Castigliano's Second Theorem

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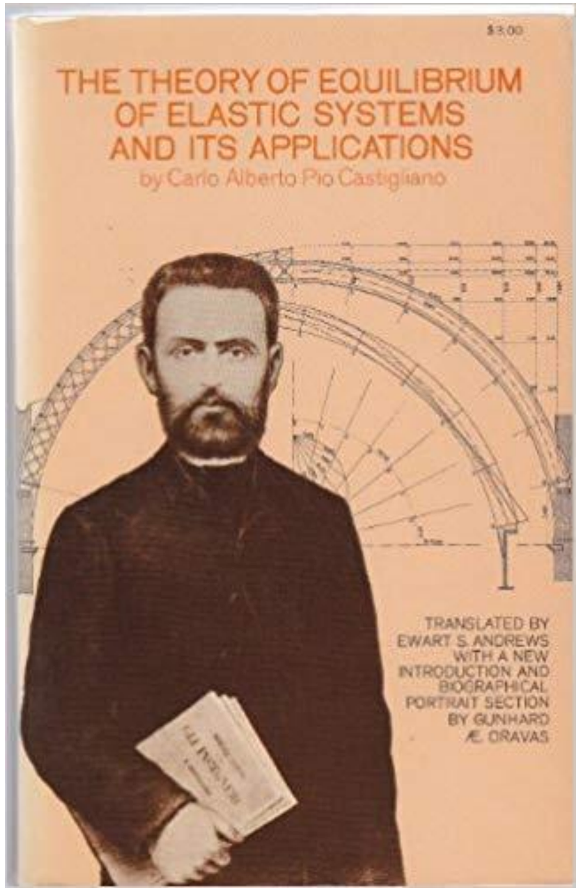
The Theorem of Least Work

The theorem of least work derives from what is known as Castigliano’s second theorem (named after Carlo Alberto Castigliano (1847-1884) who was an Italian railroad engineer).

In 1879, Castigliano published two theorems:

Castigliano’s first theorem

The first partial derivative of the total internal energy (strain energy) in a structure with respect to any particular deflection component at a point is equal to the force applied at that point and in the direction corresponding to that deflection component.



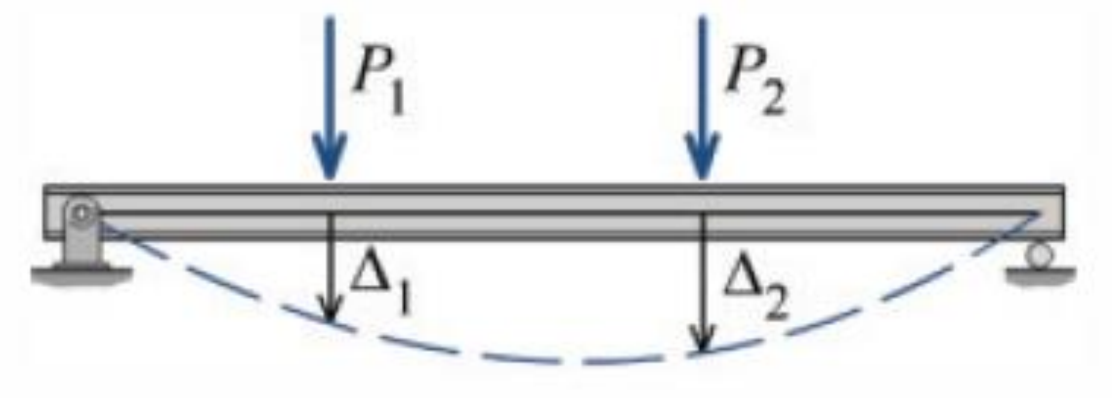
The Theorem of Least Work

Castigliano's second theorem

The first partial derivative of the total internal energy in a structure with respect to the force applied at any point is equal to the deflection at the point of application of that force in the direction of its line of action.

The second theorem of Castigliano is applicable to *linearly elastic structures* with *constant temperature* and *unyielding supports*.

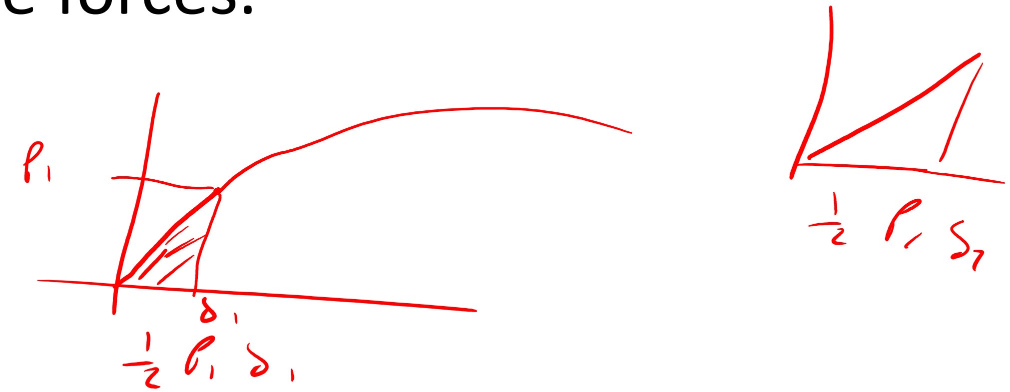
The Theorem of Least Work



If the beam shown is slowly and simultaneously loaded by two forces P_1 and P_2, with resulting deflections Δ_1 and Δ_2, the strain energy *U* of the beam is equal to the work done by the forces.

