FUNDAMENTALS OF ACOUSTICS AND NOISE CONTROL

Finn Jacobsen, Torben Poulsen, Jens Holger Rindel,
Anders Christian Gade and Mogens Ohlrich

Department of Electrical Engineering, Technical University of Denmark
August 2011 Note no 31200

CONTENTS

1 AN ELEMENTARY INTRODUCTION TO ACOUSTICS

Finn Jacobsen

1.1 INTRODUCTION

Acoustics is the science of sound, that is, wave motion in gases, liquids and solids, and the effects of such wave motion. Thus the scope of acoustics ranges from fundamental physical acoustics to, say, bioacoustics, psychoacoustics and music, and includes technical fields such as transducer technology, sound recording and reproduction, design of theatres and concert halls, and noise control.

The purpose of this chapter is to give an introduction to fundamental acoustic concepts, to the physical principles of acoustic wave motion, and to acoustic measurements.

1.2 FUNDAMENTAL ACOUSTIC CONCEPTS

One of the characteristics of fluids, that is, gases and liquids, is the lack of constraints to deformation. Fluids are unable to transmit shearing forces, and therefore they react against a change of *shape* only because of inertia. On the other hand a fluid reacts against a change in its *volume* with a change of the pressure. Sound waves are compressional oscillatory disturbances that propagate in a fluid. The waves involve molecules of the fluid moving back and forth in the direction of propagation (with no net flow), accompanied by changes in the pressure, density and temperature; see figure 1.2.1. The *sound pressure*, that is, the difference between the instantaneous value of the total pressure and the static pressure, is the quantity we hear. It is also much easier to measure the sound pressure than, say, the density or temperature fluctuations. Note that sound waves are *longitudinal waves*, unlike bending waves on a beam or waves on a stretched string, which are *transversal waves* in which the particles move back and forth in a direction perpendicular to the direction of propagation.

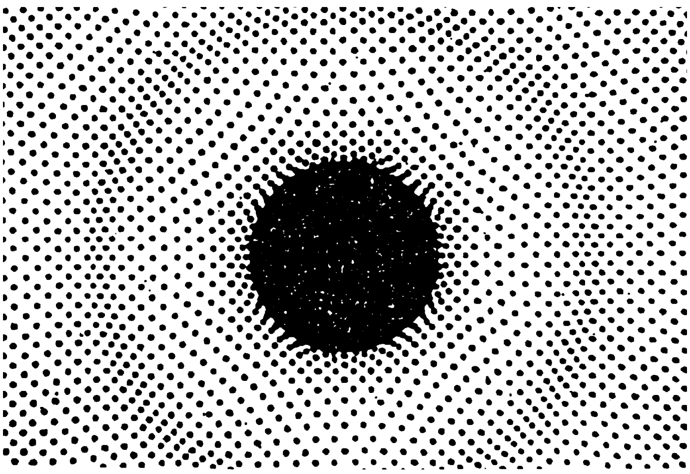


Figure 1.2.1 Fluid particles and compression and rarefaction in the propagating spherical sound field generated by a pulsating sphere. (From ref. [1].)

In most cases the oscillatory changes undergone by the fluid are extremely small. One can get an idea about the orders of magnitude of these changes by considering the variations in air corresponding to a sound pressure level[1] of 120 dB, which is a very high sound pressure level, close to the threshold of pain. At this level the fractional pressure variations (the sound pressure relative to the static pressure) are about 2×10^{-4}, the fractional changes of the density are about 1.4×10^{-4}, the oscillatory changes of the temperature are less than 0.02 °C, and the particle velocity[2] is about 50 mm/s, which at 1000 Hz corresponds to a particle displacement of less than $8\ \mu\text{m}$. In fact at 1000 Hz the particle displacement at the threshold of hearing is less than the diameter of a hydrogen atom![3]

Sound waves exhibit a number of phenomena that are characteristics of waves; see figure 1.2.2. Waves propagating in different directions *interfere*; waves will be *reflected* by a rigid surface and more or less *absorbed* by a soft one; they will be *scattered* by small obstacles; because of *diffraction* there will only partly be shadow behind a screen; and if the medium is inhomogeneous for instance because of temperature gradients the waves will be *refracted*, which means that they change direction as they propagate. The speed with which sound waves propagate in fluids is independent of the frequency, but other waves of interest in acoustics, bending waves on plates and beams, for example, are *dispersive*, which means that the speed of such waves depends on the frequency content of the waveform.

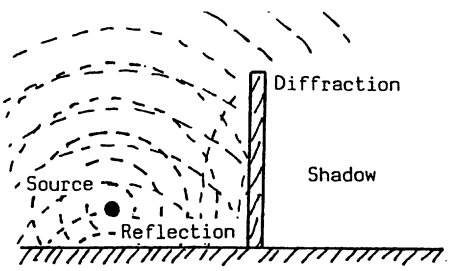


Figure 1.2.2 Various wave phenomena.

A mathematical description of the wave motion in a fluid can be obtained by combining equations that express the facts that i) mass is conserved, ii) the local longitudinal force caused by a difference in the local pressure is balanced by the inertia of the medium, and iii) sound is very nearly an adiabatic phenomenon, that is, there is no flow of heat. The observation that most acoustic phenomena involve perturbations that are several orders of magnitude smaller than the equilibrium values of the medium makes it possible to simplify the mathematical description by neglecting higher-order terms. The result is the *linearised wave equation.* This is a second-order partial differential equation that, expressed in terms of the sound

---

[1] See section 1.3.2 for a definition of the sound pressure level.

[2] The concept of fluid particles refers to a macroscopic average, not to individual molecules; therefore the particle velocity can be much less than the velocity of the molecules.

[3] At these conditions the fractional pressure variations amount to about 2.5×10^{-10}. By comparison, a change in altitude of *one metre* gives rise to a fractional change in the static pressure that is about 400000 times larger, about 10^{-4}. Moreover, inside an aircraft at cruising height the static pressure is typically only 80% of the static pressure at sea level. In short, the acoustic pressure fluctuations are *extremely* small compared with commonly occurring static pressure variations.

pressure p, takes the form

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{1.2.1}$$

in a Cartesian (x, y, z) coordinate system.[4] Here t is the time and, as we shall see later, the quantity

$$c=\sqrt{K_S/\rho} \tag{1.2.2a}$$

is the *speed of sound*. The physical unit of the sound pressure is pascal (1 Pa = 1 Nm^{-2}). The quantity K_S is the adiabatic bulk modulus, and ρ is the equilibrium density of the medium. For gases, $K_S = \gamma p_0$, where γ is the ratio of the specific heat at constant pressure to that at constant volume ($\simeq 1.401$ for air) and p_0 is the static pressure ($\simeq$ 101.3 kPa for air under normal ambient conditions). The adiabatic bulk modulus can also be expressed in terms of the gas constant R ($\simeq 287\ \text{J}\cdot\text{kg}^{-1}\text{K}^{-1}$ for air), the absolute temperature T, and the equilibrium density of the medium,

$$c=\sqrt{\gamma p_0/\rho}=\sqrt{\gamma RT}\ , \tag{1.2.2b}$$

which shows that the equilibrium density of a gas can be written as

$$\rho = p_0/RT. \tag{1.2.3}$$

At 293.15 K = 20°C the speed of sound in air is 343 m/s. Under normal ambient conditions (20°C, 101.3 kPa) the density of air is 1.204 kgm^{-3}. Note that the speed of sound of a gas depends only on the temperature, not on the static pressure, whereas the adiabatic bulk modulus depends only on the static pressure; the equilibrium density depends on both quantities.

Adiabatic compression

Because the process of sound is adiabatic, the fractional pressure variations in a small cavity driven by a vibrating piston, say, a pistonphone for calibrating microphones, equal the fractional density variations multiplied by the ratio of specific heats γ. The physical explanation for the 'additional' pressure is that the pressure increase/decrease caused by the reduced/expanded volume of the cavity is accompanied by an increase/decrease of the temperature, which increases/reduces the pressure even further. The fractional variations in the density are of course identical with the fractional change of the volume (except for the sign); therefore,

$$\frac{p}{p_0}=\gamma\frac{\Delta\rho}{\rho}=-\gamma\frac{\Delta V}{V}.$$

In section 1.4 we shall derive a relation between the volume velocity (= the volume displacement ΔV per unit of time) and the resulting sound pressure.

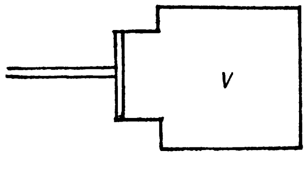


Figure 1.2.3 A small cavity driven by a vibrating piston.

[4] The left-hand side of eq. (1.2.1) is the *Laplacian* of the sound pressure, that is, the divergence of the gradient. A negative value of this quantity at a certain point implies that the gradient converges towards the point, indicating a high local value. The wave equation states that this high local pressure tends to decrease.

The linearity of eq. (1.2.1) is due to the absence of higher-order terms in p in combination with the fact that $\partial^2/\partial x^2$ and $\partial^2/\partial t^2$ are linear operators.[5] This is an extremely important property. It implies that a sinusoidal source will generate a sound field in which the pressure at all positions varies sinusoidally. It also implies linear superposition: sound waves do not interact, they simply pass through each other (see figure 1.2.5).[6]

The diversity of possible sound fields is of course enormous, which leads to the conclusion that we must supplement eq. (1.2.1) with some additional information about the sources that generate the sound field, surfaces that reflect or absorb sound, objects that scatter sound, etc. This information is known as *the boundary conditions*. The boundary conditions are often expressed in terms of the particle velocity. For example, the normal component of the particle velocity **u** is zero on a rigid surface. Therefore we need an additional equation that relates the particle velocity to the sound pressure. This relation is known as Euler's equation of motion,

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla p = \mathbf{0}, \tag{1.2.4}$$

which is simply Newton's second law of motion for a fluid. The operator ∇ is the gradient (the spatial derivative ($\partial/\partial x, \partial/\partial y, \partial/\partial z$)). Note that the particle velocity is a vector, unlike the sound pressure, which is a scalar.

Sound in liquids

The speed of sound is much higher in liquids than in gases. For example, the speed of sound in water is about 1500 ms^{-1}. The density of liquids is also much higher; the density of water is about 1000 kgm^{-3}. Both the density and the speed of sound depend on the static pressure and the temperature, and there are no simple general relations corresponding to eqs. (1.2.2b) and (1.2.3).

1.2.1 Plane sound waves

The *plane wave* is a central concept in acoustics. Plane waves are waves in which any acoustic variable at a given time is a constant on any plane perpendicular to the direction of propagation. Such waves can propagate in a duct. In a limited area at a distance far from a source of sound in free space the curvature of the spherical wavefronts is negligible and the waves can be regarded as locally plane.

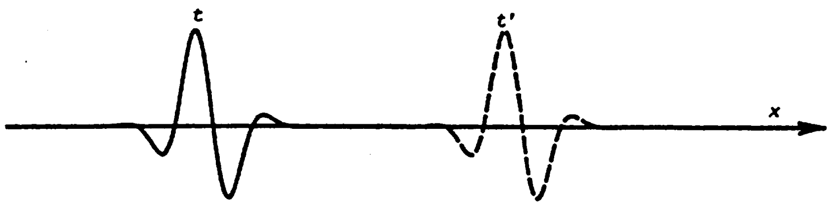


Figure 1.2.4 The sound pressure in a plane wave of arbitrary waveform at two different instants of time.

---

[5] This follows from the fact that $\partial^2(p_1 + p_2)/\partial t^2 = \partial^2 p_1/\partial t^2 + \partial^2 p_2/\partial t^2$.

[6] At very high sound pressure levels, say at levels in excess of 140 dB, the linear approximation is no longer adequate. This complicates the analysis enormously. Fortunately, we can safely assume linearity under practically all circumstances encountered in daily life.

The plane wave is a solution to the one-dimensional wave equation,

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}, \tag{1.2.5}$$

cf. eq. (1.2.1). It is easy to show that the expression

$$p = f_1(ct - x) + f_2(ct + x), \tag{1.2.6}$$

where f_1 and f_2 are arbitrary functions, is a solution to eq. (1.2.5), and it can be shown this is the general solution. Since the argument of f_1 is constant if x increases as ct it follows that the first term of this expression represents a wave that propagates undistorted and unattenuated in the positive x-direction with constant speed, c, whereas the second term represents a similar wave travelling in the opposite direction. See figures 1.2.4 and 1.2.5.

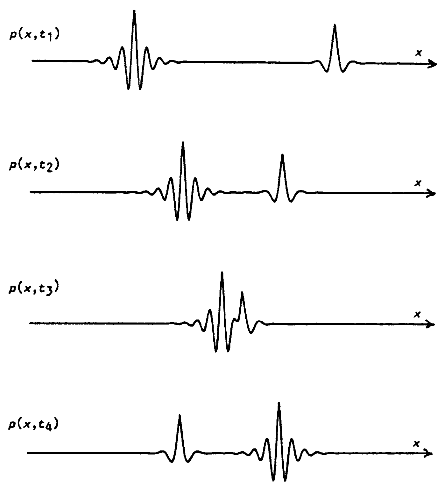


Figure 1.2.5 Two plane waves travelling in opposite directions are passing through each other.

The special case of a *harmonic* plane progressive wave is of great importance. Harmonic waves are generated by sinusoidal sources, for example a loudspeaker driven with a pure tone. A harmonic plane wave propagating in the x-direction can be written

$$p = p_1 \sin\left(\frac{\omega}{c}(ct - x) + \varphi\right) = p_1 \sin(\omega t - kx + \varphi), \tag{1.2.7}$$

where $\omega = 2\pi f$ is the angular (or radian) *frequency* and $k = \omega/c$ is the (angular) *wavenumber*. The quantity p_1 is known as the amplitude of the wave, and φ is a phase angle (the arbitrary value of the phase angle of the wave at the origin of the coordinate system at $t = 0$). At any position in this sound field the sound pressure varies sinusoidally with the angular frequency ω, and at any fixed time the sound pressure varies sinusoidally with x with the spatial period

$$\lambda = \frac{c}{f} = \frac{2\pi c}{\omega} = \frac{2\pi}{k}. \tag{1.2.8}$$

The quantity λ is the *wavelength*, which is defined as the distance travelled by the wave in one cycle. Note that the wavelength is inversely proportional to the frequency. At 1000 Hz the wavelength in air is about 34 cm. In rough numbers the audible frequency range goes from 20 Hz to 20 kHz, which leads to the conclusion that acousticians are faced with wavelengths (in air) in the range from 17 m at the lowest audible frequency to 17 mm at the highest audible frequency. Since the efficiency of a radiator of sound or the effect of an obstacle on the sound field depends very much on its size expressed in terms of the acoustic wavelength, it can be realised that the wide frequency range is one of the challenges in acoustics. It simplifies the analysis enormously if the wavelength is very long or very short compared with typical dimensions.

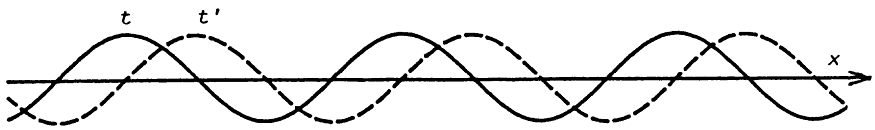


Figure 1.2.6 The sound pressure in a plane harmonic wave at two different instants of time.

Sound fields are often studied frequency by frequency. As already mentioned, linearity implies that a sinusoidal source with the frequency ω will generate a sound field that varies harmonically with this frequency at all positions.[7] Since the frequency is given, all that remains to be determined is the amplitude and phase at all positions. This leads to the introduction of the complex exponential representation, where the sound pressure is written as a complex function of the position multiplied with a complex exponential. The former function takes account of the amplitude and phase, and the latter describes the time dependence. Thus at any given position the sound pressure can be written as a complex function of the form[8]

$$\hat{p} = A\mathrm{e}^{\mathrm{j}\omega t} = |A|\mathrm{e}^{\mathrm{j}\varphi}\mathrm{e}^{\mathrm{j}\omega t} = |A|\mathrm{e}^{\mathrm{j}(\omega t+\varphi)} \tag{1.2.9}$$

(where φ is the phase of the complex amplitude A), and the real, physical, time-varying sound pressure is the real part of the complex pressure,

$$p = \mathrm{Re}\{\hat{p}\} = \mathrm{Re}\{|A|\mathrm{e}^{\mathrm{j}(\omega t+\varphi)}\} = |A|\cos(\omega t + \varphi). \tag{1.2.10}$$

Since the entire sound field varies as $\mathrm{e}^{\mathrm{j}\omega t}$, the operator $\partial/\partial t$ can be replaced by $\mathrm{j}\omega$ (because the derivative of $\mathrm{e}^{\mathrm{j}\omega t}$ with respect to time is $\mathrm{j}\omega\mathrm{e}^{\mathrm{j}\omega t}$),[9] and the operator $\partial^2/\partial t^2$ can be replaced by $-\omega^2$. It follows that Euler's equation of motion can now be written

$$\mathrm{j}\omega\rho\hat{\mathbf{u}} + \nabla\hat{p} = \mathbf{0}, \tag{1.2.11}$$

and the wave equation can be simplified to

---

[7] If the source emitted any other signal than a sinusoidal the waveform would in the general case change with the position in the sound field, because the various frequency components would change amplitude and phase relative to each other. This explains the usefulness of harmonic analysis.

[8] Throughout this note complex variables representing harmonic signals are indicated by carets.

[9] The sign of the argument of the exponential is just a convention. The $\mathrm{e}^{\mathrm{j}\omega t}$ convention is common in electrical engineering, in audio and in related areas of acoustics. The alternative convention $\mathrm{e}^{-\mathrm{j}\omega t}$ is favoured by mathematicians, physicists and acousticians concerned with outdoor sound propagation. With the alternative sign convention $\partial/\partial t$ should obviously be replaced by $-\mathrm{j}\omega$. Mathematicians and physicists also tend to prefer the symbol 'i' rather than 'j' for the imaginary unit.

$$\frac{\partial^2 \hat{p}}{\partial x^2} + \frac{\partial^2 \hat{p}}{\partial y^2} + \frac{\partial^2 \hat{p}}{\partial z^2} + k^2 \hat{p} = 0, \qquad (1.2.12)$$

which is known as the Helmholtz equation. See the Appendix (section 1.9) for further details about complex representation of harmonic signals. We note that the use of complex notation is mathematically very convenient, which will become apparent later.

Written with complex notation the equation for a plane wave that propagates in the x-direction becomes

$$\hat{p} = p_{\mathrm{i}}\, \mathrm{e}^{\mathrm{j}(\omega t - kx)}. \qquad (1.2.13)$$

Equation (1.2.11) shows that the particle velocity is proportional to the gradient of the pressure. It follows that the particle velocity in the plane propagating wave given by eq. (1.2.13) is

$$\hat{u}_x = -\frac{1}{\mathrm{j}\omega\rho}\frac{\partial \hat{p}}{\partial x} = \frac{k}{\omega\rho} p_{\mathrm{i}}\, \mathrm{e}^{\mathrm{j}(\omega t - kx)} = \frac{p_{\mathrm{i}}}{\rho c} \mathrm{e}^{\mathrm{j}(\omega t - kx)} = \frac{\hat{p}}{\rho c}. \qquad (1.2.14)$$

Thus the sound pressure and the particle velocity are in phase in a plane propagating wave (see also figure 1.2.10), and the ratio of the sound pressure to the particle velocity is ρc, the characteristic impedance of the medium. As the name implies, this quantity describes an important acoustic property of the fluid, as will become apparent later. The characteristic impedance of air at 20°C and 101.3 kPa is about 413 kg·m^{-2}s^{-1}.

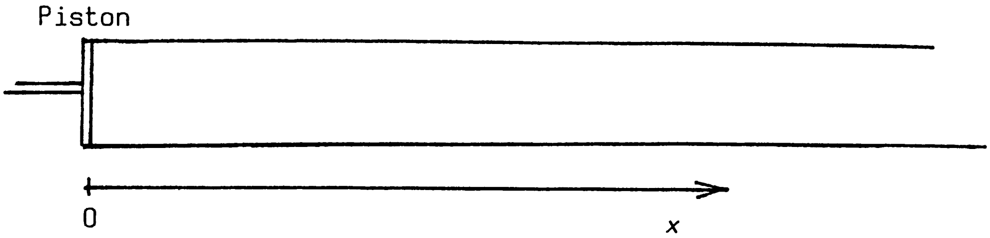


Figure 1.2.7 A semi-infinite tube driven by a piston.

Example 1.2.1

An semi-infinite tube is driven by a piston with the vibrational velocity $U\mathrm{e}^{\mathrm{j}\omega t}$ as shown in figure 1.2.7. Because the tube is infinite there is no reflected wave, so the sound field can be written

$$\hat{p}(x) = p_{\mathrm{i}}\mathrm{e}^{\mathrm{j}(\omega t - kx)}, \qquad \hat{u}_x(x) = \frac{p_{\mathrm{i}}}{\rho c}\mathrm{e}^{\mathrm{j}(\omega t - kx)}.$$

The boundary condition at the piston implies that the particle velocity equals the velocity of the piston:

$$\hat{u}_x(0) = \frac{p_{\mathrm{i}}}{\rho c}\mathrm{e}^{\mathrm{j}\omega t} = U\mathrm{e}^{\mathrm{j}\omega t}.$$

It follows that the sound pressure generated by the piston is

$$\hat{p}(x) = U\rho c\mathrm{e}^{\mathrm{j}(\omega t - kx)}.$$

The general solution to the one-dimensional Helmholtz equation is

$$\hat{p} = p_{\mathrm{i}}\, \mathrm{e}^{\mathrm{j}(\omega t - kx)} + p_{\mathrm{r}}\, \mathrm{e}^{\mathrm{j}(\omega t + kx)}, \qquad (1.2.15)$$

which can be identified as the sum of a wave that travels in the positive x-direction and a wave that travels in the opposite direction (cf. eq. (1.2.6)). The corresponding expression for the particle velocity becomes, from eq. (1.2.11),

$$\begin{aligned}\hat{u}_x &= -\frac{1}{\mathrm{j}\omega\rho}\frac{\partial\hat{p}}{\partial x} = \frac{k}{\omega\rho}p_\mathrm{i}\,\mathrm{e}^{\mathrm{j}(\omega t-kx)} - \frac{k}{\omega\rho}p_\mathrm{r}\,\mathrm{e}^{\mathrm{j}(\omega t+kx)} \\ &= \frac{p_\mathrm{i}}{\rho c}\mathrm{e}^{\mathrm{j}(\omega t-kx)} - \frac{p_\mathrm{r}}{\rho c}\mathrm{e}^{\mathrm{j}(\omega t+kx)}.\end{aligned} \tag{1.2.16}$$

It can be seen that whereas $\hat{p} = \hat{u}_x\rho c$ in a plane wave that propagates in the positive x-direction, the sign is the opposite, that is, $\hat{p} = -\hat{u}_x\rho c$, in a plane wave that propagates in the negative x-direction. The reason for the change in the sign is that the particle velocity is a vector, unlike the sound pressure, so $\hat{u}_x$ is a vector component. It is also worth noting that the general relation between the sound pressure and the particle velocity in this interference field is far more complicated than in a plane propagating wave.

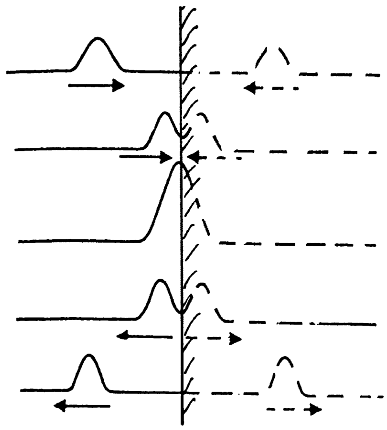


Figure 1.2.8 Instantaneous sound pressure in a wave that is reflected from a rigid surface at different instants of time. (Adapted from ref. [2].)

A plane wave that impinges on a plane rigid surface perpendicular to the direction of propagation will be reflected. This phenomenon is illustrated in figure 1.2.8, which shows how an incident transient disturbance is reflected. Note that the normal component of the gradient of the pressure is identically zero on the surface for all values of t. This is a consequence of the fact that the boundary condition at the surface implies that the particle velocity must equal zero here, cf. eq. (1.2.4).

However, it is easier to analyse the phenomenon assuming harmonic waves. In this case the sound field is given by the general expressions (1.2.15) and (1.2.16), and our task is to determine the relation between p_i and p_r from the boundary condition at the surface, say at $x = 0$. As mentioned, the rigid surface implies that the particle velocity must be zero here, which with eq. (1.2.16) leads to the conclusion that $p_\mathrm{i} = p_\mathrm{r}$, so the reflected wave has the same amplitude as the incident wave. Equation (1.2.15) now becomes

$$\hat{p} = p_\mathrm{i}\left(\mathrm{e}^{\mathrm{j}(\omega t-kx)} + \mathrm{e}^{\mathrm{j}(\omega t+kx)}\right) = p_\mathrm{i}\left(\mathrm{e}^{-\mathrm{j}kx} + \mathrm{e}^{\mathrm{j}kx}\right)\mathrm{e}^{\mathrm{j}\omega t} = 2p_\mathrm{i}\cos kx\,\mathrm{e}^{\mathrm{j}\omega t}, \tag{1.2.17}$$

and eq. (1.2.16) becomes

$$\hat{u}_x = -\mathrm{j}\frac{2p_\mathrm{i}}{\rho c}\sin kx\,\mathrm{e}^{\mathrm{j}\omega t}\,. \qquad (1.2.18)$$

Note that the amplitude of the sound pressure is doubled on the surface (cf. figure 1.2.8). Note also the nodal[10] planes where the sound pressure is zero at x = - λ/4, x = - 3λ/4, etc., and the planes where the particle velocity is zero at x = - λ/2, x = - λ, etc. The interference of the two plane waves travelling in opposite directions has produced a *standing wave pattern*, shown in figure 1.2.9.

The physical explanation of the fact that the sound pressure is identically zero at a distance of a quarter of a wavelength from the reflecting plane is that the incident wave must travel a distance of half a wavelength before it returns to the same point; accordingly the incident and reflected waves are in *antiphase* (that is, 180° out of phase), and since they have the same amplitude they cancel each other. This phenomenon is called *destructive interference*. At a distance of half a wavelength from the reflecting plane the incident wave must travel one wavelength before it returns to the same point. Accordingly, the two waves are in phase and therefore the sound pressure is doubled here (*constructive interference*). The corresponding pattern for the particle velocity is different because the particle velocity is a vector.

Another interesting observation from eqs. (1.2.17) and (1.2.18) is that the resulting sound pressure and particle velocity signals as functions of time at any position are 90° out of phase (since $\mathrm{j}\mathrm{e}^{\mathrm{j}\omega t} = \mathrm{e}^{\mathrm{j}(\omega t+\pi/2)}$). Otherwise expressed, if the sound pressure as a function of time is a cosine then the particle velocity is a sine. As we shall see later this indicates that there is no net flow of sound energy towards the rigid surface. See also figure 1.2.10.

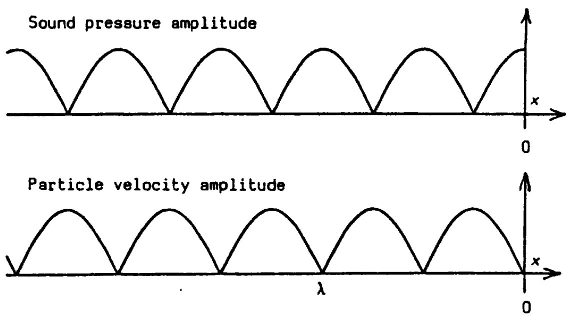


Figure 1.2.9 Standing wave pattern caused by reflection from a rigid surface at x = 0; amplitudes of the sound pressure and the particle velocity.

Example 1.2.2

The standing wave phenomenon can be observed in a tube terminated by a rigid cap. When the length of the tube, l, equals an odd-numbered multiple of a quarter of a wavelength the sound pressure is zero at the input, which means that it would take very little force to drive a piston here. This is an example of an *acoustic resonance*. In this case it occurs at the frequency

$$f_0 = \frac{c}{4l},$$

[10] A *node* on, say, a vibrating string is a point that does not move, and an *antinode* is a point with maximum displacement. By analogy, points in a standing wave at which the sound pressure is identically zero are called pressure nodes. In this case the pressure nodes coincide with velocity antinodes.

and at odd-numbered multiples of this frequency, $3f_0$, $5f_0$, $7f_0$, etc. Note that the resonances are harmonically related. This means that if some mechanism excites the tube the result will be a musical sound with the fundamental frequency f_0 and overtones corresponding to odd-numbered harmonics.[11]

Brass and woodwind instruments are based on standing waves in tubes. For example, closed organ pipes are tubes closed at one end and driven at the other, open end, and such pipes have only odd-numbered harmonics. See also example 1.4.4.

The ratio of p_r to p_i is the (complex) *reflection factor* R. The amplitude of this quantity describes how well the reflecting surface reflects sound. In the case of a rigid plane $R = 1$, as we have seen, which implies perfect reflection with no phase shift. However, in the general case of a more or less absorbing surface R will be complex and its magnitude less than unity ($|R| \le 1$), indicating partial reflection with a phase shift at the reflection plane.

If we introduce the reflection factor in eq. (1.2.15) it becomes

$$\hat{p} = p_i \left(e^{j(\omega t - kx)} + R\, e^{j(\omega t + kx)} \right), \tag{1.2.19}$$

from which it can be seen that the amplitude of the sound pressure varies with the position in the sound field. When the two terms in the parenthesis are in phase the sound pressure amplitude assumes its maximum value,

$$p_{max} = p_i \left(1 + |R|\right), \tag{1.2.20a}$$

and when they are in antiphase the sound pressure amplitude assumes the minimum value

$$p_{min} = p_i \left(1 - |R|\right). \tag{1.2.20b}$$

The ratio of p_{max} to p_{min} is called the *standing wave ratio*,

$$s = \frac{p_{max}}{p_{min}} = \frac{1 + |R|}{1 - |R|}. \tag{1.2.21}$$

From eq. (1.2.21) it follows that

$$|R| = \frac{s - 1}{s + 1}, \tag{1.2.22}$$

which leads to the conclusion that it is possible to determine the acoustic properties of a material by exposing it to normal sound incidence and measuring the standing wave ratio in the resulting interference field. See also chapter 1.5.

Figure 1.2.10 shows the instantaneous sound pressure and particle velocity at two different instants of time in a tube that is terminated by a material that does not reflect sound at all (case (a)), by a soft material that partly absorbs the incident sound wave (case (b)), and by a rigid material that gives perfect reflection (case (c)).

---

[11] A musical (or complex) tone is not a pure (sinusoidal) tone but a periodic signal, usually consisting of the fundamental and a number of its harmonics, also called partials. These pure tones occur at multiples of the fundamental frequency. The *n*'th harmonic (or partial) is also called the (*n*-1)'th overtone, and the fundamental is the first harmonic. The relative position of a tone on a musical scale is called the *pitch* [2]. The pitch of a musical tone essentially corresponds to its fundamental frequency, which is also the distance between two adjacent harmonic components. However, pitch is a subjective phenomenon and not completely equivalent to frequency. We tend to determine the pitch on the basis of the spacing between the harmonic components, and thus we can detect the pitch of a musical tone even if the fundamental is missing.

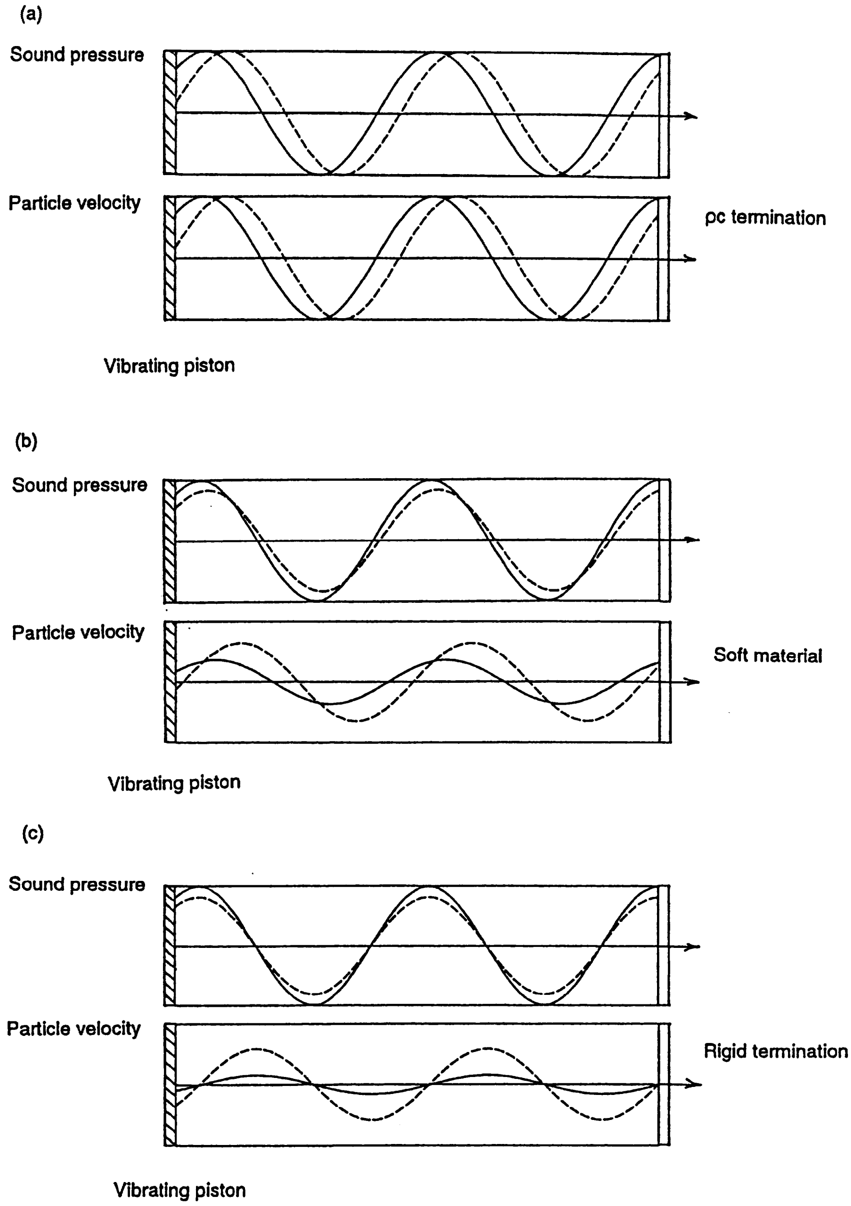


Figure 1.2.10 Spatial distributions of instantaneous sound pressure and particle velocity at two different instants of time. (a) Case with no reflection (R = 0); (b) case with partial reflection from a soft surface; (c) case with perfect reflection from a rigid surface (R = 1). (From ref. [3].)

Sound transmission between fluids

When a sound wave in one fluid is incident on the boundary of another fluid, say, a sound wave in air is incident on the surface of water, it will be partly reflected and partly transmitted. For simplicity let us assume that a plane wave in fluid 1 strikes the surface of fluid 2 at normal incidence as shown in figure 1.2.11. Anticipating a reflected wave we can write

$$\hat{p}_1 = p_i e^{j(\omega t - kx)} + p_r e^{j(\omega t + kx)}$$

for fluid 1, and

$$\hat{p}_2 = p_t e^{j(\omega t - kx)}$$

for fluid 2. There are two boundary conditions at the interface: the sound pressure must be the same in fluid 1 and in fluid 2 (otherwise there would be a net force), and the particle velocity must be the same in fluid 1 and in fluid 2 (otherwise the fluids would not remain in contact). It follows that

$$p_i + p_r = p_t \qquad \text{and} \qquad \frac{p_i - p_r}{\rho_1 c_1} = \frac{p_t}{\rho_2 c_2}.$$

Combining these equations gives

$$\frac{p_r}{p_i} = R = \frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1},$$

which shows that the wave is almost fully reflected in phase ($R \simeq 1$) if $\rho_2 c_2 >> \rho_1 c_1$, almost fully reflected in antiphase ($R \simeq -1$) if $\rho_2 c_2 << \rho_1 c_1$, and not reflected at all if $\rho_2 c_2 = \rho_1 c_1$, irrespective of the individual properties of c_1, c_2, ρ_1 and ρ_2.

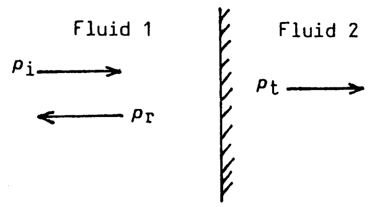


Figure 1.2.11 Reflection and transmission of a plane wave incident on the interface between two fluids.

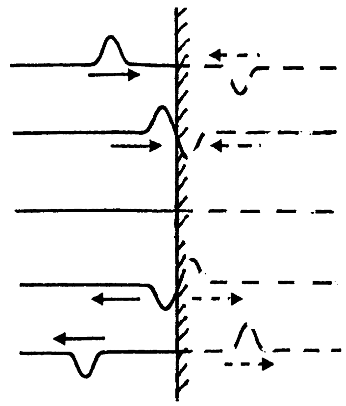


Figure 1.2.12 Reflection of a pressure wave at the interface between a medium of high characteristic impedance and a medium of low characteristic impedance. (Adapted from ref. [2].)

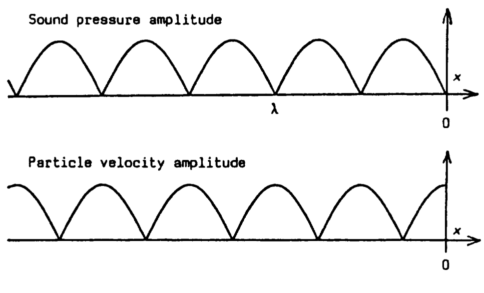


Figure 1.2.13 Standing wave pattern in a medium of high characteristic impedance caused by caused by reflection from a medium of low characteristic impedance; amplitudes of the sound pressure and the particle velocity.

Because of the significant difference between the characteristic impedances of air and water (the ratio is about 1: 3600) a sound wave in air that strikes a surface of water at normal incidence is almost completely reflected, and so is a sound wave that strikes the air-water interface from the water, but in the latter case the phase of the reflected wave is reversed, as shown in figure 1.2.12. Compare figures 1.2.8 and 1.2.12, and figures 1.2.9 and 1.2.13.

1.2.2 Spherical sound waves

The wave equation can be expressed in other coordinate systems than the Cartesian. If sound is generated by a source in an environment without reflections (which is usually referred to as a free field) it will generally be more useful to express the wave equation in a spherical coordinate system (r, θ, φ). The resulting equation is more complicated than eq. (1.2.1). However, if the source under study is spherically symmetric there can be no angular dependence, and the equation becomes quite simple,[12]

$$\frac{\partial^2 p}{\partial r^2}+\frac{2}{r}\frac{\partial p}{\partial r}=\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}. \tag{1.2.23a}$$

If we rewrite in the form

$$\frac{\partial^2 (rp)}{\partial r^2}=\frac{1}{c^2}\frac{\partial^2 (rp)}{\partial t^2}, \tag{1.2.23b}$$

it becomes apparent that this equation is identical in form with the one-dimensional wave equation, eq. (1.2.5), although p has been replaced by rp. (It is easy to get from eq. (1.2.23b) to eq. (1.2.23a); it is more difficult the other way.) It follows that the general solution to eq. (1.2.23) can be written

$$rp = f_1(ct-r)+f_2(ct+r), \tag{1.2.24a}$$

---

[12] This can be seen as follows. Since the sound pressure depends only on r we have

$$\frac{\partial p}{\partial x}=\frac{\partial p}{\partial r}\frac{\partial r}{\partial x},$$

which, with

$$r=\sqrt{x^2+y^2+z^2},$$

becomes

$$\frac{\partial p}{\partial x}=\frac{x}{r}\frac{\partial p}{\partial r}.$$

Similar considerations leads to the following expression for the second-order derivative,

$$\frac{\partial^2 p}{\partial x^2}=\frac{1}{r}\frac{\partial p}{\partial r}+x\frac{\partial}{\partial x}\left(\frac{1}{r}\frac{\partial p}{\partial r}\right)=\frac{1}{r}\frac{\partial p}{\partial r}+\frac{x^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial p}{\partial r}\right)=\frac{1}{r}\frac{\partial p}{\partial r}+\frac{x^2}{r^2}\frac{\partial^2 p}{\partial r^2}-\frac{x^2}{r^3}\frac{\partial p}{\partial r}.$$

Combining eq. (1.2.1) with this expression and the corresponding relations for y and z finally yields eq. (1.2.23a):

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}=\frac{3}{r}\frac{\partial p}{\partial r}+\frac{x^2+y^2+z^2}{r^2}\frac{\partial^2 p}{\partial r^2}-\frac{x^2+y^2+z^2}{r^3}\frac{\partial p}{\partial r}=\frac{2}{r}\frac{\partial p}{\partial r}+\frac{\partial^2 p}{\partial r^2}=\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}.$$

that is

$$p = \frac{1}{r}\left(f_1(ct-r) + f_2(ct+r)\right), \tag{1.2.24b}$$

where f_1 and f_2 are arbitrary functions. The first term is wave that travels outwards, away from the source (cf. the first term of eq. (1.2.6)). Note that the shape of the wave is preserved. However, the sound pressure is seen to decrease in inverse proportion to the distance. This is *the inverse distance law*.[13] The second term represents a converging wave, that is, a spherical wave travelling inwards. In principle such a wave could be generated by a reflecting spherical surface centred at the source, but that is a rare phenomenon indeed. Accordingly we will ignore the second term when we study sound radiation in chapter 1.6.

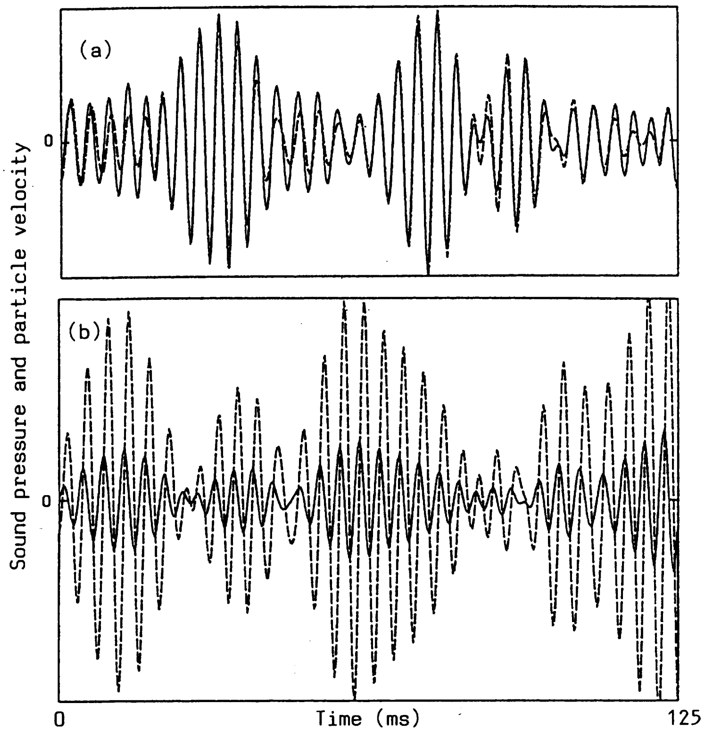


Figure 1.2.14 (a) Measurement far from a spherical source in free space; (b) measurement close to a spherical source. —, Instantaneous sound pressure; - - -, instantaneous particle velocity multiplied by ρc. (From ref. [4].)

A harmonic spherical wave is a solution to the Helmholtz equation

$$\frac{\partial^2 (r\hat{p})}{\partial r^2} + k^2 r\hat{p} = 0. \tag{1.2.25}$$

Expressed in the complex notation the diverging wave can be written

$$\hat{p} = A\frac{\mathrm{e}^{\mathrm{j}(\omega t - kr)}}{r}. \tag{1.2.26}$$

---

13 The inverse distance law is also known as the inverse square law because the sound intensity is inversely proportional to the square of the distance to the source. See chapters 1.5 and 1.6.

The particle velocity component in the radial direction can be calculated from eq. (1.2.11),

$$\hat{u}_r = -\frac{1}{\mathrm{j}\omega\rho}\frac{\partial\hat{p}}{\partial r} = \frac{A}{\rho c}\frac{\mathrm{e}^{\mathrm{j}(\omega t - kr)}}{r}\left(1+\frac{1}{\mathrm{j}kr}\right) = \frac{\hat{p}}{\rho c}\left(1+\frac{1}{\mathrm{j}kr}\right). \tag{1.2.27}$$

Because of the spherical symmetry there are no components in the other directions. Note that far[14] from the source the sound pressure and the particle velocity are in phase and their ratio equals the characteristic impedance of the medium, just as in a plane wave. On the other hand, when $kr << 1$ the particle velocity is larger than $|\hat{p}|/\rho c$ and the sound pressure and the particle velocity are almost in *quadrature*, that is, 90° out of phase. These are *near field* characteristics, and such a sound field is also known as a *reactive field*. See figure 1.2.14.

1.3 ACOUSTIC MEASUREMENTS

The most important measure of sound is the *rms* sound pressure,[15] defined as

$$p_{\mathrm{rms}} = \sqrt{\overline{p^2(t)}} = \left(\lim_{T\to\infty}\frac{1}{T}\int_0^T p^2(t)\mathrm{d}t\right)^{½}. \tag{1.3.1}$$

However, as we shall see, a frequency weighting filter[16] is usually applied to the signal before the rms value is determined. Quite often such a single value does not give sufficient information about the nature of the sound, and therefore the rms sound pressure is determined in frequency bands. The resulting sound pressures are practically always compressed logarithmically and presented in decibels.

Example 1.3.1

The fact that $\sin^2\omega t$ = ½ (1 - cos2ωt) and thus has a time average of ½ leads to the conclusion that the rms value of a sinusoidal signal with the amplitude A is $A/\sqrt{2}$.

1.3.1 Frequency analysis

Single frequency sound is useful for analysing acoustic phenomena, but most sounds encountered in practice have ‘broadband’ characteristics, which means that they cover a wide frequency range. If the sound is more or less steady, it will practically always be more useful to analyse it in the frequency domain than to look at the sound pressure as a function of time.

Frequency (or spectral) analysis of a signal involves decomposing the signal into its spectral components. This analysis can be carried out by means of digital analysers that employ the discrete Fourier transform (‘FFT analysers’). This topic is outside the scope of this note, but see, e.g., refs. [5, 6]. Alternatively, the signal can be passed through a number of

---

[14] In acoustics, dimensions are measured in terms of the wavelength, so that ‘far from’ means that $r >> \lambda$ (or $kr >> 1$), just as ‘near’ means that $r << \lambda$ (or $kr << 1$). The dimensionless quantity kr is known as the Helmholtz number.

[15] Root mean square value, usually abbreviated rms. This is the square root of the mean square value, which is the time average of the squared signal.

[16] A filter is a device that modifies a signal by attenuating some of its frequency components.

contiguous analogue or digital bandpass filters[17] with different centre frequencies, a 'filter bank.' The filters can have the same bandwidth or they can have constant relative bandwidth, which means that the bandwidth is a certain percentage of the centre frequency. Constant relative bandwidth corresponds to uniform resolution on a logarithmic frequency scale. Such a scale is in much better agreement with the subjective *pitch* of musical sounds than a linear scale, and therefore frequencies are often represented on a logarithmic scale in acoustics, and frequency analysis is often carried out with constant percentage filters. The most common filters in acoustics are octave band filters and one-third octave band filters.

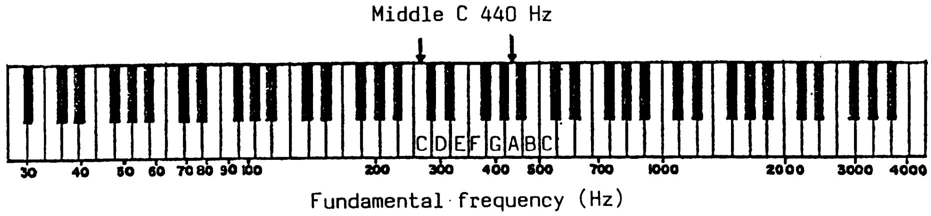


Figure 1.3.1 The keyboard of a small piano. The white keys from C to B correspond to the seven notes of the C major scale. (Adapted from ref. [7].)

An *octave*[18] is a frequency ratio of 2:1, a fundamental unit in musical scales. Accordingly, the lower limiting frequency of an octave band is half the upper frequency limit, and the centre frequency is the geometric mean, that is,

$$f_l = f_c / 2^{1/2}, \quad f_u = 2^{1/2} f_c, \quad f_c = \sqrt{f_l f_u}, \qquad \text{(1.3.2a, 1.3.2b, 1.3.2c)}$$

where f_c is the centre frequency. In a similar manner a one-third octave[19] band is a band for which $f_u = 2^{1/3} f_l$, and

$$f_l = f_c / 2^{1/6}, \quad f_u = 2^{1/6} f_c, \quad f_c = \sqrt{f_l f_u}, \qquad \text{(1.3.3a, 1.3.3b, 1.3.3c)}$$

Since $2^{10} = 1024 \simeq 10^3$ it follows that $2^{10/3} \simeq 10$ and $2^{1/3} \simeq 10^{1/10}$, that is, ten one-third octaves very nearly make a decade, and a one-third octave is almost identical with one tenth of a decade. Table 1.3.1 gives the nominal centre frequencies of standardised octave and one-third octave band filters.[20] As mentioned earlier, the human ear may respond to frequencies in the range from 20 Hz to 20 kHz, that is, a range of three decades, ten octaves or thirty one-third octaves.

---

[17] An ideal bandpass filter would allow frequency components in the passband to pass unattenuated, but would completely remove frequency components outside the passband. Real filters have, of course, a certain passband ripple and a finite stopband attenuation.

[18] Musical tones an octave apart sound very similar. The diatonic scale contains seven notes per octave corresponding to the white keys on a piano keyboard; see figure 1.3.1. Thus an octave spans eight notes, say, from C to C'; hence the name octave (from Latin *octo*: eight).

[19] A semitone is one twelfth of an octave on the equally tempered scale (a frequency ratio of $2^{1/12}$:1). Since $2^{1/3} = 2^{4/12}$ it can be seen that a one-third octave is identical with four semitones or a major third (e.g. from C to E, cf. figure 1.3.1). Accordingly, one-third octave band filters are called Terzfilters in German.

[20] Round numbers are convenient. The standardised nominal centre frequencies are based on the fact that the series 1.25, 1.6, 2, 2.5, 3.15, 4, 5, 6.3, 8, 10 is in reasonable agreement with $10^{n/10}$, with n = 1, 2,....., 10.

Table 1.3.1 Standardised one-third octave and octave (bold characters) band centre frequencies (in hertz).

