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2}a_n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)x^n a_{(n+2)} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} x^n\left(a_{n+2}(n+2)(n+1) - a_n\right) = 0$$

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \rightarrow \text{reccurrance relation}$$

$\rightarrow n=0$

$$a_2 = \frac{a_0}{2!}$$

$\rightarrow n=1$

$$a_3 = \frac{a_1}{3!}$$

$\rightarrow n=2$

$$a_4 = \frac{a_2}{4.3} = \frac{a_0}{4!}$$

$\rightarrow n=3$

$$a_5 = \frac{a_1}{5!}$$

$\rightarrow n=4$

$$a_6 = \frac{a_0}{6!}$$

$\rightarrow n=5$

$$a_7 = \frac{a_1}{7!}$$

$$a_n = \begin{cases} \frac{a_0}{(2k)!}, & n=2k \\ \frac{a_1}{(2k+1)!}, & n=2k+1 \end{cases}$$

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} \frac{a_0 x^{2k}}{2k!} + \sum_{n=0}^{\infty} \frac{a_1 x^{2k+1}}{(2k+1)!}$$

$$= a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{2k!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

for $x_0=1$ ← ordinary

$$y = a_0 \sum_{k=0}^{\infty} \frac{(x-1)^{2k}}{2k!} + a_1 \sum_{k=0}^{\infty} \frac{(x-1)^{2k+1}}{(2k+1)!}$$

اي نفس الحل لكن $(x-1)$ بدلاً من x

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Q5. find recurrance relation for $y''-y'+xy=0$ near $x=0$

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2}a_n - \sum_{n=1}^{\infty} n x^{n-1}a_n + \sum_{n=0}^{\infty} x^{n+1}a_n$$

نجعل كافة الاسس x^n

$$\rightarrow \sum_{n=0}^{\infty}(n+2)(n+1)x^n a_{n+2} - \sum_{n=0}^{\infty}(n+1)x^n a_{n+1} + \sum_{n=1}^{\infty} x^n a_{n-1} = 0$$

$$\rightarrow \sum_{n=0}^{\infty}(n+2)(n+1)x^n a_{n+2} - \sum_{n=1}^{\infty}(n+1)x^n a_{n+1} + \sum_{n=1}^{\infty} x^n a_{n-1} = 0$$

منفرد سيجما ← منفرد سيجما ← نوحد العداد

$$2a_2 - a_1 + \sum_{n=1}^{\infty}(n+2)(n+1)x^n a_{n+2} - \sum_{n=1}^{\infty}(n+1)x^n a_{n+1} + \sum_{n=1}^{\infty} x^n a_{n-1} = 0$$

$$2a_2 - a_1 = 0$$

$$\boxed{a_2 = \frac{a_1}{2}}$$

recurrance relation

$$(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_{n-1} = 0$$

$$a_{n+2} = \frac{a_{(n+1)}(n+1) - a_{n-1}}{(n+2)(n+3)}, \quad n \geq 1$$

or

$$a_{n+3} = \frac{a_{(n+2)}(n+2) - a_n}{(n+3)(n+2)}, \quad n \geq 0$$

recurrance relation

homework find P.S solution for $y''-xy'+y=0$, $x=1$

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Power Series solution near singular point ($x_0=0$)

"Forbenius Method"

G.S is $y=\sum_{n=0}^{\infty} a_n x^{n+r}$

Let $Y''+p(x)Y'+r(x)Y=0$

Indicial equation $r(r-1)+p_0 r+r_0=0$

تستخدم لايجاد قيمة r المستخدمة في الحل العام

$$\rightarrow p_0=\lim_{x\to x_0} x\,p(x)$$

$$\rightarrow r_0=\lim_{x\to x_0} x^2 r(x)$$

Q_1 the indicial equation of $x^2y''-xy'+(x^2-4)y=0$ at $x=0$ is:

A. $r^2-2r-4=0$

b. $r^2=0$

c. $r^2-4=0$

d. $r^2+4=0$

e. none

$$y''-\frac{1}{x}y'+\frac{x^2-4}{x^2}y=0$$

$$p_0=\lim_{x\to 0} x\cdot\frac{-1}{x}=-1$$

$$r_0=\lim_{x\to 0} x^2\,\frac{x^2-4}{x^2}=-4$$

$$r(r-1)-r-4=0$$

$$r^2-2r-4=0 \quad \text{ans is } \boxed{a}$$

Q_2 find the indicial equation of $x(x-1)y''+\beta(x-1)y'+y=0$ near $x=0$

$$y''+\frac{(3x-1)}{x(x-1)}y'+\frac{1}{x(x-1)}y=0$$

$$p_0=\lim_{x\to 0} x\cdot\frac{(3x-1)}{x(x-1)}=\frac{-1}{-1}=1$$

$$r_0=\lim x^2\frac{1}{x(x-1)}=0$$

$$r(r-1)+r=0$$

$$r^2-r+r=0$$

$$\boxed{r^2=0}$$

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Case 1 if $r_1 - r_2 \neq$ عدد صحيح (كسر)

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

Case 2 if $r_1 = r_2$

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_2(x) = y_1(x)\ln x + \sum_{n=1}^{\infty} b_n x^{n+r}$$

انتبه!!
دائما r_1 هو العدد الكبير و r_2 هو الصغير

Case 3 if $r_1 - r_2 =$ عدد صحيح

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}$$

$$y_2(x) = k y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

Q3 let $y_1 = \sum_{n=0}^{\infty} a_n x^{n+2}$ be the 1st solution of $a(x)y'' + b(x)y' + c(x)y = 0$ about $x_0 = 0$ and the root of indicial equation are 2 and 1, then the second solution is

a- $y_2 = k y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+1}$

b- $y_2 = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^{n+1}$

c- $y_2 = \sum_{n=0}^{\infty} b_n x^{n+1}$

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Note: we can find y_2 as we take in ch2

$$y_2 = y_1(x)\, v(x)$$

where $$v(x) = \int \frac{1}{y_1^2} e^{-\int P(x)dx}\, dx$$

Q4 let $x(x-1)y'' + (3x-1)y' + y = 0$ and $y_1 = a_0 \frac{1}{1-x}$ is the first power series solution near $x_0 = 0$ find the second solution

$$y'' + \frac{(3x-1)}{x(x-1)} y' + \frac{1}{x(x-1)} y = 0$$

$$\int P(x)dx = \int \frac{3x-1}{x(x-1)} = \int \frac{1}{x} + \frac{2}{x-1} = \ln x + 2\ln(x-1)$$

$$= \ln\left((x)(x-1)^2\right)$$

$$e^{-\int P(x)dx} = e^{\ln\left(\frac{1}{x(x-1)^2}\right)} = \frac{1}{x(x-1)^2}$$

$$v = \int \frac{1}{\left(\frac{a_0}{1-x}\right)^2} \cdot \frac{1}{x(x-1)^2}\, dx = \int \frac{(1-x)^2}{(a_0)^2} \cdot \frac{1}{x(x-1)^2} = \frac{1}{a_0^2} \ln x$$

$$y_2 = \frac{a_0}{1-x} \cdot \frac{1}{a_0^2} \ln x = \frac{\ln x}{a_0(1-x)}$$

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Q5 find a basis of solution by forbenius method