Therefore,

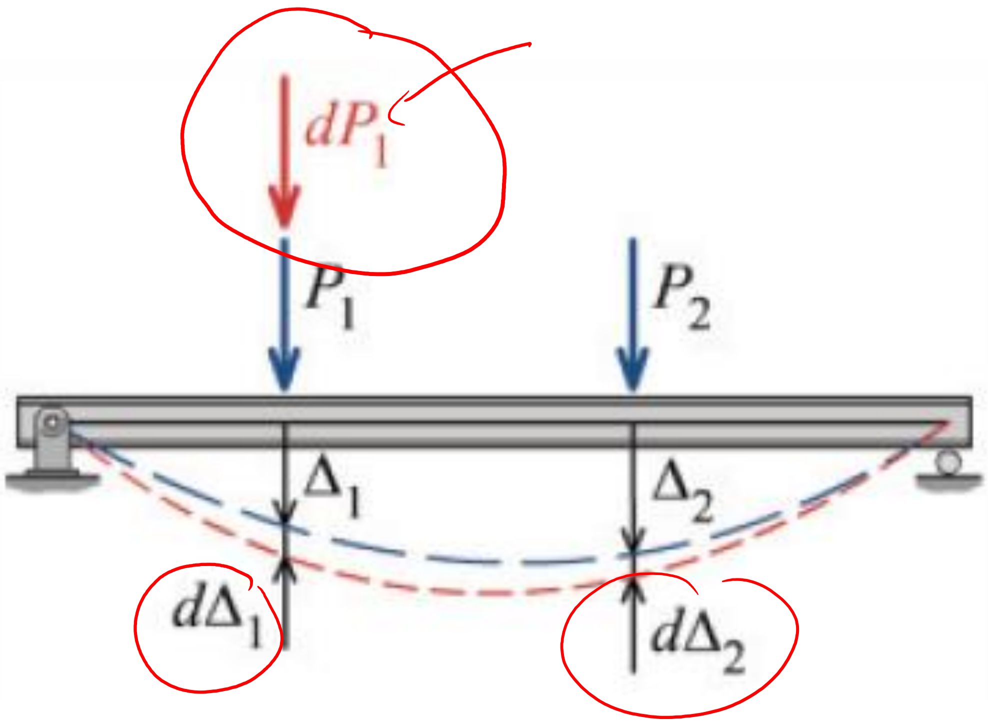
$U = \frac{1}{2} P_1\Delta_1 + \frac{1}{2} P_2 \Delta_2$



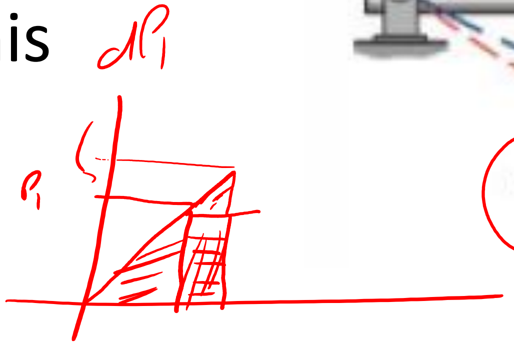
Recall that the factor ½ in each term is required because the loads build up from zero to their final magnitude.

The Theorem of Least Work

Let the force P_1 be increased by a small amount dP_1 while force P_2 remains Constant.



The changes in deflection due to this Incremental load will be denoted $d\Delta_1$ and $d\Delta_2$.

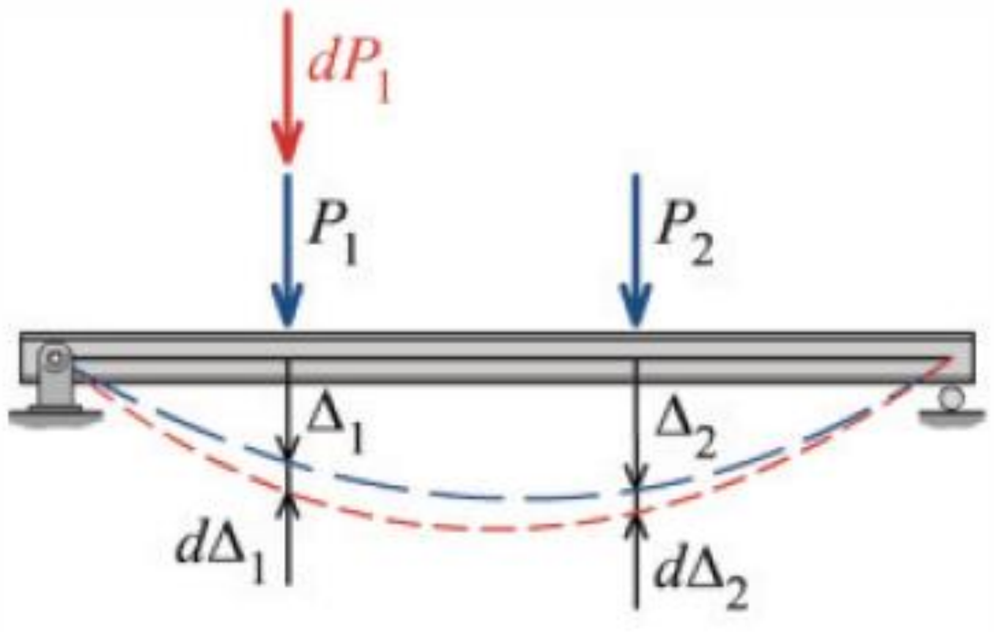


The strain energy in the beam increases by the amount ½ $P_1 \, d\Delta_1$ as the incremental force dP_1 deflects through the distance $d\Delta_1$.

The Theorem of Least Work

However, forces P_1 and P_2, which remain present on the beam, also perform work as the beam deflects.

Altogether, the increase in the strain energy due to the application of dP_1 is

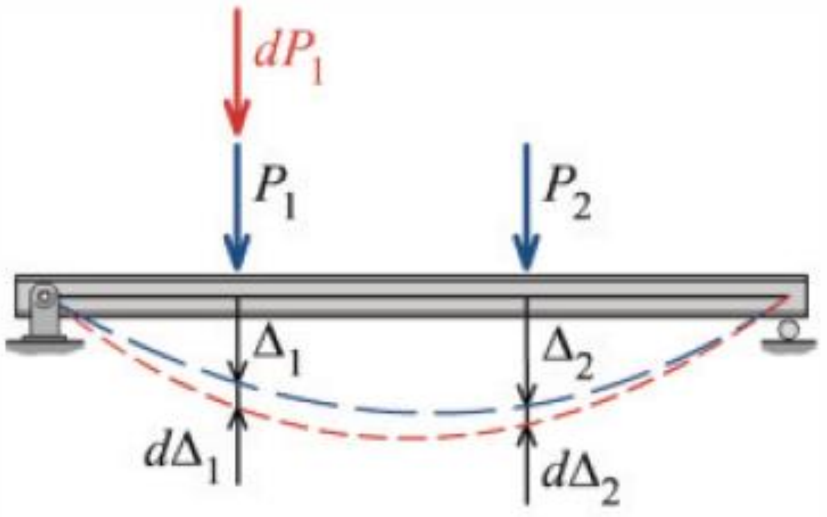


$$dU = P_1 d\Delta_1 + P_2 d\Delta_2 + \frac{1}{2} dP_1 d\Delta_1$$

The Theorem of Least Work

Therefore, the total strain energy in the beam is

$$U + dU = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + P_1 d\Delta_1 + P_2 d\Delta_2 + \frac{1}{2}dP_1 d\Delta_1$$

