Fundamentals of Heat and Mass Transfer

Chapter 2

Fourier's Law and the Heat Equation

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The Heat Diffusion Equation

- A major objective in a conduction analysis is to determine the temperature field in a medium resulting from conditions imposed on its boundaries.

- That is, we wish to know the temperature distribution, which represents **how temperature varies with position in the medium.**

- Once this distribution is known, **the conduction heat flux at any point in the medium or on its surface may be computed from Fourier's law.**

- Other important quantities of interest may also be determined:
- For a solid, knowledge of the temperature distribution could be used to ascertain structural integrity through determination of **thermal stresses**, **expansions**, and **deflections**.
- The temperature distribution could also be used **to optimize the thickness of an insulating material** or to **determine the compatibility of special coatings or adhesives used with the material.**

- We now proceed to derive a differential equation whose solution provides the temperature distribution in the medium.

- Consider a **homogeneous medium** within which there **is no bulk motion (advection**) and the temperature distribution **T(x, y, z) is expressed in Cartesian coordinates.**

- The medium is assumed to **be incompressible**, that is, its **density can be treated as constant.**

- Following the four-step methodology of applying conservation of energy (Section 1.3.1),

- we first define an infinitesimally small (differential) control volume, $dx \cdot dy \cdot dz$, as shown in Figure 2.11.

- Choosing to **formulate the first law at an instant of time,** the second step is to consider **the energy processes** that are relevant to this control volume.

- In the absence of motion, there are no changes in **mechanical energy and no work being done on the system.**

- **Only thermal forms** of energy need be considered.

- Specifically, if there are temperature gradients, conduction heat transfer will occur across each of the control surfaces.

- The conduction heat rates perpendicular to each of the control surfaces at the x-, y-, and z-coordinate locations are indicated by the terms qx, qy, and qz, respectively.

- The conduction heat rates at the opposite surfaces can then be expressed as a Taylor series expansion where, neglecting higher-order terms,

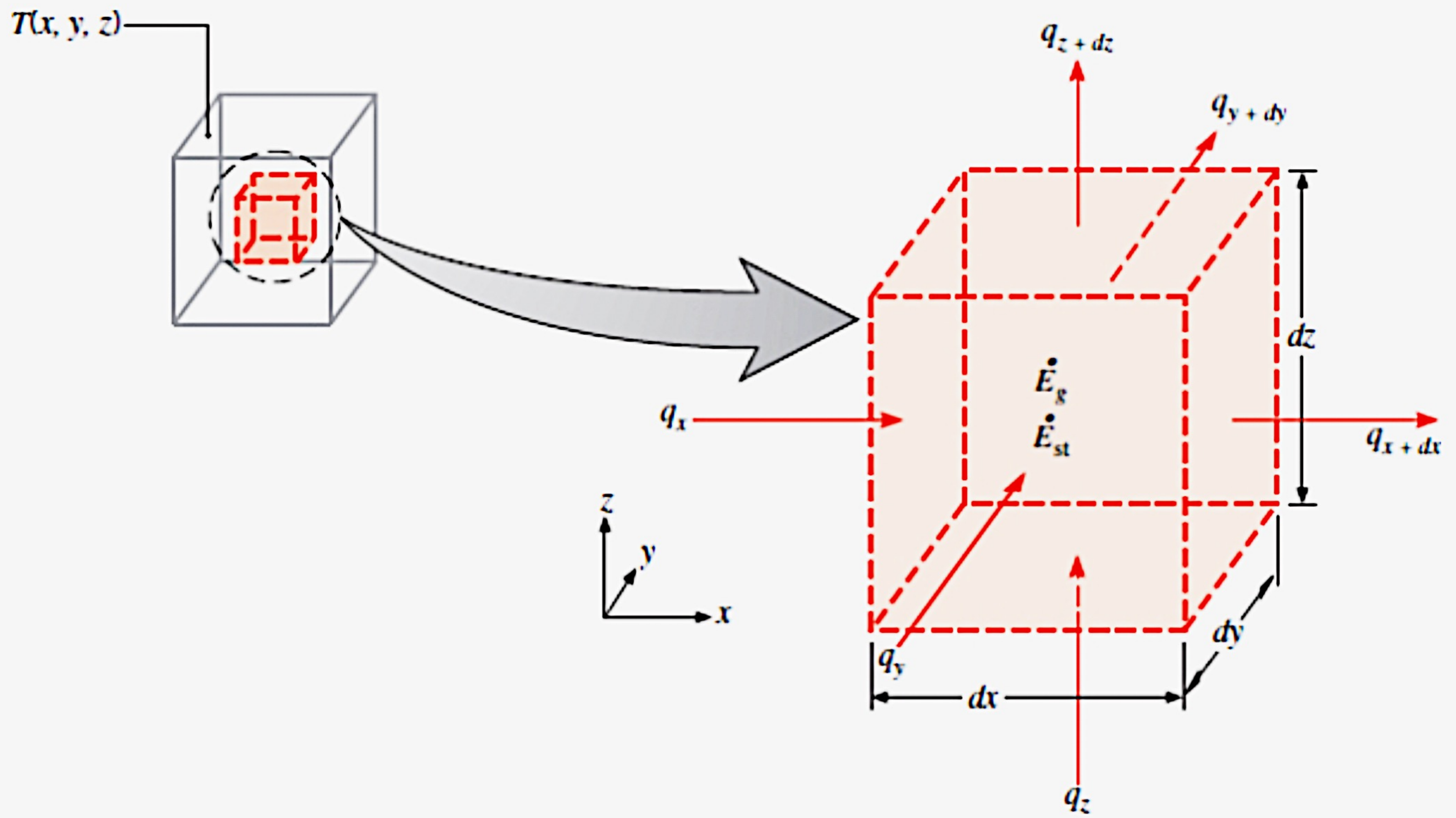


FIGURE 2.11 Differential control volume, *dx dy dz*, for conduction analysis in Cartesian coordinates.

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz$$

- In words, the first equation in this slide simply states that the x-component of the heat transfer rate at x + dx is equal to the value of this component at x plus the amount by which it changes with respect to x times dx.
- Within the medium there may also be an **energy source term** associated with the rate of thermal energy generation. This term is represented as

$$\dot{E}_g = \dot{q}\, dx\, dy\, dz$$

  - where $\dot{q}$ is the rate at which energy is generated per unit volume of the medium (W/m^3).