20	25	**31.5**	40	50	**63**	80	100	**125**	160	200	**250**	315	400	**500**	630	800	**1000**
1250	1600	**2000**	2500	3150	**4000**	5000	6300	**8000**	10000	12500	**16000**	20000					

An important property of the mean square value of a signal is that it can be partitioned into frequency bands. This means that if we analyse a signal in, say, one-third octave bands, the sum of the mean square values of the filtered signals equals the mean square value of the unfiltered signal. The reason is that products of different frequency components average to zero, so that all cross terms vanish; the different frequency components are *uncorrelated signals*. This can be illustrated by analysing a sum of two pure tones with different frequencies,

$$\begin{aligned}\overline{(A\sin\omega_1 t + B\sin\omega_2 t)^2} &= A^2\overline{\sin^2\omega_1 t} + B^2\overline{\sin^2\omega_2 t} + 2AB\overline{\sin\omega_1 t\sin\omega_2 t} \\ &= (A^2 + B^2)/2.\end{aligned} \tag{1.3.4}$$

Note that the mean square values of the two signals are added unless $\omega_1 = \omega_2$. The validity of this rule is not restricted to pure tones of different frequency; the mean square value of any stationary signal equals the sum of mean square values of its frequency components, which can be determined with a parallel bank of contiguous filters. Thus

$$p_{\text{rms}}^2 = \sum_i p_{\text{rms},i}^2, \tag{1.3.5}$$

where $p_{\text{rms},i}$ is the rms value of the output of the i'th filter. Equation (1.3.5) is known as Parseval's formula.

Random noise

Many generators of sound produce *noise* rather than pure tones. Whereas pure tones and other periodic signals are deterministic, *noise* is a stochastic or random phenomenon. *Stationary* noise is a stochastic signal with statistical properties that do not change with time.

White noise is stationary noise with a flat power spectral density, that is, constant mean square value per hertz. The term white noise is an analogy to white light. When white noise is passed through a bandpass filter, the mean square of the output signal is proportional to the bandwidth of the filter. It follows that when white noise is analysed with constant percentage filters, the mean square of the output is proportional to the centre frequency of the filter. For example, if white noise is analysed with a bank of octave band filters, the mean square values of the output signals of two adjacent filters differ by a factor of two.

Pink noise is stationary noise with constant mean square value in bands with constant relative width, e.g., octave bands. Thus compared with white noise low frequencies are emphasised; hence the name pink noise, which is an analogy to an optical phenomenon. It follows that the mean square value of a given pink noise signal in octave bands is three times larger than the mean square value of the noise in one-third octave bands.

Example 1.3.2

The fact that noise, unlike periodic signals, has a finite power spectral density (mean square value *per hertz*) implies that one can detect a pure tone in noise irrespective of the signal-to-noise ratio by analysing with sufficiently fine spectral resolution: As the bandwidth is reduced, less and less noise passes through the filter, and the tone will emerge. Compared with filter bank analysers FFT analysers have the advantage that the spectral resolution can be varied over a wide range [6]; therefore FFT analysers are particular suitable for detecting tones in noise.

When several independent sources of noise are present at the same time the mean square sound pressures generated by the individual sources are additive. This is due to the fact that independent sources generate uncorrelated signals, that is, signals whose product average to zero; therefore the cross terms vanish:

$$\overline{(p_1(t)+p_2(t))^2} = \overline{p_1^2(t)} + \overline{p_2^2(t)} + 2\overline{p_1(t)p_2(t)} = \overline{p_1^2(t)} + \overline{p_2^2(t)}. \tag{1.3.6}$$

It follows that

$$p^2_{\text{rms,tot}} = \sum_i p^2_{\text{rms},i}. \tag{1.3.7}$$

Note the similarity between eqs. (1.3.5) and (1.3.7). It is of enormous practical importance that the mean square values of uncorrelated signals are additive, because signals generated by different mechanisms are invariably uncorrelated. Almost all signals that occur in real life are mutually uncorrelated.

Example 1.3.3

Equation (1.3.7) leads to the conclusion that the mean square pressure generated by a crowd of noisy people in a room is proportional to the number of people. Thus the rms value of the sound pressure in the room is proportional to the *square root* of the number of people.

Example 1.3.4

Consider the case where the rms sound pressure generated by a source of noise is to be measured in the presence of background noise that cannot be turned off. It follows from eq. (1.3.7) that it is possible to correct the measurement for the influence of the stationary background noise; one simply subtracts the mean square value of the background noise from the total mean square pressure. For this to work in practice the background noise must not be too strong, though, and it is absolutely necessary that it is completely stationary.

1.3.2 Levels and decibels

The human auditory system can cope with sound pressure variations over a range of more than a million times. Because of this wide range, the sound pressure and other acoustic quantities are usually measured on a logarithmic scale. An additional reason is that the subjective impression of how loud noise sounds correlates much better with a logarithmic measure of the sound pressure than with the sound pressure itself. The unit is the *decibel*,[21] abbreviated dB, which is a relative measure, requiring a reference quantity. The results are called *levels*. The sound pressure level (sometimes abbreviated SPL) is defined as

$$L_p = 10\log_{10}\frac{p^2_{\text{rms}}}{p^2_{\text{ref}}} = 20\log_{10}\frac{p_{\text{rms}}}{p_{\text{ref}}}, \tag{1.3.8}$$

where p_{ref} is the reference sound pressure, and $\log_{10}$ is the base 10 logarithm, henceforth written log. The reference sound pressure is 20 μPa for sound in air, corresponding roughly to the lowest audible sound at 1 kHz.[22] Some typical sound pressure levels are given in figure 1.3.2.

[21] As the name implies, the decibel is one tenth of a bel. However, the bel is rarely used today. The use of decibels rather than bels is probably due to the fact that most sound pressure levels encountered in practice take values between 10 and 120 when measured in decibels, as can be seen in figure 1.3.2. Another reason might be that to be audible, the change of the level of a given (broadband) sound must be of the order of one decibel.

[22] For sound in other fluids than atmospheric air (water, for example) the reference sound pressure is 1 μPa. To avoid possible confusion it may be advisable to state the reference sound pressure explicitly, e.g., ‘the sound pressure level is 77 dB re 20 μPa.’

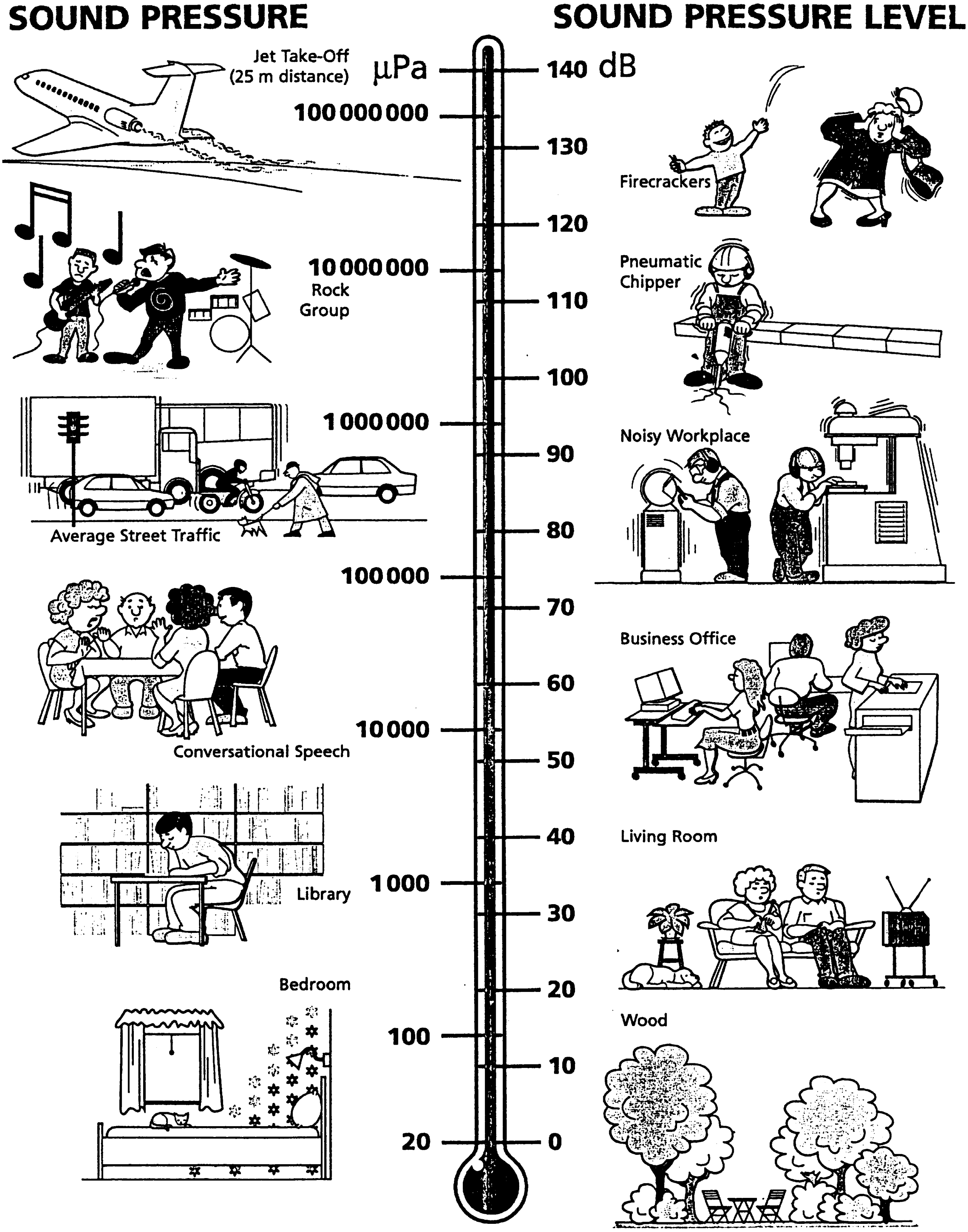


Figure 1.3.2 Typical sound pressure levels. (Source: Brüel & Kjær.)

The fact that the mean square sound pressures of independent sources are additive (cf. eq. (1.3.7)) leads to the conclusion that the levels of such sources are combined as follows:

$$L_{p,\mathrm{tot}} = 10\log\left(\sum_i 10^{0.1L_{p,i}}\right). \qquad (1.3.9)$$

Another consequence of eq. (1.3.7) is that one can correct a measurement of the sound pressure level generated by a source for the influence of steady background noise as follows:

$$L_{p,\mathrm{source}} = 10\log\left(10^{0.1L_{p,\mathrm{tot}}} - 10^{0.1L_{p,\mathrm{background}}}\right). \qquad (1.3.10)$$

This corresponds to subtracting the mean square sound pressure of the background noise from the total mean square sound pressure as described in example 1.3.5. However, since all measurements are subject to random errors, the result of the correction will be reliable only if the background level is at least, say, 3 dB below the total sound pressure level. If the background noise is more than 10 dB below the total level the correction is less than 0.5 dB.

Example 1.3.5

Expressed in terms of sound pressure levels the inverse distance law states that the level decreases by 6 dB when the distance to the source is doubled.

Example 1.3.6

When each of two independent sources in the absence of the other generates a sound pressure level of 70 dB at a certain point, the resulting sound pressure level is 73 dB (**not** 140 dB!), because $10\log 2 \simeq 3$. If one source creates a sound pressure level of 65 dB and the other a sound pressure level of 59 dB, the total level is $10\log(10^{6.5} + 10^{5.9}) \simeq 66$ dB.

Example 1.3.7

Say the task is to determine the sound pressure level generated by a source in background noise with a level of 59 dB. If the total sound pressure level is 66 dB, it follows from eq. (1.3.10) that the source would have produced a sound pressure level of $10\log(10^{6.6} - 10^{5.9}) \simeq 65$ dB in the absence of the background noise.

Example 1.3.8

When two sinusoidal sources emit pure tones of the same frequency they create an interference field, and depending on the phase difference the total sound pressure amplitude at a given position will assume a value between the sum of the two amplitudes and the difference:

$$\left||A| - |B|\right| \le \left|A\mathrm{e}^{\mathrm{j}\omega t} + B\mathrm{e}^{\mathrm{j}\omega t}\right| = |A + B| = \left||A|\mathrm{e}^{\mathrm{j}\varphi_A} + |B|\mathrm{e}^{\mathrm{j}\varphi_B}\right| \le |A| + |B|.$$

For example, if two pure tone sources of the same frequency each generates a sound pressure level of 70 dB in the absence of the other source then the total sound pressure level can be anywhere between 76 dB (constructive interference) and - ∞ dB (destructive interference). Note that eqs. (1.3.7) and (1.3.9) do **not** apply in this case because the signals are not uncorrelated. See also figure 1.9.2 in the Appendix.

Other first-order acoustic quantities, for example the particle velocity, are also often measured on a logarithmic scale. The reference velocity is 1 nm/s = 10^{-9} m/s.[23] This reference is also used in measurements of the vibratory velocities of vibrating structures.

The acoustic second-order quantities sound intensity and sound power, defined in chapter 1.5, are also measured on a logarithmic scale. The sound intensity level is

[23] The prefix n (for 'nano') represents a factor of 10^{-9}.

$$L_I = 10 \log \frac{|I|}{I_{\text{ref}}}, \tag{1.3.11}$$

where I is the intensity and I_{ref} = 1 pWm^{-2} = 10^{-12} Wm^{-2},[24] and the sound power level is

$$L_W = 10 \log \frac{P_{\text{a}}}{P_{\text{ref}}}, \tag{1.3.12}$$

where P_{a} is the sound power and P_{ref} = 1 pW. Note than levels of linear quantities (pressure, particle velocity) are defined as twenty times the logarithm of the ratio of the rms value to a reference value, whereas levels of second-order (quadratic) quantities are defined as *ten* times the logarithm, in agreement with the fact that if the linear quantities are doubled then quantities of second order are quadrupled.

Example 1.3.9

It follows from the constant spectral density of white noise that when such a signal is analysed in one-third octave bands, the level increases 1 dB from one band to the next $(10 \log(2^{1/3}) \simeq 1\,\text{dB})$.

1.3.3 Noise measurement techniques and instrumentation

A sound level meter is an instrument designed to measure sound pressure levels. Today such instruments can be anything from fairly simple devices with analogue filters and detectors and a moving coil meter to advanced digital analysers. Figure 1.3.3 shows a block diagram of a simple sound level meter. The microphone converts the sound pressure to an electrical signal, which is amplified and passes through various filters. After this the signal is squared and averaged with a detector, and the result is finally converted to decibels and shown on a display. In the following a very brief description of such an instrument will be given; see e.g. refs. [8, 9] for further details.

The most commonly used microphones for this purpose are condenser microphones, which are more stable and accurate than other types. The diaphragm of a condenser microphone is a very thin, highly tensioned foil. Inside the housing of the microphone cartridge is the other part of the capacitor, the back plate, placed very close to the diaphragm (see figure 1.3.4). The capacitor is electrically charged, either by an external voltage on the back plate or (in case of prepolarised *electret* microphones) by properties of the diaphragm or the back plate. When the diaphragm moves in response to the sound pressure, the capacitance changes, and this produces an electrical voltage proportional to the instantaneous sound pressure.

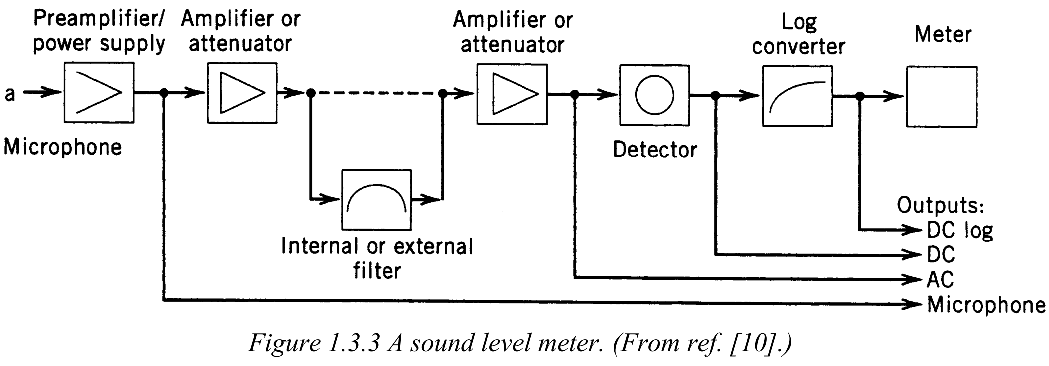


Figure 1.3.3 A sound level meter. (From ref. [10].)

[24] The prefix p (for 'pico') represents a factor of 10^{-12}.

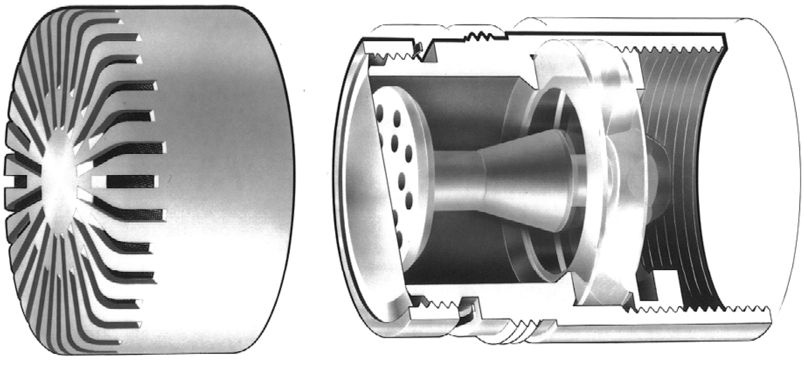


Figure 1.3.4 A condenser microphone. (From ref. [11].)

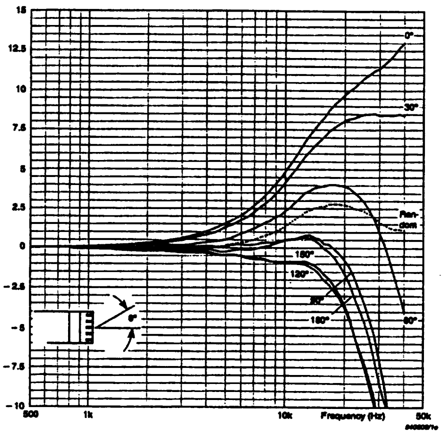


Figure 1.3.5 The 'free-field correction' of a typical measurement microphone for sound coming from various directions. The free-field correction is the fractional increase of the sound pressure (usually expressed in dB) caused by the presence of the microphone in the sound field. (From ref. [11].)

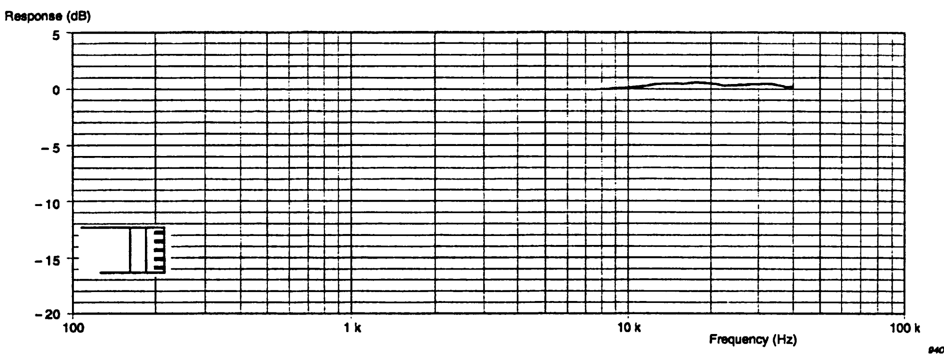


Figure 1.3.6 Free-field response of a microphone of the 'free-field' type at axial incidence. (From ref. [11].)

The microphone should be as small as possible so as not to disturb the sound field. However, this is in conflict with the requirement of a high sensitivity and a low inherent noise level, and typical measurement microphones are '½-inch' microphones with a diameter of about 13 mm. At low frequencies, say below 1 kHz, such a microphone is much smaller than the wavelength and does not disturb the sound field appreciably. In this frequency range the microphone is *omnidirectional* as of course it should be since the sound pressure is a scalar and has no direction. However, from a few kilohertz and upwards the size of the microphone is no longer negligible compared with the wavelength, and therefore it is no longer omnidirectional, which means that its response varies with the nature of the sound field; see figure 1.3.5.

One can design condenser microphones to have a flat response in as wide a frequency range as possible under specified sound field conditions. For example, 'free-field' microphones are designed to have a flat response for axial incidence (see figure 1.3.6), and such microphones should therefore be pointed towards the source. 'Random-incidence' microphones are designed for measurements in a diffuse sound field where sound is arriving from all directions, and 'pressure' microphones are intended for measurements in small cavities.

The sensitivity of the human auditory system varies significantly with the frequency in a way that changes with the level. In particular the human ear is, at low levels, much less sensitive to low frequencies than to medium frequencies. This is the background for the standardised frequency weighting filters shown in figure 1.3.7. The original intention was to simulate a human ear at various levels, but it has long ago been realised that the human auditory system is far more complicated than implied by such simple weighting curves, and B- and D-weighting filters are little used today. On the other hand the A-weighted sound pressure level is the most widely used single-value measure of sound, because the A-weighted sound pressure level correlates in general much better with the subjective effect of noise than measurements of the sound pressure level with a flat frequency response. C-weighting, which is essentially flat in the audible frequency range, is sometimes used in combination with A-weighting, because a large difference between the A-weighted level and the C-weighted level is a clear indication of a prominent content of low frequency noise. The results of measurements of the A- and C-weighted sound pressure level are denoted L_A and L_C respectively, and the unit is dB.[25] If no weighting filter is applied, the level is sometimes denoted L_Z.

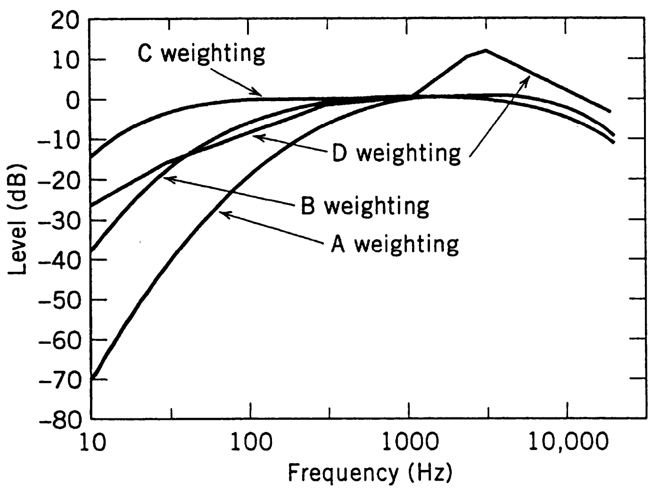


Figure 1.3.7 Standardised frequency weighting curves. (From ref. [8].)

[25] In practice the unit is often written dB (A) and dB (C), respectively.

Table 1.3.2 The response of standard A- and C-weighting filters in one-third octave bands.

Centre frequency (Hz)	A-weighting (dB)	C-weighting (dB)
8	-77.8	-20.0
10	-70.4	-14.3
12.5	-63.4	-11.2
16	-56.7	-8.5
20	-50.5	-6.2
25	-44.7	-4.4
31.5	-39.4	-3.0
40	-34.6	-2.0
50	-30.2	-1.3
63	-26.2	-0.8
80	-22.5	-0.5
100	-19.1	-0.3
125	-16.1	-0.2
160	-13.4	-0.1
200	-10.9	0.0
250	-8.6	0.0
315	-6.6	0.0
400	-4.8	0.0
500	-3.2	0.0
630	-1.9	0.0
800	-0.8	0.0
1000	0.0	0.0
1250	0.6	0.0
1600	1.0	-0.1
2000	1.2	-0.2
2500	1.3	-0.3
3150	1.2	-0.5
4000	1.0	-0.8
5000	0.5	-1.3
6300	-0.1	-2.0
8000	-1.1	-3.0
10000	-2.5	-4.4
12500	-4.3	-6.2
16000	-6.6	-8.5
20000	-9.3	-11.2

In the measurement instrument the frequency weighting filter is followed by a squaring device, a lowpass filter that smooths out the instantaneous fluctuations, and a logarithmic converter. The lowpass filter corresponds to applying a time weighting function. The most common time weighting in sound level meters is exponential, which implies that the squared signal is smoothed with a decaying exponential so that recent data are given more weight than older data:

$$L_p(t) = 10\log\left(\left(\frac{1}{\tau}\int_{-\infty}^{t} p^2(u)\,\mathrm{e}^{-(t-u)/\tau}\,\mathrm{d}u\right)\Big/ p_{\mathrm{ref}}^2\right). \tag{1.3.13}$$

Two values of the time constant τ are standardised: S (for 'slow') corresponds to a time constant of 1 s, and F (for 'fast') is exponential averaging with a time constant of 125 ms.

The alternative to exponential averaging is linear (or integrating) averaging, in which all the sound is weighted uniformly during the integration. The equivalent sound pressure level is defined as

$$L_{eq} = 10\log\left(\left(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} p^2(t)\mathrm{d}t\right)\bigg/ p_{ref}^2\right). \qquad (1.3.14)$$

Measurements of random noise with a finite integration time are subject to random errors that depend on the bandwidth of the signal and on the integration time. It can be shown that the variance of the measurement result is inversely proportional to the product of the bandwidth and the integration time [6].[26]

As can be seen by comparing with eqs. (1.3.1) and (1.3.8), the equivalent sound pressure level is just the sound pressure level corresponding to the rms sound pressure determined with a specified integration period. The A-weighted equivalent sound pressure level L_{Aeq} is the level corresponding to a similar time integral of the A-weighted instantaneous sound pressure. Sometimes the quantity is written $L_{\mathrm{Aeq},T}$ where T is the integration time.

Whereas exponential averaging corresponds to a running average and thus gives a (smoothed) measure of the sound at any instant of time, the equivalent sound pressure level (with or without A-weighting) can be used for characterising the total effect of fluctuating noise, for example noise from road traffic. Typical values of T are 30 s for measurement of noise from technical installations, 8 h for noise in a working environment and 24 h for traffic noise.

Sometimes it is useful to analyse noise signals in one-third octave bands, cf. section 1.3.1. From eq. (1.3.5) it can be seen that the total sound pressure level can be calculated from the levels in the individual one-third octave bands, L_i, as follows:

$$L_{\mathrm{Z}} = 10\log\left(\sum_i 10^{0.1L_i}\right). \qquad (1.3.15)$$

In a similar manner one can calculate the A-weighted sound pressure level from the one-third octave band values and the attenuation data given in table 1.3.2,

$$L_{\mathrm{A}} = 10\log\left(\sum_i 10^{0.1(L_i + K_i)}\right), \qquad (1.3.16)$$

where K_i is the relative response of the A-weighting filter (in dB) in the i'th band, given in table 1.3.2.

Example 1.3.10

A source gives rise to the following one-third octave band values of the sound pressure level at a certain point,

Centre frequency (Hz)	Sound pressure level (dB)
315	52
400	68
500	76
630	71
800	54

[26] In the literature reference is sometimes made to the equivalent integration time of exponential detectors. This is two times the time constant (e.g. 250 ms for 'F'), because a measurement of random noise with an exponential detector with a time constant of τ has the same statistical uncertainty as a measurement with linear averaging over a period of 2τ [9].

and less than 50 dB in all the other bands. It follows that

$$L_Z \simeq 10\log\left(10^{5.2} + 10^{6.8} + 10^{7.6} + 10^{7.1} + 10^{5.4}\right) \simeq 77.7 \text{ dB},$$

and

$$L_A \simeq 10\log\left(10^{(5.2-0.66)} + 10^{(6.8-0.48)} + 10^{(7.6-0.32)} + 10^{(7.1-0.19)} + 10^{(5.4-0.08)}\right) \simeq 74.7 \text{ dB}.$$

Noise that changes its level in a regular manner is called *intermittent noise*. Such noise could for example be generated by machinery that operates in cycles. If the noise occurs at several steady levels, the equivalent sound pressure level can be calculated from the formula

$$L_{\text{eq},T} = 10\log\left(\sum_i \frac{t_i}{T} 10^{0.1L_i}\right). \quad (1.3.17)$$

This corresponds to adding the mean square values with a weighting that reflects the relative duration of each level.

Example 1.3.11

The A-weighted sound pressure level at a given position in an industrial hall changes periodically between 84 dB in intervals of 15 minutes, 95 dB in intervals of 5 minutes and 71 dB in intervals of 20 minutes. From eq. (1.3.17) it follows that the equivalent sound pressure level over a working day is

$$L_{\text{Aeq}} = 10\log\left(\frac{15}{40}10^{8.4} + \frac{5}{40}10^{9.5} + \frac{20}{40}10^{7.1}\right) \simeq 87.0 \text{ dB}.$$

Most sound level meters have also a peak detector for determining the highest absolute value of the instantaneous sound pressure (without filters and without time weighting), p_{peak}. The *peak level* is calculated from this value and eq. (1.3.8) in the usual manner, that is,

$$L_p = 20\log\frac{p_{\text{peak}}}{p_{\text{ref}}}. \quad (1.3.18)$$

Example 1.3.12

The *crest factor* of a signal is the ratio of its peak value to the rms value (sometimes expressed in dB). From example 3.1 it follows that the crest factor of a pure tone signal is $\sqrt{2}$ or 3 dB.

The sound exposure level (sometimes abbreviated SEL) is closely related to L_{Aeq}, but instead of dividing the time integral of the squared A-weighted instantaneous sound pressure by the actual integration time one divides by t_0 = 1 s. Thus the sound exposure level is a measure of the total energy[27] of the noise, normalised to 1 s:

---

[27] In signal analysis it is customary to use the term ‘energy’ in the sense of the integral of the square of a signal, without regard to its units. This should not be confused with the potential energy density of the sound field introduced in chapter 1.5.

$$L_{\mathrm{AE}} = 10\log\left(\left(\frac{1}{t_0}\int_{t_1}^{t_2} p_{\mathrm{A}}^2(t)dt\right)\Big/ p_{\mathrm{ref}}^2\right) \tag{1.3.19}$$

This quantity is used for measuring the total energy of a ‘noise event’ (say, a hammer blow or the take off of an aircraft), independently of its duration. Evidently the measurement interval should encompass the entire event.

Example 1.3.13

It is clear from eqs. (1.3.14) and (1.3.19) that $L_{\mathrm{Aeq},T}$ of a noise event of finite duration decreases with the logarithm of T if the T exceeds its duration:

$$L_{\mathrm{Aeq},T} = 10\log\left(\left(\frac{1}{T}\int_{-\infty}^{\infty} p_{\mathrm{A}}^2(t)\mathrm{d}t\right)\Big/ p_{\mathrm{ref}}^2\right) = L_{\mathrm{AE}} - 10\log\frac{T}{t_0}.$$

Example 1.3.14

If n identical noise events each with a sound exposure level of L_{AE} occur within a period of T (e.g., one working day) then the A-weighted equivalent sound level is

$$L_{\mathrm{Aeq},T} = L_{\mathrm{AE}} + 10\log n - 10\log\frac{T}{t_0},$$

because the integrals of the squared signals are additive; cf. eq. (1.3.7).[28]

1.4 THE CONCEPT OF IMPEDANCE

By definition an *impedance* is the ratio of the complex amplitudes of two signals representing cause and effect, for example the ratio of an AC voltage across a part of an electric circuit to the corresponding current, the ratio of a mechanical force to the resulting vibrational velocity, or the ratio of the sound pressure to the particle velocity. The term has been coined from the verb ‘impede’ (obstruct, hinder), indicating that it is a measure of the opposition to the flow of current etc. The reciprocal of the impedance is the *admittance*, coined from the verb ‘admit’ and indicating lack of such opposition. Note that these concepts require complex representation of harmonic signals; it makes no sense to divide, say, the instantaneous sound pressure with the instantaneous particle velocity. There is no simple way of describing properties corresponding to a complex value of the impedance without the use of complex notation.

The mechanical impedance is perhaps simpler to understand than the other impedance concepts, since it is intuitively clear that it takes a certain vibratory force to generate mechanical vibrations. The mechanical impedance of a structure at a given point is the ratio of the complex amplitude of a harmonic point force acting on the structure to the complex amplitude of the resulting vibratory velocity at the same point,[29]

---

[28] Strictly speaking this requires that the instantaneous product of the ‘event’ and any of its time shifted versions time average to zero. In practice this will always be the case.

[29] Note that the sign of the imaginary part of the impedance changes if the $e^{-i\omega t}$ convention is used instead of the $e^{j\omega t}$ convention. Cf. footnote no 9 on p. 6.

$$Z_\mathrm{m} = \frac{\hat{F}}{\hat{v}}. \tag{1.4.1a}$$

The unit is kg/s. The mechanical admittance is the reciprocal of the mechanical impedance,

$$Y_\mathrm{m} = \frac{\hat{v}}{\hat{F}}. \tag{1.4.1b}$$

This quantity is also known as the mobility. The unit is s/kg.

Example 1.4.1

It takes a force of $F = a \cdot M$ to set a mass M into the acceleration a (Newton's second law of motion); therefore the mechanical impedance of the mass is

$$Z_\mathrm{m} = \frac{\hat{F}}{\hat{v}} = \frac{\hat{F}}{\hat{a}/\mathrm{j}\omega} = \mathrm{j}\omega M.$$

Example 1.4.2

It takes a force of $F = \xi K$ to stretch a spring with the stiffness K a length of ξ (Hooke's law); therefore the mechanical impedance of the spring is

$$Z_\mathrm{m} = \frac{\hat{F}}{\hat{v}} = \frac{\hat{F}}{\mathrm{j}\omega\hat{\xi}} = \frac{K}{\mathrm{j}\omega}.$$

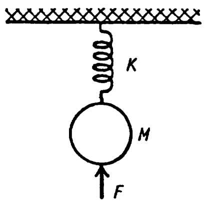


Figure 1.4.1 A mass hanging from a spring.

Example 1.4.3

A simple mechanical oscillator consists of a mass M suspended from a spring with a stiffness constant of K, as sketched in figure 1.4.1. In order to set the mass into vibrations one will have to move the mass *and* displace the spring from its equilibrium value. It follows that the mechanical impedance of this system is the sum of the impedance of the mass and the impedance of the spring,

$$Z_\mathrm{m} = \mathrm{j}\omega M + \frac{K}{\mathrm{j}\omega} = \mathrm{j}\left(\omega M - \frac{K}{\omega}\right) = \mathrm{j}\omega M\left(1 - \left(\omega_0/\omega\right)^2\right),$$

where

$$\omega_0 = \sqrt{K/M}$$

is the angular resonance frequency. Note that the impedance is zero at the resonance, indicating that even a very small harmonic force at this frequency will generate an infinite velocity. In practice there will always be some losses, of course, so the impedance is very small but not zero at the resonance frequency.

The *acoustic impedance* is associated with average properties on a surface. This quantity is mainly used under conditions where the sound pressure is more or less constant on the surface. It is defined as the complex ratio of the average sound pressure to the *volume velocity*, which is the surface integral of the normal component of the particle velocity,

$$\hat{q} = \int_S \hat{\mathbf{u}} \cdot d\mathbf{S}, \tag{1.4.2}$$

where S is the surface area. Thus the acoustic impedance is

$$Z_a = \hat{p}_{av} / \hat{q} \, . \tag{1.4.3}$$

The unit is $kgm^{-4}s^{-1}$. Since the total force acting on the surface equals the product of the average sound pressure and the area, and since $\hat{q} = S\hat{u}_n$ if the velocity is uniform, it can be seen that there is a simple relation between the two impedance concepts under such conditions:

$$Z_m = Z_a S^2. \tag{1.4.4}$$

This equation makes it possible to calculate the force it would take to drive a massless piston with the velocity $\hat{u}_n$. In other words, the acoustic impedance describes the load on a (real or fictive) piston caused by the medium. If the piston is real, the impedance is called the *radiation impedance*. This quantity is used for describing the load on, for example, a loudspeaker membrane caused by the motion of the medium.[30]

The concept of acoustic impedance is essentially associated with approximate low-frequency models. For example, it is a very good approximation to assume that the sound field in a tube is one-dimensional when the wavelength is long compared with the cross-sectional dimensions of the tube. Under such conditions the sound field can be described by eqs. (1.2.15) and (1.2.16), and a tube of a given length behaves as an acoustic two-port.[31] It is possible to calculate the transmission of sound through complicated systems of pipes using fairly simple considerations based the assumption of continuity of the sound pressure and the volume velocity at each junction [12].[32] The acoustic impedance is also useful in studying the properties of acoustic transducers. Such transducers are usually much smaller than the wavelength in a significant part of the frequency range. This makes it possible to employ so-called lumped parameter models where the system is described by an analogous electrical circuit composed of simple lumped element, inductors, resistors and capacitors, representing masses, losses and springs [13, 14]. Finally it should be mentioned that the acoustic impedance can be used for describing the acoustic properties of materials exposed to normal sound incidence.[33]

---

[30] The load of the medium on a vibrating piston can be described either in terms of the acoustic radiation impedance (the ratio of the sound pressure to the volume velocity) or the mechanical radiation impedance (the ratio of the force to the velocity).

[31] ‘Two-port’ is a term from electric circuit theory denoting a network with two terminals. Such a network is completely described by the relations between four quantities, the voltage and current at the input terminal and the voltage and current at the output terminal. By analogy, an acoustic two-port is completely described by the relations between the sound pressures and the volume velocities at the two terminals. In case of a cylindrical tube such relations can easily be derived from eqs. (1.2.15) and (1.2.16) [12].

[32] Such systems act as acoustic filters. Silencers (or mufflers) are composed of coupled tubes.

[33] In the general case we need to describe the properties of acoustic materials with the local ratio of the sound pressure on the surface to the resulting vibrational velocity. In most literature this quantity, which is used mainly in theoretical work, is called the specific acoustic impedance. In many practical applications the properties of acoustic materials are described in terms of absorption coefficients (or absorption factors), assuming either normal or diffuse sound incidence (see chapter 1.5). It is possible to calculate the absorption coefficient of a material from its specific acoustic impedance, but not the impedance from the absorption coefficient.

Example 1.4.4

The acoustic input impedance of a tube terminated by a rigid cap can be deduced from eqs. (1.2.17) and (1.2.18) (with $x = -l$),

$$Z_a = -j\frac{\rho c}{S}\cot kl,$$

where l is the length of the tube and S is its cross-sectional area. Note that the impedance goes to infinity when l equals a multiple of half a wavelength, indicating that it would take an infinitely large force to drive a piston at the inlet of the tube at these frequencies (see figure 1.4.2). Conversely, the impedance is zero when l equals an odd-numbered multiple of a quarter of a wavelength; at these frequencies the sound pressure on a vibrating piston at the inlet of the tube would vanish. Cf. example 1.2.2.

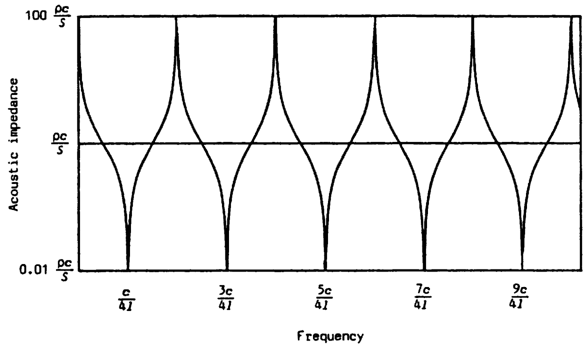


Figure 1.4.2 The acoustic input impedance of a tube terminated rigidly.

At low frequencies the acoustic impedance of the rigidly terminated tube analysed in example 1.4.4 can be simplified. The factor cot*kl* approaches 1/*kl,* and the acoustic impedance becomes

$$Z_a \simeq -j\frac{\rho c}{Slk} = \frac{\rho c^2}{j\omega V}, \tag{1.4.5}$$

where $V = Sl$ is the volume of the tube, indicating that the air in the tube acts as a spring. Thus the acoustic impedance of a cavity much smaller than the wavelength is spring-like, with a stiffness that is inversely proportional to the volume and independent of the shape of the cavity. Since, from eq. (1.2.2b),

$$\rho c^2 = \gamma p_0, \tag{1.4.6}$$

it can be seen that the acoustic impedance of a cavity at low frequencies also can be written

$$Z_a = \frac{\gamma p_0}{j\omega V}, \tag{1.4.7}$$

in agreement with the considerations on p. 3.

Example 1.4.5

A Helmholtz resonator is the acoustic analogue to the simple mechanical oscillator described in example 1.4.3; see figure 1.4.3. The dimensions of the cavity are much smaller that the wavelength; therefore it behaves as a spring with the acoustic impedance

$$Z_a = \frac{\rho c^2}{j\omega V},$$

where V is the volume; cf. eq. (1.4.5). The air in the neck moves back and forth uniformly as if it were incompressible; therefore the air in the neck behaves as a lumped mass with the mechanical impedance

$$Z_m = j\omega\rho S l_{eff},$$

where l_{eff} is the effective length and S is the cross-sectional area of the neck. (The effective length of the neck is somewhat longer than the physical length, because some of the air just outside the neck is moving along with the air in the neck.) The corresponding acoustic impedance follows from eq. (1.4.4):

$$Z_a = \frac{j\omega\rho l_{eff}}{S}.$$

By analogy with example 1.4.3 we conclude that the angular resonance frequency is

$$\omega_0 = c\sqrt{\frac{S}{V l_{eff}}}.$$

Note that the resonance frequency is independent of the density of the medium.

It is intuitively clear that a larger volume or a longer neck would correspond to a lower frequency, but it is perhaps less obvious that a smaller neck area gives a lower frequency.

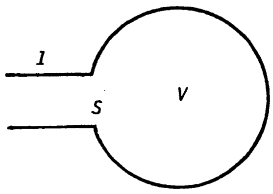


Figure 1.4.3 A Helmholtz resonator.

Yet another impedance concept, the characteristic impedance, has already been introduced. As we have seen in section 1.2.1, the complex ratio of the sound pressure to the particle velocity in a plane propagating wave equals the characteristic impedance of the medium (cf. eq. (1.2.14)), and it approximates this value in a free field far from the source (cf. eq. (1.2.27)). Thus, the characteristic impedance describes a property of the medium, as we have seen on p. 12. The unit is $kgm^{-2}s^{-1}$.

1.5 SOUND ENERGY, SOUND INTENSITY, SOUND POWER AND SOUND ABSORPTION

The most important quantity for describing a sound field is the sound pressure. However, sources of sound emit sound power, and sound fields are also energy fields in which potential and kinetic energies are generated, transmitted and dissipated. Some typical sound power levels are given in table 1.5.1.

It is apparent that the radiated sound power is a negligible part of the energy conversion of almost any source. However, energy considerations are nevertheless of great practical importance in acoustics. The usefulness is due to the fact that a statistical approach where the energy of the sound field is considered turns out to give very useful approximations in room acoustics and in noise control. In fact determining the sound power of sources is a central point in noise control engineering. The value and relevance of knowing the sound power radiated by a source is due to the fact that this quantity is largely independent of the surroundings of the source in the audible frequency range.

Table 1.5.1 Typical sound power levels.

Aircraft turbojet engine	10 kW	160 dB
Gas turbine (1 MW)	32 W	135 dB
Small airplane	5 W	127 dB
Tractor (150 hp)	100 mW	110 dB
Large electric motor (0.5 MW)	10 mW	100 dB
Vacuum cleaner	100 µW	80 dB
Office machine	32 µW	75 dB
Speech	10 µW	70 dB
Whisper	10 nW	40 dB

1.5.1 The energy in a sound field

It can be shown that the instantaneous potential energy density at a given position in a sound field (the potential sound energy per unit volume) is given by the expression

$$w_{\text{pot}}(t) = \frac{p^2(t)}{2\rho c^2}. \tag{1.5.1}$$

This quantity describes the local energy stored per unit volume of the medium because of the compression or rarefaction; the phenomenon is analogous to the potential energy stored in a compressed or elongated spring, and the derivation is similar.

The instantaneous kinetic energy density at a given position in a sound field (the kinetic energy per unit volume) is

$$w_{\text{kin}}(t) = \frac{1}{2}\rho u^2(t). \tag{1.5.2}$$

This quantity describes the energy per unit volume at the given position represented by the mass of the particles of the medium moving with the velocity u. This corresponds to the kinetic energy of a moving mass, and the derivation is similar.

The instantaneous sound intensity at a given position is the product of the instantaneous sound pressure and the instantaneous particle velocity,

$$\mathbf{I}(t) = p(t)\mathbf{u}(t). \tag{1.5.3}$$

This quantity, which is a vector, expresses the magnitude and direction of the instantaneous flow of sound energy per unit area at the given position, or the work done by the sound wave per unit area of an imaginary surface perpendicular to the vector.

Energy conservation

By combining the fundamental equations that govern a sound field (the conservation of mass, the relation between the sound pressure and density changes, and Euler's equation of motion), one can derive the equation

$$\nabla \cdot \mathbf{I}(t) = -\frac{\partial w(t)}{\partial t},$$

where $\nabla \cdot \mathbf{I}(t)$ is the divergence of the instantaneous sound intensity and $w(t)$ is the sum of the potential and kinetic energy densities. This is the equation of conservation of sound energy, which expresses the simple fact that the rate of change of the total sound energy at a given point in a sound field is equal to the flow of converging sound energy; if the sound energy density at the point increases there must be a net flow of energy towards the point, and if it decreases there must be net flow of energy diverging away from the point.

The global version of this equation is obtained using Gauss's theorem,[34]

$$\int_V \nabla \cdot \mathbf{I}(t) \mathrm{d}V = \int_S \mathbf{I}(t) \cdot \mathrm{d}\mathbf{S} = -\frac{\partial}{\partial t}\left(\int_V w(\mathrm{t})\mathrm{d}V\right) = -\frac{\partial E(t)}{\partial t},$$

where S is the area of an arbitrary, closed surface, V is the volume inside the surface, and $E(t)$ is the total instantaneous sound energy within the surface. This equation shows that the rate of change of the total sound energy within a closed surface is identical with the surface integral of the normal component of the instantaneous sound intensity.

In practice the time-averaged energy densities,

$$w_{\text{pot}} = \frac{p_{\text{rms}}^2}{2\rho c^2}, \qquad w_{\text{kin}} = \frac{1}{2}\rho u_{\text{rms}}^2, \qquad \text{(1.5.4a, 1.5.4b)}$$

are more important than the instantaneous quantities, and the time-averaged sound intensity (which is usually referred to just as the 'sound intensity'),

$$\mathbf{I} = \overline{\mathbf{I}(t)} = \overline{p(t)\mathbf{u}(t)}, \qquad \text{(1.5.5)}$$

is more important than the instantaneous intensity $\mathbf{I}(t)$. Energy conservation considerations lead to the conclusion that the integral of the normal component of the sound intensity over a closed surface is zero,

$$\int_S \mathbf{I} \cdot \mathrm{d}\mathbf{S} = 0 \qquad \text{(1.5.6)}$$

in any sound field unless there is generation or dissipation of sound power within the surface S. If, on the other hand, the surface encloses a source the integral equals the radiated sound power of the source, irrespective of the presence of other sources of noise outside the surface:

$$\int_S \mathbf{I} \cdot \mathrm{d}\mathbf{S} = P_{\text{a}} \qquad \text{(1.5.7)}$$

Often we will be concerned with harmonic signals and make use of complex notation, as in chapters 1.2 and 1.4. Expressed in the complex notation eqs. (1.5.4) and (1.5.5) become

$$w_{\text{pot}} = \frac{|\hat{p}|^2}{4\rho c^2}, \qquad w_{\text{kin}} = \frac{1}{4}\rho|\hat{u}|^2, \qquad \text{(1.5.8a, 1.5.8b)}$$

[34] According to Gauss's theorem the volume integral of the divergence of a vector equals the corresponding surface integral of the (outward pointing) normal component of the vector.

$$\mathbf{I} = \frac{1}{2}\mathrm{Re}\left\{\hat{p}\hat{\mathbf{u}}^*\right\}. \tag{1.5.9}$$

(Note that the two complex exponentials describing the time dependence of the sound pressure and the particle velocity cancel each other because one of them is conjugated; see the Appendix.) The component of the sound intensity in the *x*-direction is

$$I_x = \frac{1}{2}\mathrm{Re}\left\{\hat{p}\hat{u}_x^*\right\}. \tag{1.5.10}$$

Inserting the expressions for the sound pressure (eq. (1.2.13)) and the particle velocity (eqs. (1.2.14)) in a plane propagating wave into eq. (1.5.10) shows that

$$I_x = \frac{|\hat{p}|^2}{2\rho c} = \frac{p_{\mathrm{rms}}^2}{\rho c} \tag{1.5.11}$$

in this particular sound field. Moreover, inserting expressions for the sound pressure and the particle velocity in a simple spherical wave, eqs. (1.2.26) and (1.2.27), into eq. (1.5.10) gives the same relation for the radial sound intensity:

$$I_r = \frac{1}{2}\mathrm{Re}\left\{\hat{p}\hat{u}_r^*\right\} = \mathrm{Re}\left\{\frac{A\,\mathrm{e}^{\mathrm{j}(\omega t - kr)}}{r}\frac{A^*\,\mathrm{e}^{-\mathrm{j}(\omega t - kr)}}{\rho c r}\left(1 - \frac{1}{\mathrm{j}kr}\right)\right\} = \frac{|A|^2}{2\rho c r^2} = \frac{|\hat{p}|^2}{2\rho c} = \frac{p_{\mathrm{rms}}^2}{\rho c}. \tag{1.5.12}$$

It is apparent that there is a simple relation between the sound intensity and the mean square sound pressure in these two extremely important cases.[35] However, it should be emphasised that in the general case eq. (1.5.11) is **not** valid, and one will have to measure both the sound pressure and the particle velocity simultaneously and average the instantaneous product over time in order to measure the sound intensity. Equipment for such measurements has been commercially available since the early 1980s [3].

Example 1.5.1

It follows from eq. (1.5.11) that the sound intensity in a plane propagating wave with an rms sound pressure of 1 Pa is $(1\ \mathrm{Pa})^2/(1.2\ \mathrm{kgm^{-3}} \cdot 343\ \mathrm{ms^{-1}}) \simeq 2.4\ \mathrm{mW/m^2}$.

Example 1.5.2

The sound intensity in the interference field generated by a plane sound wave reflected from a rigid surface at normal incidence can be determined by inserting eqs. (1.2.17) and (1.2.18) into eq. (1.5.10):

$$I_x = \frac{1}{2}\mathrm{Re}\left\{2p_{\mathrm{i}}\cos kx \frac{\mathrm{j}2p_{\mathrm{i}}^*}{\rho c}\sin kx\right\} = \mathrm{Re}\left\{\frac{2\mathrm{j}|p_{\mathrm{i}}|^2}{\rho c}\sin 2kx\right\} = 0.$$

This result shows that there is no net flow of sound energy towards the rigid surface.