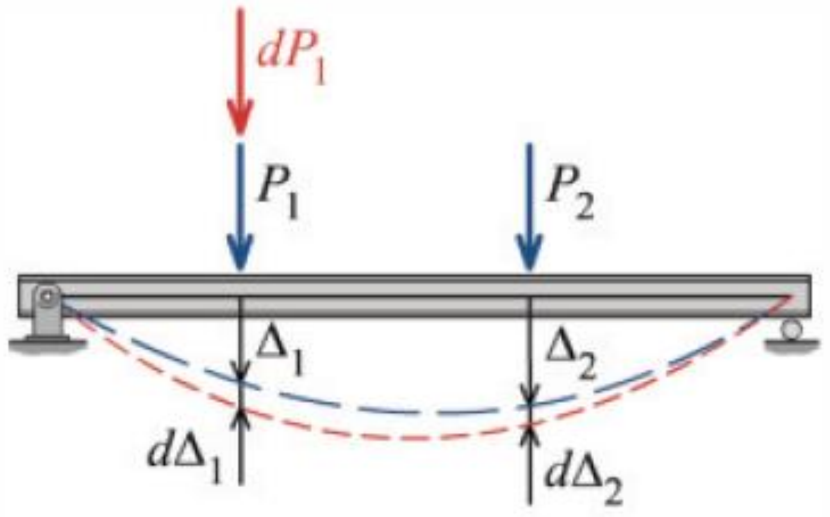


If the order of loading is reversed so that the incremental force dP_1 is applied first, followed by P_1 and P_2, the resulting strain energy will be: (**HOW?**)

$$U + dU = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + dP_1\Delta_1 + \frac{1}{2}dP_1 d\Delta_1$$

The Theorem of Least Work

The beam is linearly elastic, the loads P_1 and P_2 cause the same deflections Δ_1 and Δ_2 regardless of whether or not any other load is acting on the beam.



Because dP_1 remains constant at its point of application during the additional deflection Δ_1, the term $dP_1\Delta_1$ does not contain the factor ½.

$$U + dU = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + dP_1\Delta_1 + \frac{1}{2}dP_1 d\Delta_1$$

Remember that elastic deformation is reversible and energy losses are neglected, the resulting strain energy must be independent of the order of loading.

The Theorem of Least Work

The beam is linearly elastic, the loads P_1 and P_2

$$U + dU = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + P_1 d\Delta_1 + P_2 d\Delta_2 + \frac{1}{2}dP_1 d\Delta_1$$

$$dP_1\Delta_1 = P_1 d\Delta_1 + P_2 d\Delta_2$$

$$U + dU = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + dP_1\Delta_1 + \frac{1}{2}dP_1 d\Delta_1$$

The Theorem of Least Work

$$dP_1\Delta_1 = P_1 d\Delta_1 + P_2 d\Delta_2$$

Combine:

$$dU = dP_1\Delta_1 + \frac{1}{2} dP_1 d\Delta_1$$

$$dU = P_1 d\Delta_1 + P_2 d\Delta_2 + \frac{1}{2} dP_1 d\Delta_1$$

The Theorem of Least Work

$$dU = dP_1\Delta_1 + \frac{1}{2}dP_1 d\Delta_1$$

second order term than can be neglected

Strain energy U is a function of both P_1 and P_2; therefore, the change in strain energy, dU, due to the incremental load dP_1 is expressed by the partial derivative of U with respect to P_1 as:

$$dU = \frac{\partial U}{\partial P_1} dP_1$$

The Theorem of Least Work

$$dU = dP_1\Delta_1 + \frac{1}{2}dP_1 d\Delta_1$$

second order term than can be neglected

$$\frac{\partial U}{\partial P_1}dP_1 = dP_1\Delta_1$$

$$\frac{\partial U}{\partial P_1} = \Delta_1$$

The Theorem of Least Work

In general:

$$\frac{\partial U}{\partial P_i} = \Delta_i \quad (i = 1, \ldots, n)$$

If the strain energy of a linearly elastic structure is expressed in terms of the system of external loads, then the partial derivative of the strain energy with respect to a concentrated external load is the deflection of the structure at the point of application and in the direction of that load.

Castigliano's second theorem applies to any elastic system at constant temperature and on unyielding supports and that obeys the law of superposition.

The Theorem of Least Work

Castigliano's theorem can also be shown to be valid for applied moments and the resulting rotations (or changes in slope) of the structure.

$$\frac{\partial U}{\partial M_i} = \theta_i \,(i = 1, \; \ldots, \; n)$$

Application to Beams

$$\frac{\partial U}{\partial P_1} = \Delta_1$$

Recall:

$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

Castigliano's second theorem states:

$$\Delta = \frac{\partial}{\partial P}\int_0^L \frac{M^2}{2EI}\,dx$$

thus:

$$\frac{\partial}{\partial P}\int_0^L \frac{M^2}{2EI}\,dx = \int_0^L \left(\frac{\partial M^2}{\partial P}\right)\frac{1}{2EI}\,dx$$

But $\partial M^2/\partial P = 2M(\partial M/\partial P)$

Then:

$$\Delta = \int_0^L \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI}\,dx$$

similarly:

$$\theta = \int_0^L \left(\frac{\partial M}{\partial M'}\right)\frac{M}{EI}\,dx$$

Application to Beams

$$\Delta = \int_0^L \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI}\,dx \qquad \theta = \int_0^L \left(\frac{\partial M}{\partial M'}\right)\frac{M}{EI}\,dx$$

Δ = displacement of a point on the beam

P = external force applied to the beam in the direction of D and *expressed as a variable*

M = internal bending moment in the beam, expressed as a function of *x* and caused by both the force *P* and the loads on the beam

I = moment of inertia of the beam cross section about the neutral axis

E = elastic modulus of the beam

L = length of the beam

Θ = rotation angle (or slope) of the beam at a point

M′ = a concentrated moment applied to the beam in the direction of θ at the point of interest and *expressed as a variable*.

