Fundamentals of Heat and Mass Transfer

Chapter 5

The Lumped Capacitance Method

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Introduction

- In the treatment of conduction, we began with the simple case of one-dimensional, steady-state conduction with no internal generation, and we subsequently considered more realistic situations involving generation effects with different coordinates. Then, the analysis of fins was tackled.

- However, we have not yet considered situations for which conditions change with time.

- We now recognize that many heat transfer problems are time dependent.
- Such unsteady, or transient, problems typically arise when the boundary conditions of a system are changed.

Introduction

- For example, if the surface temperature of a system is altered, the temperature at each point in the system will also begin to change.
- The changes will continue to occur until, as is often the case, a steady-state temperature distribution is ultimately reached.
- Consider a hot metal billet that is removed from a furnace and exposed to a cool airstream.
- Energy is transferred by convection and radiation from its surface to the surroundings.
- Energy transfer by conduction also occurs from the interior of the metal to the surface, and the temperature at each point in the billet decreases until a steady-state condition is reached.

Introduction

- The objective in this chapter is to develop procedures for **determining the time dependence of the temperature distribution within a solid during a transient process, as well as for determining heat transfer between the solid and its surroundings.**

- The nature of the procedure depends on assumptions that may be made for the process.

- If, for example, **temperature gradients within the solid may be neglected, a comparatively simple approach, termed the lumped capacitance method, may be used to determine the variation of temperature with time.**

Introduction

- Under conditions for which temperature gradients are not negligible, but heat transfer within the solid is one-dimensional, exact solutions to the heat equation may be used to compute the dependence of temperature on both location and time or

- Finite-difference or finite-element methods must be used to predict the time dependence of temperatures within the solid, as well as heat rates at its boundaries.

The Lumped Capacitance Method

- A simple, yet common, **transient conduction problem is one for which a solid experiences a sudden change in its thermal environment**.

- Consider a hot metal forging that is initially at a uniform temperature Ti and is quenched by immersing it in a liquid of lower temperature $T\infty < Ti$ (Figure 5.1).

- If the quenching is said to begin at time $t = 0$, the temperature of the solid will decrease for time $t > 0$, until it eventually reaches $T\infty$.

- This reduction is due to convection heat transfer at the solid–liquid interface.

The Lumped Capacitance Method

- The essence of the lumped capacitance method is the assumption that the temperature of the solid is spatially uniform at any instant during the transient process.

- This assumption implies that temperature gradients within the solid are negligible.