- In addition, changes may occur in the amount of the internal thermal energy stored by the material in the control volume.
- If the material is not experiencing a change in phase, latent energy effects are not pertinent, and the energy storage term reduces to the rate of change of sensible energy:

$$\dot{E}_{st} = \frac{\partial U_{sens}}{\partial t} = \rho c_v \frac{\partial T}{\partial t} dx\, dy\, dz = \rho c_p \frac{\partial T}{\partial t} dx\, dy\, dz$$

- Here, use has been made of the fact that cp = cv for an incompressible substance.2
- Once again it is important to note that the terms $\dot{E}_g$ and $\dot{E}_s$ represent differenhet physical processes.

- T energy generation term is due to energy conversion process involving some forms of energy, such as chemical, electrical, or nuclear, on the other.

- **The term is positive (a source) if thermal energy is being generated in the material at the expense of some other energy form; it is negative (a sink) if thermal energy is being consumed.**

- In contrast, **the energy storage term refers to the rate of change of thermal energy stored by the matter.**

- The last step in the methodology outlined in Section 1.3.1 is to express conservation of energy using the foregoing rate equations. On a rate basis, the general form of the conservation of energy requirement is

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$$

- Hence, recognizing that the conduction rates constitute the energy inflow and outflow , and substituting the previous equations:

$$q_x + q_y + q_z + \dot{q}\,\mathrm{dx\ dy\ dz} - q_{x+\mathrm{dx}} - q_{y+\mathrm{dy}} - q_{z+\mathrm{dz}} = \rho c_p \frac{\partial \mathrm{T}}{\partial \mathrm{t}} \mathrm{dx\ dy\ dz}$$

After substitution

$$-\frac{\partial q_x}{\partial \mathrm{x}}\mathrm{dx} - \frac{\partial q_y}{\partial \mathrm{y}}\mathrm{dy} - \frac{\partial q_z}{\partial \mathrm{z}}\mathrm{dz} + \dot{q}\ \mathrm{dx\ dy\ dz} = \rho c_p \frac{\partial \mathrm{T}}{\partial \mathrm{t}} \mathrm{dx\ dy\ dz}$$

The conduction heat rates in an isotropic material may be evaluated from Fourier's law,

$$q_x = -k\ \mathrm{dy\ dz}\frac{\partial \mathrm{T}}{\partial \mathrm{x}}$$

$$q_y = -k\ \mathrm{dx\ dz}\frac{\partial \mathrm{T}}{\partial \mathrm{y}}$$

$$q_z = -k\ \mathrm{dx\ dy}\frac{\partial \mathrm{T}}{\partial \mathrm{z}}$$

Substituting the equations into the last equation and dividing out the dimensions of the control volume (dx dy dz), we obtain

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

This is the general form, in Cartesian coordinates, of the heat diffusion equation.

- This equation, often referred to as the heat equation, provides the basic tool for heat conduction analysis.
- From its solution, we can obtain the temperature distribution T(x, y, z) as a function of time.
- You should have a clear understanding of the physical significance of each term appearing in the equation.

- The heat equation states that at any point in the medium the net rate of energy transfer by conduction into a unit volume plus the volumetric rate of thermal energy generation must equal the rate of change of thermal energy stored within the volume.

- It is often possible to work with simplified versions of the heat equation. For example, if the thermal conductivity is constant, the heat equation is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where $\alpha = k/\rho c_p$ is the thermal diffusivity.

- Additional simplifications of the general form of the heat equation are often possible.

- For example, under steady-state conditions, there can be no change in the amount of energy storage; hence Equation 2.19 reduces to

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=0$$

Moreover, if the heat transfer is one-dimensional (e.g., in the x-direction) and there is no energy generation, Equation 2.22 reduces to

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

- The important implication of this result is that, under steady-state, one-dimensional conditions with no energy generation, the heat flux is a constant in the direction of transfer .

$$(dq''_x/dx = 0).$$

The heat equation may also be expressed in cylindrical and spherical coordinates. The differential control volumes for these two coordinate systems are shown in Figures 2.12 and 2.13.