1. $(x+1)^2y'' + (x+1)y' - y = 0$

$$P_0 = \lim_{x\to 0} x \cdot \frac{1}{x+1} = 0$$

$$r_0 = \lim x^2 \cdot \frac{-1}{(x+1)^2} = 0$$

$r(r-1) = 0$

$r=0$, $r=1$, case 3 $r_1 = 1$, $r_2 = 0$

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$y_2 = K y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n$$

2. $2x(x-1)y'' - (x+1)y' + y = 0$

3. $x^2y'' + \sin x y' - 4y = 0$

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Laplace Transformation

Def: Let $f(t)$ be defined on $[0,\infty)$ and cts, then the laplace transformation of $f(t)$ is defined as:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s) \rightarrow$$

الناتج دائماً اقتران بدلالة s

↳ Note that this is an improper integral so it must be made into a limit as below

Ex: find $\mathcal{L}\{e^{at}\}$?

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \lim_{b\to\infty} \int_0^b e^{-(s-a)t}\,dt$$

$$= \lim_{b\to\infty} \frac{e^{-(s-a)b}}{-(s-a)} = \lim_{b\to\infty} \left[\frac{e^{-(s-a)b}}{-(s-a)} + \frac{1}{s-a}\right] = \frac{1}{s-a}, \quad s > a$$

	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$	
1	C (constant)	$\frac{C}{s}$	$\mathcal{L}\{5\} = \frac{5}{s}$
2	t^n	$\frac{n!}{s^{n+1}}$	$\mathcal{L}\{t^2\} = \frac{2!}{s^3}$
3	e^{at}	$\frac{1}{s-a}$	$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3}$
4	$\sin at$	$\frac{a}{s^2+a^2}$	$\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$
5	$\cos at$	$\frac{s}{s^2+a^2}$	$\mathcal{L}\{\cos 3t\} = \frac{s}{s^2+9}$
6	$\sinh at$	$\frac{a}{s^2-a^2}$	$\mathcal{L}\{\sinh 3t\} = \frac{3}{s^2-9}$
7	$\cosh at$	$\frac{s}{s^2-a^2}$	$\mathcal{L}\{\cosh 3t\} = \frac{s}{s^2-9}$

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Properities of L.T

1. L.T is a linear operater → اي يوزع على الجمع والطرح

$$\mathcal{L}\{\alpha f(t) \pm \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} \pm \beta \mathcal{L}\{g(t)\}$$

2. if $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$

Examples: find L.T for

1. $f(t) = t^4$

$$\mathcal{L}\{t^4\} = \frac{4!}{s^5}$$

2. $f(t) = (t+3)^2$

$$\mathcal{L}\{(t+3)^2\} = \mathcal{L}\{t^2+6t+9\} = \frac{2!}{s^3} + \frac{6}{s^2} + \frac{9}{s}$$

3. $f(t) = \{\sin t \cos t\}$

$$\mathcal{L}\left\{\frac{1}{2}\sin 2t\right\} = \frac{1}{2}\,\frac{2}{s^2+4} = \frac{1}{s^2+4}$$

4. $f(t) = \{\cos^2 3t\}$

$$\mathcal{L}\left\{\frac{1}{2} + \frac{1}{2}\cos 6t\right\} = \frac{1/2}{s} + \frac{1}{2}\,\frac{s}{s^2+36}$$

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5. $f(t) = \mathcal{L}^{-1}\{e^{2t}\cosh 3t\}$

$\rightarrow e^{2t}\cosh 3t = e^{2t}\cdot\left(\frac{e^{3t}+e^{-3t}}{2}\right) = \frac{1}{2}e^{5t} + \frac{1}{2}e^{-t}$

$\mathcal{L}\left\{\frac{1}{2}e^{5t} + \frac{1}{2}e^{-t}\right\} = \frac{1}{2}\frac{1}{s-5} + \frac{1}{2}\frac{1}{s+1}$

Remember

$\rightarrow \cosh t = \frac{e^t + e^{-t}}{2}$

$\rightarrow \sinh t = \frac{e^t - e^{-t}}{2}$

6. $f(t) = \mathcal{L}^{-1}\{\cos(2t+\frac{\pi}{4})\}$

$\rightarrow \cos(2t+\frac{\pi}{4}) = \cos 2t \cdot \frac{1}{\sqrt{2}} - \sin 2t \cdot \frac{1}{\sqrt{2}}$

$\mathcal{L}\{f(t)\} = \frac{1}{\sqrt{2}}\left[\frac{s}{s^2+4} - \frac{2}{s^2+4}\right]$

Remember

$\rightarrow \cos(a+b) = \cos a\cos b - \sin a\sin b$

Ex: find $\mathcal{L}^{-1}\{F(s)\}$

1. $F(s) = \frac{s-3}{s^2+5}$

$F(s) = \frac{s}{s^2+5} - \frac{3}{s^2+5} \rightarrow \frac{3}{\sqrt{5}}\cdot\frac{\sqrt{5}}{s^2+5}$

$f(t) = \cos\sqrt{5}t - \frac{3}{\sqrt{5}}\sin\sqrt{5}t$

2. $F(s) = \frac{2s+1}{s^3+3s^2+2s}$

$F(s) = \frac{2s+1}{s(s+1)(s+2)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+2}$

by partial fraction $a=1, b=1, c=1$

$F(s) = \frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2}$

$f(t) = 1 + e^{-t} + e^{-2t}$

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3. $F(s) = \frac{1}{s^5}$

$$F(s) = \frac{4!}{s^5} \cdot \frac{1}{4!}$$

$$f(t) = \frac{1}{4!} t^4$$

homework

4. $F(s) = \frac{s+4}{s^2-4s+3}$

5. $\mathcal{L}^{-1}\left\{\frac{s^3+1}{s^4-16}\right\}$

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First Shifting Theorm

$$\rightarrow \mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

طريقة الحل !!

نجد قيمة $F(s)$ للاقتران $f(t)$ ثم نستبدل كل ك د $(s-a)$ اي ان e^{at} عندما تضرب باقتران يكون تأثيرها فقط ازالة ال القوس الخاص بـ ك

1. find $\mathcal{L}\{e^{3t}t\}$

step1 $\mathcal{L}\{t\} = \frac{1}{s^2}$

step2 $\mathcal{L}\{e^{3t}t\} = \frac{1}{(s-3)^2}$

2. find $\mathcal{L}\{e^{2t}\sin 5t\}$

step1 $\mathcal{L}\{\sin 5t\} = \frac{5}{s^2+25}$

step2 $\mathcal{L}\{e^{2t}\sin 5t\} = \frac{5}{(s-2)^2+25}$

3. find $\mathcal{L}^{-1}\left\{\frac{s+2}{s^2-6s+1}\right\}$

hint !!