Under conditions where the sound pressure and the particle velocity are constant over a surface in phase as well as in amplitude we can write

---

[35] Eq. (1.5.11) implies that the sound intensity level is almost identical with the sound pressure level in air at 20°C and 101.3 kPa:

$$L_I = 10\log\left(I/I_{\mathrm{ref}}\right) = 10\log\left(p_{\mathrm{rms}}^2/(\rho c)/I_{\mathrm{ref}}\right) = 10\log\left(p_{\mathrm{rms}}^2/p_{\mathrm{ref}}^2\right) - 10\log\left(\rho c I_{\mathrm{ref}}/p_{\mathrm{ref}}^2\right) \simeq L_p - 0.14\ \mathrm{dB} \simeq L_p.$$

$$\hat{p} = \hat{q} Z_{\mathrm{a}} \tag{1.5.13}$$

(cf. eq. (1.4.4)), and the sound power passing through the surface can be expressed in terms of the acoustic impedance:

$$P_{\mathrm{a}} = \frac{1}{2}\mathrm{Re}\left\{\hat{p}\hat{q}^*\right\} = \frac{1}{2}\mathrm{Re}\left\{\left|\hat{q}\right|^2 Z_{\mathrm{a}}\right\} = \frac{\left|\hat{q}\right|^2}{2}\mathrm{Re}\left\{Z_{\mathrm{a}}\right\}. \tag{1.5.14}$$

This expression demonstrates that the radiated sound power of a vibrating surface is closely related to the volume velocity and to the real part of the radiation impedance.

Equation (1.5.7) implies that one can determine the sound power radiated by a source by integrating the normal component of the sound intensity over a surface that encloses the source. This is the *sound intensity method* of measuring sound power. Note that special equipment for such measurements is required.

In an environment without reflecting surfaces the sound field generated by a source of finite extent is locally plane far from the source, as mentioned in section 1.2.2, and therefore the local sound intensity is to a good approximation given by eq. (1.5.11). With eq. (1.5.7) we now conclude that one can estimate the radiated sound power of a source by integrating the mean square pressure generated by the source over a spherical surface centred at the source:

$$P_{\mathrm{a}} = \int_S \left(p_{\mathrm{rms}}^2/(\rho c)\right)\mathrm{d}S. \tag{1.5.15}$$

However, whereas eq. (1.5.7) is valid even in the presence of sources outside the measurement surface eq. (1.5.11) is not; therefore only the source under test must be present. In practice one measures the sound pressure at a finite number of discrete points. This is the *free-field method* of measuring sound power. Note that an anechoic room (a room without any reflecting surfaces) is required.

Yet another method of measuring sound power requires a diffuse sound field in a reverberation room; see chapter 3.

1.5.2 Sound absorption

Most materials absorb sound. As we have seen in chapter 1.2 we need a precise description of the boundary conditions for solving the wave equation, which leads to a description of material properties in terms of the specific acoustic impedance, as mentioned in chapter 1.4. However, in many practical applications, for example in architectural acoustics, a simpler measure of the acoustic properties of materials, the absorption coefficient (or absorption factor), is more useful. By definition the absorption coefficient of a given material is the absorbed fraction of the incident sound power. From this definition it follows that the absorption coefficient takes values between naught and unity. A value of unity implies that all the incident sound power is absorbed.

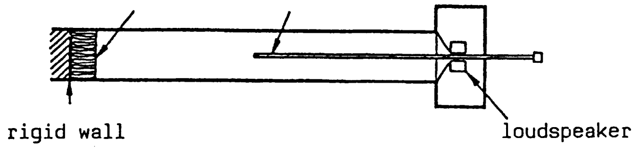


Figure 1.5.1 A standing wave tube for measuring the normal incidence absorption coefficient. (From ref. [15].)

In general the absorption coefficient of a given material depends on the structure of the sound field (plane wave incidence of a given angle of incidence, for example, or random or diffuse incidence in a room). Here we will study only the absorption for plane waves of normal incidence.

Consider the sound field in a tube driven by a loudspeaker at one end and terminated by the material under test at the other end, as sketched in figure 1.5.1. This is a one-dimensional field, which means that it has the general form given by eqs. (1.2.15) and (1.2.16). The amplitudes p_i and p_r depend on the boundary conditions, that is, the vibrational velocity of the loudspeaker and the properties of the material at the end of the tube. The sound intensity is obtained by inserting eqs. (1.2.15) and (1.2.16) into eq. (1.5.10),

$$\begin{aligned} I_x &= \mathrm{Re}\left\{\left(p_\mathrm{i}\,\mathrm{e}^{-\mathrm{j}kx}+p_\mathrm{r}\,\mathrm{e}^{\mathrm{j}kx}\right)\frac{\left(p_\mathrm{i}^*\,\mathrm{e}^{\mathrm{j}kx}-p_\mathrm{r}^*\,\mathrm{e}^{-\mathrm{j}kx}\right)}{2\rho c}\right\} = \frac{|p_\mathrm{i}|^2-|p_\mathrm{r}|^2}{2\rho c} \\ &= \frac{\left(|p_\mathrm{i}|+|p_\mathrm{r}|\right)\left(|p_\mathrm{i}|-|p_\mathrm{r}|\right)}{2\rho c} = \frac{p_\mathrm{max}\,p_\mathrm{min}}{2\rho c}, \end{aligned} \tag{1.5.16}$$

where the last equation sign follows from eq. (1.2.20). (Note that p_max and p_min are amplitudes.) The *incident* sound intensity is the value associated with the incident wave, that is,

$$I_\mathrm{inc} = \frac{|p_\mathrm{i}|^2}{2\rho c}. \tag{1.5.17}$$

The absorption coefficient is the ratio of I_x to I_inc,

$$\alpha = \frac{I_x}{I_\mathrm{inc}} = \frac{|p_\mathrm{i}|^2-|p_\mathrm{r}|^2}{|p_\mathrm{i}|^2} = 1-|R|^2 = 1-\left(\frac{s-1}{s+1}\right)^2 = \frac{4s}{(1+s)^2}, \tag{1.5.18}$$

where we have introduced the reflection factor and the standing wave ratio (cf. eqs. (1.2.19) and (1.2.22)). Note that the absorption coefficient is independent of the phase angle of R, which shows that there is more information in the complex reflection factor than in the absorption coefficient. Equation (1.5.18) demonstrates that one can determine the normal incidence absorption coefficient of a material by exposing it to normal sound incidence in a tube and measuring the standing wave ratio of the resulting interference field.

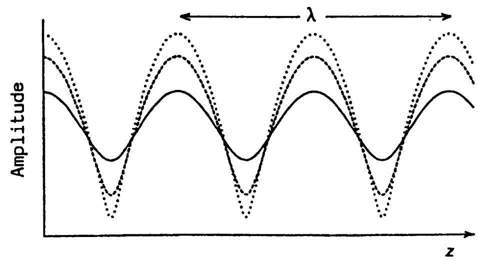


Figure 1.5.2 Standing wave pattern for various absorption coefficients: 0.9 (——); 0.6 (– –); 0.3 (···).

Example 1.5.3

If the material under test is completely reflecting then $|R| = 1$, corresponding to an absorption coefficient of zero. In this case the standing wave ratio is infinitely large. If the material is completely absorbing, $R = 0$, corresponding to an absorption coefficient of unity. In the latter case there is no reflected wave, so the sound pressure amplitude is constant in the tube, corresponding to a standing wave ratio of one.

1.6 RADIATION OF SOUND

Sound can be generated by many different mechanisms. In this note we will study only the simplest one, which is also the most important: that of a solid vibrating surface. As we shall see, the most efficient mechanism for radiation of sound involves a net volume displacement.

1.6.1 Point sources

The simplest source to describe mathematically is a sphere that expands and contracts harmonically with spherical symmetry. In free space such a source generates the simple spherical sound field we studied in section 1.2.2. Say the source has a radius of *a*. From eq. (1.2.27) we know that the particle velocity on the surface of the source is

$$\hat{u}_r(a) = \frac{A}{\rho c}\frac{e^{j(\omega t - ka)}}{a}\left(1 + \frac{1}{jka}\right). \tag{1.6.1}$$

The boundary condition on the surface implies that the vibrational velocity $U e^{j\omega t}$ must equal the normal component of the particle velocity; therefore

$$A = \frac{j\rho cka^2 U e^{jka}}{1 + jka} = \frac{j\rho\omega Q e^{jka}}{4\pi(1 + jka)}, \tag{1.6.2}$$

where we have introduced the volume velocity of the pulsating sphere,

$$Q = 4\pi a^2 U, \tag{1.6.3}$$

by multiplying with the surface area of the sphere. Inserting into eq. (1.2.26) gives an expression for the sound pressure generated by the source,

$$\hat{p} = \frac{j\rho\omega Q e^{j(\omega t - k(r-a))}}{4\pi r(1 + jka)}. \tag{1.6.4}$$

We can now calculate the radiation impedance of the pulsating sphere. This is the ratio of the sound pressure on the surface of the sphere to the volume velocity (cf. eq. (1.4.3)):

$$Z_{\mathrm{a,r}} = \frac{\hat{p}(a)}{Q e^{j\omega t}} = \frac{j\rho\omega}{4\pi a(1 + jka)} \simeq \frac{\rho ck^2}{4\pi} + \frac{j\omega\rho}{4\pi a}, \tag{1.6.5}$$

where the approximation to the right is based on the assumption that $ka \ll 1$. Note that the imaginary part of the radiation impedance is much larger than the real part at low frequencies, indicating that most of the force it takes to expand and contract the sphere goes to moving the mass of the air in a region near the sphere (cf. example 1.4.1). This air moves back and forth almost as if it were incompressible.

In the limit of a vanishingly small sphere the source becomes a *monopole*, also known as a *point source* or a *simple source*. With $ka \ll 1$, the expression for the sound pressure generated by a point source with the volume velocity $Q e^{j\omega t}$ becomes

$$\hat{p} = \frac{j\rho\omega Q e^{j(\omega t - kr)}}{4\pi r}. \tag{1.6.6}$$

A vanishingly small sphere with a finite volume velocity[36] may seem to be a rather academic source. However, the monopole is a central concept in theoretical acoustics. At low frequencies it is a good approximation to any source that produces a net displacement of volume, that is, any source that is small compared with the wavelength and changes its volume as a function of time, irrespective of its shape and the way it vibrates. An enclosed loudspeaker is to a good approximation a monopole at low frequencies. A source that injects fluid, the outlet of an engine exhaust system, for example, is also in effect a monopole.

The sound intensity generated by the monopole can be determined from eq. (1.5.10):

$$I_r = \frac{1}{2}\mathrm{Re}\left\{\hat{p}\hat{u}_r^*\right\} = \frac{1}{2}\mathrm{Re}\left\{\frac{\mathrm{j}\rho\omega Q}{4\pi r}\frac{-\mathrm{j}\rho\omega Q^*}{4\pi r\rho c}\left(1-\frac{1}{\mathrm{j}kr}\right)\right\} = \frac{\left(\rho\omega|Q|\right)^2}{32\pi^2 r^2\rho c}. \tag{1.6.7}$$

By multiplying with the surface of the area of a sphere with the radius r we get the sound power radiated by the monopole,

$$P_\mathrm{a} = \frac{\left(\rho\omega|Q|\right)^2}{32\pi^2 r^2\rho c}4\pi r^2 = \frac{\rho c k^2|Q|^2}{8\pi}. \tag{1.6.8}$$

We could also obtain this result from eqs. (1.5.14) and (1.6.5), of course. Note that the sound power is proportional to the square of the frequency, indicating that a small pulsating sphere is not a very efficient radiator of sound at low frequencies.

Reciprocity

The *reciprocity principle* states that if a monopole source at a given point generates a certain sound pressure at a another point then the monopole would generate the same sound pressure if we interchange listener and source position, irrespective of the presence of reflecting or absorbing surfaces. This is a strong statement with many practical implications.

It is easy to take account of a large reflecting plane surface, say, at $z = 0$, if one makes use of the concept of *image sources*. If the surface is rigid the boundary condition implies that $u_z = 0$ at $z = 0$, and simple symmetry considerations show that this is satisfied if we replace the rigid plane with an image source; see figure 1.6.1. The resulting sound pressure is simply the sum of the sound pressures generated by the source and the image source,

$$\hat{p} = \frac{\mathrm{j}\rho\omega Q\,\mathrm{e}^{\mathrm{j}(\omega t-kR_1)}}{4\pi R_1} + \frac{\mathrm{j}\rho\omega Q\,\mathrm{e}^{\mathrm{j}(\omega t-kR_2)}}{4\pi R_2} = \frac{\mathrm{j}\rho\omega Q}{4\pi R_1}\mathrm{e}^{\mathrm{j}(\omega t-kR_1)}\left(1+\frac{R_1}{R_2}\mathrm{e}^{\mathrm{j}k(R_1-R_2)}\right). \tag{1.6.9}$$

The parenthesis shows the effect of the reflecting plane, that is, it represents the sound pressure normalised by the free field value. The normalised equation can be used for studying outdoor sound propagation over a hard surface, and it is common practice to present the ‘ground effect’, that is, the effect of reflections from the ground on outdoor sound propagation, in this form.

At very low frequencies $k(R_1 - R_2) << 1$, and the rigid surface can be seen to have the effect of increasing the sound pressure by a factor of $1+ R_1/R_2$. Destructive interference occurs when the second term in the parenthesis is real and negative, and the first interference

[36] The volume velocity of the monopole is sometimes referred to as the source strength. However, some authors use other definitions of the source strength. The term ‘volume velocity’ is unambiguous.

dip occurs when $k(R_1 - R_2) = \pi$, corresponding to $(R_1 - R_2)$ being half a wavelength. Figure 1.6.2 shows the sound pressure relative to free field for sound propagation over a rigid plane surface.

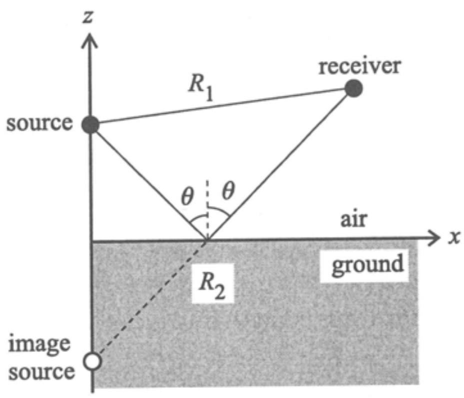


Figure 1.6.1 The sound pressure generated by a monopole above a rigid plane is the sum of two terms: direct sound and the contribution from the image source.(From ref. [16].)

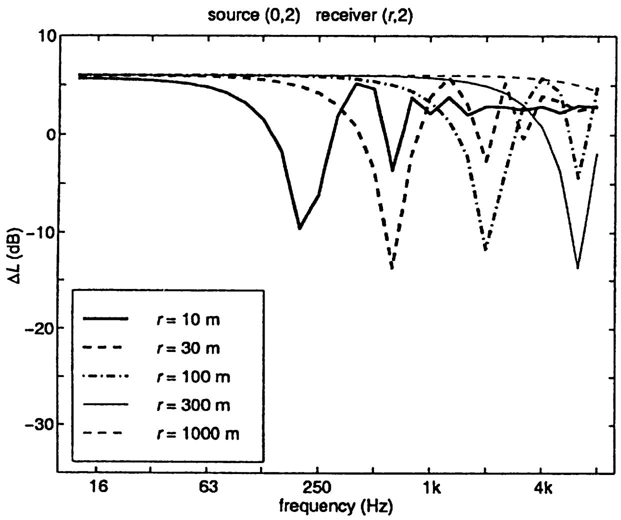


Figure 1.6.2 The sound pressure in one-third octave bands generated by a monopole above a rigid plane and shown relative to free field for five different source-receiver distances.(From ref. [16].)

If the distance between the source and the observation point is much longer than the distance between the source and the reflecting plane (see figure 1.6.3) we can make use of the *far-field approximation* and let $r_1 \simeq r_2 \simeq r$ in the denominator of eq, (1.6.6). However, the two contributions will arrive with a different phase no matter how far from the source we are. If the observation point is sufficiently far we can approximate the two distances by $r_1 \simeq r - h\cos\theta$ and $r_2 \simeq r + h\cos\theta$ in the complex exponentials. The resulting sound pressure now becomes

$$\hat{p} = \frac{\mathrm{j}\rho\omega Q\,\mathrm{e}^{\mathrm{j}(\omega t - kr_1)}}{4\pi r_1} + \frac{\mathrm{j}\rho\omega Q\,\mathrm{e}^{\mathrm{j}(\omega t - kr_2)}}{4\pi r_2}$$
$$\simeq \frac{\mathrm{j}\rho\omega Q}{4\pi r}\left(\mathrm{e}^{\mathrm{j}(\omega t - k(r - h\cos\theta))} + \mathrm{e}^{\mathrm{j}(\omega t - k(r + h\cos\theta))}\right) = \frac{\mathrm{j}\rho\omega Q}{2\pi r}\cos(kh\cos\theta)\,\mathrm{e}^{\mathrm{j}(\omega t - kr)}. \quad (1.6.10)$$

Inspection of eq. (1.6.10) leads to the conclusion that the sound pressure in the far field depends on kh and on θ unless $kh << 1$, in which case the sound pressure is simply doubled.

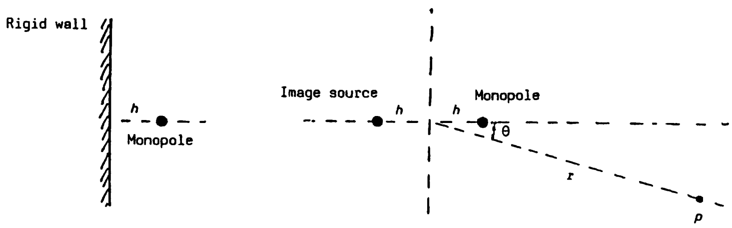


Figure 1.6.3 Far field sound pressure generated by a monopole near a rigid plane surface.

The sound power of the monopole is affected by the presence of the reflecting surface unless it is far away, $kh >> 1$. We can calculate the sound power by integrating the sound intensity over a hemisphere, cf. eq. (1.5.7). (Since the normal component of the particle velocity is zero at all points on the plane between the source and the image source, the normal component of the intensity is also zero, so this surface does not contribute to the integral.) Moreover, the considerations that lead to eq. (1.5.15) are also valid for combinations of sources. It follows that

$$P_{\mathrm{a}} = \int_0^{\pi/2}\int_0^{2\pi} \frac{|\hat{p}|^2}{2\rho c} r^2 \sin\theta \mathrm{d}\varphi \mathrm{d}\theta = \frac{\rho c k^2 |Q|^2}{4\pi} \int_0^{\pi/2} \cos^2(kh\cos\theta)\sin\theta \mathrm{d}\theta$$
$$= \frac{\rho c k^2 |Q|^2}{4\pi kh} \int_0^{kh} \cos^2 x \mathrm{d}x = \frac{\rho c k^2 |Q|^2}{8\pi}\left(1 + \frac{\sin(2kh)}{2kh}\right). \tag{1.6.11}$$

Figure 1.6.4 shows the factor in parentheses. It is apparent that the sound power is doubled if the source is very close to the surface, and that the rigid surface has an insignificant influence on the sound power output of the source when h exceeds a quarter of a wavelength, corresponding to $kh = \pi/2$.

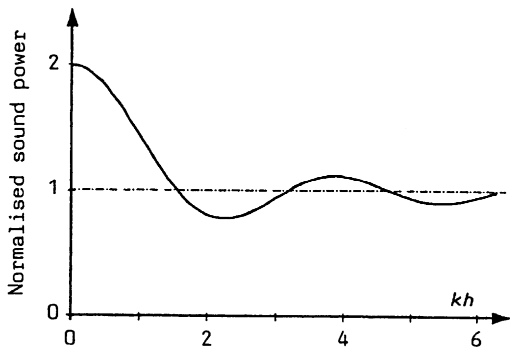


Figure 1.6.4 The influence of a rigid surface on the sound power of a monopole.

Example 1.6.1

It can be deduced from eq. (1.6.11) that two identical monopoles in close proximity (two enclosed loudspeakers driven with the same signal, for example) at very low frequencies will radiate twice as much sound power as they do when they are far from each other. The physical explanation is that the radiation load on each source is doubled; the sound pressure on each source is not only generated by the source itself but also by the neighbouring source. Alternatively one might regard the two loudspeakers as one compound source with twice the volume velocity of each loudspeaker. Because of the quadratic relation between volume velocity and power (cf. eq. (1.6.8)) this source will radiate four times more sound power than one single loudspeaker in isolation.

Two monopoles of the same volume velocity but vibrating in antiphase constitute a point dipole if the distance between them is much less than the wavelength; see figure 1.6.5. It is clear that the combined source has no net volume velocity. A point dipole is a good approximation to a small vibrating body that does not change its volume as a function of time. Such a source exerts a force on the fluid. The oscillating sphere shown in figure 1.6.6, for example, is in effect a dipole, and so is an unenclosed loudspeaker unit. Other examples include vibrating beams and wires.

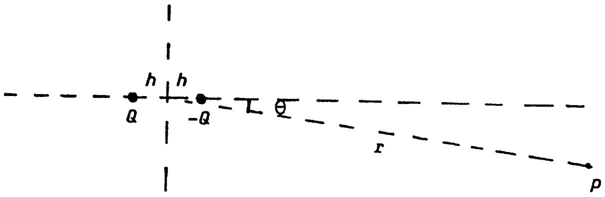


Figure 1.6.5 A point dipole.

The sound pressure generated by the two monopoles is

$$\hat{p} = \frac{\mathrm{j}\rho\omega Q\,\mathrm{e}^{\mathrm{j}(\omega t - kr_1)}}{4\pi r_1} - \frac{\mathrm{j}\rho\omega Q\,\mathrm{e}^{\mathrm{j}(\omega t - kr_2)}}{4\pi r_2}. \tag{1.6.12}$$

The near field of this combination of sources is fairly complicated. However, the far field is relatively simple. We can calculate the sound pressure in the far field in the same way we used in deriving eq. (1.6.10),

$$\begin{aligned} \hat{p} &\simeq \frac{\mathrm{j}\rho\omega Q}{4\pi r}\left(\mathrm{e}^{\mathrm{j}(\omega t - k(r + h\cos\theta))} - \mathrm{e}^{\mathrm{j}(\omega t - k(r - h\cos\theta))}\right) = \frac{\rho\omega Q}{2\pi r}\sin(kh\cos\theta)\,\mathrm{e}^{\mathrm{j}(\omega t - kr)} \\ &\simeq \frac{\rho c h k^2 Q}{2\pi r}\cos\theta\,\mathrm{e}^{\mathrm{j}(\omega t - kr)}. \end{aligned} \tag{1.6.13}$$

Note that the sound pressure is proportional to $h|Q|$, varies as $\cos\theta$ and is identically zero in the plane between the two monopoles.[37]

The sound power of the dipole is calculated by integrating the mean square sound pressure over a spherical surface centred midway between the two monopoles:

$$\begin{aligned} P_{\mathrm{a}} &= \int_0^{\pi}\int_0^{2\pi}\frac{|\hat{p}|^2}{2\rho c} r^2 \sin\theta\,\mathrm{d}\varphi\,\mathrm{d}\theta = \frac{\rho c h^2 k^4 |Q|^2}{4\pi}\int_0^{\pi}\cos^2\theta\sin\theta\,\mathrm{d}\theta \\ &= \frac{\rho c h^2 k^4 |Q|^2}{4\pi}\int_{-1}^{1} x^2\,\mathrm{d}x = \frac{\rho c h^2 k^4 |Q|^2}{6\pi}. \end{aligned} \tag{1.6.14}$$

Note that the sound power of the dipole is proportional to the fourth power of the frequency, indicating very poor sound radiation at low frequencies. The physical explanation of the poor radiation efficiency of the dipole is of course that the two monopoles almost cancel each other.

---

[37] The quantity *2hQ* is referred to by some authors as the dipole strength. However, other authors use other definitions.

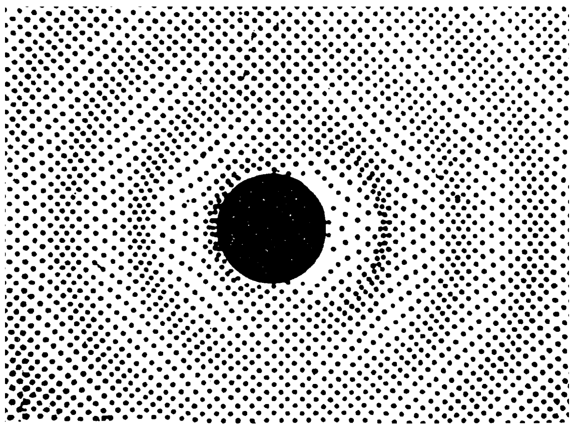


Figure 1.6.6 Fluid particles in the sound field generated by an oscillating sphere. (From ref. [1].)

1.6.2 Sound radiation from a circular piston in an infinite baffle

Apart from the pulsating sphere, a vibrating circular piston in an infinite, rigid baffle is one of the simplest cases of a spatially extended sound source that can be dealt with analytically. It is often used in connection with loudspeaker modelling.

The basic approach to extended sound sources is to consider them as composed of many simple sources, just as a dipole is made up of two monopoles. Thus, the piston is the sum of many monopoles that all radiate in phase. Because of the infinite baffle each monopole gives rise to an image source which coincides with the monopole, cf. eqs. (1.6.9) and (1.6.10)); in other words, the baffle has the effect of doubling the volume velocity of each monopole. Let the piston vibrate with the velocity $U\mathrm{e}^{\mathrm{j}\omega t}$. It follows that the volume velocity of each elementary monopole is $U\mathrm{d}S$. By linear superposition we conclude that the sound pressure radiated by the piston can be evaluated at any position in front of the baffle simply by integrating over the surface of the piston,

$$\hat{p} = \mathrm{j}\omega\rho\int_S \frac{\mathrm{e}^{\mathrm{j}(\omega t - kh)}}{2\pi h} U\mathrm{d}S\,, \tag{1.6.15}$$

where h is the distance between the observation point and the running position on the piston, and S is the surface of the piston of radius a (see figure 1.6.7).This is a special case of what is known as Rayleigh's integral, which can be used for computing the sound radiation into half space of any plane infinite surface with a given vibrational velocity [17]. Note the factor of two in the denominator instead of four for the monopole, which is due to the contribution of the image sources.

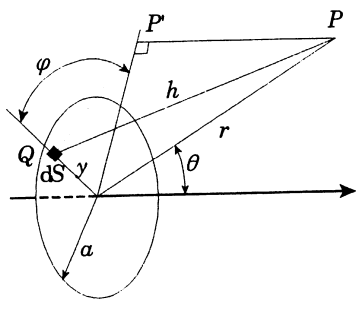


Figure 1.6.7 Definition of the variables. (From ref. [18].)

The far field sound pressure, that is, the sound pressure at long distances from the centre of the piston compared with the radius and the wavelength, can be calculated by expanding h in the complex exponential,

$$h = \sqrt{r^2 + y^2 - 2ry\sin\theta\cos\varphi} \simeq r\sqrt{1 - 2\frac{y}{r}\sin\theta\cos\varphi} \simeq r - y\sin\theta\cos\varphi, \tag{1.6.16}$$

while retaining only the first term of eq. (1.6.16) in the denominator (cf. eq. (1.6.10)). Thus the expression for the sound pressure becomes

$$\hat{p}(r,\theta) \simeq \frac{\mathrm{j}\omega\rho U}{2\pi r}\mathrm{e}^{\mathrm{j}(\omega t - kr)}\int_0^{2\pi}\int_0^a \mathrm{e}^{\mathrm{j}ky\sin\theta\cos\varphi} y\mathrm{d}y\mathrm{d}\varphi. \tag{1.6.17}$$

The calculation makes use of the Bessel functions $J_0(z)$ and $J_1(z)$, defined by

$$\mathrm{J}_0(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{j}z\cos\beta}\mathrm{d}\beta \tag{1.6.18}$$

and

$$\mathrm{J}_1(z) = \frac{1}{z}\int_0^{z} \beta\,\mathrm{J}_0(\beta)\mathrm{d}\beta \tag{1.6.19}$$

(see figure 1.6.8), and leads to the following expression for the far field sound pressure,

$$\hat{p}(r,\theta) = \frac{\mathrm{j}\omega\rho U}{r}\frac{a\,\mathrm{J}_1(ka\sin\theta)}{k\sin\theta}\mathrm{e}^{\mathrm{j}(\omega t - kr)} = \frac{\mathrm{j}\omega\rho Q}{2\pi r}\left[\frac{2\,\mathrm{J}_1(ka\sin\theta)}{ka\sin\theta}\right]\mathrm{e}^{\mathrm{j}(\omega t - kr)}, \tag{1.6.20}$$

where we have introduced the volume velocity of the piston, $Q = \pi a^2 U$. The factor in brackets is called the directivity of the piston, which is a frequency dependent function that describes the directional characteristics of the source in the far field,

$$D(f,\theta) = \left[\frac{2\,\mathrm{J}_1(ka\sin\theta)}{ka\sin\theta}\right]. \tag{1.6.21}$$

This function has its maximum value, unity, when $\theta = 0$, indicating maximum radiation in the axial direction all frequencies. Figure 1.6.9 shows the directivity for different values of the normalised frequency ka. Note that the piston is an omnidirectional source (a monopole placed on a rigid surface) at low frequencies, just as one would expect. At high frequencies the radiation of the piston is concentrated in a beam near the axial direction.

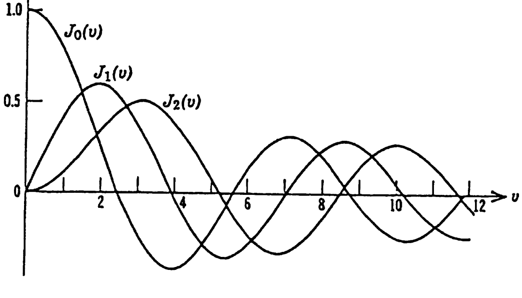


Figure 1.6.8 Bessel functions.

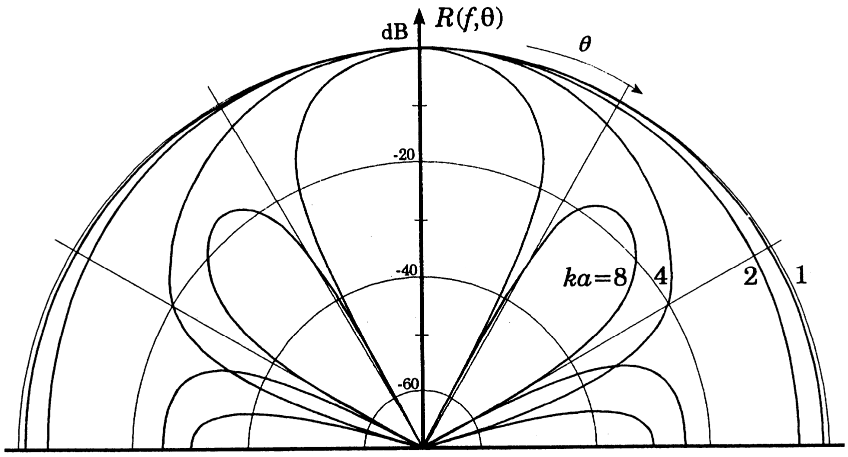


Figure 1.6.9 Directivity of the piston as a function of the normalised frequency ka. (From ref. [18].)

The sound pressure on the axis of the piston can be evaluated fairly easily. Since sin θ = 0 on the axis, the expression for the distance h reduces to

$$h = \sqrt{r^2 + y^2}, \tag{1.6.22}$$

from which,

$$\mathrm{d}h = \frac{y\mathrm{d}y}{\sqrt{r^2 + y^2}} = \frac{y}{h}\mathrm{d}y. \tag{1.6.23}$$

Thus the sound pressure on the axis is given by

$$\hat{p} = \frac{\mathrm{j}\omega\rho U}{2\pi}\mathrm{e}^{\mathrm{j}\omega t}\int_0^{2\pi}\int_r^{\sqrt{r^2+a^2}}\mathrm{e}^{-\mathrm{j}kh}\mathrm{d}h\mathrm{d}\varphi = \rho c U \mathrm{e}^{\mathrm{j}\omega t}\left[\mathrm{e}^{-\mathrm{j}kr} - \mathrm{e}^{-\mathrm{j}k\sqrt{r^2+a^2}}\right]. \tag{1.6.24}$$

If we introduce the quantity

$$\Delta = \left(\sqrt{r^2 + a^2} - r\right)/2, \tag{1.6.25}$$

the sound pressure can be written

$$\hat{p} = 2\mathrm{j}\rho c U\,\mathrm{e}^{\mathrm{j}(\omega t - k[r+\Delta])}\sin(k\Delta). \tag{1.6.26}$$

It can be seen that the sound pressure is zero when $k\Delta$ is a multiple of π, that is, when Δ is a multiple of half a wavelength, corresponding to the positions

$$r = a\left[\frac{1}{2n}\frac{a}{\lambda} - \frac{n}{2}\frac{\lambda}{a}\right] \tag{1.6.27}$$

on the axis, where n is a positive integer. In a similar way, the sound pressure assumes a maximum value for

$$2\Delta = \sqrt{r^2 + a^2} - r = (2m+1)\lambda/2 \tag{1.6.28}$$

(where m is a positive integer), that is, for

$$r = a\left[\frac{1}{2m+1}\frac{a}{\lambda} - \frac{2m+1}{4}\frac{\lambda}{a}\right]. \tag{1.6.29}$$

Figure 1.6.10 shows the normalised sound pressure on the axis of the piston as a function of the distance, which for a given frequency is defined by the corresponding *ka*-factor.

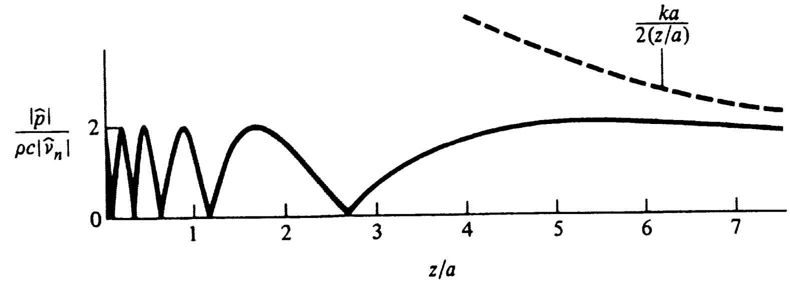


Figure 1.6.10 Sound pressure on the axis of a baffled piston for ka/2π = 5.5. (From ref. [19].)

It may seem surprising that the sound pressure is zero at some positions right in front of the vibrating piston. The explanation is destructive interference, caused by the fact that the distance from such a position to the various parts of the piston varies in such a manner that the contributions cancel out.

Example 1.6.2

In the far field, when $r >> a$ and $r >> a^2/\lambda$, one obtains

$$\Delta \simeq \frac{1}{2}\left[r\left(1+\frac{a^2}{2r^2}\right)-r\right]=\frac{a^2}{4r},$$

and the sound pressure reduces to

$$\hat{p} = \mathrm{j}\rho c U\left(\frac{ka^2}{2r}\right)\mathrm{e}^{\mathrm{j}(\omega t-kr)} = \frac{\mathrm{j}\rho ckQ}{2\pi r}\mathrm{e}^{\mathrm{j}(\omega t-kr)}.$$

This expression agrees with eq. (1.6.20) for $\theta = 0$ ($D(f) = 1$), as of course it should. This asymptotic expression is plotted as a dashed line in figure 1.6.10.

Example 1.6.3

The distances at which the minima occur, normalised by the radius of the piston, are given in terms of normalised frequencies by

$$r/a = \left[\frac{ka}{4\pi n} - \frac{\pi n}{ka}\right].$$

Minima of order n only occur for $ka \geq 2\pi n > 6$. Thus for a loudspeaker with a radius of 50 mm, minima only occur at frequencies higher than 6900 Hz, that is, far above the frequencies at which the piston approximation is valid. It follows that the minima are never observed in front of loudspeakers in real life.

In the near field there is no possible approximation except on the axis. However, by developing the spherical monopole field in cylindrical coordinates, the *force* exerted on the piston can be calculated analytically. The calculations are rather complicated (see ref. [19] or

[20] for a complete treatment), and lead to an expression in terms of special functions such as Bessel and Struve functions. The result is,

$$\hat{F} = \int_S \hat{p}\mathrm{d}S = \rho c \pi a^2 U\,\mathrm{e}^{\mathrm{j}\omega t}\left[1 - \frac{\mathrm{J}_1(2ka)}{ka} + \mathrm{j}\frac{\mathrm{H}_1(2ka)}{ka}\right], \tag{1.6.30}$$

where H_1 is the first Struve function.

The radiation impedance is the impedance seen by the piston, that is, the ratio of the average sound pressure to the volume velocity,

$$Z_{\mathrm{a,r}} = \frac{<\hat{p}>}{Q\,\mathrm{e}^{\mathrm{j}\omega t}} = \frac{\hat{F}}{SQ\,\mathrm{e}^{\mathrm{j}\omega t}}. \tag{1.6.31}$$

Combining eqs.(1.6.30) and (1.6.31) gives

$$Z_{\mathrm{a,r}} = \frac{\rho c}{\pi a^2}\left[1 - \frac{\mathrm{J}_1(2ka)}{ka} + \mathrm{j}\frac{\mathrm{H}_1(2ka)}{ka}\right]. \tag{1.6.32}$$

Figure 1.6.11 shows the normalised, dimensionless radiation impedance (the bracket in eq. (1.6.32)),

$$\frac{Z_{\mathrm{a,r}}\pi a^2}{\rho c} = R_1 + \mathrm{j}X_1. \tag{1.6.33}$$

At low frequencies and at high frequencies the radiation impedance takes simple expressions:

$$ka << 1 \qquad Z_{\mathrm{a,r}} = r_{\mathrm{a,r}} + \mathrm{j}\omega m_{\mathrm{a,r}} = \frac{1}{2\pi}\rho c k^2 + \mathrm{j}\omega\rho\frac{8}{3a\pi^2}, \tag{1.6.34a}$$

$$ka >> 1 \qquad Z_{\mathrm{a,r}} = \frac{\rho c}{\pi a^2} + \mathrm{j}\omega\rho\frac{2}{\pi^2 k^2 a^3} = \frac{\rho c}{\pi a^2}\left(1 + \mathrm{j}\frac{4/\pi}{2ka}\right). \tag{1.6.34b}$$

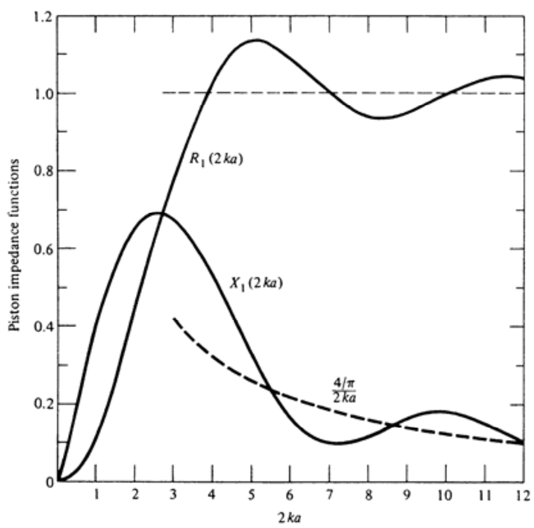


Figure 1.6.11 Radiation impedance of a piston as a function of the normalised frequency. (From ref. [19].)

The first expression is fundamental for designing loudspeakers. Note that the real part of the radiation impedance equals that of a small pulsating sphere, eq. (1.6.5), multiplied by a factor of two because of the rigid plane. The quantity $m_{a,r}$ can be interpreted as the acoustic mass of the air driven along by the piston. Interference effects in the near field make it different from the imaginary part of impedance of the pulsating sphere. However, as in the case of the pulsating sphere, eq. (1.6.5), the imaginary part of the acoustic radiation impedance diverges when the radius a goes to zero.

Example 1.6.4

The mechanical radiation impedance is given by eqs. (1.4.4) and (1.6.33) as $Z_{m,r} = \rho c\pi a^2(R_1+jX_1)$. Its low frequency approximation is therefore:

$$Z_{m,r} = \frac{\pi a^4 \rho c k^2}{2} + j\omega\rho\frac{8a^3}{3}.$$

The imaginary part of this impedance is the impedance of the mass of a layer of air in front of the piston. This layer of air is moving back and forth as if it were incompressible.

The radiated sound power is defined in chapter 1.5 as the integral of the normal component of the sound intensity over a surface than encloses the source. This method can also be used for computing the sound power of a piston in an infinite baffle. However, by far the simplest approach is to use eq. (1.5.14), which expresses the sound power in terms of the mean square volume velocity and the real part of the acoustic radiation impedance:

$$P_a = \tfrac{1}{2}|Q|^2 \operatorname{Re}\{Z_{a,r}\} = \tfrac{1}{2}|Q|^2 \frac{\rho c}{\pi a^2} R_1 = \tfrac{1}{2}|Q|^2 \frac{\rho c}{\pi a^2}\left[1 - \frac{J_1(2ka)}{ka}\right]. \tag{1.6.35}$$

At low frequencies this becomes, with eq. (1.6.34a),

$$P_a = \frac{\rho c k^2 |Q|^2}{4\pi}, \tag{1.6.36}$$

which is just what we would expect since the piston acts as a monopole on a rigid plane in this frequency range (cf. eq. (1.6.11)).

Example 1.6.5

Instead of using the volume velocity and the acoustic impedance we could equally well compute the sound power from the mean square velocity and the real part of the mechanical radiation impedance, since, with eq. (1.4.4),

$$P_a = \tfrac{1}{2}|Q|^2 \operatorname{Re}\{Z_{a,r}\} = \tfrac{1}{2}|U|^2 \operatorname{Re}\{Z_{m,r}\}.$$

Example 1.6.6

Equation (1.6.36) shows that the sound power of the piston is proportional to $|\omega Q|^2$ at low frequencies, that is, the sound power is independent of the frequency if the volume *acceleration* is independent of the frequency. This implies that the displacement of the piston should be inversely proportional to the *square* of the frequency if we want the sound power to be independent of the frequency. In other words, it implies very large displacements at low frequencies. Since mechanical systems such as loudspeakers only allow a limited excursion, the low frequency sound power output of a loudspeaker is always limited: the only way to increase the sound power is to increase the size of the membrane. This explains why very large loudspeakers are found in subwoofers.

The *directivity factor* of a source is defined as the sound intensity on the axis in the far field normalised by the sound intensity of an omnidirectional source with the same sound power. From eq. (1.6.20) the sound intensity on the axis is

$$I_r = \frac{1}{2}\rho c k^2 \left(\frac{|Q|}{2\pi r} \right)^2 \tag{1.6.37}$$

(see also example 1.6.2). Normalising with $P_a/4\pi r^2$ (eq. (1.6.35)) gives the directivity factor *Q(f)*,

$$Q(f) = \frac{(ka)^2}{R_1} = \frac{(ka)^2}{1 - \frac{J_1(2ka)}{ka}}. \tag{1.6.38}$$

The directivity factor of the piston is plotted in figure 1.6.12 as a function of the normalised frequency *ka*. Note that the directivity factor approaches two at low frequencies rather than one, reflecting the fact that all the sound power is radiated in only half a sphere.

In practice, one often uses the directivity index, defined by

$$DI(f) = 10 \log Q(f). \tag{1.6.40}$$

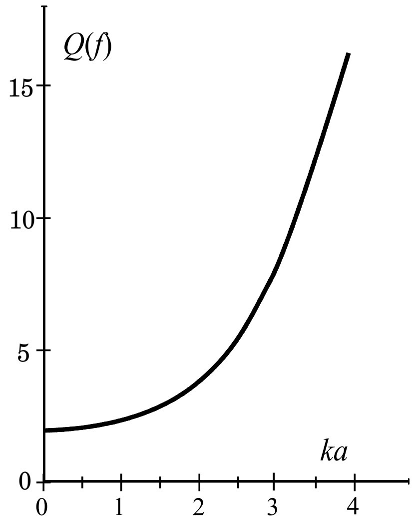


Figure 1.6.12 Directivity factor of a piston in a baffle. (From ref. [18].)

REFERENCES

1 L. Cremer and M. Hubert: *Vorlesungen über Technische Akustik* (3rd edition). Springer-Verlag, Berlin, 1985.

2 T.D. Rossing, F.R. Moore and P.A. Wheeler: *The Science of Sound* (3nd edition). Addison Wesley, San Francisco, CA, 2002.

3 M.J. Crocker and F. Jacobsen: Sound intensity. Chapter 156 in *Encyclopedia of Acoustics*, ed. M.J. Crocker. John Wiley & Sons, New York, 1997.

4 F. Jacobsen: A note on instantaneous and time-averaged active and reactive sound intensity. *Journal of Sound and Vibration* **147**, 1991, 489-496.

5 F. Jacobsen: An elementary introduction to applied signal analysis. Acoustic Technology, Department of Electrical Engineering, Technical University of Denmark, Note no 7001, 2010.

6 R.B. Randall: *Frequency analysis* (3rd edition). Brüel & Kjær, Nærum, 1987.

7 D.W. Martin and W.D. Ward: Subjective evaluation of musical scale temperament in pianos. *Journal of the Acoustical Society of America* **33**, 1961, 582-585.

8 R.W. Krug: Sound level meters. Chapter 155 in *Encyclopedia of Acoustics*, ed. M.J. Crocker, John Wiley & Sons, New York, 1997.

9 J. Pope: Analyzers. Chapter 107 in *Handbook of Acoustics*, ed. M.J. Crocker. John Wiley & Sons, New York, 1998.

10 P.V. Brüel, J. Pope and H.K. Zaveri: Introduction to acoustical measurement and instrumentation. Chapter 154 in *Encyclopedia of Acoustics*, ed. M.J. Crocker. John Wiley & Sons, New York, 1997.

11 Anon.: *Microphone Handbook*. Brüel & Kjær, Nærum, 1996.

12 F. Jacobsen: Propagation of sound waves in ducts. Acoustic Technology, Department of Electrical Engineering, Technical University of Denmark, Note no 31260, 2010.

13 K. Rasmussen: *Analogier mellem mekaniske, akustiske og elektriske systemer* (4th edition). Polyteknisk Forlag, Lyngby, 1994.

14 W. Marshall Leach, Jr.: *Introduction to Electroacoustics and Audio Amplifier Design* (2nd edition). Kendall/Hunt Publishing Company, Dubuque, IA, 1999.

15 Z. Maekawa and P. Lord: Environmental and Architectural Acoustics. E & FN Spon, London, 1994.

16 E.M. Salomons: *Computational Atmospheric Acoustics*. Kluwer Academic Publishers, Dordrecht, 2001.

17 W.S. Rayleigh: On the passage of waves through apertures in plane screens, and allied theorems. *Philosophical Magazine* **43**, 1897, 259-272.

18 K. Rasmussen: Lydfelter. Department of Acoustic Technology, Technical University of Denmark, Note no 2107, 1996.

19 A.D. Pierce: *Acoustics. An Introduction to Its Physical Principles and Applications*. The Acoustical Society of America, New York, 1989.

20 P.M. Morse and K.U. Ingard: *Theoretical Acoustics*. McGraw-Hill, New York, 1968.

21 L.E. Kinsler, A.R. Frey, A.B. Coppens and J.V. Sanders: *Fundamentals of Acoustics* (4th edition). John Wiley & Sons, New York, 2000.

BIBLIOGRAPHY

Recommended reading includes the following. The book by Everest and Pohlmann manages to deal with many acoustic phenomena practically without mathematics. The books by Fahy, Beranek, and Kinsler *et al*. are also introductory. More advanced treatments can be found in the Nelson's chapter, in the books by Morse, Morse & Ingard, and Filippi *et al*., and in Pierce's chapters and book.

1. F. Alton Everest and K.C. Pohlmann: *Master Handbook of Acoustics* (5th edition). McGraw-Hill, New York, 2009.
2. L.L. Beranek: *Acoustics* (2nd edition). The American Institute of Physics for the Acoustical Society of America, Cambridge, MA, 1986.
3. F. Fahy: *Foundations of Engineering Acoustics*. Academic Press, San Diego, 2000.
4. L.E. Kinsler, A.R. Frey, A.B. Coppens and J.V. Sanders: *Fundamentals of Acoustics* (4th edition). John Wiley & Sons, New York, 2000.
5. P.A. Nelson: An introduction to acoustics. Chapter 1 in *Fundamentals of Noise and Vibration*, ed. F.J. Fahy and J. Walker. E & FN Spon, London, 1998.
6. P.M. Morse: *Vibration and Sound* (2nd edition). The American Institute of Physics, New York, 1983.
7. P.M. Morse and K.U. Ingard: *Theoretical Acoustics*. McGraw-Hill, New York, 1968.
8. A.D. Pierce: *Acoustics. An Introduction to Its Physical Principles and Applications*. The Acoustical Society of America, New York, 1989.
9. A.D. Pierce: Mathematical theory of wave propagation. Chapter 2 in *Encyclopedia of Acoustics*, ed. M.J. Crocker, John Wiley & Sons, New York, 1997.
10. A.D. Pierce: Basic linear acoustics. Chapter 3 in *Springer Handbook of Acoustics*, ed. T.D. Rossing, Springer, New York, 2007.
11. P. Filippi, D. Habault, J.P. Lefebre and A. Bergassoli: *Acoustics: Basic Physics, Theory and Methods.* Academic Press, London, 1999.

1.9 APPENDIX: COMPLEX NOTATION

In a harmonic sound field the sound pressure at any point is a function of the type $\cos(\omega t + \varphi)$. It is common practice to use *complex notation* in such cases. This is a symbolic method that makes use of the fact that complex exponentials give a more condensed notation that trigonometric functions because of the complicated multiplication theorems of sines and cosines.

We recall that a complex number A can be written either in terms of its real and imaginary part or in terms of its magnitude (also called absolute value or modulus) and phase angle,

$$A = A_{\mathrm{r}} + \mathrm{j}A_{\mathrm{i}} = |A|\mathrm{e}^{\mathrm{j}\varphi_A}, \tag{1.9.1}$$

where

$$\mathrm{j} = \sqrt{-1} \tag{1.9.2}$$

is the imaginary unit, and

$$A_{\mathrm{r}} = \mathrm{Re}\{A\} = |A|\cos\varphi_A, \qquad A_{\mathrm{i}} = \mathrm{Im}\{A\} = |A|\sin\varphi_A, \tag{1.9.3, 1.9.4}$$

$$|A| = \sqrt{A_{\mathrm{r}}^2 + A_{\mathrm{i}}^2} \tag{1.9.5}$$

(see figure 1.9.1). The complex conjugate of A is

$$A^* = A_{\mathrm{r}} - \mathrm{j}A_{\mathrm{i}} = |A|\mathrm{e}^{-\mathrm{j}\varphi_A}; \tag{1.9.6}$$

therefore the magnitude can also be written

$$|A| = \sqrt{A \cdot A^*}. \tag{1.9.7}$$

Multiplication and division of two complex numbers are most conveniently carried out if they are given in terms of magnitudes and phase angles,

$$AB = |A||B|\mathrm{e}^{\mathrm{j}(\varphi_A + \varphi_B)}, \qquad A/B = \frac{|A|}{|B|}\mathrm{e}^{\mathrm{j}(\varphi_A - \varphi_B)}. \tag{1.9.8, 1.9.9}$$

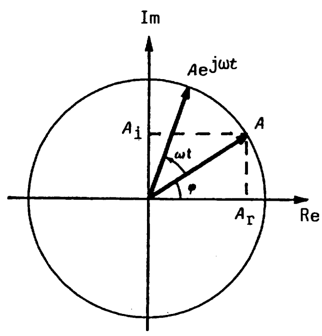


Figure 1.9.1. Complex representation of a harmonic signal.