The Lumped Capacitance Method

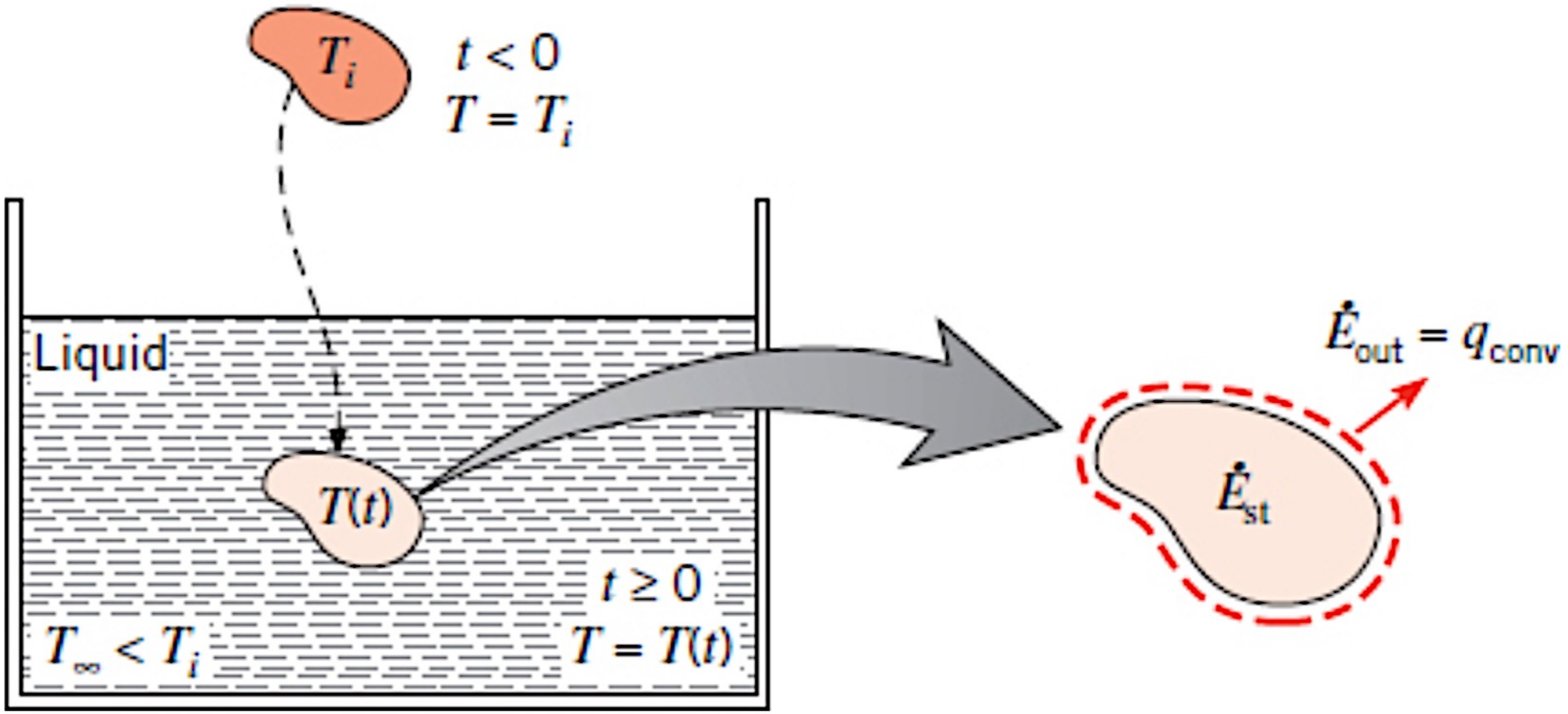


FIGURE 5.1 Cooling of a hot metal forging.

The Lumped Capacitance Method

- From Fourier's law, heat conduction in the absence of a temperature gradient implies **the existence of infinite thermal conductivity**.
- **Such a condition is clearly impossible. However, the condition is closely approximated if the resistance to conduction within the solid is small compared with the resistance to heat transfer between the solid and its surroundings.**
- For now we assume that this is, in fact, the case.
- In neglecting temperature gradients within the solid, we can no longer consider the problem from within the framework of the heat equation, since the heat equation is a differential equation governing the spatial temperature distribution within the solid.

The Lumped Capacitance Method

- Instead**, the transient temperature response is determined by formulating an overall energy balance on the entire solid**.

- This balance must relate the rate of heat loss at the surface to the rate of change of the internal energy. Applying Equation 1.12c to the control volume of Figure 5.1, this requirement takes the form

The Lumped Capacitance Method

$$-\dot{E}_{out} = \dot{E}_{st}$$

$$-hA_s(T - T_\infty) = \rho V c \frac{dT}{dt}$$

$$\theta \equiv T - T_\infty$$

$$\frac{\rho V c}{hA_s}\frac{d\theta}{dt} = -\theta$$

$$\frac{\rho V c}{hA_s}\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} = -\int_{o}^{t} dt$$

$$\frac{\rho V c}{hA_s}\ln\frac{\theta_i}{\theta} = t$$

$$\theta_i \equiv T_i - T_\infty$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho V c}\right)t\right]$$

The Lumped Capacitance Method

It is also evident that the quantity (ρVc/hAs) may be interpreted as a thermal time constant expressed as