اذا كان المقام اقتران تربيعي لا يحلل نقوم بعمل اكمال مربع

$$\frac{s+2}{s^2-6s+9-9+1} = \frac{s+2}{(s-3)^2-8}$$

$$= \frac{(s-3)+3+2}{(s-3)^2-8} = \frac{(s-3)+5}{(s-3)^2-8} = \frac{(s-3)}{(s-3)^2-8} + \frac{5}{(s-3)^2-8}$$

$$= \frac{(s-3)}{(s-3)^2-8} + \frac{5}{\sqrt{8}}\frac{\sqrt{8}}{(s-3)^2-8}$$

$$f(t) = e^{3t}\cosh\sqrt{8}t + \frac{5}{\sqrt{8}}e^{3t}\sinh\sqrt{8}t$$

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4. Find $\mathcal{L}^{-1}\left\{\frac{s}{s^2-6s+10}\right\}$?

$$\frac{s}{s^2-6s+9-9+10} = \frac{s}{(s-3)^2+1} = \frac{s-3+3}{(s-3)^2+1} = \frac{(s-3)}{(s-3)^2+1} + \frac{+3\cdot 1}{(s-3)^2+1}$$

$$f(t) = e^{3t}\cos t + e^{3t}\sin t$$

5. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2-4s}\right\}$?

Note!!

يمكن حل هذا السؤال تحليل او اكمال مربع

$$\frac{1}{s^2-4s+4-4} = \frac{1}{(s-2)^2-4} = \frac{1}{2}\,\frac{2}{(s-2)^2-4}$$

$$f(t) = \frac{1}{2}e^{2t}\sinh(2t)$$

homework

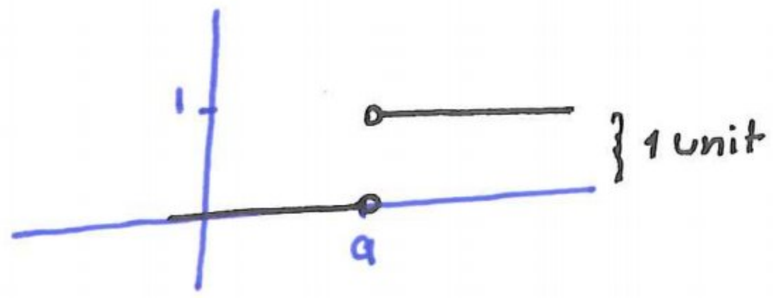
6. $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}$?

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Unit Step function

Def: let $a \geq 0$ be any real number, then the unit step function at a is defined by:

$$U_a(t) = U(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$$



$$\mathcal{L}\{U(t-a)\} = \frac{e^{-as}}{s}, \quad s>0$$

Second Shifting Theorm

$$\mathcal{L}\{f(t-a)\,U(t-a)\} = e^{-as}F(s)$$

→ $\mathcal{L}\{f(t)\}$ not $\mathcal{L}\{f(t-a)\}$

Ex: find L.T For

1. $f(t) = U(t-1)\cos(t-1)$

→ step1 $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$

→ step2 $\mathcal{L}\{U(t-1)\cos(t-1)\} = e^{-s}\cdot\frac{s}{s^2+1}$

2. $f(t) = U(t-4)(t-4)^2$

→ step1 $\mathcal{L}\{t^2\} = \frac{2!}{s^3}$

→ step2 $\mathcal{L}\{U(t-4)(t-4)^2\} = e^{-4s}\cdot\frac{2!}{s^3}$

3. $f(t) = U(t-2)\,t^2$

→ step0 $f(t) = U(t-2)((t-2)+2)^2 = U(t-2)[(t-2)^2 - 4(t-2)+4]$

$f(t) = U(t-2)(t-2)^2 - 4U(t-2)(t-2) + 4U(t-2)$

→ step1 $\mathcal{L}\{f(t)\} = e^{-2s}\frac{2!}{s^3} - 4e^{-2s}\frac{1}{s^2} + 4\frac{e^{-2s}}{s}$ ↳ $\mathcal{L}\{U(t-a)\}$

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4. $f(t) = \sin 2t\, u(t-\pi)$

hint!!

$\sin(x+2\pi) = \sin x$

→ step 0 $\sin 2t = \sin 2(t-\pi+\pi) = \sin[2(t-\pi)+2\pi]$

$= \sin 2(t-\pi)$

step 1 $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

step 2 $\mathcal{L}\{\sin 2(t-\pi)\, u(t-\pi)\} = \frac{2}{s^2+4} \cdot e^{-\pi s}$

5. $f(t) = e^{2t}\cos t\, u(t-\frac{\pi}{2})$

hint!!

$\cos(x+\frac{\pi}{2}) = -\sin x$

→ step 0 $e^{2t} = e^{2(t-\frac{\pi}{2}+\frac{\pi}{2})} = e^{2(t-\frac{\pi}{2})+\pi} = e^{\pi} e^{2(t-\frac{\pi}{2})}$

$\cos t = \cos(t-\frac{\pi}{2}+\frac{\pi}{2}) = -\sin(t-\frac{\pi}{2})$

→ step 1 $\mathcal{L}\{-e^{\pi} e^{2t} \sin t\} = -e^{\pi} \frac{1}{(s-2)^2+1}$

→ step 2 $\mathcal{L}\{-e^{\pi} e^{2(t-\frac{\pi}{2})} \sin(t-\frac{\pi}{2})\, u(t-\frac{\pi}{2})\} = -e^{\pi} \frac{1}{(s-2)^2+1} \cdot e^{-\frac{\pi}{2}s}$

6. $f(t) = e^{3(t-1)}(t-1)^2\, u(t-1)$

$\mathcal{L}\{f(t)\} = \frac{2!}{(s-3)^3} e^{-s}$

7. $f(t) = \begin{cases} 0, & t<2 \\ t^2, & t>2 \end{cases}$

⇐ لحل هذا النوع من الاسئلة نحاول جعل الافتران على شكل $u(t-a)$ أي $\begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$

$f(t) = t^2 \begin{cases} 0, & t<2 \\ 1, & t>2 \end{cases} = t^2\, u(t-2)$

↳ continue as ex. 3

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8. $f(t)=\begin{cases}1, & t<4\\ 0, & t>4\end{cases}$

Remember $u(t-4)=\begin{cases}0, & t<4\\ 1, & t>4\end{cases}$

$\rightarrow f(t) = 1-u(t-4)$

$\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-4s}}{s}$

9. $f(t)=\begin{cases}1, & t<4\\ 3, & t>4\end{cases}$

hint !!

$2u(t-4)=\begin{cases}0, & t<4\\ 2, & t>4\end{cases}$

$\rightarrow f(t) = 1+2u(t-4)$

$\rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{2e^{-4s}}{s} = \frac{1+2e^{-4s}}{s}$

10. $f(t)=\begin{cases}t, & t<2\\ 0, & t>2\end{cases}$

hint !!

