Torsion of Noncircular Sections

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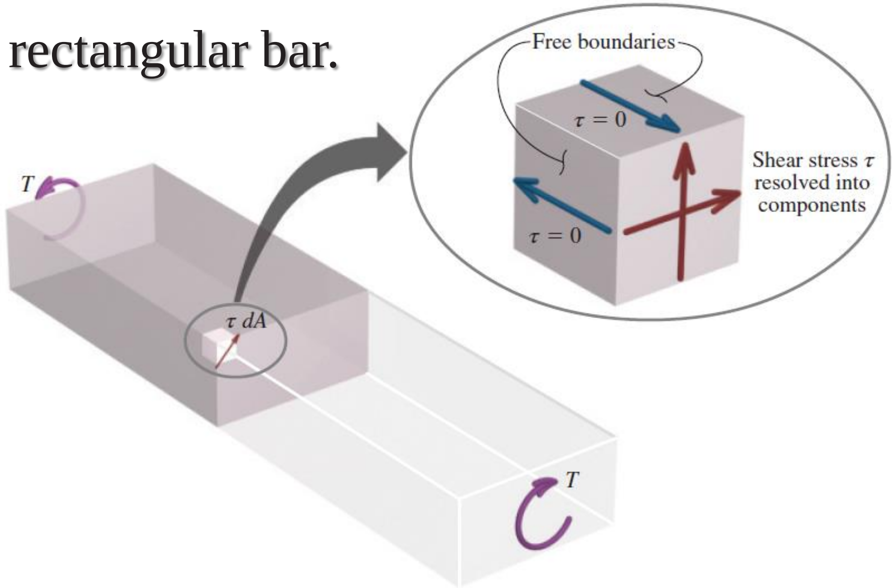
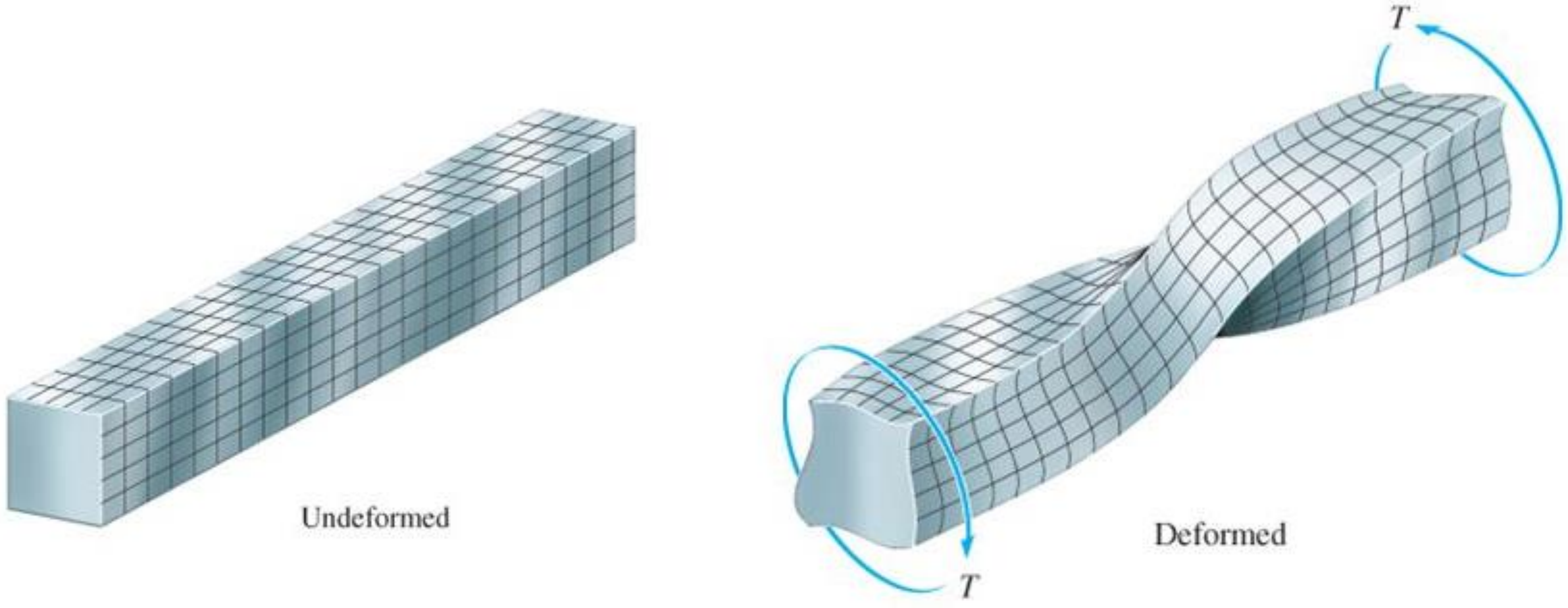
Introduction

In general, every section will warp or bulge (i.e., not remain plane) when twisted, *except for members with circular cross sections*.