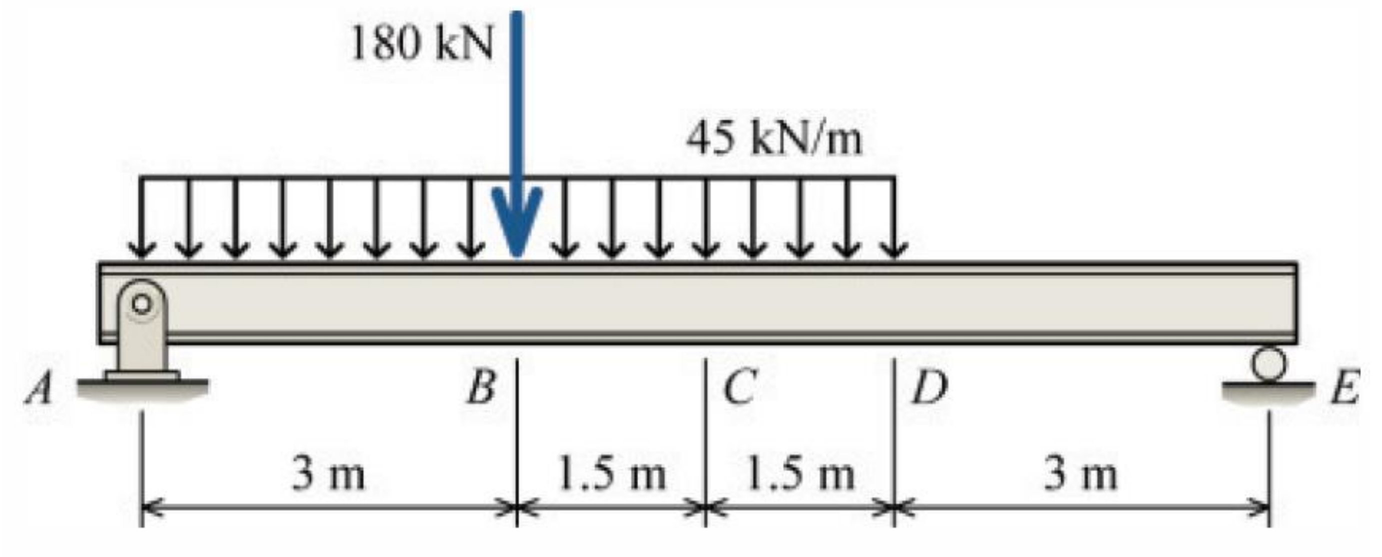
Procedure for Analysis

Please refer to the handout entitled "*Castigliano's second theorem*" for the full procedure.

Example

Compute the deflection at point C for the simply supported beam shown.

Assume that $EI = 3.4 \times 10^5$ kN·m^2.



Procedure for Analysis

Please refer to the handout entitled “*Castigliano’s second theorem*” for the full procedure.

Example

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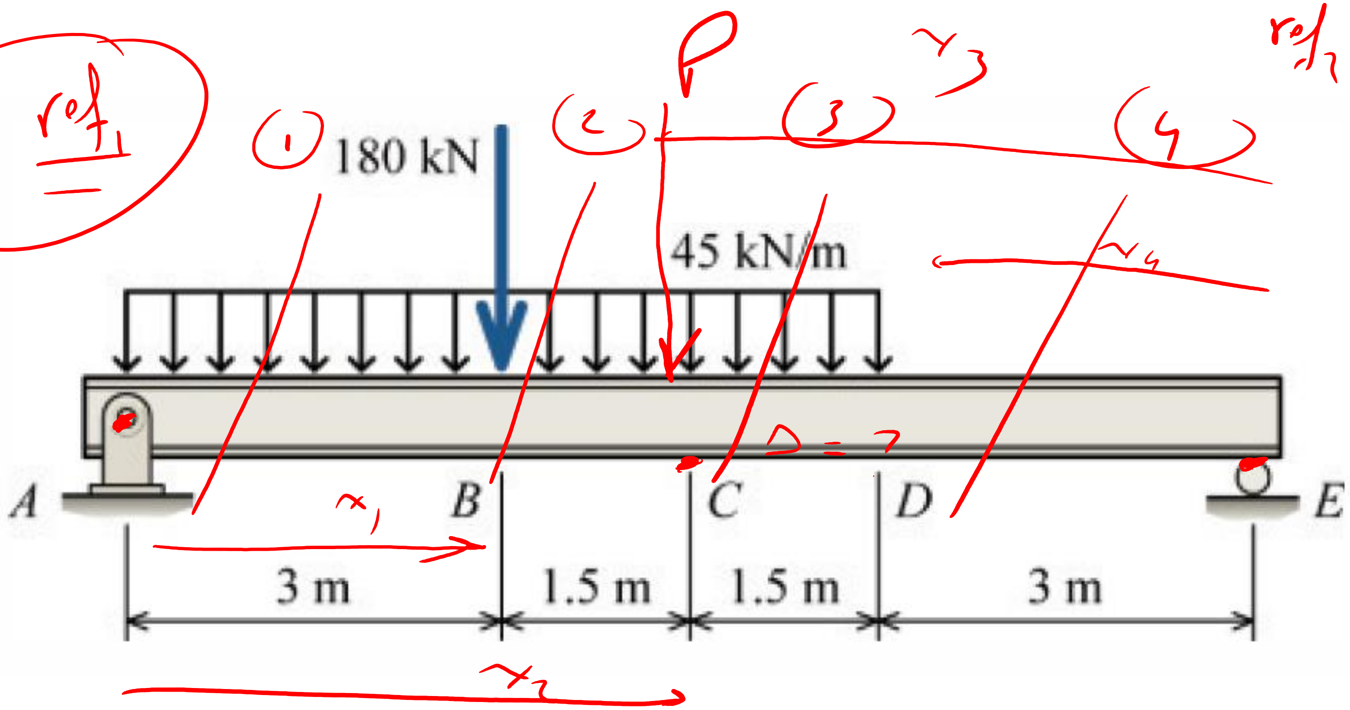