$tu(t-2)=\begin{cases}0, & t<2\\ t, & t>2\end{cases}$

$\rightarrow f(t) = t - tu(t-2)$

$\rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s^2} - \mathcal{L}\{tu(t-2)\}$

$\mathcal{L}\{((t-2)+2)u(t-2)\}$

$\mathcal{L}\{(t-2)u(t-2)+2u(t-2)\}$

$= \frac{1}{s^2} - \frac{1}{s^2}e^{-2s} + \frac{2e^{-2s}}{s}$

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11. $f(t) = \begin{cases} t^2 & , t < \frac{\pi}{2} \\ \cos t & , t > \frac{\pi}{2} \end{cases}$

$$f(t) = t^2 \begin{cases} 1 & , t < \frac{\pi}{2} \\ 0 & , t > \frac{\pi}{2} \end{cases} + \cos t \begin{cases} 0 & , \pi < \frac{\pi}{2} \\ 1 & , \pi > \frac{\pi}{2} \end{cases}$$

$$f(t) = t^2 - t^2 u(t-\tfrac{\pi}{2}) + \cos t\, u(t-\tfrac{\pi}{2})$$

then we can continue as examples before

Rule If $f(t) = \begin{cases} g(t) & , a < t < b \\ 0 & , \text{otherwise} \end{cases}$

then $f(t) = g(t)[u(t-a) - u(t-b)]$

12. $f(t) = \begin{cases} e^{3t} & , 1 < t < 3 \\ 0 & , o.w \end{cases}$

$$f(t) = e^{3t} u(t-1) - e^{3t} u(t-3)$$

$* e^{3t} = e^{3(t-1+1)} = e^{3(t-1)} \cdot e^3$ $\quad * e^{3t} = e^{3(t-3+3)} = e^{3(t-3)} \cdot e^9$

$$\mathcal{L}[f(t)] = e^3 \frac{1}{s-3} e^{-s} - e^9 \frac{1}{s-3} e^{-3s}$$

13. $f(t) = \begin{cases} t^2 & , 3 < t < 4 \\ 0 & , o.w \end{cases}$

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Theorm

$$\mathcal{L}\{f(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$$

Ex : 1. find $\mathcal{L}\{t^2 u(t-1)\}$

$$\mathcal{L}\{t^2 u(t-1)\} = e^{-s}\,\mathcal{L}\{(t+1)^2\}$$
$$= e^{-s}\,\mathcal{L}\{t^2+2t+1\}$$
$$= e^{-s}\left(\frac{2!}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$$

* يمكن حل هذا السؤال حسب الطرق السابقة او باستخدام القانون الجديد اعلاه

2. find $\mathcal{L}\{\sin t\, u(t-\frac{\pi}{2})\}$

hint !!
$\sin(a+b) = \sin a\cos B + \sin b\cos a$

$$= \mathcal{L}\{\sin(t+\tfrac{\pi}{2})\}\,e^{-\frac{\pi}{2}s}$$
$$= \mathcal{L}\{\cos t\}\,e^{-\frac{\pi}{2}s} = \frac{s}{1+s^2}\cdot e^{-\frac{\pi}{2}s}$$

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Ex: find $\mathcal{L}^{-1}$ for

1. $\mathcal{L}^{-1}\left\{\frac{e^{-2s}\cdot s}{s^2-4}\right\}$

$$\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{s}{s^2-4}\right\} = u(t-2)\cosh 2(t-2)$$

Remember !!

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

2. $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^6}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{5^6}\right\} = \frac{1}{5!}\mathcal{L}^{-1}\left\{e^{-2s}\frac{5!}{5^6}\right\} = \frac{1}{5}u(t-2)\,(t-2)^5$$

3. $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2-3s+2}\right\}$

by Partial fraction

$$\mathcal{L}^{-1}\left\{e^{-2s}\cdot\frac{1}{(s+2)(s-1)}\right\} = \mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{1}{s-2}-\frac{1}{s-1}\right)\right\}$$

$$= u(t-2)\,e^{2(t-2)} - u(t-2)\,e^{(t-2)}$$

$$\frac{1}{(s-2)(s-1)} = \frac{a}{s-2}+\frac{b}{s-1}$$

$$1 = a(s-1)+b(s-2)$$

$b=-1$

$a=1$

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Differentiation of laplace transformation

$$\mathcal{L}\{f(t)\} = F(s)$$

$\rightarrow$ $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ $\longrightarrow y(0)$ since $f(t) = y(t)$

$\rightarrow$ $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

$\rightarrow$ $\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$

Ex: Solve the IVP using L.T $y'' + y = 0$, $y(0) = y'(0) = 1$

$\rightarrow$ Step 1: $\mathcal{L}$ للطرفين

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 0$$

$$s^2F(s) - sf(0) - f'(0) + F(s) = 0$$

$$s^2F(s) - s - 1 + F(s) = 0$$

$\rightarrow$ step 2: $F(s)$ نجد

$$F(s)(s^2+1) = 1 + s$$

$$F(s) = \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$\rightarrow$ step 3 $y(t) = \mathcal{L}^{-1}\{F(s)\}$

$$y(t) = \cos t + \sin t$$

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Ex: if $\frac{dy}{dx} = u(t-3)$, $y(0)=2$ then $y=$

A) $(t-3)u(t-3)+2$

b) $(t-3)$

c) $(t-3)u(t-3)$

d) $2u(t-3)$

e) $\frac{e^{-3t}}{t^2}$

Step 1 $\mathcal{L}\{y'\} = \mathcal{L}\{u(t-3)\}$

$sF(s) - \vec{y(0)}^{2} = \frac{e^{-3s}}{s}$

$sF(s) = \frac{e^{-3s}}{s} + 2$

Step 2 $F(s) = \frac{e^{-3s}}{s^2} + \frac{2}{s}$

Step 3 $\mathcal{L}^{-1}\{F(s)\} = u(t-3)(t-3) + 2$ [A]

Ex: The solution of $y''+y=t$, $y(0)=1$, $y'(0)=1$ using L.T

Step 1 $\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{t\}$

$s^2F(s) - sF(0) - F'(0) - F(s) = \frac{1}{s^2}$

$s^2F(s) - s - 1 - F(s) = \frac{1}{s^2}$

Step 2 $F(s) = \frac{1}{s^2(s^2-1)} + \frac{s}{s^2-1} + \frac{1}{s^2-1}$

Step 3 $y(t) = \mathcal{L}^{-1}\{F(s)\}$

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homework solve $y''+3y'+2y=U(t-1)-U(t-2), y(0)=y'(0)=0$

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Integration of laplace transformation

$$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{F(s)}{s}$$, where $F(s) = \mathcal{L}\{f(t)\}$ ← same f(x)

Ex : 1. find $\mathcal{L}\left\{\int_0^t \sin 3x\,dx\right\}$

step1 $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$

Step2 $\mathcal{L}\left\{\int_0^t \sin 3x\,dx\right\} = \frac{3}{s(s^2+9)}$

2. find $\mathcal{L}^{-1}\left\{\frac{1}{s^3+s}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3+s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$$

$$f(t) = \int_0^t \sin x\,dx = -\cos x\Big]_0^t$$

$$= -\cos t + 1$$

Note !!