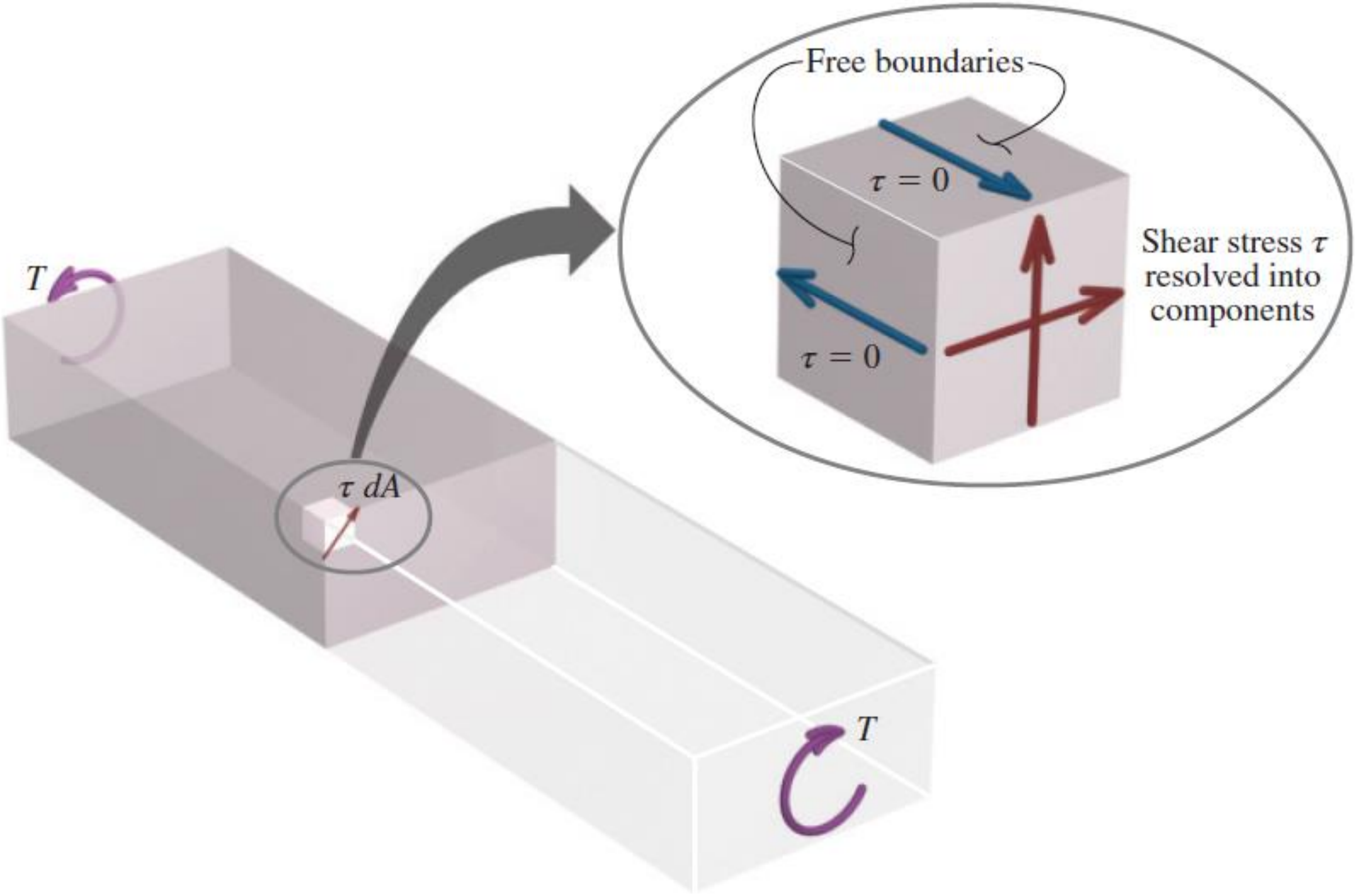
Torsional shear stresses in a rectangular bar.





Shear stresses in any torsionally loaded member were proportional to the distance from the longitudinal axis of the member.

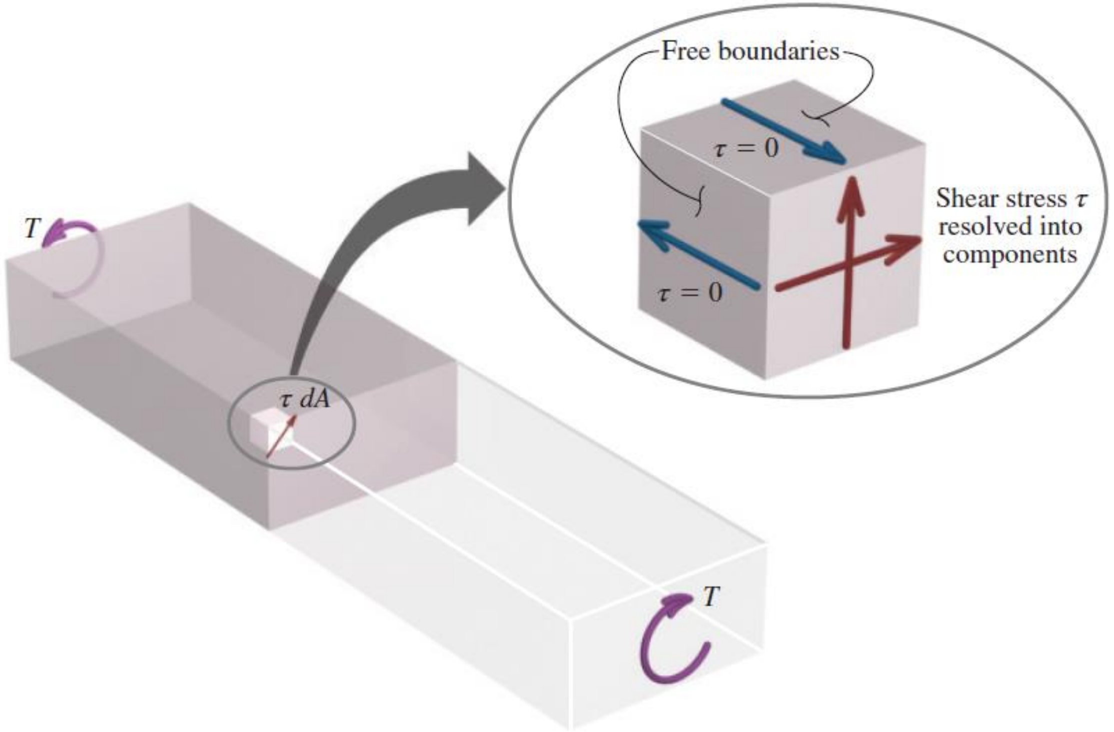
Duleau proved experimentally that relationship does not hold for rectangular cross sections.

