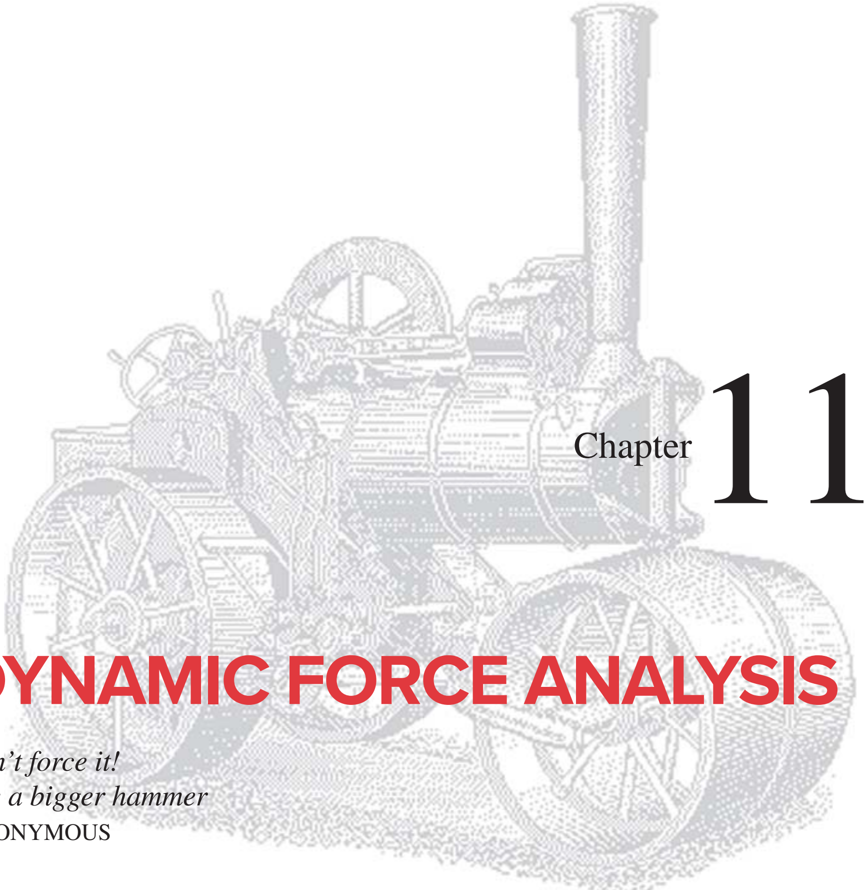

Chapter 11

# DYNAMIC FORCE ANALYSIS

*Don't force it!*
*Use a bigger hammer*
ANONYMOUS

## 11.0 INTRODUCTION *Watch the first lecture video for this chapter (27:28)**

When kinematic synthesis and analysis have been used to define a geometry and set of motions for a particular design task, it is logical and convenient to then use a **kinetostatic**, or **inverse dynamics,** solution to determine the forces and torques in the system. We will take that approach in this chapter and concentrate on solving for the forces and torques that result from, and are required to drive, our kinematic system in such a way as to provide the designed accelerations. Numerical examples are presented throughout this chapter. These examples are also downloadable as disk files for input to either program MATRIX or LINKAGES. These programs are described in Appendix A. The reader is encouraged to open the referenced files in these programs and investigate the examples in more detail. The file names are noted in the discussion of each example.

* http://www.designofmachinery.com/DOM/Dynamic_Force_Analysis.mp4



## 11.1 NEWTONIAN SOLUTION METHOD

Dynamic force analysis can be done by any of several methods. The one which gives the most information about forces internal to the mechanism requires only the use of Newton's law as defined in equations 10.1 and 10.4. These can be written as a summation of all forces and torques in the system.

$$\sum \mathbf{F} = m\mathbf{a} \qquad \sum \mathbf{T} = I_G \alpha \tag{11.1a}$$

It is also convenient to separately sum force components in *X* and *Y* directions, with the coordinate system chosen for convenience. The torques in our two-dimensional system are all in the *Z* direction. This lets us break the two vector equations into three scalar equations:

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum T = I_G \alpha \tag{11.1b}$$

These three equations must be written for each moving body in the system which will lead to a set of linear simultaneous equations for any system. The set of simultaneous equations can most conveniently be solved by a matrix method as was shown in Chapter 5. These equations do not account for the gravitational force (weight) on a link. If the kinematic accelerations are large compared to gravity, which is often the case, then the weight forces can be ignored in the dynamic analysis. If the machine members are very massive or moving slowly with small kinematic accelerations, or both, the weight of the members may need to be included in the analysis. The weight can be treated as an external force acting on the *CG* of the member at a constant angle.

## 11.2 SINGLE LINK IN PURE ROTATION *Watch a short video* (15:30)[†]

[†] http://www.designofmachinery.com/DOM/Single_Link_in_Rotation.mp4

As a simple example of this solution procedure, consider the single link in pure rotation shown in Figure 11-1a. In any of these kinetostatic dynamic force analysis problems, the kinematics of the problem must first be fully defined. That is, the angular accelerations of all rotating members and the linear accelerations of the *CG*s of all moving members must be found for all positions of interest. The mass of each member and the mass moment of inertia $I_G$ with respect to each member's *CG* must also be known. In addition there may be external forces or torques applied to any member of the system. These are all shown in the figure.

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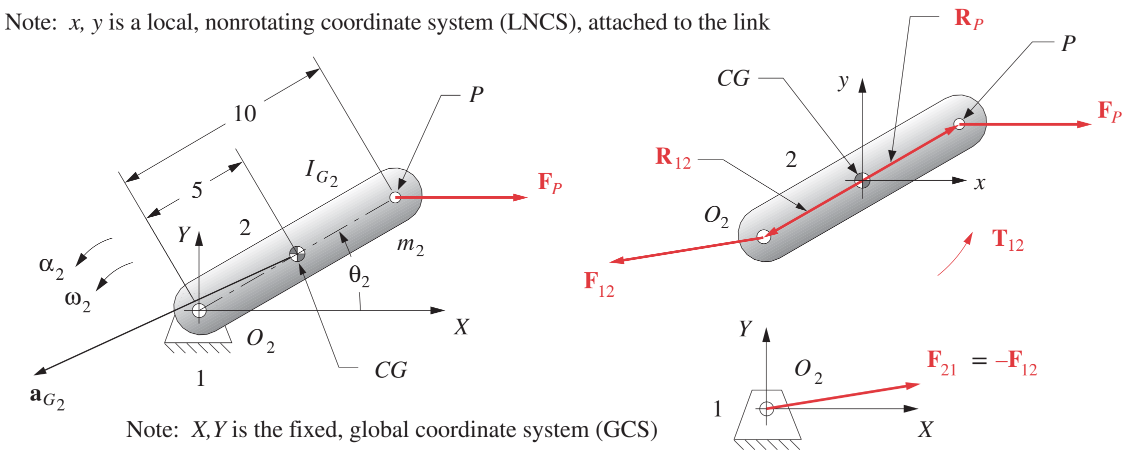
While this analysis can be approached in many ways, it is useful for the sake of consistency to adopt a particular arrangement of coordinate systems and stick with it. We present such an approach here which, if carefully followed, will tend to minimize the chances of error. The reader may wish to develop his or her own approach once the principles are understood. The underlying mathematics is invariant, and one can choose coordinate systems for convenience. The vectors which are acting on the dynamic system in any loading situation are the same at a particular time regardless of how we may decide to resolve them into components for the sake of computation. The solution result will be the same.

We will first set up a nonrotating, local coordinate system on each moving member, located at its *CG*. (In this simple example we have only one moving member.) All externally applied forces, whether due to other connected members or to other systems must then have their points of application located in this local coordinate system. Figure 11-1b shows a free-body diagram of the moving link 2. The pin joint at $O_2$ on link 2 has a force $\mathbf{F}_{12}$ due to the mating link 1, the *x* and *y* components of which are $F_{12x}$ and $F_{12y}$. These subscripts are read "force of link 1 on 2" in the *x* or *y* direction. This subscript notation scheme will be used consistently to indicate which of the "action-reaction" pair of forces at each joint is being solved for.




(*a*) Kinematic diagram (*b*) Force (free-body) diagrams

**FIGURE 11-1**

Dynamic force analysis of a single link in pure rotation

There is also an externally applied force $\mathbf{F}_P$ shown at point $P$, with components $F_{P_x}$ and $F_{P_y}$. The points of application of these forces are defined by position vectors $\mathbf{R}_{12}$ and $\mathbf{R}_P$, respectively. These position vectors are defined with respect to the local coordinate system at the *CG* of the member. We will need to resolve them into $x$ and $y$ components. There will have to be a source torque available on the link to drive it at the kinematically defined accelerations. This is one of the unknowns to be solved for. The source torque is the torque delivered *from the ground to the driver link 2* and so is labeled $\mathbf{T}_{12}$. The other two unknowns in this example are the force components at the pin joint $F_{12x}$ and $F_{12y}$.



We have three unknowns and three equations, so the system can be solved. Equations 11.1 can now be written for the moving link 2. Any applied forces or torques whose directions are known must retain the proper signs on their components. We will assume all unknown forces and torques to be positive. Their true signs will "come out in the wash."

$$\begin{aligned}\sum \mathbf{F} &= \mathbf{F}_P + \mathbf{F}_{12} = m_2 \mathbf{a}_G \\ \sum \mathbf{T} &= \mathbf{T}_{12} + \left(\mathbf{R}_{12} \times \mathbf{F}_{12}\right) + \left(\mathbf{R}_P \times \mathbf{F}_P\right) = I_G \alpha\end{aligned} \tag{11.2}$$

The force equation can be broken into its two components. The torque equation contains two cross product terms which represent torques due to the forces applied at a distance from the *CG*. When these cross products are expanded, the system of equations becomes:

$$\begin{aligned}F_{P_x} + F_{12_x} &= m_2 a_{G_x} \\ F_{P_y} + F_{12_y} &= m_2 a_{G_y} \\ T_{12} + \left(R_{12_x} F_{12_y} - R_{12_y} F_{12_x}\right) + \left(R_{P_x} F_{P_y} - R_{P_y} F_{P_x}\right) &= I_G \alpha\end{aligned} \tag{11.3}$$

This can be put in matrix form with the coefficients of the unknown variables forming the **A** matrix, the unknown variables the **B** vector, and the constant terms the **C** vector and then solved for **B**.

$$[\mathbf{A}] \times [\mathbf{B}] = [\mathbf{C}]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -R_{12_y} & R_{12_x} & 1 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} m_2 a_{G_x} - F_{P_x} \\ m_2 a_{G_y} - F_{P_y} \\ I_G \alpha - \left( R_{P_x} F_{P_y} - R_{P_y} F_{P_x} \right) \end{bmatrix} \tag{11.4}$$

Note that the **A** matrix contains all the geometric information and the **C** matrix contains all the dynamic information about the system. The **B** matrix contains all the unknown forces and torques. We will now present a numerical example to reinforce your understanding of this method.

## EXAMPLE 11-1

Dynamic Force Analysis of a Single Link in Pure Rotation. (See Figure 11-1)

***Given:*** The 10-in-long link shown weighs 4 lb. Its *CG* is on the line of centers at the 5-in point. Its mass moment of inertia about its *CG* is 0.08 lb-in-sec$^2$. Its kinematic data are:

| $\theta_2$ deg | $\omega_2$ rad/sec | $\alpha_2$ rad/sec$^2$ | $a_{G_2}$ in/sec$^2$ |
|---|---|---|---|
| 30 | 20 | 15 | 2001 @ 208° |

An external force of 40 lb at 0° is applied at point *P*.

***Find:*** The **force F**$_{12}$ at pin joint $O_2$ and the driving **torque T**$_{12}$ needed to maintain motion with the given acceleration for this instantaneous position of the link.

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***Solution:***

1 Convert the given weight to proper mass units, in this case blobs:

$$mass = \frac{weight}{g} = \frac{4 \text{ lb}}{386 \text{ in/sec}^2} = 0.0104 \text{ blob} \tag{a}$$

2 Set up a local coordinate system at the *CG* of the link and draw all applicable vectors acting on the system as shown in the figure. Draw a free-body diagram as shown.

3 Calculate the *x* and *y* components of the position vectors $\mathbf{R}_{12}$ and $\mathbf{R}_P$ in this coordinate system:

$$\begin{aligned} \mathbf{R}_{12} &= 5 \text{ in} @ \angle 210°; \quad R_{12_x} = -4.33, \quad R_{12_y} = -2.50 \\ \mathbf{R}_P &= 5 \text{ in} @ \angle 30°; \quad R_{P_x} = +4.33, \quad R_{P_y} = +2.50 \end{aligned} \tag{b}$$

4 Calculate the *x* and *y* components of the acceleration of the *CG* in this coordinate system:

$$\mathbf{a}_G = 2001 @ \angle 208°; \qquad a_{G_x} = -1766.78, \qquad a_{G_y} = -939.41 \tag{c}$$

5 Calculate the $x$ and $y$ components of the external force at $P$ in this coordinate system:

$$\mathbf{F}_P = 40 @ \angle 0°; \qquad F_{P_x} = 40, \qquad F_{P_y} = 0 \tag{d}$$

6 Substitute these given and calculated values into the matrix equation 11.4:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2.50 & -4.33 & 1 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} (0.01)(-1766.78) - 40 \\ (0.01)(-939.41) - 0 \\ (0.08)(15) - \{(4.33)(0) - (2.5)(40)\} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2.50 & -4.33 & 1 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} -57.67 \\ -9.39 \\ 101.2 \end{bmatrix} \tag{e}$$

7 Solve this system either by inverting matrix **A** and premultiplying that inverse times matrix **C** using a pocket calculator with matrix capability; using *Mathcad* or *Matlab*; or by putting the values for matrices **A** and **C** into program MATRIX downloadable with this text.

Program MATRIX gives the following solution:

$$F_{12_x} = -57.67 \text{ lb}, \qquad F_{12_y} = -9.39 \text{ lb}, \qquad T_{12} = 204.72 \text{ lb-in} \tag{f}$$

Converting the force to polar coordinates:

$$\mathbf{F}_{12} = 58.43 @ \angle 189.25° \tag{g}$$

Open the disk file E11-01.mtr in program MATRIX to exercise this example.

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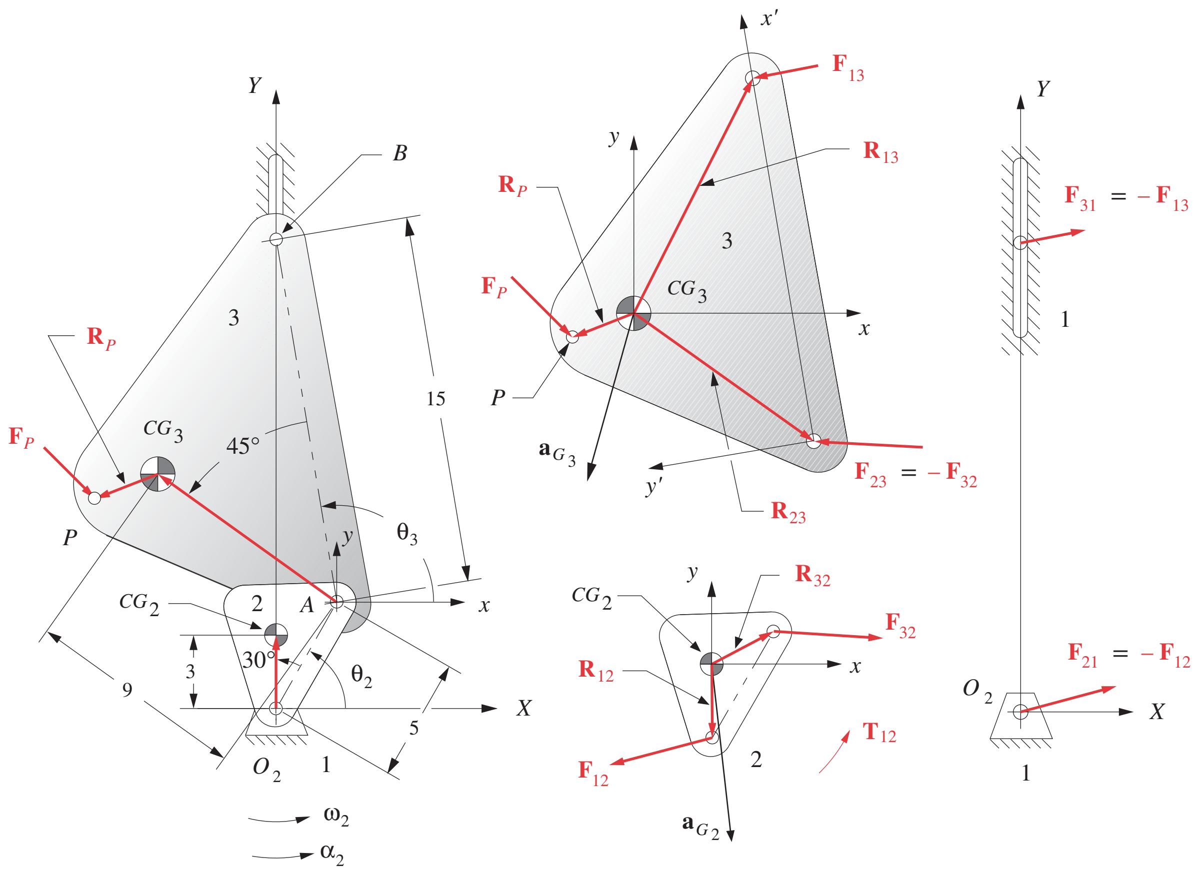
## 11.3 FORCE ANALYSIS OF A THREEBAR CRANK-SLIDE LINKAGE

When there is more than one link in the assembly, the solution simply requires that the three equations 11.1b be written for each link and then solved simultaneously. Figure 11-2a shows a threebar crank-slide linkage. This linkage has been simplified from the fourbar crank-slider (see Figure 11-4) by replacing the kinematically redundant slider block (link 4) with a half joint as shown. This linkage transformation reduces the number of links to three with no change in degree of freedom (see Section 2.10). Only links 2 and 3 are moving. Link 1 is ground. Thus we should expect to have six equations in six unknowns (three per moving link).

Figure 11-2b shows the linkage "exploded" into its three separate links, drawn as free bodies. A kinematic analysis must have been done in advance of this dynamic force analysis in order to determine, for each moving link, its angular acceleration and the linear acceleration of its *CG*. For the kinematic analysis, only the link lengths from pin to pin




(*a*) Linkage and dimensions

(*b*) Free-body diagrams



**FIGURE 11-2**

Dynamic force analysis of a crank-slide linkage

were required. For a dynamic analysis the mass ($m$) of each link, the location of its *CG*, and its mass moment of inertia ($I_G$) about that *CG* are also needed.

The *CG* of each link is initially defined by a position vector rooted at one pin joint whose angle is measured with respect to the line of centers of the link in the local, rotating coordinate system (LRCS) $x'$, $y'$. This is the most convenient way to establish the *CG* location since the link line of centers is the kinematic definition of the link. However, we will need to define the link's dynamic parameters and force locations with respect to a local, nonrotating coordinate system (LNCS) $x$, $y$ located at its *CG* and which is always parallel to the global coordinate system (GCS) *XY*. The position vector locations of all attachment points of other links and points of application of external forces must be defined with respect to the link's LNCS. Note that these kinematic and applied force data must be available for all positions of the linkage for which a force analysis is desired. In the following discussion and examples, only one linkage position will be addressed. The process is identical for each succeeding position and only the calculations must be repeated. Obviously, a computer will be a valuable aid in accomplishing the task.

Link 2 in Figure 11-2b shows forces acting on it at each pin joint, designated $\mathbf{F}_{12}$ and $\mathbf{F}_{32}$. By convention their subscripts denote the force that the adjoining link is exerting *on* the link being analyzed; that is, $\mathbf{F}_{12}$ is the force of 1 *on* 2 and $\mathbf{F}_{32}$ is the force of 3 *on* 2. Obviously there is also an equal and opposite force at each of these pins which would be designated as $\mathbf{F}_{21}$ and $\mathbf{F}_{23}$, respectively. The choice of which of the members of these pairs of forces to be solved for is arbitrary. As long as proper bookkeeping is done, their identities will be maintained.

When we move to link 3, we maintain the same convention of showing forces acting *on* the link in its free-body diagram. Thus at instant center $I_{23}$ we show $\mathbf{F}_{23}$ acting on link 3. However, because we showed force $\mathbf{F}_{32}$ acting at the same point on link 2, this introduces an additional unknown to the problem for which we need an additional equation. The equation is available from Newton's third law:

$$\mathbf{F}_{23} = -\mathbf{F}_{32} \tag{11.5}$$

Thus we are free to substitute the negative reaction force for any action force at any joint. This has been done on link 3 in the figure in order to reduce the unknown forces at that joint to one, namely $\mathbf{F}_{32}$. The same procedure is followed at each joint with one of the action-reaction forces arbitrarily chosen to be solved for and its negative reaction applied to the mating link.

The naming convention used for the position vectors ($\mathbf{R}_{ap}$) which locate the pin joints with respect to the *CG* in the link's nonrotating local coordinate system is as follows. The first subscript ($a$) denotes the adjoining link to which the position vector points. The second subscript ($p$) denotes the parent link to which the position vector belongs. Thus in the case of link 2 in Figure 11-2b, vector $\mathbf{R}_{12}$ locates the attachment point of link 1 to link 2, and $\mathbf{R}_{32}$ the attachment point of link 3 to link 2. Note that in some cases these subscripts will match those of the pin forces shown acting at those points, but where the negative reaction force has been substituted as described above, the subscript order of the force and its position vector will not agree. This can lead to confusion and must be carefully watched for typographical errors when setting up the problem. 11

Any external forces acting on the links are located in similar fashion with a position vector to a point on the line of application of the force. This point is given the same letter subscript as that of the external force. Link 3 in the figure shows such an external force $\mathbf{F}_P$ acting on it at point $P$. The position vector $\mathbf{R}_P$ locates that point with respect to the *CG*. It is important to note that the *CG* of each link is consistently taken as the point of reference for all forces acting on that link. Left to its own devices, an unconstrained body in complex motion will spin about its own *CG*; thus we analyze its linear acceleration at that point and apply the angular acceleration about the *CG* as a center.