يمكن حل هذا السؤال عن طريق
الكسور الجزئية والطرق السابقة
او من خلال قاعدة التكامل

3. find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+s^4}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2(1+s^2)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{s}\cdot\frac{1}{s^2+1}\right)\right\}$$

$$f(t) = \int_0^t\int_0^t \sin x\,dx = \int_0^t -\cos t + 1$$

$$= -\sin t + t$$

Note !!

$\mathcal{L}^{-1}\left\{\frac{1}{s^2}F(s)\right\} \rightarrow$ تكامل مرتين $\int_0^t\int_0^t$

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Dirac Delta function "Impulse function"

The derec delta function is defined as :

$$\delta(t-a) = \begin{cases} \infty, & t=a \\ 0, & t\neq a \end{cases}$$

Examples:

1. let $f(t) = 2+3u(t-2)+5\delta(t-7)$, find $f(5)$

$$u(t-2) = \begin{cases} 0, & t<2 \\ 1, & t>2 \end{cases}$$

$$\delta(t-7) = \begin{cases} \infty, & t=7 \\ 0, & t\neq 7 \end{cases}$$

so $f(5) = 2 + 3(1) + 5(0)$

$= 5$

Rule: $\mathcal{L}[\delta(t-a)] = e^{-as}$

2. find $\mathcal{L}[\delta(t-2)]$

$$\mathcal{L}[\delta(t-2)] = e^{-2s}$$

3. the solution of $y''+y = \delta(t-1), y(0)=y'(0)=0$ is $y(t)=$

A) $u(t-1)\sin t$

b) $\sin(t-1)$

c) $u(t-1)\cos(t-1)$

d) $u(t-1)\sin(t-1)$

e) None

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step 1 $\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-1)\}$

$$s^2F(s) - sf(0) - f'(0) + F(s) = e^{-s}$$

$$F(s)(s^2+1) = e^{-s}$$

step 2 $$F(s) = e^{-s} \cdot \frac{1}{s^2+1}$$

step 3 $\mathcal{L}^{-1}\{F(s)\} = u(t-1)\sin(t-1)$ **d**

4. Solve $y'' + 3y' + 2y = \delta(t-1)$, $y(0) = y'(0) = 0$

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Rule: $\rightarrow \int_0^\infty \delta(t-a)\,dt = 1$

$\rightarrow \int_0^\infty g(t)\,\delta(t-a)\,dt = g(a)$

5. Find $\mathcal{L}\{t^2\delta(t-4)\}$?

بما انه لا يوجد قاعدة عامة لهذا الشكل اذا استخدم القانون

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt$$

$$\rightarrow \mathcal{L}\{t^2\delta(t-4)\} = \int_0^\infty t^2\,\delta(t-4)\,e^{-st}\,dt$$

$$= t^2 e^{-st}\Big|_{t=4} = 16\,e^{-4s}$$

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Theorm $\mathcal{L}\{t\,f(t)\} = -F'(s)$

$\mathcal{L}\{t^2 f(t)\} = F''(s)$

Examples 1. find $\mathcal{L}\{t\sin t\}$

For $f(t)=\sin t \to F(s) = \frac{1}{s^2+1}$

$\mathcal{L}\{t\sin t\} = -F'(s) = \frac{2s}{(s^2+1)^2}$

2. find $\mathcal{L}\{t^2\delta(t-4)\}$

for $f(t) = \delta(t-4) \to F(s) = e^{-4s}$

$\mathcal{L}\{t^2\delta(t-4)\} = F''(s)$

$\to F'(s) = -4e^{-4s}$

$\to F''(s) = 16e^{-4s}$

Theorm $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

3. find $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$

for $f(t) = \sin t \longrightarrow F(s) = \frac{1}{s^2+1}$

$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds = \tan^{-1}s \Big]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}s$

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QUESTIONS



Calculus final Exam

sum 2019/2020

By Anas Abu Zahra

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Question 17
Not yet answered
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The indicial equation of $x^2y'' - xy' + (x^2 - 4)y = 0$ at $x = 0$ is :

(A) $r^2 - 2r - 4 = 0$
(B) $r^2 = 0$
(C) $r^2 - 4 = 0$
(D) $r^2 + 4 = 0$
(E) None

solution

$y'' - \frac{1}{x}y' + \frac{(x^2-4)}{x^2}y = 0$

$P_0 = \lim_{x\to 0} x \cdot \frac{-1}{x} = -1$

$r_0 = \lim_{x\to 0} x^2 \cdot \frac{(x^2-4)}{x^2} = -4$

$r(r-1) - r - 4 = 0$

$r^2 - 2r - 4 = 0$ [A]

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Question 11
Not yet answered
Marked out of 2.00
Flag question

$\mathcal{L}\{\sin^2 t\} =$

(A) $\frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$
(B) $\frac{1}{s} + \frac{s}{s^2+4}$
(C) $\frac{1}{2}\left[\frac{1}{s} + \frac{2}{s^2+4}\right]$
(D) $\frac{1}{s} + \frac{2}{s^2+4}$
(E) None

solution

$\mathcal{L}\{\sin^2 t\} = \mathcal{L}\{\frac{1}{2} - \frac{1}{2}\cos 2t\}$

$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$

[A]

Select one:

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Question 12

Not yet answered

Marked out of 2.00

Flag question

$$\mathcal{L}\{e^{1-t}\} =$$

(A) $\frac{e}{s-1}$

(B) $\frac{1}{s+1}$

(C) $\frac{1}{s-1}$

(D) $\frac{e}{s+1}$

(E) None

Solution

$\mathcal{L}\{e \cdot e^{-t}\} = e \cdot \frac{1}{s+1}$

D

Select one:

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Question 16

Not yet answered

Marked out of 3.00

Flag question

Consider $(x^3 - 2x^2)y'' + xy' + y = 0$. Then the set of all regular singular points is

(A) $\{0, \sqrt{2}\}$

(B) $\{0\}$

(C) $\{\sqrt{2}\}$

(D) $\{0, 2\}$

(E) $\{2\}$

solution

$y'' + \frac{1}{x^2-2x} y' + \frac{1}{x^3-2x^2} y = 0$

singular point $x_0 = 0, 2$

$x = 0$

$\lim_{x\to 0} x \cdot \frac{1}{x(x-2)} = \frac{-1}{2}$

$\lim_{x\to 0} x^2 \frac{1}{x^2(x-2)} = \frac{-1}{2}$

$x = 0$ regular

$x = 2$

$\lim_{x\to 2} (x-2) \frac{1}{x(x-2)} = \frac{1}{2}$

$\lim_{x\to 2} (x-2)^2 \frac{1}{x^2(x-2)} = 0$

$x = 2$ regular

D

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Question 8

Not yet answered

Marked out of 3.00

Flag question

If $\sum_{n=1} na_n x^{n-1} + \sum_{n=0} a_n x^n = 0$, then $a_3 =$

(A) $\frac{a_0}{3}$

(B) $-\frac{a_0}{6}$

(C) $\frac{a_0}{2}$

(D) $-\frac{a_0}{2}$

(E) None

solution

$$\sum_{n=0}^{\infty} (n+1)\,a_{(n+1)}\,x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} x^n\left((n+1)\,a_{(n+1)} + a_n\right) = 0$$

$$a_{n+1} = \frac{-a_n}{n+1}$$

$\rightarrow n = 0$

$a_1 = \frac{-a_0}{1}$

$\rightarrow n = 1$

$a_2 = \frac{-a_1}{2}$

$\rightarrow n = 2$

$a_3 = \frac{-a_2}{3} = \frac{a_1}{6} = \frac{-a_0}{6}$ [B]