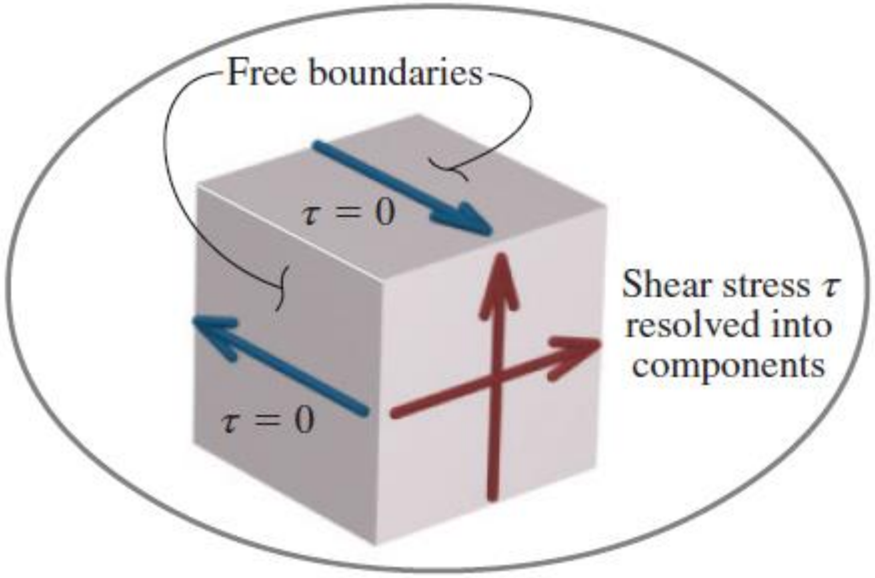


Torsional shear stresses in a rectangular bar.

If the stresses in the rectangular bar were proportional to the distance from its axis, the maximum stress would occur at the corners.

However, if there is a stress of any magnitude at a corner, as indicated in the figure, it could be resolved into the components indicated.

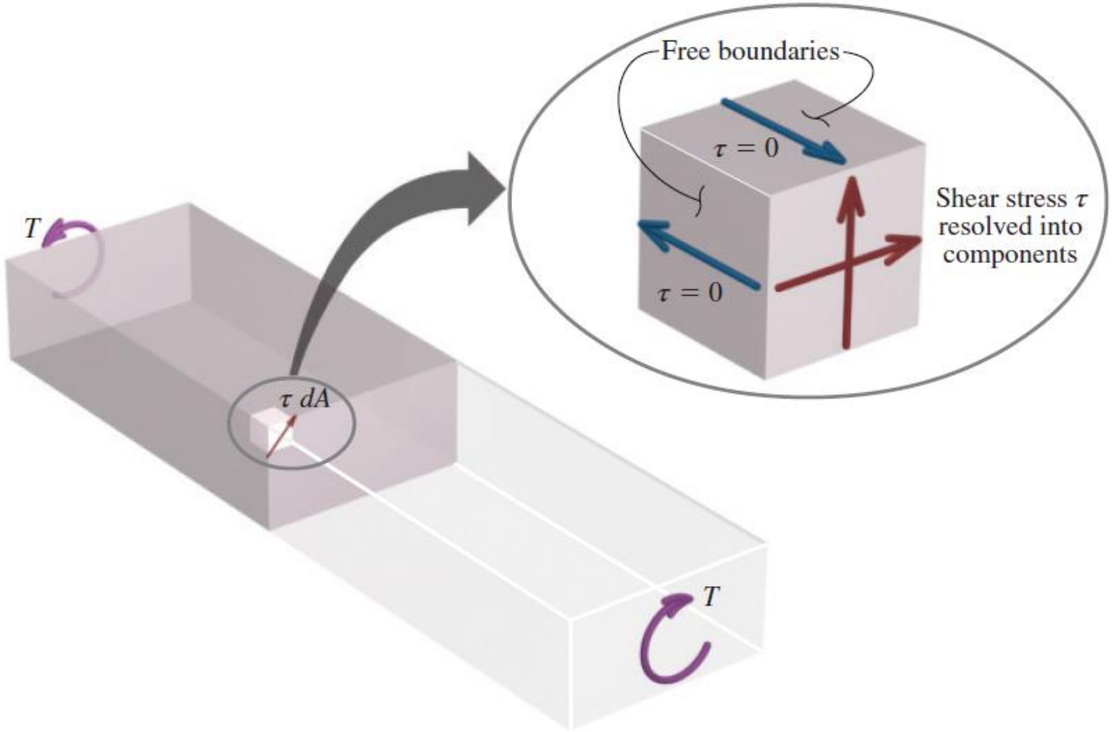


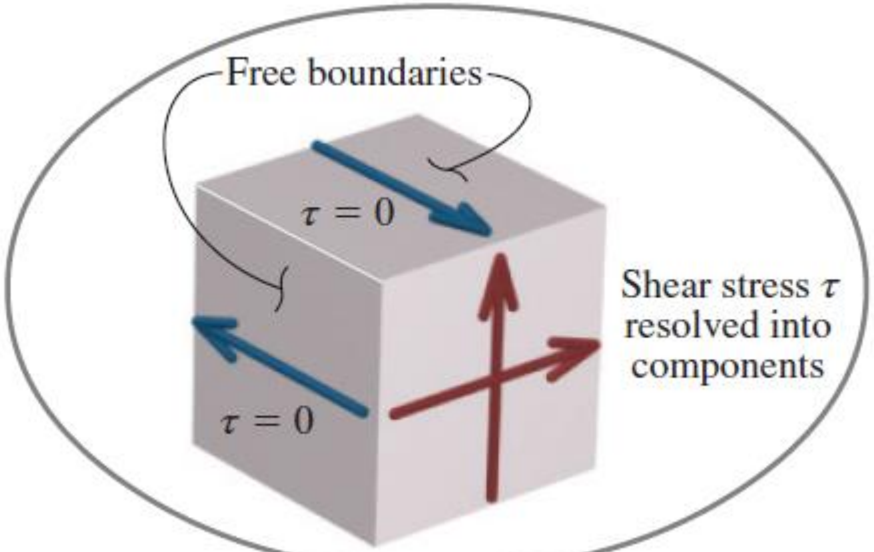


If these components existed, the two components shown by the blue arrows would also exist.