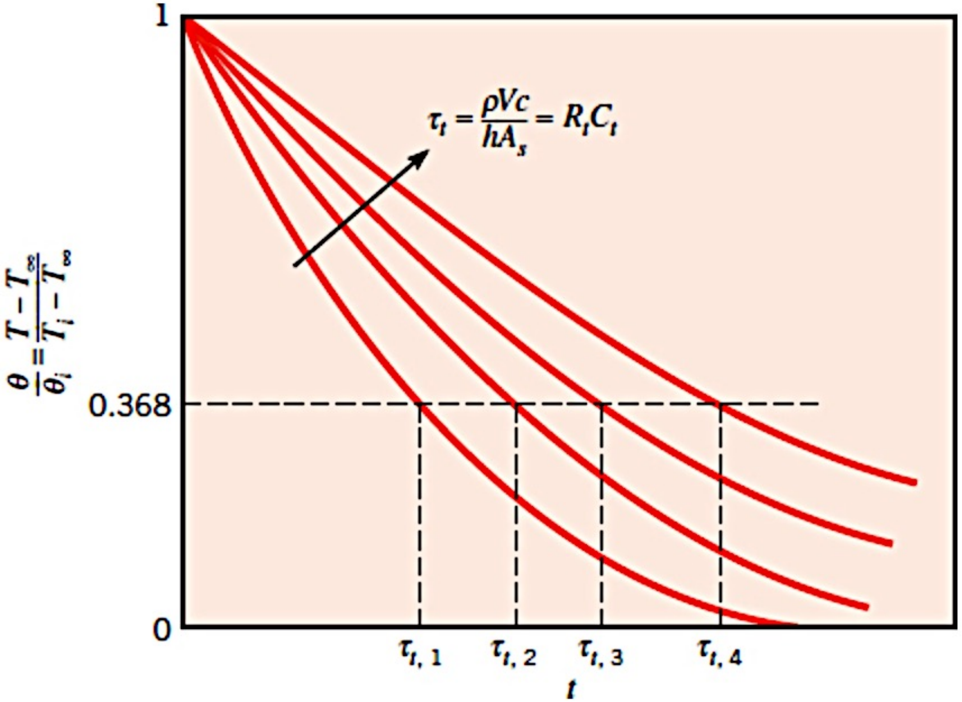
$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho V c) = R_t C_t$$

where R_t is the resistance to convection heat transfer and C_t is the lumped thermal capacitance of the solid.

Any increase in R_t or C_t will cause the solid to respond more slowly to changes in its thermal environment.

This behavior is analogous to the voltage decay that occurs when a capacitor is discharged through a resistor in an electrical RC circuit.

The Lumped Capacitance Method



These results indicate that the difference between the solid and fluid temperatures decays exponentially to zero as t approaches infinity.

This behavior is shown in Figure 5.2.

FIGURE 5.2 Transient temperature response of lumped capacitance solids for different thermal time constants τ_i.

The Lumped Capacitance Method

To determine the total energy transfer Q occurring up to some time t, we simply write

$$Q = \int_0^t q\,\mathrm{dt} = hA_s \int_0^t \theta\,\mathrm{dt}$$

$$Q = (\rho V c)\theta_i\,[1 - \exp(-\frac{t}{\tau_t})]$$

The quantity Q is, of course, related to the change in the internal energy of the solid

$$-Q = \Delta E_{\mathrm{st}}$$

Validity of the Lumped Capacitance Method

- The foregoing results demonstrate that the lumped capacitance method is **a simple and convenient method for solving transient heating and cooling problems.**

- In this section, we determine under **what conditions it may be used with reasonable accuracy.**

- To develop a suitable criterion, consider steady-state conduction through the plane wall of area A (Figure 5.3).

- Although we are assuming steady-state conditions, the following criterion is readily extended to transient processes.

Validity of the Lumped Capacitance Method

- One surface is maintained at a temperature Ts, 1 and the other surface is exposed to a fluid of temperature $T_\infty < T_{s,1}$.
- The temperature of this surface will be some intermediate value Ts, 2, for which $T_\infty < T_{s,2} < T_{s,1}$.
- Hence under steady-state conditions the surface energy balance, Equation 1.13, reduces to