Equations 11.1 are now written for each moving link. For link 2, with the cross products expanded:

$$\begin{aligned} F_{12_x} + F_{32_x} &= m_2 a_{G_{2x}} \\ F_{12_y} + F_{32_y} &= m_2 a_{G_{2y}} \end{aligned} \tag{11.6a}$$

$$T_{12} + \left( R_{12_x} F_{12_y} - R_{12_y} F_{12_x} \right) + \left( R_{32_x} F_{32_y} - R_{32_y} F_{32_x} \right) = I_{G_2} \alpha$$

For link 3, with the cross products expanded, note the substitution of the reaction force $-\mathbf{F}_{32}$ for $\mathbf{F}_{23}$:

$$\begin{gathered}F_{13_x} - F_{32_x} + F_{P_x} = m_3 a_{G_{3x}} \\ F_{13_y} - F_{32_y} + F_{P_y} = m_3 a_{G_{3y}} \\ \left(R_{13_x} F_{13_y} - R_{13_y} F_{13_x}\right) - \left(R_{23_x} F_{32_y} - R_{23_y} F_{32_x}\right) + \left(R_{P_x} F_{P_y} - R_{P_y} F_{P_x}\right) = I_{G_3} \alpha_3\end{gathered} \tag{11.6b}$$

Note also that $\mathbf{T}_{12}$, the source torque, only appears in the equation for link 2 as that is the driver crank to which the motor is attached. Link 3 has no externally applied torque but does have an external force $\mathbf{F}_P$ which might be due to whatever link 3 is pushing on to do its external work.

There are seven unknowns present in these six equations, $F_{12x}$, $F_{12y}$, $F_{32x}$, $F_{32y}$, $F_{13x}$, $F_{13y}$, and $T_{12}$. But, $F_{13y}$ is due only to friction at the joint between link 3 and link 1. We can write a relation for the friction force $f$ at that interface such as $f = \pm\mu N$, where $\pm\mu$ is a known coefficient of coulomb friction. The friction force always opposes motion. The kinematic analysis will provide the velocity of the link at the sliding joint. The direction of $f$ will always be the opposite of this velocity. Note that $\mu$ is a nonlinear function which has a discontinuity at zero velocity; thus at the linkage positions where velocity is zero, the inclusion of $\mu$ in these linear equations is not valid. (See Figure 10-7a.) In this example, the normal force $N$ is equal to $F_{13x}$ and the friction force $f$ is equal to $F_{13y}$. For linkage positions with nonzero velocity, we can eliminate $F_{13y}$ by substituting into equation 11.6b,

$$F_{13_y} = -\mu\, SGN\left(V_{31}\right)\left|F_{13_x}\right| \tag{11.6c}$$

where $\mu$ is negated and multiplied by the sign of the velocity at that point. The absolute value on $F_{13x}$ is needed to prevent reversal of $F_{13y}$ with the sign of $F_{13x}$. Friction doesn't care which side of the pin $B$ is being forced against the slot by $F_{13x}$.

We are then left with six unknowns in equations 11.6 and can solve them simultaneously. We also rearrange equations 11.6a and 11.6b to put all known terms on the right side.

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$$\begin{gathered}F_{12_x} + F_{32_x} = m_2 a_{G_{2x}} \\ F_{12_y} + F_{32_y} = m_2 a_{G_{2y}} \\ T_{12} + R_{12_x} F_{12_y} - R_{12_y} F_{12_x} + R_{32_x} F_{32_y} - R_{32_y} F_{32_x} = I_{G_2} \alpha_2 \\ F_{13_x} - F_{32_x} = m_3 a_{G_{3x}} - F_{P_x} \\ -\mu\, SGN\left(V_{31}\right)\left|F_{13_x}\right| - F_{32_y} = m_3 a_{G_{3y}} - F_{P_y} \\ \left(-\mu R_{13_x} - R_{13_y}\right) F_{13_x} - R_{23_x} F_{32_y} + R_{23_y} F_{32_x} = I_{G_3} \alpha_3 - R_{P_x} F_{P_y} + R_{P_y} F_{P_x}\end{gathered} \tag{11.6d}$$

Putting these six equations in matrix form, we get:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -R_{12_y} & R_{12_x} & -R_{32_y} & R_{32_x} & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -\mu SGN(V_{31}) & 0 \\ 0 & 0 & R_{23_y} & -R_{23_x} & \left(\mu R_{13_x} - R_{13_y}\right) & 0 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{13_x} \\ T_{12} \end{bmatrix} = \begin{bmatrix} m_2 a_{G_{2x}} \\ m_2 a_{G_{2y}} \\ I_{G_2}\alpha_2 \\ m_3 a_{G_{3x}} - F_{P_x} \\ m_3 a_{G_{3y}} - F_{P_y} \\ I_{G_3}\alpha_3 - R_{P_x}F_{P_y} + R_{P_y}F_{P_x} \end{bmatrix} \tag{11.7}$$

This system can be solved by using program MATRIX or any other matrix solving calculator. As an example of this solution consider the following linkage data.

## EXAMPLE 11-2

Dynamic Force Analysis of a Threebar Crank-Slide Linkage with Half Joint. (See Figure 11-2.)

**Given:** The 5-in long crank (link 2) shown weighs 2 lb. Its *CG* is at 3 in and 30° from the line of centers. Its mass moment of inertia about its *CG* is 0.05 lb-in-sec$^2$. Its acceleration is defined in its LNCS, *x,y*. Its kinematic data are:

| $\theta_2$ deg | $\omega_2$ rad/sec | $\alpha_2$ rad/sec$^2$ | $a_{G_2}$ in/sec$^2$ |
|---|---|---|---|
| 60 | 30 | –10 | 2700.17 @ –89.4° |

The coupler (link 3) is 15 in long and weighs 4 lb. Its *CG* is at 9 in and 45° from the line of centers. Its mass moment of inertia about its *CG* is 0.10 lb-in-sec$^2$. Its acceleration is defined in its LNCS, *x,y*. Its kinematic data are:

| $\theta_3$ deg | $\omega_3$ rad/sec | $\alpha_3$ rad/sec$^2$ | $a_{G_3}$ in/sec$^2$ |
|---|---|---|---|
| 99.59 | –8.78 | –136.16 | 3453.35 @ 254.4° |

The sliding joint on link 3 has a velocity of 96.95 in/sec in the +*Y* direction.

There is an external force of 50 lb at – 45°, applied at point *P* which is located at 2.7 in and 101° from the *CG* of link 3, measured in the link's embedded, rotating coordinate system or LRCS *x'*, *y'* (origin at *A* and *x* axis from *A* to *B*). The coefficient of friction μ is 0.2.

**Find:** The **forces** $\mathbf{F}_{12}$, $\mathbf{F}_{32}$, $\mathbf{F}_{13}$ at the joints and the driving **torque** $\mathbf{T}_{12}$ needed to maintain motion with the given acceleration for this instantaneous position of the link.

**Solution:**

1 Convert the given weights to proper mass units, in this case blobs:

$$mass_{link2} = \frac{weight}{g} = \frac{2 \text{ lb}}{386 \text{ in/sec}^2} = 0.0052 \text{ blob} \tag{a}$$

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$$mass_{link3} = \frac{weight}{g} = \frac{4 \text{ lb}}{386 \text{ in/sec}^2} = 0.0104 \text{ blob} \tag{b}$$

2 Set up a local, nonrotating $xy$ coordinate system (LNCS) at the *CG* of each link, and draw all applicable position and force vectors acting within or on that system as shown in Figure 11-2. Draw a free-body diagram of each moving link as shown.

3 Calculate the $x$ and $y$ components of the position vectors $\mathbf{R}_{12}$, $\mathbf{R}_{32}$, $\mathbf{R}_{23}$, $\mathbf{R}_{13}$, and $\mathbf{R}_P$ in the LNCS coordinate system:

$$\begin{array}{lll}
\mathbf{R}_{12} = 3.00 \;@\; \angle\; 270.0°; & R_{12_x} = 0.000, & R_{12_y} = -3.0 \\
\mathbf{R}_{32} = 2.83 \;@\; \angle\; 28.0°; & R_{32_x} = 2.500, & R_{32_y} = 1.333 \\
\mathbf{R}_{23} = 9.00 \;@\; \angle\; 324.5°; & R_{23_x} = 7.329, & R_{23_y} = -5.224 \\
\mathbf{R}_{13} = 10.72 \;@\; \angle\; 63.14°; & R_{13_x} = 4.843, & R_{13_y} = 9.563 \\
\mathbf{R}_{P} = 2.70 \;@\; \angle\; 201.0°; & R_{P_x} = -2.521, & R_{P_y} = -0.968
\end{array} \tag{c}$$

These position vector angles are measured with respect to the LNCS which is always parallel to the global coordinate system (GCS), making the angles the same in both systems.

4 Calculate the $x$ and $y$ components of the acceleration of the *CG*s of all moving links in the global coordinate system:

$$\begin{array}{lll}
\mathbf{a}_{G_2} = 2700.17 \;@\; \angle\; -89.4°; & a_{G_{2x}} = 28.28, & a_{G_{2y}} = -2700 \\
\mathbf{a}_{G_3} = 3453.35 \;@\; \angle\; 254.4°; & a_{G_{3x}} = -930.82, & a_{G_{3y}} = -3325.54
\end{array} \tag{d}$$

5 Calculate the $x$ and $y$ components of the external force at $P$ in the global coordinate system:

$$\mathbf{F}_P = 50 @ \angle -45°; \qquad F_{P_x} = 35.36, \qquad F_{P_y} = -35.36 \tag{e}$$

6 Substitute these given and calculated values into the matrix equation 11.7.

$$\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
3 & 0 & -1.333 & 2.5 & 0 & 1 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -0.2 & 0 \\
0 & 0 & -5.224 & -7.329 & [(0.2)4.843-(9.563)] & 0
\end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{13_x} \\ T_{12} \end{bmatrix} =$$

$$\begin{bmatrix}
(0.005)(28.28) \\
(0.005)(-2700) \\
(0.05)(-10) \\
(0.01)(-930.82) - 35.36 \\
(0.01)(-3325.54) - (-35.36) \\
(0.1)(-136.16) - (-2.521)(-35.36) + (-0.968)(35.36)
\end{bmatrix} = \begin{bmatrix} 0.141 \\ -13.500 \\ -0.500 \\ -44.668 \\ 2.105 \\ -136.987 \end{bmatrix} \tag{f}$$

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7 Solve this system either by inverting matrix **A** and premultiplying that inverse times matrix **C** using a pocket calculator with matrix capability; using *Mathcad* or *Matlab*; or by inputting the values for matrices **A** and **C** to program MATRIX downloadable with this text which gives the following solution:

$$\begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{13_x} \\ T_{12} \end{bmatrix} = \begin{bmatrix} -39.232 \\ -10.336 \\ 39.373 \\ -3.164 \\ -5.295 \\ 177.590 \end{bmatrix} \tag{g}$$

Converting the forces to polar coordinates:

$$\begin{aligned} \mathbf{F}_{12} &= 40.57 \text{ lb} \quad @ \angle\ 194.76° \\ \mathbf{F}_{32} &= 39.50 \text{ lb} \quad @ \angle\ -4.60° \\ \mathbf{F}_{13} &= 5.40 \text{ lb} \quad @ \angle\ 191.31° \end{aligned} \tag{h}$$

---

Open the disk file E11-02.mtr in program MATRIX to exercise this example.

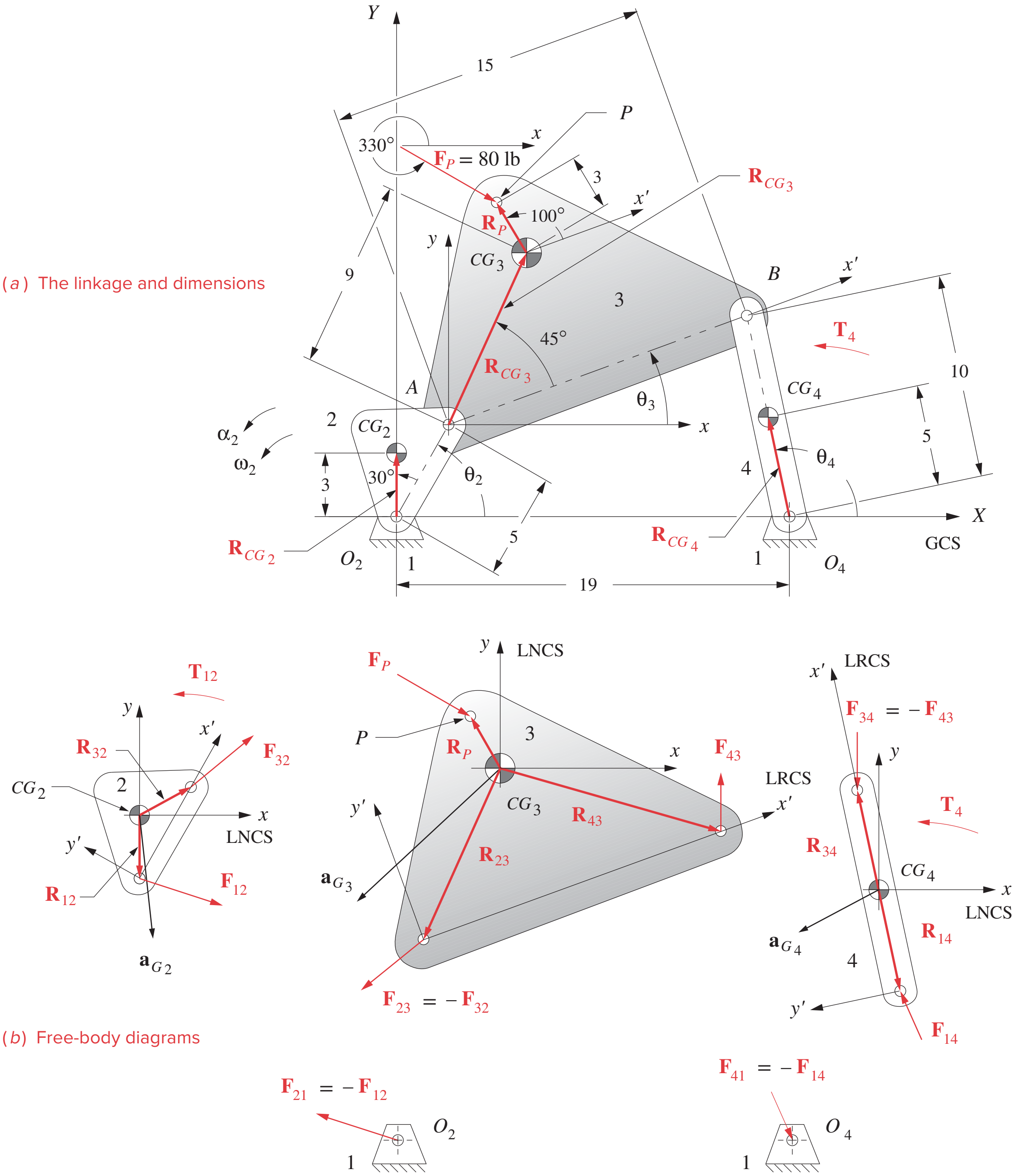
## 11.4 FORCE ANALYSIS OF A FOURBAR LINKAGE *Watch a video (12:19)*†

Figure 11-3a shows a fourbar linkage. All dimensions of link lengths, link positions, locations of the links' *CG*s, linear accelerations of those *CG*s, and link angular accelerations and velocities have been previously determined from a kinematic analysis. We now wish to find the forces acting at all the pin joints of the linkage for one or more positions. The procedure is exactly the same as that used in the previous two examples. This linkage has three moving links. Equation 11.1 provides three equations for any link or rigid body in motion. We should expect to have nine equations in nine unknowns for this problem.

Figure 11-3b shows the free-body diagrams for all links, with all forces shown. Note that an external force $\mathbf{F}_P$ is shown acting on link 3 at point *P*. Also an external torque $\mathbf{T}_4$ is shown acting on link 4. These external loads are due to some other mechanism (device, person, thing, etc.) pushing or twisting against the motion of the linkage. Any link can have any number of external loads and torques acting on it. Only one external torque and one external force are shown here to serve as examples of how they are handled in the computation. (Note that a more complicated force system, if present, could also be reduced to the combination of a single force and torque on each link.)

To solve for the pin forces, it is necessary that these applied external forces and torques be defined for all positions of interest. We will solve for one member of the pair of action-reaction forces at each joint, and also for the driving torque $\mathbf{T}_{12}$ needed to be supplied at link 2 in order to maintain the kinematic state as defined. The force subscript convention is the same as that defined in the previous example. For example, $\mathbf{F}_{12}$ is the force of 1 *on* 2 and $\mathbf{F}_{32}$ is the force of 3 *on* 2. The equal and opposite forces at each of

† http://www.designofmachinery.com/DOM/Fourbar_Force_Analysis.mp4




11

**FIGURE 11-3**

Dynamic force analysis of a fourbar linkage. (See also Figure P11-2) these pins are designated $\mathbf{F}_{21}$ and $\mathbf{F}_{23}$, respectively. All the unknown forces in the figure are shown at arbitrary angles and lengths as their true values are still to be determined.

The linkage kinematic parameters are defined with respect to a global *XY* system (GCS) whose origin is at the driver pivot $O_2$ and whose *X* axis goes through link 4's fixed pivot $O_4$. The mass ($m$) of each link, the location of its *CG*, and its mass moment of inertia ($I_G$) about that *CG* are also needed. The *CG* of each link is initially defined within each link with respect to a local moving and rotating axis system (LRCS) embedded in the link because the *CG* is an unchanging physical property of the link. The origin of this *x'*, *y'* axis system is at one pin joint and the *x'* axis is the line of centers of the link. The *CG* position within the link is defined by a position vector in this LRCS. The instantaneous location of the *CG* can easily be determined for each dynamic link position by adding the angle of the internal *CG* position vector to the current GCS angle of the link.

We need to define each link's dynamic parameters and force locations with respect to a local, moving, but nonrotating axis system (LNCS) *x,y* located at its *CG* as shown on each free-body diagram in Figure 11-3b. The position vector locations of all attachment points of other links and points of application of external forces must be defined with respect to this LNCS axis system. These kinematic and applied force data differ for each position of the linkage. In the following discussion and examples, only one linkage position will be addressed. The process is identical for each succeeding position.

Equations 11.1 are written for each moving link. For link 2, the result is identical to that done for the crank-slider example in equation 11.6a.

$$
\begin{aligned}
F_{12_x} + F_{32_x} &= m_2 a_{G_{2x}} \\
F_{12_y} + F_{32_y} &= m_2 a_{G_{2y}} \\
T_{12} + \left(R_{12_x} F_{12_y} - R_{12_y} F_{12_x}\right) + \left(R_{32_x} F_{32_y} - R_{32_y} F_{32_x}\right) &= I_{G_2} \alpha_2
\end{aligned}
\tag{11.8a}
$$

For link 3, with substitution of the reaction force $-\mathbf{F}_{32}$ for $\mathbf{F}_{23}$, the result is similar to equation 11.6b with some subscript changes to reflect the presence of link 4.

$$
\begin{aligned}
F_{43_x} - F_{32_x} + F_{P_x} &= m_3 a_{G_{3x}} \\
F_{43_y} - F_{32_y} + F_{P_y} &= m_3 a_{G_{3y}} \\
\left(R_{43_x} F_{43_y} - R_{43_y} F_{43_x}\right) - \left(R_{23_x} F_{32_y} - R_{23_y} F_{32_x}\right) + \left(R_{P_x} F_{P_y} - R_{P_y} F_{P_x}\right) &= I_{G_3} \alpha_3
\end{aligned}
\tag{11.8b}
$$

For link 4, substituting the reaction force $-\mathbf{F}_{43}$ for $\mathbf{F}_{34}$, a similar set of equations 11.1 can be written:

$$
\begin{aligned}
F_{14_x} - F_{43_x} &= m_4 a_{G_{4x}} \\
F_{14_y} - F_{43_y} &= m_4 a_{G_{4y}} \\
\left(R_{14_x} F_{14_y} - R_{14_y} F_{14_x}\right) - \left(R_{34_x} F_{43_y} - R_{34_y} F_{43_x}\right) + T_4 &= I_{G_4} \alpha_4
\end{aligned}
\tag{11.8c}
$$

Note that $\mathbf{T}_{12}$, the source torque, only appears in the equation for link 2, the motor-driven crank. Link 3, in this example, has no externally applied torque (though it could have) but does have an external force $\mathbf{F}_P$. Link 4, in this example, has no external force acting on it (though it could have) but does have an external torque $\mathbf{T}_4$. (The driving link 2 could also have an externally applied force on it though it usually does not.) There are nine unknowns present in these nine equations, $F_{12x}$, $F_{12y}$, $F_{32x}$, $F_{32y}$, $F_{43x}$, $F_{43y}$, $F_{14x}$,

$F_{14y}$, and $T_{12}$, so we can solve them simultaneously. We rearrange terms in equations 11.8 to put all known constant terms on the right side and then put them in matrix form.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -R_{12_y} & R_{12_x} & -R_{32_y} & R_{32_x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & R_{23_y} & -R_{23_x} & -R_{43_y} & R_{43_x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & R_{34_y} & -R_{34_x} & -R_{14_y} & R_{14_x} & 0 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_x} \\ F_{14_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} m_2 a_{G_{2x}} \\ m_2 a_{G_{2y}} \\ I_{G_2} \alpha_2 \\ m_3 a_{G_{3x}} - F_{P_x} \\ m_3 a_{G_{3y}} - F_{P_y} \\ I_{G_3} \alpha_3 - R_{P_x} F_{P_y} + R_{P_y} F_{P_x} \\ m_4 a_{G_{4x}} \\ m_4 a_{G_{4y}} \\ I_{G_4} \alpha_4 - T_4 \end{bmatrix} \tag{11.9}$$

This system can be solved by using program MATRIX or any matrix solving calculator. As an example of this solution consider the following linkage data.

## EXAMPLE 11-3

Dynamic Force Analysis of a Fourbar Linkage. (See Figure 11-3)

**Given:** The 5-in-long crank (link 2) shown weighs 1.5 lb. Its *CG* is at 3 in @ +30° from the line of centers (LRCS). Its mass moment of inertia about its *CG* is 0.4 lb-in-sec$^2$. Its kinematic data are:

| $\theta_2$ deg | $\omega_2$ rad/sec | $\alpha_2$ rad/sec$^2$ | $a_{G_2}$ in/sec$^2$ |
|---|---|---|---|
| 60 | 25 | –40 | 1878.84 @ 273.66° |

The coupler (link 3) is 15 in long and weighs 7.7 lb. Its *CG* is at 9 in @ 45° off the line of centers (LRCS). Its mass moment of inertia about its *CG* is 1.5 lb-in-sec$^2$. Its kinematic data are:

| $\theta_3$ deg | $\omega_3$ rad/sec | $\alpha_3$ rad/sec$^2$ | $a_{G_3}$ in/sec$^2$ |
|---|---|---|---|
| 20.92 | –5.87 | 120.9 | 3646.1 @ 226.5° |

The ground link is 19 in long. The rocker (link 4) is 10 in long and weighs 5.8 lb. Its *CG* is at 5 in @ 0° on the line of centers (LRCS). Its mass moment of inertia about its *CG* is 0.8 lb-in-sec$^2$. There is an external torque on link 4 of 120 lb-in (GCS). An external force of 80 lb @ 330° acts on link 3 in the GCS, applied at point *P* at 3 in @ 100° from the *CG* of link 3 (LRCS). The kinematic data are:

| $\theta_4$ deg | $\omega_4$ rad/sec | $\alpha_4$ rad/sec$^2$ | $a_{G_4}$ in/sec$^2$ |
|---|---|---|---|
| 104.41 | 7.93 | 276.29 | 1416.8 @ 207.2° |

***Find:*** Forces $\mathbf{F}_{12}$, $\mathbf{F}_{32}$, $\mathbf{F}_{43}$, and $\mathbf{F}_{14}$ at the joints and the driving torque $\mathbf{T}_{12}$ needed to maintain motion with the given acceleration for this instantaneous position of the link.