Complex representation of harmonic signals makes use of the fact that

$$e^{jx} = \cos x + j\sin x \tag{1.9.10}$$

(Euler's equation) or, conversely,

$$\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right), \quad \sin x = -j\frac{1}{2}\left(e^{jx} - e^{-jx}\right). \tag{1.9.11a, 1.9.11b}$$

In a harmonic sound field the sound pressure at a given position can be written

$$\hat{p} = Ae^{j\omega t}, \tag{1.9.12}$$

where A is the *complex amplitude* of the sound pressure. The real, physical sound pressure is of course a real function of the time,

$$p = \text{Re}\{\hat{p}\} = \text{Re}\left\{|A|e^{j(\omega t+\varphi_A)}\right\} = |A|\cos(\omega t + \varphi_A), \tag{1.9.13}$$

which is seen to be an expression of the form $\cos(\omega t + \varphi)$. The magnitude of the complex quantity $|A|$ is called the *amplitude* of the pressure, and φ_A is its phase. It can be concluded that complex notation implies the mathematical trick of adding another solution, an expression of the form $\sin(\omega t + \varphi)$, multiplied by a constant, the imaginary unit j. This trick relies on linear superposition.

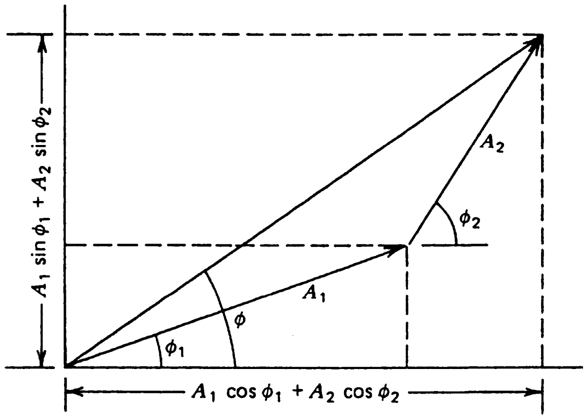


Figure 1.9.2. Two simple harmonic signals with identical frequencies. (From ref. [21].)

The mathematical convenience of the complex representation of harmonic signals can be illustrated by an example. A sum of two harmonic signals of the same frequency, $A_1e^{j\omega t}$ and $A_2e^{j\omega t}$, is yet another harmonic signal with an amplitude of $|A_1 + A_2|$ (see figure 1.9.2). Evidently, this can also be derived without complex notation,

$$\begin{aligned} p &= |A_1|\cos(\omega t + \varphi_1) + |A_2|\cos(\omega t + \varphi_2) \\ &= \left(|A_1|\cos\varphi_1 + |A_2|\cos\varphi_2\right)\cos\omega t - \left(|A_1|\sin\varphi_1 + |A_2|\sin\varphi_2\right)\sin\omega t \\ &= \left[\left(|A_1|\cos\varphi_1 + |A_2|\cos\varphi_2\right)^2 + \left(|A_1|\sin\varphi_1 + |A_2|\sin\varphi_2\right)^2\right]^{1/2}\cos(\omega t + \varphi), \end{aligned} \tag{1.9.14}$$

where

$$\varphi = \arctan\frac{|A_1|\sin\varphi_1 + |A_2|\sin\varphi_2}{|A_1|\cos\varphi_1 + |A_2|\cos\varphi_2}, \tag{1.9.15}$$

but the expedience and convenience of the complex method seems indisputable.

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{j}\omega t} = \mathrm{j}\omega\mathrm{e}^{\mathrm{j}\omega t}, \tag{1.9.16}$$

it follows that differentiation with respect to time corresponds to multiplication by a factor of jω. Conversely, integration with respect to time corresponds to division with jω. If, for example, the vibrational velocity of a surface is, in complex representation,

$$\hat{v} = B\mathrm{e}^{\mathrm{j}\omega t} = |B|\,\mathrm{e}^{\mathrm{j}(\omega t+\varphi_B)}, \tag{1.9.17}$$

which means that the real, physical velocity is

$$v = \mathrm{Re}\{\hat{v}\} = |B|\cos(\omega t + \varphi_B), \tag{1.9.18}$$

then the acceleration is written

$$\hat{a} = \mathrm{j}\omega\hat{v}, \tag{1.9.19}$$

which means that the physical acceleration is

$$a = \mathrm{Re}\{\hat{a}\} = \mathrm{Re}\{\mathrm{j}\omega B\mathrm{e}^{\mathrm{j}\omega t}\} = -\omega|B|\sin(\omega t + \varphi_B), \tag{1.9.20}$$

and this is seen to agree with the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\cos(\omega t + \varphi_B) = -\omega\sin(\omega t + \varphi_B). \tag{1.9.21}$$

In a similar manner we find the displacement,

$$\hat{\xi} = \frac{\hat{v}}{\mathrm{j}\omega}, \tag{1.9.22}$$

which means that

$$\xi = \mathrm{Re}\{\hat{\xi}\} = \mathrm{Re}\left\{\frac{1}{\mathrm{j}\omega}B\mathrm{e}^{\mathrm{j}\omega t}\right\} = \frac{1}{\omega}|B|\sin(\omega t + \varphi_B), \tag{1.9.23}$$

in agreement with the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\omega}\sin(\omega t+\varphi_B)\right)=\cos(\omega t+\varphi_B). \tag{1.9.24}$$

Acoustic second-order quantities involve time averages of squared harmonic signals and, more generally, products of harmonic signals. Such quantities are dealt with in a special way, as follows. Expressed in terms of the complex pressure amplitude $\hat{p}$, the mean square pressure becomes

$$\overline{p^2}=p_{\mathrm{rms}}^2=|\hat{p}|^2/2, \tag{1.9.25}$$

in agreement with the fact that the average value of a squared cosine is ½. Note that it is the squared *magnitude* of $\hat{p}$ that enters into the expression, not the square of $\hat{p}$, which in general would be a complex number proportional to $\mathrm{e}^{2\mathrm{j}\omega t}$.

The time average of a product is given by the following expression

$$\overline{xy}=\frac{1}{2}\mathrm{Re}\left\{\hat{x}\hat{y}^*\right\}=\frac{1}{2}\mathrm{Re}\left\{\hat{x}^*\hat{y}\right\}. \tag{1.9.26}$$

This can be seen as follows,

$$\frac{1}{2}\mathrm{Re}\left\{\hat{x}\hat{y}^*\right\}=\frac{1}{2}\mathrm{Re}\left\{|\hat{x}|\mathrm{e}^{\mathrm{j}(\omega t+\phi_x)}|\hat{y}|\mathrm{e}^{-\mathrm{j}(\omega t+\phi_y)}\right\}=\frac{1}{2}|\hat{x}||\hat{y}|\cos(\phi_x-\phi_y), \tag{1.9.27}$$

which is seen to in agree with

$$\overline{xy}=\overline{|\hat{x}|\cos(\omega t+\varphi_x)|\hat{y}|\cos(\omega t+\varphi_y)}=\frac{1}{2}|\hat{x}||\hat{y}|\cos(\varphi_x-\varphi_y). \tag{1.9.28}$$

Note that either $\hat{x}$ or $\hat{y}$ must be conjugated.½

2 Ear, Hearing and Speech

Torben Poulsen

2.1 Introduction

The aim of the present chapter is to give the student a basic understanding of the function of the ear, the perception of sound and the consequences for speech understanding. The content covers the basic psychoacoustic aspects of a situation where two persons speak to each other. The major topics are: the ear and its functional principles, basic psychoacoustics (hearing threshold, loudness, masking) and speech intelligibility.

2.2 The Ear

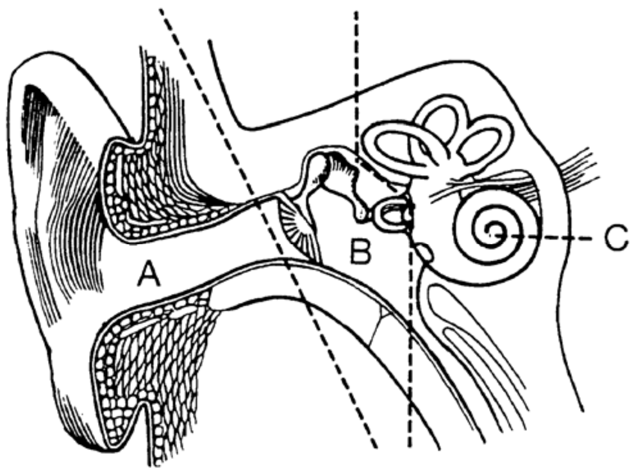


Figure 2.2.1 Drawing of the ear. A is the outer ear. B is the middle ear. C is the inner ear. From [1]

The ear may be divided into four main parts: The outer ear, the middle ear, the inner ear and the nerve connection to the brain. The first three parts (the peripheral parts) are shown in Figure 2.2.1. Part A being the outer ear, B is the middle ear and C is the inner ear. The sound will reach the outer ear, progress through the outer ear canal, reach the tympanic membrane (the ear drum), transmit the movements to the bones in the middle ear, and further transmit the movements to the fluid in the inner ear. The fluid movements will be transformed to nerve impulses from the hair cells in the inner ear and the impulses are transmitted to the brain through the auditory nerve.

2.2.1 The outer ear

The outer ear consists of the pinna (or the auricle) and the ear canal. The Pinna plays an important role for our localisation of sounds sources. The special shape of pinna produces reflections and diffraction so that the signal that reaches the ear will be dependent on the direction to the sound. The pinna has common features from person to person but there are big individual differences in the details. Localisation of sound sources is difficult if a hearing protector or a crash helmet covers the pinna. The outer part of the ear canal is relatively soft whereas the inner part is stiff and bony. At the end of the ear canal the tympanic membrane is situated. The length of the ear canal is approximately 25 mm and the diameter is approximately 7 mm. The area is approximately 1 cm^2. These numbers are approximate and vary from person to person.

The ear canal may be looked upon as a tube that is closed in one end and open in the other. This will give resonances for frequencies where the length of the ear canal corresponds to 1/4 of the wavelength of the sound. With a length of 25 mm and a speed of sound of 340 m/s the resonance frequency will be

$$f_{res} = \frac{340\, m/s}{4 * 0{,}025\, m} \quad = 3{,}4\, kHz$$

This calculation is correct if the ear canal was a cylindrical tube. Most ear canals will have one or two bends. This implies that it is usually not possible from the outside to see the tympanic membrane at the end of the ear canal. It’s necessary to make the canal straighter, which may be done by pulling pinna upward and backwards.

The tympanic membrane is found at the end of the canal. The membrane is not perpendicular to the axis of the ear canal but tilted approx. 30 degrees. The tympanic membrane is shaped like a cone with the top of the cone pointing inwards into the middle ear. The thickness is approx. 0.1 mm.

2.2.2 The middle ear

The middle ear consists of three small bones: hammer, anvil and stirrup. The Latin names are also often used: Malleus, Incus and Stapes. These bones are the smallest bones in the human body. A drawing is shown in Figure 2.2.2. The function of the middle ear is to transmit the vibrations of the tympanic membrane to the fluid in the inner ear. From Figure 2.2.2 it is seen that the hammer (Maleus, M) is fixed to the tympanic membrane (1) from the edge and into the centre of the membrane (the top of the cone). The anvil (Incus, I, 2) connects the hammer and the stirrup (Stapes, S) and the footplate of the stirrup makes the connection into the inner ear. This connection is sometimes called the oval window. The footplate rotates around the point marked (3). The middle ear is filled with air and is connected to the nose cavity (and thus the atmospheric pressure) through The Eustachian tube (ET, 4). The fluid in the inner ear is incompressible and an inwards movement of the stirrup will be equalised by a corresponding outward movement by the round window (5).

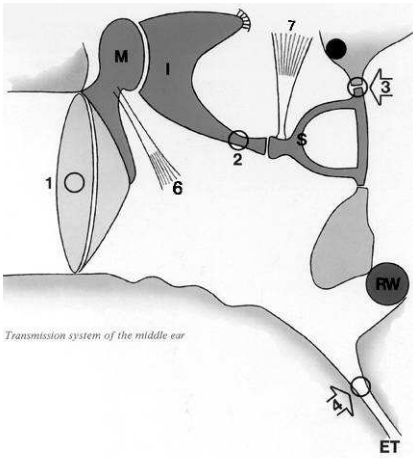


Figure 2.2.2 Drawing of the middle ear. See text for details. From [2]

Usually the Eustachian tube is closed but opens up when you swallow or yawn. When the tube is open, the pressure at the two sides of the tympanic membrane is equalised. If the Eustachian tube becomes blocked (which is typically the case when you catch a cold) the equalisation will not take place and after some time the oxygen in the middle ear will be assimilated by the tissue and an under-pressure will build up in the middle ear. This causes the tympanic membrane to be pressed inwards and thus the sensitivity of the hearing is reduced.

The chain of middle ear bones forms a lever function that - together with the area ratio between the tympanic membrane and the footplate of stapes - makes an impedance match between the air in the outer ear and the liquid in the inner ear. The lever ratio is approx. 1.3 and the area ratio is approx. 14. The total ratio is thus 18, which corresponds to approx. 25 dB.

Two small muscles, tensor tympani (6) and stapedius (7), see Figure 2.2.2, are attached to the bones and will be activated by the so-called middle ear reflex. The reflex is elicited when the ear is exposed to sounds above approx. 70 dB SPL whereby the transmission through the middle ear is reduced. The reduction is about 20 dB at 125 Hz, 10 dB at 1000 Hz and less than 5 dB at frequencies above 2000 Hz. The middle ear reflex can to some extent protect the inner ear from excessive exposure. Because the reflex is activated by a signal from the brain there will be a delay of about 25 to 150 ms before the effect is active. The reflex has therefore no protective effect on impulsive sounds.

2.2.3 The inner ear

The inner ear consists of a snail-shell shaped structure in the temporal bone called Cochlea. The cochlea is filled with lymph and is closely connected to the balance organ that contains the three semicircular canals. There are 2.75 turns in the snail shell and the total length from the base to the top is 32 mm. A cross section of one of the turns is shown in Figure 2.2.3.

This figure shows that the cochlea is divided into three channels (latin: Scala) called scala vestibuli (1), scala media (2), and scala tympani (3).

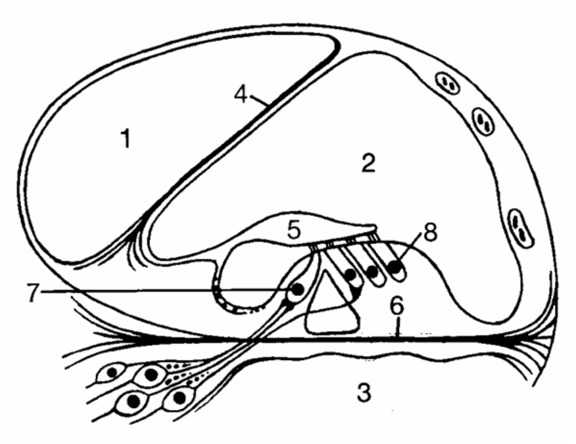


Figure 2.2.3 Cross section of a cochlea turn. See text for details. From [1]

There are two connections (windows) from cochlea to the middle ear cavity. The oval window is the footplate of the stirrup and is connected to Scala Vestibuli (1). The round window is connected to Scala Tympani (3). The round window prevents an over-pressure to build up when the oval window moves inwards. Scala Vestibuli and Scala Tympani are connected at the top of the cochlea with a hole called Helicotrema.

The Basilar membrane (6 in Figure 2.2.3) divides scala tympani from scala media. The width of the basilar membrane (BM) changes from about 0.1 mm at the base of the cochlea to about 0.5 mm at the top of the cochlea (at helicotrema). The change of the BM-width is thus the opposite of the width of the snail shell. The function of the BM is very important for the understanding of the function of the ear.

A structure - called the organ of Corti - is positioned on top of the Basilar Membrane in Scala Media. The organ of Corti consists of one row of inner hair cells (7 in Figure 2.2.3) and three rows of outer hair cells (8 in Figure 2.2.3). The designations ‘inner’ and ‘outer’ refer to the centre axis of the snail shell which is to the left in Figure 2.2.3. The hair cells are special nerve cells where small hairs protrude from the top of the cells. There are approx. 3000 inner hair cells and about 12000 outer hair cells. A soft membrane (5 in Figure 2.2.3) covers the top of the hair cells. The organ of Corti transforms the movements of the Basilar membrane to nerve impulses that are then transmitted to the hearing centre in the brain.

The inner hair cells are the main sensory cells. Most of the nerve fibres are connected to the inner hair cells. When sound is applied to the ear, the basilar membrane and the organ of Corti will vibrate and the hairs on the top of the hair cells will bend back and forth. This will trigger the (inner) hair cells to produce nerve impulses.

The outer hair cells contain muscle tissue and these cells will amplify the vibration of the basilar membrane when the ear is exposed to weak sounds so that the vibrations are big enough for the inner hair cells to react. The amplification function of the outer hair cells is nonlinear which means that they have an important effect at low sound levels whereas they are of almost no importance at high sound levels. The amplifier function - sometimes called the cochlear amplifier - is destroyed if the ear is exposed to loud sounds such as gunshots or heavy industrial noise. This is called a noise induced hearing loss. The amplifier function also deteriorates with age. This is called an age related hearing loss.

2.2.4 The frequency analyzer at the Basilar membrane

The basilar membrane acts like a frequency analyser. When the ear is exposed to a pure tone the movement of the basilar membrane will show a certain pattern and the pattern is connected to a certain position on the basilar membrane. If the frequency is changed, the pattern will *not* change but the position of the pattern will move along the basilar membrane. This is illustrated in Figure 2.2.4 for the frequencies 400 Hz, 1600 Hz and 6400 Hz. The 400 Hz component produce BM-movement close to the top of the cochlea. 6400 Hz produces a similar pattern but close to the base of the cochlea. Note that a *single* frequency produces movements of the basilar membrane over a broad area. This means that even for a single frequency many hair cells are active at the same time. Note also that the deflection of the BM is asymmetrical. The envelope of the deflection (shown dotted in Figure 2.2.4) has a steep slope towards the low frequency side and a much less steep slope towards the high frequency side. The same different slopes are also found in masking thresholds and it can be shown that masking is closely related to the basilar membrane movements.

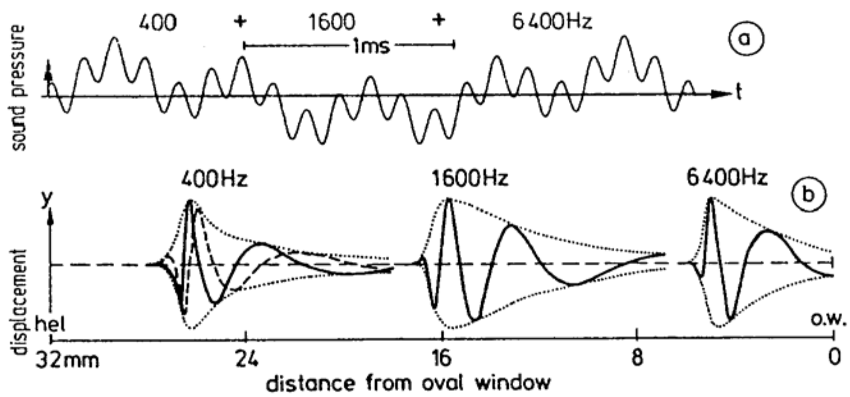


Figure 2.2.4 Movement of the basilar membrane (b) when the ear is exposed to a combination of 400 Hz, 1600 Hz and 6400 Hz (a). O.W.: Owal window (base of cochlea). Hel: Helicotrema (top of cochlea). From [3]

The non-linear behaviour of the outer hair cells and their influence on the BM movement is illustrated in Figure 2.2.5. This figure shows the BM-amplitude at a certain position of the basilar membrane as a function of the stimulus frequency. (Note that this is different from Figure 2.2.4 where the amplitude is shown as a function of basilar membrane position for different frequencies). There are at least three nonlinear phenomena illustrated in the figure.

1) At low exposure levels (20 dB) the amplitude is very selective and a 'high' amplitude is achieved only in a very narrow frequency range. For high exposure levels (80 dB) the

'high' amplitude is achieved at a much wider frequency range. Thus, the filter bandwidth of the auditory analyser changes with the level of the incoming sound.

2) The frequency where the maximum amplitude is found change with level. At high levels it is almost one octave below the max-amplitude frequency at low levels.

3) The maximum amplitude grows non-linearly with level. At low levels (20 dB) the maximum BM-amplitude is about 60 dB (with some arbitrary reference). At an input level of 80 dB the maximum BM amplitude is about 85 dB. In other words the change in the outside level from 20 dB to 80 dB, i.e., 60 dB, is reduced (compressed) to a change in the maximum BM-amplitude of only 25 dB.

These non-linear phenomena are caused by the function of the outer hair cells. The increase of amplitude at low levels is sometimes called 'the cochlear amplifier'. In a typical cochlear hearing loss, the outer hair cells are not functioning correctly or may be destroyed. In other words: The cochlear amplifier does not work. This will be seen as an elevated hearing threshold and this is called a hearing loss.

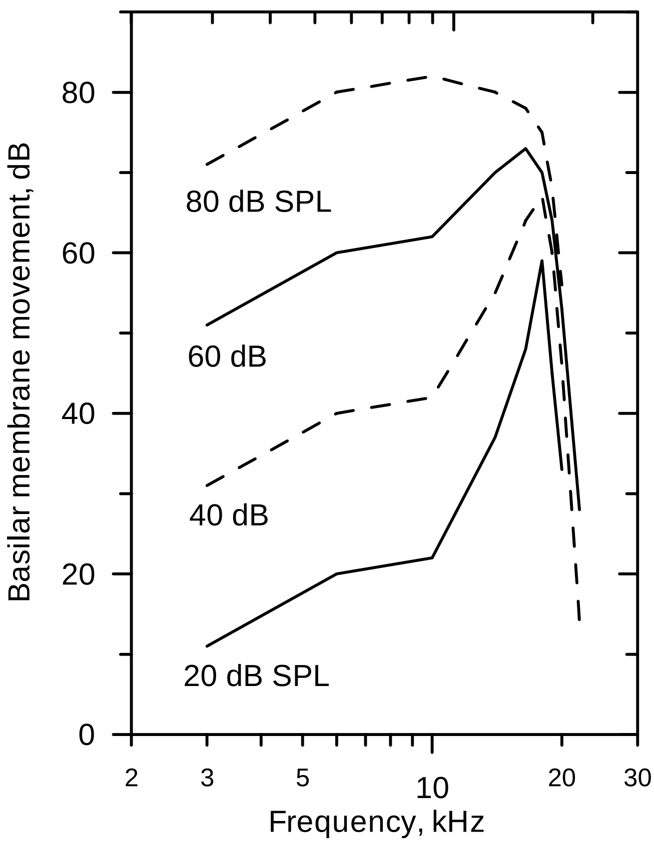


Figure 2.2.5 Movement of the Basilar membrane at a fixed point for stimulus levels from 20 dB SPL to 80 dB SPL. Redrawn from [4]

2.3 Human hearing

The human hearing can handle a wide range of frequencies and sound pressure levels. The weakest audible sound level is called the hearing threshold and the sound level of the loudest sound is called the threshold of discomfort or the threshold of pain.

2.3.1 The hearing threshold

The hearing threshold is frequency dependent, see Figure 2.3.1. At 1000 Hz the threshold is about 2 dB SPL whereas it is about 25 dB SPL at 100 Hz and about 15 dB at 10000 Hz.

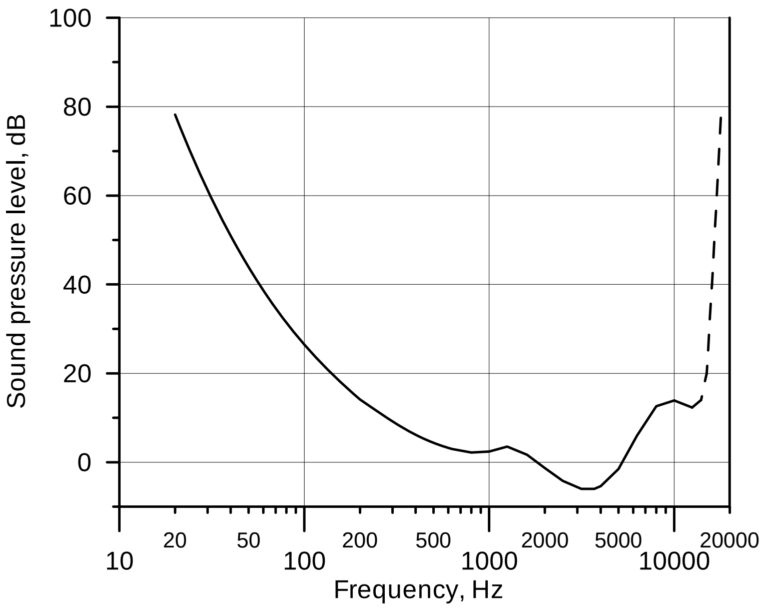


Figure 2.3.1 The binaural hearing threshold in a free field. From [5]

The threshold curve in Figure 2.3.1 is measured under the following conditions:

- Free field (no reflections from walls, floor, ceiling)
- Frontally incoming sound (called frontal incidence)
- signals are single pure tones
- binaural listening (i.e. listening with both ears)
- no background noise
- test subjects between 18 and 25 years of age
- the threshold is determined by means of either the ascending or the bracketing method

The curve is the median value (not the mean) over the subject's data. The sound pressure level, which is shown in the figure, is the level in the room at the position of the test subject's head but measured *without* the presence of the test subject. This curve is also called the absolute threshold (in a free field) and data for the curve may be found in ISO 389-7 [6] and in ISO 226 [5].

In ISO 389-7 also threshold data for narrow band noise in a diffuse sound field are found. The threshold curve is similar to the curve in Figure 2.3.1 and deviates from the pure tone curve only by a few dB (-2 to +6) in the frequency range 500 Hz to 16 kHz.

2.3.2 Audiogram

For practical use it is not convenient to measure the hearing threshold in a free or a diffuse sound field in the way described in the previous section. For practical and clinical purposes, usually only the deviation from normal hearing is of interest. Such deviations are determined by means of a calibrated audiometer and the result of the measurement is called an audiogram.

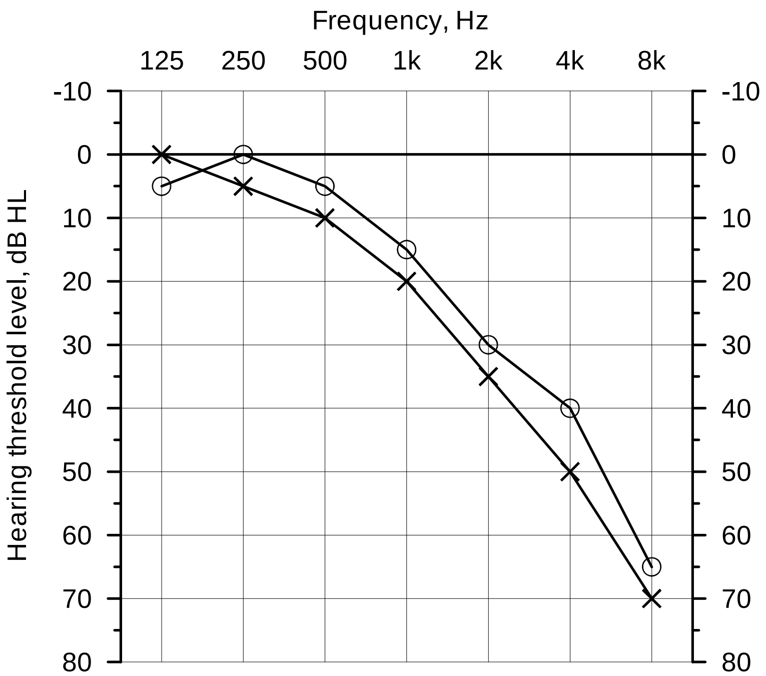


Figure 2.3.2 Audiogram for a typical age related hearing loss.

Figure 2.3.2 shows an audiogram for a person in the frequency range 125 Hz til 8000 Hz. The zero line indicates the average threshold for young persons and a normal audiogram will give data points within 10 to 15 dB from the zero line. An elevated hearing threshold (i.e. a hearing loss) is indicated downwards in an audiogram and the values are given in dB HL. The term ‘HL’ (hearing level) is used to emphasise that it is the *deviation* from the average normal hearing threshold.

The measurements are performed with headphones for each ear separately. The results from the left ear are shown with '×' and the results from the right ear are shown with '○'.

Sound pressure level, dB SPL, and hearing level, dB HL, is not the same. An example: From Figure 2.3.1 it can be seen that the hearing threshold at 125 Hz is 22 dB SPL (measured in the way described previously). If a person has a hearing loss of 5 dB HL at this frequency the threshold would be 27 dB SPL. In an audiogram the 5 dB hearing loss will be shown as a point 5 dB below the zero line (e.g. right ear, Figure 2.3.2). Another example: At 4000 Hz the free field threshold is -6 dB (see Figure 2.3.1). A hearing loss of 50 dB HL (e.g. left ear, Figure 2.3.2) will give a threshold of 44 dB SPL.

In order for the audiometry to give correct results, the audiometer must be calibrated according to the ISO 389 series of standards. These standards specify the SPL values that shall be measured in a specific coupler (an artificial ear) when the audiometer is set to 0 dB HL. The values in the standards are headphone specific, which means that the audiometer must be recalibrated if the headphone is exchanged with another headphone.
Table 2.3.1 shows reference values for two headphones commonly used in audiometry.

F, Hz	125	250	500	1k	2k	3k	4k	6k	8k	10k	12,5k	14k	16k
TDH 39	45,0	25,5	11,5	7,0	9,0	10,0	9,5	15,5	13,0	-	-	-	-
HDA 200	30,5	18,0	11,0	5,5	4,5	2,5	9,5	17,0	17,5	22,0	28,0	36,0	56,0

Table 2.3.1. Calibration values in dB SPL for a Telephonics TDH 39 earphone and a Sennheiser HDA 200 earphone. The TDH 39 earphone can not be used above 8 kHz. The TDH 39 data are from ISO 389-1 [7]. The HDA 200 data are from ISO 389-5 [8] and ISO 389-8 [9].

2.3.3 Loudness Level

The definition of loudness levels is as follows: For a given sound, A, the loudness level is defined as the sound pressure level (SPL) of a 1000-Hz tone which is perceived equally loud as sound A. The unit for loudness level is Phon (or Phone). In order to measure loudness level a 1 kHz tone is needed and this tone should then be adjusted up and down in level until it is perceived just as loud as the other sound. When this situation is achieved, the sound pressure level of the 1 kHz tone is per definition equal to the loudness level in phone. For a 1000-Hz tone the value in dB SPL and in Phone will be the same.

The loudness level for pure tones has been measured for a great number of persons with normal hearing under the same conditions as for the absolute threshold (Figure 2.3.1). The result is shown in Figure 2.3.3.

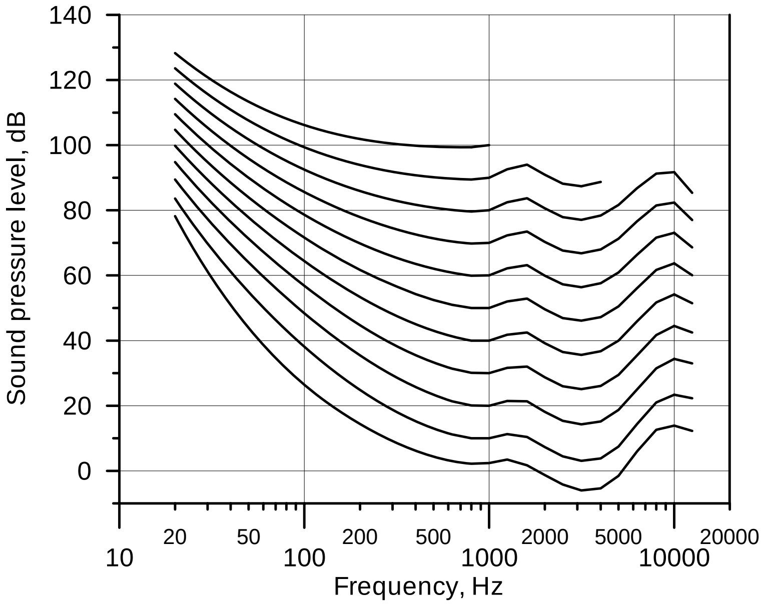


Figure 2.3.3 Equal loudness level contours. Redrawn from [5]

Some examples, see Figure 2.3.3: A 4000-Hz tone at 26 dB SPL will be perceived with the same loudness as a 1000-Hz tone at 30 dB SPL and thus the loudness level of the 4000 Hz tone is 30 Phone. A 125-Hz tone at 90 dB SPL will have a loudness level of 80 Phone.

The curves in Figure 2.3.3 are - in principle - valid only for the special measurement situation where the tones are presented one at a time. They should not be used directly to predict the perception of more complicated signals such as music and speech because the curves do not take masking and temporal matters into account. Reflections in a room are not taken into account either.

Translations of Loudness Level:

Danish:	Hørestyrkeniveau (enhed: Phon)
German:	Lautstärkepegel (Einheit: Phon)
French:	Niveau de Sonie.

2.4 Masking

The term 'Masking' is used about the phenomenon that the presence of a given sound (sound A) can make another sound (sound B) inaudible, in other words A masks B or B is masked by A. Masking is a very common phenomenon which is experienced almost every day, e.g. when you need to turn down the radio in order to be able to use the telephone.

The situation described above is also called simultaneous masking because both the masking signal and the masked signal are present at the same time. This is not the case in backward and forward masking. Backward and forward refer to time. E.g. forward masking means masking after a signal has stopped (i.e. forward in time). Simultaneous masking is best described in the frequency domain and is closely related to the movements of the Basilar membrane in the inner ear.

The masking phenomenon is usually investigated by determining the hearing threshold for a pure tone when various masking signals are present. The threshold determined in this situation is called the masked threshold contrary to the absolute threshold.

2.4.1 Complete masking

If the ear is exposed to white noise, the hearing threshold (i.e. masked threshold) will be as shown in Figure 2.4.1 where also the absolute threshold is shown. The masked threshold is shown for different levels of the white noise.

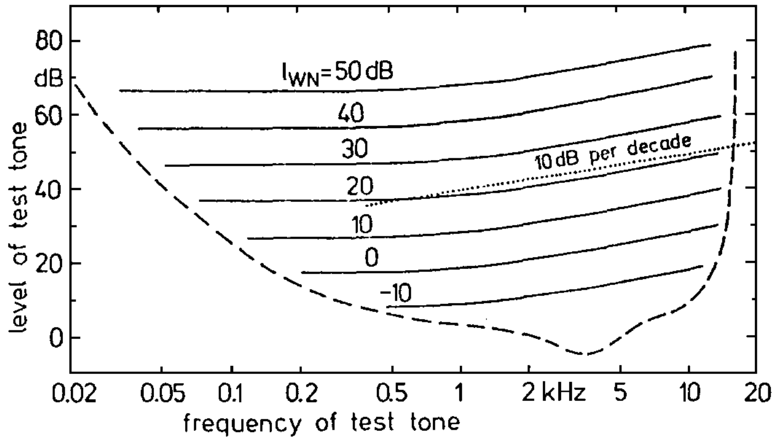


Figure 2.4.1 Masking from white noise. The curves show the masked threshold for different spectrum levels of white noise. From [3]

The masked thresholds are almost independent of frequency up to about 500 Hz. Above 500 Hz the threshold increases by about 10 dB per decade (= 3 dB/octave). A 10-dB change in the level of the noise will also change the masked threshold by 10 dB.

If a narrow band signal is used instead of the white noise, the masked threshold will be as shown in Figure 2.4.2. Here the masked threshold is shown for a narrow band signal centred at 250 Hz, 1 kHz and 4 kHz respectively. Generally the masking curves have steep slopes (about 100 dB/octave) towards the low frequency side and less steep slopes (about 60 dB/octave) towards the high frequency side.

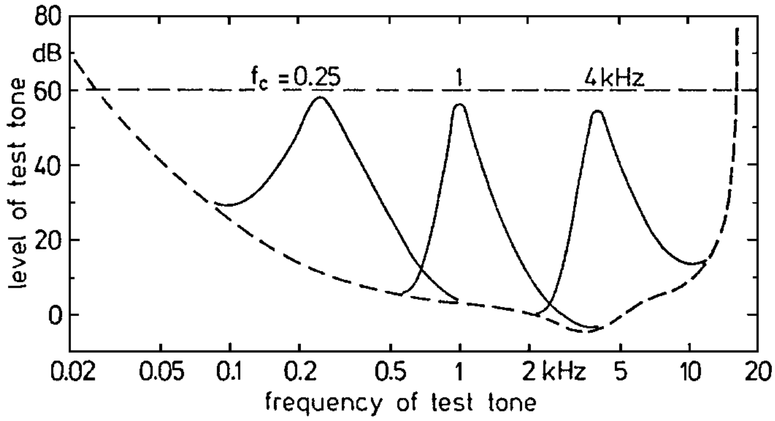


Figure 2.4.2 Masking from narrow band noise. The curves show the masked threshold when the ear is exposed to narrow band noise (critical band noise) at 250 Hz, 1 kHz and 4 kHz respectively. From [3]

The masking curves for narrow band noise are very level dependent. This is illustrated in Figure 2.4.3. The slope at the low frequency side is almost independent of level but the slope at the high frequency side depends strongly on the level of the narrow band noise. The dotted lines near the top of the curves indicate experimental difficulties due to interference between the noise itself and the pure tone used to determine the masked threshold.

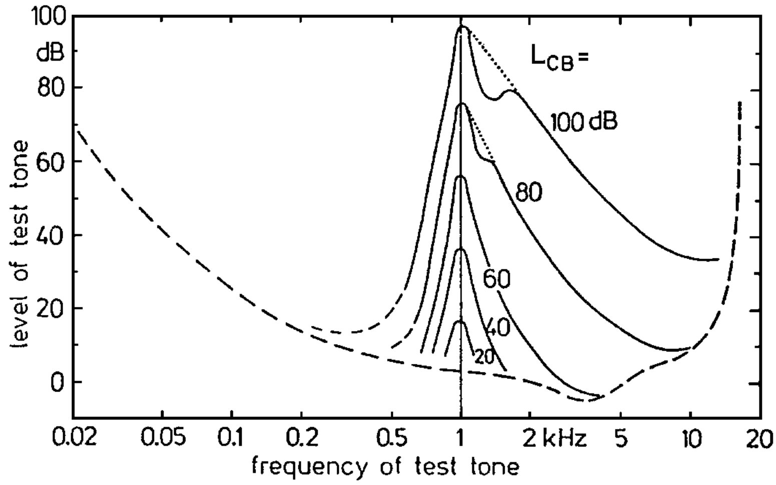


Figure 2.4.3 The influence of level on the masked threshold. The slope towards higher frequencies decreases with increasing level, i.e. masking increases non-linearly with level. From [3]

The masked threshold for narrow band noise is mainly caused by the basilar membrane motion. The different slopes towards the low and the high frequency side are also seen here and also the nonlinear level dependency is seen. Compare with Figure 2.2.4.

2.4.2 Partial masking

The term 'Complete masking' is used when the presence of a given sound (sound A) can make another sound (sound B) inaudible. Partial masking is a situation where sound A influences the perception of sound B even though sound B is still audible. The influence is mainly seen in the loudness of sound B.

An example: When you listen to a standard car-radio while you are driving at, e.g. 100 km/h, you will adjust the level of the radio to a comfortable level. There will be some background noise from the engine, the tires, and the wind around the car (at least in ordinary cars). Then, when you come to a crossing or a traffic light and have to stop you will hear that the radio-volume is much too high. This is an example of partial masking where the background noise masks part of the radio signal and when the background noise disappears the masking disappears too and the radio signal becomes louder than before. (Some modern car radios are equipped with a speed dependent automatic level control. The example above is therefore not fully convincing in this situation.)

2.4.3 Forward masking

It has been shown that a strong sound signal can mask another (weak) signal which is presented after the strong signal. This kind of masking goes forward in time and is therefore called forward masking. The effect lasts for about 200 ms after the end of the strong signal.

Forward masking is also called post-masking.

2.4.4 Backward masking

It has been shown that a strong sound signal can mask another (weak) signal which appears *before* the strong signal. This kind of masking goes back in time and is therefore called backward masking. The effect is restricted to about 20 ms before the start of the strong signal.

Backward masking is also called pre-masking.

2.5 Loudness

The term ‘loudness’ denotes the subjective perception of strength or powerfulness of the sound signal. The unit for loudness is Son or Sone. Note that ‘loudness’ and ‘loudness level’ are two different concepts. Translation of terms:

	Loudness	**Loudness Level**
Danish	Hørestyrke	Hørestyrkeniveau
German	Lautheit	Lautstärkepegel
French	Sonie	Niveau de Sonie

2.5.1 The loudness curve

The Sone scale was established in order to avoid the confusion between dB SPL values and the perception of loudness: A 1 kHz tone at 80 dB SPL is *not* perceived double as loud as the same tone at 40 dB SPL. Figure 2.5.1 shows the relation between the Sone and the Phone scales. (Hint: for a 1 kHz tone, phone and dB SPL is the same number). Arbitrarily it has been decided that *one sone should correspond to 40 phones*. The curve is based on a great number of loudness comparisons. The curve is called a loudness curve.

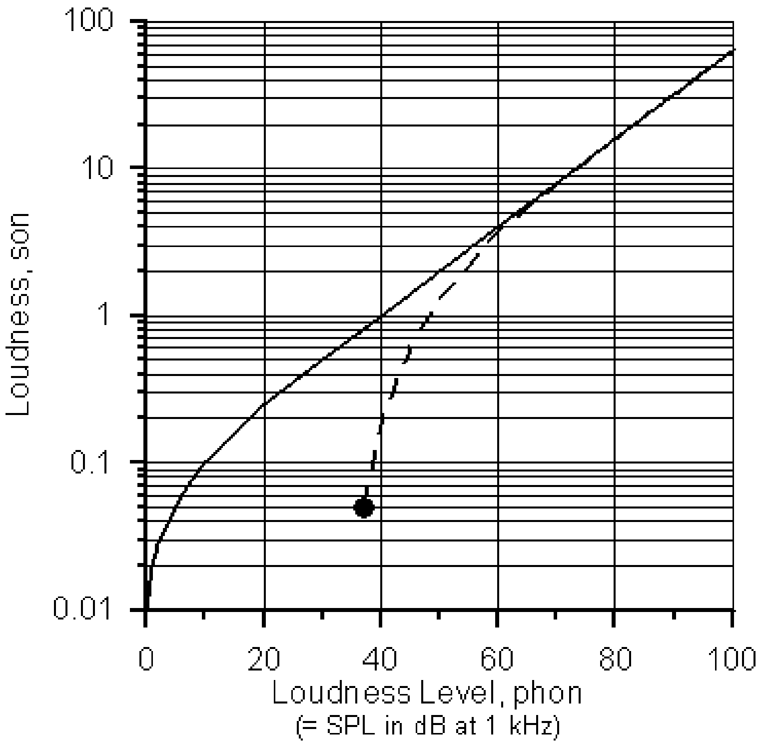


Figure 2.5.1 The loudness curve for a normal hearing test subject (solid line) and for a person with a cochlear hearing loss (dashed)

The straight part of the solid line in Figure 2.5.1 corresponds to Stevens' power law:

$$N = 2^{(L-40)/10}$$

where N is the loudness (in sone) and L is the loudness level (in phones). The curve shows that a doubling of the loudness corresponds to a 10-phone increase in loudness level (or a 10-dB increase in SPL if we are dealing with a 1 kHz tone). For many daily life sounds a rule of thumb says that a 10-dB increase is needed in order to perceive a doubling of the loudness.

The loudness curve becomes steeper near the hearing threshold. This is also the case for a person with a cochlear hearing loss (e.g., the very common hearing impairment caused by age). An example of such a hearing loss is shown by the dashed curve in Figure 2.5.1 where the threshold (1 kHz) is a little less than 40 dB SPL. The steeper slope means that - near the threshold - the loudness increases rapidly for small changes in the sound level. This effect is called loudness recruitment. Recent research have shown that – for this kind of hearing loss – the loudness at threshold has a value significantly different from nil as indicated in the figure [10]. In other words, listeners with cochlear hearing loss have *softness imperception*, rather than loudness recruitment. Note that at higher sound levels the loudness perception is the same for both normal and impaired listeners.

2.5.2 Temporal integration

The perception of loudness needs some time to build up. This means that short duration sounds (less than one second) are perceived as less loud than the same sound with longer duration. The growth of loudness as a function of duration is called temporal integration. The growth resembles the exponential growth of a time constant. It has been shown that the time constant is about 100 ms.

Short sounds - like a pistol shot, fireworks, handclap, etc. - are perceived as weak sounds although their peak sound pressure levels may be well above 150 dB SPL. This is one of the reasons why impulsive sounds generally are more dangerous than other sounds.

2.5.3 Measurement of loudness

Many years ago it was thought that a sound level meter with filters corresponding to the ears' sensitivity (described by the equal loudness level contours (Figure 2.3.3)) could be used to measure loudness. This is not the case.
Figure 2.5.2 show the characteristics for the commonly used A- and C- filters, but due to masking and other phenomena these filters will not give a result that corresponds to loudness. For the determination of loudness, special calculation software is needed. For stationary sounds two procedures can be found in [11]. For non-stationary sound, loudness calculations are found in professional Sound Quality calculation software. For research purposes loudness models (software) can be found on the Internet (e.g. at http://hearing.psychol.cam.ac.uk/Demos/demos.html)

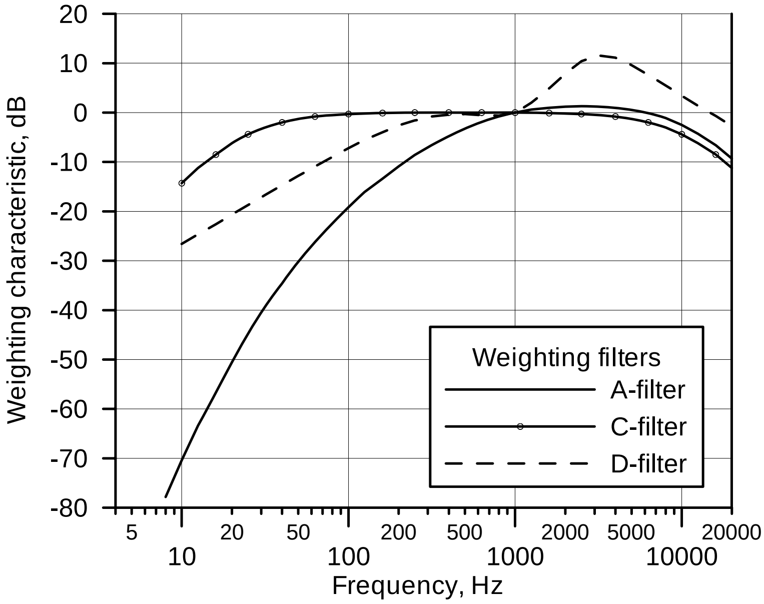


Figure 2.5.2 Filter characteristics for the A, C and D filter. The data for the A and the C filter are from [12]. The data for the D filter is from [13].

The main effect of the A-filter is that it attenuates the low frequency part of the signal. The attenuation is e.g. 20 dB at 100 Hz and 30 dB at 50 Hz. Wind noise and other low frequency components are attenuated by the A-filter and is therefore very practical for many noise measurement situations.

The C-filter is 'flat' in the major part of the audible frequency range. It may me used as an approximation to a measurement with linear characteristic.

The D-filter is mainly used in connection with evaluation of aircraft noise. The frequency range around 3 kHz is known to be annoying and therefore this frequency range is given a higher weight.

2.6 The auditory filters

The movements of the basilar membrane in the inner ear constitute a frequency analyser where the peak of the envelope moves along the basilar membrane as a function of frequency. See Figure 2.2.4. The width of the envelope peak may be seen as an indication of the selectivity of the analyser filter and it has been common practice to describe the frequency selectivity of the ear as a set of filters, a filter bank, which cover the audible frequency range. It should be noted though that the concept of a filter bank is a very coarse description and should be seen as a typical engineering approximation to the real situation.

Frequency selectivity is important for the perception of the different frequencies in complex sound signals such as speech and music. We rely e.g. on our frequency selectivity when we distinguish different vowels from each other.

The concept of frequency discrimination is different from frequency selectivity. Frequency discrimination is the ability to hear the difference between two tones that are close in frequency (one frequency at a time).

2.6.1 Critical bands

The bandwidth of the filters in the filter bank can be determined by means of various psychoacoustic experiments. Many of these are masking experiments and led to the formulation of the *critical band* model. It is outside the scope of the present text to go into the background and the details of this model.

The results of the investigations are shown in Figure 2.6.1. It is seen that the bandwidth (Critical Bands) is almost constant at 100 Hz up to a centre frequency of about 500 Hz and above this frequency the bandwidth increases. The increase in bandwidth above 500 Hz is similar to the increase in bandwidth for one-third-octave filters.

The critical bandwidth may be calculated from the empirical formula:

$$CB = 25 + 75\,(1 + 1{,}4\,f^{\,2})^{0{,}69}$$

where *CB* is the bandwidth in Hz of the critical band and *f* is the frequency in kHz (not in Hz).

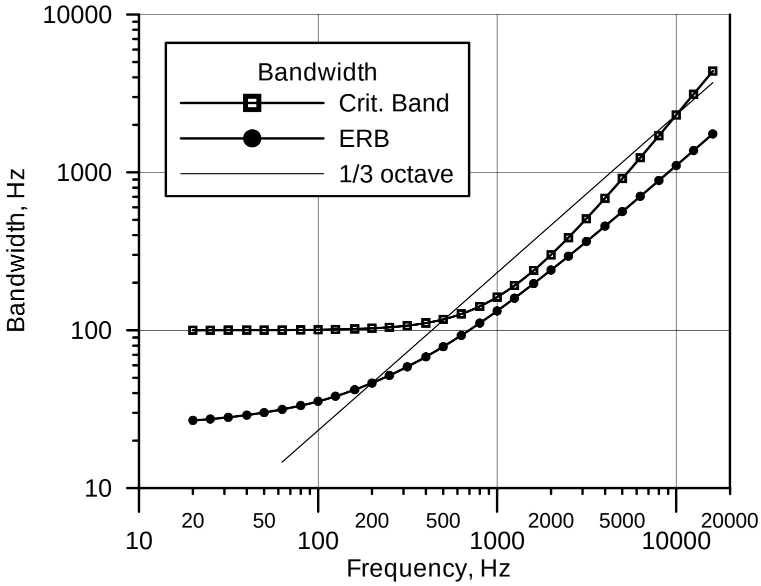


Figure 2.6.1 Bandwidth of critical bands and Equivalent Rectangular bandwidth, ERB. The bandwidth of 1/3-octave filters (straight line) is shown for comparison. The curves are computed from the formulas given in the text.