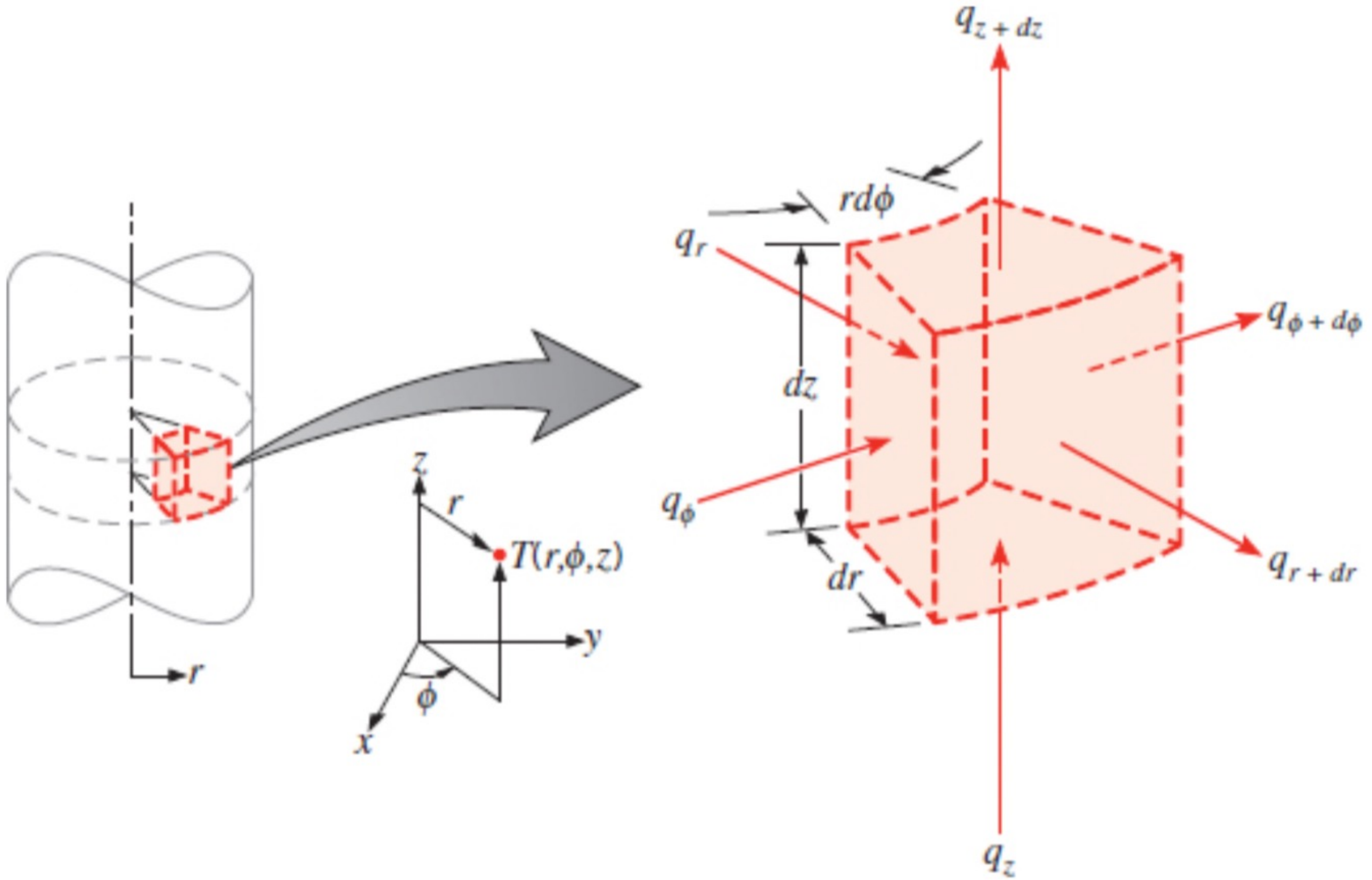


FIGURE 2.12 Differential control volume, dr · r dφ · dz, for conduction analysis in cylindrical coordinates (r, φ, z).

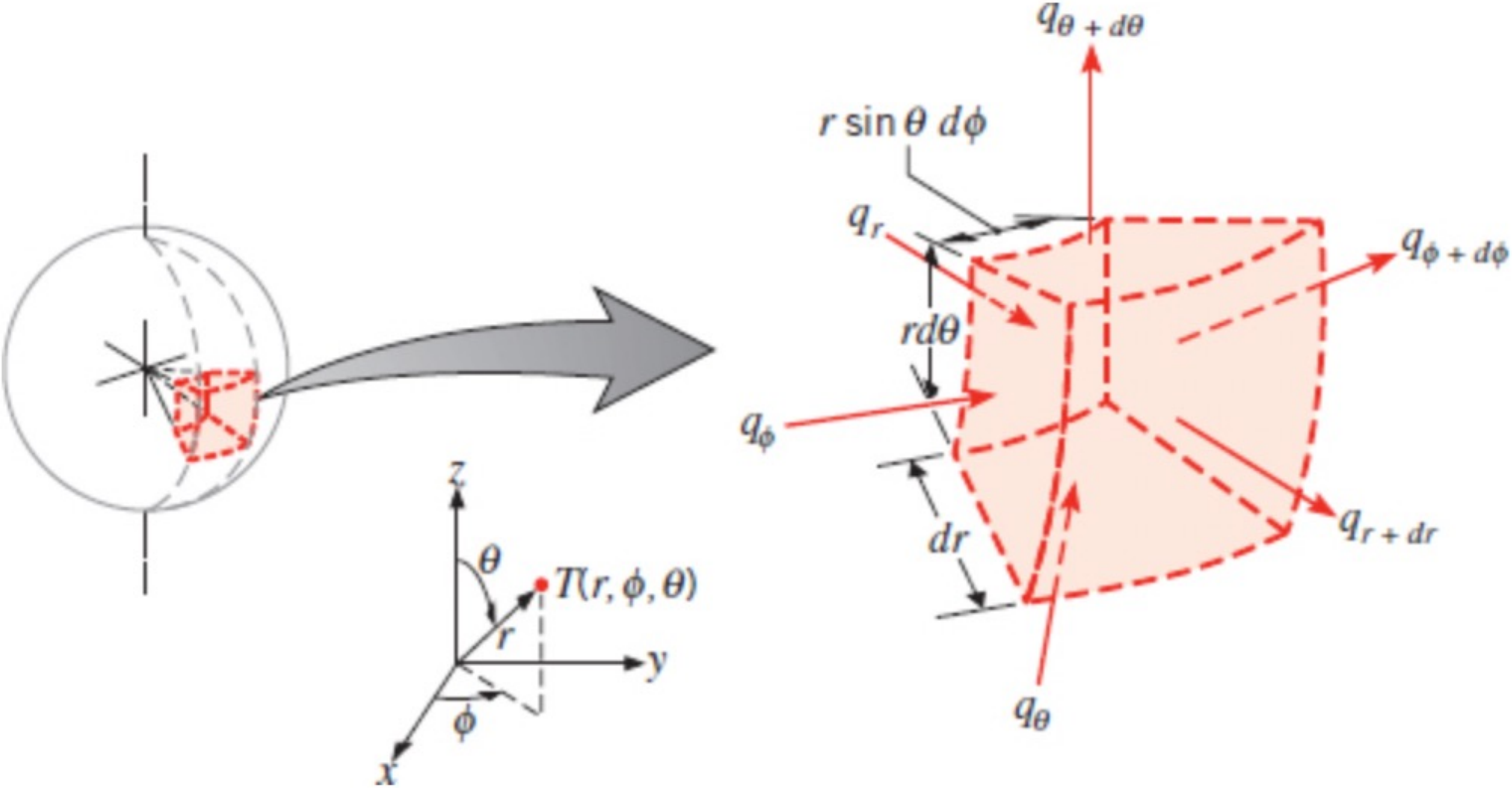


FIGURE 2.13 Differential control volume, dr · r sinθ dφ · r dθ, for conduction analysis in spherical coordinates (r, φ, θ).

Cylindrical Coordinates

When the del operator ∇ is expressed in cylindrical coordinates, with i, j, and k representing the unit vectors in the r, ø, and z directions, the general form of the heat flux vector and hence of Fourier's law is

$$\boldsymbol{q}'' = -k\nabla T = -k\left(\boldsymbol{i}\frac{\partial T}{\partial r} + \boldsymbol{j}\frac{1}{r}\frac{\partial T}{\partial ø} + \boldsymbol{k}\frac{\partial T}{\partial z}\right)$$

$$q_r'' = -k\frac{\partial T}{\partial r} \quad q_ø'' = -\frac{k}{r}\frac{\partial T}{\partial ø} \quad q_z'' = -k\frac{\partial T}{\partial z}$$

are heat flux components in the radial, circumferential, and axial directions, respectively.