***Solution:***

1 Convert the given weight to proper mass units, in this case blobs:

$$mass_{link2} = \frac{weight}{g} = \frac{1.5 \text{ lb}}{386 \text{ in/sec}^2} = 0.004 \text{ blob} \tag{a}$$

$$mass_{link3} = \frac{weight}{g} = \frac{7.7 \text{ lb}}{386 \text{ in/sec}^2} = 0.020 \text{ blob} \tag{b}$$

$$mass_{link4} = \frac{weight}{g} = \frac{5.8 \text{ lb}}{386 \text{ in/sec}^2} = 0.015 \text{ blob} \tag{c}$$

2 Set up an LNCS *xy* coordinate system at the *CG* of each link, and draw all applicable vectors acting on that system as shown in the figure. Draw a free-body diagram of each moving link.

3 Calculate the *x* and *y* components of the position vectors $\mathbf{R}_{12}$, $\mathbf{R}_{32}$, $\mathbf{R}_{23}$, $\mathbf{R}_{43}$, $\mathbf{R}_{34}$, $\mathbf{R}_{14}$, and $\mathbf{R}_P$ in the link's LNCS. $\mathbf{R}_{43}$, $\mathbf{R}_{34}$, and $\mathbf{R}_{14}$ will have to be calculated from the given link geometry data using the law of cosines and law of sines. Note that the current value of link 3's position angle ($\theta_3$) in the GCS must be added to the angles of all position vectors before creating their *x,y* components in the LNCS if their angles were originally measured with respect to the link's embedded, local rotating coordinate system (LRCS).

$$\begin{array}{llll}
\mathbf{R}_{12} = 3.00 \;@\; \angle\; 270.00°; & R_{12_x} = 0.000, & R_{12_y} = -3 & \\
\mathbf{R}_{32} = 2.83 \;@\; \angle\; 28.00°; & R_{32_x} = 2.500, & R_{32_y} = 1.333 & \\
\mathbf{R}_{23} = 9.00 \;@\; \angle\; 245.92°; & R_{23_x} = -3.672, & R_{23_y} = -8.217 & \\
\mathbf{R}_{43} = 10.72 \;@\; \angle\; -15.46°; & R_{43_x} = 10.332, & R_{43_y} = -2.858 & (d)\\
\mathbf{R}_{34} = 5.00 \;@\; \angle\; 104.41°; & R_{34_x} = -1.244, & R_{34_y} = 4.843 & \\
\mathbf{R}_{14} = 5.00 \;@\; \angle\; 284.41°; & R_{14_x} = 1.244, & R_{14_y} = -4.843 & \\
\mathbf{R}_{P} = 3.00 \;@\; \angle\; 120.92°; & R_{P_x} = -1.542, & R_{P_y} = 2.574 &
\end{array}$$



4 Calculate the *x* and *y* components of the acceleration of the *CG*s of all moving links in the global coordinate system (GCS):

$$\begin{array}{llll}
\mathbf{a}_{G_2} = 1878.84 \;@\; \angle 273.66°; & a_{G_{2x}} = 119.94, & a_{G_{2y}} = -1875.01 & \\
\mathbf{a}_{G_3} = 3646.10 \;@\; \angle 226.51°; & a_{G_{3x}} = -2509.35, & a_{G_{3y}} = -2645.23 & (e)\\
\mathbf{a}_{G_4} = 1416.80 \;@\; \angle 207.24°; & a_{G_{4x}} = -1259.67, & a_{G_{4y}} = -648.50 &
\end{array}$$

5 Calculate the *x* and *y* components of the external force at *P* in the GCS:

$$\mathbf{F}_{P3} = 80 \;@\; \angle\; 330°; \qquad F_{P3_x} = 69.28, \qquad F_{P3_y} = -40.00 \tag{f}$$

6 Substitute these given and calculated values into the matrix equation 11.9.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & -1.330 & 2.5 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -8.217 & 3.673 & 2.861 & 10.339 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4.843 & 1.244 & 4.843 & 1.244 & 0 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_x} \\ F_{14_y} \\ T_{12} \end{bmatrix} = \tag{g}$$

$$\begin{bmatrix} (0.004)(119.94) \\ (0.004)(-1875.01) \\ (0.4)(-40) \\ (0.02)(-2509.35)-(69.28) \\ (0.02)(-2645.23)-(-40) \\ (1.5)(120.9)-\left[(-1.542)(-40)-(2.574)(69.28)\right] \\ (0.015)(-1259.67) \\ (0.015)(-648.50) \\ (0.8)(276.29)-(120) \end{bmatrix} = \begin{bmatrix} 0.480 \\ -7.500 \\ -16.000 \\ -119.465 \\ -12.908 \\ 298.003 \\ -18.896 \\ -9.727 \\ 101.031 \end{bmatrix}$$

7 Solve this system either by inverting matrix **A** and premultiplying that inverse times matrix **C** using a pocket calculator with matrix capability, or by inputting the values for matrices **A** and **C** to program MATRIX downloadable with this text, which gives the following solution:

$$\begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_x} \\ F_{14_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} -117.65 \\ -107.84 \\ 118.13 \\ 100.34 \\ -1.34 \\ 87.43 \\ -20.23 \\ 77.71 \\ 243.23 \end{bmatrix} \tag{h}$$

Converting the forces to polar coordinates:

$$\begin{aligned} \mathbf{F}_{12} &= 159.60 \text{ lb} \quad @ \angle \quad 222.52^\circ \\ \mathbf{F}_{32} &= 154.99 \text{ lb} \quad @ \angle \quad 40.35^\circ \\ \mathbf{F}_{43} &= 87.44 \text{ lb} \quad @ \angle \quad 90.88^\circ \\ \mathbf{F}_{14} &= 80.30 \text{ lb} \quad @ \angle \quad 104.59^\circ \end{aligned} \qquad (i)$$

8 The pin-force magnitudes in (*i*) are needed to size the pivot pins and links against failure and to select pivot bearings that will last for the required life of the assembly. The driving torque $T_{12}$ defined in (*h*) is needed to select a motor or other device capable of supplying the power to drive the system. See Section 2.19 for a brief discussion of motor selection. Issues of stress calculation and failure prevention are beyond the scope of this text, but note that those calculations cannot be done until a good estimate of the dynamic forces and torques on the system has been made by methods such as those shown in this example.

---

This solves the linkage for one position. A new set of values can be put into the **A** and **C** matrices for each position of interest at which a force analysis is needed. Open the disk file E11-03.mtr in program MATRIX to exercise this example. The disk file E11-03.4br can also be opened in program LINKAGES and will run the linkage through a series of positions starting with the stated parameters as initial conditions. The linkage will slow to a stop and then run in reverse due to the negative acceleration. The matrix for equation (*g*) can be seen within LINKAGES using *Dynamics/Solve/Show Matrix.*

It is worth noting some general observations about this method at this point. The solution is done using cartesian coordinates of all forces and position vectors. Before being placed in the matrices, these vector components must be defined in the global coordinate system (GCS) or in nonrotating, local coordinate systems, parallel to the global coordinate system, with their origins at the links' *CG*s (LNCS). Some of the linkage parameters are normally expressed in such coordinate systems, but others are not, and so must be transformed to the proper coordinate system. The kinematic data should all be computed in the global system or in parallel, **nonrotating**, local systems placed at the *CG*s of individual links. Any external forces on the links must also be defined in the global system.

However, the position vectors that define intralink locations, such as the pin joints versus the *CG*, or which locate points of application of external forces versus the *CG* are defined in local, **rotating** coordinate systems embedded in the links (LRCS). Thus these position vectors must be redefined in a **nonrotating**, parallel system before being used in the matrix. An example of this is vector $\mathbf{R}_p$, which was initially defined as 3 in at 100° in link 3's embedded, **rotating** coordinate system. Note in Example 11-3 that its cartesian coordinates for use in the equations were calculated after adding the current value of $\theta_3$ to its angle. This redefined $\mathbf{R}_p$ as 3 in at 120.92° in the **nonrotating** local system. The same was done for position vectors $\mathbf{R}_{12}$, $\mathbf{R}_{32}$, $\mathbf{R}_{23}$, $\mathbf{R}_{43}$, $\mathbf{R}_{34}$, and $\mathbf{R}_{14}$. In each case the **intralink angle** of these vectors (which is independent of linkage position) was added to the current link angle to obtain its position in the *xy* system at the link's *CG*. The proper definition of these position vector components is critical to the solution, and it is very easy to make errors in defining them.

To further confuse things, even though the position vector $\mathbf{R}_p$ is initially measured in the link's embedded, rotating coordinate system, the force $\mathbf{F}_p$, which it locates, is not. The force $\mathbf{F}_p$ is not part of the link, as is $\mathbf{R}_p$, but rather is part of the external world, so it is defined in the global system.

## 11.5 FORCE ANALYSIS OF A FOURBAR CRANK-SLIDER LINKAGE

The approach taken for the pin-jointed fourbar is equally valid for a fourbar crank-slider linkage. The principal difference will be that the slider block will have no angular acceleration. Figure 11-4 shows a fourbar crank-slider with an external force on the slider block, link 4. This is representative of the mechanism used extensively in piston pumps and internal combustion engines. We wish to determine the forces at the joints and the driving torque needed on the crank to provide the specified accelerations. A kinematic analysis must have previously been done in order to determine all position, velocity, and acceleration information for the positions being analyzed. Equations 11.1 are written for each link. For link 2:

$$\begin{aligned} F_{12_x} + F_{32_x} &= m_2 a_{G_{2x}} \\ F_{12_y} + F_{32_y} &= m_2 a_{G_{2y}} \\ T_{12} + \left(R_{12_x} F_{12_y} - R_{12_y} F_{12_x}\right) + \left(R_{32_x} F_{32_y} - R_{32_y} F_{32_x}\right) &= I_{G_2} \alpha_2 \end{aligned} \tag{11.10a}$$

This is identical to equation 11.8a for the "pure" fourbar linkage. For link 3:

$$\begin{aligned} F_{43_x} - F_{32_x} &= m_3 a_{G_{3x}} \\ F_{43_y} - F_{32_y} &= m_3 a_{G_{3y}} \\ \left(R_{43_x} F_{43_y} - R_{43_y} F_{43_x}\right) - \left(R_{23_x} F_{32_y} - R_{23_y} F_{32_x}\right) &= I_{G_3} \alpha_3 \end{aligned} \tag{11.10b}$$

This is similar to equation 11.8b, lacking only the terms involving $\mathbf{F}_p$ since there is no external force shown acting on link 3 of our example crank-slider. For link 4:

$$\begin{aligned} F_{14_x} - F_{43_x} + F_{P_x} &= m_4 a_{G_{4x}} \\ F_{14_y} - F_{43_y} + F_{P_y} &= m_4 a_{G_{4y}} \\ \left(R_{14_x} F_{14_y} - R_{14_y} F_{14_x}\right) - \left(R_{34_x} F_{43_y} - R_{34_y} F_{43_x}\right) + \left(R_{P_x} F_{P_y} - R_{P_y} F_{P_x}\right) &= I_{G_4} \alpha_4 \end{aligned} \tag{11.10c}$$

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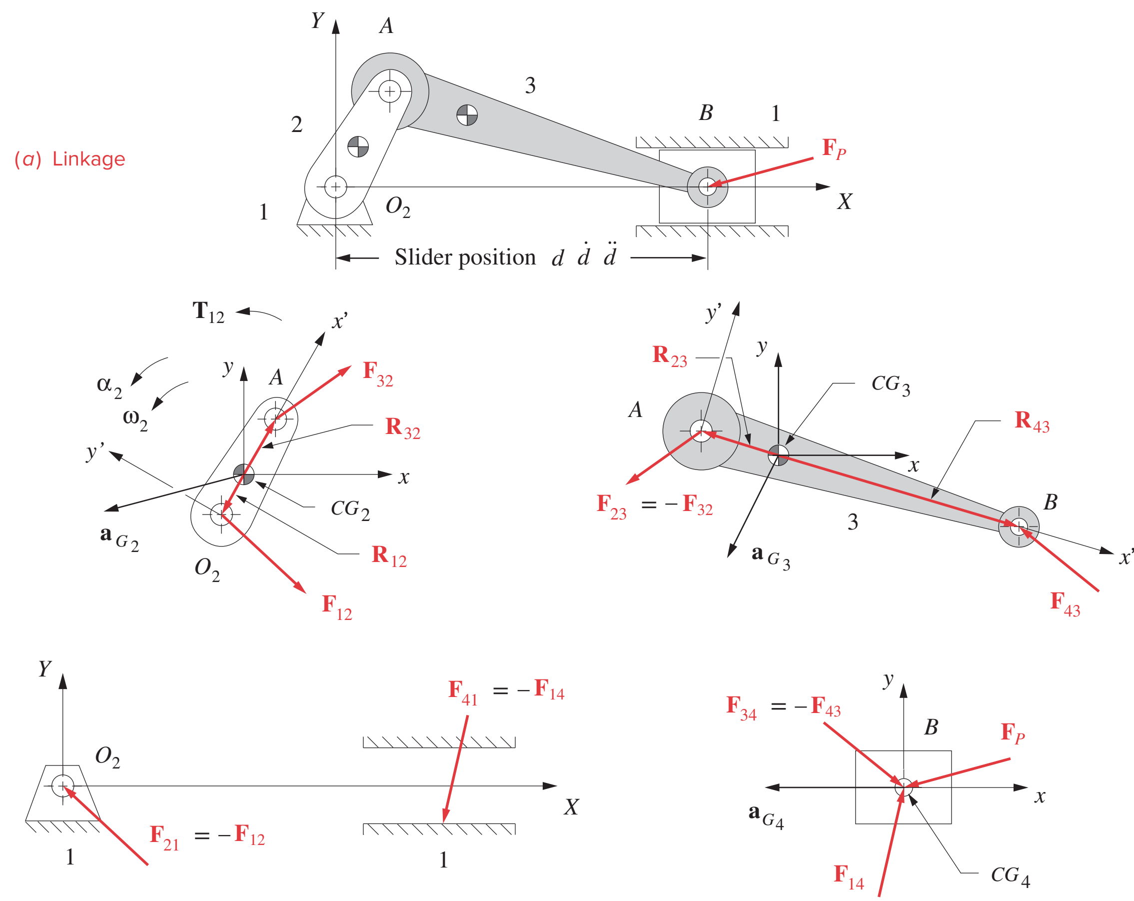
These contain the external force $\mathbf{F}_p$ shown acting on link 4.

For the inversion of the crank-slider shown, the slider block, or piston, is in pure translation against the stationary ground plane; thus it can have no angular acceleration or angular velocity. Also, the position vectors in the torque equation (equation 11.10c) are all zero as the force $\mathbf{F}_p$ acts at the *CG*. Thus the torque equation for link 4 (third expression in equation 11.10c) is zero for this inversion of the crank-slider linkage. Its linear acceleration also has no *y* component.

$$\alpha_4 = 0, \qquad a_{G_{4_y}} = 0 \tag{11.10d}$$

The only *x* directed force that can exist at the interface between links 4 and 1 is friction. Assuming coulomb friction, the *x* component can be expressed in terms of the *y* component of force at this interface. We can write a relation for the friction force *f* at that interface such as $f = \pm\mu\mathbf{N}$, where $\pm\mu$ is a known coefficient of friction. The plus and minus signs on the coefficient of friction are to recognize the fact that the friction force always opposes motion. The kinematic analysis will provide the velocity of the link at the sliding joint. The sign on $\mu$ will always be the opposite of the sign of this velocity.




(b) Free-body diagrams

**FIGURE 11-4**

Dynamic force analysis of the fourbar slider-crank linkage



$$F_{14_x} = -\mu \, SGN\left(\dot{d}\right)\left|F_{14_y}\right| \tag{11.10e}$$

The *SGN* function returns the sign of its argument. The absolute value on $F_{14y}$ is needed to prevent reversal of $F_{14x}$ with the sign of $F_{14y}$. Friction doesn't care which side of the piston is being forced against the cylinder by $F_{14y}$.

Substituting equations 11.10d and 11.10e into the reduced equation 11.10c yields:

$$\begin{aligned} -\mu \, SGN\left(\dot{d}\right)\left|F_{14_y}\right| - F_{43_x} + F_{P_x} &= m_4 a_{G_{4x}} \\ F_{14_y} - F_{43_y} + F_{P_y} &= 0 \end{aligned} \tag{11.10f}$$

This last substitution has reduced the unknowns to eight, $F_{12x}$, $F_{12y}$, $F_{32x}$, $F_{32y}$, $F_{43x}$, $F_{43y}$, $F_{14y}$, and $T_{12}$; thus we need only eight equations. We can now use the eight equations in 11.10a, b, and f to assemble the matrices for solution.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -R_{12_y} & R_{12_x} & -R_{32_y} & R_{32_x} & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & R_{23_y} & -R_{23_x} & -R_{43_y} & R_{43_x} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -\mu SGN\left(\dot{d}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_y} \\ T_{12} \end{bmatrix}$$

$$= \begin{bmatrix} m_2 a_{G_{2x}} \\ m_2 a_{G_{2y}} \\ I_{G_2}\alpha_2 \\ m_3 a_{G_{3x}} \\ m_3 a_{G_{3y}} \\ I_{G_3}\alpha_3 \\ m_4 a_{G_{4x}} - F_{P_x} \\ -F_{P_y} \end{bmatrix} \tag{11.10g}$$

Solution of this matrix equation 11.10g plus equation 11.10e will yield complete dynamic force information for the fourbar crank-slider linkage.

## 11.6 FORCE ANALYSIS OF THE INVERTED CRANK-SLIDER

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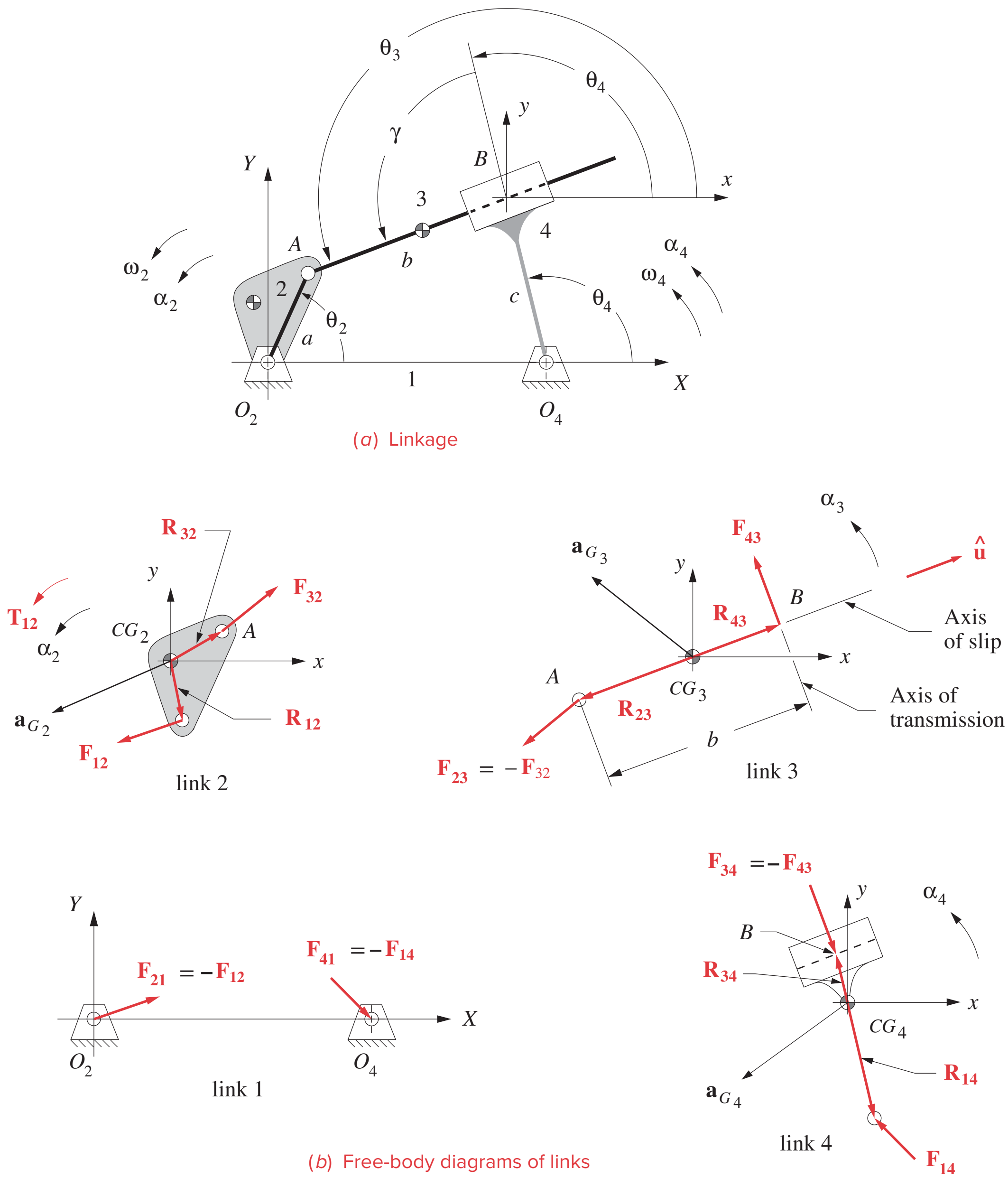
Another inversion of the fourbar crank-slider was also analyzed kinematically in Part I. It is shown in Figure 11-5. Link 4 does have an angular acceleration in this inversion. In fact, it must have the same angle, angular velocity, and angular acceleration as link 3 because they are rotationally coupled by the sliding joint. We wish to determine the forces at all pin joints and at the sliding joint as well as the driving torque needed to create the desired accelerations. Each link's joints are located by position vectors referenced to nonrotating local *xy* coordinate systems at each link's *CG* as before. The sliding joint is located by the position vector $\mathbf{R}_{43}$ to the center of the slider, point *B*. The instantaneous position of point *B* was determined from the kinematic analysis as length *b* referenced to instant center $I_{23}$ (point *A*). See Sections 4.8, 6.7, and 7.3 to review the position, velocity, and acceleration analysis of this mechanism. Recall that this mechanism has a nonzero Coriolis component of acceleration. The force between link 3 and link 4 within the sliding joint is distributed along the unspecified length of the slider block. For this analysis the distributed force can be modeled as a force concentrated at point *B* within the sliding joint. We will neglect friction in this example.