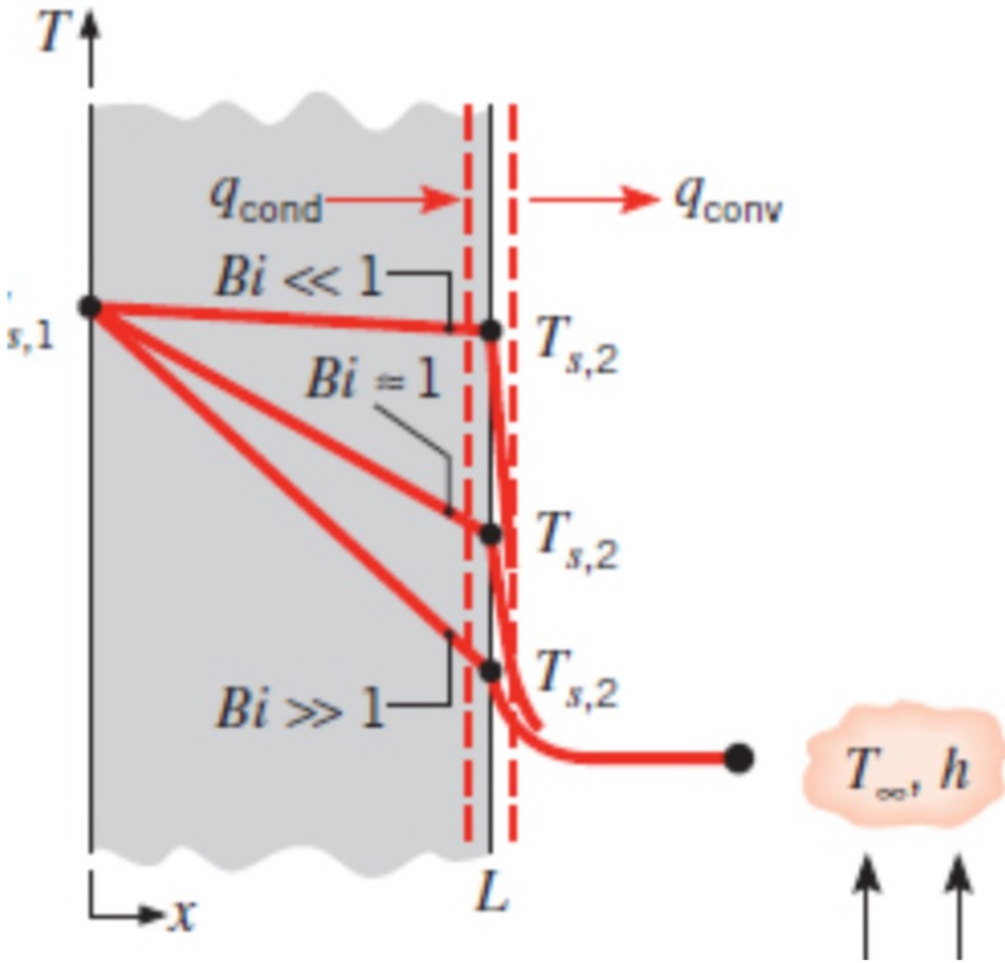


FIGURE 5.3 Effect of Biot number on steady-state temperature distribution in a plane wall with surface convection.

$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_{\infty})$$

where k is the thermal conductivity of the solid. Rearranging, we then obtain

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_{\infty}} = \frac{(L/kA)}{(1/hA)} = \frac{R_{t,\text{cond}}}{R_{t,\text{conv}}} = \frac{hL}{k} \equiv \text{Bi}$$

- The quantity (hL/k) is a dimensionless parameter. It is termed the Biot number, and it plays a fundamental role in conduction problems that involve surface convection effects.

- The Biot number provides a measure of the temperature drop in the solid relative to the temperature difference between the solid's surface and the fluid.

- The Biot number may be interpreted as a ratio of thermal resistances.

- In particular, if $Bi \ll 1$, the resistance to conduction within the solid is much less than the resistance to convection across the fluid boundary layer.

- Hence, the assumption of a uniform temperature distribution within the solid is reasonable if the Biot number is small.

- Although we have discussed the Biot number in the context of steady-state conditions, we are reconsidering this parameter because of its significance to transient conduction problems.

- Consider the plane wall of Figure 5.4, which is initially at a uniform temperature T_i and experiences convection cooling when it is immersed in a fluid of $T_\infty < T_i$.

- The problem may be treated as one-dimensional in x, and we are interested in the temperature variation with position and time, $T(x, t)$.