But these last components cannot exist, since the surfaces on which they are shown are free boundaries.

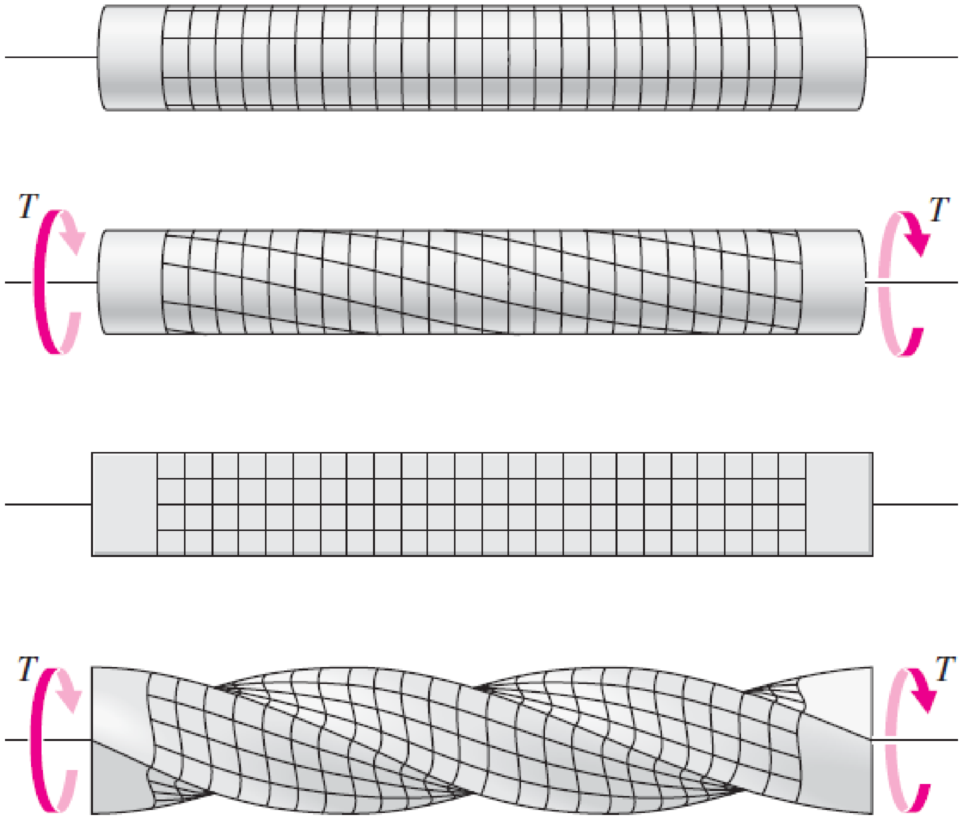
Therefore, the shear stresses at the corners of the rectangular bar must be zero.





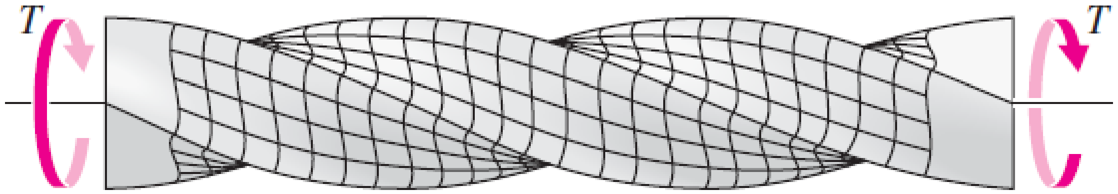
For the case of the rectangular bar shown, the distortion of the small squares is greatest at the midpoint of a side of the cross section and disappears at the corners.

Since this distortion is a measure of shear strain, Hooke’s law requires that the shear stress be largest at the midpoint of a side of the cross section and zero at the corners.



Torsional deformations illustrated by rubber models with circular and square cross sections.

Equations for the maximum shear stress and the angle of twist for a rectangular section obtained from Saint-Venant's theory are:

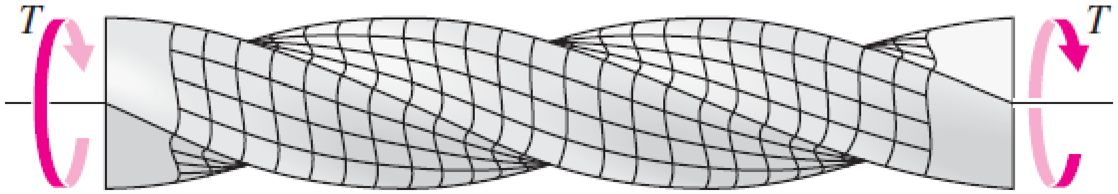


$$\tau_{\max} = \frac{T}{\alpha a^2 b} \quad \text{and} \quad \phi = \frac{TL}{\beta a^3 b G}$$

where *a* and *b* are the lengths of the short and long sides of the rectangle, respectively.

$$\tau_{max} = \frac{T}{\alpha a^2 b}$$

$$\phi = \frac{TL}{\beta a^3 b G}$$



The numerical constants α and β can be obtained from:

Ratio b/a	α	β
1.0	0.208	0.1406
1.2	0.219	0.166
1.5	0.231	0.196
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

Narrow Rectangular Cross Sections

Values for α and β are equal for $b/a \geq 5$.
For aspect ratios $b/a \geq 5$, the coefficients α and β:

$$\alpha = \beta = \frac{1}{3}\left(1 - 0.630\frac{a}{b}\right)$$

Ratio b/a	α	β
1.0	0.208	0.1406
1.2	0.219	0.166
1.5	0.231	0.196
2.0	0.246	0.229
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Narrow Rectangular Cross Sections