If the audible frequency range is 'filled up' with consecutive critical bands from the lowest frequency to the highest frequency, it is seen that 24 critical bands will cover the whole frequency range. Each of the 'filters' has been given a number called Bark. Bark number one is the band from zero to 100 Hz; Bark number two is the band from 100 Hz to 200 Hz, etc. Band no. 8 has a centre frequency of 1000 Hz and goes from 920 Hz to 1080 Hz. The band around 4000 Hz is no. 17 and has a bandwidth of 700 Hz.

The critical bands are not fixed filters similar to the filters in a physical filter bank as the numbers given above may indicate. The critical bands are a result of the incoming sound signal and as such much more 'flexible' than physical filters would be.

2.6.2 Equivalent Rectangular Bands

The auditory filters have also been determined by means of notched noise measurements where the threshold of a pure tone is determined in the notch of a broadband noise as a function of the width of the notch. This leads to the concept of equivalent rectangular bandwidth, i.e. the bandwidth of a rectangular filter that transmits the same amount of energy as the auditory filter. The bandwidth of such rectangular filters is shown in Figure 2.6.1 as a function of centre frequency.

The rectangular bandwidth may be calculated from the empirical formula:

$$ERB = 24{,}7\,(4{,}37\,f + 1)$$

where *ERB* is the bandwidth in Hz and *f* is the centre frequency in kHz.

2.7 Speech

A speech signal is produced in the following way. Air is pressed from the lungs up through the vocal tract, through the mouth cavities and/or the nose cavities and the sound is radiated from the mouth and the nose. The vocal folds will vibrate when voiced sounds are produced.

2.7.1 Speech production

A schematic illustration of the production of voiced sounds is given in Figure 2.7.1 where the vocal folds vibrate. The source spectrum is a line spectrum where the distance between the lines corresponds to the fundamental frequency. The fundamental frequency is around 125 Hz for men, around 250 Hz for woman and around 300 for children, but there are big individual variations. There are thus more lines in a male spectrum compared to a female. The source spectrum decreases with the square of the frequency ($1/f^2$). The source spectrum is transformed by the 'tube' consisting of trachea, throat (pharynx) and the mouth. This structure is simulated in Figure 2.7.1 by a cylindrical tube of length 17 cm.

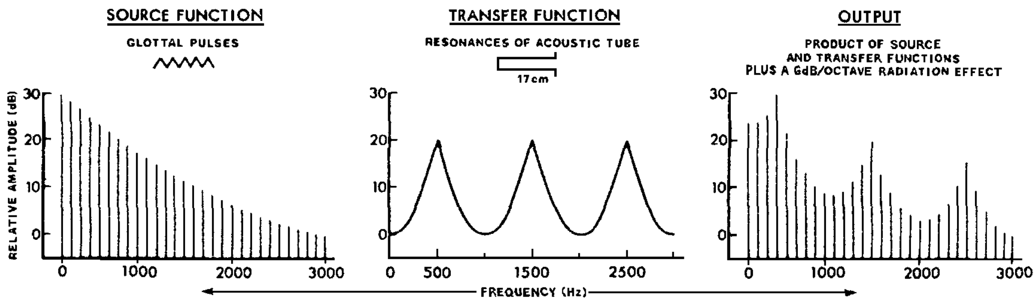


Figure 2.7.1 The principle of vowel generation. From [14]

The tube has pronounced resonances (where the length of the tube corresponds to the odd multiples of 1/4 wavelength) indicated by the peaks at 500, 1500 and 2500 Hz. The final spectrum radiated from the mouth is then the product of the two spectra. The final spectrum is a line spectrum with characteristic peaks caused by the transfer function. The peaks are called *formants* and the formants are positioned differently for each vowel. Table 2.7.1 shows the formants frequencies (in round numbers) for the three most different vowels. The sounds are /i/: as in eve, /a/ as in father, /u/ as in moon. There are individual differences from person to person.

	/i/	/a/	/u/
1. formant	225	700	250
2. formant	2200	1200	700
3. formant	3000	2500	2200

Table 2.7.1 Formant frequencies in Hz of the vowels /i/, /a/ and /u/.

The *un*voiced sounds are produced in many different ways, e.g. by pressing air out through the teeth /s/, out between the lips and the teeth /f/, by sudden opening of the lips /p/, sudden opening between tongue and teeth /t/ and between tongue and palate /k/. These sounds are called unvoiced because the vocal folds do not vibrate but stays open in order for the air to pass.

2.7.2 Speech spectrum, speech level

A general long-term speech spectrum is shown in Figure 2.7.2 that is based on the average of 18 speech samples from 12 languages.

The spectrum is a one-third octave spectrum which means that the curves are tilted 3 dB/octave compared to the result of a FFT-calculation. (The result of a FFT is a density spectrum).

It is worth to note that the speech spectrum is almost independent of the language. This is not surprising when the speech production mechanism is taken into account. The spectrum in Figure 2.7.2 is based on English (several dialects), Swedish, Danish, German, French (Canadian), Japanese, Cantonese, Mandarin, Russian, Welsh, Singhalese and Vietnamese. A total of 392 talkers participated in the investigation.

The spectrum for women falls off below 200 Hz because their fundamental frequency typically is around 250 Hz. The maximum is found around 500 Hz for both gender and above 500 Hz the two curves are almost identical. The slope above 500 Hz is approximately minus 10 dB per decade (or -3 dB/octave).

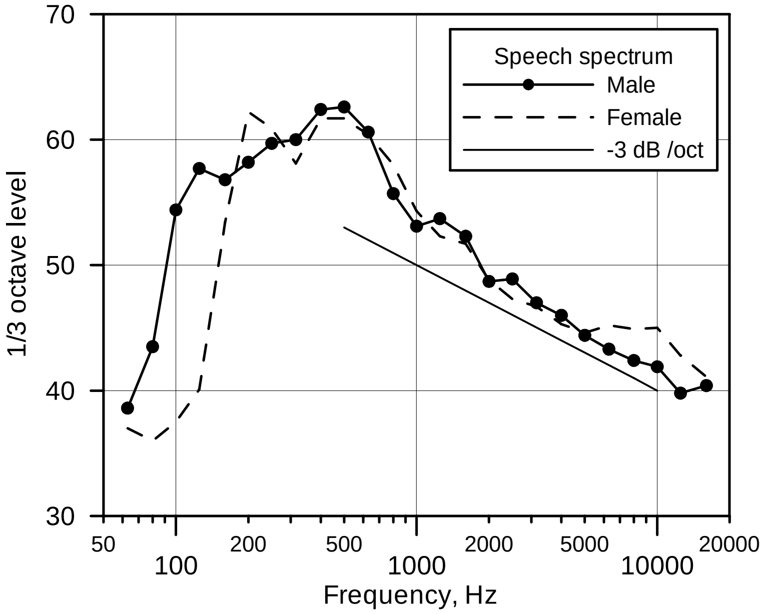


Figure 2.7.2 The long-term speech spectrum for male and female speech shown as a 1/3-octave spectrum. For comparison a line with slope –3 dB per octave (= –10 dB per decade) is shown. Redrawn from [15]

The average level of male speech is about 65 dB SPL, measured at 1 m in front of the mouth. For women the level is typically 3 dB lower, i.e. 63 dB. (Compare the number of lines in the spectrum). During normal speech the level will vary ±15 dB around the mean value.

2.7.3 Speech intelligibility

The speech intelligibility of a transmission system is usually measured by means of a list of words (or sentences) where the percentage of correctly understood words gives the intelligibility score. The transmission system could be almost anything, e.g. a telephone line or a room. The intelligibility depends on the word material (sentences, single words, numbers, etc.), the speaker, the listener, the scoring method and the quality of the transmission system.

Often the intelligibility score is given as a function of the signal-to-noise ratio. An example of this is shown in Figure 2.7.3 for the word-material on the *Dantale* CD. This CD contains eight tracks of 25 words each. The words are common Danish single-syllable words that are distributed phonetically balanced over the eight lists so that the lists can be regarded as equivalent. The words are recorded on the left channel of the CD and on the right channel a noise signal is recorded with (almost) the same spectrum as the words. The noise signal is amplitude modulated in order to make it resemble normal speech. The Dantale CD is described in [16]

The result in Figure 2.7.3 is obtained with the words and the noise on the Dantale CD with untrained Danish normal hearing listeners. It is seen that even at a signal-to-noise ratio of

0 dB almost all words are understood. It is also seen that an increase of just 10 dB in SNR can change the situation from impossible to reasonable, e.g. from -15 dB (10%) to - 5 dB (70%). It is a general finding that such a relatively small improvement of the signal-to-noise ratio can improve the intelligibility situation dramatically. In other words, if the background noise in a room is a problem for the understanding of speech in the room, then just a small reduction of the background noise will be beneficial.

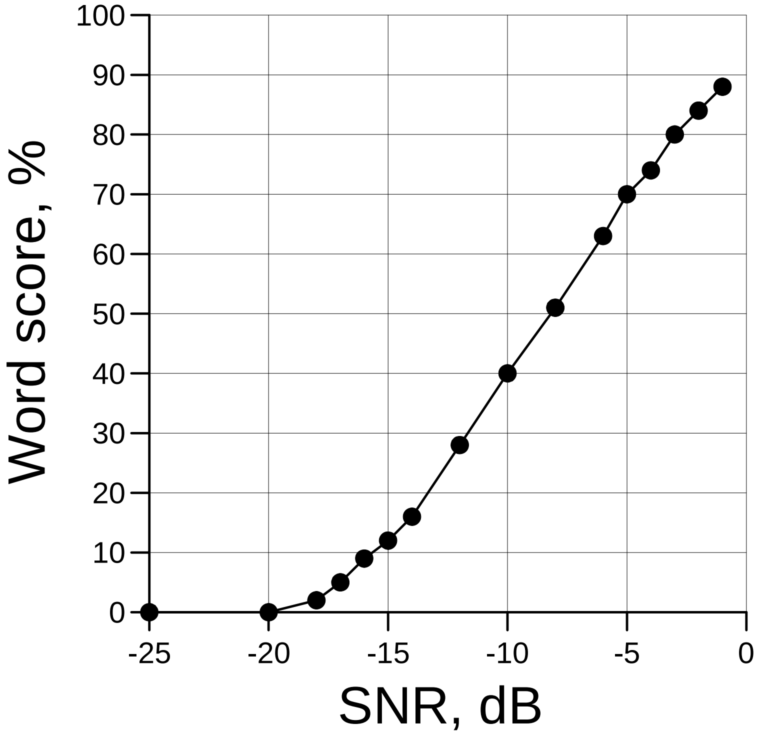


Figure 2.7.3 Word score for the speech material DANTALE as a function of speech-to-noise ratio (SNR). Redrawn from [17]

It is time consuming and complicated to measure speech intelligibility with test subjects. Therefore measurement and calculation methods have been developed for the estimation of the expected speech intelligibility in a room or on a transmission line.

Articulation Index, AI [18]: Determination of the signal-to-noise ratio in frequency bands (usually one octave or one-third octave). The SNR values are weighted according to the importance of the frequency band. The weighted values are added and the result normalised to give an index between zero and one. The index can then be translated to an expected intelligibility score for different speech materials.

Speech Intelligibility Index, SII [19]: This method is based on the AI principle, but the weighting functions are changed and a number of 'corrections' to the AI-method are implemented. One of these is the correction for the change in speech spectrum according to the vocal effort (shouting, raised voice, low voice).

Speech Transmission Index, STI [20]: In this method the modulation transfer function, MTF, from the source (the speaker) to the receiver (the listener) is determined. The MTF is determined for octave bands of noise (125 Hz to 8 kHz) and for a number of modulation frequencies (0,63 Hz to 12,5 Hz). The reduction in modulation is transformed to an equivalent signal-to-noise ratio and as in the AI method these values are added and normalised in order to yield an index between zero and one. The index can then be translated to an expected intelligibility score for different speech materials.

Rapid Speech Transmission Index, RASTI [21]: This is an abbreviated version of STI. Only the frequency bands 500 Hz and 2 kHz and only nine different modulation frequencies are used. The result is an index which is used in the same way as in STI.

2.8 References

1. Hougaard, S., et al., *Sound and Hearing*. 2 ed. 1995: Widex.

2. Engström, H. and Engström, B., *A short survey of some common or important ear diseases*. 1979: Widex.

3. Zwicker, E. and Fastl, H., *Psychoacoustics. Facts and models.* 2 ed. 1999: Springer.

4. Kemp, D.T., *Developments in cochlear mechanics and techniques for noninvasive evaluation.* Adv Audiol, 1988. 5: p. 27-45.

5. ISO-226, *Acoustics - Normal equal-loudness-level contours*, in *FDIS, May 2002, N327*. 2002, International Standardization Organization: Geneva.

6. ISO-389-7, *Acoustics - Reference zero for the calibration of audiometric equipment-Part 7: Reference threshold of hearing under free-field and diffuse-field listening conditions*. 1996, International Organization for Standardization: Geneva, Switzerland.

7. ISO-389-1, *Acoustics - Reference zero for the calibration of audiometric equipment - Part 1: Reference equivalent threshold sound pressure levels for pure tones and supra-aural earphones*. 1991, International Organization for Standardisation: Geneva, Switzerland.

8. ISO-389-5, *Acoustics - Reference zero for the calibration of audiometric equipment - Part 5: Reference equivalent threshold sound pressure levels for pure tones in the frequency range 8 kHz to 16 kHz"*. 1998, International Organization for Standardization: Geneva, Switzerland.

9. ISO-389-8, *Acoustics - Reference zero for the calibration of audiometric equipment - Part 8: Reference equivalent threshold sound pressure levels for pure tones and circumaural earphones (ISO/DIS)*. 2001, International Organization for Standardization: Geneva, Switzerland.

10. Florentine, M. and Buus, S. *Evidence for normal loudness growth near threshold in cochlear hearing loss*. in *19 Danavox Symposium*. 2001. Kolding, Denmark. p. xx-yy.

11. ISO-532, *Acoustics - Method for calculating loudness level*. 1975, International Organisation for Standardisation: Geneva, Switzerland.

12. IEC-651, *Sound level meters*. 1979, International Electrotecnical Commission: Geneva, Switzerland.

13. IEC-537, *Frequency weighting for the measurement of aircraft noise (D-weighting)*. 1976, International Electrotechnical Commission: Geneva, Switzerland.

14. Borden, G. and Harris, K., *Speech science primer*. 1980: Williams & Wilkins.

15. Byrne, D., Ludvigsen, C., and al., e., *Long-term average speech spectra ...* J. Acoust. Soc. Am., 1994. 96(no. 4): p. 2110?-2120?

16. Elberling, C., Ludvigsen, C., and Lyregaard, P.E., *DANTALE, a new Danish Speech material.* Scandinavian Audiology, 1989. 18: p. 169-175.

17. Keidser, G., *Normative data in quiet and in noise for DANTALE - a Danish speech material.* Scandinavian Audiology, 1993. 22: p. 231-236.

18. ANSI-S3.5, *American National Standard methods for the calculation of the Articulation Index*. 1969, American National Standards Institute, Inc.: New York.

19. ANSI-S3.5, *American National Standard methods for the calculation of the Speech Intelligibility Index*. 1997, American National Standards Institute, Inc.: New York.

20. Steeneken, H. and Houtgast, T., *A physical method for measuring speech-transmission quality*. J. Acoust. Soc. Am., 1980. 67: p. 318-326.

21. IEC-268-16, *Sound system equipment - Part 16: The objective rating of speech intelligibility in auditoria by the RASTI method.* 1988, International Electrotechnical Commission.

Further reading:

Plack, C. J. (2005). The sense of hearing. Lawrence Earlbaum Associates. ISBN: 0-8058-4884-3

Moore, B. C. J. (2003). An introduction to the psychology of hearing. 5th Edition. Academic press ISBN: 0-12-505628-1

Yost, W. A. (2000). Fundamentals of hearing. An introduction. 4th Edition. Academic press. ISBN: 0-12-775695-7

About standardized audiological, clinical tests see
http://www.phon.ucl.ac.uk/home/andyf/natasha/

3. An introduction to room acoustics

Jens Holger Rindel

3.1 SOUND WAVES IN ROOMS

3.1.1 Standing waves in a rectangular room

A rectangular room has the dimensions l_x, l_y, and l_z. The wave equation can then be written

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = 0 \qquad (3.1.1)$$

where p is the sound pressure and $k = \omega / c$ is the angular wave number, ω is the angular frequency and c is the speed of sound in air. The equation can be solved by separation of the variables and it is assumed that the solution can be written in the form:

$$p = X(x) \cdot Y(y) \cdot Z(z) \cdot \mathrm{e}^{\mathrm{j}\omega t}$$

Insertion in (3.1.1) and division by p gives

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + k^2 = 0$$

This can be separated, and for the x-direction it yields

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_x^2 = 0$$

Similar equations hold for the y- and z-directions. The angular wave number k has been divided into three

$$k^2 = k_x^2 + k_y^2 + k_z^2 \qquad (3.1.2)$$

The general solution to the above one-dimensional equation is

$$X(x) = C_x \cos(k_x x + \varphi_x)$$

in which the constants C_x and φ_x are determined from the boundary conditions.

The room surfaces are now assumed to be rigid, i.e. the normal component of the particle velocity is zero at the boundaries

$$u_x = -\frac{1}{\mathrm{j}\omega\rho}\frac{\partial p}{\partial x} = 0 \quad \text{for } x = 0 \text{ and } x = l_x$$

This means that $\varphi_x = 0$ and

$$k_x = \frac{\pi}{l_x} \cdot n_x \quad \text{where} \quad n_x = 0, 1, 2, 3, \dots \qquad (3.1.3)$$

Two similar boundary conditions hold for the y- and z-directions. With these conditions the solution to (3.1.1) is

$$p = p_0 \cdot \cos\left(\pi n_x \frac{x}{l_x}\right) \cdot \cos\left(\pi n_y \frac{y}{l_y}\right) \cdot \cos\left(\pi n_z \frac{z}{l_z}\right) \qquad (3.1.4)$$

The time factor $\mathrm{e}^{\mathrm{j}\omega t}$ is understood. The amplitude of the sound pressure does not move with time, so the waves that are solutions to (3.1.4) are called *standing waves*. They are also called the *modes* of the room, and each of them is related to a certain *natural frequency* (or eigenfrequency) given by

$$f_n = \frac{\omega_n}{2\pi} = \frac{c\,k}{2\pi} = \frac{c}{2\pi}\sqrt{k_x^2 + k_y^2 + k_z^2}$$

$$f_n = \frac{c}{2}\sqrt{\left(\frac{n_x}{l_x}\right)^2 + \left(\frac{n_y}{l_y}\right)^2 + \left(\frac{n_z}{l_z}\right)^2} \qquad (3.1.5)$$

The modes can be divided into three groups:
Axial modes are one-dimensional, only one of n_x, n_y, n_z is > 0.
Tangential modes are two-dimensional, two of n_x, n_y, n_z are > 0.
Oblique modes are three-dimensional, all three of n_x, n_y, n_z are > 0.

Some examples are shown in Fig. 3.1.1. It is observed that the set of numbers (n_x, n_y, n_z) indicate the number of *nodes* (places with $p = 0$) along each coordinate axis.

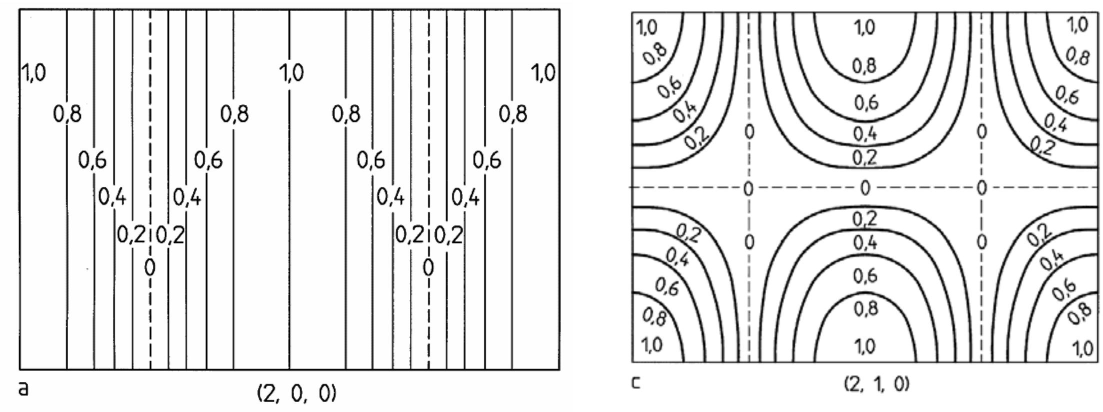


Figure 3.1.1. Examples of room modes. (2,0,0) is one-dimensional and (2,1,0) is two-dimensional. The lines are iso-sound pressure amplitude curves.

n_x	n_y	n_z	f_n (Hz)
0	1	0	25
1	0	0	30
0	0	1	36
1	1	0	39
0	1	1	43
1	0	1	47
0	2	0	49
1	1	1	53
1	2	0	58
2	0	0	60
0	2	1	61
2	1	0	65
1	2	1	68
2	0	1	70
0	0	2	72

Table 3.1.1. Calculated natural frequencies at low frequencies using (3.1.5) in a rectangular room with dimensions 5.7 m, 7.0 m, 4.8 m.

3.1.2 Transfer function in a room

The transfer function is the frequency response from a source position to a receiver position in a room. A measured transfer function is shown in Fig. 3.1.2. It fluctuates very much with frequency and the maxima can be identified as the natural frequencies of the room. The example in Fig. 3.1.2 has the same room dimensions as was used for the calculations in Table 3.1.1.

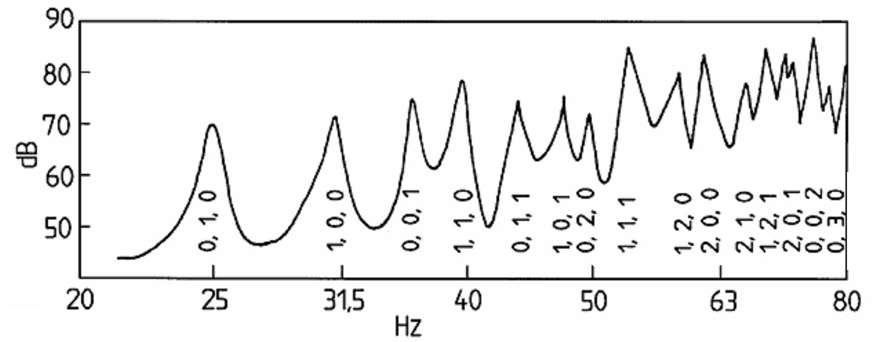


Figure 3.1.2. Transfer function in a rectangular room. At low frequencies it is possible to identify the modes by their modal numbers.

3.1.3 Density of natural frequencies

A closer inspection of equation (3.1.5) shows that the natural frequencies of a rectangular room may be interpreted in a geometrical way. A three-dimensional frequency space is shown in Fig. 3.1.3. The natural frequencies of the one-dimensional modes are marked on each of the axes, representing the axial modes of the length, the width and the height, respectively. The interesting observation is now that the points in the grid represent the oblique modes, and the distance to each point from the origin is the natural frequency of that mode. So, the number of oblique modes below a certain frequency f is equal to the number of grid points inside the sphere with radius f.

The volume is 1/8 of the sphere with radius f, i.e. $(4\,\pi f^3 / 3) / 8 = \pi f^3 / 6$. Each mode occupies a volume $c^3 / (8\, l_x\, l_y\, l_z) = c^3 / (8\, V)$. So, the number of oblique modes below f is approximately:

$$N_{obl} = \frac{\pi\, f^3}{6} \frac{8V}{c^3} = \frac{4\pi V}{3} \frac{f^3}{c^3}$$

The tangential modes are found in the plane between two of the axes. If these and the axial modes are also taken into account, the number of modes with natural frequencies below the frequency f is:

$$N = \frac{4\pi V}{3}\left(\frac{f}{c}\right)^3 + \frac{\pi S}{4}\left(\frac{f}{c}\right)^2 + \frac{L}{8}\frac{f}{c} \qquad (3.1.6)$$

V is the volume of the room, $S = 2(\, l_x\, l_y + l_x\, l_z + l_y\, l_z)$ is the total area of the surfaces, and $L = 4\,(l_x + l_y + l_z)$ is the total length of all edges. It should be noted that the modal points of the tangential and axial modes in Fig. 3.1.3 are located on the coordinate planes and axes, respectively. Therefore we count the tangential points only as halves and those on the axes only as quarters.

At high frequencies the oblique modes dominate, and the first term in (3.1.6) is a good approximation for any room, not only for rectangular rooms.

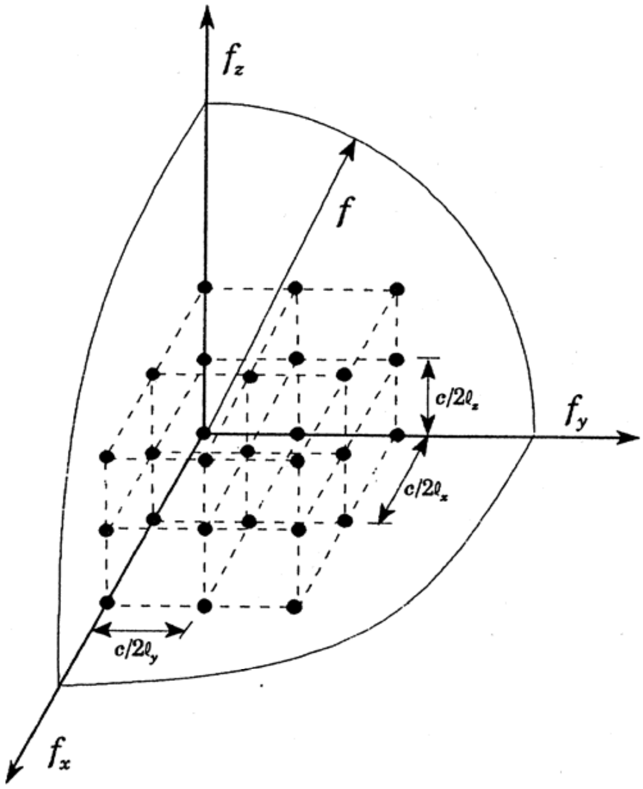


Figure 3.1.3. Frequency-grid, in which each grid point represents a room mode.

The *modal density* is the average number of modes per hertz.

$$\frac{\mathrm{d}N}{\mathrm{d}f} = 4\pi \frac{V}{c^3} f^2 + \frac{\pi S}{2c^2} f + \frac{L}{8c} \tag{3.1.7}$$

In Fig. 3.1.4 this is compared to the actual modal density in a room. For high frequencies it is sufficient to use the first term (oblique modes) for the modal density:

$$\frac{\mathrm{d}N}{\mathrm{d}f} \cong 4\pi \frac{V}{c^3} f^2 \tag{3.1.8}$$

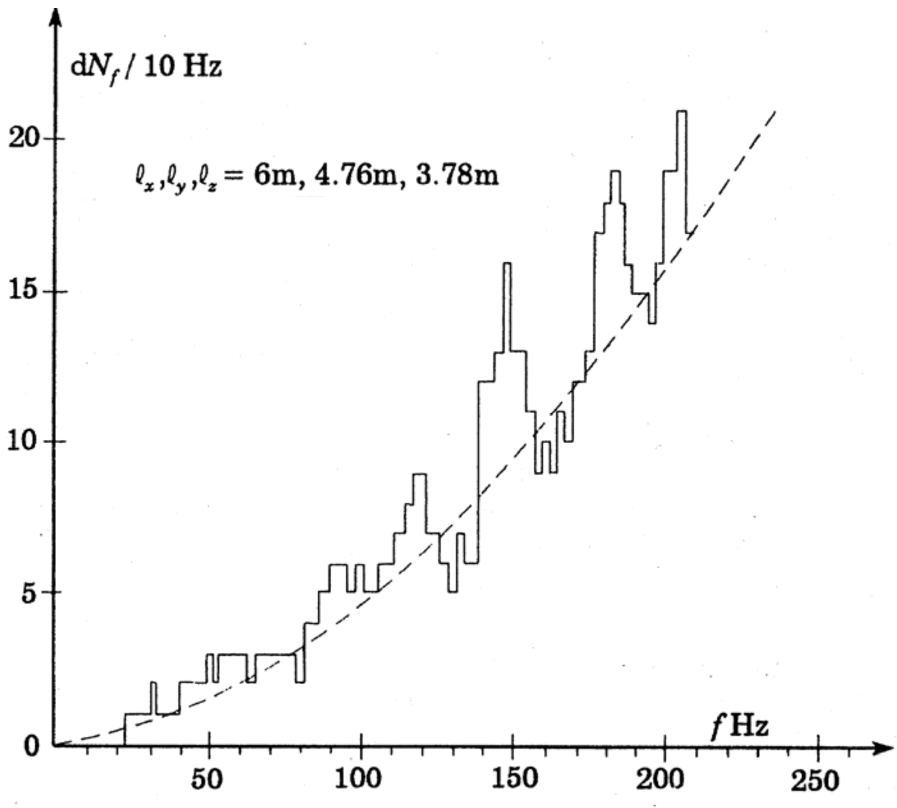


Figure 3.1.4. Modal density as a function of frequency. Actual number of modes per 10 Hz in a rectangular room and estimated by (3.1.7).

3.2 STATISTICAL ROOM ACOUSTICS

3.2.1 The diffuse sound field

In this chapter the acoustical behaviour of a room is treated from a statistical point of view, based on energy balance considerations. It is assumed that the modal density is high enough, so the influence of single modes in the room can be neglected. It is also assumed that the reflection density is high enough, so the phase relations between individual reflections can be neglected. This means that the reflections in the room are assumed to be uncorrelated and their contribution can be added on an energy basis.

The diffuse sound field is defined as a sound field in which:
The energy density is the same everywhere
All directions of sound propagation occur with the same probability

It is obvious that the direct sound field near a sound source is not included in the diffuse sound field. Neither are the special interference phenomena that are known to give increased energy density near the room boundaries and corners. The diffuse sound field is an ideal sound field that does not exist in any room. However, in many cases the diffuse sound field can be a good and very practical approximation to the real sound field.

3.2.2 Incident sound power on a surface

In a plane propagating sound wave the relation between rms sound pressure p_1 and sound intensity I_1 is

$$p_1^2 = I_1 \cdot \rho c$$

In a diffuse sound field the rms sound pressure p_{diff} is the result of sound waves propagating in all directions, and all having the sound intensity I_1. By integration over a sphere with the solid angle $\psi = 4\pi$ the rms sound pressure in the diffuse sound field is

$$p_{diff}^2 = \int_{\psi=4\pi} I_1 \cdot \rho c \, \mathrm{d}\psi = 4\pi \cdot I_1 \cdot \rho c \qquad (3.2.1)$$

In the case of a plane wave with the angle of incidence θ relative to the normal of the surface, the incident sound power per unit area on the surface is

$$I_\theta = I_1 \cos\theta = \frac{p_{diff}^2}{4\pi \rho c} \cos\theta \qquad (3.2.2)$$

where p_{diff} is the rms sound pressure in the diffuse sound field. This is just the sound intensity in the plane propagating wave multiplied by the cosine, which is the projection of a unit area as seen from the angle of incidence, see Fig. 3.2.1.

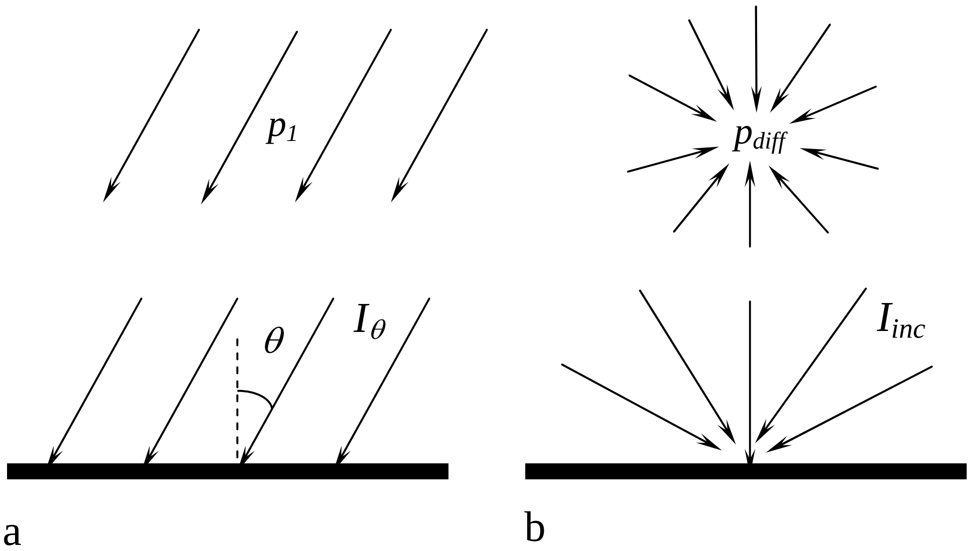


Figure 3.2.1. a: Plane wave at oblique incidence on a surface. b: Diffuse incidence on a surface.

The total incident sound power per unit area is found by integration over all angles of incidence covering a half sphere in front of the surface, see Fig. 3.2.2. The integration covers the solid angle $\psi = 2\pi$.

$$I_{inc} = \int_{\psi=2\pi} I_\theta \, \mathrm{d}\psi = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi/2} \frac{p_{diff}^2}{\rho c} \cos\theta \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi$$

$$= \frac{1}{4\pi} \cdot 2\pi \cdot \frac{p_{diff}^2}{\rho c} \int_0^1 \sin\theta \, \mathrm{d}(\sin\theta) = \frac{1}{2} \cdot \frac{p_{diff}^2}{\rho c} \cdot \frac{1}{2}$$

$$I_{inc} = \frac{p_{diff}^2}{4 \rho c} \qquad (3.2.3)$$

It is noted that this is four times less than in the case of a plane wave of normal incidence.

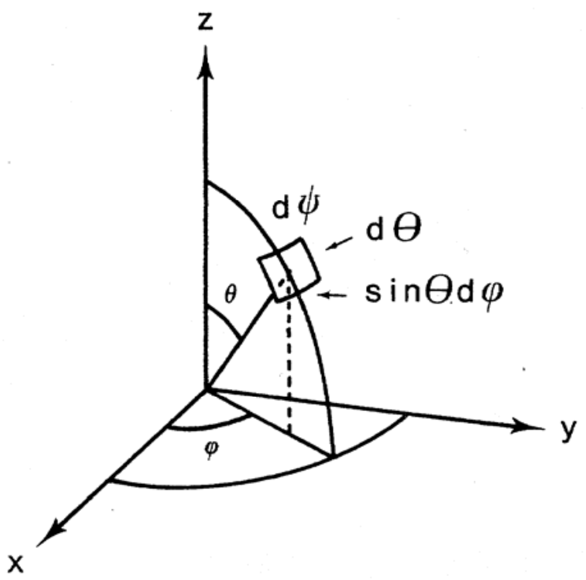


Figure 3.2.2. Definition of angles of incidence in a diffuse sound field.

3.2.3 Equivalent absorption area

The *absorption coefficient* α is defined as the ratio of the non-reflected sound energy to the incident sound energy on a surface. It can take values between 0 and 1, and $\alpha = 1$ means that all incident sound energy is absorbed in the surface. An example of a surface with absorption coefficient, $\alpha = 1$ is an open window.

The product of area and absorption coefficient of a surface material is the *equivalent absorption area* of that surface, i.e. the area of open windows giving the same amount of sound absorption as the actual surface. The equivalent absorption area of a room is

$$A = \sum_i S_i \alpha_i = S_1\alpha_1 + S_2\alpha_2 + \quad ..... \quad = S\alpha_m \qquad (3.2.4)$$

where S is the total surface area of the room and α_m is the *mean absorption coefficient*. The unit of A is m^2. In general, the equivalent absorption area may also include sound absorption due to the air and due to persons or other objects in the room.

3.2.4 Energy balance in a room

The total acoustic energy in a room is the sum of potential energy and kinetic energy, or twice the potential energy, since the time average of the two parts must be equal. The total energy E is the energy density multiplied by the room volume V:

$$E = (w_{pot} + w_{kin})\,V = 2w_{pot}\,V = \frac{p^2}{\rho\, c^2} V \qquad (3.2.5)$$

Here and in the following, p denotes the rms sound pressure in the diffuse sound field (called p_{diff} in section 3.2.2). The energy absorbed in the room is the incident sound power per unit area (3.2.3) multiplied by the total surface area and the mean absorption coefficient, i.e. the equivalent absorption area (3.2.4),

$$P_{a,abs} = I_{inc} S\alpha_m = I_{inc} A = \frac{p^2}{4\rho\, c} A \qquad (3.2.6)$$

If P_a is the sound power of a source in the room, the energy balance equation of the room is

$$P_a - P_{a,abs} = \frac{\mathrm{d}E}{\mathrm{d}t} \tag{3.2.7a}$$

$$P_a - \frac{p^2}{4\rho\, c} A = \frac{V}{\rho\, c^2} \frac{\mathrm{d}}{\mathrm{d}t}(p^2) \tag{3.2.7b}$$

With a constant sound source a steady state situation is reached after some time, and the right side of the equation is zero. So, the absorbed power equals the power emitted from the source, and the steady state sound pressure in the room is

$$p_s^2 = \frac{4\, P_a}{A}\, \rho\, c \tag{3.2.8}$$

This equation shows that the sound power of a source can be determined by measuring the sound pressure generated by the source in a room, provided that the equivalent absorption area of the room is known. It also shows how the absorption area in a room has a direct influence on the sound pressure in the room. For some cases it is more convenient to express eq. (3.2.8) in terms of the sound pressure level L_p and the sound power level L_W ,

$$L_p \cong L_W + 10 \log\left(\frac{4A_0}{A}\right) \qquad \text{(dB)} \tag{3.2.9}$$

where A_0 = 1 m^2 is a reference area. The approximation comes from neglecting the term with the constants and reference values

$$10 \log \frac{\rho\, c\, P_{ref}}{A_0\, p_{ref}^2} = 10 \log \frac{1.204 \cdot 343 \cdot 10^{-12}}{1 \cdot (20 \cdot 10^{-6})^2} = 0.14 \text{ dB} \cong 0 \text{ dB}$$

3.2.5 Reverberation time. Sabine´s formula

If the sound source is turned off after the sound pressure has reached the stationary value, the first term in the energy balance equation (3.2.7b) is zero, and the rms sound pressure is now a function of time:

$$\frac{A}{4\rho\, c}\, p^2(t) \; + \; \frac{V}{\rho\, c^2} \frac{\mathrm{d}}{\mathrm{d}t}\left(p^2(t)\right) \; = \; 0 \tag{3.2.10}$$

The solution to this equation can be written

$$p^2(t) \;\; = \;\; p_s^2 \; \mathrm{e}^{-\frac{cA}{4V}t} \tag{3.2.11}$$

where p_s^2 is the mean square sound pressure in the steady state and t = 0 is the time when the source is turned off. It is seen that the mean square sound pressure, and hence the sound energy, follows an exponential decay function. On a logarithmic scale the decay is linear, and this is called the *decay curve*, see Fig. 3.2.3.

If instead the source is turned on at the time t = 0, the sound build-up in the room follows a similar exponential curve, also shown in Fig. 3.2.3.

$$p^2(t) \;\; = \;\; p_s^2 \left(1 - \mathrm{e}^{-\frac{cA}{4V}t}\right) \tag{3.2.12}$$

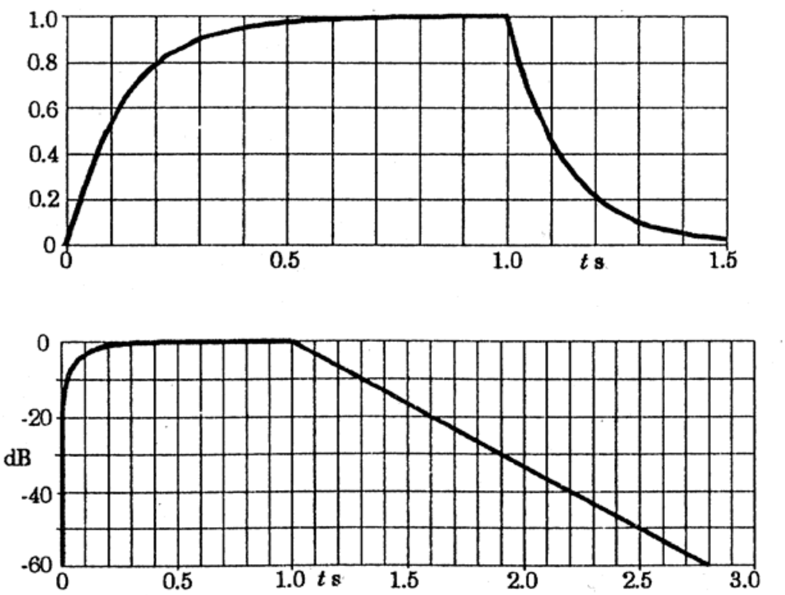


Figure 3.2.3. Build-up and decay of sound in a room. Here the source is turned on a t = 0 and turned off at t = 1 s. Top: linear scale (sound pressure squared). Bottom: logarithmic scale (dB).

The *reverberation time* T_{60} is defined as the time it takes for the sound energy in the room to decay to one millionth of the initial value, i.e. a 60 dB decay of the sound pressure level. Hence, for $t = T_{60}$,

$$p^2(t) \quad = \quad p_s^2\, 10^{-6} \quad = \quad p_s^2\; \mathrm{e}^{-\frac{c\,A}{4\,V}T_{60}}$$

So, the reverberation time is

$$T_{60} \quad = \quad 6 \cdot \ln(10) \cdot \frac{4\,V}{c\;A} \quad = \quad \frac{55.3\,V}{c\;A} \tag{3.2.13}$$

This is *Sabine's formula* named after Wallace C. Sabine, who introduced the reverberation time concept around 1896. He was the first to demonstrate that T_{60} is inversely proportional to the equivalent absorption area *A*.

Note: Sabine's formula is often written as $T_{60} = 0.16\ V/A$. However, this implies that *V* must be in m^3 and *A* in m^2.

3.2.6 Stationary sound field in a room. Reverberation distance

A *reverberation room* is a special room with long reverberation time and a good diffusion. In such a room the diffuse sound field is a good approximation, and the results for stationary conditions (3.2.8) and for sound decay (3.2.13) can be applied to measure the sound power of a sound source:

$$P_a \quad = \quad \frac{p_s^2}{4\rho\, c} \cdot \frac{55.3\,V}{c\,T_{60}} \tag{3.2.14}$$

The reverberation time and the average sound pressure level in the reverberation room are measured, and the sound power level is calculated from

$$L_W = L_p + 10\log\frac{p_{ref}^2 \cdot 55.3 \cdot V}{P_{ref} \cdot 4\,\rho\, c^2 \cdot T_{60}}$$
$$= L_p + 10\log\frac{V}{V_0} - 10\log\frac{T_{60}}{t_0} - 14\ dB \qquad (3.2.15)$$

where V_0 = 1 m^3 and t_0 = 1 s.

In most ordinary rooms the diffuse sound field is not a good approximation. Each of the following conditions may indicate that the sound field is not diffuse

An uneven distribution of sound absorption on the surfaces, e.g. only one surface is highly absorbing
A lack of diffusing or sound scattering elements in the room
The ratio of longest to shortest room dimension is higher than three
The volume is very large, say more than 5000 m^3

A rather simple modification to the stationary sound field is to separate the direct sound. The sound power radiated by an omni-directional source is the sound intensity at the distance *r* in a spherical sound field multiplied by the surface area of a sphere with radius *r*

$$P_a = I_r \cdot 4\pi\, r^2 \qquad (3.2.16)$$

Thus, the sound pressure squared of direct sound in the distance *r* from the source is

$$p_{dir}^2 = \frac{P_a}{4\pi\, r^2}\,\rho\, c \qquad (3.2.17)$$

The stationary sound is described by (3.2.8)

$$p_s^2 = \frac{4\,P_a}{A}\,\rho\, c$$

The *reverberation distance* r_{rev} is defined as the distance where $p_{dir}^2 = p_s^2$ when an omni-directional point source is placed in a room. It is a descriptor of the amount of absorption in a room, since the reverberation distance depends only on the equivalent absorption area

$$r_{rev} = \sqrt{\frac{A}{16\pi}} = 0.14\sqrt{A} \qquad (3.2.18)$$

At a distance closer to the source than the reverberation distance, the direct sound field dominates, and this is called the *direct field*. At longer distances the reverberant sound field dominates, and in this so-called *far field* the stationary, diffuse sound field may be a usable approximation.

An expression for the combined direct and diffuse sound field can derived by simple addition of the squared sound pressures of the two sound fields. However, since the direct sound is treated separately, it should be extracted from the energy balance equation, which was used to describe the diffuse sound field. To do this, the sound power of the source should be reduced by a factor of (1 - α_m), which is the fraction of the sound power emitted to the room after the first reflection. So, the squared sound pressure in the total sound field is

$$p_{total}^2 = p_{dir}^2 + p_s^2(1-\alpha_m) = p_s^2\left(\frac{r_{rev}^2}{r^2} + 1 - \alpha_m\right) \qquad (3.2.19)$$

$$p_{total}^2 = P_a \cdot \rho\, c\left(\frac{1}{4\pi\, r^2} + \frac{4}{A}(1-\alpha_m)\right) \qquad (3.2.20)$$

Normal sound sources like a speaking person, a loudspeaker or a musical instrument radiate sound with different intensity in different directions. The *directivity factor Q* is the ratio of the intensity in a certain direction to the average intensity,

$$Q = I \cdot \frac{4\pi\, r^2}{P_a} \qquad (3.2.21)$$

So, the squared sound pressure of the direct sound is

$$p_{dir}^2 = \frac{Q \cdot P_a}{4\pi\, r^2}\, \rho\, c \qquad (3.2.22)$$

This leads to a general formula for the sound pressure level as a function of the distance from a sound source in room.

$$L_p \cong L_W + 10 \log \frac{4A_0}{A} + 10 \log \left(Q \frac{r_{rev}^2}{r^2} + 1 - \alpha_m \right) \quad \text{(dB)} \qquad (3.2.23)$$

where A_0 = 1 m^2. In a reverberant room with little sound absorption (say, $\alpha_m < 0.1$) the sound pressure level in the far field will be approximately as predicted by the diffuse field theory, i.e. the last term will be close to zero. In the case of a highly directive sound source like a trumpet ($Q >> 1$) the direct field can be extended to distances much longer than the reverberation distance. In the latter situation the last term in (3.2.23) raises the sound pressure level above the diffuse field value.

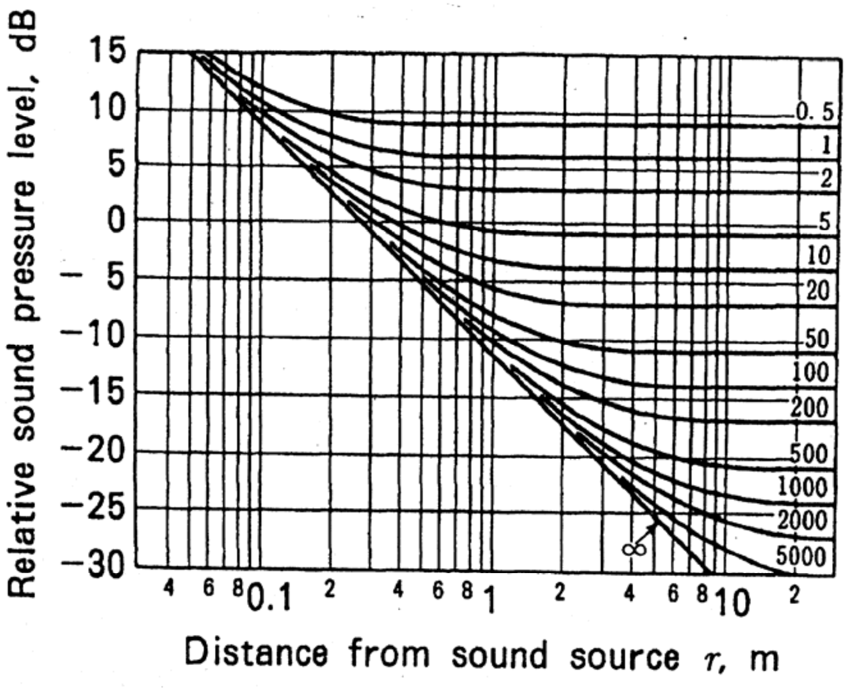


Figure 3.2.4. Relative sound pressure level as a function of distance in a room with approximately diffuse sound field. The source has a directivity factor of one. The parameter on the curves is $A / (1 - \alpha_m)$ in m^2.

In large rooms with medium or high sound absorption (say, $\alpha_m > 0.2$) the sound pressure level will continue to decrease as a function of the distance, because the diffuse field theory is not valid in such a room. Instead, the slope of the spatial decay curve may be taken as a measure of the degree of acoustic attenuation in a room. So, in large industrial halls the attenuation in dB per doubling of the distance may be a better descriptor than the reverberation time.

3.3 GEOMETRICAL ROOM ACOUSTICS

3.3.1 Sound rays and a general reverberation formula

In geometrical acoustics rays are used to describe the sound propagation. The concept of rays implies that the wavelength and the phase of the sound are neglected, and only the direction of sound energy propagation is treated in geometrical acoustics.

The sound decay shall now be studied by following a plane wave travelling as a ray from wall to wall, see Fig. 3.3.1. The energy of the wave is gradually decreased due to absorption at the surfaces, all of which are assumed to have the mean absorption coefficient α_m.

The ray representing a plane wave may start in any direction and it is assumed that the decay of energy in the ray is representative for the decay of energy in the room. The room may have any shape.

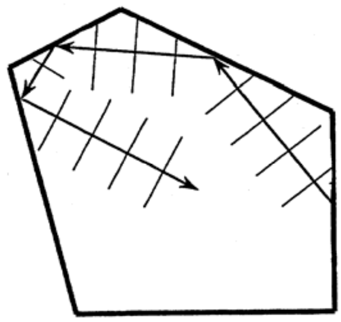


Figure 3.3.1. A plane wave travelling as a ray from wall to wall in a room.