(a) Linkage

(b) Free-body diagrams of links

**FIGURE 11-5**

Dynamic forces in the inverted slider-crank fourbar linkage

The equations for links 2 and 3 are identical to those for the noninverted crank-slider (equations 11.10a and b). The equations for link 4 are the same as equations 11.10c except for the absence of the terms involving $\mathbf{F}_p$ since no external force is shown acting on link 4 in this example. The slider joint can only transmit force from link 3 to link 4 or vice versa along a line perpendicular to the axis of slip. This line is called the axis of transmission. In order to guarantee that the force $\mathbf{F}_{34}$ or $\mathbf{F}_{43}$ is always perpendicular to the axis of slip, we can write the following relation:

$$\hat{\mathbf{u}} \cdot \mathbf{F}_{43} = 0 \tag{11.11a}$$

which expands to:

$$u_x F_{43_x} + u_y F_{43_y} = 0 \tag{11.11b}$$

The dot product of two vectors will be zero when the vectors are mutually perpendicular. The unit vector $\hat{\mathbf{u}}$ is in the direction of link 3 which is defined from the kinematic analysis as $\theta_3$.

$$u_x = \cos\theta_3, \qquad u_y = \sin\theta_3 \tag{11.11c}$$

Equation 11.11b provides a tenth equation, but we have only nine unknowns, $F_{12x}$, $F_{12y}$, $F_{32x}$, $F_{32y}$, $F_{43x}$, $F_{43y}$, $F_{14x}$, $F_{14y}$, and $T_{12}$, so one of our equations is redundant. Since we must include equation 11.11, we will combine the torque equations for links 3 and 4 rewritten here in vector form and without the external force $\mathbf{F}_p$.

$$\begin{gathered}\left(\mathbf{R}_{43} \times \mathbf{F}_{43}\right) - \left(\mathbf{R}_{23} \times \mathbf{F}_{32}\right) = I_{G_3}\alpha_3 = I_{G_3}\alpha_4 \\ \left(\mathbf{R}_{14} \times \mathbf{F}_{14}\right) - \left(\mathbf{R}_{34} \times \mathbf{F}_{43}\right) = I_{G_4}\alpha_4\end{gathered} \tag{11.12a}$$

Note that the angular acceleration of link 3 is the same as that of link 4 in this linkage. Adding these equations gives:

$$\left(\mathbf{R}_{43} \times \mathbf{F}_{43}\right) - \left(\mathbf{R}_{23} \times \mathbf{F}_{32}\right) + \left(\mathbf{R}_{14} \times \mathbf{F}_{14}\right) - \left(\mathbf{R}_{34} \times \mathbf{F}_{43}\right) = \left(I_{G_3} + I_{G_4}\right)\alpha_4 \tag{11.12b}$$

Expanding and collecting terms:

$$\begin{aligned}&\left(R_{43x} - R_{34_x}\right)F_{43_y} + \left(R_{34y} - R_{43_y}\right)F_{43_x} - R_{23_x}F_{32_y} \\ &\quad + R_{23_y}F_{32_x} + R_{14_x}F_{14_y} - R_{14_y}F_{14_x} = \left(I_{G_3} + I_{G_4}\right)\alpha_4\end{aligned} \tag{11.12c}$$

Equations 11.10a, 11.11b, 11.12c, and the four force equations from equations 11.10b and 11.10c (excluding the external force $\mathbf{F}_P$) give us nine equations in the nine unknowns which we can put in matrix form for solution.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -R_{12_y} & R_{12_x} & -R_{32_y} & R_{32_x} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & R_{23_y} & -R_{23_x} & \left(R_{34_x} - R_{43_y}\right) & \left(R_{43_y} - R_{34_x}\right) & -R_{14_y} & R_{14_x} & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & u_x & u_y & 0 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} F_{12_x} \\ F_{12_y} \\ F_{32_x} \\ F_{32_y} \\ F_{43_x} \\ F_{43_y} \\ F_{14_x} \\ F_{14_y} \\ T_{12} \end{bmatrix} = \begin{bmatrix} m_2 a_{G_{2x}} \\ m_2 a_{G_{2y}} \\ I_{G_2} \alpha_2 \\ m_3 a_{G_{3x}} \\ m_3 a_{G_{3y}} \\ \left(I_{G_3} + I_{G_4}\right)\alpha_4 \\ m_4 a_{G_{4x}} \\ m_4 a_{G_{4y}} \\ 0 \end{bmatrix} \tag{11.13}$$

## 11.7 FORCE ANALYSIS—LINKAGES WITH MORE THAN FOUR BARS

This matrix method of force analysis can easily be extended to more complex assemblages of links. The equations for each link are of the same form. We can create a more general notation for equations 11.1 to apply them to any assembly of $n$ pin-connected links. Let $j$ represent any link in the assembly. Let $i = j - 1$ be the previous link in the chain and $k = j + 1$ be the next link in the chain; then, using the vector form of equations 11.1:

$$\mathbf{F}_{ij} + \mathbf{F}_{jk} + \sum \mathbf{F}_{ext_j} = m_j \, \mathbf{a}_{G_j} \tag{11.14a}$$

$$\left(\mathbf{R}_{ij} \times \mathbf{F}_{ij}\right) + \left(\mathbf{R}_{jk} \times \mathbf{F}_{jk}\right) + \sum \mathbf{T}_j + \left(\mathbf{R}_{ext_j} \times \sum \mathbf{F}_{ext_j}\right) = I_{G_j} \alpha_j \tag{11.14b}$$

where:

$$j = 2, 3, \ldots, n; \qquad i = j - 1; \qquad k = j + 1, \; j \neq n; \qquad \text{if } j = n, k = 1$$

and

$$\mathbf{F}_{ji} = -\mathbf{F}_{ij}; \qquad \mathbf{F}_{kj} = -\mathbf{F}_{jk} \tag{11.14c}$$

The sum of forces vector equation 11.14a can be broken into its two $x$ and $y$ component equations and then applied, along with the sum of torques equation 11.14b, to each of the links in the chain to create the set of simultaneous equations for solution. Any link may have external forces and/or external torques applied to it. All will have pin forces.

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Since the $n$th link in a closed chain connects to the first link, the value of $k$ for the $n$th link is set to 1. In order to reduce the number of variables to a tractable quantity, substitute the negative reaction forces from equation 11.14c where necessary as was done in the examples in this chapter. When sliding joints are present, it will be necessary to add constraints on the allowable directions of forces at those joints as was done in the inverted crank-slider derivation above.

## 11.8 SHAKING FORCE AND SHAKING MOMENT

It is usually of interest to know the net effect of the dynamic forces as felt on the ground plane as this can set up vibrations in the structure that supports the machine. For our simple examples of three- and fourbar linkages, there are only two points at which the dynamic forces can be delivered to link 1, the ground plane. More complicated mechanisms will have more joints with the ground plane. The forces delivered by the moving links to the ground at the fixed pivots $O_2$ and $O_4$ are designated $\mathbf{F}_{21}$ and $\mathbf{F}_{41}$ by our subscript convention as defined in Section 11.1. Since we chose to solve for $\mathbf{F}_{12}$ and $\mathbf{F}_{14}$ in our solutions, we simply negate those forces to obtain their equal and opposite counterparts (see also equation 11.5).

$$\mathbf{F}_{21} = -\mathbf{F}_{12} \qquad \mathbf{F}_{41} = -\mathbf{F}_{14} \tag{11.15a}$$

The *sum of all the forces acting on the ground plane* is called the **shaking force** ($\mathbf{F}_s$) as shown in Figure 11-6.* In these simple examples it is equal to:

$$\mathbf{F}_s = \mathbf{F}_{21} + \mathbf{F}_{41} \tag{11.15b}$$

The *reaction moment felt by the ground plane* is called the **shaking moment** ($\mathbf{M}_s$) as shown in Figure 11-7.* This is the negative of the source torque ($\mathbf{T}_{21} = -\mathbf{T}_{12}$) plus the cross products of the ground-pin forces and their distances from the reference point. The shaking moment about the crank pivot $O_2$ is:

* The LINKAGES files (F11-06.4br & F11-07.4br) that generated the plots in Figures 11-6 and 11-7 may be downloaded and opened in that program to see more details on the linkage's dynamics.

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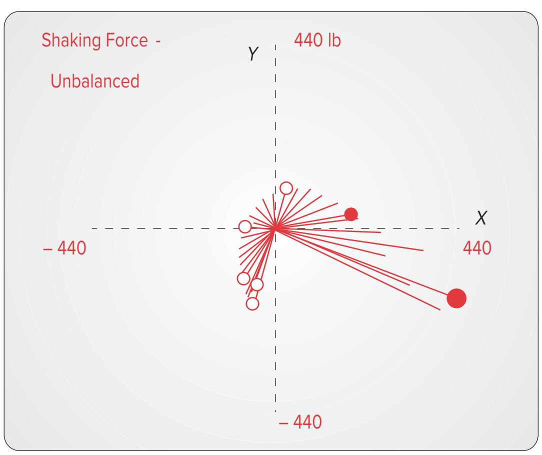



| Link No. | Length in | Mass Units | Inertia Units | CG Posit | at Deg | Ext. Force lb | at Deg |
|---|---|---|---|---|---|---|---|
| 1 | 5.5 | | | | | | |
| 2 | 2.0 | 0.002 | 0.004 | 1. 0 | 0 | | |
| 3 | 6.0 | 0.030 | 0.060 | 2.5 | 30 | 12 | 270 |
| 4 | 3.0 | 0.010 | 0.020 | 1. 5 | 0 | 60 | –45 |

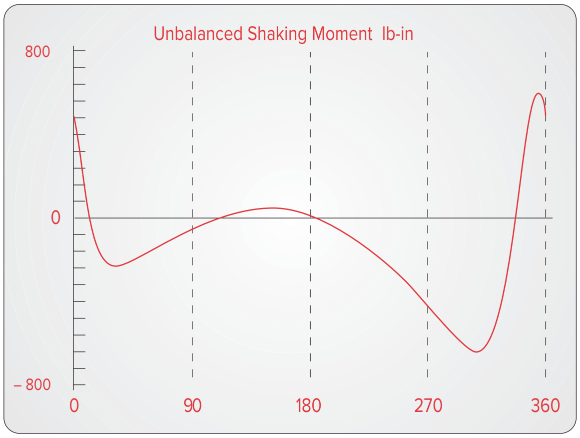
Coupler pt. = 3 in @ 45°
Open/Crossed = open

Ext. Force 3 acts at 5 in @ 30° vs. CG of Link 3
Ext. Force 4 acts at 5 in @ 90° vs. CG of Link 4
Ext. Torque 3 = – 20 lb-in
Ext. Torque 4 = 25 lb-in

Start Alpha2 = 0 rad/sec$^2$
Start Omega2 = 50 rad/sec
Start Theta2 = 0°
Final Theta2 = 360°
Delta Theta2 = 10°

**FIGURE 11-6**

Linkage data and polar plot of shaking force for an unbalanced crank-rocker fourbar linkage from program LINKAGES




| Link No. | Length in | Mass Units | Inertia Units | CG Posit | at Deg | Ext. Force lb | at Deg |
|---|---|---|---|---|---|---|---|
| 1 | 5.5 | | | | | | |
| 2 | 2.0 | 0.002 | 0.004 | 1. 0 | 0 | | |
| 3 | 6.0 | 0.030 | 0.060 | 2.5 | 30 | 12 | 270 |
| 4 | 3.0 | 0.010 | 0.020 | 1. 5 | 0 | 60 | –45 |

Coupler pt. = 3 in @ 45°
Open/Crossed = open

Ext. Force 3 acts at 5 in @ 30° vs. CG of Link 3
Ext. Force 4 acts at 5 in @ 90° vs. CG of Link 4
Ext. Torque 3 = – 20 lb-in
Ext. Torque 4 = 25 lb-in

Start Alpha2 = 0 rad/sec$^2$
Start Omega2 = 50 rad/sec
Start Theta2 = 0°
Final Theta2 = 360°
Delta Theta2 = 2°

**FIGURE 11-7**

Linkage data and shaking moment curve for an unbalanced crank-rocker fourbar linkage from program LINKAGES

$$\mathbf{M}_s = \mathbf{T}_{21} + \left(\mathbf{R}_1 \times \mathbf{F}_{41}\right) \tag{11.15c}$$

The shaking force will tend to move the ground plane back and forth, and the shaking moment will tend to rock the ground plane about the driveline axis. Both will cause vibrations. We are usually looking to minimize the effects of the shaking force and shaking moment on the frame. This can sometimes be done by balancing, sometimes by the addition of a flywheel to the system, and sometimes by shock mounting the frame to isolate the vibrations from the rest of the assembly. Most often we will use a combination of all three approaches. We will investigate some of these techniques in Chapter 12.

## 11.9 PROGRAM LINKAGES *Second lecture video for this chapter (34:51)**

The matrix methods introduced in the preceding sections all provide force and torque information for one position of the linkage assembly as defined by its kinematic and geometric parameters. To do a complete force analysis for multiple positions of a machine requires that these computations be repeated with new input data for each position. A computer program is the obvious way to accomplish this. The program LINKAGES computes the kinematic parameters for those linkages over changes in time or driver (crank) angle plus the forces and torques concomitant with the linkage kinematics and link geometry. Examples of its output are shown in Figures 11-6 and 11-7. Please refer to Appendix A for more information about this and other programs.

## 11.10 TORQUE ANALYSIS BY AN ENERGY METHOD *Watch a video (10:53)*†

In Section 10.15 the method of virtual work was presented. We will now use that approach to solve the linkage from Example 11-3 as a check on its solution by the newtonian method used in that example. The kinematic data given in Example 11-3 did not include information on the angular velocities of all the links, the linear velocities of the centers of

* http://www.designofmachinery.com/DOM/Virtual_Work_and_Flywheels.mp4

† http://www.designofmachinery.com/DOM/Virtual_Work.mp4

gravities of the links, and the linear velocity of the point $P$ of application of the external force on link 3. Velocity data were not needed for the newtonian solution but are needed for the virtual work approach and are detailed below. Equation 10.28a is repeated here and renumbered.

$$\sum_{k=2}^{n} \mathbf{F}_k \cdot \mathbf{v}_k + \sum_{k=2}^{n} \mathbf{T}_k \cdot \omega_k = \sum_{k=2}^{n} m_k \mathbf{a}_k \cdot \mathbf{v}_k + \sum_{k=2}^{n} I_k \alpha_k \cdot \omega_k \tag{11.16a}$$

Expanding the summations, still in vector form:

$$\begin{aligned}&\left(\mathbf{F}_{P_3} \cdot \mathbf{v}_{P_3} + \mathbf{F}_{P_4} \cdot \mathbf{v}_{P_4}\right) + \left(\mathbf{T}_{12} \cdot \omega_2 + \mathbf{T}_3 \cdot \omega_3 + \mathbf{T}_4 \cdot \omega_4\right)\\ &= \left(m_2 \mathbf{a}_{G_2} \cdot \mathbf{v}_{G_2} + m_3 \mathbf{a}_{G_3} \cdot \mathbf{v}_{G_3} + m_4 \mathbf{a}_{G_4} \cdot \mathbf{v}_{G_4}\right)\\ &\quad + \left(I_{G_2} \alpha_2 \cdot \omega_2 + I_{G_3} \alpha_3 \cdot \omega_3 + I_{G_4} \alpha_4 \cdot \omega_4\right)\end{aligned} \tag{11.16b}$$

Expanding the dot products to create a scalar equation:

$$\begin{aligned}&\left(F_{P_{3_x}} V_{P_{3_x}} + F_{P_{3_y}} V_{P_{3_y}}\right) + \left(F_{P_{4_x}} V_{P_{4_x}} + F_{P_{4_y}} V_{P_{4_y}}\right) + \left(T_{12}\omega_2 + T_3\omega_3 + T_4\omega_4\right)\\ &= m_2\left(a_{G_{2_x}} V_{G_{2_x}} + a_{G_{2_y}} V_{G_{2_y}}\right) + m_3\left(a_{G_{3_x}} V_{G_{3_x}} + a_{G_{3_y}} V_{G_{3y}}\right)\\ &\quad + m_4\left(a_{G_{4_x}} V_{G_{4_x}} + a_{G_{4_y}} V_{G_{4_y}}\right) + \left(I_{G_2}\alpha_2\omega_2 + I_{G_3}\alpha_3\omega_3 + I_{G_4}\alpha_4\omega_4\right)\end{aligned} \tag{11.16c}$$

### EXAMPLE 11-4

Analysis of a Fourbar Linkage by the Method of Virtual Work. (See Figure 11-3.)

**Given:** The 5-in-long crank (link 2) shown weighs 1.5 lb. Its $CG$ is at 3 in at +30° from the line of centers. Its mass moment of inertia about its $CG$ is 0.4 lb-in-sec$^2$. Its kinematic data are:

| $\theta_2$ deg | $\omega_2$ rad/sec | $\alpha_2$ rad/sec$^2$ | $V_{G_2}$ in/sec |
|---|---|---|---|
| 60 | 25 | –40 | 75 @ 180° |

The coupler (link 3) is 15 in long and weighs 7.7 lb. Its $CG$ is at 9 in at 45° off the line of centers. Its mass moment of inertia about its $CG$ is 1.5 lb-in-sec$^2$. Its kinematic data are:

| $\theta_3$ deg | $\omega_3$ rad/sec | $\alpha_3$ rad/sec$^2$ | $V_{G_3}$ in/sec |
|---|---|---|---|
| 20.92 | –5.87 | 120.9 | 72.66 @ 145.7° |

There is an external force on link 3 of 80 lb at 330°, applied at point $P$ which is located 3 in @ 100° from the $CG$ of link 3. The linear velocity of that point is 67.2 in/sec at 131.94°.

The rocker (link 4) is 10-in long and weighs 5.8 lb. Its $CG$ is at 5 in at 0° off the line of centers. Its mass moment of inertia about its $CG$ is 0.8 lb-in-sec$^2$. Its data are:

| $\theta_4$ deg | $\omega_4$ rad/sec | $\alpha_4$ rad/sec$^2$ | $V_{G_4}$ in/sec |
|---|---|---|---|
| 104.41 | 7.93 | 276.29 | 39.66 @ 194.41° |

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There is an external torque on link 4 of 120 lb-in. The ground link is 19-in long.

**Find:** The driving torque $T_{12}$ needed to maintain motion with the given acceleration for this instantaneous position of the link.

**Solution:**

1 The torque, angular velocity, and angular acceleration vectors in this two-dimensional problem are all directed along the $Z$ axis, so their dot products each have only one term. Note that in this particular example there is no force $\mathbf{F}_{P_4}$ and no torque $\mathbf{T}_3$.

2 The cartesian coordinates of the acceleration data were calculated in Example 11-3.

$$\begin{aligned}
\mathbf{a}_{G_2} &= 1878.84 \ @ \angle \ 273.66°; & a_{G_{2_x}} &= 119.94, & a_{G_{2_y}} &= -1875.01 \\
\mathbf{a}_{G_3} &= 3646.10 \ @ \angle \ 226.51°; & a_{G_{3_x}} &= -2509.35, & a_{G_{3_y}} &= -2645.23 \\
\mathbf{a}_{G_4} &= 1416.80 \ @ \angle \ 207.24°; & a_{G_{4_x}} &= -1259.67, & a_{G_{4_y}} &= -648.50
\end{aligned} \tag{a}$$

3 The $x$ and $y$ components of the external force at $P$ in the global coordinate system were also calculated in Example 11-3:

$$\mathbf{F}_{P_3} = 80 \ @ \ \angle 330°; \qquad F_{P_{3_x}} = 69.28, \qquad F_{P_{3_y}} = -40.00 \tag{b}$$

4 Converting the velocity data for this example to cartesian coordinates:

$$\begin{aligned}
\mathbf{V}_{G_2} &= 75.00 \ @ \angle \ 180.00°; & V_{G_{2_x}} &= -75.00, & V_{G_{2_y}} &= 0 \\
\mathbf{V}_{G_3} &= 72.66 \ @ \angle \ 145.70°; & V_{G_{3_x}} &= -60.02, & V_{G_{3_y}} &= 40.95 \\
\mathbf{V}_{G_4} &= 39.66 \ @ \angle \ 194.41°; & V_{G_{4_x}} &= -38.41, & V_{G_{4_y}} &= -9.87 \\
\mathbf{V}_{P_3} &= 67.20 \ @ \angle \ 131.94°; & V_{P_{3_x}} &= -44.91, & V_{P_{3_y}} &= 49.99
\end{aligned} \tag{c}$$

5 Substituting the example data into equation 11.16c:

$$\begin{aligned}
&\left[(69.28)(-44.91)+(-40)(49.99)\right]+\left[0\right]+\left[25T_{12}+(0)+(120)(7.93)\right] \\
&= \frac{1.5}{386}\left[(119.94)(-75)+(-1875.01)(0)\right] \\
&\quad + \frac{7.7}{386}\left[(-2509.35)(-60.02)+(-2645.23)(40.95)\right] \\
&\quad + \frac{5.8}{386}\left[(-1259.67)(-38.41)+(-648.50)(-9.87)\right] \\
&\quad + \left[(0.4)(-40)(25)+(1.5)(120.9)(-5.87)+(0.8)(276.29)(7.93)\right]
\end{aligned} \tag{d}$$

6 The only unknown in this equation is the input torque $\mathbf{T}_{12}$ which calculates to:

$$\mathbf{T}_{12} = 243.2 \ \hat{\mathbf{k}} \tag{e}$$

which is the same as the answer obtained in Example 11-3.

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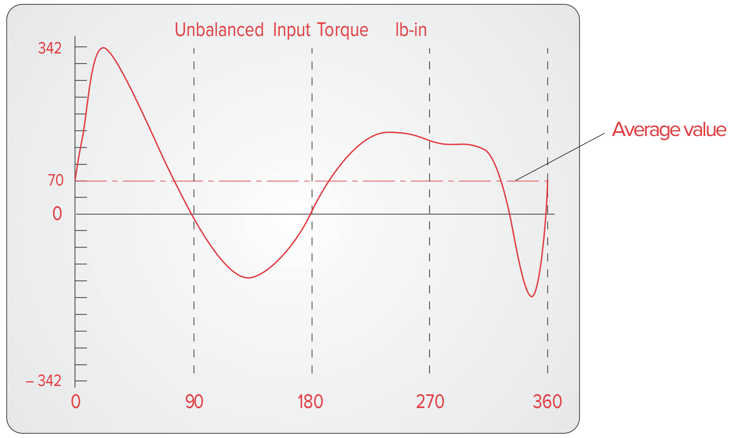


**FIGURE 11-8**

Input torque curve for an unbalanced crank-rocker fourbar linkage

This method of virtual work is useful if a quick answer is needed for the input torque, but it does not give any information about the joint forces.