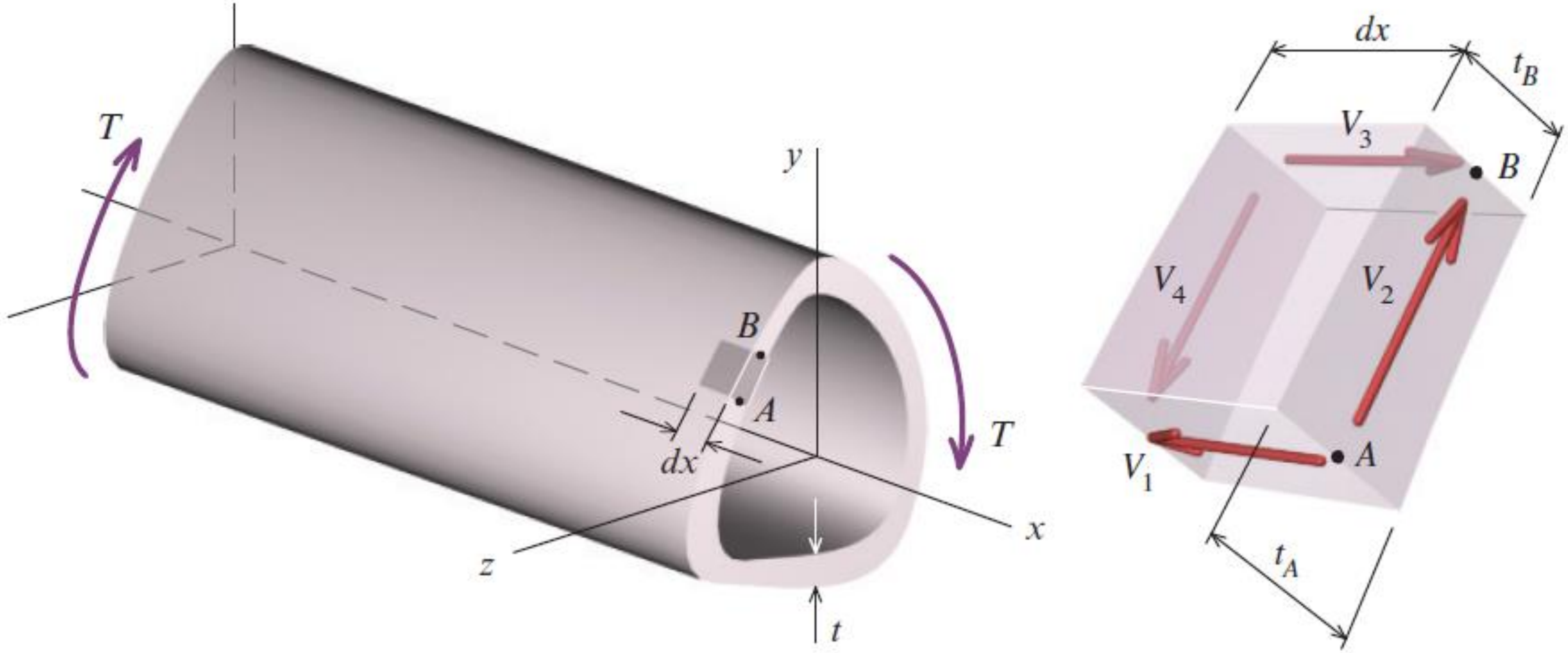
As a practical matter, an aspect ratio $b/a \geq 21$ is sufficiently large that values of $\alpha = \beta = 0.333$ can be used to calculate maximum shear stresses and deformations in narrow rectangular bars within an accuracy of 3 percent.

Accordingly, equations for the maximum shear stress and angle of twist in narrow rectangular bars can be expressed as

Ratio b/a	α	β
1.0	0.208	0.1406
1.2	0.219	0.166
1.5	0.231	0.196
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

$$\tau_{max} = \frac{3T}{a^2 b} \quad \text{and} \quad \phi = \frac{3TL}{a^3 bG}$$

Torsion of Thin-Walled Tubes: Shear Flow



Noncircular section with a wall of variable thickness

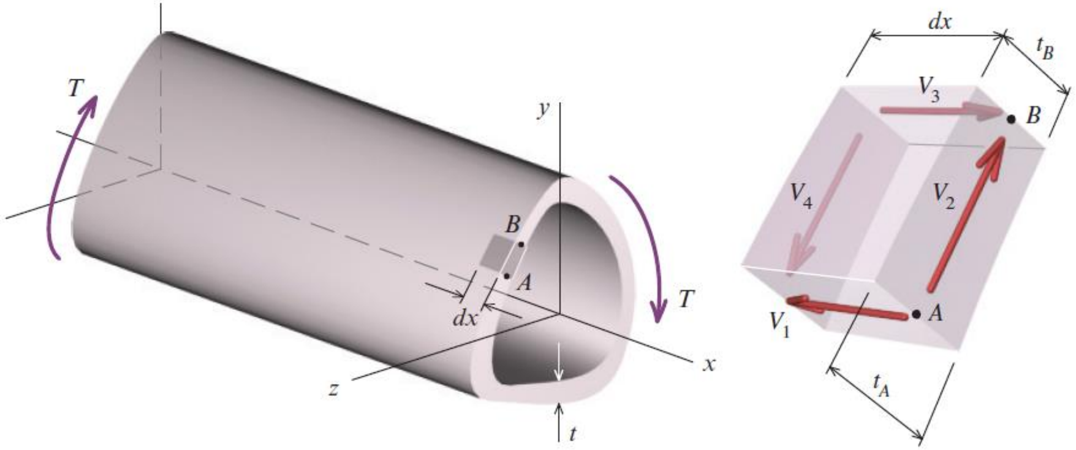
Torsion of Thin-Walled Tubes: Shear Flow

A useful concept associated with the analysis of thin-walled sections is the **shear flow** *q*, defined as the internal shearing force per unit of length of the thin section.