- This variation is a strong function of the Biot number, and three conditions are shown in Figure 5.4. Again, for $Bi \ll 1$ the temperature gradients in the solid are small and the assumption of a uniform temperature distribution, $T(x, t) \approx T(t)$ is reasonable.

- Virtually all the temperature difference is between the solid and the fluid, and the solid temperature remains nearly uniform as it decreases to T_∞.

- For moderate to large values of the Biot number, however, the temperature gradients within the solid are significant. Hence $T = T(x, t)$.

- Note that for $Bi \gg 1$, the temperature difference across the solid is much larger than that between the surface and the fluid.

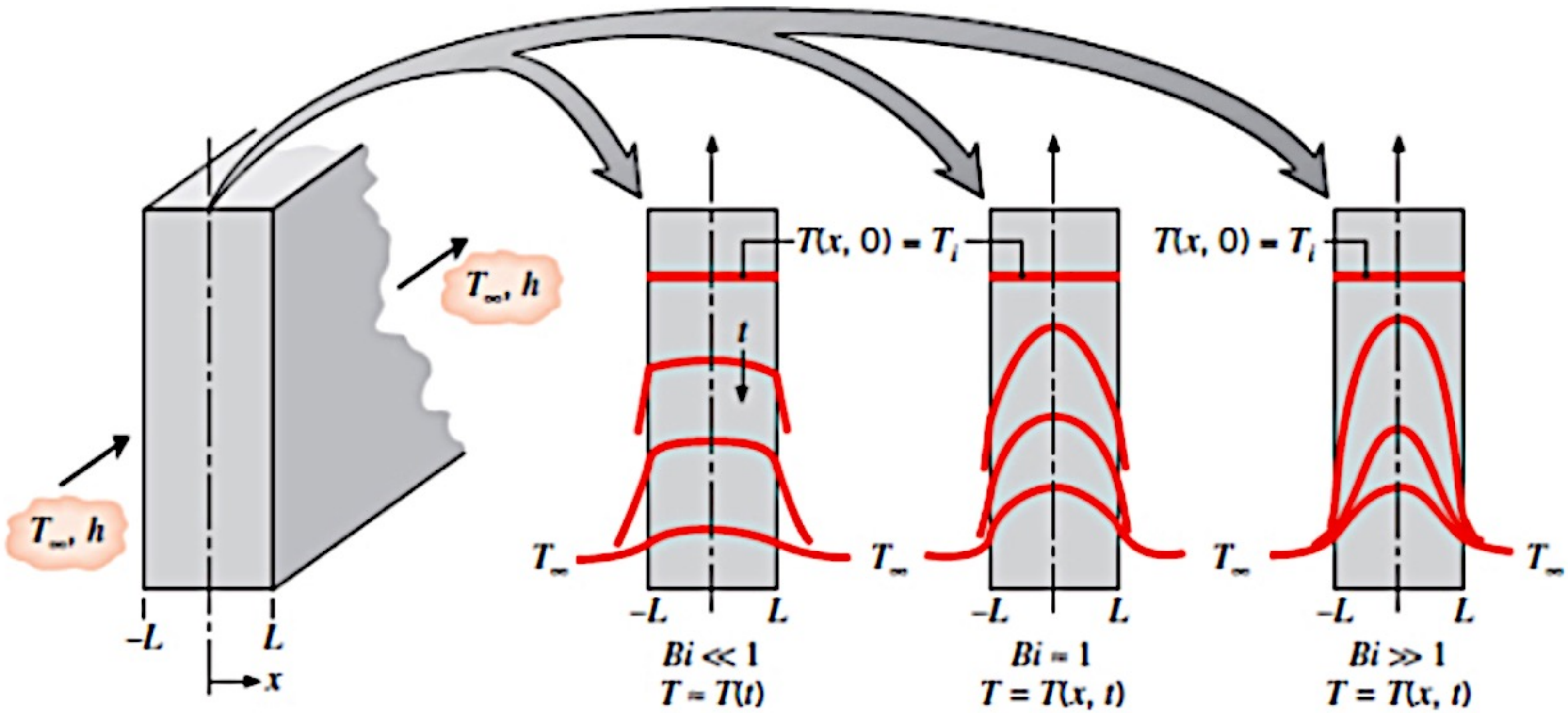


FIGURE 5.4 Transient temperature distributions for different Biot numbers in a plane wall symmetrically cooled by convection.