## 11.11 CONTROLLING INPUT TORQUE—FLYWHEELS *Watch a video (24:07)*†

† http://www.designofmachinery.com/DOM/Flywheels.mp4

The typically large variation in accelerations within a mechanism can cause significant oscillations in the torque required to drive it at a constant or near constant speed. The peak torques needed may be so high as to require an overly large motor to deliver them. However, the average torque over the cycle, due mainly to losses and external work done, may often be much smaller than the peak torque. We would like to provide some means to smooth out these oscillations in torque during the cycle. This will allow us to size the motor to deliver the average torque rather than the peak torque. One convenient and relatively inexpensive means to this end is the addition of a **flywheel** to the system.

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**Torque Variation** Figure 11-8 shows the variation in the input torque for a crank-rocker fourbar linkage over one full revolution of the drive crank. It is running at a constant angular velocity of 50 rad/sec. The torque varies a great deal within one cycle of the mechanism, going from a positive peak of 341.7 lb-in to a negative peak of –166.4 lb-in. The **average value of this torque** over the cycle is only 70.2 lb-in, being due to the *external work done plus losses*. This linkage has only a 12-lb external force applied to link 3 at the *CG* and a 25 lb-in external torque applied to link 4. These small external loads cannot account for the large variation in input torque required to maintain constant crank speed. What then is the explanation? The large variations in torque are evidence of the kinetic energy that is stored in the links as they move. We can think of the positive pulses of torque as representing energy delivered by the driver (motor) and stored temporarily in the moving links, and the negative pulses of torque as energy attempting to return from the links to the driver. Unfortunately most motors are designed to deliver energy but not to take it back. Thus the "returned energy" has no place to go.

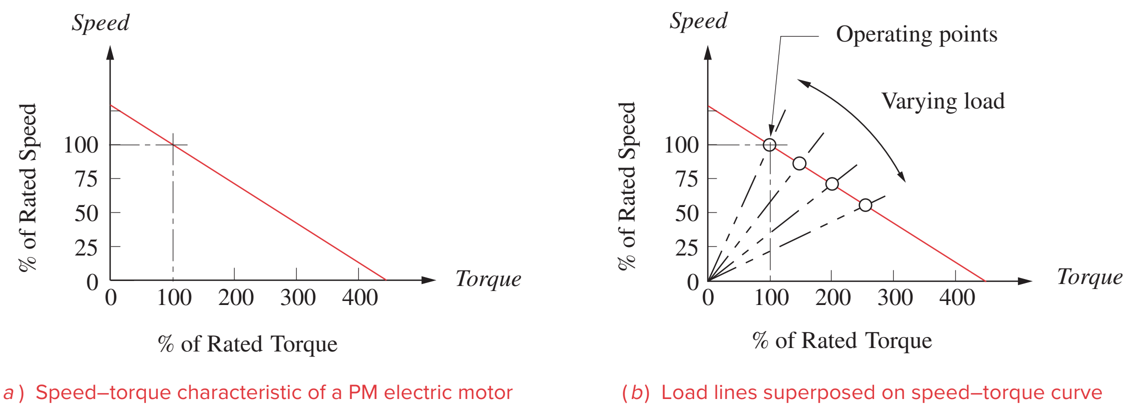


**FIGURE 11-9**

DC permanent magnet (PM) electric motor's typical speed-torque characteristic

Figure 11-9 shows the speed-torque characteristic of a permanent magnet (PM) DC electric motor. Other types of motors will have differently shaped functions that relate motor speed to torque as shown in Figures 2-41 and 2-42, but all drivers (sources) will have some such characteristic curve. As the torque demands on the motor change, the motor's speed must also change according to its inherent characteristic. This means that the torque curve being demanded in Figure 11-8 will be very difficult for a standard motor to deliver without drastic changes in its speed.

The computation of the torque curve in Figure 11-8 was made on the assumption that the crank (thus the motor) speed was a constant value. All the kinematic data used in the force and torque calculation were generated on that basis. With the torque variation shown we would have to use a large-horsepower motor to provide the power required to reach that peak torque at the design speed:

$$\begin{aligned} \textit{Power} &= \textit{torque} \times \textit{angular velocity} \\ \textit{Peak power} &= 341.7 \text{ lb-in} \times 50 \frac{\text{rad}}{\text{sec}} = 17\,085 \frac{\text{in-lb}}{\text{sec}} = 2.59 \text{ hp} \end{aligned}$$

The power needed to supply the average torque is much smaller.

$$\textit{Average power} = 70.2 \text{ lb-in} \times 50 \frac{\text{rad}}{\text{sec}} = 3510 \frac{\text{in-lb}}{\text{sec}} = 0.53 \text{ hp}$$

It would be extremely inefficient to specify a motor based on the peak demand of the system, as most of the time it will be underutilized. We need something in the system which is capable of storing kinetic energy. One such kinetic energy storage device is called a **flywheel**.

FLYWHEEL ENERGY Figure 11-10 shows a **flywheel**, designed as a flat circular disk, attached to a motor shaft which might also be the driveshaft for the crank of our linkage. The motor supplies a torque magnitude $T_M$ which we would like to be as constant as possible, i.e., to be equal to the average torque $T_{avg}$. The load (our linkage), on the other

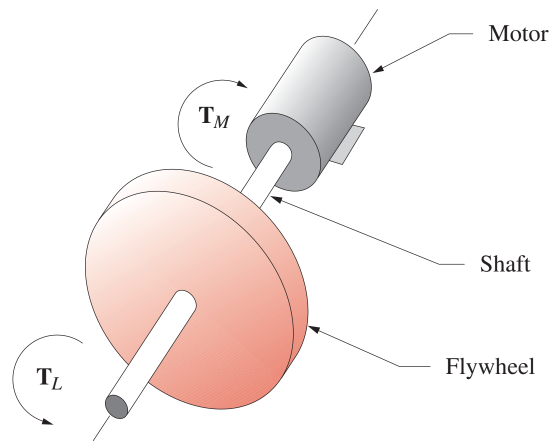


**FIGURE 11-10**

Flywheel on a driveshaft

side of the flywheel, demands a torque $T_L$ which is time varying as shown in Figure 11-8. The kinetic energy in a rotating system is:

$$E = \frac{1}{2} I \omega^2 \tag{11.17}$$

where $I$ is the moment of inertia of all rotating mass on the shaft. This includes the $I$ of the motor rotor and of the linkage crank plus that of the flywheel. We want to determine how much $I$ we need to add in the form of a flywheel to reduce the speed variation of the shaft to an acceptable level. We begin by writing Newton's law for the free-body diagram in Figure 11-10.



$$\sum T = I\alpha$$

$$T_L - T_M = I\alpha$$

but we want:

$$T_M = T_{avg}$$

so:

$$T_L - T_{avg} = I\alpha \tag{11.18a}$$

substituting:

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{dt}\left(\frac{d\theta}{d\theta}\right) = \omega \frac{d\omega}{d\theta}$$

gives:

$$T_L - T_{avg} = I\omega \frac{d\omega}{d\theta}$$

$$\left(T_L - T_{avg}\right) d\theta = I\omega\, d\omega \tag{11.18b}$$

and integrating:

$$\int_{\theta @ \omega_{min}}^{\theta @ \omega_{max}} \left(T_L - T_{avg}\right) d\theta = \int_{\omega_{min}}^{\omega_{max}} I \omega \, d\omega$$

$$\int_{\theta @ \omega_{min}}^{\theta @ \omega_{max}} \left(T_L - T_{avg}\right) d\theta = \frac{1}{2} I \left(\omega_{max}^2 - \omega_{min}^2\right) \tag{11.18c}$$

The left side of this expression represents the change in energy $E$ between the maximum and minimum shaft ω's and is equal to the *area under the torque-time diagram** (Figures 11-8, and 11-11) between those extreme values of ω. The right side of equation 11.18c is the change in energy stored in the flywheel. The only way we can extract energy from the flywheel is to slow it down as shown in equation 11.17. Adding energy will speed it up. Thus it is impossible to obtain exactly constant shaft velocity in the face of changing energy demands by the load. The best we can do is to minimize the speed variation $(\omega_{max} - \omega_{min})$ by providing a flywheel with sufficiently large $I$.

## EXAMPLE 11-5

Determining the Energy Variation in a Torque-Time Function.

**Given:** An input torque-time function which varies over its cycle. Figure 11-11 shows the input torque curve from Figure 11-8. The torque is varying during the 360° cycle about its average value.

**Find:** The total energy variation over one cycle.

**Solution:**

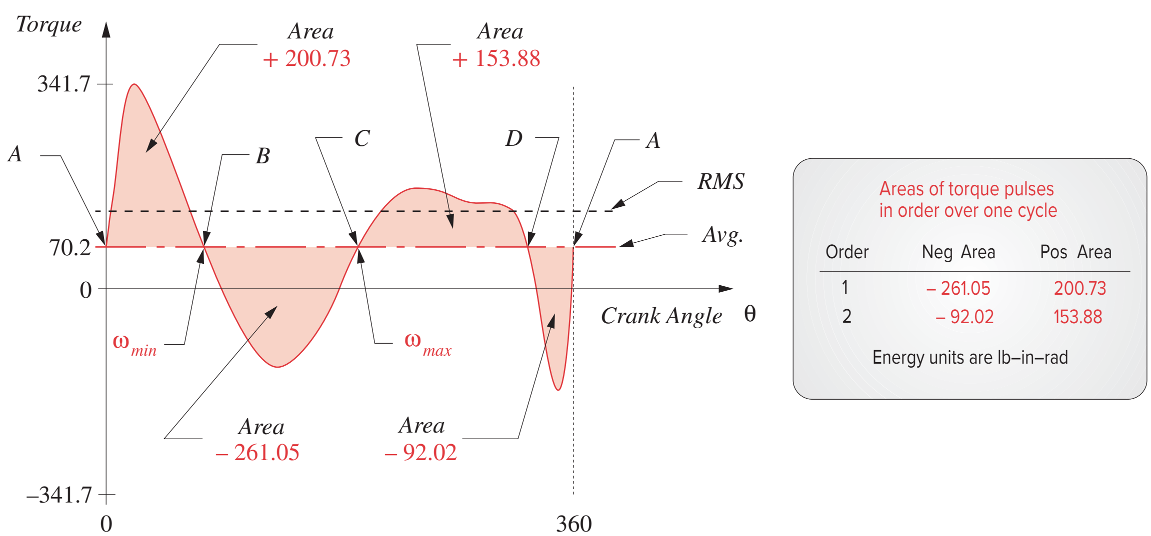


**FIGURE 11-11**

Integrating the pulses above and below the average value in the input torque function

* There is often confusion between torque and energy because they appear to have the same units of *lb-in* (*in-lb*) or *N-m* (*m-N*). This leads some students to think that they are the same quantity, but they are not. Torque ≠ energy. The **integral** of torque with respect to angle, measured in radians, **is** equal to energy. This integral has the units of *in-lb-rad*. The radian term is usually omitted since it is in fact unity. Power in a rotating system is equal to torque x angular velocity (measured in *rad/sec*), and the power units are then *(in-lb-rad)/sec*. When power is integrated versus time to get energy, the resulting units are *in-lb-rad*, the same as the integral of torque versus angle. The radians are again usually dropped, contributing to the confusion.

**TABLE 11-1 Integrating the Torque Function**

| From | Area = $E$ | Accum. Sum = $E$ | |
|---|---|---|---|
| $A$ to $B$ | +200.73 | +200.73 | $\omega_{min}@B$ |
| $B$ to $C$ | −261.05 | −60.32 | $\omega_{max}@C$ |
| $C$ to $D$ | +153.88 | +93.56 | |
| $D$ to $A$ | −92.02 | +1.54 | |
| | Total Energy | $= E\ @\ \omega_{max} - E\ @\ \omega_{min}$ | |
| | | $= (-60.32) - (+200.73) = -261.05$ in-lb | |

1 Calculate the average value of the torque-time function over one cycle, which in this case is 70.2 lb-in. (Note that in some cases the average value may be zero.)

2 Note that the *integration on the left side of equation 11.18c is done with respect to the average line of the torque function, not with respect to the* θ *axis.* (From the definition of the average, the sum of positive area above an average line is equal to the sum of negative area below that line.) The integration limits in equation 11.18 are from the shaft angle θ at which the shaft ω is a minimum to the shaft angle θ at which ω is a maximum.

3 The minimum ω will occur after the maximum positive energy has been delivered from the motor to the load, i.e., at a point (θ) where the summation of positive energy (area) in the torque pulses is at its largest positive value.

4 The maximum ω will occur after the maximum negative energy has been returned to the load, i.e., at a point (θ) where the summation of energy (area) in the torque pulses is at its largest negative value.

5 To find these locations in θ corresponding to the maximum and minimum ω's and thus find the amount of energy needed to be stored in the flywheel, we need to numerically integrate each pulse of this function from crossover to crossover with the average line. The crossover points in Figure 11-11 have been labeled *A, B, C,* and *D*. (Program LINKAGES does this integration for you numerically, using a trapezoidal rule.)

6 The LINKAGES program prints the table of areas shown in Figure 11-11. The positive and negative pulses are separately integrated as described above. Reference to the plot of the torque function will indicate whether a positive or negative pulse is the first encountered in a particular case. The first pulse in this example is a positive one.

7 The remaining task is to accumulate these pulse areas beginning at an arbitrary crossover (in this case point *A*) and proceeding pulse by pulse across the cycle. Table 11-1 shows this process and the result.

8 Note in Table 11-1 that the minimum shaft speed occurs after the largest accumulated positive energy pulse (+200.73 in-lb) has been delivered from the driveshaft to the system. Delivery of energy slows the motor down. Maximum shaft speed occurs after the largest accumulated negative energy pulse (–60.32 in-lb) has been returned from the system by the driveshaft. This return of stored energy will speed up the motor. The total energy variation is the algebraic difference between these two extreme values, which in this example is –261.05 in-lb. This

negative energy coming out of the system needs to be absorbed by the flywheel and then returned to the system *during each cycle* to smooth the variations in shaft speed.

---

**SIZING THE FLYWHEEL** We now must determine how large a flywheel is needed to absorb this energy with an acceptable change in speed. The change in shaft speed during a cycle is called its *fluctuation (Fl)* and is equal to:

$$Fl = \omega_{max} - \omega_{min} \tag{11.19a}$$

We can normalize this to a dimensionless ratio by dividing it by the average shaft speed. This ratio is called the *coefficient of fluctuation (k)*.

$$k = \frac{\omega_{\text{max}} - \omega_{\text{min}}}{\omega_{avg}} \tag{11.19b}$$

This *coefficient of fluctuation* is a design parameter to be chosen by the designer. It typically is set to a value between 0.01 and 0.05, which corresponds to a 1 to 5% fluctuation in shaft speed. The smaller this chosen value, the larger the flywheel will have to be. This presents a design trade-off. A larger flywheel will add more cost and weight to the system, which factors have to be weighed against the smoothness of operation desired.

We found the required change in energy $E$ by integrating the torque curve

$$\int_{\theta @ \omega_{min}}^{\theta @ \omega_{max}} \left(T_L - T_{avg}\right) d\theta = E \tag{11.20a}$$

and can now set it equal to the right side of equation 11.18c:

$$E = \frac{1}{2} I \left(\omega_{max}^2 - \omega_{min}^2\right) \tag{11.20b}$$

Factoring this expression:

$$E = \frac{1}{2} I \left(\omega_{max} + \omega_{min}\right)\left(\omega_{max} - \omega_{min}\right) \tag{11.20c}$$

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If the torque-time function were a pure harmonic, then its average value could be expressed exactly as:

$$\omega_{avg} = \frac{\omega_{max} + \omega_{min}}{2} \tag{11.21}$$

Our torque functions will seldom be pure harmonics, but the error introduced by using this expression as an approximation of the average is acceptably small. We can now substitute equations 11.19b and 11.21 into equation 11.20c to get an expression for the mass moment of inertia $I_s$ of the system flywheel needed.

$$E = \frac{1}{2} I_s \left(2\omega_{avg}\right)\left(k\,\omega_{avg}\right)$$

$$I_s = \frac{E}{k\,\omega_{avg}^2} \tag{11.22}$$

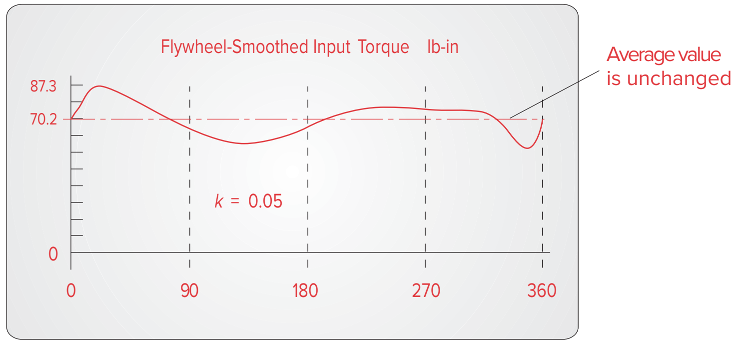



**FIGURE 11-12**

Input torque curve for the linkage in Figure 11-8 after smoothing with a flywheel

Equation 11.22 can be used to design the physical flywheel by choosing a desired coefficient of fluctuation $k$, and using the value of $E$ from the numerical integration of the torque curve (see Table 11-1) and the average shaft $\omega$ to compute the needed system $I_s$. The physical flywheel's mass moment of inertia $I_f$ is then set equal to the required system $I_s$. But if the moments of inertia of the other rotating elements on the same driveshaft (such as the motor) are known, the physical flywheel's required $I_f$ can be reduced by those amounts.

The most efficient flywheel design in terms of maximizing $I_f$ for minimum material used is one in which the mass is concentrated in its rim and its hub is supported on spokes, like a carriage wheel. This puts the majority of the mass at the largest radius possible and minimizes the weight for a given $I_f$. Even if a flat, solid circular disk flywheel design is chosen, either for simplicity of manufacture or to obtain a flat surface for other functions (such as an automobile clutch), the design should be done with an eye to reducing weight and thus cost. Since in general $I = mr^2$, a thin disk of large diameter will need fewer pounds of material to obtain a given $I$ than will a thicker disk of smaller diameter. Dense materials such as cast iron and steel are the obvious choices for a flywheel. Aluminum is seldom used. Though many metals (lead, gold, silver, platinum) are more dense than iron and steel, one can seldom get the accounting department's approval to use them in a flywheel.

11

Figure 11-12 shows the change in the input torque $\mathbf{T}_{12}$ for the linkage in Figure 11-8 after the addition of a flywheel sized to provide a coefficient of fluctuation of 0.05. The oscillation in torque about the unchanged average value is now 5%, much less than what it was without the flywheel. A much smaller-horsepower motor can now be used because the flywheel is available to absorb the energy returned from the linkage during its cycle.

## 11.12 A LINKAGE FORCE TRANSMISSION INDEX

The transmission angle was introduced in Chapter 2 and used in subsequent chapters as an index of merit to predict the kinematic behavior of a linkage. A too-small transmission angle predicts problems with motion and force transmission in a fourbar linkage. Unfortunately, the transmission angle has limited application. It is only useful for fourbar linkages and then only when the input and output torques are applied to links that are pivoted to ground (i.e., the crank and rocker). When external forces are applied to the coupler link, the transmission angle tells nothing about the linkage's behavior.

Holte and Chase[1] define a joint-force index (JFI) which is useful as an indicator of any linkage's ability to smoothly transmit energy regardless of where the loads are applied on the linkage. It is applicable to higher-order linkages as well as to the fourbar linkage. The JFI at any instantaneous position is defined as the ratio of the maximum static force in any joint of the mechanism to the applied external load. If the external load is a force, then it is:

$$JFI = MAX\left|\frac{F_{ij}}{F_{ext}}\right| \qquad \text{for all pairs } i, j \tag{11.23a}$$

If the external load is a torque, then it is:

$$JFI = MAX\left|\frac{F_{ij}}{T_{ext}}\right| \qquad \text{for all pairs } i, j \tag{11.23b}$$

where, in both cases, $F_{ij}$ is the force in the linkage joint connecting links $i$ and $j$.

The $F_{ij}$ are calculated from a static force analysis of the linkage. Dynamic forces can be much greater than static forces if speeds are high. However, if this static force transmission index indicates a problem in the absence of any dynamic forces, then the situation will obviously be worse at speed. The largest joint force at each position is used rather than a composite or average value on the assumption that high friction in any one joint is sufficient to hamper linkage performance regardless of the forces at other joints.

Equation 11.23a is dimensionless and so can be used to compare linkages of different design and geometry. Equation 11.23b has dimensions of reciprocal length, so caution must be exercised when comparing designs when the external load is a torque. Then the units used in any comparison must be the same, and the compared linkages should be similar in size.

Equations 11.23 apply to any one instantaneous position of the linkage. As with the transmission angle, this index must be evaluated for all positions of the linkage over its expected range of motion and the largest value of that set found. The peak force may move from pin to pin as the linkage rotates. If the external loads vary with linkage position, they can be accounted for in the calculation.

Holte and Chase suggest that the JFI be kept below a value of about 2 for linkages whose output is a force. Larger values may be tolerable especially if the joints are designed with good bearings that are able to handle the higher loads.

There are some linkage positions in which the JFI can become infinite or indeterminate as when the linkage reaches an immovable position, defined as the input link or input joint being inactive. This is equivalent to a stationary configuration as described in earlier chapters provided that the input joint is inactive in the particular stationary configuration. These positions need to be identified and avoided in any event, independent of the determination of any index of merit. In some cases the mechanism may be immovable but still capable of supporting a load. See reference [1] for more detailed information on these special cases.

**TABLE P11-0**
**Topic/Problem Matrix**

**11.4 Force Analysis of a Fourbar**
Instantaneous
11-8, 11-9, 11-10, 11-11, 11-12, 11-20
Continuous
11-13, 11-15, 11-21, 11-26, 11-29, 11-32, 11-35, 11-38

**11.5 Force Analysis of a Crank-Slider or Slider-Crank**
11-16, 11-17, 11-18, 11-45

**11.7 Linkages with More Than Four Bars**
11-1, 11-2

**11.8 Shaking Forces and Torques**
11-3, 11-5, 11-47 to 11-51

**11.10 Torque Analysis by Energy Methods**
11-4, 11-6, 11-22, 11-23, 11-24, 11-25, 11-27, 11-28, 11-30, 11-31, 11-33, 11-34, 11-36, 11-37, 11-39, 11-46

**11.11 Flywheels** 11-7, 11-19, 11-40 to 11-44

**11.12 Linkage Force Transmission Index**
11-14, 11-52



## 11.13 PRACTICAL CONSIDERATIONS

This chapter has presented some approaches to the computation of dynamic forces in moving machinery. The newtonian approach gives the most information and is necessary in order to obtain the forces at all pin joints so that stress analyses of the members can be done. Its application is really quite straightforward, requiring only the creation of correct free-body diagrams for each member and the application of the two simple vector equations which express Newton's second law to each free body. Once these equations are expanded for each member in the system and placed in standard matrix form, their solution (with a computer) is a trivial task.