Typical units for *q* are pounds per inch or newtons per meter.

$$q = \tau \times t$$

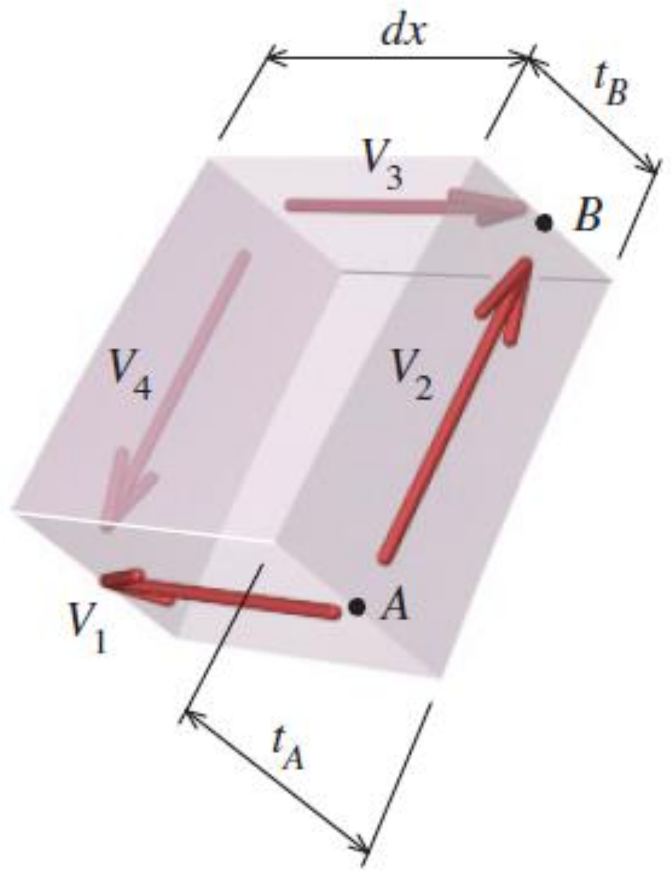
where τ is the average shear stress across the thickness *t*.



Noncircular section with a wall of variable thickness

Torsion of Thin-Walled Tubes: Shear Flow

Since the member is subjected to pure torsion, the shear forces V_1, V_2, V_3, and V_4 alone are necessary and sufficient for equilibrium.



$$V_1 = V_3$$

Or

$$q_1\,dx = q_3\,dx$$

and

$$\tau_1 t_A = \tau_3 t_B$$

Torsion of Thin-Walled Tubes: Shear Flow

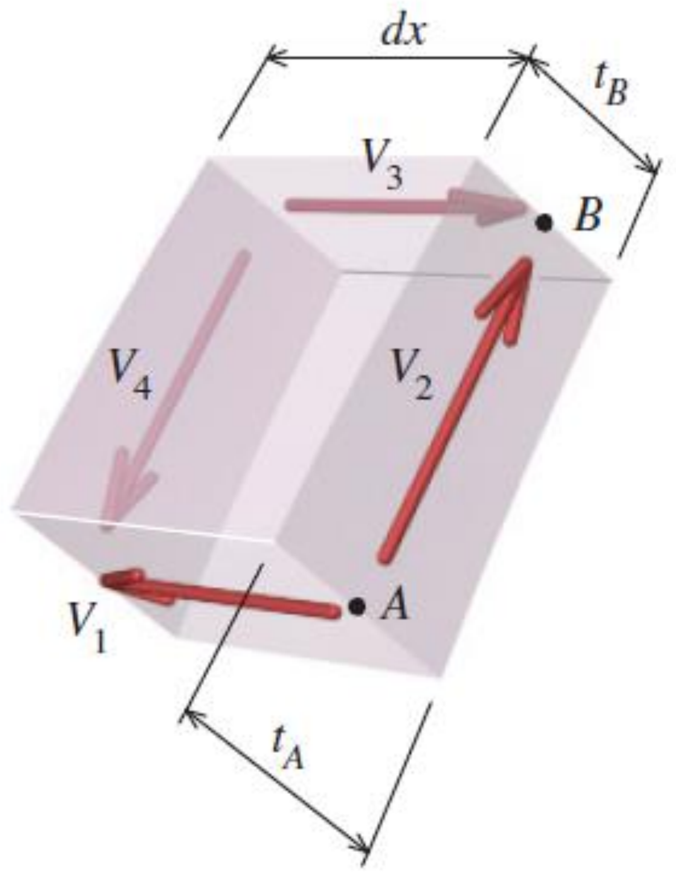
$$\tau_A t_A = \tau_B t_B$$

or

$$q_A = q_B$$

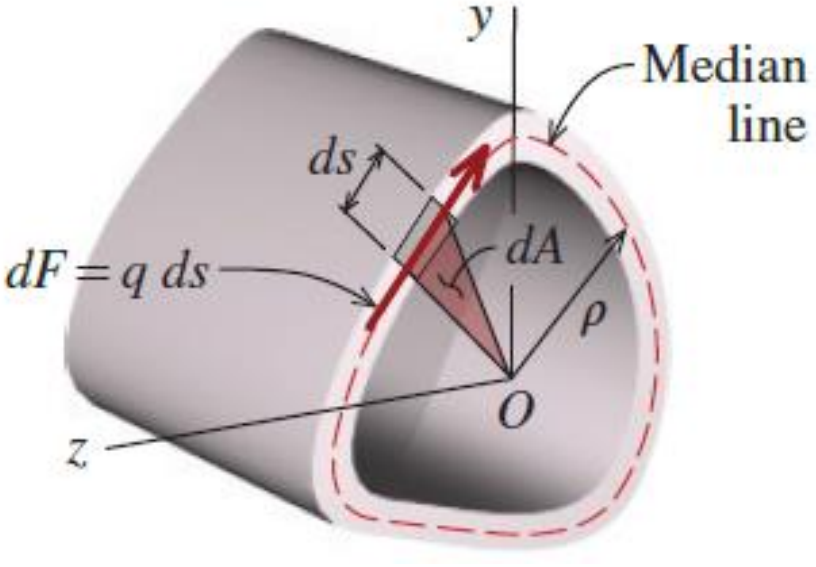
which demonstrates that the *shear flow on a cross section is constant* even though the wall thickness of the section varies.