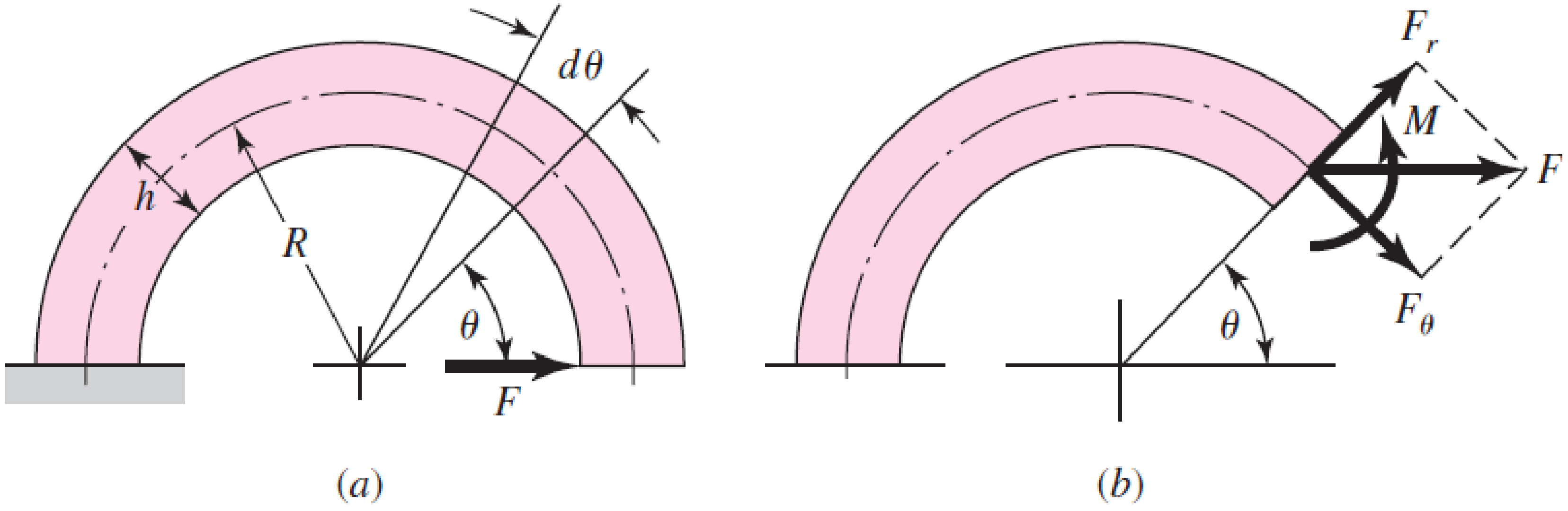
1) FBD
2) R_x :
3) $M(x)$

x_1 : 0 → 3
x_2 3 → 4.5
x_3 3 → 4.5
x_4 : 0 → 3

Deflection of Curved Members

Machine frames, springs, clips, fasteners, and the like frequently occur as curved shapes.

Consider, for example, the curved frame. We are interested in finding the deflection of the frame due to F and in the direction of F.

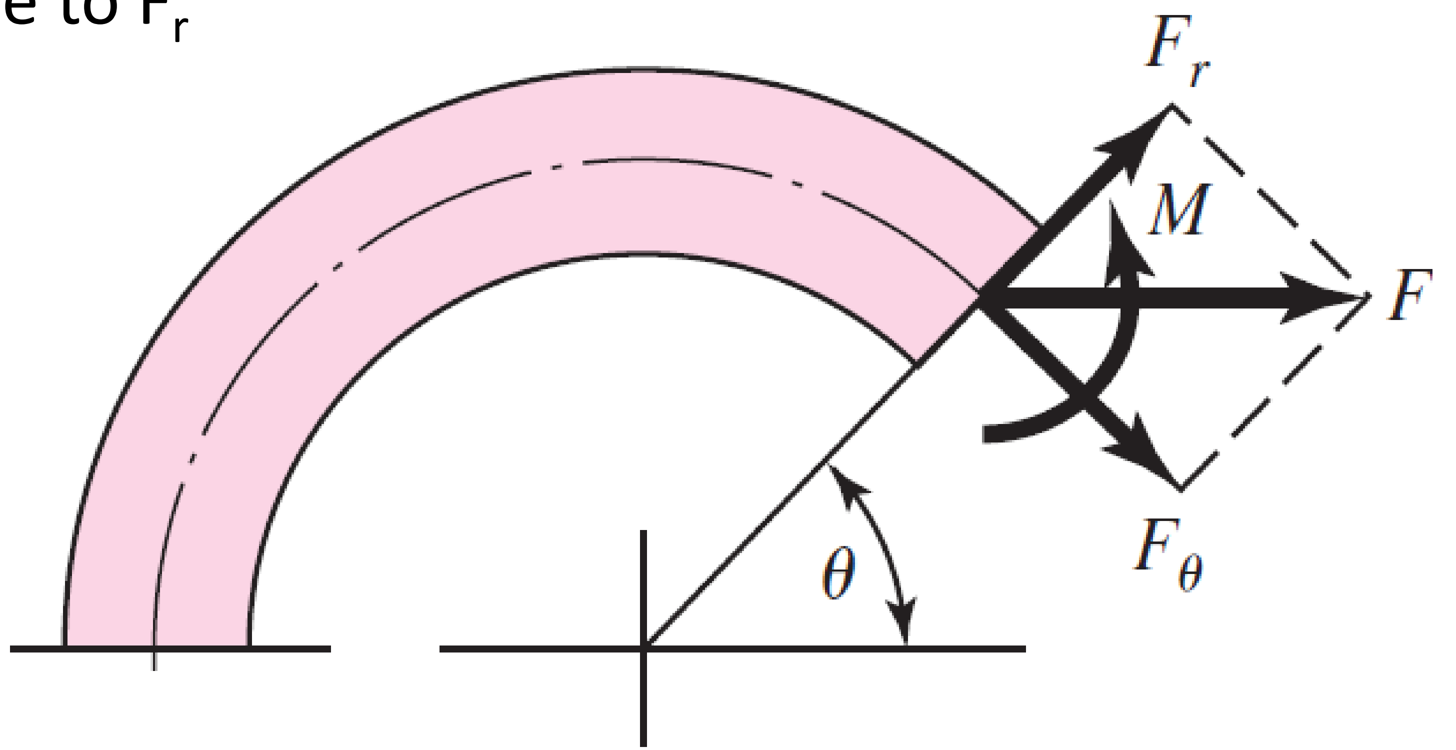


(a) Curved bar loaded by force F. R = radius to centroidal axis of section; h = section thickness. (b) Diagram showing forces acting on section taken at angle θ. $F_r = V$ = shear component of F; F_θ is component of F normal to section; M is moment caused by force F.