The real work in designing these mechanisms comes in the determination of the shapes and sizes of the members. In addition to the kinematic data, the force computation requires only the masses, *CG* locations, and mass moments of inertia versus those *CG*s for its completion. These three geometric parameters completely characterize the member for dynamic modeling purposes. Even if the link shapes and materials are completely defined at the outset of the force analysis process (as with the redesign of an existing system), it is a tedious exercise to calculate the dynamic properties of complicated shapes. Current solids modeling CAD systems make this step easy by computing these parameters automatically for any part designed within them.

If, however, you are starting from scratch with your design, the *blank-paper syndrome* will inevitably rear its ugly head. A first approximation of link shapes and selection of materials must be made in order to create the dynamic parameters needed for a "first pass" force analysis. A stress analysis of those parts, based on the calculated dynamic forces, will invariably find problems that require changes to the part shapes, thus requiring recalculation of the dynamic properties and recomputation of the dynamic forces and stresses. This process will have to be repeated in circular fashion (*iteration*—see Chapter 1) until an acceptable design is reached. The advantage of using a computer to do these repetitive calculations is obvious and cannot be overstressed. An equation solver program such as *Mathcad, Matlab, or TKSolver* will be a useful aid in this process by reducing the amount of computer programming necessary.

Students with no design experience are often not sure how to approach this process of designing parts for dynamic applications. The following suggestions are offered to get you started. As you gain experience, you will develop your own approach.

It is often useful to create complex shapes from a combination of simple shapes, at least for first approximation dynamic models. For example, a link could be considered to be made up of a hollow cylinder at each pivot end, connected by a rectangular prism along the line of centers. It is easy to calculate the dynamic parameters for each of these simple shapes and then combine them. The steps would be as follows (repeated for each link):

1. Calculate the volume, mass, *CG* location, and mass moments of inertia with respect to the local *CG* of each separate part of your built-up link. In our example link these parts would be the two hollow cylinders and the rectangular prism.
2. Find the location of the composite *CG* of the assembly of the parts into the link by the method shown in Section 10.4 and equations 10.3. See also Figure 10-2.
3. Use the *parallel axis theorem* (equation 10.8) to transfer the mass moments of inertia of each part to the common, composite *CG* for the link. Then add the individual,

transferred $I$'s of the parts to get the total $I$ of the link about its composite $CG$. See Section 10.6.

Steps 1 to 3 will create the link geometry data for each link needed for the dynamic force analysis as derived in this chapter.

4 Do the dynamic force analysis.

5 Do a dynamic stress and deflection analysis of all parts.

6 Redesign the parts and repeat steps 1 to 5 until a satisfactory result is achieved.

Remember that lighter (lower-mass) links will have smaller inertial forces on them and thus could have lower stresses despite their smaller cross sections. Also, smaller mass moments of inertia of the links can reduce the driving torque requirements, especially at higher speeds. But be cautious about the dynamic deflections of thin, light links becoming too large. We are assuming rigid bodies in these analyses. That assumption will not be valid if the links are too flexible. Always check the deflections as well as the stresses in your designs.

## 11.14 REFERENCE

1 **Holte, J. E., and T. R. Chase**. (1994). "A Force Transmission Index for Planar Linkage Mechanisms." *Proc. of 23rd Biennial Mechanisms Conference*, Minneapolis, MN, p. 377.

## 11.15 PROBLEMS§

11-1 Draw free-body diagrams of the links in the geared fivebar linkage shown in Figure 4-11 and write the dynamic equations to solve for all forces plus the driving torque. Assemble the symbolic equations in matrix form for solution.

11-2 Draw free-body diagrams of the links in the sixbar linkage shown in Figure 4-12 and write the dynamic equations to solve for all forces plus the driving torque. Assemble the symbolic equations in matrix form for solution.

*†‡11-3 Table P11-1 shows kinematic and geometric data for several crank-slider linkages of the type and orientation shown in Figure P11-1. The point locations are defined as described in the text. For the row(s) in the table assigned, use the matrix method of Section 11.5 and program MATRIX, *Mathcad*, *Matlab*, *TKSolver*, or a matrix solving calculator to solve for forces and torques at the position shown. Also compute the shaking force and shaking torque. Consider the coefficient of friction $\mu$ between slider and ground to be zero. You may check your solution by opening the solution files (located in the downloadable Solutions folder) named P11-03x (where x is the row letter) in program LINKAGES.

*†11-4 Repeat Problem 11-3 using the method of virtual work to solve for the input torque on link 2. Additional data for corresponding rows are given in Table P11-2.

*†11-5 Table P11-3 shows kinematic and geometric data for several pin-jointed fourbar linkages of the type and orientation shown in Figure P11-2. All have $\theta_1 = 0$. The point locations are defined as described in the text. For the row(s) in the table assigned, use the matrix method of Section 11.4 and program MATRIX or a matrix solving calculator to solve for forces and torques at the position shown. You may check your solution by

§ All problem figures are downloadable as PDF files, and some are also downloadable as animated Working Model files. PDF filenames are the same as the figure number. Run the file *Animations.html* to access and run the animations.



* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

‡ These problems are suited to solution using program LINKAGES.

## TABLE P11-1 Data for Problem 11-3 (See Figure P11-1 for Nomenclature)

Part 1 Lengths in inches, angles in degrees, mass in blobs, angular velocity in rad/sec

| Row | link 2 | link 3 | offset | $\theta_2$ | $\omega_2$ | $\alpha_2$ | $m_2$ | $m_3$ | $m_4$ |
|---|---|---|---|---|---|---|---|---|---|
| a | 4 | 12 | 0 | 45 | 10 | 20 | 0.002 | 0.020 | 0.060 |
| b | 3 | 10 | 1 | 30 | 15 | –5 | 0.050 | 0.100 | 0.200 |
| c | 5 | 15 | –1 | 260 | 20 | 15 | 0.010 | 0.020 | 0.030 |
| d | 6 | 20 | 1 | – 75 | –10 | – 10 | 0.006 | 0.150 | 0.050 |
| e | 2 | 8 | 0 | 135 | 25 | 25 | 0.001 | 0.004 | 0.014 |
| f | 10 | 35 | 2 | 120 | 5 | – 20 | 0.150 | 0.300 | 0.050 |
| g | 7 | 25 | –2 | – 45 | 30 | – 15 | 0.080 | 0.200 | 0.100 |

Part 2 Angular acceleration in rad/sec$^2$, moments of Inertia in blob-in$^2$, torque in lb-in

| Row | $I_2$ | $I_3$ | $Rg_2$ mag | $\delta_2$ ang | $Rg_3$ mag | $\delta_3$ ang | $F_{P_3}$ mag | $\delta F_{P_3}$ ang | $R_{P_3}$ mag | $\delta R_P$ ang | $T_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| a | 0.10 | 0.2 | 2 | 0 | 5 | 0 | 0 | 0 | 0 | 0 | 20 |
| b | 0.20 | 0.4 | 1 | 20 | 4 | – 30 | 10 | 45 | 4 | 30 | – 35 |
| c | 0.05 | 0.1 | 3 | – 40 | 9 | 50 | 32 | 270 | 0 | 0 | – 65 |
| d | 0.12 | 0.3 | 3 | 120 | 12 | 60 | 15 | 180 | 2 | 60 | – 12 |
| e | 0.30 | 0.8 | 0.5 | 30 | 3 | 75 | 6 | – 60 | 2 | 75 | 40 |
| f | 0.24 | 0.6 | 6 | 45 | 15 | 135 | 25 | 270 | 0 | 0 | – 75 |
| g | 0.45 | 0.9 | 4 | – 45 | 10 | 225 | 9 | 120 | 5 | 45 | – 90 |

Part 3 Forces in lb, linear accelerations in in/sec$^2$

| Row | $\theta_3$ | $\alpha_3$ | $a_{g_2}$ mag | $a_{g_2}$ ang | $a_{g_3}$ mag | $a_{g_3}$ ang | $a_{g_4}$ mag | $a_{g_4}$ ang |
|---|---|---|---|---|---|---|---|---|
| a | 166.40 | – 2.40 | 203.96 | 213.69 | 371.08 | 200.84 | 357.17 | 180 |
| b | 177.13 | 34.33 | 225.06 | 231.27 | 589.43 | 200.05 | 711.97 | 180 |
| c | 195.17 | –134.76 | 1200.84 | 37.85 | 2088.04 | 43.43 | 929.12 | 0 |
| d | 199.86 | – 29.74 | 301.50 | 230.71 | 511.74 | 74.52 | 23.97 | 180 |
| e | 169.82 | 113.12 | 312.75 | –17.29 | 976.79 | –58.13 | 849.76 | 0 |
| f | 169.03 | 3.29 | 192.09 | 23.66 | 302.50 | –29.93 | 301.92 | 0 |
| g | 186.78 | – 172.20 | 3600.50 | 90.95 | 8052.35 | 134.66 | 4909.27 | 180 |



## TABLE P11-2 Data for Problem 11-4

See also Table P11-1. Unit system is the same as in that table.

| Row | $\omega_3$ | $Vg_2$ mag | $Vg_2$ ang | $Vg_3$ mag | $Vg_3$ ang | $Vg_4$ mag | $Vg_4$ ang | $V_{P_3}$ mag | $V_{P_3}$ ang |
|---|---|---|---|---|---|---|---|---|---|
| a | – 2.43 | 20.0 | 135 | 35.24 | 152.09 | 35.14 | 180 | 35.24 | 152.09 |
| b | – 3.90 | 15.0 | 140 | 40.35 | 140.14 | 24.45 | 180 | 26.69 | 153.35 |
| c | 1.20 | 60.0 | 310 | 89.61 | –8.23 | 93.77 | 0 | 89.61 | – 8.23 |
| d | 0.83 | 30.0 | 315 | 69.10 | 191.15 | 63.57 | 180 | 70.63 | 191.01 |
| e | 4.49 | 12.5 | 255 | 56.02 | 211.93 | 29.01 | 180 | 61.36 | 204.87 |
| f | 0.73 | 30.0 | 255 | 60.89 | 210.72 | 38.46 | 180 | 60.89 | 210.72 |
| g | –5.98 | 120.0 | 0 | 211.46 | 61.31 | 166.14 | 0 | 208.60 | 53.19 |

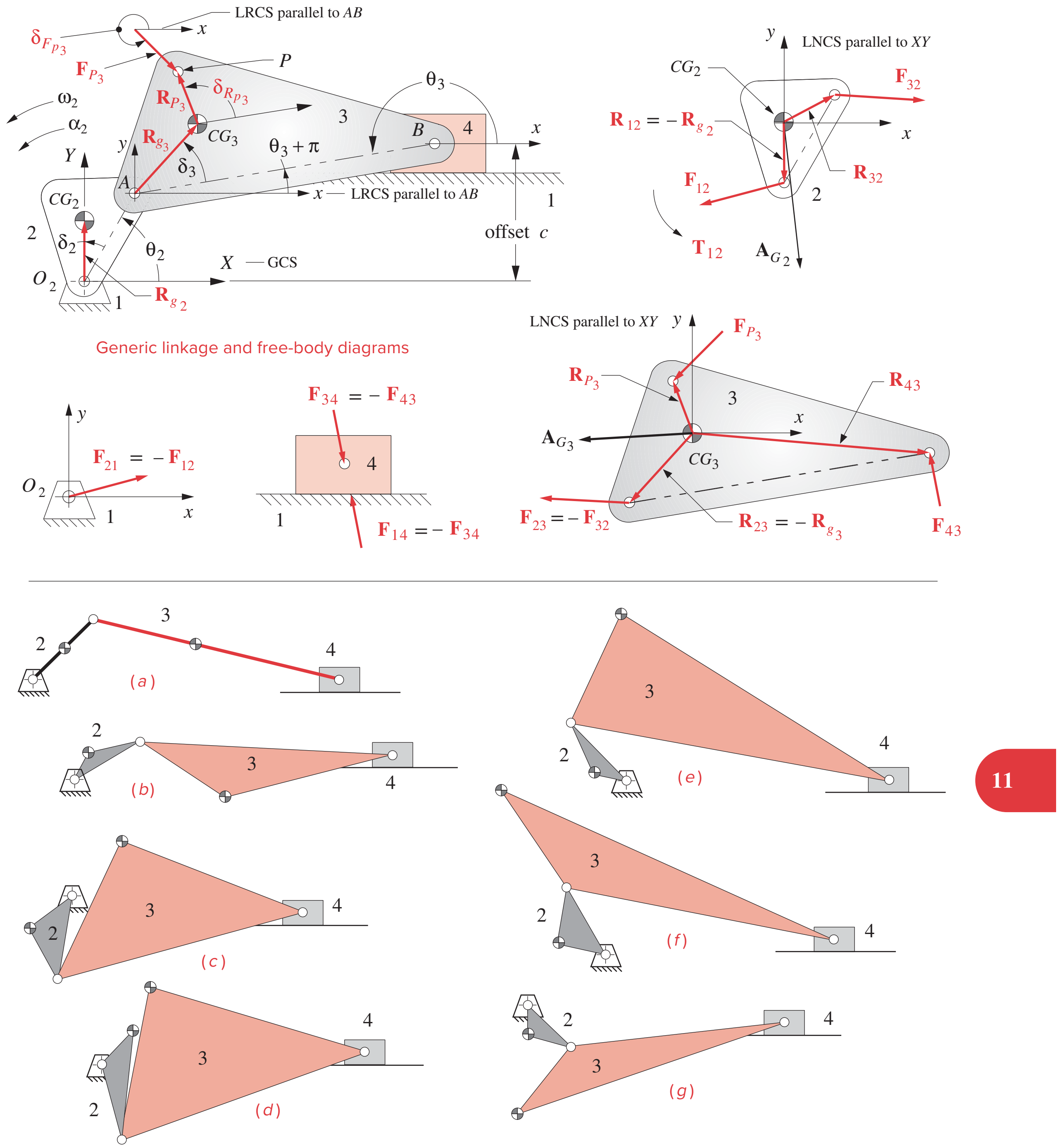
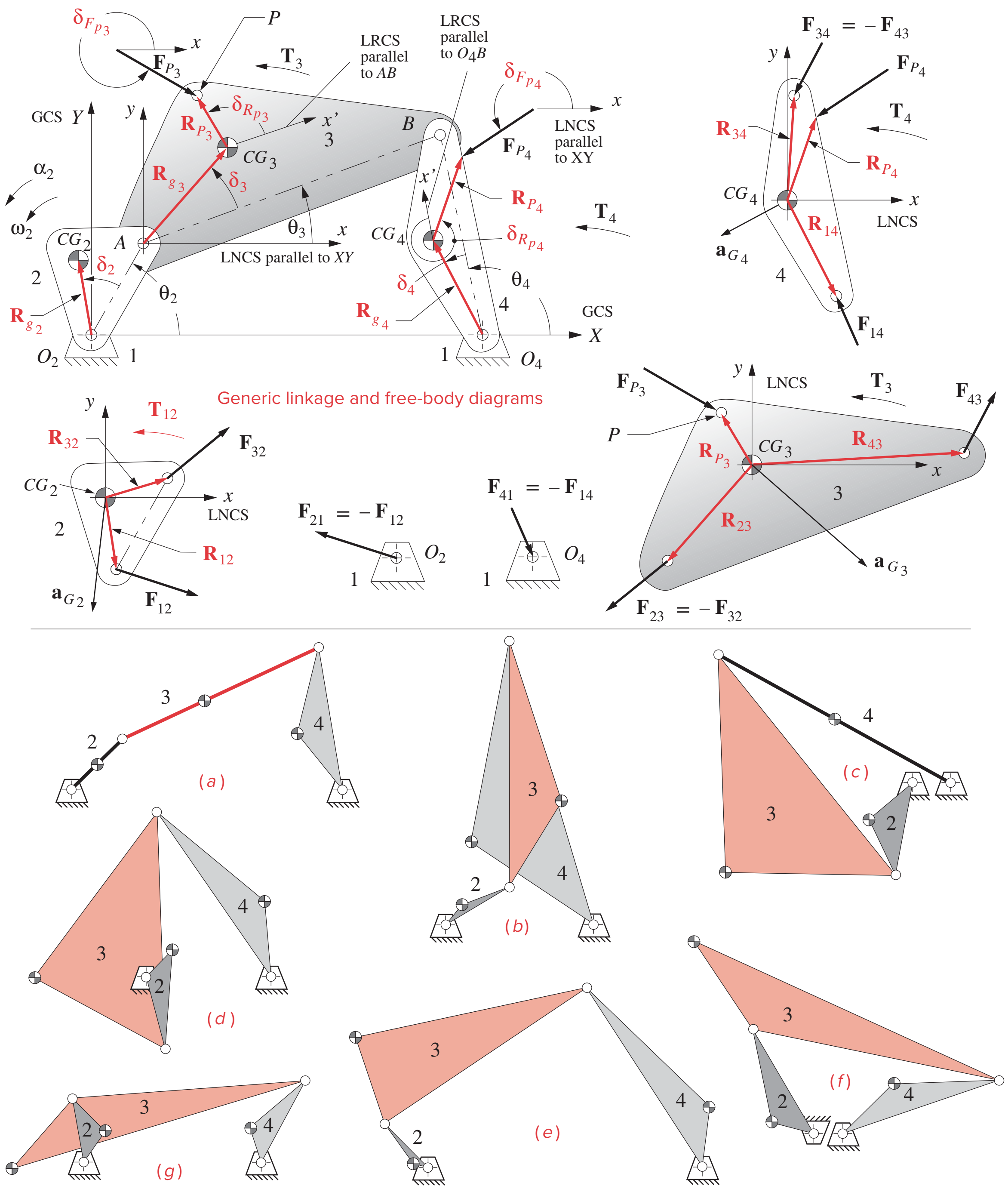





**FIGURE P11-1**

Linkage geometry, notation, and free-body diagrams for problems 11-3 to 11-4




Generic linkage and free-body diagrams

Sketches of the linkages in Table P11-3

**FIGURE P11-2**

Linkage geometry, notation, and free-body diagrams for Problems 11-5 to 11-7

## TABLE P11-3 Data for Problems 11-5 and 11-7 (See Figure P11-2 for Nomenclature)

Part 1 Lengths in inches, angles in degrees, angular acceleration in rad/sec$^2$

| Row | link 2 | link 3 | link 4 | link 1 | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ |
|---|---|---|---|---|---|---|---|---|---|---|
| *a* | 4 | 12 | 8 | 15 | 45 | 24.97 | 99.30 | 20 | 75.29 | 244.43 |
| *b* | 3 | 10 | 12 | 6 | 30 | 90.15 | 106.60 | –5 | 140.96 | 161.75 |
| *c* | 5 | 15 | 14 | 2 | 260 | 128.70 | 151.03 | 15 | 78.78 | 53.37 |
| *d* | 6 | 19 | 16 | 10 | – 75 | 91.82 | 124.44 | – 10 | – 214.84 | – 251.82 |
| *e* | 2 | 8 | 7 | 9 | 135 | 34.02 | 122.71 | 25 | 71.54 | – 14.19 |
| *f* | 17 | 35 | 23 | 4 | 120 | 348.08 | 19.01 | – 20 | – 101.63 | – 150.86 |
| *g* | 7 | 25 | 10 | 19 | 100 | 4.42 | 61.90 | – 15 | – 17.38 | –168.99 |

Part 2 Angular velocity in rad/sec, mass in blobs, moment of Inertia in blob-in$^2$, torque in lb-in

| Row | $\omega_2$ | $\omega_3$ | $\omega_4$ | $m_2$ | $m_3$ | $m_4$ | $I_2$ | $I_3$ | $I_4$ | $T_3$ | $T_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *a* | 20 | – 5.62 | 3.56 | 0.002 | 0.02 | 0.10 | 0.10 | 0.20 | 0.50 | – 15 | 25 |
| *b* | 10 | – 10.31 | – 7.66 | 0.050 | 0.10 | 0.20 | 0.20 | 0.40 | 0.40 | 12 | 0 |
| *c* | 20 | 16.60 | 14.13 | 0.010 | 0.02 | 0.05 | 0.05 | 0.10 | 0.13 | – 10 | 20 |
| *d* | 20 | 3.90 | – 3.17 | 0.006 | 0.15 | 0.07 | 0.12 | 0.30 | 0.15 | 0 | 30 |
| *e* | 20 | 1.06 | 5.61 | 0.001 | 0.04 | 0.09 | 0.30 | 0.80 | 0.30 | 25 | 40 |
| *f* | 20 | 18.55 | 21.40 | 0.150 | 0.30 | 0.25 | 0.24 | 0.60 | 0.92 | 0 | – 25 |
| *g* | 20 | 4.10 | 16.53 | 0.080 | 0.20 | 0.12 | 0.45 | 0.90 | 0.54 | 0 | 0 |

Part 3 Lengths in inches, angles in degrees, linear accelerations in in/sec$^2$

| Row | $R_{g_2}$ mag | $R_{g_2}$ ang | $R_{g_3}$ mag | $R_{g_3}$ ang | $R_{g_4}$ mag | $R_{g_4}$ ang | $a_{g_2}$ mag | $a_{g_2}$ ang | $a_{g_3}$ mag | $a_{g_3}$ ang |
|---|---|---|---|---|---|---|---|---|---|---|
| *a* | 2 | 0 | 5 | 0 | 4 | 30 | 801.00 | 222.14 | 1691.49 | 208.24 |
| *b* | 1 | 20 | 4 | – 30 | 6 | 40 | 100.12 | 232.86 | 985.27 | 194.75 |
| *c* | 3 | – 40 | 9 | 50 | 7 | 0 | 1200.84 | 37.85 | 3120.71 | 22.45 |
| *d* | 3 | 120 | 12 | 60 | 6 | – 30 | 1200.87 | 226.43 | 4543.06 | 81.15 |
| *e* | 0.5 | 30 | 3 | 75 | 2 | – 40 | 200.39 | 341.42 | 749.97 | 295.98 |
| *f* | 6 | 45 | 15 | 135 | 10 | 25 | 2403.00 | 347.86 | 12 064.20 | 310.22 |
| *g* | 4 | – 45 | 10 | 225 | 4 | 45 | 1601.12 | 237.15 | 2562.10 | –77.22 |

Part 4 Linear accelerations in in/sec$^2$, forces in lb, lengths in inches, angles in degrees

| Row | $a_{g_4}$ mag | $a_{g_4}$ ang | $F_{P_3}$ mag | $\delta F_{P_3}$ ang | $R_{P_3}$ mag | $\delta R_{P_3}$ ang | $F_{P_4}$ mag | $\delta F_{P_4}$ ang | $R_{P_4}$ mag | $\delta R_{P_4}$ ang |
|---|---|---|---|---|---|---|---|---|---|---|
| *a* | 979.02 | 222.27 | 0 | 0 | 0 | 0 | 40 | – 30 | 8 | 0 |
| *b* | 1032.32 | 256.52 | 4 | 30 | 10 | 45 | 15 | – 55 | 12 | 0 |
| *c* | 1446.58 | 316.06 | 0 | 0 | 0 | 0 | 75 | 45 | 14 | 0 |
| *d* | 1510.34 | 2.15 | 2 | 45 | 15 | 180 | 20 | 270 | 16 | 0 |
| *e* | 69.07 | 286.97 | 9 | 0 | 6 | – 60 | 16 | 60 | 7 | 0 |
| *f* | 4820.72 | 242.25 | 0 | 0 | 0 | 0 | 23 | 0 | 23 | 0 |
| *g* | 1284.55 | –41.35 | 12 | – 60 | 9 | 120 | 32 | 20 | 10 | 0 |

**TABLE P11-4 Data for Problem 11-6**

| Row | $V_{g_2}$ mag | $V_{g_2}$ ang | $V_{g_3}$ mag | $V_{g_3}$ ang | $V_{g_4}$ mag | $V_{g_4}$ ang | $V_{P_3}$ mag | $V_{P_3}$ ang | $V_{P_4}$ mag | $V_{P_4}$ ang |
|---|---|---|---|---|---|---|---|---|---|---|
| *a* | 40.00 | 135.00 | 54.44 | 145.19 | 14.23 | 219.30 | 54.44 | 145.19 | 41.39 | –160.80 |
| *b* | 10.00 | 140.00 | 21.46 | 14.74 | 45.94 | 56.60 | 122.10 | 40.04 | 130.51 | 29.68 |
| *c* | 60.00 | –50.00 | 191.94 | 299.70 | 98.91 | 241.03 | 191.94 | –60.30 | 296.73 | –118.97 |
| *d* | 60.00 | 135.00 | 94.36 | 353.80 | 19.03 | 4.44 | 152.51 | –3.13 | 67.86 | 26.38 |
| *e* | 10.00 | 255.00 | 42.89 | 223.13 | 11.22 | 172.71 | 37.01 | –140.37 | 48.41 | –155.86 |
| *f* | 120.00 | 255.00 | 618.05 | 211.39 | 213.98 | 134.01 | 618.03 | –148.61 | 692.08 | 116.52 |
| *g* | 80.00 | 145.00 | 118.29 | 205.52 | 66.10 | 196.90 | 154.85 | –152.36 | 217.15 | 164.33 |

opening the solution files named P11-05x (where x is the row letter) in program LINKAGES.

* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

‡ These problems are suited to solution using program LINKAGES.

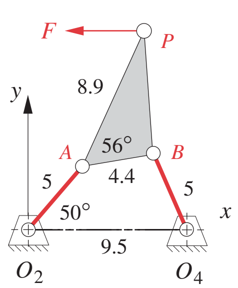
*†11-6 Repeat Problem 11-5 using the method of virtual work to solve for the input torque on link 2. Additional data for corresponding rows are given in Table P11-4.

*‡11-7 For the row(s) assigned in Table P11-3 (*a-f*), input the associated disk file to program LINKAGES, calculate the linkage parameters for crank angles from zero to 360° by 5° increments with $\alpha_2 = 0$, and design a steel disk flywheel to smooth the input torque using a coefficient of fluctuation of 0.05. Minimize the flywheel weight.

‡11-8 Figure P11-3 shows a fourbar linkage and its dimensions. The steel crank and rocker have uniform cross sections 1 in wide by 0.5 in thick. The aluminum coupler is 0.75 in thick. In the instantaneous position shown, the crank $O_2A$ has $\omega = 40$ rad/sec and $\alpha = -20$ rad/sec$^2$. There is a horizontal force at $P$ of $F = 50$ lb. Find all pin forces and the torque needed to drive the crank at this instant.

‡11-9 Figure P11-4a shows a fourbar linkage and its dimensions in meters. The steel crank and rocker have uniform cross sections of 50 mm wide by 25 mm thick. The aluminum coupler is 25 mm thick. In the instantaneous position shown, the crank $O_2A$ has $\omega = 10$ rad/sec and $\alpha = 5$ rad/sec$^2$. There is a vertical force at $P$ of $F = 100$ N. Find all pin forces and the torque needed to drive the crank at this instant.

11




Dimensions in inches

**FIGURE P11-3**

Problem 11-8

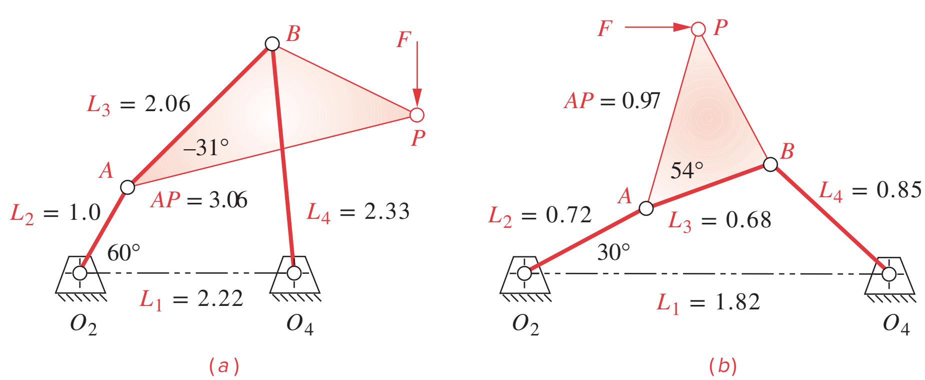



**FIGURE P11-4**

Problems 11-9 to 11-10

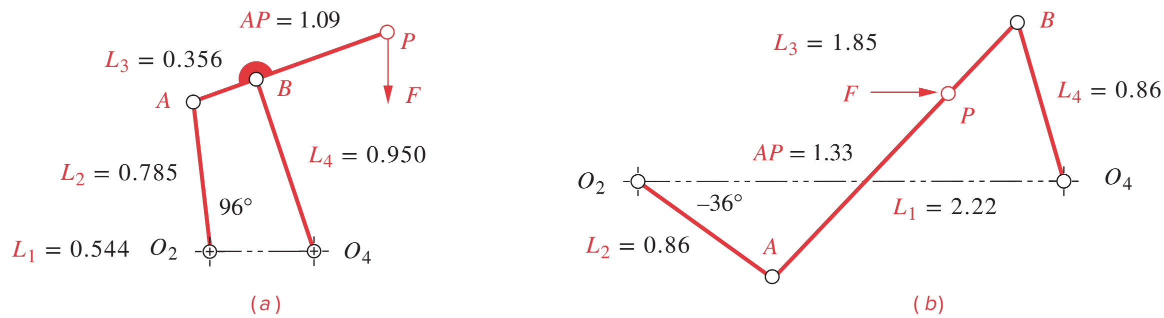



**FIGURE P11-5**

Problems 11-11 to 11-12

‡11-10 Figure P11-4b shows a fourbar linkage and its dimensions in meters. The steel crank and rocker have uniform cross sections of 50 mm wide by 25 mm thick. The aluminum coupler is 25 mm thick. In the instantaneous position shown, the crank $O_2A$ has $\omega = 15$ rad/sec and $\alpha = -10$ rad/sec$^2$. The horizontal force applied at point $P$ is $F = 500$ N. Find all pin forces and the torque needed to drive the crank at this instant.

‡11-11 Figure P11-5a shows a fourbar linkage and its dimensions in meters. The steel crank, coupler, and rocker have uniform cross sections of 50 mm wide by 25 mm thick. In the instantaneous position shown, the crank $O_2A$ has $\omega = 15$ rad/sec and $\alpha = -10$ rad/sec$^2$. There is a vertical force at $P$ of $F = 500$ N. Find all pin forces and the torque needed to drive the crank at this instant.

*†‡11-12 Figure P11-5b shows a fourbar linkage and its dimensions in meters. The steel crank, coupler, and rocker have uniform cross sections of 60-mm diameter. In the instantaneous position shown, the crank $O_2A$ has $\omega = -10$ rad/sec and $\alpha = 10$ rad/sec$^2$. There is a horizontal force at $P$ of $F = 500$ N. Find all pin forces and the torque needed to drive the crank at this instant.

*†‡11-13 Figure P11-6 shows a water-jet loom laybar drive mechanism driven by a pair of Grashof crank-rocker fourbar linkages. The crank rotates at 500 rpm. The laybar is carried between the coupler-rocker joints of the two linkages at their respective instant centers $I_{3,4}$. The combined weight of the reed and laybar is 29 lb. A 540-lb beat-up force from the cloth is applied to the reed as shown. The steel links have a 2 x 1-in uniform cross section. Find the forces on the pins for one revolution of the crank. Find the torque-time function required to drive the system.

*†11-14 Figure P11-7 shows a crimping tool. Find the force $F_{hand}$ needed to generate a 2000-lb $F_{crimp}$. Find the pin forces. What is this linkage's joint force transmission index (JFI) in this position?

†11-15 Figure P11-8 shows a walking-beam conveyor mechanism that operates at slow speed (25 rpm). The boxes being pushed each weigh 50 lb. Determine the pin forces in the linkage and the torque required to drive the mechanism through one revolution. Neglect the masses of the links.

†11-16 Figure P11-9 shows a surface grinder table crank-slider drive that operates at 120 rpm. The crank radius is 22 mm, the coupler is 157 mm, and its offset is 40 mm. The mass of table and workpiece combined is 50 kg. Find the pin forces, slider side loads, and driving torque over one revolution. Neglect the mass of the crank and connecting rod.

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* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

‡ These problems are suited to solution using program LINKAGES.

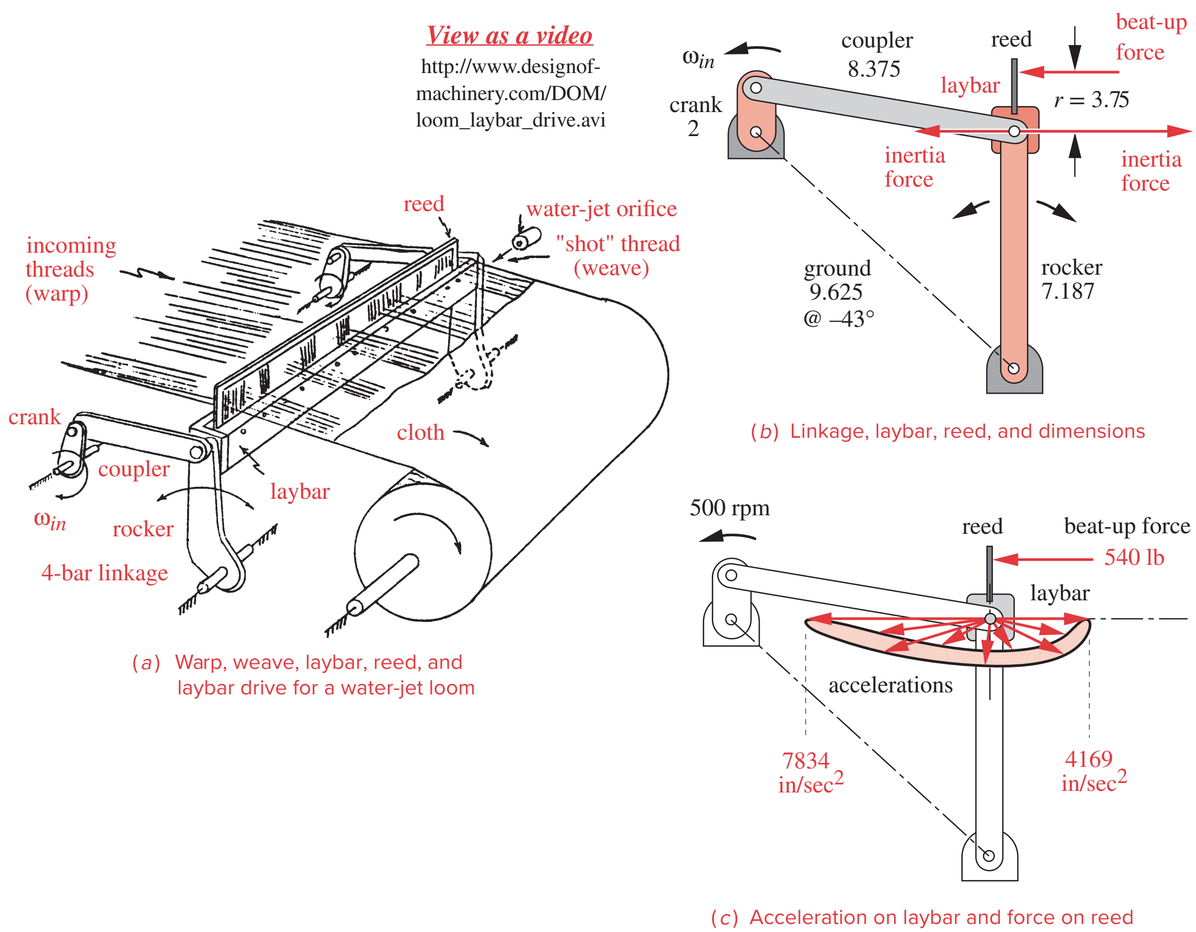


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**FIGURE P11-6**



Problem 11-13 - Fourbar linkage for laybar drive, showing forces and accelerations

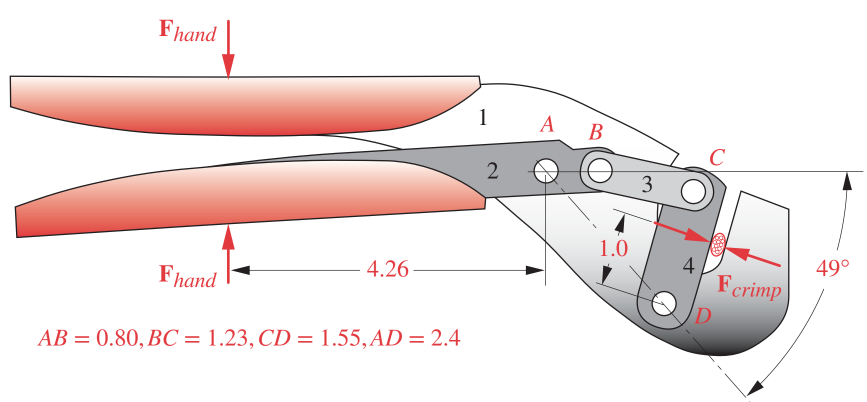


**FIGURE P11-7**

Problem 11-14

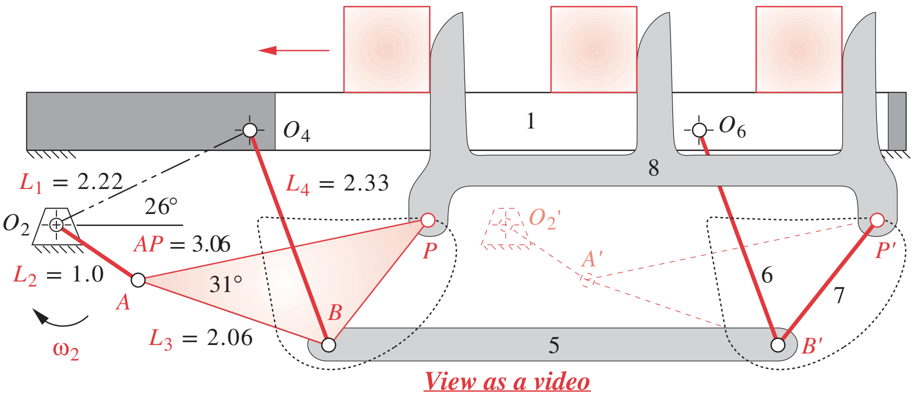


**FIGURE P11-8**

Problem 11-15

†11-17 Figure P11-10 shows a crank-slider power hacksaw that operates at 50 rpm. The balanced crank is 75 mm; the uniform cross section coupler is 170 mm long, weighs 2 kg, and its offset is 45 mm. Link 4 weighs 15 kg. Find the pin forces, slider side loads, and driving torque over one revolution for a cutting force of 250 N in the forward direction and 50 N during the return stroke.

†11-18 Figure P11-11 shows a crank-slider paper roll off-loading station. The paper rolls have a 0.9-m *OD* and 0.22-m *ID*, are 3.23 m long, and have a density of 984 kg/m$^3$. The forks that support the roll are 1.2 m long. The motion is slow so inertial loading can be neglected. Find the force required of the air cylinder to rotate the roll through 90°.

†11-19 Derive an expression for the relationship between flywheel mass and the dimensionless parameter radius/thickness ($r/t$) for a solid disk flywheel of moment of inertia $I$. Plot this function for an arbitrary value of $I$ and determine the optimum $r/t$ ratio to minimize flywheel weight for that $I$.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

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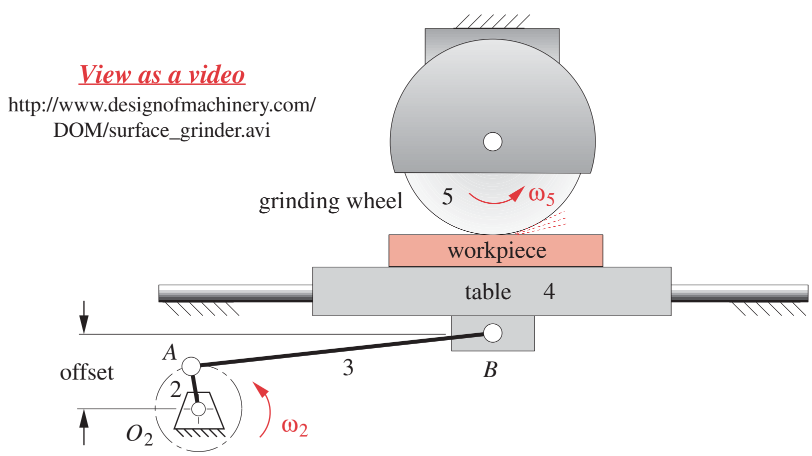


**FIGURE P11-9**

Problem 11-16

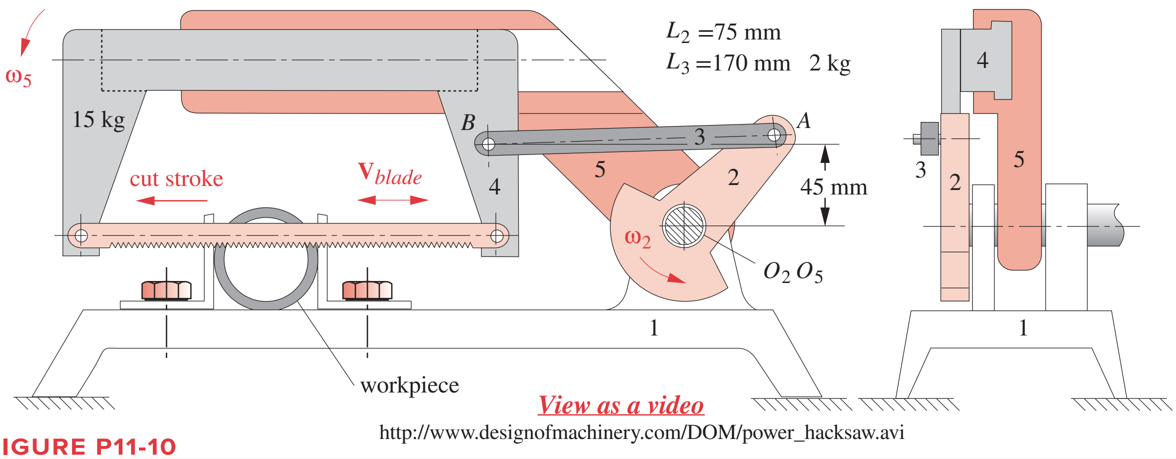


**FIGURE P11-10**

Problem 11-17 Power hacksaw

11-20 Figure P11-12 shows an oil field pump mechanism. The head of the rocker arm is shaped such that the lower end of a flexible cable attached to it will always be directly over the well head regardless of the position of the rocker arm 4. The pump rod, which connects to the pump in the well casing, is connected to the lower end of the cable. The force in the pump rod on the up stroke is 2970 lb and the force on the down stroke is 2300 lb. Link 2 weighs 598.3 lb and has a mass moment of inertia of 11.8 lb-in-sec$^2$ (blob-in$^2$); both include the counterweight. Its *CG* is on the link centerline, 13.2 in from $O_2$. Link 3 weighs 108 lb and its *CG* is on the link centerline, 40 in from *A*. It has a mass moment of inertia of 150 lb-in-sec$^2$ (blob-in$^2$). Link 4 weighs 2706 lb and has a mass moment of inertia of 10 700 lb-in-sec$^2$ (blob-in$^2$); both include the counterweight. Its *CG* is on the link centerline where shown. The crank turns at a constant



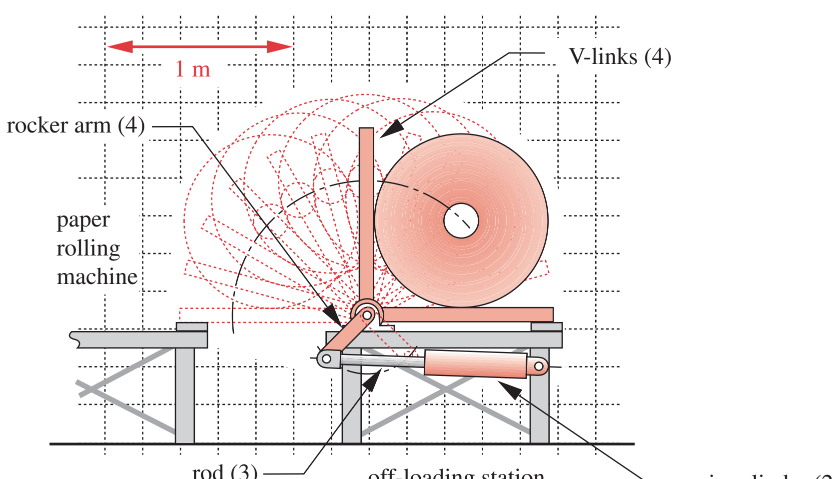


**FIGURE P11-11**

Problem 11-18

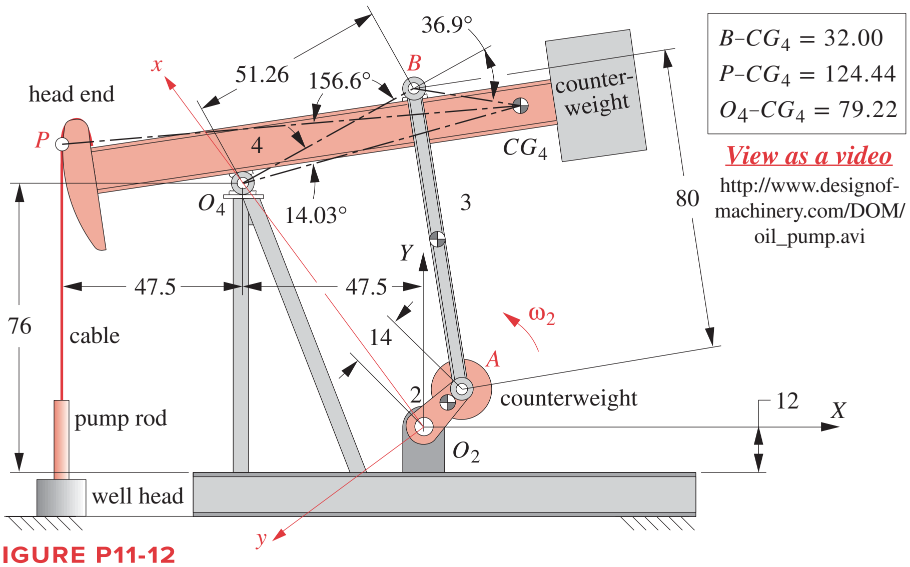


**FIGURE P11-12**

Problems 11-20 to 11-23 An oil field pump - dimensions in inches

speed of 4 rpm CCW. At the instant shown in the figure the crank angle is at 45° with respect to the global coordinate system. Find all pin forces and the torque needed to drive the crank for the position shown. Include gravity forces because the links are heavy and the speed is low.