Since q is constant over a cross section, the *largest* average
shear stress will occur where the wall thickness is the *smallest*.



Torsion of Thin-Walled Tubes: Shear Flow

Consider the force *dF* acting through the center of a differential length of perimeter *ds*, as shown.



The differential moment produced by *dF* about the origin *O* is simply $\rho \times dF$, where ρ is the mean radial distance from the perimeter element to the origin.

The internal torque equals the resultant of all of the differential moments; that is:

$$T = \int (dF)\rho = \int (q\ ds)\rho = q \int \rho\ ds$$

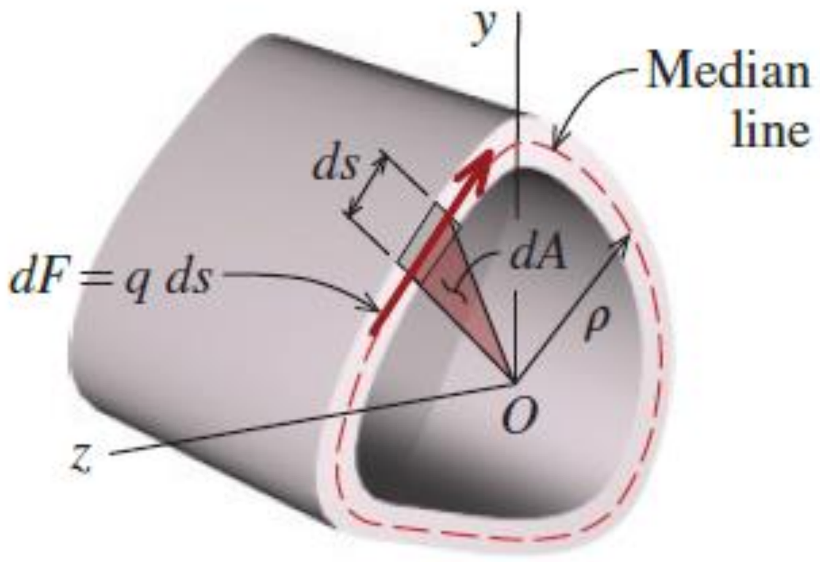
Torsion of Thin-Walled Tubes: Shear Flow

The quantity ρ *ds* is twice the area of the triangle shown shaded in the figure.
Thus:

$$T = q(2A_m)$$

and

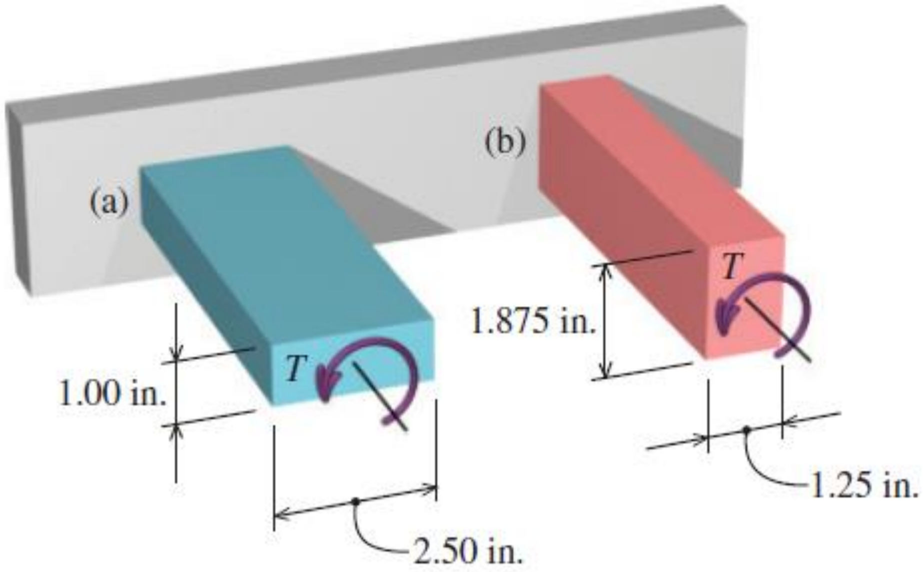
$$\tau = \frac{T}{2A_m t}$$



Examples

The two rectangular polymer bars shown are each subjected to a torque $T = 2{,}000$ lb·in. For each bar, determine

(a) the maximum shear stress.
(b) the rotation angle at the free end if the bar has a length of 12 in. Assume that $G = 500$ ksi for the polymer material.



The maximum shear stress produced in bar (a) by a torque $T = 2{,}000$ lb·in. is

$$\tau_{max} = \frac{T}{\alpha a^2 b} = \frac{2{,}000 \text{ lb}\cdot\text{in.}}{(0.258)(1.00 \text{ in.})^2(2.50 \text{ in.})} = 3{,}100 \text{ psi}$$

and the angle of twist for a 12 in. long bar is

$$\phi = \frac{TL}{\beta a^3 b G} = \frac{(2{,}000 \text{ lb}\cdot\text{in.})(12 \text{ in.})}{(0.249)(1.00 \text{ in.})^3(2.50 \text{ in.})(500{,}000 \text{ psi})} = 0.0771 \text{ rad}$$

<u>Examples</u>

For bar (b), the long side of the bar is $b = 1.875$ in. and the short side is $a = 1.25$ in.