We conclude this section by emphasizing the importance of the lumped capacitance method.

Its inherent simplicity renders it the preferred method for solving transient heating and cooling problems.

Hence, when confronted with such a problem, the very first thing that one should do is calculate the Biot number. If the following condition is satisfied

$$\mathrm{Bi} = \frac{hL_c}{k} < 0.1$$

the error associated with using the lumped capacitance method is small. For convenience, it is **customary to define the characteristic length of Equation 5.10 as the ratio of the solid's volume to surface area Lc ≡ V/As.**

- **Such a definition facilitates calculation of Lc for solids of complicated shape and reduces to the half-thickness L for a plane wall of thickness 2L (Figure 5.4), to ro/2 for a long cylinder, and to ro/3 for a sphere.**

- However, if one wishes to implement the criterion in a conservative fashion, Lc should be associated with the length scale corresponding to the maximum spatial temperature difference.

- Accordingly, for a symmetrically heated (or cooled) plane wall of thickness 2L, Lc would remain equal to the half-thickness L.

- However, for a long cylinder or sphere, Lc would equal the actual radius ro, rather than ro/2 or ro/3.

- Finally, we note that, with Lc ≡ V/As, the exponent of Equation 5.6 may be expressed as

$$\frac{hA_s t}{\rho V c} = \frac{ht}{\rho c L_c} = \frac{hL_c}{k} \frac{k}{\rho c} \frac{t}{L_c^2} = \frac{hL_c}{k} \frac{\alpha t}{L_c^2}$$

$$\frac{hA_s t}{\rho V c} = \mathrm{Bi} \cdot \mathrm{Fo}$$

$$\mathrm{Fo} \equiv \frac{\alpha t}{L_c^2}$$

Fo is termed the Fourier number. It is a dimensionless time, which, with the Biot number, characterizes transient conduction problems. Substitutiing in the equation, we obtain

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-\mathrm{Bi} \cdot \mathrm{Fo})$$

EXAMPLE 5.1

A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. The convection coefficient between the junction surface and the gas is h = 400 W/m2 · K, and the junction thermophysical properties are k = 20 W/m · K, c = 400 J/kg · K, and ρ = 8500 kg/m3. Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?