Deflection of Curved Members

The total strain energy consists of four terms:

- Bending moment.
- Normal force F_θ
- The force F_θ also produces a moment
- Transverse shear energy due to F_r



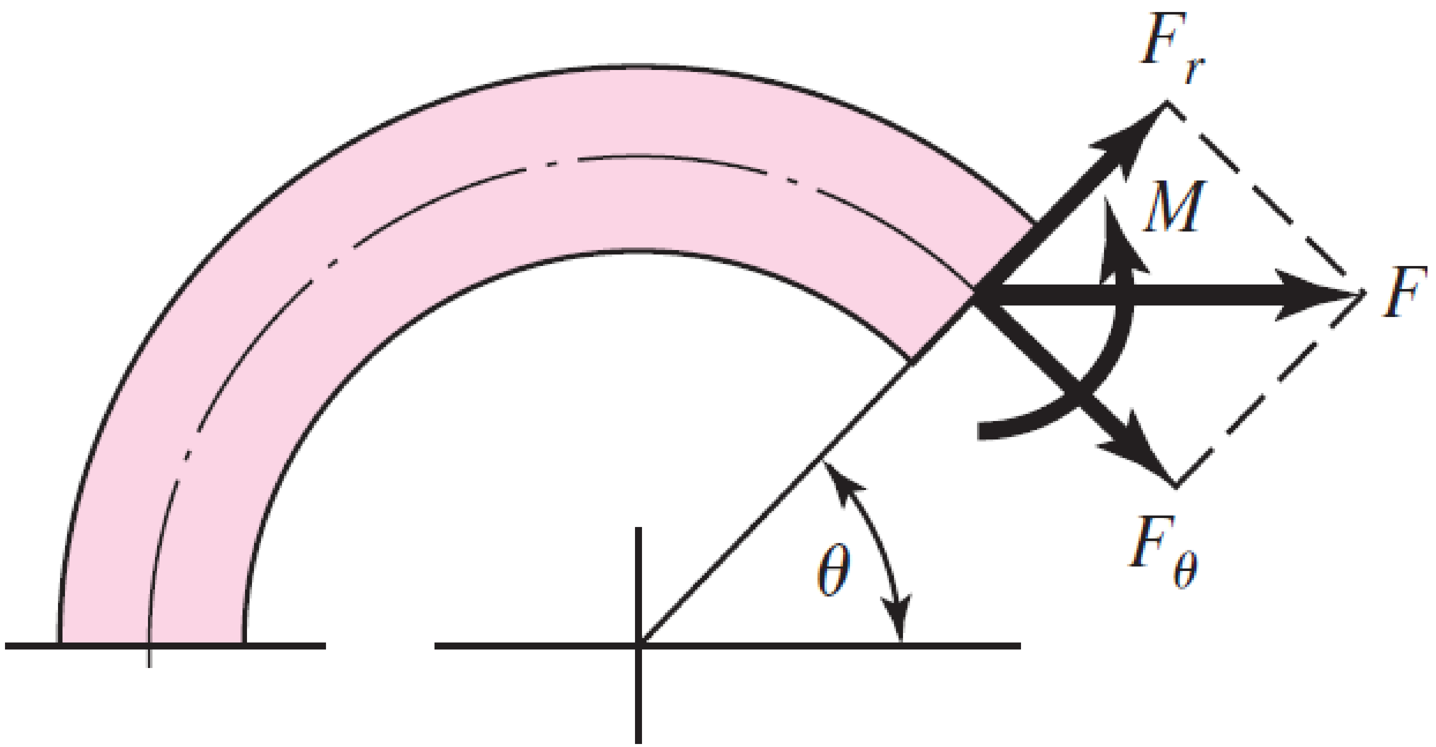
Deflection of Curved Members

$$U_1 = \int \frac{M^2\, d\theta}{2AeE} \quad e = R - r_n$$

$$U_2 = \int \frac{F_\theta^2 R\, d\theta}{2AE}$$

$$U_3 = -\int \frac{M F_\theta\, d\theta}{AE}$$

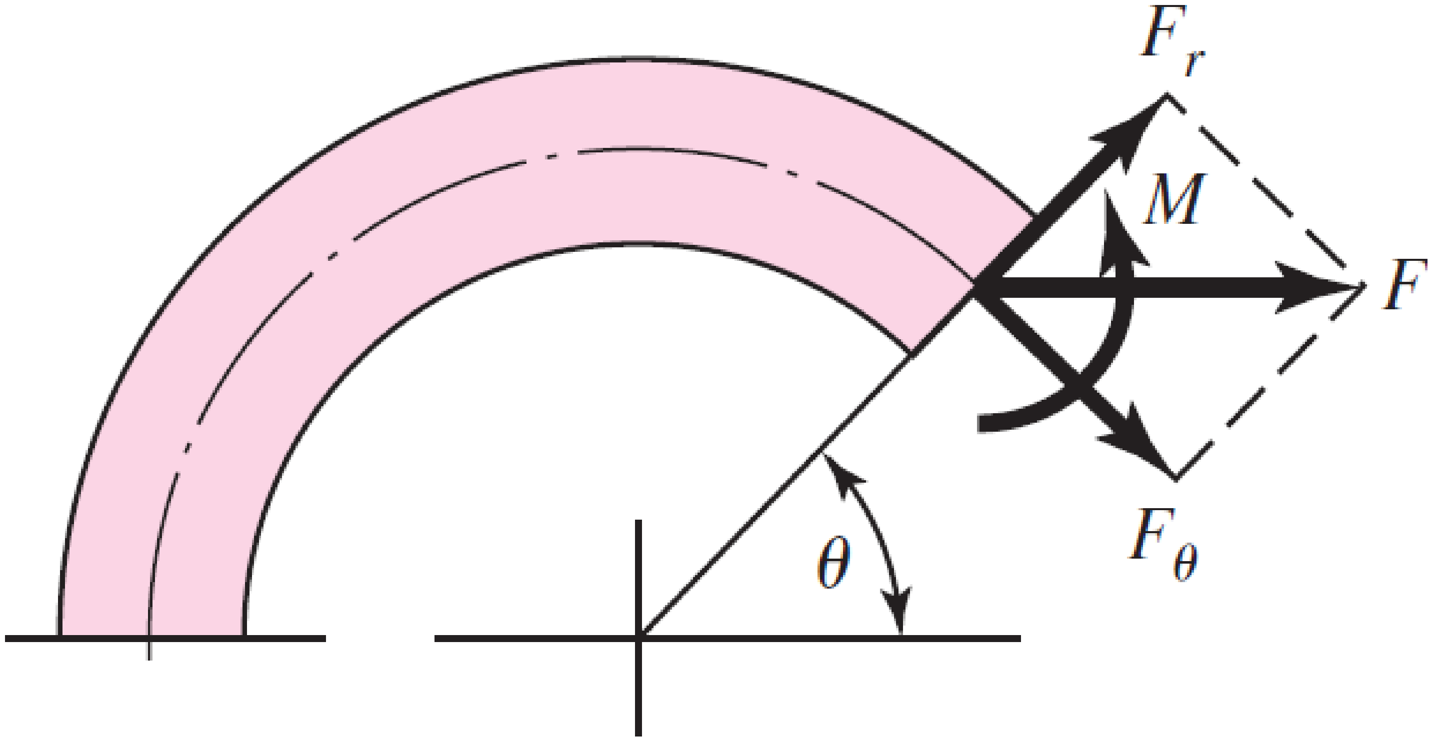
$$U_4 = \int \frac{C F_r^2 R\, d\theta}{2AG}$$