The maximum shear stress produced in bar (b) by a torque $T = 2{,}000$ lb · in. is

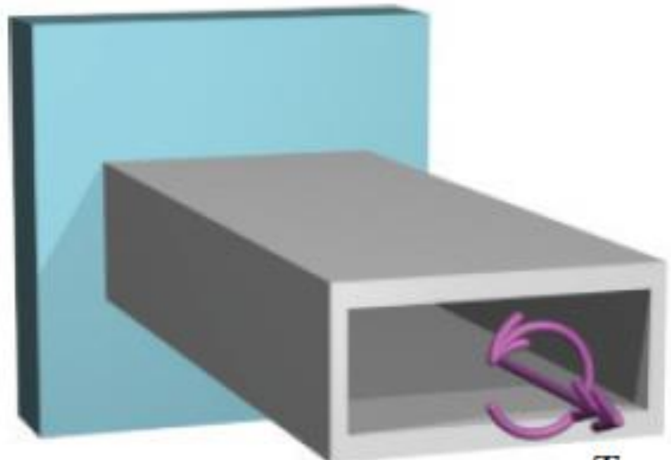
$$\tau_{max} = \frac{T}{\alpha a^2 b} = \frac{2{,}000 \text{ lb}\cdot\text{in.}}{(0.231)(1.25 \text{ in.})^2(1.875 \text{ in.})} = 2{,}960 \text{ psi}$$

and the angle of twist for a 12 in. long bar is

$$\phi = \frac{TL}{\beta a^3 bG} = \frac{(2{,}000 \text{ lb}\cdot\text{in.})(12 \text{ in.})}{(0.196)(1.25 \text{ in.})^3(1.875 \text{ in.})(500{,}000 \text{ psi})} = 0.0669 \text{ rad}$$

Examples

A rectangular box section of aluminum alloy has outside dimensions of 100 mm by 50 mm. The plate thickness is 2 mm for the 50 mm sides and 3 mm for the 100 mm sides. If the maximum shear stress must be limited to 95 MPa, determine the maximum torque T that can be applied to the section.

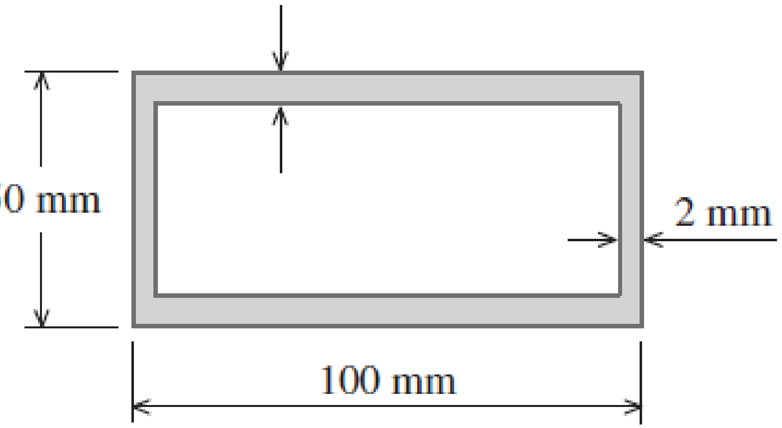


The maximum shear stress will occur in the thinnest plate; therefore, the critical shear flow q is

$$q = \tau t = (95\ \text{N/mm}^2)(2\ \text{mm}) = 190\ \text{N/mm}$$

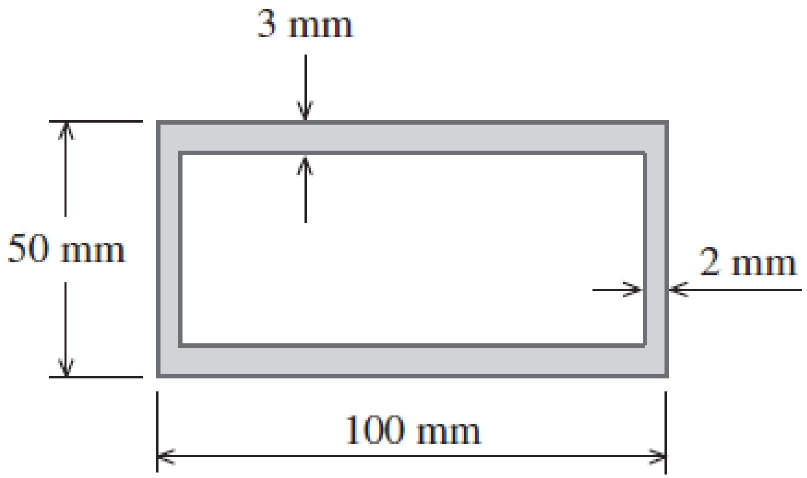
The area enclosed by the median line is

$$A_m = (100\ \text{mm} - 2\ \text{mm})(50\ \text{mm} - 3\ \text{mm}) = 4{,}606\ \text{mm}^2$$



Cross-sectional dimensions.

Examples



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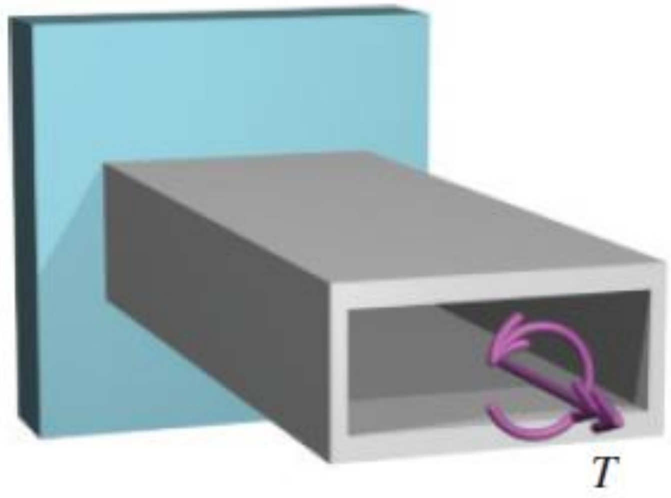

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The area enclosed by the median line is

$$A_m = (100\ \text{mm} - 2\ \text{mm})(50\ \text{mm} - 3\ \text{mm}) = 4{,}606\ \text{mm}^2$$

$$T = q(2A_m) = (190\ \text{N/mm})(2)(4{,}606\ \text{mm}^2) = 1{,}750{,}280\ \text{N}\cdot\text{mm} = 1{,}750\ \text{N}\cdot\text{m}$$