ENGINEERING MATHEMATICS (1)

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Question 5

Not yet answered

Marked out of 3.00

Flag question

If $r^3 - 2r^2 + r = 0$ is the characteristic equation of a homogeneous D.E. with constant coefficients, then we can write the general solution as $y(x) =$

(A) $c_1 + c_2 x + c_3 x^2$

(B) $c_1 + c_2 x + c_3 x \ln x$

(C) $c_1 + c_2 e^x + c_3 x e^x$

(D) $x(c_1 + c_2 \ln x)$

(E) $c_1 + c_2 e^{-x} + c_3 x e^{-x}$

Solution

$r(r^2 - 2r + 1) = 0$

$r(r-1)(r-1) = 0$

$r = 0, r = 1, r = 1$

ans is [C]

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Question 2
Not yet answered
Marked out of 2.00
Flag question

The solution of $y'' + y = \delta(t-1)\,,\ y(0) = y'(0) = 0$ **is** $y(t) =$

$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-1)\}$

$s^2F(s) - s f(0) - f'(0) + F(s) = e^{-s}$

$F(s) = e^{-s} \cdot \frac{1}{s^2+1}$

$y(t) = u(t-1)\sin(t-1)$

(d)

(A) $u(t-1)\sin t$

(B) $\sin(t-1)$

(C) $u(t-1)\cos(t-1)$

(D) $u(t-1)\sin(t-1)$

(E) None

ENGINEERING MATHEMATICS (1)

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Question 1
Not yet answered
Marked out of 3.00
Flag question

If $\lambda_1 = 2\,,\ \lambda_2 = 4$ **are two eigenvalues of** $A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix}$.

Then the particular solution of

$$y_1' = 5y_1 + 3y_2 - 2e^{-t} + 1$$
$$y_2' = -y_1 + y_2 + e^{-t} - 5t$$

can be written as $\underline{y}^{(p)} =$

$Y' = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} Y + \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ -5 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Y_h = c_1 x_{h_1} e^{2t} + c_2 x_{h_2} e^{4t}$

$Y_p = \vec{a}e^{-t} + \vec{b}t + \vec{c}$

[c]

(A) $(\vec{a}t + \vec{b})\,e^{t} + \vec{c}$

(B) $(\vec{a}t + \vec{b})\,e^{-t} + \vec{c}$

(C) $\vec{a}\,e^{-t} + \vec{b}t + \vec{c}$

(D) $\vec{a}\,e^{-t} + \vec{c}$

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Question 3
Not yet answered
Marked out of 3.00
Flag question

Consider $y^{(5)} + y^{(4)} = 1 + t + \sin 2t$. Then a suitable form of the particular solution can be written as $y_p(t) =$

$r^5 + r^4 = 0$
$r^4(r+1) = 0$
$y_h = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_5 e^{-t}$

(A) $t^2(At + B) + C\sin 2t + D\cos 2t$
(B) $t(At + B) + C\sin 2t + D\cos 2t$
(C) $At + B + C\sin 2t + D\cos 2t$
(D) $t^3(At + B) + C\sin 2t + D\cos 2t$
(E) $t^4(At + B) + C\sin 2t + D\cos 2t$

$Y_{p_1} = at+b \rightarrow t^4(at+b)$
$Y_{p_2} = \alpha\sin 2t + \alpha\cos 2t$
E

ENGINEERING MATHEMATICS (1)

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Question 4
Not yet answered
Marked out of 2.00
Flag question

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 2s + 2}\right\} =$$

(A) $e^{-t}[\cos t - \sin t]$
(B) $\cos 2t - \frac{1}{2}\sin 2t$
(C) $e^{t}[\cos t - \sin t]$
(D) $e^{-t}[\cos 2t - \sin 2t]$
(E) None

solution
$s^2+2s+1-1+2 =$
$(s+1)^2+1$
$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+1}\right\} =$
$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$
$= e^{-t}\cos t - e^{-t}\sin t$
$= e^{-t}(\cos t - \sin t)$
A

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Question 19

Not yet answered

Marked out of 2.00

Flag question

$$\mathcal{L}\left\{\delta(t-\frac{\pi}{6})\,\cos t\right\} =$$

$\int_0^\infty \cos t\,\delta(t-\frac{\pi}{6})\,e^{-st}\,dt$

Rule: $\int_0^\infty g(t)\delta(t-a) = g(a)$

$\cos\frac{\pi}{6}\cdot e^{-\frac{\pi}{6}s} = \frac{\sqrt{3}}{2}e^{-\frac{\pi}{6}s}$

A

(A) $\frac{\sqrt{3}}{2}e^{-\frac{\pi}{6}s}$

(B) $-e^{-\frac{\pi}{3}s}$

(C) $e^{-\frac{\pi}{3}s}$

(D) $\frac{\sqrt{3}}{2}e^{-s}$

(E) $\frac{1}{2}e^{-\frac{\pi}{6}s}$

Select one:

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Question 20

Not yet answered

Marked out of 3.00

Flag question

The eigenvalues of $A = \begin{bmatrix} 7 & -4 \\ 5 & -2 \end{bmatrix}$ are

$A-\lambda I = \begin{bmatrix} 7-\lambda & -4 \\ 5 & -2-\lambda \end{bmatrix}$

$-14-7\lambda+2\lambda+\lambda^2+20=0$

$\lambda^2-5\lambda+6=0$

$(\lambda-3)(\lambda-2)=0$

$\lambda=3 \quad \lambda=2$

A

(A) $\lambda_1 = 2,\ \lambda_2 = 3$

(B) $\lambda_1 = 2,\ \lambda_2 = 4$

(C) $\lambda_1 = 5,\ \lambda_2 = 1$

(D) $\lambda_1 = -1,\ \lambda_2 = 6$

(E) None

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Question 7
Not yet answered
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Flag question

The solution of $y' - \frac{1}{x}y = 2x^2$, $y(1) = 2$ is

$M_x = e^{\int P(x)} = e^{\int -\frac{1}{x}} = e^{-\ln x} = \frac{1}{x}$

(A) $y = x^3 + x$

(B) $y = x^2 + x$

(C) $y = x^2 + 3x$

(D) $y = 3x^2 - 1$

(E) $y = x^2 + 1$

$y = \frac{1}{M_x}\left[\int M_x\, g(x)\,dx + c\right]$

$y = x \int 2x\,dx = x\left[x^2 + c\right]$

$2 = [1 + c]$

$c = 1$

$y = x(x^2+1)$

$= x^3 + x$ [A]