By each reflection the energy is reduced by a factor (1 - α_m). The initial sound pressure is p_0 and after n reflections the squared sound pressure is

$$p^2(t) = p_0^2 \cdot (1-\alpha_m)^n \quad = p_0^2 \cdot \mathrm{e}^{n \cdot \ln(1-\alpha_m)} \tag{3.3.1}$$

The distance of the ray from one reflection to the next is l_i and the total distance traveled by the ray up to the time t is

$$\sum_i l_i = c \cdot t = n \cdot l_m \tag{3.3.2}$$

where l_m is the *mean free path*. So, the squared sound pressure is

$$p^2(t) = p_0^2 \cdot \mathrm{e}^{\frac{c}{l_m} \cdot \ln(1-\alpha_m) \cdot t} \tag{3.3.3}$$

When the squared sound pressure has dropped to 10^{-6} of the initial value, the time t is by definition the reverberation time T_{60}:

$$10^{-6} = \mathrm{e}^{\frac{c}{l_m} \cdot \ln(1-\alpha_m) \cdot T_{60}} \quad \Rightarrow \quad -6 \cdot \ln(10) = \frac{c}{l_m} \cdot \ln(1-\alpha_m) \cdot T_{60}$$

This leads to an interesting pair of general reverberation formulas:

$$T_{60} = \frac{13.8 \cdot l_m}{-c \cdot \ln(1-\alpha_m)} \quad \approx \frac{13.8 \cdot l_m}{c \cdot \alpha_m} \tag{3.3.4}$$

The last approximation is valid if $\alpha_m < 0.3$, i.e. only in rather reverberant rooms. The approximation comes from:

$$-\ln(1-\alpha_m) = \ln\left(\frac{1}{1-\alpha_m}\right) = \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{3} + \cdots$$

With the assumption that all directions of sound propagation appear with the same probability, it can be show (Kosten, 1960) that the mean free path in a three-dimensional room is

$$l_m = \frac{4\,V}{S} \qquad \text{(3-dimensional)} \tag{3.3.5}$$

where V is the volume and S is the total surface area.

Similarly, the mean free path in a two-dimensional room can be derived. This could be the narrow air space in a double wall, or structure-borne sound in a plate. The height or thickness must be small compared to the wavelength. In this case the mean free path is

$$l_m = \frac{\pi\, S_x}{U} \qquad \text{(2-dimensional)} \tag{3.3.6}$$

where S_x is the area and U is the perimeter. The one-dimensional case is just the sound travelling back and forth between two parallel surfaces with the distance $l = l_m$.

Insertion of (3.3.5) in the last part of (3.3.4) gives the Sabine formula (3.2.13), whereas insertion in the first part of (3.3.4) leads to the so-called *Eyring's formula* for reverberation time in a room:

$$T_{60} = \frac{55.3 \cdot V}{-c \cdot S \cdot \ln(1-\alpha_m)} \tag{3.3.7}$$

In a reverberant room ($\alpha_m < 0.3$) it gives the same result as Sabine's formula, but in highly absorbing rooms Eyring's formula is theoretically more correct. In practice the absorption coefficients are not the same for all surfaces and the mean absorption coefficient is calculated as in (3.2.4):

$$\alpha_m = \frac{1}{S} \cdot \sum_i S_i \alpha_i \tag{3.3.8}$$

In the extreme case of an anechoic room ($\alpha_m = 1$) Eyring's formula gives correctly a reverberation time of zero, whereas Sabine's formula is obviously wrong, giving the value T_{60} = 55.3 $V/c\,S$. However, in normal rooms with a mixture of different absorption coefficients it is recommended to use Sabine's formula.

3.3.2 Sound absorption in the air

A sound wave travelling through the air is attenuated by a factor m, which depends on the temperature and the relative humidity of the air, see Fig. 3.3.2. The unit of the air attenuation factor is m^{-1}. If this attenuation is included in (3.3.3) the squared sound pressure in the decay is

$$p^2(t) = p_0^2 \cdot \mathrm{e}^{\frac{c}{l_m}\cdot \ln(1-\alpha_m)\cdot t} \cdot \mathrm{e}^{-m\,c\,t} = p_0^2 \cdot \mathrm{e}^{\frac{c\,t}{l_m}\cdot\left(\ln(1-\alpha_m) - m\cdot l_m\right)} \tag{3.3.9}$$

The general reverberation formula then becomes

$$T_{60} = \frac{13.8 \cdot l_m}{c\,(-\ln(1-\alpha_m) + m \cdot l_m)} \approx \frac{13.8 \cdot l_m}{c\,(\alpha_m + m \cdot l_m)} \tag{3.3.10}$$

In the three-dimensional case with (3.3.5) we then have

$$T_{60} = \frac{55.3 \cdot V}{c\,(-S \cdot \ln(1-\alpha_m) + 4\,mV)} \approx \frac{55.3 \cdot V}{c\,(S \cdot \alpha_m + 4\,mV)} \tag{3.3.11}$$

These two expressions are the Eyring and the Sabine formula, respectively, with the air absorption included. By comparison with (3.2.13) it is seen that the equivalent absorption area including air absorption is

$$A = \sum_i S_i \alpha_i + 4\,mV \tag{3.3.12}$$

Some typical values of m are found later in Table 3.4.3.

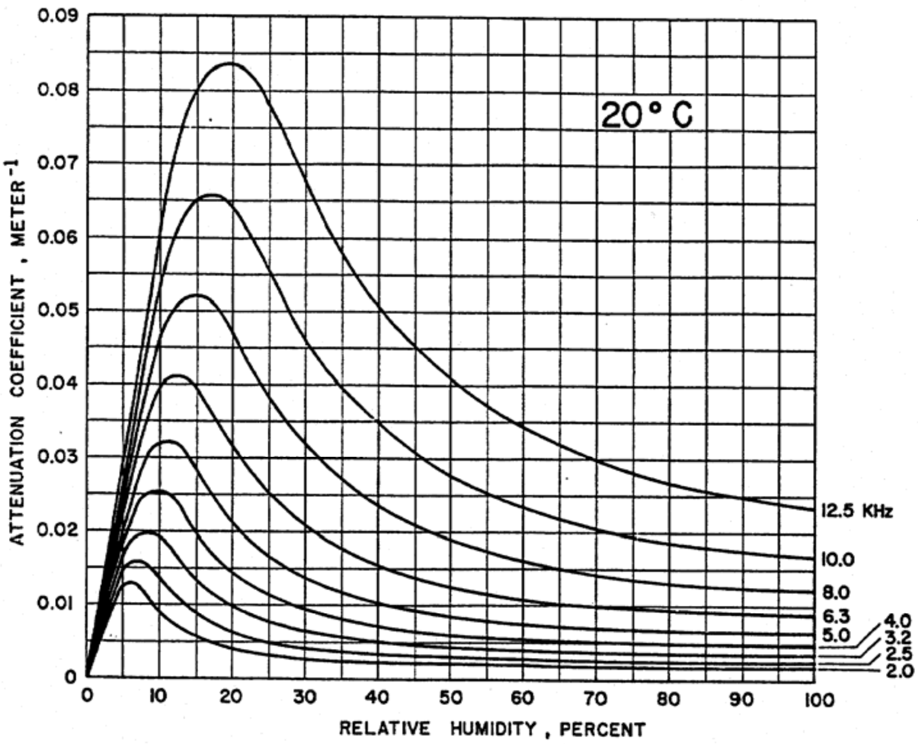


Figure 3.3.2. The air attenuation factor m as a function of the relative humidity. The air temperature is 20 °C. (Ref.: Harris1966).

3.3.3 Sound reflections and image sources

The direction of a sound reflection from a large plane surface follows the same geometrical law, as known from optics, i.e. the angle of reflection is equal to the angle of incidence. This means that the reflected sound can be interpreted as sound coming from an image source behind the reflecting surface, see Fig. 3.3.3. This principle can be extended to higher order reflections.

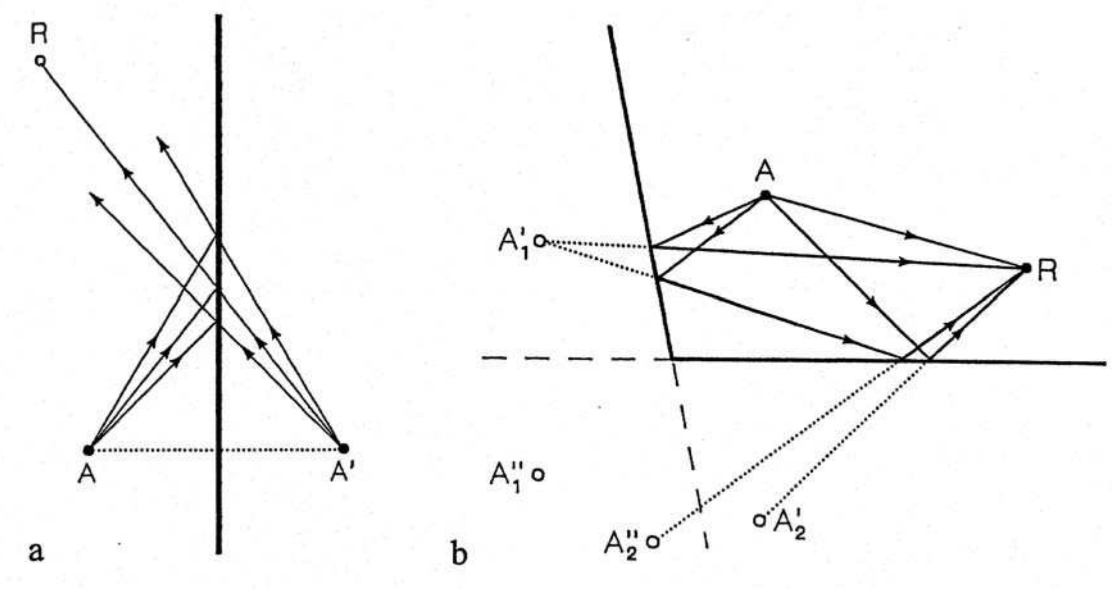


Figure 3.3.3. Reflection in one surface (a) and in two surfaces (b). A is the source and R is the receiver. First order image sources are indicated by A' and second order image sources by A''.

Echo is a well-known acoustic phenomenon. It is defined as a single sound reflection that is clearly audible as separate from the direct sound. The human ear is able to hear a reflection as an echo if the time delay is approximately 50 ms. The so-called echo-ellipse is shown in Fig. 3.3.4. Any point E on the ellipse represents a potential reflection with a delay of 50 ms, i.e. the distance LE + EP = 17 m. Reflections from room surfaces outside the ellipse (as R_2 on the figure) are delayed more than 50 ms and may cause an echo at the receiver point.

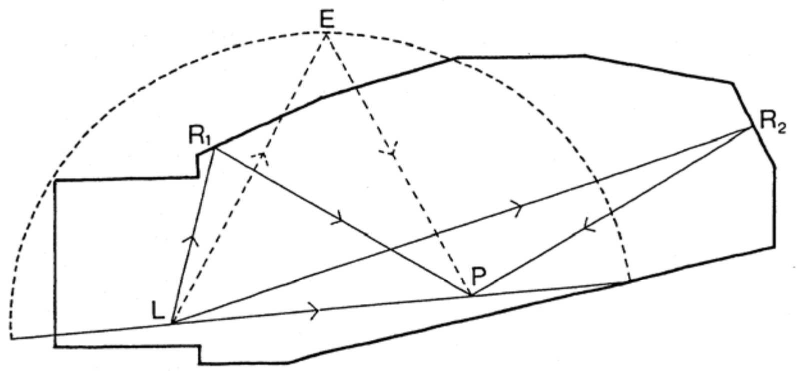


Figure 3.3.4. The echo-ellipse in the longitudinal section of an auditorium. L is the source and P the receiver. (Ref.: Petersen 1984).

3.3.4 Reflection density in a room

The image source principle can easily be applied to higher order reflections in a rectangular room. An infinite number of image rooms make a grid, and each cell in the grid is an image room containing an image source. The principle is shown for the two-dimensional case in Fig. 3.3.5.

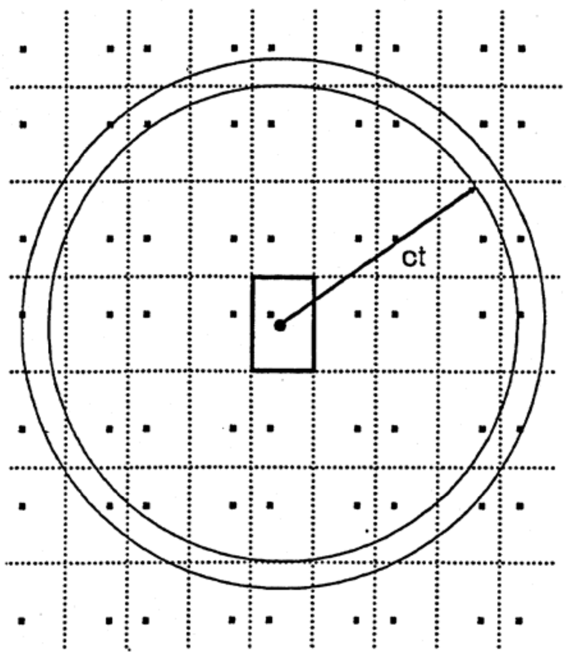


Figure 3.3.5. Rectangular room with a sound source and image sources, here shown in two dimensions. Image sources located inside the circle with radius ct will contribute reflections up to time t.

If an impulse sound is emitted the number of reflections that will arrive within the time t can be calculated as the volume of a sphere with radius ct divided by the room volume V:

$$N(t) = \frac{\frac{4}{3}\pi (ct)^3}{V} \tag{3.3.13}$$

The reflection density is then the number of reflections within a small time interval dt, and by differentiation:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 4\pi \frac{c^3}{V} t^2 \tag{3.3.14}$$

The reflection density increases with the time squared, so the higher order reflections are normally so dense in arrival time that it is impossible to distinguish separate reflections. If (3.3.14) is compared to (3.1.8), it is striking to observe the analogy between reflection density in the time domain and modal density in the frequency domain.

3.4 ROOM ACOUSTICAL DESIGN

3.4.1 Choice of room dimensions

The room dimensions determine the natural frequencies of a room. A good acoustical design of a room implies that the transfer function should be as smooth as possible. With reference to Fig. 3.1.2 is clear that the room dimensions of a rectangular room should not be identical, because in a cubic room many modes will have the same natural frequency, and thus there will be bigger gaps in the transfer function. This would be very unfortunate, especially at low frequencies in small rooms for speech, music or acoustic measurements. The dimensions of such rooms should be designed after calculations of the normal modes below 100 Hz, see also Table 3.1.1.

3.4.2 Reflection control

In room with an audience it is very important to design the room surfaces with respect to the early reflections. First of all in order to avoid problems with echo and focusing, but also to ensure a good distribution of reflections to the audience area, see Fig. 3.4.1. In rooms for speech the ceiling reflections are most important, whereas rooms for music should not give too much reflection directly from the ceiling. In such room the ceiling should rather give diffuse reflections, but the side walls are important because lateral reflections contribute to the acoustic of a concert hall, see Fig. 3.4.2.

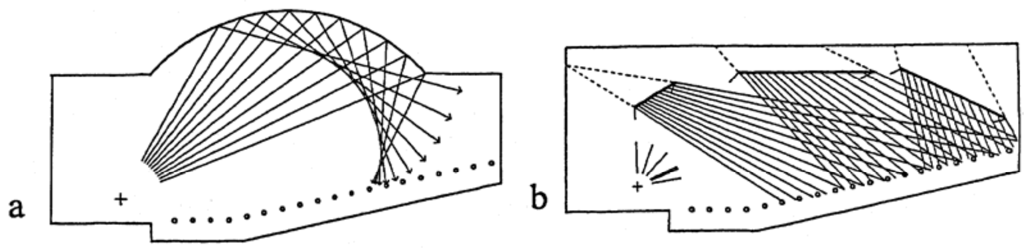


Figure 3.4.1. Ceiling reflections in auditoriums. a) concave ceiling causing focusing and uneven sound distribution. b) plane reflectors causing an even sound distribution. (Ref.: Petersen 1984).

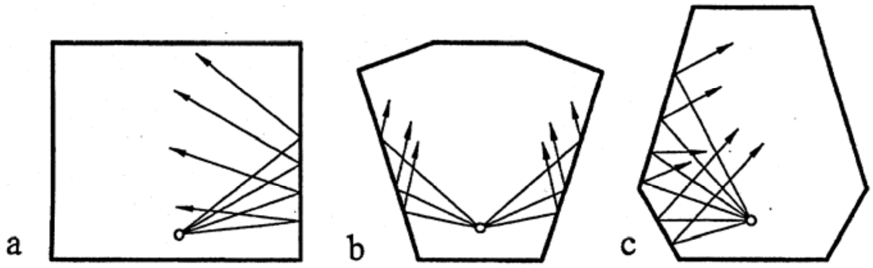


Figure 3.4.2. Wall reflections in auditoriums. a) rectangular room, b) fan shape room, c) inverse fan shape room.

3.4.3 Calculation of reverberation time

Sabine's formula (3.2.13) is the most well known and simple method for calculation of reverberation time in a room

$$T_{60} = \frac{55.3\,V}{c\,A} \cong \frac{0.16\,V}{A} \qquad (3.4.1)$$

with volume V in m^3 and A in m^2. The equivalent absorption area is calculated as in (3.3.12), but in addition to absorption from surfaces and air, the absorption from persons or other items in the room should be included, if relevant

$$A = \sum_i S_i \alpha_i + \sum_j n_j A_j + 4\,mV \qquad (3.4.2)$$

Here n_j is the number of items, each contributing with an absorption area A_j. Examples of absorption coefficients of common materials and absorption areas for persons are given in Table 3.4.1 and 3.4.2, respectively. The air attenuation can be taken from Table 3.4.3.

	Frequency (Hz)					
Material	125	250	500	1000	2000	4000
Brick, bare concrete	0.01	0.02	0.02	0.02	0.03	0.04
Parquet floor on studs	0.16	0.14	0.11	0.08	0.08	0.07
Needle-punch carpet	0.03	0.04	0.06	0.10	0.20	0.35
Window glass	0.35	0.25	0.18	0.12	0.07	0.04
Curtain draped to half its area, 100 mm air space	0.10	0.25	0.55	0.65	0.70	0.70

Table 3.4.1. Typical values of the absorption coefficient α for some common materials.

	Frequency (Hz)					
Persons	125	250	500	1000	2000	4000
Standing, normal clothing	0.12	0.24	0.59	0.98	1.13	1.12
Standing, with overcoat	0.17	0.41	0.91	1.30	1.43	1.47
Sitting musician with instrument	0.60	0.95	1.06	1.08	1.08	1.08

Table 3.4.2. Typical values of absorption area A in m^2 for persons.

Relative humidity (%)	Frequency 1 kHz	2 kHz	4 kHz	8 kHz
40	0.0011	0.0026	0.0072	0.0237
50	0.0010	0.0024	0.0061	0.0192
60	0.0009	0.0023	0.0056	0.0162
70	0.0009	0.0021	0.0053	0.0143
80	0.0008	0.0020	0.0051	0.0133

Table 3.4.3. Examples of air attenuation factor m (m^{-1}) at a temperature of 20°C.

3.4.4 Reverberation time in non-diffuse rooms

In a room with the sound absorption unequally distributed on the surfaces the assumption of a diffuse sound field is not fulfilled, and thus Sabine's formula will not be reliable. The measured reverberation time may be either shorter or longer than predicted by Sabine's formula.

A shorter reverberation time will appear in a room in which the first reflections are directed towards the most absorbing surface. In an auditorium this is typically the floor with the audience, see Fig. 3.4.1 b.

In a rectangular room without sound scattering surfaces or elements, there is a possibility of prolonged decay in certain directions. In order to give an idea of the problem it is possible to calculate the different reverberation times associated to one-dimensional decays in each of the three main directions using the general reverberation formula (3.3.4).

$$T_{60} \approx \frac{13.8 \cdot l_m}{c \cdot \alpha_m} \approx 0.04 \cdot \frac{l_m}{\alpha_m} \qquad (l_m \text{ in m}) \qquad (3.4.3)$$

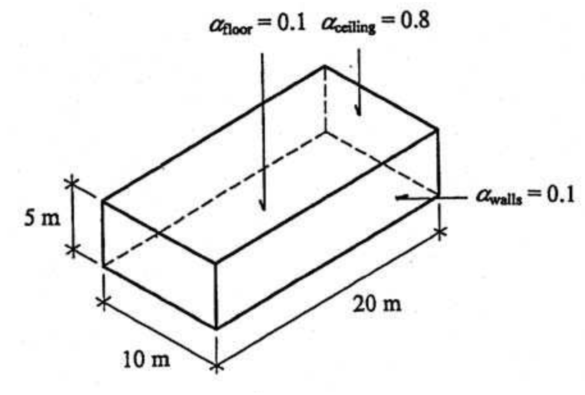


Figure 3.4.3. A rectangular room with indicated absorption coefficients.

As an example the room in Fig. 3.4.3 is considered. The ceiling has a high absorption coefficient (α = 0.8), but all other surfaces are acoustically hard (α = 0.1).

Volume $V = 5 \cdot 10 \cdot 20 = 1000$ m^3
Surface area $S = 700$ m^2
Equivalent absorption area $A = 200 \cdot 0.8 + 500 \cdot 0.1 = 210$ m^2
Mean absorption coefficient $\alpha_m = A / S = 210 / 700 = 0.30$
Mean absorption coefficient (height) $\alpha_m = (0.8 + 0.1) / 2 = 0.45$
Mean free path (3-dim.) $l_m = 4\ V / S = 4 \cdot 1000 / 700 = 5.7$ m
Mean free path (2-dim.) $l_m = \pi\ S_x / U = \pi \cdot 200 / 60 = 10.5$ m

The results are shown in Table 3.4.4. A two-dimensional reverberation in the horizontal plane between the walls has also been calculated (4.2 s). The one-dimensional decays are the extreme cases with the longest reverberation time being 20 times the shortest one, 8.0 s and 0.4 s, respectively!

Direction	l_m (m)	α_m	T_{60} (s)
3-dim. (Sabine)	5.7	0.30	0.8
3-dim. (Eyring)	5.7	0.30	0.6
2-dim. (horizontal)	10.5	0.10	4.2
1-dim. (length)	20	0.10	8.0
1-dim. (width)	10	0.10	4.0
1-dim. (height)	5	0.45	0.4

Table 3.4.4. Calculation of the one-dimensional reverberation times of the rectangular room in Fig. 3.4.3.

The real decay that is measured in the room will be a mixture of these different decays, and the reverberation time will be considerably longer than predicted from Sabine's formula. Eyring's formula is even worse. The measured decay curve will be bent, and thus the measuring result depends on which part of the decay curve is considered for the evaluation of reverberation time.

In a room with long reverberation time due to non-diffuse conditions and at least one sound-absorbing surface, introducing some sound scattering elements in the room can have a significant effect. It could be furniture or machines on the floor or some diffusers on the walls. This will make the sound field more diffuse, and the reverberation time will be reduced, i.e. it will come closer to the Sabine value. In other words: The sound absorption available in the room becomes more efficient when scattering elements are introduced to the room.

Note. In the one-dimensional case it is strictly not correct to use the arithmetic average of the absorption coefficients, if one of them is high. By inspection of (3.3.1) it is seen that the mean absorption coefficient should be calculated from

$$(1-\alpha_m) = \sqrt{(1-\alpha_1)(1-\alpha_2)} \qquad (3.4.4)$$

So, if one of the surfaces is reflective and the other is totally absorbing, α_m = 1 and hence the reverberation time is zero.

3.4.5 Optimum reverberation time and acoustic regulation of rooms

The optimum reverberation time depends of the activities in the room. It is important to choose the room volume and the surface materials with such sound absorbing properties that the reverberation time can get the right value for the purpose. In workshops with noise sources it is important to have a reverberation time as short as possible. In schools the classrooms should have a reverberation time between 0.6 and 0.9 s and independent of the frequency between 100 and 4000 Hz in order to obtain good acoustical conditions for speech. In concert halls the reverberation time should be around 1.5 to 2.2 s at mid frequencies (500 – 1000 Hz) with the longer values in the bigger halls. For music the reverberation time may be up to 50% longer at low frequencies (125 Hz) and somewhat shorter at high frequencies. The latter is unavoidable in a big hall due to the air attenuation.

Use of room	Optimum reverberation time, s (500 – 1000 Hz)
Cinema	0,4 – 1,0
Rock concert	0,8 – 1,1
Lecture	0,8 – 1,2
Theatre	1,0 – 1,2
Opera	1,3 – 1,7
Symphony concert	1,5 – 2,2
Choir concert	1,7 – 2,5
Organ music	2,0 – 3,0

Table 3.4.5. Optimum reverberation time at mid frequencies for various purposes in rooms with an audience.

3.4.6 Measurement of reverberation time

The reverberation time in a room can be measured with a noise signal or with an impulse. The traditional method uses white noise emitted by a loudspeaker and a microphone to measure the sound pressure level as a function of time after the source is switched off. This gives a decay curve and a typical example is shown in Fig. 3.4.4.

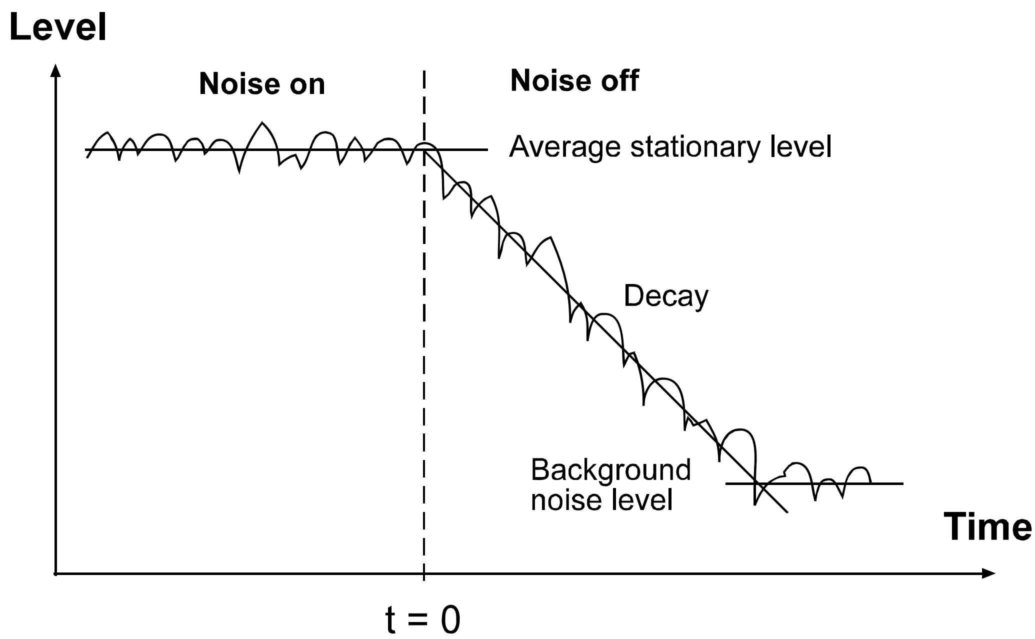


Figure 3.4.4. Typical decay curve measured with noise interrupted at the time t = 0.

From the microphone the signal is led to a frequency filter, which is either an octave filter of a one-third octave filter. If the sound in the room is sufficiently diffuse and a sufficient large number of modes are excited the decay curve is close to a straight line between the excitation level and the background level. The dynamic range is seldom more than around 50 dB and the whole range of the measured decay curve is not used. The lower part of the decay curve is influenced by the background noise and the upper part may be influenced by the direct sound, which gives a steeper start of the curve. So, the part of the decay curve used for evaluation begins 5 dB below the average stationary level and ends normally 35 dB below the same level. The evaluation range is thus 30 dB and the slope is determined by fitting a straight line or automatically by calculating the slope of a linear regression line. From the slope of the decay curve in dB per second is calculated the reverberation time, which is the time for a 60 dB drop following the straight line. The result is sometimes denoted T_{30} in order to make it clear that the actually used evaluation range is 30 dB.

If the background noise is too high and a sufficient dynamic range is not available the reverberation time can instead be measured as T_{20}. In this case the slope of the decay curve is evaluated between –5 dB and –25 dB below the excitation level.

The reverberation time is measured in a number of source- and receiver positions, and in each position the decay is determined as an average of a number of excitations. White noise is a random noise signal and thus the measured decay curves are always a little different.

Sometimes the decay curves are not nice and straight and it is difficult to measure a certain reverberation time. One reason can be that it is a measurement at low frequencies in a small room and maybe only two or three modes are excited within the frequency band of the measurement. In this case there may be interference between the modes causing very irregular decay curves.

Another difficult situation is coupled rooms, i.e. a room divided into sections with different reverberation times. A typical example is a theatre with a reverberant stage house and a rather dead auditorium. In this case the decay curve will be bent, i.e. the upper part shows a short reverberation time and the lower part shows a longer reverberation time. It might be possible to determine both of these reverberation times, however, the shorter one representing the initial decay is the most important one, because the subjective evaluation of the reverberation is related to the initial decay.

3.5 REFERENCES

C.M. Harris (1966). Absorption of sound in air versus humidity and temperature. JASA **40,** pp 148-159.
C.W. Kosten (1960). The mean free path in room acoustics. Acoustica **10**, pp 245-250.
J. Petersen 1984). Rumakustik (in Danish). SBI-anvisning 137. Danish Building Research Institute.
W.C. Sabine (1922). Collected papers on acoustics. Dover Publications, Inc. 1964, New York.

4 Sound absorbers and their application in room design

Anders Chr. Gade

4.1 Introduction

The reverberation time T_{60} as defined in Section 3.2.5 is the most important descriptor of the acoustics of a room. Therefore, calculating predictions of T_{60} (e.g. according to Equation 3.4.1) is a very basic part of room acoustical design which in turn calls for the availability of reliable data on the frequency dependant sound absorption characteristics of materials used for room surface cladding and for furnishing of rooms (such as furniture, people and machinery).

In Table 3.4.1 absorption coefficients per octave band were listed for some materials generally found in rooms. The values indicate that some of these, e.g. windows and wooden floors on studs, primarily absorb low frequency sounds. On the other hand, curtains and persons (see Table 3.4.2) mainly absorb middle and high frequencies. In order to obtain a well balanced T_{60} versus frequency for a given type of room it is therefore important to mix properly different types of materials when designing the room.

In this chapter we will give a basic introduction to the physical mechanisms involved in sound absorption and present some types of sound absorption materials well suited for - or specifically designed for - sound absorption and reverberation control. The absorption properties will be described in terms of the sound absorption coefficient as defined in Section 1.5.2.

For certain types of rooms, such as schools and work rooms, general demands on reverberation control exist. Therefore the last section in this chapter is devoted to examples on how sound absorbing materials can be applied in the design of such rooms.

4.2 The room method for measurement of sound absorption.

In Section 1.5.2, a method for measuring the absorption coefficient, the tube method, was presented which reveals the absorption coefficient for a single angle of incidence (usually normal incidence as illustrated to the left in Figure 4.2.1). However, the absorption will normally depend on the direction of sound incidence[1]. Materials applied in rooms with a (more or less) diffuse sound field will be exposed to sound arriving from many different directions as illustrated in Fig. 4.2.1(c). Therefore we will start this chapter by presenting a method for measurement of sound absorption, which provides the relevant diffuse field absorption coefficient: the reverberation room method.

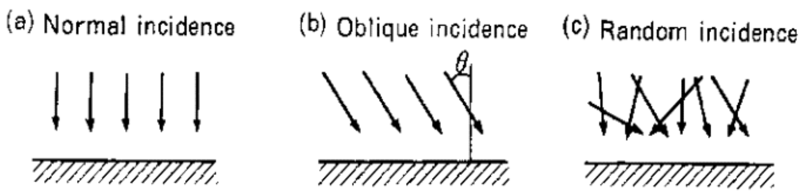


Figure 4.2.1 Different conditions for sound incidence on a surface. From [1]

[1] The absorption for oblique incidence as illustrated in case (b) in Figure 4.2.1 – or as a function of angle of incidence - can be measured using various techniques using separation in time or subtraction of incident and reflected sound pulses. However, these techniques are not always very reliable.

The measurement takes place in a reverberation room, with highly irregular or non parallel surfaces and/or suspended, sound diffusing elements. Hereby it can be assumed that the sound field will fulfil the requirements for application of the Sabine reverberation equation. Assume the room has a volume V, total surface area S and that α_{empty} is the absorption coefficient of the room surfaces (which ideally should all be made from the same, acoustically hard material). In this case equations 3.4.1 and 3.4.2 (disregarding air absorption) yields:

$$T_{60,empty} = \frac{0.16V}{S_{Room}\,\alpha_{empty}} \quad (4.1)$$

If now we place a test sample of a material with area S_{sample} (usually 10 m^2) in the room, the equation changes into:

$$T_{60,sample} = \frac{0.16V}{S_{sample}\,\alpha_{sample} + \left(S_{Room} - S_{sample}\right)\alpha_{empty}} \quad (4.2)$$

in which we have considered that an area, S_{sample}, of the room surface has now been covered by the sample. Combining equations 4.1 and 4.2 by eliminating S yields for the unknown absorption coefficient, α_{sample}, of the test sample:

$$\alpha_{sample} = \frac{0,16V}{S_{sample}}\left[\frac{1}{T_{60,sample}} - \frac{1}{T_{60,empty}}\right] + \alpha_{empty} \quad (4.3)$$

The measurement is normally carried out in 1/1 or 1/3 octave bands from 100 to 5000 Hz.

If absorption measurements using the room method is carried out on small sized samples, these sometimes appear to have a absorption coefficient larger than 1.0, as seen in Figure 4.2.2. Of course this is not logical, if the absorption power should be related solely to the physical area of the sample. The phenomenon is probably due to diffraction of sound around the edges of the sample, which dominates the behaviour in cases where the linear dimension of the sample approaches the wave length of the sound, i.e. the effect is more pronounced at low frequencies.

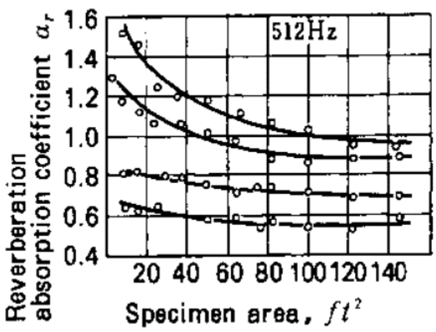
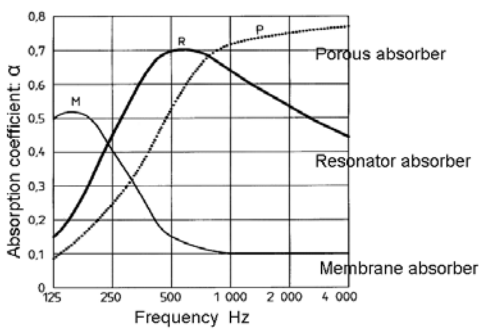


Fig. 4.2.2 Absorption coefficients of different materials versus area (measured in square feet). From [1].

Although a complication in documentation of absorption properties, this phenomenon can be applied successfully in practice by providing increased absorption effect, if the available absorption material can be provided in smaller pieces and spread out over the room surfaces.

4.3 Different types of sound absorbers



In this section the three most common types of sound absorbing constructions will be described, each with its own characteristic frequency dependency of the absorption coefficient as sketched in Fig. 4.3.1.

Fig. 4.3.1 Typical behaviour of absorption versus frequency for Porous, resonating and membrane absorbers respectively.

4.3.1 Porous absorbers

Porous absorbers are present in rooms in the form of textiles like curtains, carpets and furniture upholstery, porous mortar in (unpainted !) brick walls and not least as a wide variety of dedicated sound absorbing products for suspended ceilings.

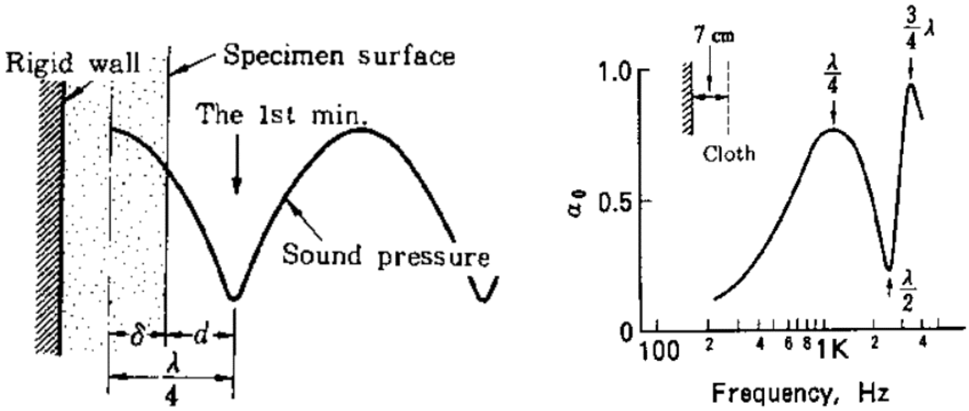


Figure 4.3.2 Left: Standing wave pattern formed by an incident and a reflected sound wave in front of a porous material of a certain thickness flush mounted on a heavy and hard surface. Right: Absorption versus frequency of a thin, porous sheet placed in front of a hard surface. From [1].

Porous materials are characterized by having an open structure of e.g. of fibres glued or woven together which is accessible by the air. Thus, air can be pressed through the material more or less easily depending on the flow resistance (determined e.g. by how densely a fabric is woven – try for yourself by blowing through clothing or curtains !). The absorption properties are caused by viscous friction between the moving air molecules in the sound waves and the often huge internal surface area of the structure whereby the (kinetic) sound energy is converted into heat.

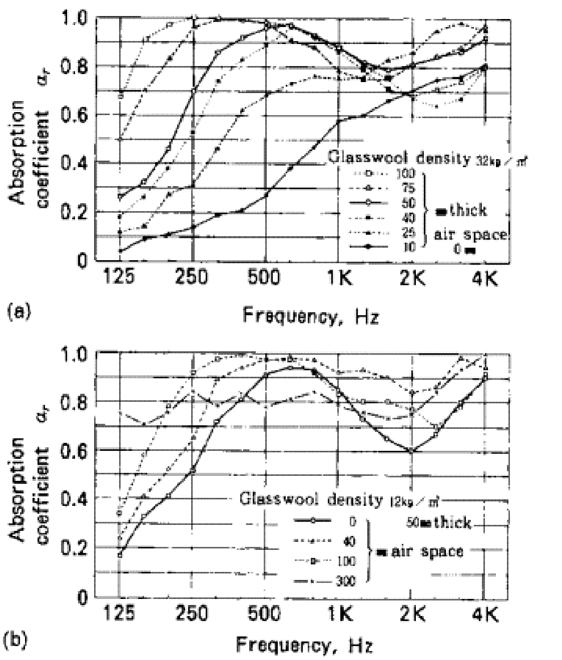


Fig. 4.3.3 Absorption coefficients for mineral wool (glasswool) with thickness as parameter (a) and with wall distance as parameter (b).

If a porous sheet of a certain thickness is placed flush on a rigid surface and hit by an normal incidence sound wave a standing wave pattern will be created with pressure amplitude as indicated to the left in Fig. 4.3.2. As seen from Figure 1.2.10 (c), with a rigid termination, the particle velocity and so the kinetic energy of the sound field will be high where the pressure amplitude (the potential energy) is low. In other words, for the absorber to be efficient (with normal incidence of the sound wave), the thickness of the porous layer need to be at least $\lambda/4$, so that friction takes place where the energy is primarily kinetic. In other words, for a given thickness of the material, there is a lower limiting frequency below which the absorption drops off because the material can no longer "reach" the region of high kinetic energy. On the other hand, as the absorber is not absorbing the potential energy anyway, one can save material and just place a thin sheet (but still with a suitable flow resistance) at a certain distance from the rigid wall (like a curtain in front of a window). In the case of normal incidence, applying a thin sheet will cause the absorption to drop again at a higher frequency where the distance between sheet and hard wall equals λ/2; but with diffuse field incidence this dip will not be very pronounced. Diffuse field incidence also causes the absorbers to be effective ($\alpha > 0{,}8$) if just the thickness/distance is > λ/8.

Fig. 4.3.3 shows how the absorption coefficient varies with frequency for mineral wool mats of different thickness (upper graph) and different distances to the rigid wall (lower graph). It is seen that more low frequencies are absorbed as the thickness or the wall distance increases.

Mineral wool consists of thin fibres pressed and glued together. The fibres are made from melted glass (Glasswool) or stone (Rockwool) much like "Candy Floss". Mineral wool is used as porous sound absorbers, very often in the form of tiles which can be mounted in a suspended ceiling system. Such ceilings will often be placed below ventilation ducts and other technical installations, whereby a large distance (typically between 20 cm and one metre) is ensured to the hard concrete deck behind. Hereby the ceiling can absorb efficiently over a wide frequency range – as well as hide the installations. Mineral wool ceiling tiles are normally given a carefully controlled layer of special paint from the factory to make them look like normal (white) plaster ceilings as much as possible. However, if one tried to repaint them, the porous properties and so the absorption normally disappears.

4.3.2 Membrane absorbers

A membrane absorber is characterized by consisting of a non porous sheet or panel placed at a certain distance from a hard backing whereby an air filled cavity is formed. This system can resonate at frequency determined by the mass per unit area of the plate, *m*, and the spring function of the enclosed air, which is determined by the depth, *L*, of the cavity:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{\rho c^2}{m L}} \tag{4.4}$$

However, Equation 4.4 only apply if the plate is completely limp. Normally, the resonance frequency is also determined by the plate stiffness and mode of plate vibration, of which a few are illustrated in Fig. 4.3.4, with p and q being integers determining the shape of the two dimensional oscillation pattern of the plate.

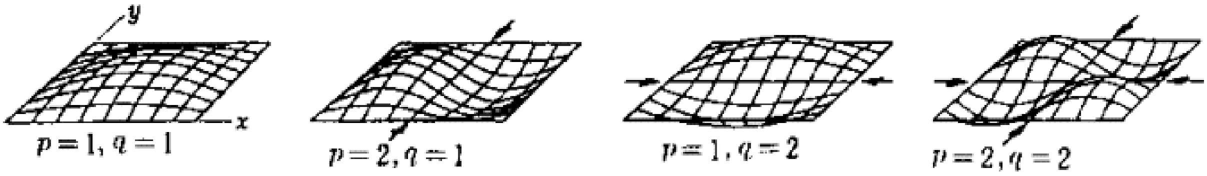


Fig. 4.3.4 Different modes of vibration in a stiff plate.

In this case the resonance frequency can be described as follows:

$$f_r = \frac{1}{2\pi}\sqrt{\frac{\rho c^2}{m L} + \frac{\pi^4}{m}\left[\left(\frac{p}{a}\right)^2 + \left(\frac{q}{b}\right)^2\right]^2 \frac{E h^3}{12\left(1-\nu^2\right)}} \tag{4.5}$$

in which *a* and *b* are the dimensions of the plate (or the distance between studs supporting the plate), *h* is the thickness while *E* and *ν* are the Young's modulus and the Poisson ratio respectively.

From this formula it is seen that a resonance frequency is determined completely by the stiffness if the depth of the cavity is infinitely deep – as is the case e.g. with a single pane window.

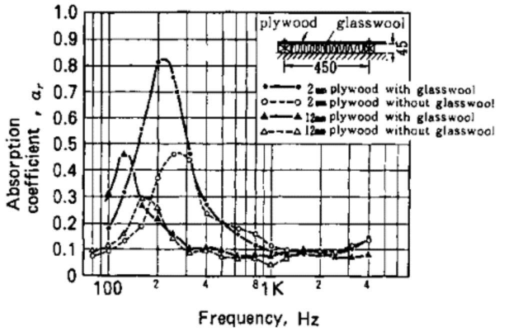


Fig. 4.3.5 absorption versus frequency of membrane absorber for two different plate thicknesses and with and without mineral wool in the cavity. From [1]

Fig. 4.3.5 show absorption versus frequency for two different thickness of plywood placed 45 mm from a hard backing – with and without mineral wool in the cavity. As expected it is seen that the thicker and heavier plate result in the lowest resonance frequency as expected from equations 4.4 and 4.5. Besides, it is observed that the mineral wool inlay, which increases the internal damping of the construction causes a significant improvement in the absorption around the resonance frequency and also causes the resonance frequency to become lower.

Membrane absorbers are often found in rooms in the form of wooden floors on joists or as gypsum board or wood panel walls. The effect is a controlled low frequency T_{60} value as opposed to rooms made entirely from heavy concrete or masonry which causes the sound to be "dark" and blurred at low frequencies.

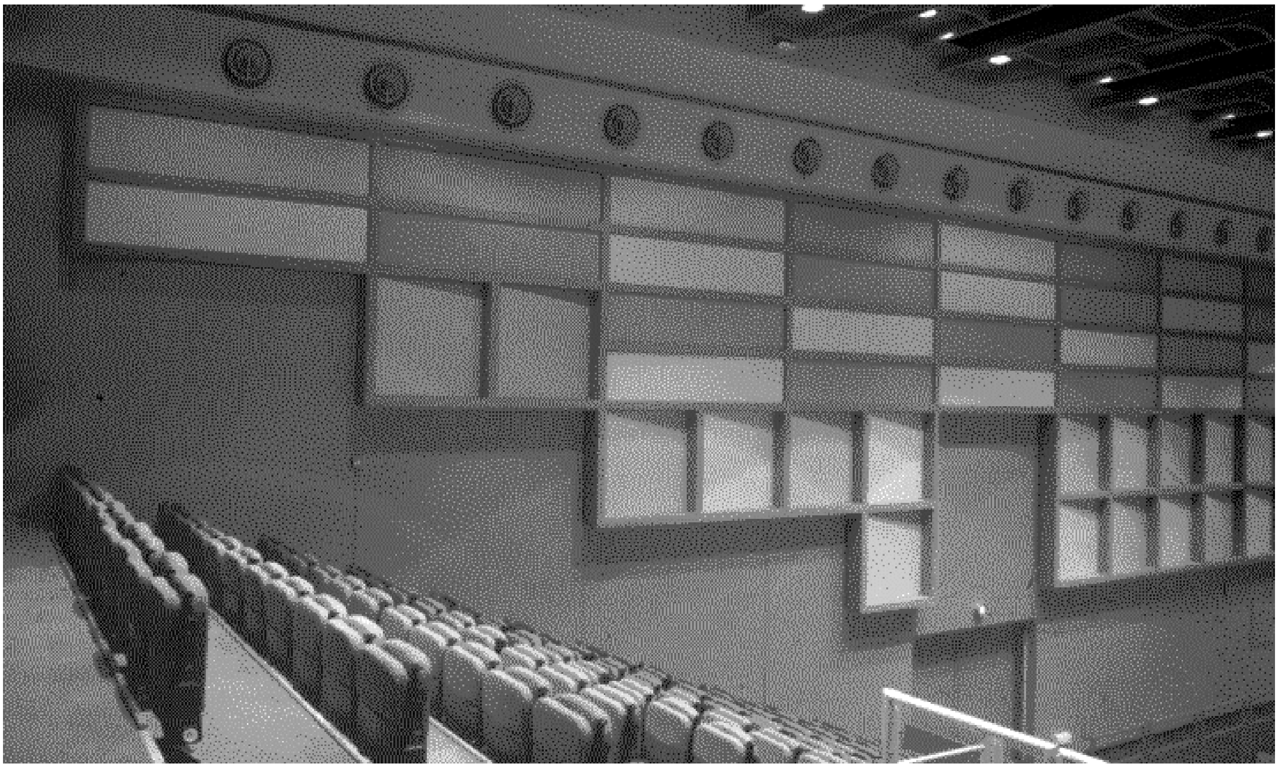


Fig. 4.3.6 Example of membrane absorbers attached to the concrete side wall in the multi purpose hall (Kolding Teater). Besides controlling low frequency reverberation, the panels also provide some diffusion of the sound.

4.3.3 Resonator absorbers
In stead of having a plate forming the mass of the resonating system, the mass can be oscillating air in an opening between a closed cavity and the open atmosphere. Also in this case, the enclosed air

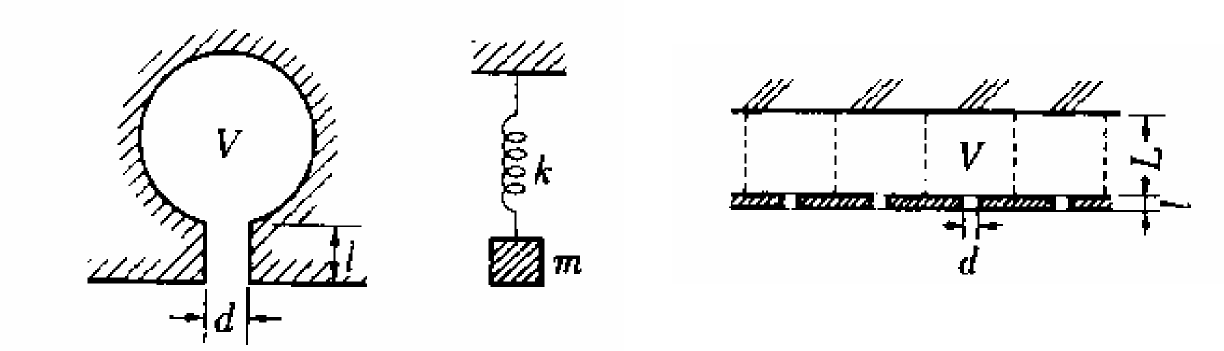


Fig. 4.3.7 Single resonator (left) and resonating panel (right). From [1].

in the cavity provides the spring function. An example of such a single resonator, called a Helmholz resonator, is illustrated in Figure 4.3.7. The resonance frequency (which can be experienced by blowing across the opening of a bottle) is given by:

$$f_0 = \frac{c}{2\pi}\sqrt{\frac{S}{V(l+\delta)}} \tag{4.6}$$

with S being the area of the opening, V being the enclosed volume, l the length of the neck and δ a correction to the neck length which is due to the fact that the oscillating air mass - often moving with very high velocity - is not confined to the physical length of the neck; but some of the air outside both ends of the neck will be moving as well.

Resonators like the build in "bottle" in the left side of Fig. 4.3.7 are not very practical, as the frequency range of the absorption is normally very limited around the sharp resonance frequency. However, if a perforated panel is placed in front of a cavity as seen to the right in Fig. 4.3.7, then this construction can be regarded as a large number of single resonators put together, and the physical proportions in this case often causes a much more useful frequency range of absorption. For the resonance frequency of the panel we have:

$$f_0 = \frac{c}{2\pi}\sqrt{\frac{P}{L(l+\delta)}} \tag{4.7}$$

which is almost identical with Equation 4.6 except for the opening area being replaced by the degree of perforation, P, of the panel and the volume V being changed into the depth of the cavity L. If the holes are circular with diameter d, we have for the end correction: $\delta \approx 0.8\ d$. Resonating panels will often have a higher resonance frequency and absorb efficiently in a wider frequency range than the membrane absorbers.

Regarding damping, the viscous damping can be significant if the hole/slit dimensions are small; but often the absorption can be optimised by placing a thin layer of mineral wool or glass felt (called vlies) in the cavity. Like in the case of the membrane absorber, it is important to adjust the damping to achieve optimal absorption.