Applying an energy balance to the differential control volume of Figure 2.12, the following general form of the heat equation is obtained:

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

Spherical Coordinates:

In spherical coordinates, with i, j, and k representing the unit vectors in the r, θ, and ø directions, the general form of the heat flux vector and Fourier's law is

$$q'' = -k\nabla T = -k\left(\mathbf{i}\frac{\partial T}{\partial r} + \mathbf{j}\frac{1}{r}\frac{\partial T}{\partial \theta} + \mathbf{k}\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)$$

$$q''_r = -k\frac{\partial T}{\partial r} \quad q''_\theta = -\frac{k}{r}\frac{\partial T}{\partial \theta} \quad q''_\phi = -\frac{k}{r\sin\theta}\frac{\partial T}{\partial \phi}$$

are heat flux components in the radial, polar, and azimuthal directions, respectively. Applying an energy balance to the differential control volume of Figure 2.13, the following general form of the heat equation is obtained:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{q} = \rho c_p\frac{\partial T}{\partial t}$$

- **You should attempt to do the derivation to gain experience in applying conservation principles to differential control volumes (see Problems 2.27 and 2.28).**

- Note that the temperature gradient in Fourier's law must have units of K/m. Hence, when evaluating the gradient for an angular coordinate, it must be expressed in terms of the differential change in arc length.

For example, the heat flux component in the circumferential direction of a cylindrical coordinate system is

$$q''_{\phi} = -(k/r)(\partial T/\partial \phi), \textit{ not } q''_{\phi} = -k(\partial T/\partial \phi).$$

$$\frac{\partial}{\partial \mathrm{x}}(k\frac{\partial \mathrm{T}}{\partial \mathrm{x}}) + \frac{\partial}{\partial \mathrm{y}}\left(k\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\right) + \frac{\partial}{\partial \mathrm{z}}(k\frac{\partial \mathrm{T}}{\partial \mathrm{z}}) + \dot{q} = \rho c_p \frac{\partial \mathrm{T}}{\partial \mathrm{t}}$$

$$\frac{1}{r}\frac{\partial}{\partial \mathrm{r}}(kr\frac{\partial \mathrm{T}}{\partial \mathrm{r}}) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial \mathrm{T}}{\partial \phi}\right) + \frac{\partial}{\partial \mathrm{z}}(k\frac{\partial \mathrm{T}}{\partial \mathrm{z}}) + \dot{q} = \rho c_p \frac{\partial \mathrm{T}}{\partial \mathrm{t}}$$

$$\frac{1}{r^2}\frac{\partial}{\partial \mathrm{r}}(kr^2\frac{\partial \mathrm{T}}{\partial \mathrm{r}}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial \mathrm{T}}{\partial \phi}\right)$$
$$+ \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(k \sin\theta\frac{\partial \mathrm{T}}{\partial \theta}) + \dot{q} = \rho c_p \frac{\partial \mathrm{T}}{\partial \mathrm{t}}$$

EXAMPLE 2.3

The temperature distribution across a wall 1 m thick at a certain instant of time is given as

$$T(x) = a + bx + cx^2$$

where T is in degrees Celsius and x is in meters, while a = 900°C, b = −300°C/m, and c = −50°C/m². A uniform heat generation, $\dot{q} = 1000 \text{ W/m}^3$ the wall of area 10 m² having the properties ρ = 1600 kg/m³, k = 40 W/m · K, and cp = 4 kJ/kg · K.

1. Determine the rate of heat transfer entering the wall (x = 0) and leaving the wall (x = 1 m).
2. Determine the rate of change of energy storage in the wall.
3. Determine the time rate of temperature change at x = 0, 0.25, and 0.5 m.