EXAMPLE 5.1

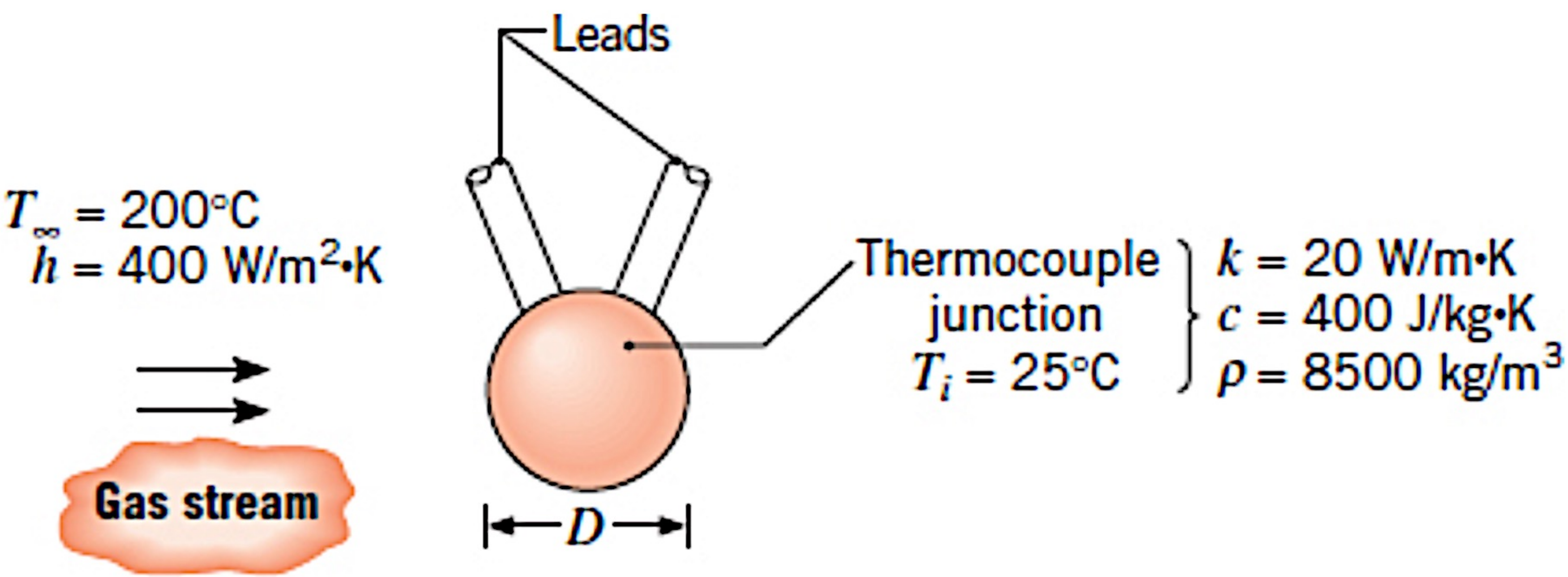
SOLUTION

Known: Thermophysical properties of thermocouple junction used to measure temperature of a gas stream.

Find:

1. Junction diameter needed for a time constant of 1 s.
2. Time required to reach 199°C in gas stream at 200°C.

Schematic:



Leads
$T_\infty = 200°C$
$h = 400\ W/m^2 \cdot K$
Gas stream
D
Thermocouple junction
$T_i = 25°C$
$k = 20\ W/m \cdot K$
$c = 400\ J/kg \cdot K$
$\rho = 8500\ kg/m^3$

EXAMPLE 5.1

Assumptions:

1. Temperature of junction is uniform at any instant.
2. Radiation exchange with the surroundings is negligible.
3. Losses by conduction through the leads are negligible.
4. Constant properties.

Analysis:

1. Because the junction diameter is unknown, it is not possible to begin the solution by determining whether the criterion for using the lumped capacitance method, Equation 5.10, is satisfied. However, a reasonable approach is to use the method to find the diameter and to then determine whether the criterion is satisfied. From Equation 5.7 and the fact that As = πD2 and V = πD3/6 for a sphere, it follows that

$$\tau_t = \frac{1}{h\pi D^2} \times \frac{\rho\pi D^3}{6} c$$

Rearranging and substituting numerical values,

$$D = \frac{6h\tau_t}{\rho c} = \frac{6 \times 400 \ \mathrm{W/m^2 \cdot K} \times 1 \ \mathrm{s}}{8500 \ \mathrm{kg/m^3} \times 400 \ \mathrm{J/kg \cdot K}} = 7.06 \times 10^{-4} \ \mathrm{m} \qquad \triangleleft$$

With $L_c = r_o/3$ it then follows from Equation 5.10 that

$$\mathrm{Bi} = \frac{h(r_o/3)}{k} = \frac{400 \ \mathrm{W/m^2 \cdot K} \times 3.53 \times 10^{-4} \ \mathrm{m}}{3 \times 20 \ \mathrm{W/m \cdot K}} = 2.35 \times 10^{-3}$$

Accordingly, Equation 5.10 is satisfied (for Lc = ro, as well as for Lc = ro/3) and the lumped capacitance method may be used to an excellent approximation.

2. From Equations 5.5 and 5.6 the time required for the junction to reach a temperature T can be written as

$$t = \frac{\rho V_c}{hA_s} \ln\left(\frac{T_i - T_\infty}{T - T_\infty}\right) = \tau_t \ln\left(\frac{T_i - T_\infty}{T - T_\infty}\right)$$

Thus, the time required to reach $T = 199°\mathrm{C}$ is

$$t = \tau_t \ln\left(\frac{25 - 200}{199 - 200}\right) = 5.2\tau_t = 5.2 \times 1 \ \mathrm{s} = 5.2 \ \mathrm{s} \qquad \triangleleft$$

Comments: Heat transfer due to radiation exchange between the junction and the surroundings and conduction through the leads would affect the time response of the junction and would, in fact, yield an equilibrium temperature that differs from T_∞.