Deflection of Curved Members

Combining the four terms gives the total strain energy

$$U = \int \frac{M^2\, d\theta}{2AeE} + \int \frac{F_\theta^2 R\, d\theta}{2AE} - \int \frac{M F_\theta\, d\theta}{AE} + \int \frac{C F_r^2 R\, d\theta}{2AG}$$



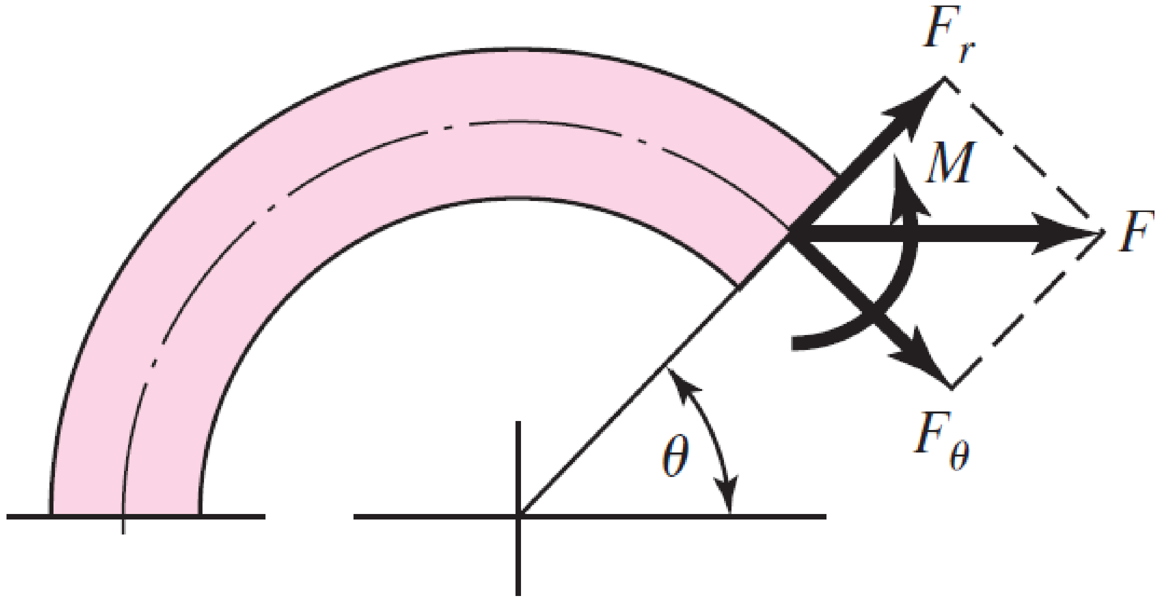
Deflection of Curved Members

The deflection produced by the force *F* can now be found.

$$\delta = \frac{\partial U}{\partial F} = \int \frac{M}{AeE}\left(\frac{\partial M}{\partial F}\right) d\theta + \int \frac{F_\theta R}{AE}\left(\frac{\partial F_\theta}{\partial F}\right) d\theta$$

$$- \int \frac{1}{AE}\frac{\partial (M F_\theta)}{\partial F} d\theta + \int \frac{C F_r R}{AG}\left(\frac{\partial F_r}{\partial F}\right) d\theta$$

This equation is general and may be applied to any section of a thick-walled circular curved beam with application of appropriate limits of integration.



Deflection of Curved Members

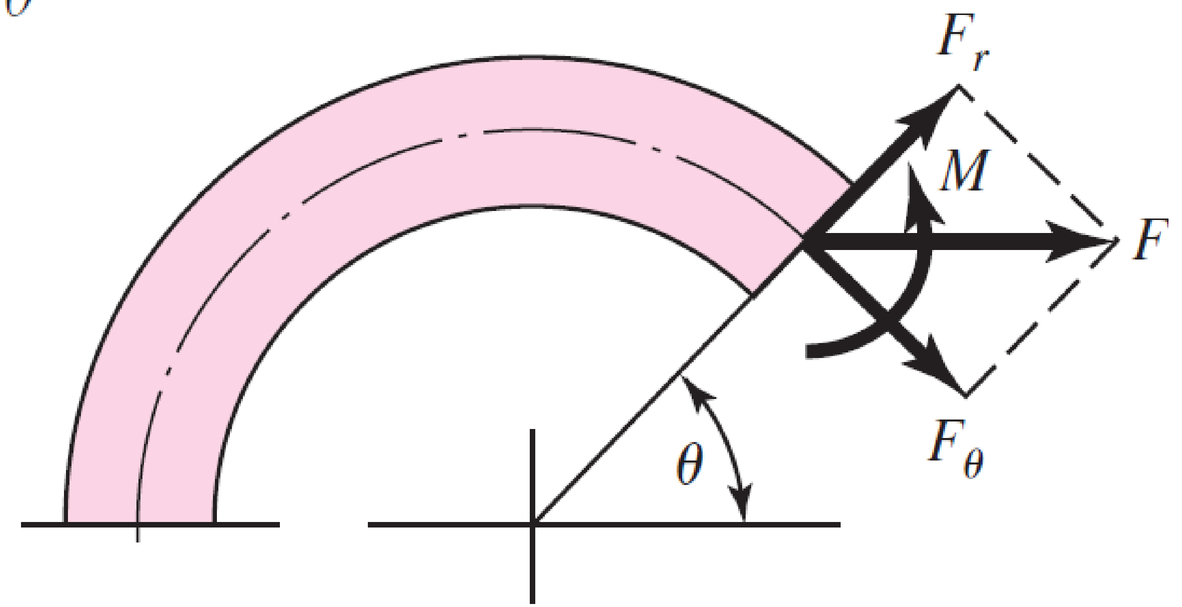
For this specific curved beam, the integrals are evaluated from 0 to π.