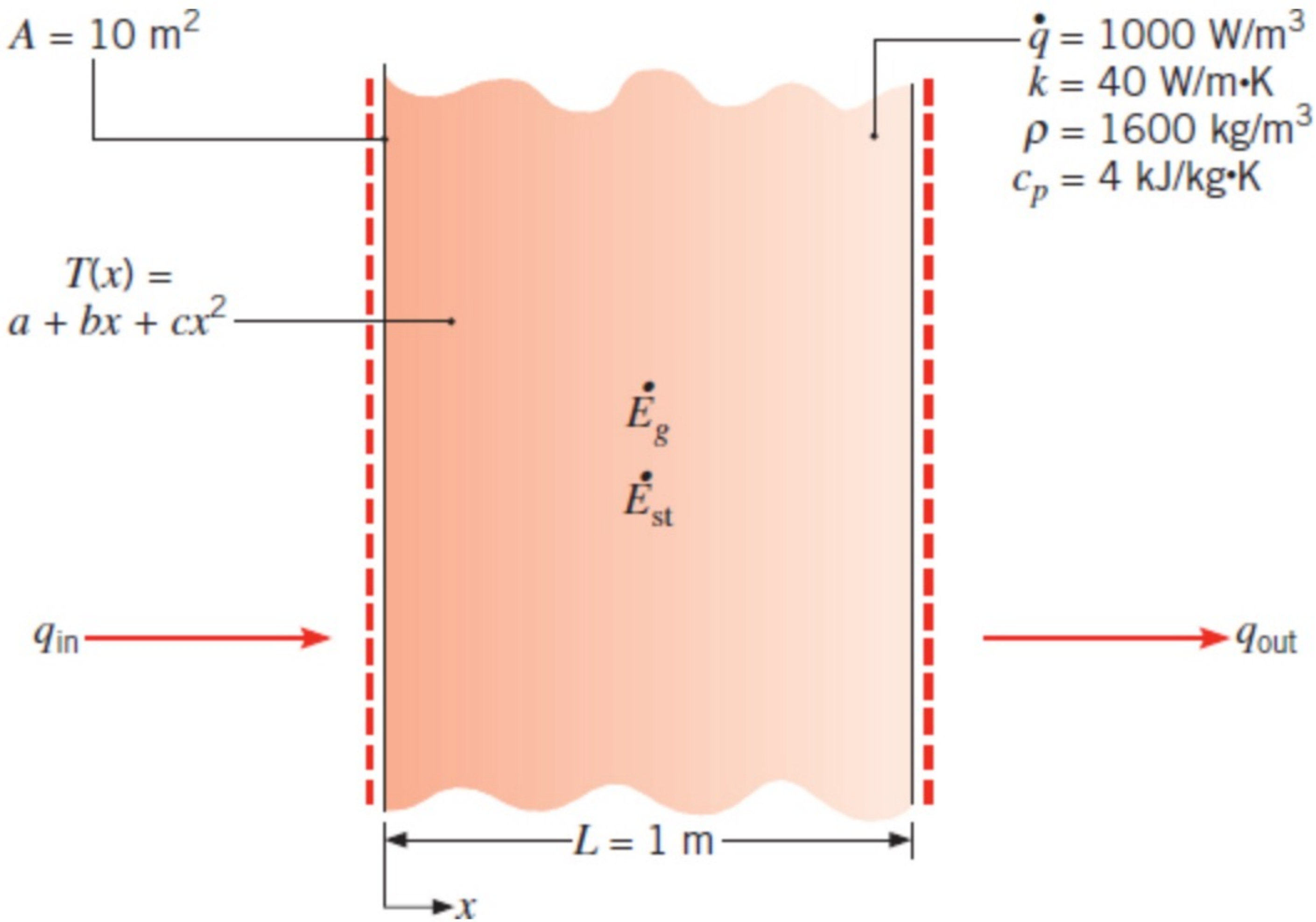
SOLUTION

Known: Temperature distribution T(x) at an instant of time t in a one-dimensional wall with uniform heat generation.

Find:

1. Heat rates entering, qin (x = 0), and leaving, qout (x = 1 m), the wall.
2. Rate of change of energy storage in the wall, .
3. Time rate of temperature change at x = 0, 0.25, and 0.5 m

Schematic:



A = 10 m2
q̇ = 1000 W/m3
k = 40 W/m·K
ρ = 1600 kg/m3
cp = 4 kJ/kg·K
T(x) =
a + bx + cx2
Ėg
Ėst
qin
qout
L = 1 m
x

Assumptions:

1. One-dimensional conduction in the x-direction.
2. Incompressible, isotropic medium with constant properties.
3. Uniform internal heat generation, .

Analysis:

1. Recall that once the temperature distribution is known for a medium, it is a simple matter to determine the conduction heat transfer rate at any point in the medium or at its surfaces by using Fourier's law. Hence the desired heat rates may be determined by using the prescribed temperature distribution with Equation 2.1. Accordingly,

$$q_{in} = q_x(0) = -kA\frac{\partial T}{\partial x}\bigg|_{x=0} = -kA(b+2cx)_{x=0}$$

$$q_{in} = -bkA = 300°C/m \times 40\ W/m \cdot K \times 10\ m^2 = 120\ kW$$

$$q_{out} = q_x(L) = -kA\frac{\partial T}{\partial x}\bigg|_{x=L} = -kA(b+2cx)_{x=L}$$

$$q_{out} = -(b+2cL)kA = -[-300°C/m$$

$$+2(-50°C/m^2) \times 1\ m] \times 40\ W/m \cdot K \times 10\ m^2 = 160\ kW$$

2. The rate of change of energy storage in the wall may be determined by applying an overall energy balance to the wall. Using Equation 1.12c for a control volume about the wall,

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$$

where $\dot{E}_g = \dot{q}AL$, it follows that

$$\dot{E}_{st} = \dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = q_{in} + \dot{q}AL - q_{out}$$

$$\dot{E}_{st} = 120\ \text{kW} + 1000\ \text{W/m}^3 \times 10\ \text{m}^2 \times 1\ \text{m} - 160\ \text{kW}$$

$$\dot{E}_{st} = -30\ \text{kW}$$

3. The time rate of change of the temperature at any point in the medium may be determined from the heat equation, Equation 2.21, rewritten as

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

From the prescribed temperature distribution, it follows that

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(b + 2cx) = 2c = 2(-50°C/m^2) = -100°C/m^2$$

Note that this derivative is independent of position in the medium. Hence the time rate of temperature change is also independent of position and is given by

$$\frac{\partial T}{\partial t} = \frac{40 \text{ W/m} \cdot \text{K}}{1600 \text{ kg/m}^3 \times 4 \text{ kJ/kg} \cdot \text{K}} \times (-100°C/m^2)$$

$$+ \frac{1000 \text{ W/m}^3}{1600 \text{ kg/m}^3 \times 4 \text{ kJ/kg} \cdot \text{K}}$$

$$\frac{\partial T}{\partial t} = -6.25 \times 10^{-4} \text{ °C/s} + 1.56 \times 10^{-4} \text{ °C/s}$$

$$= -4.69 \times 10^{-4} \text{ °C/s}$$

Comments:

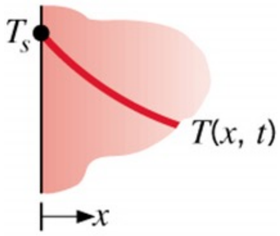
1. From this result, it is evident that the temperature at every point within the wall is decreasing with time.

2. Fourier's law can always be used to compute the conduction heat rate from knowledge of the temperature distribution, even for unsteady conditions with internal heat generation.

Boundary and Initial Conditions

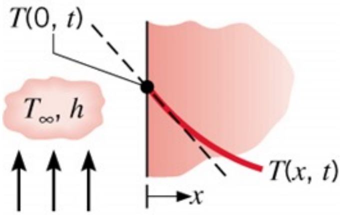
- For transient conduction, heat equation is first order in time, requiring specification of an initial temperature distribution:
- Since heat equation is second order in space, two boundary conditions must be specified for each coordinate direction. Some common cases:

Constant Surface Temperature:



$$T(0, t) = T_s$$

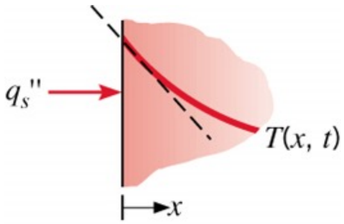
Convection:



$$-k \frac{\partial T}{\partial x}\Big|_{x=0} = h\left[T_\infty - T(0,t)\right]$$

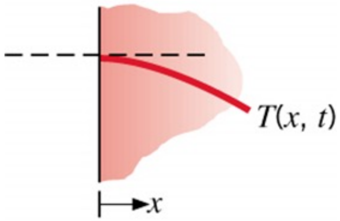
Constant Heat Flux:

Applied Flux



$$-k \frac{\partial T}{\partial x}\Big|_{x=0} = q_s''$$

Insulated Surface



$$\frac{\partial T}{\partial x}\Big|_{x=0} = 0$$

Typical Methodology of a Conduction Analysis

- Coordinates? Cartesian, Cylindrical, or Spherical?
- Solve appropriate form of heat equation to obtain the temperature distribution.
- Knowing the temperature distribution, apply Fourier's law to obtain the heat flux at any time, location and direction of interest.
- Applications:

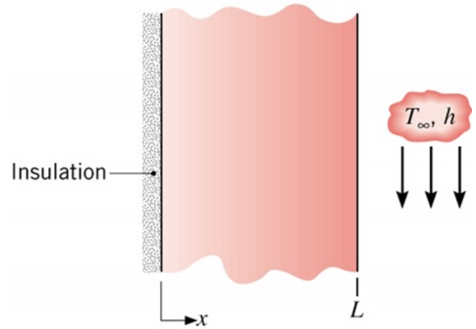
  Chapter 3: One-Dimensional, Steady-State Conduction

  Chapter 4: Two-Dimensional, Steady-State Conduction

  Chapter 5: Transient Conduction

Problem: Thermal Response of Plane Wall

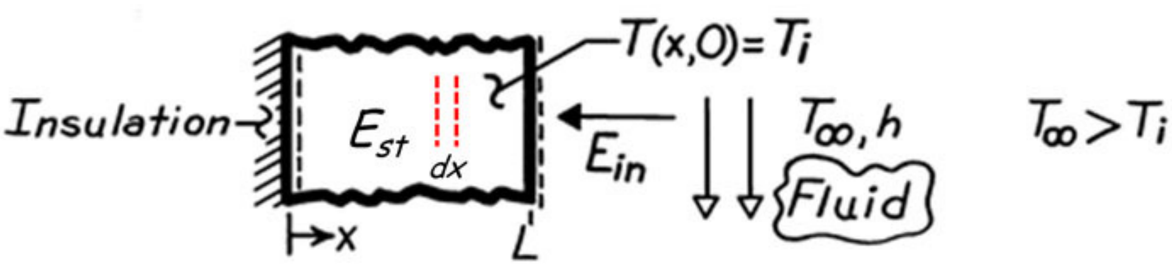
Problem 2.43 Thermal response of a plane wall to convection heat transfer.



KNOWN: Plane wall, initially at a uniform temperature, is suddenly exposed to convective heating.

FIND: (a) Differential equation and initial and boundary conditions which may be used to find the temperature distribution, $T(x, t)$; (b) Sketch $T(x, t)$ for the following conditions: initial ($t \leq 0$), steady-state ($t \to \infty$), and two intermediate times; (c) Sketch heat fluxes as a function of time at the two surfaces; (d) Expression for total energy transferred to wall per unit volume $[\mathrm{J/m^3}]$.

SCHEMATIC:



Problem: Thermal Response (1 of 3)

ASSUMPTIONS: (1) One-dimensional conduction, (2) constant properties, (3) No internal heat generation.

ANALYSIS: (a) For one –dimensional conduction with constant properties, the heat equation has the form,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}$$

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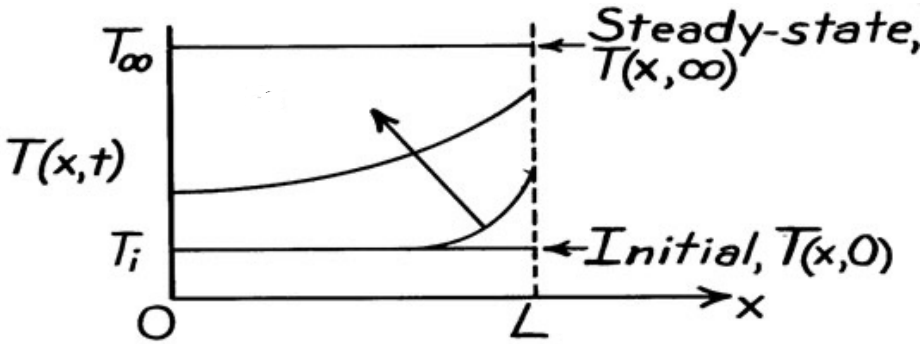
and the conditions are:

Initial:	$t < 0$	$T(x,\ 0) = T_i$	uniform temperature
Boundaries:	$x = 0$	$\partial T / \partial x\big\vert_0 = 0$	adiabatic surface
	$x = L$	$-k\partial T / \partial x\big\vert_L = h\left[T(L,t) - T_\infty\right]$	surface convection

Problem: Thermal Response (2 of 3)

(b) The temperature distributions are shown on the sketch.



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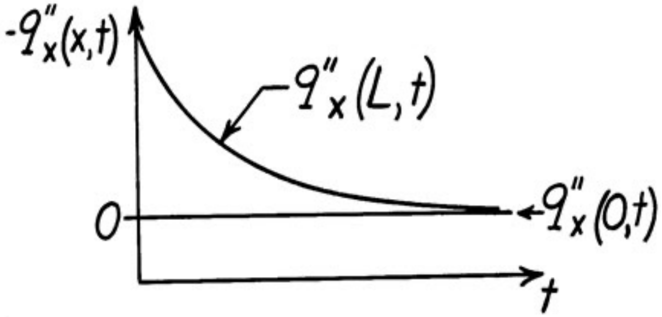
Note that the gradient at $x = 0$ is always zero, since this boundary is adiabatic. Note also that the gradient at $x = L$ decreases with time.

Problem: Thermal Response (3 of 3)

c) The heat flux , $q''_x\ (x, t)$ as a function of time, is shown on the sketch for the surfaces $x = 0$ and $x = L$.