Perforated panels are found in the form of perforated gypsum board or steel plates (used e.g. for suspended ceilings)[2], or as panels made of wooden boards with slits between the individual boards as illustrated to the left in Figure 4.3.8. Other possibilities are walls made from perforated tiles, which make use of the cavity already present in a double masonry wall as shown to the right in the same Figure.

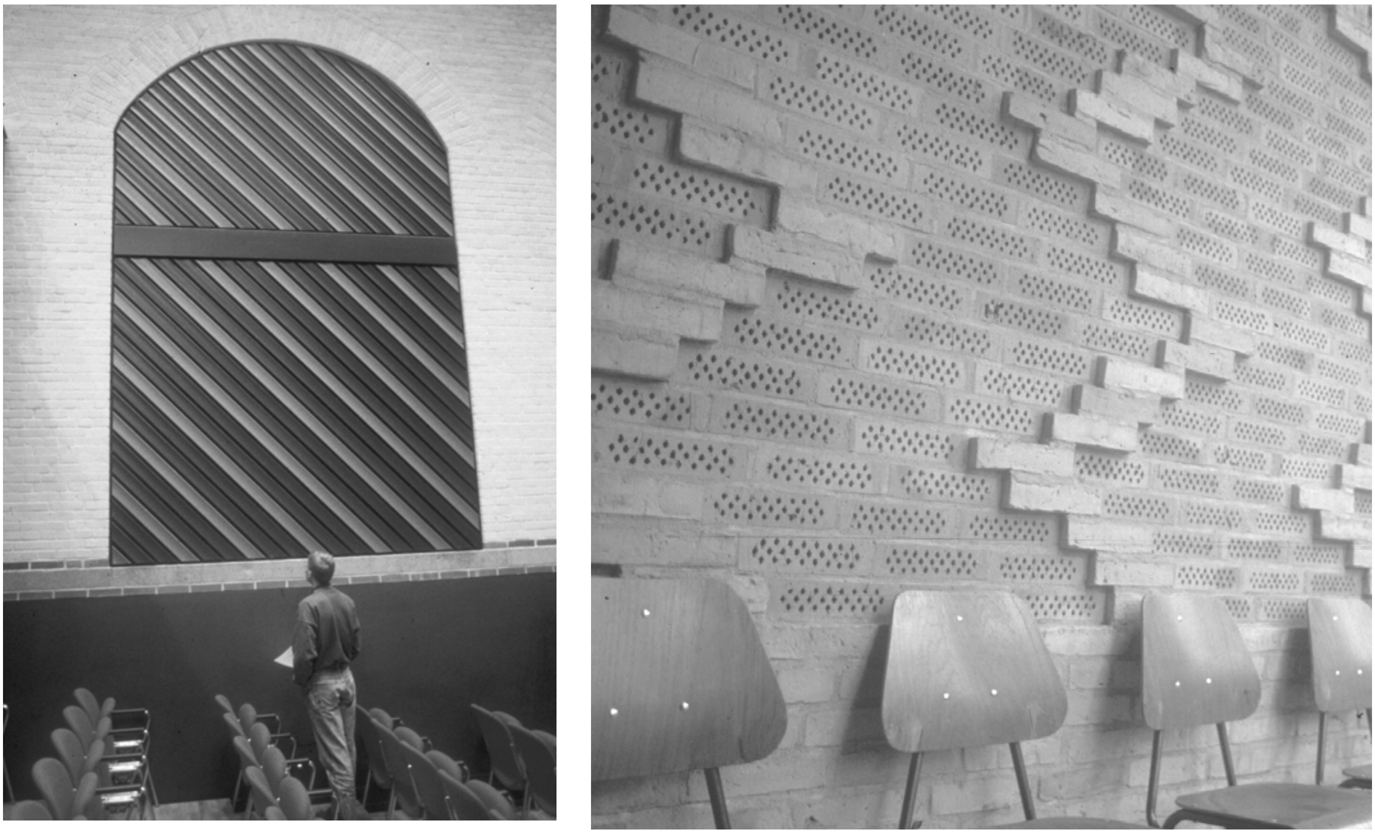


Fig. 4.3.8 Resonating panel constructions in practice. Left: Wooden boards separated by controlled gaps in front of a former window niche filled with mineral wool. The panel controls low frequency reverberation in a former power plant building made from heavy masonry converted into a concert hall (Værket, Randers). Right: Perforated bricks on the rear wall in a sports and multi purpose hall. By making this wall absorbing, echoes back to the stage placed more than 50 m away are avoided (Frihedshallen, Sønderborg).

4.4 Application of sound absorbers in room acoustic design

The main purpose of introducing absorption for reverberation control in rooms is to reduce noise levels (see Fig. 3.2.4) and in some cases to increase intelligibility. The Danish Building Law (Bygningsreglementet af 1995, BR95) [2] contains demands on maximum T_{60} values in school class rooms, day care institutions and apartment buildings, whereas the Danish Working Environment Agency have issued rules for industrial buildings and offices [3]. These current Danish rules are

[2] It should be added that in many cases with perforated gypsum or steel plates used as suspended ceilings, the combinations of perforation and cavity depth causes the absorber to act more like a porous absorber but with reduced performance at high frequencies due to the panel shielding off the porous layer to some degree.

briefly listed in Figure 4.4.1. Recommendable values for other types of rooms – including auditoria and concert halls were listed in Table 3.4.5. Special standards exist for design of cinemas and studio control rooms and listening rooms. In Denmark, no rules exist for other public spaces like traffic terminals, sports arenas and restaurants - although the acoustic conditions in these places are often horrible. However, acoustic concerns a generally included in modern design of these spaces as well.

INDUSTRY #

Rooms < 200 m³ :	T < 0.6 Sec. (avg. 125 - 2000 Hz)
Rooms 200-1000 m³ :	T dep. on volume up to 1.3 Sec.
Rooms > 1000 m³ :	Abs. area > 0.6 x floor area
- and room height > 5 m :	Abs. area > 0.7 x floor area

OFFICE SPACES

One person offices :	T < 0.6 Sec. (avg. 125 - 2000 Hz)
Open plan 75-300 m² :	Abs. area > 0.8 x floor area #
Open plan > 300 m² :	Abs. area > 0.9 x floor area #
Corridors:	T < 1.0 Sec. (avg. 500 - 2000 Hz)

Danish working environment agency requirement

SCHOOLS *

Normal class rooms :	T < 0.9 Sec. (avg. 125 - 2000 Hz)
Special class rooms : + day care rooms	T < 0.6 Sec. (e.g. in rooms with hard of hearing pupils)
Open plan areas:	Abs. area > 0.9 x floor area
Gym halls:	1.6 Sec. (avg. 125 - 2000 Hz)
Swim halls:	2.0 Sec. (avg. 125 - 2000 Hz)
Corridors/Staircases:	T < 0.9/1.3 Sec.(avg 500-2000Hz)

DWELLINGS *

Corridors/Staircases:	T < 0.9/1.3 Sec.(avg 500-2000Hz)

* Danish building law

Fig. 4.4.1 Listing of Danish rules regarding maximum values of reverberation time in buildings. (The values listed for single person offices and corridors in office buildings just reflect common design practice.)

As indicated in Figure 4.4.1 the rules for large industrial halls as well as open plan areas in offices and schools are specified in terms of a required minimum absorption area. The reason for this is that often calculation as well as measurement of T_{60} is often questionable in these rooms.

In most cases the ceiling is the most obvious surface to treat with absorption, as it constitutes a large area which is normally available apart from a few light or ventilation fixtures and because here the often delicate absorption materials are not subject to mechanical damage.

Fig. 4.4.2 Examples of acoustic treatment mounted in ceiling in industrial halls. Left: suspended ceiling of mineral wool tiles with integrated light fixtures. Right: Vertical Mineral wool baffles.

In Figure 4.4.2 are shown two examples of acoustic treatment of ceilings. To the left a normal suspended ceiling of mineral wool tiles with integrating lighting and ventilation. This type of ceiling is often found in offices, schools, shops etc.. The vertical mineral wool baffles shown to the right can be a solution when the ceiling is already heavily occupied by technical installations.

In rooms where practically all the absorption is placed in the ceiling, the reverberation time basically becomes a function of the room height as shown in Figure 4.4.3. In high rooms, it is not always sufficient to place the absorption in the ceiling surface alone; but also available wall areas must be used as illustrated by the mineral wool tiles to the right in Figure 4.4.3.

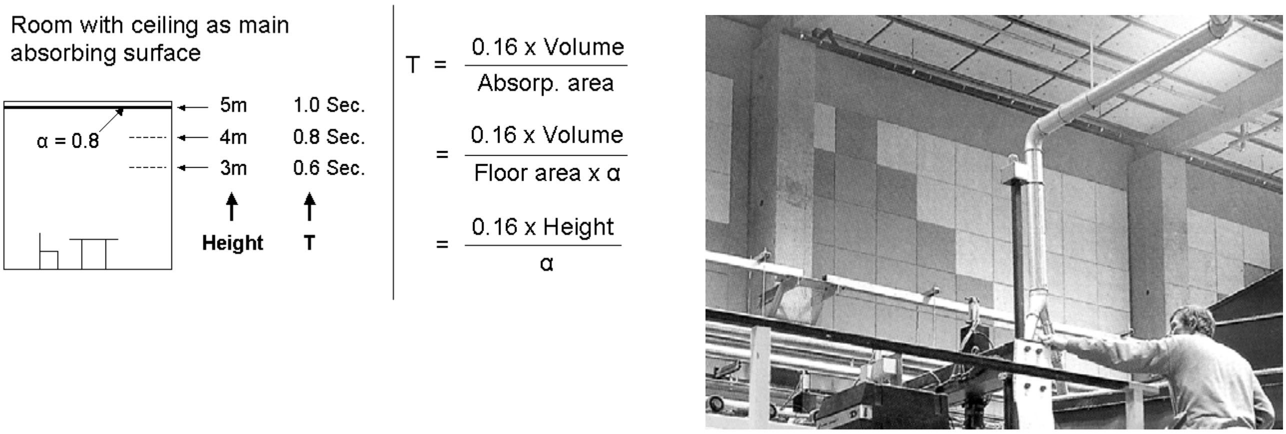


Fig. 4.4.3 Simplified calculation of T_{60} in room with all absorption placed on the ceiling surface(left) shows the need for additional absorption on walls in tall rooms (right).

In many public places like traffic terminals, department stores, sports halls etc., the room acoustic absorption treatment is not only done with the purpose of reducing noise but also to ensure proper intelligibility of speech (often emitted through loudspeakers). In Figure 4.4.4 is illustrated how a long room decay can cover (mask) the weak phonems illustrated schematically as vertical bars. In speech the consonant sounds are often the weaker elements; but they contain most of the information. Therefore, a long reverberation can seriously deteriorate intelligibility.

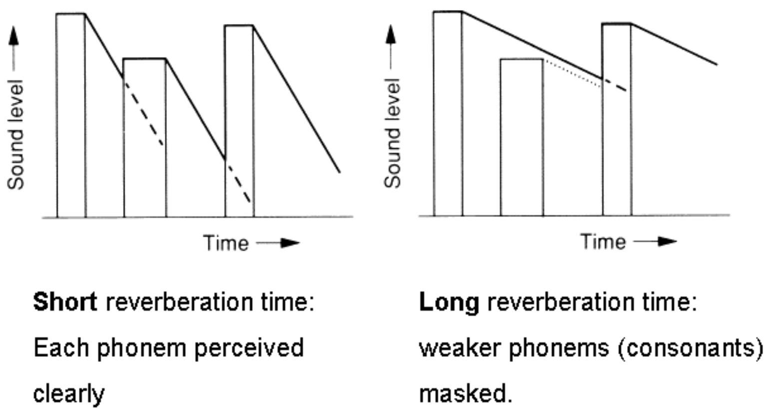


Fig. 4.4.4 Schematic illustration of the influence of reverberation on the intelligibility of speech.

In rooms dedicated for speech like auditoria, class rooms and theatres, the room acoustic design not only consists of reverberation control by absorption treating of the room surfaces. In these rooms also the design of the room geometry is important to ensure proper propagation of sound from the source to the listeners through reflection of the sound waves off non absorbing room surfaces. And in order to support intelligibility, these reflections must arrive not long (up to 40 ms) after the direct sound.

Even in normal sized class rooms this concern about supporting reflections may be applied by leaving a central part of the ceiling reflective (given that enough other surface areas can be found to provide the required reverberation control). Thus, Fig. 4.4.5 illustrates such a case in which the ODEON programme was used to balance the application of absorbing and reflective part of the ceiling for a school project and to predict reverberation time and the intelligibility in terms of the Speech Transmission Index mentioned in Section 2.7.

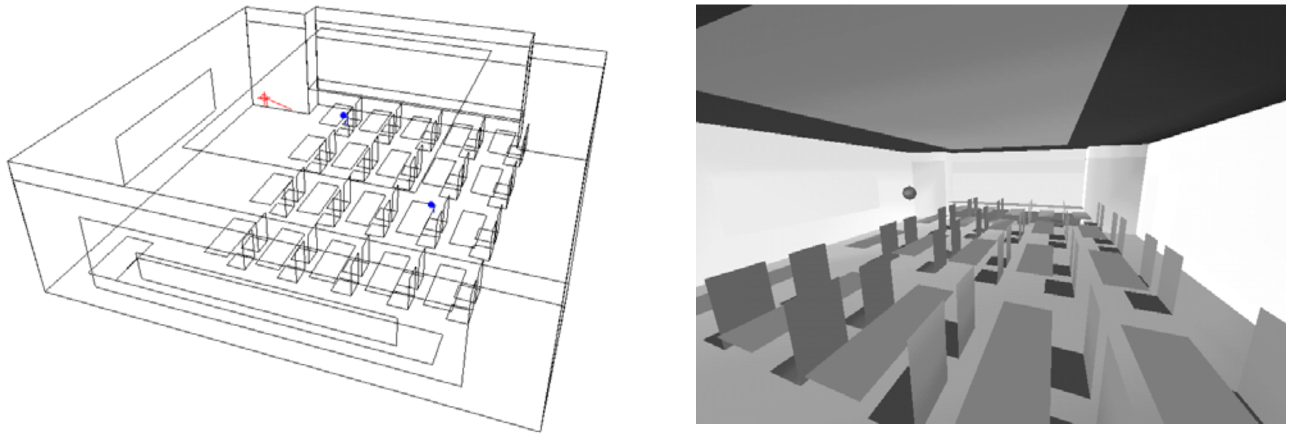


Fig. 4.4.5 Illustrations from the room acoustic simulation programme ODEON of a class room design with a partly absorbing (dark) and reflective (lighter grey) ceiling.

References

[1] Z. Maekawa and P. Lord: Environmental and Architectural Acoustics. E & FN Spon, London, 1994.

[2] Bygningsreglementet; Bygge- og Boligstyrelsen 1995. Publications from the Danish National Agency for Enterprise and Construction (in Danish) can be found at: http://www.ebst.dk/pub_lydforhold/0/8/0

[3] AT anvisning nr.1.1.0.1, november 1995; Akustik i arbejdsrum (Acoustics in work places) http://www.at.dk/sw5110.asp

5. An introduction to sound insulation

Jens Holger Rindel

5.1 THE SOUND TRANSMISSION LOSS

5.1.1 Definition

A sound wave incident on a wall or any other surface separating two adjacent rooms partly reflects back to the source room, partly dissipates as heat within the material of the wall, partly propagates to other connecting structures, and partly transmits into the receiving room.

The power incident on the wall is P_1 and the power transmitted into the receiving room is P_2. The *sound transmission coefficient* τ is defined as the ratio of transmitted to incident sound power

$$\tau = \frac{P_2}{P_1} \qquad (5.1.1)$$

However, the sound transmission coefficients are typically very small numbers, and it is more convenient to use the sound *transmission loss* R with the unit deciBel (dB). It is defined as

$$R = 10\log\frac{P_1}{P_2} = 10\log\frac{1}{\tau} = -10\log\tau \quad \text{(dB)} \qquad (5.1.2)$$

Another name for the same term is the *sound reduction index*.

5.1.2 Sound insulation between two rooms

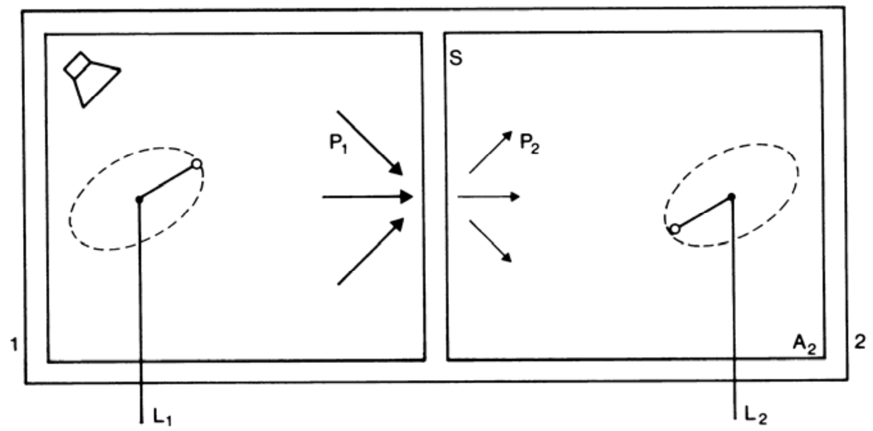


Figure 5.1.1. Airborne sound transmission from source room (1) to receiving room (2)

The most common case is the sound insulation between two rooms. With the assumption of diffuse sound fields in both rooms it is possible to derive a simple relation between the transmission loss and the sound pressure levels in the two rooms. The rooms are called the source room and the receiving room, respectively. In the first room is a sound source that generates the average sound pressure p_1. The sound power incident on the wall is, see eq. (3.2.6)

$$P_1 = I_{inc} S = \frac{p_1^2 \, S}{4\rho c} \qquad (5.1.3)$$

The area of the wall is S. In the receiving room the average sound pressure p_2 is generated from the sound power P_2 radiated into the room, see eq. (3.2.8)

$$p_2^2 = \frac{4 P_2}{A_2} \rho c \qquad (5.1.4)$$

Here A_2 denotes the absorption area in the receiving room. Insertion in the definition (5.1.2) gives

$$R = 10 \log \frac{p_1^2 \, S}{p_2^2 \, A_2} = L_1 - L_2 + 10 \log \frac{S}{A_2} \quad \text{(dB)} \qquad (5.1.5)$$

Here L_1 and L_2 are the sound pressure levels in the source and receiving room, respectively. This important result is the basis for transmission loss measurements.

5.1.3 Measurement of sound insulation

Sound insulation is measured in one-third octave bands covering the frequency range from 100 Hz to 3150 Hz. In recent years the international standards for measurement of sound insulation have been revised and it is recommended to extend the frequency range down to 50 Hz and up to 5000 Hz. One reason for this is that the low frequencies 50 – 100 Hz are very important for the subjective evaluation of the sound insulation properties of lightweight constructions. In recent years lightweight constructions have been more commonly used in new building technology, whereas heavy constructions have traditionally been used for sound insulation.

The sound pressure levels are measured as the average of a number of microphone positions or as the average from microphones slowly moving on a circular path. The results are averaged over two different source positions. More details are given in ISO 140 Part 3 and 4.

In addition to the two sound pressure levels it is also necessary to measure the reverberation time in the receiving room in order to calculate the absorption area. Sabine's equation is used for this, see eq. (3.2.13)

$$A_2 = \frac{55.3 \, V_2}{c \, T_2} \qquad (5.1.6)$$

Only under special laboratory conditions it is possible to measure the transmission loss of a wall without influence from other transmission paths. In a normal building the sound will not only be transmitted through the separating construction, but the flanking constructions will also influence the result, see later in section 5.5.4.

For measurements of sound insulation in buildings the *apparent sound transmission loss* is

$$R' = L_1 - L_2 + 10 \log \frac{S}{A_2} \quad \text{(dB)} \qquad (5.1.7)$$

The apostrophe after the symbol indicates that flanking transmission can be assumed to influence the result.

5.1.4 Multi-element partitions and apertures

A partition is often divided into elements with different sound insulation properties, e.g. a wall with a door. Each element is described by the area S_i and the transmission coefficient τ_i . If the sound intensity incident on the surfaces of the source room is denoted I_{inc} the total incident sound power on the partition is

$$P_1 \quad = \quad \sum_{i=1}^{n} S_i \, I_{inc} \quad = \quad S \, I_{inc}$$

The total area is called *S*. The total sound power transmitted through the partition is

$$P_2 \quad = \quad \sum_{i=1}^{n} \tau_i \, S_i \, I_{inc}$$

Thus, the transmission coefficient of the partition is

$$\tau_{res} \quad = \quad \frac{P_2}{P_1} \quad = \quad \frac{1}{S} \sum_{i=1}^{n} \tau_i \, S_i \tag{5.1.8}$$

The same result can also be written in terms of the transmission losses R_i of each element

$$R_{res} \quad = \quad -10 \log \tau_{res} \quad = \quad -10 \log \left(\frac{1}{S} \sum_{i=1}^{n} S_i \, 10^{-0,1 R_i} \right) \tag{5.1.9}$$

In the simple case on only two elements the graph in Fig. 5.1.2 may be used.

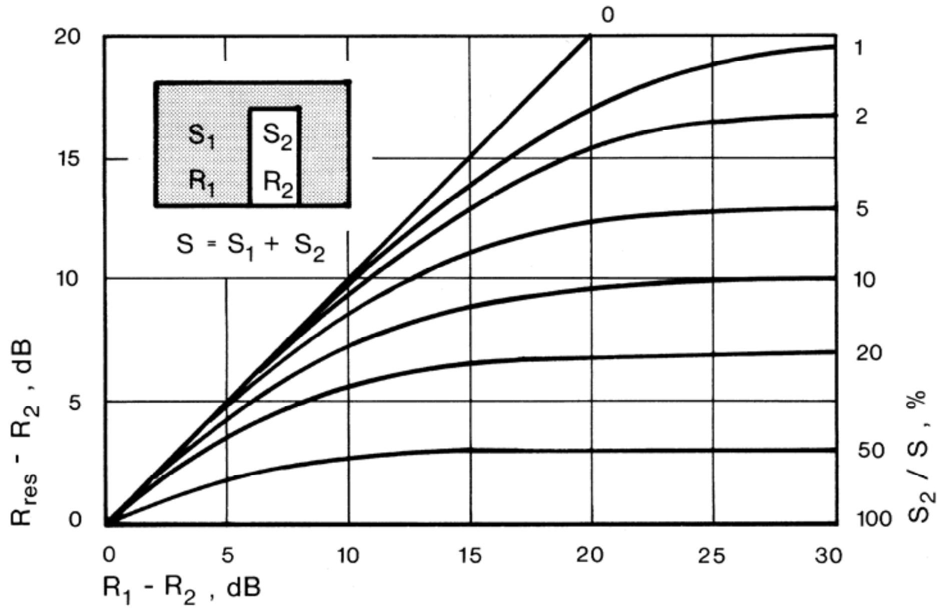


Figure 5.1.2. Graph for estimating the transmission loss of a multi-element partition

An aperture in a wall is a special example of an element with different transmission properties. As an approximation it can be assumed that the transmission coefficient of the aperture is 1. If also the area of the aperture S_{ap} is very small compared to the total area, this leads to the following result for the resulting transmission loss of the wall with aperture:

$$R_{res} \quad = \quad -10 \log \left(\frac{1}{S} \left(S_1 \, 10^{-0,1 R_1} + S_{ap} \right) \right) \quad \cong \quad -10 \log \left(10^{-0,1 R_1} + \frac{S_{ap}}{S} \right) \tag{5.1.10}$$

Fig. 5.1.3 can illustrate the result. It is seen that the relative area of the aperture defines an upper limit of the sound insulation that can be achieved.

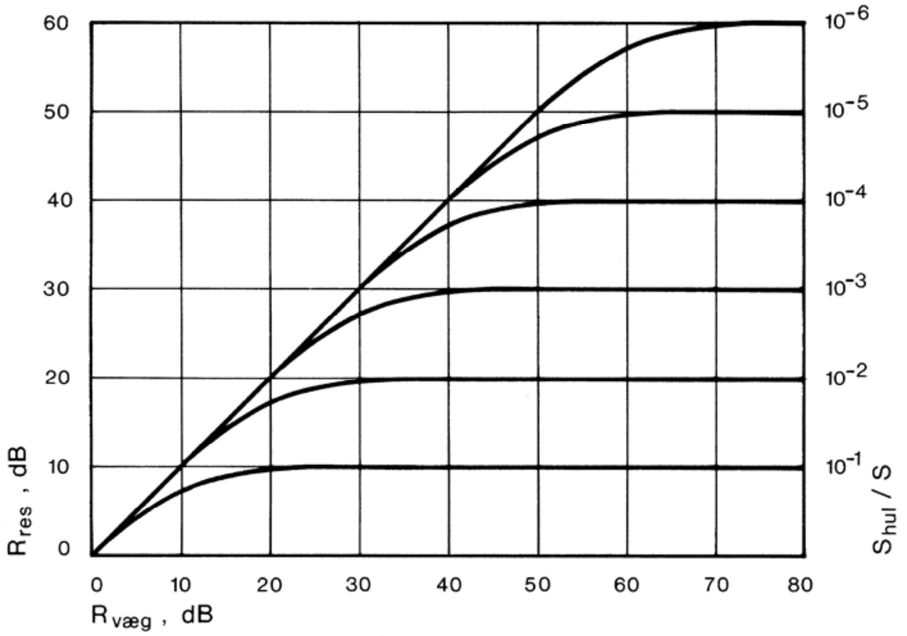


Figure 5.1.3. Graph for estimating the transmission loss of a construction with an aperture

5.2 SINGLE LEAF CONSTRUCTIONS

5.2.1 Sound transmission through a solid material

The solid material is supposed to have the shape of a large plate with thickness h. The material is characterised by the density ρ_m and the speed of longitudinal waves c_L . The surface of the material defines two transition planes where the sound waves change from one medium to another. It is assumed that the medium on either side is air with the density ρ and the speed of sound c (also longitudinal waves). The symbols and notation are explained in Fig. 5.2.1.

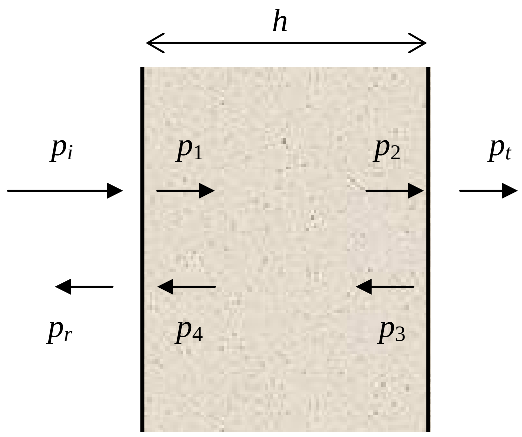


Figure 5.2.1. Thick wall with incident, reflected and transmitted sound waves

The sound pressure is equal on either side of the two transition planes:

$$\begin{aligned} p_i + p_r &= p_1 + p_4 \\ p_t &= p_2 + p_3 \end{aligned} \qquad (5.2.1)$$

Also the particle velocity is equal on either side of the two transition planes:

$$\begin{aligned} u_i - u_r &= u_1 - u_4 \\ u_t &= u_2 - u_3 \end{aligned} \qquad (5.2.2)$$

The characteristic impedance in the surrounding medium (air) is denoted Z_0 and that in solid material is denoted Z_m. Thus the ratio of sound pressure to particle velocity in each of the plane propagating waves is:

$$\begin{aligned} \frac{p_i}{u_i} = \frac{p_r}{u_r} = \frac{p_t}{u_t} &= Z_0 = \rho c \\ \frac{p_1}{u_1} = \frac{p_2}{u_2} = \frac{p_3}{u_3} = \frac{p_4}{u_4} &= Z_m = \rho_m c_L \end{aligned} \qquad (5.2.3)$$

Using (5.2.3) in (5.2.2) leads to:

$$\begin{aligned} p_i - p_r &= \frac{Z_0}{Z_m}\left(p_1 - p_4\right) \\ p_t &= \frac{Z_0}{Z_m}\left(p_2 - p_3\right) \end{aligned} \qquad (5.2.4)$$

Assuming propagation from one side of the material to the other without losses means that there is only a phase difference between the pressure at the two intersections:

$$p_2 = p_1\,\mathrm{e}^{-\mathrm{j}k_m h}$$
$$p_4 = p_3\,\mathrm{e}^{-\mathrm{j}k_m h} \qquad (5.2.5)$$

Here $k_m = \omega / c_L$ is the angular wave number for longitudinal sound propagation in the solid material.

From the above equations (5.2.1), (5.2.4) and (5.2.5) can be derived the ratio between the sound pressures p_i and p_t and thus the transmission loss can be expressed by:

$$R_0 = 10\log\left|\frac{p_i}{p_t}\right|^2 = 10\log\left(\cos^2(k_m h) + \frac{1}{4}\left(\frac{Z_0}{Z_m} + \frac{Z_m}{Z_0}\right)^2 \sin^2(k_m h)\right) \qquad (5.2.6)$$

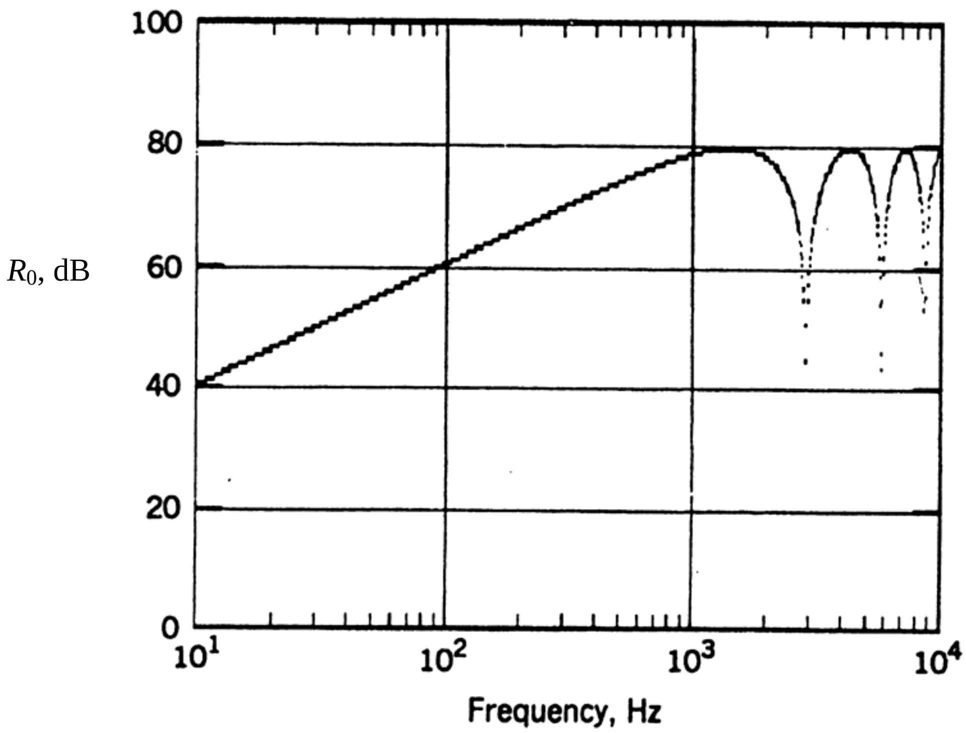


Fig. 5.2.2. Transmission loss at normal incidence of sound on a 600 mm thick concrete wall.

At high frequencies some dips can be observed in the transmission loss curve. They occur at frequencies where the thickness is equal to half a wavelength in the solid material, or a multiple of half wavelengths. However, the dips are very narrow and they are mainly of theoretical interest.

Two special cases can be studied. First the case of a thin wall: $Z_m >> Z_0$ and $k_m h << 1$

$$R_0 \cong 10\log\left(1 + \left(\frac{Z_m}{2Z_0}\right)^2 \sin^2(k_m h)\right) \cong 10\log\left(1 + \left(\frac{\omega\,\rho_m h}{2\rho c}\right)^2\right) \qquad (5.2.7)$$

The other special case is a very thick wall: $Z_m >> Z_0$ and $k_m h >> 1$

$$R_0 \cong 10\log\left(\frac{Z_m}{2Z_0}\right)^2 \cong 20\log\left(\frac{\rho_m c_L}{2\rho c}\right) \qquad (5.2.8)$$

The cross-over frequency from (5.2.7) to (5.2.8) is the frequency f_h at which $k_m h = 1$:

$$f_h = \frac{c_L}{2\pi h} \qquad (5.2.9)$$

This is the frequency at which the thickness is approximately one sixth of the longitudinal wavelength λ_L in the material:

$$h = \frac{c_L}{2\pi f} = \frac{\lambda_L}{2\pi}$$

The result for the thin wall is the so-called mass law, which will de derived in a different way in the next section. The result for a very thick wall (5.2.8) means that there is an upper limit on the sound insulation that can be achieved by a single-leaf construction, and this limit depends on the density of the material. For wood it is 68 dB, for concrete 80 dB and for steel 94 dB. (These numbers should be reduced by 5 dB in the case of random incidence instead of normal incidence, see section 5.2.3).

5.2.2 The mass law

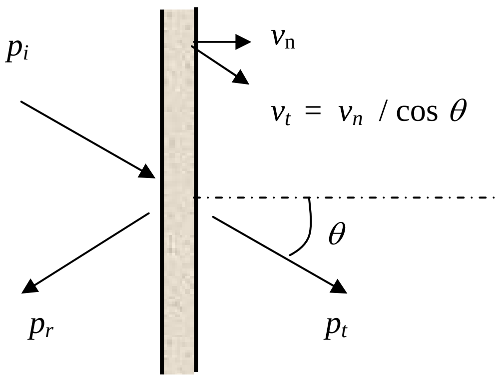


Figure 5.2.3. Thin wall with sound pressures and particle velocities

A thin wall with the mass per unit area *m* is considered, see Fig. 5.2.3. The application of Newton's second law (force = mass · acceleration) gives:

$$\Delta p = p_i + p_r - p_t = m\frac{\mathrm{d}v_n}{\mathrm{d}t} = \mathrm{j}\omega m v_n \qquad (5.2.10)$$

where v_n is the velocity of the wall vibrations (in the direction normal to the wall). The separation impedance Z_w is introduced:

$$Z_w = \frac{\Delta p}{v_n} = \mathrm{j}\omega m \qquad (5.2.11)$$

The separation impedance will be more complicated if the bending stiffness of the wall is also taken into account, see below.

The particle velocities in the sound waves are called *u* with the same indices as the corresponding sound pressures. Due to the continuity requirement the normal component of the velocity on both sides of the wall is:

$$v_n = u_t \cos\theta = (u_i - u_r)\cos\theta \qquad (5.2.12)$$

which leads to

$$p_t \quad = \quad p_i - p_r \quad = \quad Z_0 \frac{v_n}{\cos\theta} \qquad (5.2.13)$$

The sound transmission loss R_θ at a certain angle of incidence θ is:

$$R_\theta \quad = \quad 10\log\left|\frac{p_i}{p_t}\right|^2 \quad = \quad 10\log\left|1+\frac{Z_w\cos\theta}{2Z_0}\right|^2 \quad (\mathrm{dB}) \qquad (5.2.14)$$

In the special case of normal sound incidence ($\theta = 0$) the insertion of (5.2.11) gives the important *mass law* of sound insulation:

$$R_0 \quad = \quad 10\log\left|1+\mathrm{j}\frac{\omega m}{2\rho c}\right|^2 \quad \cong \quad 20\log\left(\frac{\pi f m}{\rho c}\right) \quad (\mathrm{dB}) \qquad (5.2.15)$$

Since $m = \rho_m h$ this result is the same as derived above in (5.2.7).

5.2.3 Sound insulation at random incidence

The transmission coefficient at the angle of incidence θ is from (5.2.14)

$$\tau(\theta) \quad = \quad \frac{1}{1+\left(\dfrac{\omega m}{2\rho c}\right)^2\cos^2\theta} \qquad (5.2.16)$$

Random incidence means that the sound field on the source side of the partition is approximately a diffuse sound field. In a diffuse sound field the incident sound power P_1 on a surface is found by integration over the solid angle $\psi = 2\pi$ assuming the same sound intensity I_1 in all directions. The principle is the same as used in section 3.2.2. Since, in each direction the transmitted sound power is equal to the incident sound power multiplied by the transmission coefficient, the ratio between transmitted and incident power is:

$$\tau \quad = \quad \frac{P_2}{P_1} \quad = \quad \frac{\int_{\psi=2\pi}\tau(\theta)I_1 S\,\mathrm{d}\psi}{\int_{\psi=2\pi}I_1 S\,\mathrm{d}\psi} \quad = \quad \frac{\int_0^{\pi/2}\tau(\theta)\cos\theta\sin\theta\,\mathrm{d}\theta}{\int_0^{\pi/2}\cos\theta\sin\theta\,\mathrm{d}\theta}$$

$$\tau \quad = \quad 2\int_0^{\pi/2}\tau(\theta)\cos\theta\sin\theta\,\mathrm{d}\theta \quad = \quad \int_0^1\tau(\theta)\,\mathrm{d}(\cos^2\theta)$$

$$\tau \quad = \quad \int_0^1\frac{\mathrm{d}(\cos^2\theta)}{1+(\omega m/2\rho c)^2\cos^2\theta} \quad = \quad \left(\frac{2\rho c}{\omega m}\right)^2 \ln\left(1+(\omega m/2\rho c)^2\right)$$

$$R \quad = \quad -10\log\tau \quad = \quad R_0 - 10\log(0{,}23R_0) \quad (\mathrm{dB}) \qquad (5.2.17)$$

This is the theoretical result for random incidence, and for typical values (R_0 between 30 and 60 dB) it means that R is 8 to 11 dB lower than R_0. However, in real life this is not true and it can be shown that the result is related to partitions of infinite size. Taking the finite size into account the result is approximately:

$$R \quad \cong \quad R_0 - 5\,\mathrm{dB} \qquad (5.2.18)$$

This is in good agreement with measuring results on real walls.

5.2.4 The critical frequency

The bending stiffness per unit length of a plate with thickness h is:

$$B = \frac{E\,h^3}{12\,(1-\nu^2)} \qquad (5.2.19)$$

where E is Young's modulus of the material and ν is Poisson's ratio. ($\nu \cong 0.3$ for most rigid materials).

The speed of propagation of bending waves in a plate with bending stiffness per unit width B and mass per unit area m is (see section 6.3.3):

$$c_b = \sqrt{\omega}\,\sqrt[4]{\frac{B}{m}} = c\sqrt{\frac{f}{f_c}} \qquad (5.2.20)$$

Here f_c is introduced as the *critical frequency*. It is defined as the frequency at which the speed of bending waves equals the speed of sound in air, $c_b = c$.

The critical frequency is:

$$f_c = \frac{c^2}{2\pi}\sqrt{\frac{m}{B}} \qquad (5.2.21)$$

A sound wave with the angle of incidence θ propagates across the wall with the phase speed c / sin θ, i.e. the phase speed is in general higher than c, see Fig. 5.2.4. If the bending wave speed happens to be equal to the phase speed of the incident sound wave, this is called *coincidence*:

$$c_b = c / \sin\theta$$

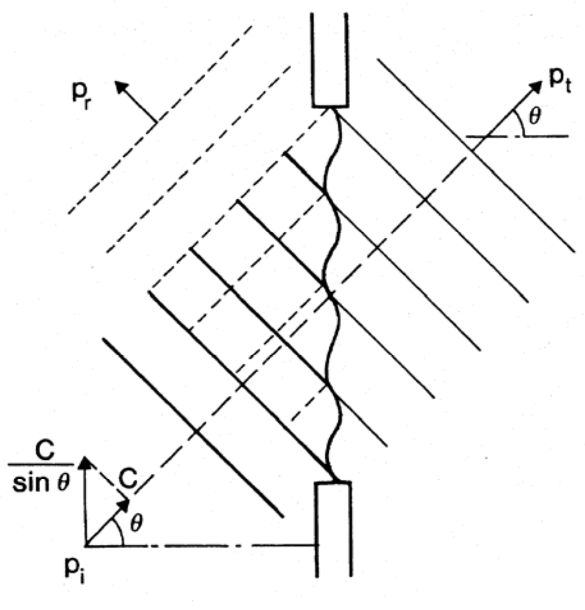


Figure 5.2.4. Thin wall with bending wave and indication of speed of propagation along the wall

The coincidence leads to a significant dip in the sound transmission loss. The coincidence dip will be at a frequency higher than or equal to the critical frequency:

$$f_{co} = f_c / \sin^2 \theta \qquad (5.2.22)$$

The separation impedance (5.2.11) is replaced by:

$$Z_w = \mathrm{j}\omega m \left(1 - \left(\frac{f}{f_c} \right)^2 \sin^4 \theta \right) \qquad (5.2.23)$$

Insertion in the general equation (5.2.14) leads to the sound transmission loss at a certain angle of incidence:

$$R_\theta = R_0 + 20 \log \left| \cos \theta \right| + 20 \log \left| 1 - \left(f / f_c \right)^2 \sin^4 \theta \right| \quad \text{(dB)} \qquad (5.2.24)$$

5.2.5 A general model of sound insulation of single constructions

The general model of sound insulation is based on mass law as given in (5.2.15). However, the following results are valid for sound insulation between rooms with approximately diffuse sound fields. In the frequency range below the critical frequency, $f < f_c$:

$$R \cong R_0 + 20 \log \left| 1 - \left(f / f_c \right)^2 \right| - 5 \,\text{dB} \qquad (5.2.25)$$

In the frequency range above the critical frequency, $f \geq f_c$:

$$R \cong R_0 + 10 \log \frac{2 \eta f}{\pi f_c} \quad \text{(dB)} \qquad (5.2.26)$$

where η is the loss factor (see section 6.2.2.3).

The upper limit for sound insulation of a single-leaf construction is, according to (5.2.8):

$$R \leq 20 \log \left(\frac{\rho_m c_L}{2 \rho c} \right) - 5 \,\text{dB} \qquad (5.2.27)$$

A sketch of the transmission loss as a function of frequency is shown in Fig. 5.2.5

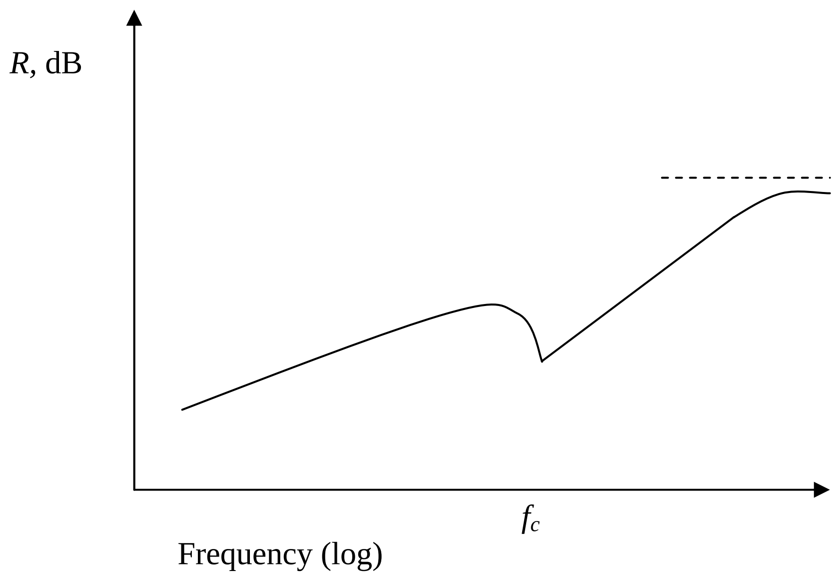


Figure 5.2.5. Sound insulation of a single-leaf construction, f_c is the critical frequency and the upper limit is the dotted line.

5.3 DOUBLE LEAF CONSTRUCTIONS

5.3.1 Sound transmission through a double construction

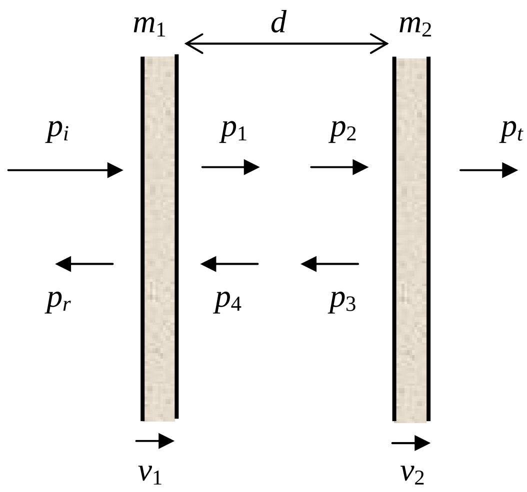


Fig. 5.3.1. A double construction with indication of sound pressures and particle velocities

A double construction with two plates in the distance *d* is considered, see Fig. 5.3.1. The separation impedance of the two plates is denoted Z_1 and Z_2, respectively. As for the single construction in (5.2.10) the movement of each wall is:

$$\begin{aligned} p_i + p_r - (p_1 - p_4) &= Z_1 v_1 \\ p_2 + p_3 - p_t &= Z_2 v_2 \end{aligned} \qquad (5.3.1)$$

The velocity of each wall equals the particle velocity on either side:

$$\begin{aligned} v_1 &= u_i - u_r = \frac{1}{Z_0}(p_i - p_r) \\ v_1 &= u_1 - u_4 = \frac{1}{Z_0}(p_1 - p_4) \\ v_2 &= u_2 - u_3 = \frac{1}{Z_0}(p_2 - p_3) \\ v_2 &= u_t = \frac{1}{Z_0} p_t \end{aligned} \qquad (5.3.2)$$

Assuming propagation from one side of the cavity to the other without losses means that there is only a phase difference between the pressure at the two intersections:

$$\begin{aligned} p_2 &= p_1 \, \mathrm{e}^{-\mathrm{j}k\,d} \\ p_4 &= p_3 \, \mathrm{e}^{-\mathrm{j}kd} \end{aligned} \qquad (5.3.3)$$

From the above equations (5.3.1), (5.3.2) and (5.3.3) can be derived the ratio between the sound pressures p_i and p_t and thus the transmission loss can be expressed by:

$$
\begin{aligned}
R_0 &= 10\log\left|\frac{p_i}{p_t}\right|^2 \\
&= 10\log\left|\left(1+\frac{Z_1+Z_2}{2Z_0}\right)\cos(kd)+\mathrm{j}\left(1+\frac{Z_1+Z_2}{2Z_0}+\frac{Z_1Z_2}{2Z_0^2}\right)\sin(kd)\right|^2
\end{aligned}
\tag{5.3.4}
$$

If only the mass of each wall is taken into account the separation impedances are:

$$
\begin{aligned}
Z_1 &= \mathrm{j}\omega m_1 \\
Z_2 &= \mathrm{j}\omega m_2
\end{aligned}
\tag{5.3.5}
$$

Neglecting the smaller parts and inserting $Z_0 = \rho\, c$ together with (5.3.5) yields:

$$
R_0 \cong 10\log\left[\left(\frac{\omega(m_1+m_2)}{2\rho c}\sin(kd)\right)^2+\left(\frac{\omega(m_1+m_2)}{2\rho c}\cos(kd)-\frac{\omega^2 m_1 m_2}{2(\rho c)^2}\sin(kd)\right)^2\right] \tag{5.3.6}
$$

This result will be discussed and simplified below.

5.3.2 The mass-air-mass resonance frequency

The transmission loss is minimum when the last term is zero, i.e.

$$
\mathrm{tg}(kd) = \frac{m_1+m_2}{m_1 m_2}\frac{\rho c}{\omega} \tag{5.3.7}
$$

For a cavity that is narrow compared to the wave length ($kd << 1$) we get:

$$
\mathrm{tg}(kd) \approx kd = \frac{\omega\, d}{c} = \frac{m_1+m_2}{m_1 m_2}\frac{\rho c}{\omega}
$$

The solution is the mass-air-mass resonance frequency $f_0 = \omega_0 / 2\pi$

$$
f_0 = \frac{c}{2\pi}\sqrt{\frac{\rho}{d}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)} \tag{5.3.8}
$$

If the depth d of the cavity is comparable to the wavelength there are many solutions to (5.3.7) and they are approximately $kd = n\,\pi$. The dips in the sound insulation occur at frequencies at which the cavity depth equals one or more half wavelenghts: $d = n\,\lambda/2$.

However, more important than these dips is the shift from low- to high-frequency behaviour of the air cavity. The cross-over frequency has no particular physical meaning, but it is the frequency f_d at which $kd = 1$:

$$
f_d = \frac{c}{2\pi\, d} \tag{5.3.9}
$$

This is quite similar to the result (5.2.9) found for the sound transmission through a solid material. Only, in this case the transmission is through air. The spring-like behaviour of the air cavity changes from that of a simple spring below the cross-over frequency to that of a transmission channel at higher frequencies.

5.3.3 A general model of sound insulation of double constructions

The result (5.3.6) can be simplified in different way depending on the frequency range. In the frequency range below the resonance frequency, $f < f_0$:

$$
R_0 \approx 20\log\left(\frac{\omega(m_1+m_2)}{2\rho c}\right) = R_{(1+2)} \tag{5.3.10}
$$

This means that the construction behaves as a single construction with the mass per unit area ($m_1 + m_2$). In the frequency range above the resonance frequency, $f_0 < f < f_d$:

$$R_0 \quad \approx \quad 20\log\left(\frac{\omega^3 m_1 m_2 d}{2\rho^2 c^3}\right) \quad \approx \quad R_1 + R_2 + 20\log(2kd) \qquad (5.3.11)$$

In this a much better sound insulation can be obtained, and it depends on the product of the three parameters m_1, m_2 and d. At frequencies above f_d where the cavity is wide compared to the wavelength, sin (kd) is replaced by its maximum value 1, and for $f \geq f_d$:

$$R_0 \quad \approx \quad 20\log\left(\frac{\omega^2 m_1 m_2}{2(\rho c)^2}\right) \quad \approx \quad R_1 + R_2 + 6\,\text{dB} \qquad (5.3.12)$$

In this high-frequency range, d is no longer an important parameter.
A sketch of the transmission loss as a function of frequency is shown in Fig. 5.3.2.

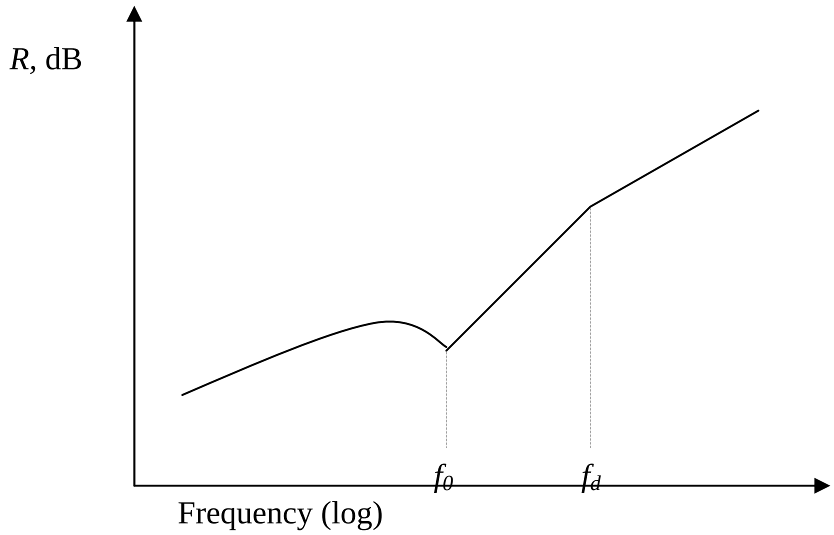


Figure 5.3.2. Sound insulation of a double-leaf construction, f_0 is the resonance frequency and f_d is the cross-over frequency of the cavity.