$$M = FR\sin\theta \qquad \frac{\partial M}{\partial F} = R\sin\theta$$

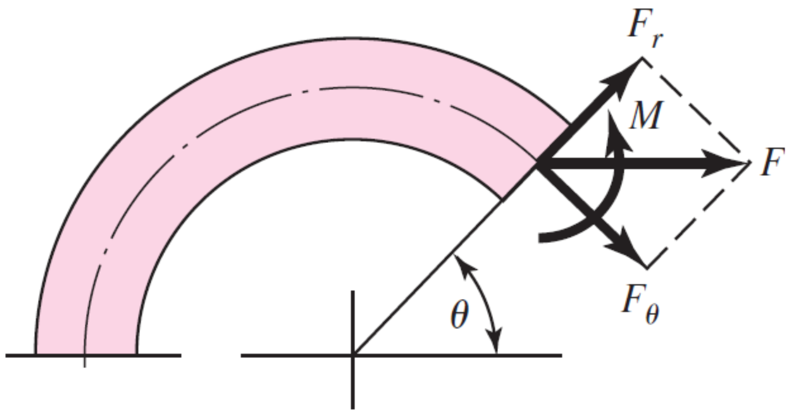
$$F_\theta = F\sin\theta \qquad \frac{\partial F_\theta}{\partial F} = \sin\theta$$

$$MF_\theta = F^2 R\sin^2\theta \qquad \frac{\partial (MF_\theta)}{\partial F} = 2FR\sin^2\theta$$

$$F_r = F\cos\theta \qquad \frac{\partial F_r}{\partial F} = \cos\theta$$



Deflection of Curved Members



Substituting yields:

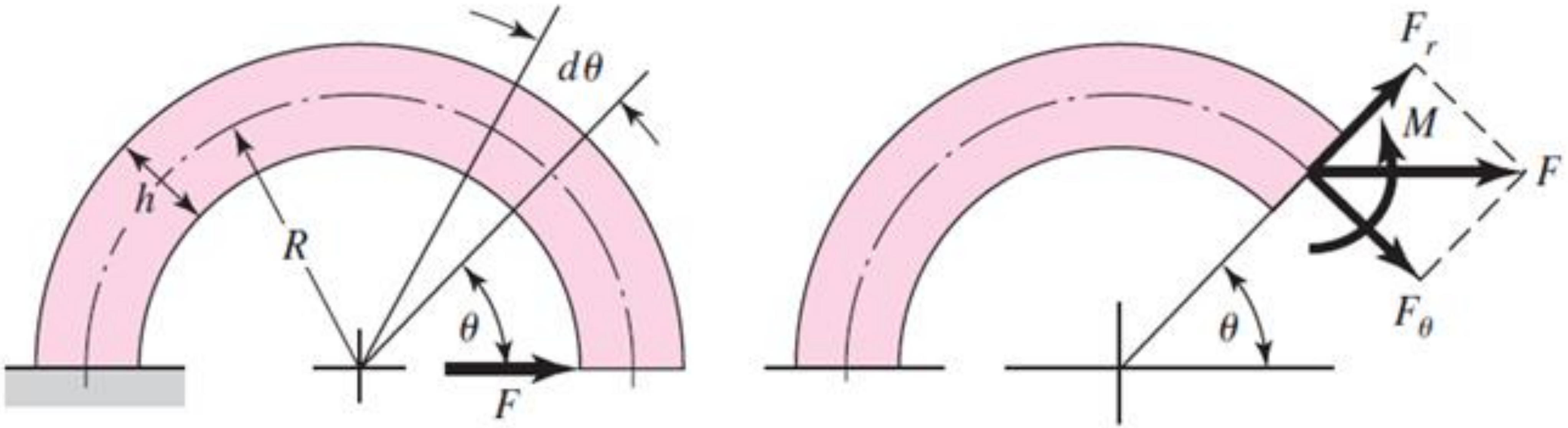
$$\delta = \frac{FR^2}{AeE}\int_0^\pi \sin^2\theta\,d\theta + \frac{FR}{AE}\int_0^\pi \sin^2\theta\,d\theta - \frac{2FR}{AE}\int_0^\pi \sin^2\theta\,d\theta + \frac{CFR}{AG}\int_0^\pi \cos^2\theta\,d\theta$$

$$= \frac{\pi FR^2}{2AeE} + \frac{\pi FR}{2AE} - \frac{\pi FR}{AE} + \frac{\pi CFR}{2AG} = \frac{\pi FR^2}{2AeE} - \frac{\pi FR}{2AE} + \frac{\pi CFR}{2AG}$$

Because the first term contains the square of the radius, the second two terms will be small if the frame has a large radius.

Deflection of Curved Members

For curved sections in which the radius is significantly larger than the thickness, say $R/h > 10$, the effect of the eccentricity is negligible, so that the strain energies can be approximated directly with a substitution of $Rd\theta$ for dx.



Deflection of Curved Members

Further, as *R* increases, the contributions to deflection from the normal force and tangential force becomes negligibly small compared to the bending component.

Therefore, an approximate result can be obtained for a thin circular curved member as

$$U \doteq \int \frac{M^2}{2EI} R\, d\theta \qquad R/h > 10$$

$$\delta = \frac{\partial U}{\partial F} \doteq \int \frac{1}{EI} \left(M \frac{\partial M}{\partial F} \right) R\, d\theta \qquad R/h > 10$$