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d) The total energy transferred to the wall may be expressed as

$$E_{in} = \int_0^\infty q''_{\text{conv}} A_s dt$$

$$E_{in} = hA_s \int_0^\infty \left(T_\infty - T(L,t)\right) dt$$

Dividing both sides by A,L, the energy transferred per unit volume is

$$\frac{E_{in}}{V} = \frac{h}{L} \int_0^\infty \left[T_\infty - T(L,t)\right] dt \qquad \left[\text{J/m}^3\right]$$

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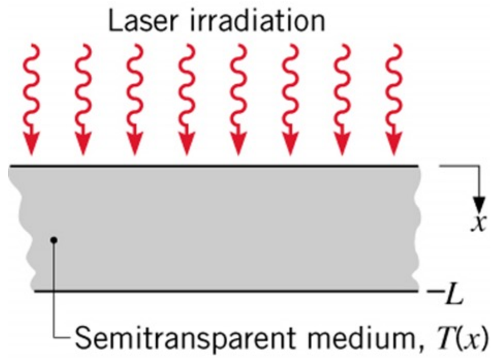
Problem 2.29: Non-uniform Generation due to Radiation Absorption

The steady-state temperature distribution in a semi-transparent material of thermal conductivity k and thickness L exposed to laser irradiation is of the form



$$T(x) = -\frac{A}{ka^2}e^{-ax} + Bx + C$$

where A, a, B, and C are known constants. For this situation, radiation absorption in the material is manifested by a distributed heat generation term, $\dot{q}(x)$,

(a) Obtain expressions for the conduction heat fluxes at the front and rear surfaces.
(b) Derive an expression for $\dot{q}(x)$,
(c) Derive an expression for the rate at which radiation is absorbed in the entire material, per unit surface area.

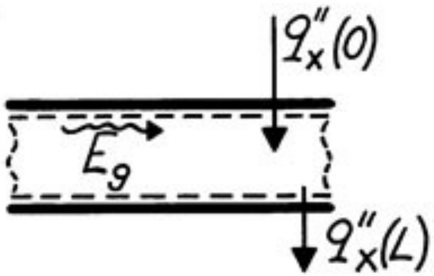
Express your result in terms of the known constants for the temperature distribution, the thermal conductivity of the material, and its thickness.

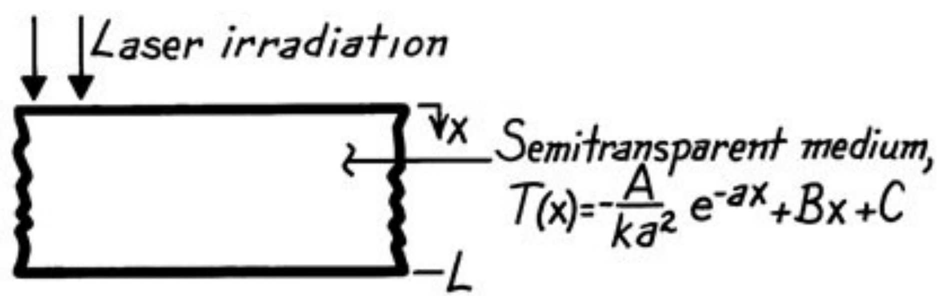
Problem 2.29: Non-uniform Generation due to Radiation Absorption

KNOWN: Temperature distribution in a semi-transparent medium subjected to radiative flux.

FIND: (a) Expressions for the heat flux at the front and rear surfaces, (b) The heat generation rate $\dot{q}(x)$, and (c) Expression for absorbed radiation per unit surface area.

SCHEMATIC:





Problem: Non-uniform Generation (1 of 2)

ASSUMPTIONS: (1) Steady-state conditions, (2) One-dimensional conduction in medium, (3) Constant properties, (4) All laser irradiation is absorbed and can be characterized by an internal volumetric heat generation

ANALYSIS: (a) Knowing the temperature distribution, the surface heat fluxes are found using Fourier's law,

$$q''_x = -k\left[\frac{dT}{dx}\right] = -k\left[\frac{A}{ka}e^{-ax} + B\right]$$

$$\textit{Front surface}, x = 0: \quad q''_x(0) = -k\left[\frac{A}{ka} + B\right] = -\left[\frac{A}{a} + kB\right] \qquad <$$

$$\textit{Rear surface}, x = L: \quad q''_x(L) = -k\left[\frac{A}{ka}e^{-aL} + B\right] = -\left[\frac{A}{a}e^{-aL} + kB\right] \qquad <$$

(b) The heat diffusion equation for the medium is

$$\frac{d}{dx}\left(\frac{dT}{dx}\right) + \frac{\dot{q}}{k} = 0 \quad \text{or} \quad \dot{q} = -k\frac{d}{dx}\left(\frac{dT}{dx}\right)$$

$$\dot{q}(x) = -k\frac{d}{dx}\left[+\frac{A}{ka}e^{-ax} + B\right] = Ae^{-ax}. \qquad <$$

(c) Performing an energy balance on the medium,

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \dot{E}_g = 0$$

Problem: Non-uniform Generation (2 of 2)

On a unit area basis

$$\dot{E}''_g = -\dot{E}''_{\text{in}} + \dot{E}''_{\text{out}} = -q''_x(0) + q''_x(L) = +\frac{A}{a}\left(1 - e^{-aL}\right).$$

$<$

Alternatively, evaluate $\dot{E}''_g$ by integration over the volume of the medium,