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Question 14
Not yet answered
Marked out of 3.00
Flag question

If $(3x^2 + 2by^2)dx + (2xy - y^2)dy = 0$ is exact, then $b =$

$M_y = 4by$

$N_x = 2y$

(A) $\frac{2}{3}$

(B) 1

(C) $\frac{1}{2}$

(D) 0

(E) None

$\frac{2y}{4y} = \frac{4by}{4y}$

$\boxed{b = \frac{1}{2}}$ [C]

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Question 13

Not yet answered

Marked out of 2.00

Flag question

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2-1}\right\} =$$

(A) $u(t-1)\sin(t-1)$

(B) $u(t-1)\sinh(t-1)$

ans is [B]

(C) $u(t-3)\sinh(t-3)$

(D) $u(t-3)\sin(t-3)$

(E) None

Select one:

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Question 6

Not yet answered

Marked out of 2.00

Flag question

Let $f(t) = 2 + 3u(t-2)$. **Then** $f(5) =$

$$U(t-2) = \begin{cases} 0, & t<2 \\ 1, & t>2 \end{cases}$$

(A) 2

(B) 5

$f(5) = 2+3(1) = 5$

[b]

(C) 0

(D) 3

(E) None

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Question 18
Not yet answered
Marked out of 2.00
Flag question

$$\mathcal{L}^{-1}\left\{\frac{3}{s(s^2+9)}\right\} =$$

(A) $\frac{1}{3}[1-\cos 3t]$

(B) $1-\cos 3t$

(C) $\frac{1}{3}[1-\sin 3t]$

(D) $1-\sin 3t$

(E) None

$\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \sin 3t$

$\mathcal{L}^{-1}\left\{\frac{3}{s(s^2+9)}\right\} = \int_0^t \sin 3x\,dx$

$= \left[\frac{-\cos 3x}{3}\right]_0^t$

$= \frac{-\cos 3t}{3} + \frac{1}{3}$

$= \frac{1}{3}[1-\cos 3t]$

A

Select one:

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Question 9
Not yet answered
Marked out of 1.00
Flag question

Let $r_1 = 0,\ r_2 = 0$ be the roots of the indicial equation at $x_0 = 0$ of the D.E. $xy'' + y' + y = 0$. If y_1 is the solution corresponding to $r = 0$, then a second linearly independent solution can be written as $y_2 =$

(A) $c\,y_1 \ln x + \sum_{n=0} a_n x^{n+1}$, c is constant

(B) $c\,y_1 \ln x + \sum_{n=0} a_n x^{n}$, c is constant

(C) $y_1 \ln x + \sum_{n=1} a_n x^{n}$

(D) $y_1 \ln x + \sum_{n=1} a_n x^{n+1}$

(E) None

ans is C

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Question 10
Not yet answered
Marked out of 3.00
Flag question

One of the following is <u>not a Bernoulli</u> D.E.

(A) $x^2y' + 2xy = y^3$

(B) $y' + 2xy = \frac{x}{y}$

(C) $y' + 2xy = x^2y^2$

(D) $y' + y = y^2$

(E) $y' + y = y^2 \sin y$

ans is (e)

Select one:

- A
- B

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Question 15
Not yet answered
Marked out of 2.00
Flag question

Let $f(t) = \begin{cases} 0 & 0 \le t < 2, \\ t & t \ge 2. \end{cases}$ **Then** $\mathcal{L}\{f(t)\} =$

(A) $e^{-s}\left[\frac{1}{s} + \frac{1}{s^2}\right]$

(B) $e^{-2s}\left[\frac{1}{s} + \frac{2}{s^2}\right]$

(C) $e^{-2s}\left[\frac{2}{s^2} + \frac{1}{s}\right]$

(D) $e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$

(E) $\frac{1}{s^2} - e^{-2s}\left[\frac{1}{s^2} + \frac{2}{s}\right]$

$f(t) = t\,u(t-2)$

$= e^{-2s}\,\mathcal{L}\{t+2\}$

$= e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$

(D)

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QUESTIONS



Calculus final Exam

First sem. 2020/2021

By Anas Abu Zahra

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Using partial fractions:

$$\mathcal{L}^{-1}\left\{\frac{-s+9}{s^2-3s-4}\right\} =$$

a) $-2e^{-t}+e^{4t}$

b) $-2e^{-t}-e^{-4t}$

c) $-e^{-t}+2e^{4t}$

d) $e^{t}-2e^{-4t}$

e) $2e^{t}+e^{4t}$

○ a

○ b

Using $u=\frac{y}{x}$, the ODE $\frac{dy}{dx}=\frac{3y^2}{x^2-xy}$ becomes

a) $\frac{du}{dx}=\frac{5u^2-u}{x(1-u)}$

b) $\frac{du}{dx}=\frac{4u^2-u}{x(1-u)}$

c) $\frac{du}{dx}=\frac{3u^2-u}{x(1-u)}$

d) $\frac{du}{dx}=\frac{2u^2-u}{x(1-u)}$

e) $\frac{du}{dx}=\frac{u^2-u}{x(1-u)}$

○ a

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Let $f(t) = \begin{cases} \sin 2t\,, & 0 \le t < \pi/2, \\ 0, & t \ge \pi/2. \end{cases}$

Then $\mathcal{L}\{f(t)\} =$

a) $\frac{2+2e^{-\pi s/2}}{s^2+4}$

b) $\frac{s+se^{-\pi s/2}}{s^2+4}$

c) $\frac{2+2e^{\pi s/2}}{s^2+4}$

d) $\frac{s+se^{\pi s/2}}{s^2+4}$

e) $\frac{2-2e^{-\pi s/2}}{s^2+4}$

a

Time left 0:54:1

Consider the ODE $(3y - x^3e^{3x})dx + (3e^{3x} - 1)dy = 0$.
A suitable integrating factor to convert the above equation to exact is

a) e^{-x}
b) e^{-2x}
c) e^{-3x}
d) e^{-y}
e) e^{-2y}

a
b
c

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Time left

The regular singular point(s) of

$$(x^3 - 4x^2)y'' + 3y' + (x+1)y = 0$$

is (are) $x =$

a) 0
b) 0, 4
c) -1
d) 4
e) None

○ a
○ b

Time le

$$\mathcal{L}^{-1}\left\{\ln\left(\frac{s-2}{s+3}\right)^4\right\} =$$

a) $-4te^{2t} + 4te^{-3t}$

b) $-4te^{-2t} + 4te^{3t}$

c) $\frac{-4e^{2t}+4e^{-3t}}{t}$

d) $\frac{-4e^{-2t}+4e^{3t}}{t}$

e) $-4e^{2t} + 4e^{-3t}$

○ a

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Given $x = 0$ is a regular singular point for the ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

with the indicial equation $r^2 + r - 2 = 0$. Let $y_1(x)$ be the series solution of this ODE with respect to the largest root of the indicial equation. Then a second linearly independent series solution of this ODE will be of the form $y_2(x) =$

a) $ay_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n-2}$
b) $\sum_{n=0}^{\infty} b_n x^{n-2}$
c) $y_1(x)\ln x + \sum_{n=1}^{\infty} b_n x^{n-2}$
d) $y_1(x)\ln x + \sum_{n=1}^{\infty} b_n x^{n+1}$
e) $ay_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+1}$

○ a
○ b
○ c

Time left 0:51:50

Given $x = 0$ is a regular singular point for

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

with $\lim_{x\to 0} \frac{Q(x)}{P(x)} x = -\frac{1}{2}$, $\lim_{x\to 0} \frac{R(x)}{P(x)} x^2 = -1$.