†11-21 Use the information in Problem 11-20 to find and plot all pin forces and the torque needed to drive the crank for one revolution of the crank.

†11-22 Use the information in Problem 11-20 to find the torque needed to drive the crank for the position shown using the method of virtual work.

†11-23 Use the information in Problem 11-20 to find and plot the torque needed to drive the crank for one revolution of the crank using the method of virtual work.

†11-24 In Figure P11-13, links 2 and 4 each weigh 2 lb and there are 2 of each (another set behind). Their *CG*s are at their midpoints. Link 3 weighs 10 lb. The mass moments of inertia of links 2, 3, and 4 are 0.071, 0.430, and 0.077 lb-in-sec$^2$ (blob-in$^2$), respectively. Find the torque needed to begin a slow CCW rotation of link 2 from the position shown using the method of virtual work. Include gravity forces because the links are heavy and the speed is low.

†*11-25 The linkage in Figure P11-14 has $L_1 = 9.5$, $L_2 = 5.0$, $L_3 = 7.4$, $L_4 = 8.0$, and $AP = 8.9$ in. The steel crank and rocker have uniform cross sections 1 in wide by 0.5 in thick. The aluminum coupler is 0.75 in thick. In the instantaneous position shown, the crank $O_2A$ has $\omega = 40$ rad/sec and $\alpha = -20$ rad/sec$^2$. There is a horizontal force at $P$ of $F = 50$ lb. Find the torque needed to drive the crank at the position shown using the method of virtual work.

11-26 For the linkage defined in Problem 11-25 use program LINKAGES to find and plot all pin forces and the torque needed to drive the crank at a constant speed of 40 rad/sec for one revolution of the crank.

* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

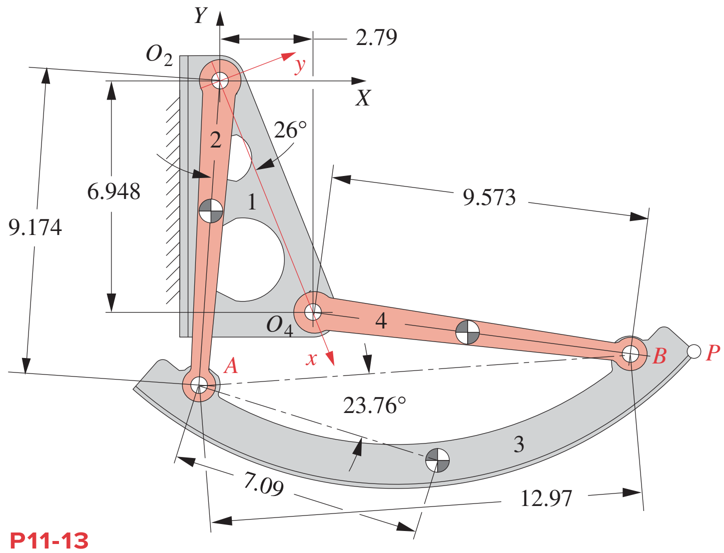


**FIGURE P11-13**

Problem 11-24 An aircraft overhead bin mechanism - dimensions in inches

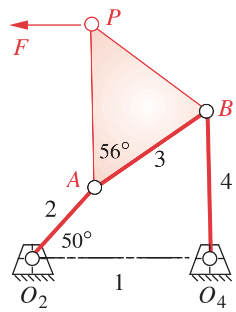


**FIGURE P11-14**

Problems 11-25 to 11-27

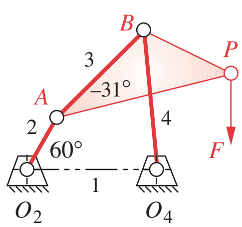


**FIGURE P11-15**

Problems 11-28 to 11-30



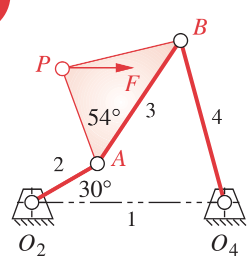


**FIGURE P11-16**

Problems 11-31 to 11-33

† 11-27 For the linkage defined in Problem 11-25 find and plot the torque needed to drive the crank at a constant speed of 40 rad/sec for one revolution of the crank using the method of virtual work.

† 11-28 The linkage in Figure P11-15 has $L_1 = 2.22$, $L_2 = 1.0$, $L_3 = 2.06$, $L_4 = 2.33$, and $AP = 3.06$ m. The steel crank and rocker have uniform cross sections of 50 mm wide by 25 mm thick. The aluminum coupler is 25 mm thick. In the instantaneous position shown, the crank $O_2A$ has $\omega = 10$ rad/sec and $\alpha = 5$ rad/sec$^2$. There is a vertical force at $P$ of $F = 100$ N. Find the torque needed to drive the crank at the position shown using the method of virtual work.

11-29 For the linkage defined in Problem 11-28 use program LINKAGES to find and plot all pin forces and the torque needed to drive the crank at a constant speed of 10 rad/sec for one revolution of the crank.

† 11-30 For the linkage defined in Problem 11-28 find and plot the torque needed to drive the crank at a constant speed of 10 rad/sec for one revolution of the crank using the method of virtual work.

† 11-31 The linkage in Figure P11-16 has $L_1 = 1.82$, $L_2 = 0.72$, $L_3 = 1.43$, $L_4 = 1.60$, and $AP = 0.97$ m. The steel crank and rocker have uniform cross sections 50 mm wide by 25 mm thick. The aluminum coupler is 25 mm thick. In the instantaneous position shown, the crank $O_2A$ has $\omega = 15$ rad/sec and $\alpha = -10$ rad/sec$^2$. There is a horizontal force at $P$ of $F = 200$ N. Find the torque needed to drive the crank at the position shown using the method of virtual work.

11-32 For the linkage defined in Problem 11-31 use program LINKAGES to find and plot all pin forces and the torque needed to drive the crank at a constant speed of 15 rad/sec for one revolution of the crank using the method of virtual work.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

†11-33 For the linkage defined in Problem 11-31 find and plot the torque needed to drive the crank at a constant speed of 15 rad/sec for one revolution of the crank using the method of virtual work.

†11-34 The linkage in Figure P11-17 has $L_1 = 1.0$, $L_2 = 0.356$, $L_3 = 0.785$, $L_4 = 0.95$, and $AP = 1.09$ m. The steel crank, coupler, and rocker have uniform cross sections of 50 mm wide by 25 mm thick. In the instantaneous position shown, the crank $O_2A$ has $\omega = 15$ rad/sec and $\alpha = -10$ rad/sec$^2$. The vertical force at $P$ is $F = 500$ N. Find the torque needed to drive the crank at the position shown using the method of virtual work.

11-35 For the linkage defined in Problem 11-34 use program LINKAGES to find and plot all pin forces and the torque needed to drive the crank at a constant speed of 15 rad/sec for one revolution of the crank using the method of virtual work.

†11-36 For the linkage defined in Problem 11-34 find and plot the torque needed to drive the crank at a constant speed of 15 rad/sec for one revolution of the crank using the method of virtual work.

†11-37 The linkage in Figure P11-18 has $L_1 = 2.22$, $L_2 = 0.86$, $L_3 = 1.85$, $L_4 = 1.86$, and $AP = 1.33$ m. The steel crank, coupler, and rocker have uniform cross sections of 50-mm diameter. In the instantaneous position shown, the crank $O_2A$ has $\omega = -10$ rad/sec and $\alpha = 10$ rad/sec$^2$. There is a horizontal force at $P$ of $F = 300$ N. Find the torque needed to drive the crank at the position shown using the method of virtual work.

11-38 For the linkage defined in Problem 11-37 use program LINKAGES to find and plot all pin forces and the torque needed to drive the crank at a constant speed of 10 rad/sec for one revolution of the crank.

†11-39 For the linkage defined in Problem 11-37 find and plot the torque needed to drive the crank at a constant speed of 10 rad/sec for one revolution of the crank using the method of virtual work.

†*11-40 Design a steel disk flywheel to smooth the input torque for the crank of Problem 11-26 using a coefficient of fluctuation of 0.05 while minimizing flywheel weight.

†11-41 Design a steel disk flywheel to smooth the input torque for the crank of Problem 11-29 using a coefficient of fluctuation of 0.05 while minimizing flywheel weight.

†11-42 Design a steel disk flywheel to smooth the input torque for the crank of Problem 11-32 using a coefficient of fluctuation of 0.07 while minimizing flywheel weight.

†11-43 Design a steel disk flywheel to smooth the input torque for the crank of Problem 11-35 using a coefficient of fluctuation of 0.05 while minimizing flywheel weight.

†11-44 Design a steel disk flywheel to smooth the input torque for the crank of Problem 11-38 using a coefficient of fluctuation of 0.06 while minimizing flywheel weight.

11-45 Table P11-5 gives kinematic and geometric data for a crank-slider linkage of the type and orientation shown in Figure 11-4. For the row(s) in the table assigned, solve for the three pin forces and the torque available at the crank for the position shown.

11-46 Table P11-5 gives kinematic and geometric data for a crank-slider linkage of the type and orientation shown in Figure 11-4. For the row(s) assigned in the table, solve for the torque available at the crank using the method of virtual work for the position shown, assuming no friction losses.

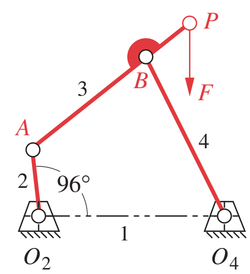


**FIGURE P11-17**

Problems 11-34 to 11-36

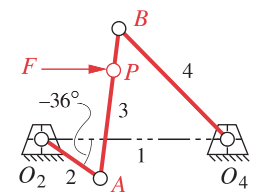


**FIGURE P11-18**

Problems 11-37 to 11-39



* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or, *TKSolver* equation solver programs.

**TABLE P11-5 Data for Problems 11-45 to 11-46 (See Figure 11-4 for Nomenclature)**

Part 1 Lengths (inches), velocity (in/sec), acceleration (in/sec$^2$)

| Row | link 2 | link 3 | $d$ | $\dot{d}$ | $\ddot{d}$ | $R_{12}$ | $R_{23}$ | $\mu$ |
|---|---|---|---|---|---|---|---|---|
| *a* | 4 | 12 | 14 | 400 | –22 760 | 1.3 | 3.0 | 0.15 |
| *b* | 3 | 10 | 8 | 375 | 67 350 | 1.0 | 2.5 | 0.00 |
| *c* | 5 | 15 | 12 | 390 | 36 400 | 1.7 | 3.8 | 0.10 |
| *d* | 6 | 20 | 18 | 700 | 45 430 | 2.0 | 5.0 | 0.18 |
| *e* | 2 | 8 | 8 | 225 | 3 010 | 0.7 | 2.0 | 0.08 |
| *f* | 10 | 35 | 35 | –900 | 69 750 | 3.3 | 8.8 | 0.12 |
| *g* | 7 | 25 | 25 | –935 | 209 900 | 2.3 | 6.2 | 0.14 |

Part 2 force (lbf, deg), mass (blobs), moments of Inertia (blob-in$^2$)

| Row | $F_P$ mag | $F_P$ ang | $m_2$ | $m_3$ | $m_4$ | $I_{G_2}$ | $I_{G_3}$ |
|---|---|---|---|---|---|---|---|
| *a* | 60 | 180 | 0.002 | 0.020 | 0.060 | 0.10 | 0.2 |
| *b* | 45 | 180 | 0.050 | 0.100 | 0.200 | 0.20 | 0.4 |
| *c* | 75 | 180 | 0.010 | 0.020 | 0.030 | 0.05 | 0.1 |
| *d* | 90 | 180 | 0.006 | 0.150 | 0.050 | 0.12 | 0.3 |
| *e* | 30 | 180 | 0.001 | 0.004 | 0.014 | 0.30 | 0.8 |
| *f* | 150 | 180 | 0.150 | 0.300 | 0.050 | 0.24 | 0.6 |
| *g* | 110 | 180 | 0.080 | 0.200 | 0.100 | 0.45 | 0.9 |

11-47 For the linkage in Problem 11-25 find and plot the shaking force and torque for one revolution of the crank when it is driven at a constant speed of 40 rad/sec.

11-48 For the linkage in Problem 11-28 find and plot the shaking force and torque for one revolution of the crank when it is driven at a constant speed of 10 rad/sec.

11-49 For the linkage in Problem 11-31 find and plot the shaking force and torque for one revolution of the crank when it is driven at a constant speed of 15 rad/sec.

11-50 For the linkage in Problem 11-34 find and plot the shaking force and torque for one revolution of the crank when it is driven at a constant speed of 15 rad/sec.

11-51 For the linkage in Problem 11-37 find and plot the shaking force and torque for one revolution of the crank when it is driven at a constant speed of –10 rad/sec.

11-52 Determine the joint-force index (JFI) for the linkage in Problem 11-9.

## 11.16 VIRTUAL LABORATORY *View the video (35:38)*† *View the lab* §

L11-1 View the downloadable video *Fourbar Linkage Virtual Laboratory*. Open the file *Virtual Fourbar Linkage Lab 11-1.doc* and follow the instructions as directed by your professor. For this lab it is suggested that you analyze only the data for the unbalanced conditions of the linkage.

† http://www.designofmachinery.com/DOM/Fourbar_Machine_Virtual_laboratory.mp4

§ http://www.designofmachinery.com/DOM/Fourbar_Virtual_Lab.zip

11

## 11.17 PROJECTS

**The following problem statement applies to all the projects listed.**

*These larger-scale project statements deliberately lack detail and structure and are loosely defined. Thus, they are similar to the kind of "identification of need" or problem statement commonly encountered in engineering practice. It is left to the student to structure the problem through* ***background research*** *and to create a* ***clear goal statement*** *and set of* ***performance specifications*** *before attempting to design a solution. This design process is spelled out in Chapter 1 and should be followed in all of these examples. All results should be documented in a professional engineering report. See the Bibliography in Chapter 1 for references on report writing.*

*Some of these project problems are based on the kinematic design projects in Chapter 3. Those kinematic devices can now be designed more realistically with consideration of the dynamic forces that they generate. The strategy in most of the following project problems is to keep the dynamic pin forces and thus the shaking forces to a minimum and also keep the input torque-time curve as smooth as possible to minimize power requirements.* **All these problems can be solved with a pin-jointed fourbar linkage**. *This fact will allow you to use program* LINKAGES *to do the kinematic and dynamic computations on a large number and variety of designs in a short time. There are infinities of viable solutions to these problems.* **Iterate to find the best one!** *All links must be designed in detail as to their geometry (mass, moment of inertia, etc.). An equation solver such as* Mathcad, Matlab, *or* TKSolver *will be useful here. Determine all pin forces, shaking force, shaking torque, and input horsepower required for your designs.*

P11-1 The tennis coach needs a better tennis ball server for practice. This device must fire a sequence of standard tennis balls from one side of a standard tennis court over the net such that they land and bounce within each of the three court areas defined by the court's white lines. The order and frequency of a ball's landing in any one of the three court areas must be random. The device should operate automatically and unattended except for the refill of balls. It should be capable of firing 50 balls between reloads. The timing of ball releases should vary. For simplicity, a motor-driven pin-jointed linkage design is preferred. This project asks you to design such a device to be mounted upon a tripod stand of 5-foot height. Design it, and the stand, for stability against tip-over due to the shaking forces and shaking torques which should also be minimized in the design of your linkage. Minimize the input torque.

P11-2 The "Save the Skeet" foundation has requested a more humane skeet launcher be designed. While they have not yet succeeded in passing legislation to prevent the wholesale slaughter of these little devils, they are concerned about the inhumane aspects of the large accelerations imparted to the skeet as it is launched into the sky for the sportsperson to shoot down. The need is for a skeet launcher that will smoothly accelerate the clay pigeon onto its desired trajectory. Design a skeet launcher to be mounted upon a child's "little red wagon." Control your design parameters so as to minimize the shaking forces and torques so that the wagon will remain as nearly stationary as possible during the launch of the clay pigeon.

P11-3 The coin-operated "kid bouncer" machines found outside supermarkets typically provide a very unimaginative rocking motion to the occupant. There is a need for a superior "bouncer" which will give more interesting motions while remaining safe for small children. Design it for mounting in the bed of a pickup truck. Keep the shaking forces to a minimum and the input torque-time curve as smooth as possible.

P11-4 NASA wants a zero-*g* machine for astronaut training to carry one person and provide a negative 1-*g* acceleration for as long as possible. Design this device and mount it to the ground plane so as to minimize dynamic forces and driving torque.

P11-5 The Amusement Machine Co. Inc. wants a portable "WHIP" ride which will give two or four passengers a thrilling but safe ride and which can be trailed behind a pickup truck from one location to another. Design this device and its mounting hardware to the truck bed minimizing the dynamic forces and driving torque.

P11-6 The Air Force has requested a pilot training simulator that will give potential pilots exposure to *g* forces similar to those they will experience in dogfight maneuvers. Design this device and mount it to the ground plane so as to minimize dynamic forces and driving torque.

P11-7 Cheers needs a better "mechanical bull" simulator for their "yuppie" bar in Boston. It must give a thrilling "bucking bronco" ride but be safe. Design this device and mount it to the ground plane so as to minimize dynamic forces and driving torque.

P11-8 Gargantuan Motors Inc. is designing a new light military transport vehicle. Their current windshield wiper linkage mechanism develops such high shaking forces when run at its highest speed that the engines are falling out! Design a superior windshield wiper mechanism to sweep the 20-lb armored wiper blade through a 90° arc while minimizing both input torque and shaking forces. The wind load on the blade, perpendicular to the windshield, is 50 lb. The coefficient of friction of the wiper blade on glass is 0.9.

P11-9 The Army's latest helicopter gunship is to be fitted with the Gatling gun, which fires 50-mm-diameter, 2-cm-long spent uranium slugs at a rate of 10 rounds per second. The reaction (recoil) force may upset the helicopter's stability. A mechanism is needed that can be mounted to the frame of the helicopter and which will provide a synchronous shaking force, 180° out of phase with the recoil force pulses, to counteract the recoil of the gun. Design such a linkage and minimize its torque and power drawn from the aircraft's engine. Total weight of your device should also be minimized.

P11-10 Steel pilings are universally used as foundations for large buildings. These are often driven into the ground by hammer blows from a "pile driver." In certain soils (sandy, muddy) the piles can be "shaken" into the ground by attaching a "vibratory driver" that imparts a vertical, dynamic shaking force at or near the natural frequency of the pile-earth system. The pile can literally be made to "fall into the ground" under optimal conditions. Design a fourbar linkage-based pile shaker mechanism which, when its ground link is firmly attached to the top of a piling (supported from a crane hook), will impart a dynamic shaking force that is predominantly directed along the piling's long, vertical axis. Operating speed should be in the vicinity of the natural frequency of the pile-earth system.



P11-11 Paint can shaker mechanisms are common in paint stores. While they do a good job of mixing the paint, they are also noisy and transmit their vibrations to the shelves and counters. A better design of the paint can shaker is possible using a balanced fourbar linkage. Design such a portable device to sit on the floor (not bolted down) and minimize the shaking forces and vibrations while still effectively mixing the paint.

P11-12 Convertible automobiles are once again popular. While offering the pleasure of open-air motoring, they offer little protection to the occupants in a rollover accident. Permanent roll bars are ugly and detract from the open feeling of a true convertible. An automatically deployable roll bar mechanism is needed that will be out of sight until needed. In the event that sensors in the vehicle detect an imminent rollover, the mechanism should deploy within 250 ms. Design a collapsible/deployable roll bar mechanism to retrofit to the convertible of your choice.

P11-13 Design a superior hand-held sanding/polishing machine. Many such devices exist on the market. Some have a simple pure-rotation motion which creates an undesireable pattern of rotary scratches on the affected surface. Others have an ineffective random vibration motion of very small amplitude. Still others have more complicated motions. What is desired in this product is a more sophisticated motion pattern which will provide a superior finish. It is also desireable that this new machine provide smoother and quieter operation than any non-rotary devices now on the market. Most current non-rotary polishing machines deliver significant vibratory forces to the user's hands. The new design should minimize the effects of vibratory forces as felt by the user. In addition, it should require the smallest possible input torque (and thus power) from its electric motor.

P11-14 NASA has requested the design of a Spacecraft Compatible Ambulatory Machine, or SCAM. Proposed interplanetary travel in this century will require that the astronaut crews spend years in micro-gravity. Research on extended micro-gravity exposure has shown that the lack of gravity-bound exercise results in significant bone-density loss in astronauts who spend long periods in space. It is believed that the key to preventing this debilitating condition is to provide the astronauts with an artificial-gravity exercise environment. NASA desires the design and analysis of a machine that can be installed on an interplanetary spacecraft that will, when activated, provide realistic earth-bound levels of walking and/or jogging forces to the feet and legs of the astronaut. They envision a compact machine into which the astronaut can be placed and secured, and which, when run, will cause realistic (physiologic) forces and motions to be imparted to the feet and legs of the victim astronaut that simulate walking and/or running on Earth in a 1-g environment.

P11-15 The Autoroll Co. makes bottle-printing machines. These use a silk-screen process to apply label information to oval bottles in an automatic assembly machine. A **Video is downloadable** for viewing that shows one of their machines in operation. A new machine is being designed. A mechanism is needed that will move the squeegee (also called a knife) in an approximate straight line across the top of the silk screen while the oval bottle is rolled against the underside of the screen. It is also preferred that the velocity of the knife be as uniform as possible during the print stroke. The useable print stroke is a maximum of 6 inches long. The knife is 5 inches wide, 1 inch high and can flex up to 0.1 inches in the vertical direction. Its spring constant is 20 lb/in. It only needs to wipe in one direction. There is an effective coefficient of friction between knife and screen of about 1.5. The desired production rate is 80 bottles per minute. The bottle-motion mechanism is not a part of this project..