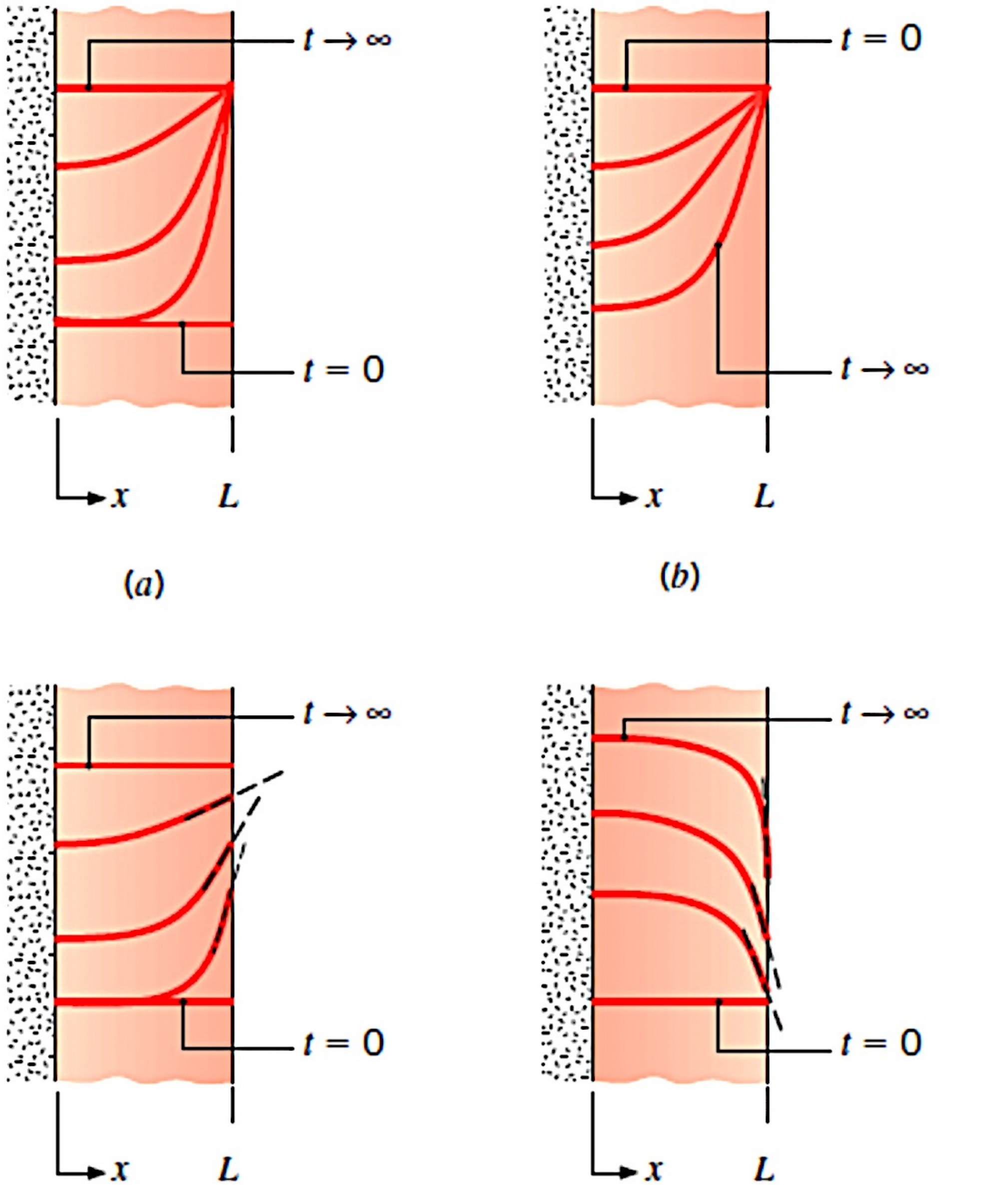
$$\dot{E}''_g = \int_0^L \dot{q}(x)dx = \int_0^L Ae^{-ax}dx = -\frac{A}{a}\left[e^{-ax}\right]_0^L = \frac{A}{a}\left(1 - e^{-aL}\right).$$

Problem 2.24

2.24 Temperature distributions within a series of one-dimensional plane walls at an initial time, at steady state, and at several intermediate times are as shown

Problem 2.24

For each case, write the appropriate form of the heat diffusion equation. Also write the equations for the initial condition and the boundary conditions that are applied at x = 0 and x = L. If volumetric generation occurs, it is uniform throughout the wall. The properties are constant.



Problem 2.24

ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) Negligible radiation.

<

ANALYSIS: The general form of the heat equation in Cartesian coordinates for constant *k* is
Equation 2.21. For one-dimensional conduction it reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

At steady state this becomes

$$\frac{d^2 T}{dx^2} + \frac{\dot{q}}{k} = 0$$

If there is no thermal energy generation the steady-state temperature distribution is linear (or could be constant). If there is uniform thermal energy generation the steady-state temperature distribution must be parabolic.

Problem 2.24

In case (a), the steady-state temperature distribution is constant, therefore there must not be any thermal energy generation. The heat equation is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

The initial temperature is uniform throughout the solid, thus the initial condition is

$$T(x,0) = T_i$$

At $x = 0$, the slope of the temperature distribution is zero at all times, therefore the heat flux is zero (insulated condition). The boundary condition is

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

At $x = L$, the temperature is the same for all $t > 0$. Therefore the surface temperature is constant:

$$T(L,t) = T_s$$

Problem 2.24

For case (b), the steady-state temperature distribution is not linear and appears to be parabolic, therefore there is thermal energy generation. The heat equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad <$$

The initial temperature is uniform, the temperature gradient at $x = 0$ is zero, and the temperature at $x = L$ is equal to the initial temperature for all $t > 0$, therefore the initial and boundary conditions are

$$T(x,0) = T_i, \quad \left.\frac{\partial T}{\partial x}\right|_{x=0} = 0, \quad T(L,t) = T_i \qquad <$$

With the left side insulated and the right side maintained at the initial temperature, the cause of the decreasing temperature must be a negative value of thermal energy generation.

Problem 2.24

In case (c), the steady-state temperature distribution is constant, therefore there is no thermal energy generation. The heat equation is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Problem 2.24

The initial temperature is uniform throughout the solid. At $x = 0$, the slope of the temperature distribution is zero at all times. Therefore the initial condition and boundary condition at $x = 0$ are

$$T(x,0) = T_i, \qquad \left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

At $x = L$, neither the temperature nor the temperature gradient are constant for all time. Instead, the temperature gradient is decreasing with time as the temperature approaches the steady-state temperature. This corresponds to a convection heat transfer boundary condition. As the surface temperature approaches the fluid temperature, the heat flux at the surface decreases. The boundary condition is:

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left[T(L,t) - T_\infty\right]$$

The fluid temperature, T_∞, must be higher than the initial solid temperature to cause the solid temperature to increase.

Problem 2.24

For case (*d*), the steady-state temperature distribution is not linear and appears to be parabolic, therefore there is thermal energy generation. The heat equation is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$ <

Since the temperature is increasing with time and it is *not* due to heat conduction due to a high surface temperature, the energy generation must be positive.

The initial temperature is uniform and the temperature gradient at $x = 0$ is zero. The boundary condition at $x = L$ is convection. The temperature gradient and heat flux at the surface are *increasing* with time as the thermal energy generation causes the temperature to rise further and further above the fluid temperature. The initial and boundary conditions are:

$$T(x,0)=T_i,\quad \left.\frac{\partial T}{\partial x}\right|_{x=0}=0,\quad -k\left.\frac{\partial T}{\partial x}\right|_{x=L}=h\left[T(L,t)-T_\infty\right]$$ <

COMMENTS: 1. You will learn to solve for the temperature distribution in transient conduction in Chapter 5. 2. Case (b) might correspond to a situation involving a spatially-uniform endothermic chemical reaction. Such situations, although they can occur, are not common.