The two linearly independent solutions of this ODE are of the form

a) $y_1 = \sum_{n=0}^{\infty} a_n x^{n+2}$, $y_2 = \sum_{n=0}^{\infty} b_n x^{n-1/2}$
b) $y_1 = \sum_{n=0}^{\infty} a_n x^{n-1/2}$, $y_2 = \sum_{n=0}^{\infty} b_n x^{n-1}$
c) $y_1 = \sum_{n=0}^{\infty} a_n x^{n+1/2}$, $y_2 = \sum_{n=0}^{\infty} b_n x^{n+1}$
d) $y_1 = \sum_{n=0}^{\infty} a_n x^{n+1/2}$, $y_2 = \sum_{n=0}^{\infty} b_n x^{n-1/4}$
e) $y_1 = \sum_{n=0}^{\infty} a_n x^{n-1/2}$, $y_2 = \sum_{n=0}^{\infty} b_n x^{n-1/4}$

○ a
○ b

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A suitable form of the particular solution of

$$y^{(6)} - 9y^{(4)} = xe^{-3x}$$

is $y_p(x) =$

a) $(Ax + B)e^{-3x}$
b) $(Ax^2 + Bx + C)e^{-3x}$
c) $(Ax^5 + Bx^4)e^{-3x}$
d) Ax^5e^{-3x}
e) $(Ax^2 + Bx)e^{-3x}$

○ a

$$\mathcal{L}\left\{(\sin t)^2\delta\left(t - \frac{\pi}{6}\right)\right\} =$$

a) $\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)e^{-\pi s/6}$
b) $\frac{1}{4}e^{\pi s/6}$
c) $\frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)e^{-\pi s/6}$
d) $\frac{1}{4}e^{-\pi s/6}$
e) $\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)e^{\pi s/6}$

○ a
○ b

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Putting $y(x) = \sum_{n=0}^{\infty} a_n x^n$ into the ODE $y'' - 4x^3 y = 0$ we get

a) $a_2 = 0,\ a_{n+2} = \frac{4a_{n-1}}{(n+2)(n+1)}, n \geq 1$

b) $a_2 = a_3 = 0,\ a_{n+2} = \frac{4a_{n-2}}{(n+2)(n+1)}, n \geq 2$

c) $a_2 = a_3 = a_4 = 0,\ a_{n+2} = \frac{4a_{n-3}}{(n+2)(n+1)}, n \geq 3$

d) $a_{n+2} = \frac{4a_n}{(n+2)(n+1)}, n \geq 0$

e) $a_2 = 0,\ a_{n+2} = \frac{-4a_{n-2}}{(n+2)(n+1)}, n \geq 1$

○ a

○ b

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Let $f(t) = \begin{cases} \cos 2t, 0 \leq t < \pi/2, \\ 0, \quad t \geq \pi/2. \end{cases}$

Then $\mathcal{L}\{f(t)\} =$

a) $\frac{2+2e^{-\pi s/2}}{s^2+4}$

b) $\frac{2+2e^{\pi s/2}}{s^2+4}$

c) $\frac{s+se^{-\pi s/2}}{s^2+4}$

d) $\frac{s+se^{\pi s/2}}{s^2+4}$

e) $\frac{s-se^{-\pi s/2}}{s^2+4}$

○ a

○ b

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Time left 0:23:0

The solution of the initial value problem

$$y'' + 9y = \delta(t-2), \qquad y(0) = 0, y'(0) = 0,$$

is $y(t) =$

a) $\frac{1}{3}u(t-2)\cos 3t$

b) $\frac{1}{3}u(t-2)\cos(3t-6)$

c) $\frac{1}{3}u(t-2)\sin 3t$

d) $\frac{1}{3}u(t-2)\sin(3t-6)$

e) $\frac{1}{3}u(t-2)\sin(3t+6)$

○ a

○ b

Time left

The general solution of the system of ODEs $\begin{matrix} y_1' = 2y_1 + y_2 \\ y_2' = -y_1 \end{matrix}$ is

a) $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \right)$

b) $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ -2 \end{bmatrix} e^t \right)$

c) $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \right)$

d) $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \right)$

e) $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} e^t \right)$

○ a

○ b

○ c

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Time left 0:2

If the solution of the ODE $y'' + 3xy' = 0$
is given by $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with a_0 and a_1 are arbitrary,
$a_2 = 0,\ a_{n+2} = \frac{-3na_n}{(n+2)(n+1)}, n \geq 1.$
In such a case the general solution of the above ODE is $y(x) =$

a) $a_0\left(1 - \frac{1}{4}x^3 + \frac{1}{16}x^5 - \cdots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{9}{40}x^5 - \cdots\right)$

b) $a_0 + a_1\left(x + \frac{1}{2}x^3 - \frac{9}{20}x^5 - \cdots\right)$

c) $a_0 + a_1\left(x - \frac{1}{2}x^3 + \frac{9}{40}x^5 - \cdots\right)$

d) $a_0\left(1 + \frac{1}{4}x^3 - \frac{1}{16}x^5 - \cdots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{9}{40}x^5 - \cdots\right)$

e) $a_0\left(1 + \frac{1}{2}x^3 - \frac{1}{3}x^5 - \cdots\right) + a_1\left(x - \frac{1}{2}x^3 - \frac{9}{20}x^5 - \cdots\right)$

a

b

c

Tin

A particular solution of the nonhomogeneous ODE

$$y'' - 8y' + 16y = 1/x$$

is given by

a) $y_p(x) = e^{4x}\int e^{-4x}\,dx + xe^{4x}\int \frac{1}{x}e^{4x}dx$

b) $y_p(x) = -e^{-4x}\int e^{-4x}\,dx + xe^{-4x}\int \frac{1}{x}e^{-4x}dx$

c) $y_p(x) = e^{-4x}\int e^{-4x}\,dx + xe^{-4x}\int \frac{1}{x}e^{-4x}dx$

d) $y_p(x) = -e^{4x}\int e^{4x}\,dx - xe^{4x}\int \frac{1}{x}e^{4x}dx$

e) $y_p(x) = -e^{4x}\int e^{-4x}\,dx + xe^{4x}\int \frac{1}{x}e^{-4x}dx$

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