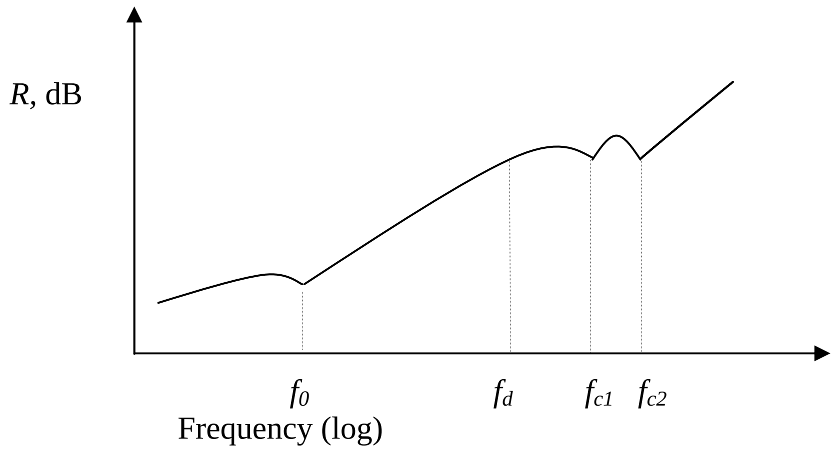


Figure 5.3.3. Sound insulation of an asymmetric double-leaf construction with two thin plates having different critical frequencies, f_{c1} and f_{c2}, respectively.

5.4 FLANKING TRANSMISSION

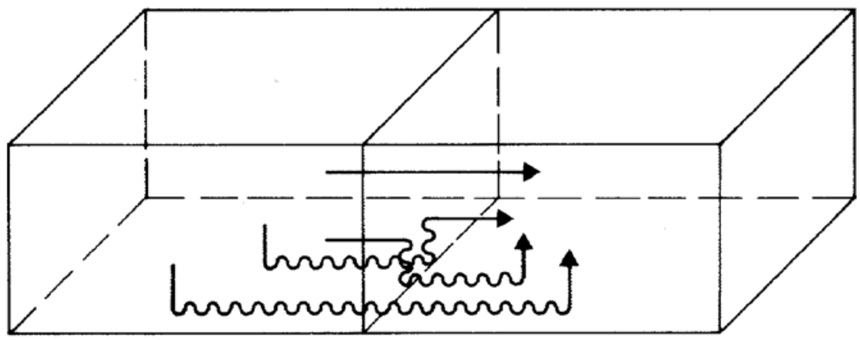


Fig. 5.4.1. Direct transmission and three flanking transmission paths via the floor.

The transmission of sound from a source room to a receiver room can be via flanking constructions like the floor, the ceiling or the façade. When all relevant transmission paths are considered the sound insulation is described by the *apparent sound transmission loss*:

$$R' = 10\log\frac{P_1}{P_2+P_3} = L_1 - L_2 + 10\log\frac{S}{A_2} \quad \text{(dB)} \qquad (5.4.1)$$

where P_2 is the sound power transmitted through the partition wall to the receiver room and P_3 is the sound power radiated to the receiver room from the flanking surfaces and other flanking paths:

$$P_3 = \sum_i P_{F,i} \qquad (5.4.2)$$

Each single flanking transmission path *i* can be characterised by the *flanking transmission loss,* $R_{F,i}$:

$$R_{F,i} = 10\log\frac{P_1}{P_{F,i}} \quad \text{(dB)} \qquad (5.4.3)$$

It is convenient to keep the incident sound power *P*1 on the partition wall as a reference for all the flanking transmission losses. In this way it is very simple to add all the contributions together, and the apparent transmission loss is calculated from:

$$R' = -10\log\left(10^{-0,1R} + \sum_i 10^{-0,1R_{F,i}}\right) \quad \text{(dB)} \qquad (5.4.4)$$

In the typical case of horizontal transmission through a wall the will be 12 flanking paths, namely three possible paths for each of the four surrounding flanking constructions, see Fig. 5.4.1.

5.5 ENCLOSURES

A noise source is supposed to radiate the sound power P_a. The noise source is totally covered by an enclosure with surface area S, absorption coefficient α on the inside, and the enclosure is made from a plate with transmission loss R or transmission coefficient τ. The average sound pressure in the enclosure p_{encl} can be estimated, if a diffuse sound field is assumed:

$$p_{encl}^2 = \frac{4\,P_a}{\alpha\,S}\,\rho\,c \qquad (5.5.1)$$

The sound power incident on the inner surface of the enclosure is (still with the assumption of a diffuse sound field):

$$P_{inc} = \frac{p_{encl}^2\,S}{4\rho c} \qquad (5.5.2)$$

The sound power transmitted through the enclosure is then:

$$P_{out} = \tau\,P_{inc} = \frac{\tau}{\alpha}\,P_a \qquad (5.5.3)$$

The *insertion loss* of the enclosure is the difference in radiated sound power level without and with the enclosure:

$$\Delta L = 10\log\frac{P_a}{P_{out}} = 10\log\frac{\alpha}{\tau} = R + 10\log\alpha \quad \text{(dB)} \qquad (5.5.4)$$

This result cannot be considered to be very accurate. Especially the assumption of a diffuse sound field inside the enclosure is doubtful. However, the result is not bad as a rough estimate for the design of an enclosure. It is clearly seen from (5.5.4) that both transmission loss and absorption coefficient are important for an efficient reduction of noise by an enclosure.

5.6 IMPACT SOUND INSULATION

The noise generated from footsteps on floors is characterised by the impact noise level. It is measured according to ISO 140 Part 6 and 7 by a standardised tapping machine. The main data for the tapping machine are:

- The noise is generated by steel hammers with a fall height of 40 mm
- Each steel hammer has a mass of 500 g
- The number of taps per second is 10.

In the source room the tapping machine is placed on the floor in a number of positions. In the room below - or any other room in the building – the calibrated sound pressure level L_2 is measured. The reverberation time in the receiving room must also be measured in order to calculate the absorption area A_2. The impact sound pressure level is the sound pressure level in dB re 20 μPa that would be measured if the absorption area is A_0 = 10 m^2:

$$L_n = L_2 + 10\log\frac{A_2}{A_0} \quad \text{(dB)} \qquad A_0 = 10\ \text{m}^2 \qquad (5.6.1)$$

The frequency range is the same as for airborne sound insulation, i.e. the 16 one-third octave bands from 100 Hz to 3150 Hz. However, it is recommended to extend the frequency range down to 50 Hz, especially in the case of lightweight floor constructions.

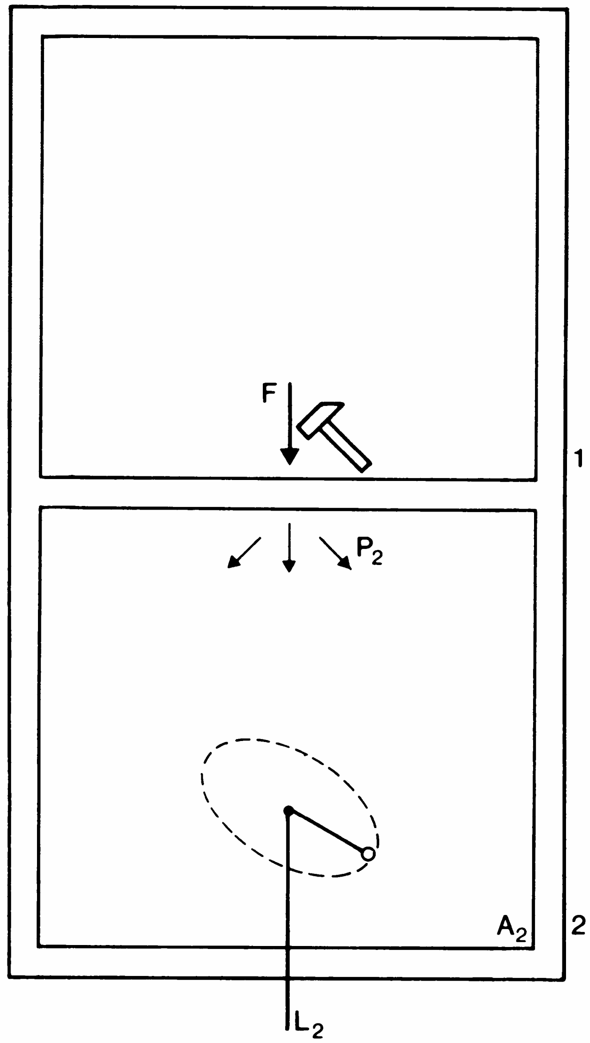


Fig. 5.6.1. Principle of measuring the impact sound pressure level from a floor to a receiving room (2)

5.7 SINGLE-NUMBER RATING OF SOUND INSULATION

5.7.1 The weighted sound reduction index

The single-number rating of sound insulation is practical for several purposes:

- to characterise the measuring result of a building construction,
- for quick comparison of the sound insulation obtained with different constructions, and
- to specify requirements for sound insulation.

The weighted sound reduction index R_w is based on a standardised reference curve that is defined in one-third octaves in the frequency range 100 Hz – 3150 Hz. The reference curve is made from three straight lines with a slope of 9 dB per octave from 100 to 400 Hz, 3 dB per octave from 400 to 1250 Hz, and 0 dB per octave from 1250 to 3150 Hz.

The measured transmission loss is compared to the reference curve, and the sum of *unfavourable deviations* is calculated. An unfavourable deviation is the deviation between the reference curve and the measured curve if the measured sound insulation is *lower* than the value of the reference curve.

The reference curve is shifted up or down in steps of 1 dB, and the correct position of the reference curve is found when the sum of unfavourable deviations is as large as possible, but do not exceed 32 dB. The value of the reference curve at 500 Hz is taken as the single-number value of the measuring result. The method is also shown in Fig. 5.7.1.

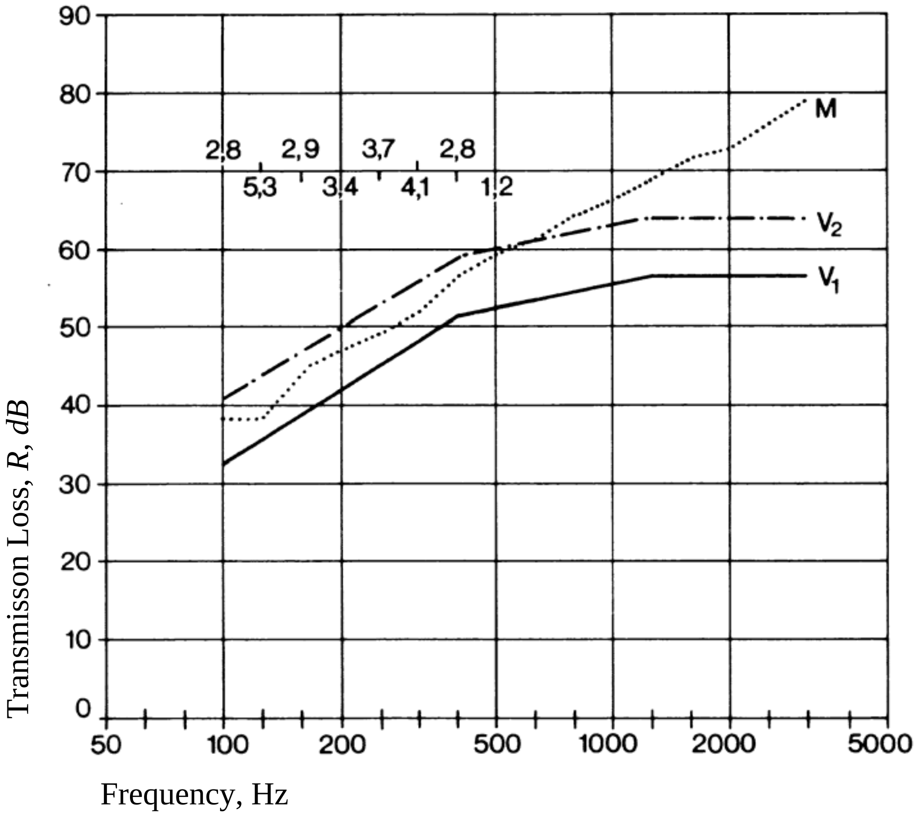


Fig. 5.7.1. Determination of the weighted sound reduction index. M is the measured curve, V_1 is the reference curve in position 52 dB, and V_2 is the shifted reference curve. The result is R_w = 60 dB.

5.7.2 The weighted impact sound pressure level

The weighted impact sound pressure level $L_{n,w}$ is very similar to the weighted sound reduction index. It is based on a standardised reference curve that is defined in one-third octaves in the frequency range 100 Hz – 3150 Hz. The reference curve is made from three straight lines with a slope of 0 dB per octave from 100 to 315 Hz, -3 dB per octave from 315 to 1000 Hz, and -9 dB per octave from 1000 to 3150 Hz.

The measured impact sound pressure level is compared to the reference curve, and the sum of *unfavourable deviations* is calculated. An unfavourable deviation is the deviation between the reference curve and the measured curve if the measured impact sound pressure level is *higher* than the value of the reference curve.

The reference curve is shifted up or down in steps of 1 dB, and the correct position of the reference curve is found when the sum of unfavourable deviations is as large as possible, but do not exceed 32 dB. The value of the reference curve at 500 Hz is taken as the single-number value of the measuring result. The method is also shown in Fig. 5.7.2.

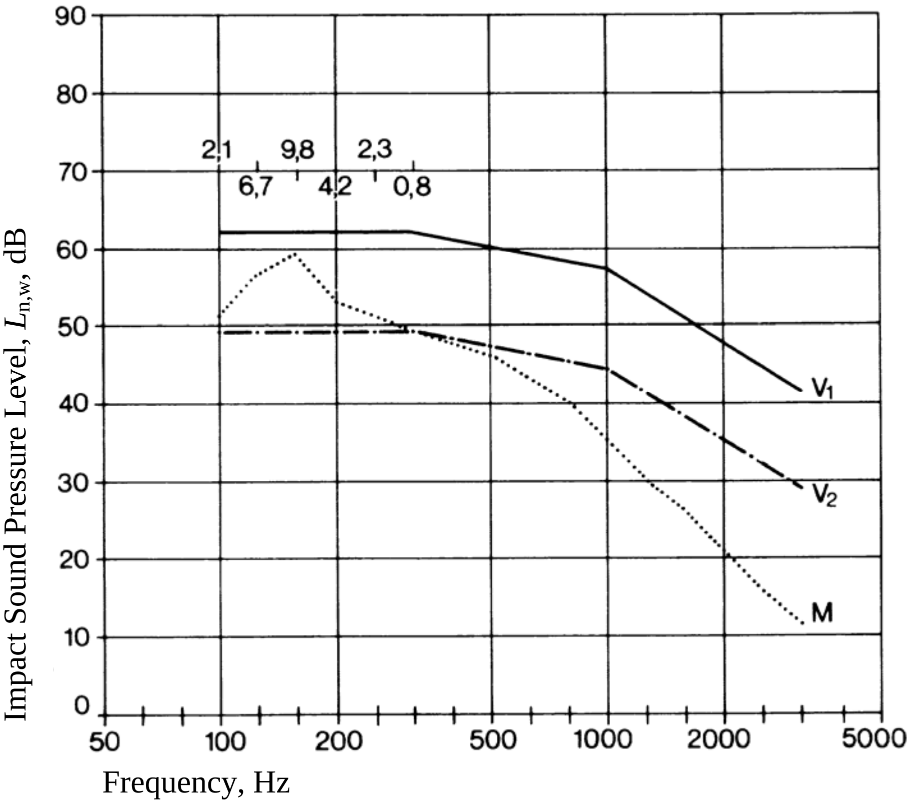


Fig. 5.7.2. Determination of the weighted impact sound pressure level. M is the measured curve, V_1 is the reference curve in position 60 dB, and V_2 is the shifted reference curve. The result is $L_{n,w}$ = 47 dB.

5.8 REQUIREMENTS FOR SOUND INSULATION

The Danish requirements for new buildings are laid down in "Bygningsreglement 1995" (BR-95) and in "Bygningsreglement for småhuse 1998" (BR-S 98).

For dwellings in multi-storey houses and for hotels the main requirements are:

- The airborne sound insulation shall be $R'_w \geq 52$ dB in horizontal directions and $R'_w \geq 53$ dB in vertical directions.
- The impact sound pressure level shall be $L'_{n,w} \leq 58$ dB.
- Between rooms for common service or commercial use and dwellings the airborne sound insulation shall be $R'_w \geq 60$ dB and the impact sound pressure level shall be $L'_{n,w} \leq 48$ dB.

For row-houses or semi-detached houses the main requirements are:

- The airborne sound insulation shall be $R'_w \geq 55$ dB.
- The impact sound pressure level shall be $L'_{n,w} \leq 53$ dB.

In schools the main requirements are:

- Between classrooms the airborne sound insulation shall be $R'_w \geq 48$ dB in horizontal directions and $R'_w \geq 51$ dB in vertical directions.
- The impact sound pressure level in classrooms shall be $L'_{n,w} \leq 63$ dB.
- From rooms for music or workshops to classrooms the airborne sound insulation shall be $R'_w \geq 60$ dB and the impact sound pressure level shall be $L'_{n,w} \leq 53$ dB.

The sound insulation of facades is not specified directly, but in buildings where then outdoor traffic noise exceeds $L_{\mathrm{Aeq,\ 24}} \geq 55$ dB, the indoor noise in living rooms shall not exceed $L_{\mathrm{Aeq,\ 24}} \leq$ 30 dB.

5.9 REFERENCES

ISO 140-3 (1995): Acoustics. Measurement of sound insulation in buildings and of building elements. Part 3: Laboratory measurements of airborne sound insulation of building elements.

ISO 140-4 (1998): Acoustics. Measurement of sound insulation in buildings and of building elements. Part 4: Field measurements of airborne sound insulation between rooms.

ISO 140-6 (1998): Acoustics. Measurement of sound insulation in buildings and of building elements. Part 6: Laboratory measurements of impact sound insulation of floors.

ISO 140-7 (1998): Acoustics. Measurement of sound insulation in buildings and of building elements. Part 7: Field measurements of impact sound insulation of floors.

ISO 717-1 (1996): Acoustics. Rating of sound insulation in buildings and of building elements. Part 1: Airborne sound insulation.

ISO 717-2 (1996): Acoustics. Rating of sound insulation in buildings and of building elements. Part 2: Impact sound insulation.

BR-95. (1995). Bygningsreglement (Building regulations, in Danish). Bygge-og Boligstyrelsen, Copenhagen.

BR-S 98. (1998). Bygningsreglement for småhuse (Building regulations for small houses, in Danish). Bygge-og Boligstyrelsen, Copenhagen.

6 MECHANICAL VIBRATION AND STRUCTUREBORNE SOUND

Mogens Ohlrich

6.1 INTRODUCTION

Audio frequency vibration of mechanical systems and waves in solid structures form an integral part of engineering acoustics in describing the dynamic phenomena in solids and fluids, and their interaction. This subject, referred to as phenomena of *structureborne sound* or *vibro-acoustics*, is important because sound or noise is very often generated directly by mechanical vibration of solid bodies or by waves transmitted in solid structures, and eventually radiated into the fluid as audible sound. Examples are musical sound from a string instrument or noise from a pump in a central heating system.

Vibration of simple resonant systems (resonators) is characterised by mass and stiffness properties and by some form of damping mechanism, which dissipate vibrational energy. The simplest description of dynamic behaviour applies to resonators that can be modelled as a (minimal) combination of discrete or ‘lumped’ elements. If the response of the resonator primarily occurs in only one direction, ie in a single motion coordinate, then the system is said to have a single degree of freedom (sdof). Figure 6.1.1 shows examples of sdof-resonators. The mathematical description of the vibration of such systems is governed by an ordinary second-order differential equation. This is usually derived from a force balance of the mass element. Solution of the equation shows that such systems have a single preferred ‘natural’ frequency of vibration, which can exist in the absence of external excitation.

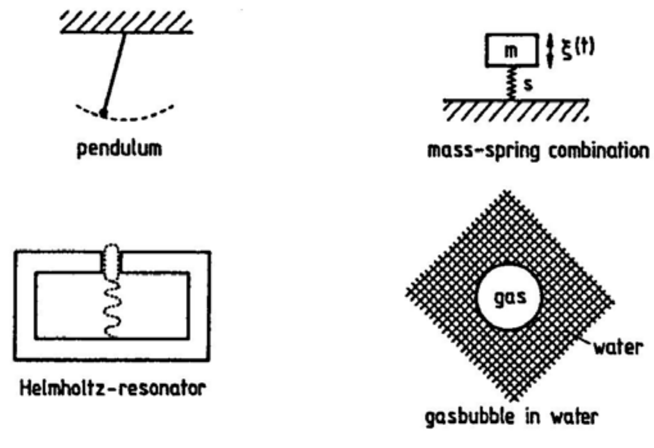


Figure 6.1.1 *Examples of single degree of freedom resonators. After ref. [1].*

Vibration of more complex systems requires more than one motion coordinate for a complete description. For example, in the case of a loudspeaker three degrees of freedom are required for describing the designed translational motion of the ‘piston cone’ and its unintentional rocking motions, which can occur in two planes. In general such motions will be governed by three coupled, second-order differential equations. However, by using a special set of coordinates these equations can be uncoupled and solved independently, as is the case for the sdof-resonator.

Vibration of different phase, ie, structural wave motion, can occur when the wavelength of vibration in a solid structure is less than one of its typical dimensions. If this is the case it is natural to threat the system as a continuous one. The response of such a system is governed by a partial differential equation, because the response depends upon both time and a spatial position coordinate that specifies the location at which the response is to be determined.

6.1.1 SOURCES OF VIBRATION

There are many types of excitation mechanisms that generate vibration and waves in solid structures. Such *sources* are associated with nature or they involve the employment of machines in the broadest sense, that is, devices that do work, ranging from a miniature loud-speaker in a hearing aid to a combustion engine of a truck, say. The sources can be classified by their temporal variations for which there are two types, transient and continuous that includes time variation of either deterministic (periodic) or random nature.

Examples of sources of vibration are shown in Figure 6.1.2. Transient sources representing local impact are very common both as a single impact and in repetition, in which case the excitation time-history becomes periodic. The hammer impact symbolizes a variety of excitation mechanisms such as musical percussion (drums, xylophones), impulsive sources of vibration and noise in buildings (foot-falls, door slamming), impacts in production machinery (punch presses, forge hammers) and periodic impacts in combustion engines (valves, piston slab). Figure 6.1.2b illustrates force excitation caused by an unbalanced rotating mass; such excitation is often of a harmonic (pure tone) nature. Other sources of vibration and noise are random variation of surface roughness, eg in wheel/surface contacts, or distributed excitation of a structure, eg caused by a sound field.

Figure 6.1.2 *Examples of sources that generate vibration and structure-borne sound.*

6.1.2 MEASUREMENT QUANTITIES

Investigations of vibration in solid structures are usually carried out by measuring a local quantity at a specific position on the structure. Distribution of vibration over a larger area can be determined by measurements in a number of discrete positions. The local measurement quantity is either a *motion* (displacement, velocity or acceleration) or a *force*. Both types of variables are vectors, and thus assigned to a certain orientation or direction.

Vibratory motion is usually measured uni-directional with a small transducer of the *accelerometer*-type that is fastened to the structure's surface. The accelerometer is based upon the piezo-electric principle with an output signal proportional to the acceleration $a = a(t)$ of the vibrating surface. Accelerometers are available with different sensitivities. The velocity v or displacement ξ of the vibratory motion is obtained by integration of the acceleration signal.

A localised (point) force $F = F(t)$ is mostly measured with a piezo-electric force transducer, which produces an output proportional to the force. The measurement is carried out by inserting the transducer between a source (eg a vibration exciter) and the measurement object. This arrangement is mostly used for measuring the dynamic properties of structures, for example, the impedances or the mobilities.

6.1.3 LINEAR MECHANICAL SYSTEMS

The dynamic properties of a physical system depend upon its mass and stiffness distribution and damping losses. These properties are attempted described by mathematical models in the form of one or more differential equations of motion. The system is said to be *linear* if the dependent response variables are of first order. When this is the case, one can use the very important superposition principle. This means that the response contributions from independent excitations can be superimposed or summed as vectors.

Herein we assume that systems considered are linear, which is often the case when vibration or waves have small amplitudes. System dynamics can therefore be described by linear differential equations. These can be based either on a *discrete* model or on a *continuous* model. In the *discrete* model the properties of system components are described by discrete ('lumped') quantities, represented by ideal masses, massless springs and dampers, see Figure 6.1.3. The physical properties of the *continuous* model are functions of the spatial coordinates. Dynamic properties of the system are therefore described by partial differential equations.

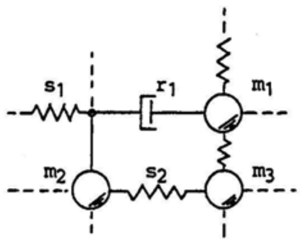


Figure 6.1.3 *Lumped model of a physical system, where the physical properties are represented by ideal discrete elements of point masses, massless springs and dampers.*

The choice between the two models depends upon a number of factors such as frequency range of interest, structural shape and forms of excitation. However, the actual decision of the type of model is usually not strictly scientific, but is often based on intuition and practical experience. In this note we shall focus mainly on the analysis of discrete models, whereas only a brief summary will be given of wave motion in continuous structures (structure-borne sound).

Figure 6.1.4a shows the basic lumped elements; the quantity s represents the *spring constant* (*stiffness*), m is the mass and r is the *damping constant* of a viscous damper; for translatory motion these quantities have units of [N/m], [kg] and [kg/s], respectively. The viscous damper represents a velocity proportional resistance that results in energy losses. Symbolically, the viscous damping is thought caused by motion of a piston in a fluid-filled cylinder.

The properties of the elements are independent of time t, and there is a linear relation between forces $F_i = F_i(t)$ and changes in, respectively, displacement $\xi = \xi(t)$, velocity $v = v(t)$

and acceleration $a = a(t)$ over the terminals of the elements. Thus, for the ideal spring there is proportionality between force and deformation according to Hooke's law. The viscous damping force is proportional to the velocity of the 'deformation' in the massless damper.

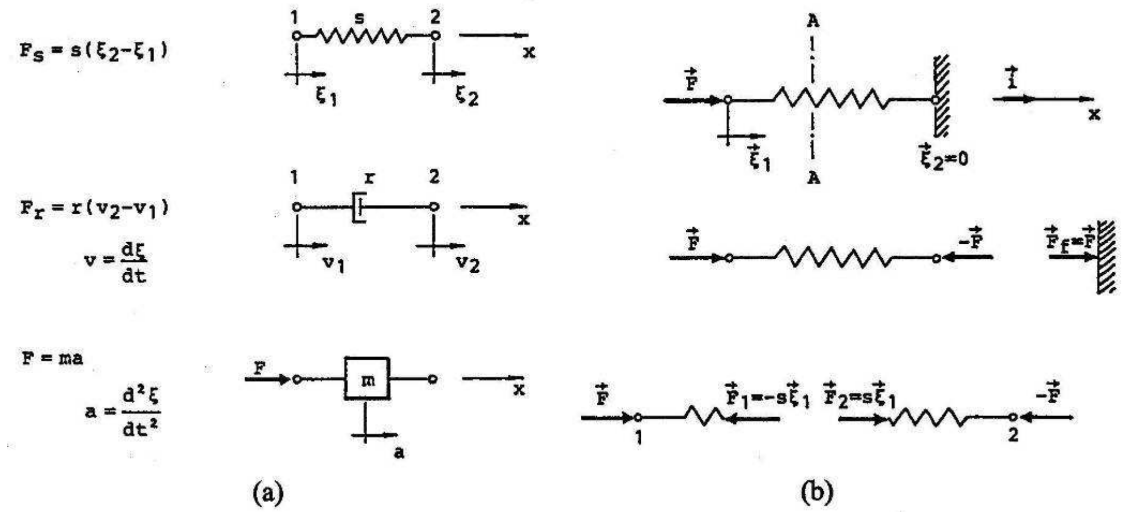


Figure 6.1.4 *(a) Force-response-relations for ideal lumped elements.(b) Excitation (action) and reaction by compression of spring.*

Note that both motion and force variables are vector quantities, as shown by the example in Figure 6.1.4b . Both quantities are defined as positive in the direction of the vector; the motion variables are thus defined as positive in the x-direction. In Figure 6.1.4a, the positive force F required for accelerating the mass m is therefore $F = ma$, which is Newton's second law of motion in its simplest form.

6.2 SIMPLE MECHANICAL RESONATORS

Figure 6.2.1a shows a model of a single degree of freedom system that is connected to a rigid foundation. The system consists of a mass m , a spring of spring constant s , and a velocity proportional viscous damper of damping constant r.

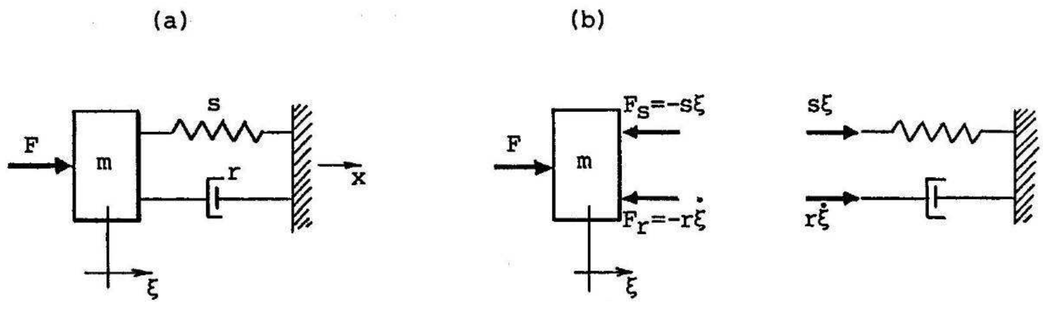


Figure 6.2.1 *(a) Viscously damped simple resonator driven by an external force F ; (b) diagram which shows the forces acting on the mass m .*

6.2.1 EQUATION OF MOTION FOR SIMPLE RESONATOR

The system is assumed excited by a time-varying external force $F = F(t)$ and it is understood that the system can vibrate only translatory, to and fro, in the direction of the force, that is, in the horizontal plane in this example. The motion of the mass from its equilibrium position is denoted by the displacement $\xi = \xi(t)$, and this is taken positive towards the right-hand side.

The vibration response caused by the external force is uniquely defined by the instantaneous value ξ . This displacement of the mass results in a compression of the spring that produces a restoring, elastic spring force

$$F_s = -s\xi . \tag{6.2.1}$$

Thus, the reaction on the mass that is caused by the spring force, acts in the opposite direction of the displacement imposed by the external force. If viscous damping is assumed as illustrated by the parallel-coupled dashpot in Figure 6.2.1 then this element will exert a corresponding restoring damping force

$$F_r = -r\frac{\mathrm{d}\xi}{\mathrm{d}t} , \tag{6.2.2}$$

that is, a force which is also directed opposite to that of the motion of the mass and in proportion to its vibration velocity $v = \mathrm{d}\xi/\mathrm{d}t$.

The vector sum of forces that act on the mass, that is, $F + F_s + F_r = F - s\xi - rv$, thus serves to accelerate the mass. So, according to Newton's second law of motion, this sum must be equal to the product of mass m and acceleration $a = \mathrm{d}^2\xi/\mathrm{d}t^2$, ie

$$\sum F_i = ma = m\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} . \tag{6.2.3}$$

The equation of motion for the system therefore becomes

$$m\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + r\frac{\mathrm{d}\xi}{\mathrm{d}t} + s\xi = F . \tag{6.2.4a}$$

This equation is often written in a reduced form as

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + \frac{r}{m}\frac{\mathrm{d}\xi}{\mathrm{d}t} + \omega_0^2\xi = \frac{F}{m} , \tag{6.2.4b}$$

where ω_0 is the *natural* angular frequency in [rad/s] of the corresponding *undamped* system ($r = 0$), defined as

$$\omega_0 = \sqrt{\frac{s}{m}} . \tag{6.2.5}$$

In the literature the fraction r/m in eq. (6.2.4b) is often replaced either by 2δ or by $2\zeta\omega_0$ where δ is the *damping coefficient* and ζ is the non-dimensional *viscous damping ratio*. Their definitions are respectively

$$\delta = \frac{r}{2m} \quad \text{and} \quad \zeta = \frac{r}{2m\omega_0} = \frac{r}{2\sqrt{ms}} . \tag{6.2.6a,b}$$

Moreover, from Figure 6.2.1b it is seen that the total force F_f acting on the rigid foundation is equal to the sum of the spring force and the damping force, that is,

$$F_f = r\frac{\mathrm{d}\xi}{\mathrm{d}t} + s\xi . \tag{6.2.7}$$

6.2.2 FORCED HARMONIC RESPONSE OF SIMPLE RESONATOR

Let us assume that the excitation force F in eq. (6.2.4) varies harmonically with time as $F = |F_1|\cos\omega t$ with angular frequency ω. After a certain built-up of vibration the mass will then also execute stationary, harmonic vibration with the same angular frequency ω. Herein we shall only deal with the stationary vibration of the system, since it is assumed that the initial built-up of vibration caused by 'starting' the force has completely decayed because of damping effects, see Figure 6.2.2.

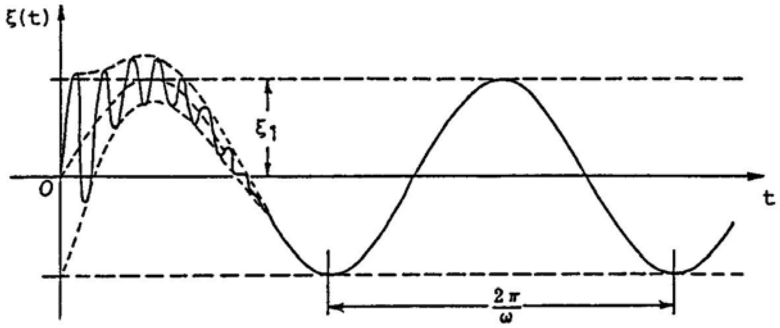


Figure 6.2.2 *Time history of vibration built-up in the case of harmonic force excitation of a simple, damped resonator when $\omega < \omega_0$. The vibration built-up response is succeeded by a stationary vibration at the angular frequency ω of the excitation.*

6.2.2.1 Undamped system

Initially, we shall disregard the damping of the considered system by setting $r = 0$. Thus for harmonic excitation the equation of motion (6.2.4) reduces to

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + \omega_0^2\xi = \frac{|F_1|}{m}\cos\omega t\,. \qquad (6.2.8)$$

The complete solution for $\xi = \xi(t)$ of such a differential equation has the well-known form

$$\xi = \xi_1\cos\omega t + \text{solutions to the } \textit{homogeneous} \text{ equation} \qquad (6.2.9)$$

where the first term represents the stationary harmonic vibration and the second term represents the above-mentioned phenomenon of vibration built-up or decay.

The displacement amplitude ξ_1 of the stationary vibration is obtained directly from eq. (6.2.8) by substituting the assumed solution $\xi = \xi_1\cos\omega t$:

$$\left(-\omega^2 + \omega_0^2\right)\xi_1\cos\omega t = \frac{|F_1|}{m}\cos\omega t\,,$$

which gives

$$\xi_1 = \frac{|F_1|}{m}\frac{1}{\omega_0^2-\omega^2} = \frac{|F_1|}{s}\frac{1}{1-\omega^2/\omega_0^2} = \xi_{stat}\frac{1}{1-\omega^2/\omega_0^2} \qquad (6.2.10\text{ a,b,c})$$

where eq. (6.2.10b) follows from eq. (6.2.5). Furthermore, the quantity ξ_{stat} represents the so-called *static* displacement, which is the compression or extension of the spring caused by the force $F = |F_1|\cos\omega t$ when $\omega = 0$:

$$\xi_{stat} = \frac{|F_1|}{s}\,. \qquad (6.2.11)$$

The stationary part of the solution (6.2.9), which describes the forced harmonic motion of the resonator, is thus given by

$$\xi = \xi_1 \cos\omega t = \xi_{stat} \frac{1}{1-\omega^2/\omega_0^2} \cos\omega t\,. \qquad (6.2.12)$$

The fraction $1/(1-\omega^2/\omega_0^2)$ represents the variation of the vibration amplitude with respect to the excitation frequency ω and it is sometimes referred to as the response amplification factor; this quantity also reveals the phase relation between the displacement response and excitation force. Figure 6.2.3 shows the variation of this quantity ξ_1/ξ_{stat} with angular frequency; in Figure 6.2.3b the same quantity is shown as absolute value (modulus) and phase.

From the figure it can be seen that the vibration amplitude grows towards infinity when the excitation frequency ω approaches the undamped natural frequency ω_0 of the system; this excitation condition is called *resonant excitation*, and the frequency at which $\omega = \omega_0$ is the *resonance frequency*. At $\omega = \omega_0$, the response ξ_1 is also seen to undergo a change in sign, which corresponds to a phase change of π radians. Physically, this simply means that the quantities ξ_1 and $|F_1|$ are in-phase at low frequencies, that is, for $\omega < \omega_0$ where the system behaves spring-like, whereas they are in anti-phase for $\omega > \omega_0$ where the response is lagging the harmonic force excitation by 180 degrees because of the system mass (inertia).

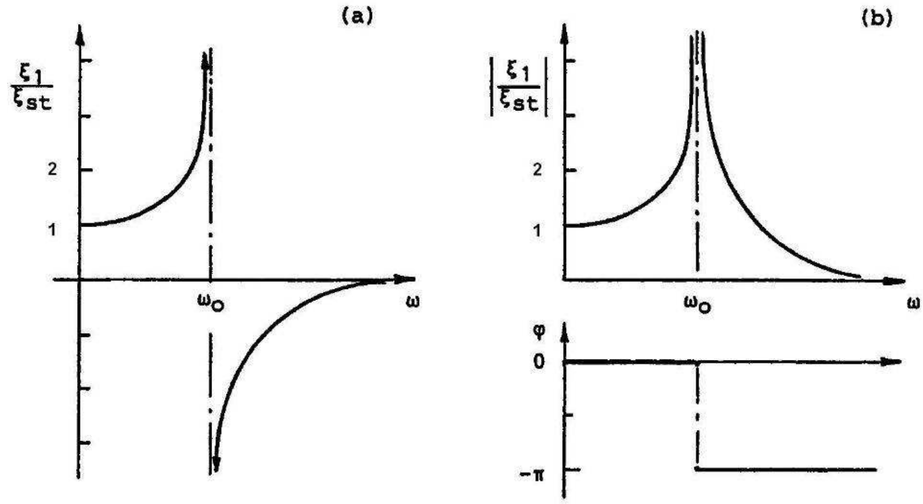


Figure 6.2.3 *(a) Relative displacement response ξ_1/ξ_{stat} for an undamped simple resonator; (b) the same response function plotted as modulus and phase.*

For this *undamped* case the force F_f that is transmitted to the foundation is caused by the spring force and is given by $F_f = s\xi$, which follows from eq. (6.2.7) for $r = 0$. The disturbance force on the foundation thus follows directly by substituting the solution eq. (6.2.12)

$$F_f = |F_f| \cos\omega t = |F_1| \frac{1}{1-\omega^2/\omega_0^2} \cos\omega t\,. \qquad (6.2.13)$$

This force ratio $F_f/|F_1|$ has the same frequency variation as the motion ratio ξ_1/ξ_{stat} shown in Figure 6.2.3. For excitation frequencies below the natural frequency of the system, that is for $\omega < \omega_0$, the mass has a negligible influence. This means that the excitation force is in equilibrium with the spring force, which is transmitted unchanged to the foundation. Thus, if the force on the foundation is to be reduced by vibration isolation it is required that natural frequency of the system is designed in such a way that $\omega_0 << \omega/\sqrt{2}$ is fulfilled. For a set excitation frequency and system mass this is accomplished by selecting a 'soft' spring element with an appropriately small spring constant s .

6.2.2.2 Viscously damped system

The influence of damping is now being considered. When damping losses are assumed to be of the viscous type as in Figure 6.2.1 then eq. (6.2.4) applies.

By using complex notation the harmonic excitation force $F(t) = |F_1|\cos\omega t$ can be expressed as $F(t) = \mathrm{Re}\{F_1 e^{i\omega t}\}$, where F_1 is the complex amplitude of the force. The solution of the equation of motion is assumed to be of the same form $\xi(t) = \mathrm{Re}\{\xi_1 e^{i\omega t}\}$, where $\xi_1 = |\xi_1|e^{i\varphi}$ is the complex amplitude of the harmonic displacement with φ being the phase angle between the displacement response and the driving force. Physical quantities are of course always real, and it is therefore necessary to take the real part of the mathematical solution when we want the time variation of the physical motion. This yields

$$\xi(t) = \mathrm{Re}\left\{\xi_1 e^{i\omega t}\right\} = |\xi_1|\cos(\omega t + \varphi). \tag{6.2.14}$$

By performing in eq. (6.2.4a) substitutions of $F(t) \equiv F_1 e^{i\omega t}$ and $\xi(t) \equiv \xi_1 e^{i\omega t}$ result in the solution for the stationary, harmonic vibration[1] :

$$\left(-\omega^2 m + i\omega r + s\right)\xi_1 e^{i\omega t} = F_1 e^{i\omega t} \tag{6.2.15}$$

$$\Leftrightarrow \qquad \xi_1 = \frac{F_1}{(s-\omega^2 m) + i\omega r} = \frac{F_1}{m(\omega_0^2 - \omega^2) + i\omega r}. \tag{6.2.16a, b}$$

Hereby, the problem is basically solved. (If the time variation of the response is sought then this is obtained by substitution in eq. (6.2.14).) Furthermore, since the squared modulus is given by $\xi_1\xi_1^* = |\xi_1|^2$, we get

$$|\xi_1|^2 = \frac{|F_1|^2}{m^2(\omega_0^2-\omega^2)^2 + \omega^2 r^2}. \tag{6.2.16c}$$

Thus, $|\xi_1|$ is obtained by simply taking the square-root of the expression (6.2.16c).

The force transmitted to the foundation follows similarly from eq. (6.2.7)

$$F_f e^{i\omega t} = (i\omega r + s)\xi_1 e^{i\omega t}, \tag{6.2.17}$$

which by substituting eq. (6.2.16b) gives

$$F_f = \frac{F_1(s + i\omega r)}{m(\omega_0^2-\omega^2) + i\omega r}, \quad \text{and} \quad |F_f|^2 = \frac{|F_1|^2(s^2 + \omega^2 r^2)}{m^2(\omega_0^2-\omega^2)^2 + \omega^2 r^2}. \tag{6.2.18a,b}$$

---

[1]) Here the symbol Re{··} is left out. This does not result in any trouble as long as one is strictly dealing with field quantities (displacement, velocity, force etc). However, when dealing with energy or power quantities, one must *only* include the *real part* of the field quantity. The time variation $e^{i\omega t}$ is also often left out in the analyses, but it is of course to be recalled and taken into account when necessary.

Solution in sum form. The solution (6.2.16) for the complex displacement can also be written in terms of its real and imaginary parts

$$\xi_1 = \xi_{re} + \mathrm{i}\xi_{im} . \tag{6.2.19a}$$

In the following we shall assume that the arbitrary phase of F_1 is set equal to zero by a suitable choice of time-reference (t = 0); this means that the force amplitude is assumed to be real, ie $F_1 = |F_1|$. Thus, by transforming the denominator in eq. (6.2.16b) to a real quantity this yields

$$\xi_1 = \frac{m(\omega_0^2 - \omega^2)F_1}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2} + \mathrm{i}\frac{(-\omega r)F_1}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2} . \tag{6.2.19b}$$

The frequency variations of this solution are sketched in Figure 6.2.4a. Shown is the real and imaginary parts of the displacement response of the viscously damped resonator when this is driven by a harmonic force of constant amplitude F_1 . The damping is seen to limit the displacement response in the frequency range around $\omega \sim \omega_0$ where the response ξ_1 is controlled largely by its imaginary part ξ_{im} .

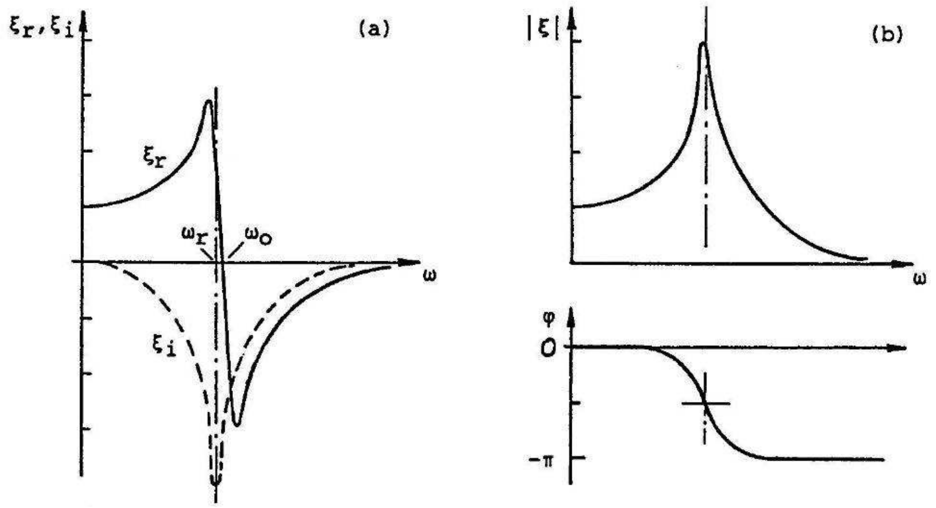


Figure 6.2.4 *Frequency variation of displacement ξ_1 for a viscously damped simple resonator driven by a harmonic force of constant amplitude. (a) Real and imaginary parts; (b) Modulus and phase.*

Solution in product form. The solution for the complex displacement response eq. (6.2.16) or (6.2.19) is often written in the alternative 'product form'

$$\xi_1 = |\xi_1| e^{\mathrm{i}\varphi} \tag{6.2.20a}$$

where the modulus $|\xi_1|$ and phase angle φ as usual are determined from eq. (6.2.19):

$$|\xi_1|^2 = \xi_{re}^2 + \xi_{im}^2 \qquad \text{and} \qquad \tan\varphi = \xi_{im}/\xi_{re} .$$

The squared modulus of the displacement is already given by eq. (6.2.16c), whereas the phase angle is found directly from eq. (6.2.19b), ie

$$\tan\phi = \frac{-\omega r}{m(\omega_0^2 - \omega^2)} . \tag{6.2.20b}$$

Note that the phase angle becomes $\varphi = -\pi/2$ at resonant excitation. As previously, the actual physical time variation of the vibration response follows from eq. (6.2.14)

$$\xi(t) = \mathrm{Re}\{\xi_1 e^{\mathrm{i}\omega t}\} = \mathrm{Re}\{|\xi_1|\, e^{\mathrm{i}\varphi} e^{\mathrm{i}\omega t}\} = |\xi_1| \cos(\omega t + \varphi)$$

$$\Leftrightarrow \qquad \xi(t) = \frac{|F_1|}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 r^2}} \cos(\omega t + \varphi) . \qquad (6.2.20c)$$

Figure 6.2.4b shows how the modulus and phase of the displacement varies with frequency for harmonic force excitation. This type of graph is the most commonly used form of presentation for frequency response functions.

The vibration *velocity* $v(t)$ of the resonator is often of interest and this follows simply by taking the time derivative of the displacement response, eq. (6.2.16) or (6.2.20):

$$v(t) = \frac{\mathrm{d}\xi(t)}{\mathrm{d}t} = -\omega |\xi_1| \sin(\omega t + \varphi) ,$$

or (6.2.21)

$$v(t) = \mathrm{Re}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\xi_1 e^{\mathrm{i}\omega t}\right)\right\} = \mathrm{Re}\{v_1 e^{\mathrm{i}\omega t}\}, \quad \text{where} \quad v_1 = \mathrm{i}\omega\xi_1 .$$

So, with respect to the complex amplitudes a differentiation is simply archived by a multiplication with $\mathrm{i}\omega$; evidently integration is performed by a division by $\mathrm{i}\omega$. Moreover, the *acceleration* $a(t)$ of the motion is obtained similarly by the time derivative of velocity or by the second derivative of displacement.

<u>Non-dimensional form.</u> It is often convenient to introduce non-dimensional parameters that enable solutions for a class of systems to be presented in a general form. For simple resonators the *frequency ratio* Ω is readily used as frequency parameter

$$\Omega = \omega / \omega_0 . \qquad (6.2.22)$$

By substituting this as well as the dimensionless viscous damping ratio ζ into eqs. (6.2.16c) and (6.2.20b) we obtain the general expressions for the displacement ratio $|\xi_1|/\xi_{stat}$ and for the phase angle φ :

$$\frac{|\xi_1|^2}{\xi_{stat}^2} = \frac{1}{(1-\Omega^2)^2 + 4\Omega^2\zeta^2} \qquad \text{and} \qquad \tan\varphi = \frac{-2\Omega\zeta}{1 - \Omega^2} , \qquad (6.2.23a,b)$$

here, it is recalled that the static displacement is $\xi_{stat} = |F_1|/s$. Amplitude and phase characteristics for the displacement ratio (6.2.23), are shown logarithmically in Figure 6.2.5a for different values of damping ratio ζ . It is clearly seen that the damping has a dominant influence on the response in the frequency range $\Omega \sim 1$, which is close to the natural frequency of the system.

Similar expressions for the force ratio $F_f \,/\, |F_1|$ are obtained by substituting the non-dimensional parameters in eq. (6.2.18). Amplitude and phase characteristics for this ratio between transmitted force and driving force are shown in Figure 6.2.5b.

In forced harmonic vibration the displacement response of the system reaches its maximum value $|\xi_{max}|$ at, say, $\Omega_r = \omega_r / \omega_0$ where ω_r is the resonance frequency. The actual value of Ω_r is determined by differentiating eq. (6.2.23a) with respect to Ω and by setting the obtained expression equal to zero. This gives the value

$$\Omega \equiv \Omega_r = \sqrt{1-2\zeta^2} \qquad (6.2.24a)$$

$$\Leftrightarrow \qquad \Omega_r \cong 1 - \zeta^2, \qquad \text{when} \qquad 2\zeta^2 << 1; \qquad (6.2.24b)$$

in the last approximate expression use have been made of the truncated series: $(1-x)^{½} \cong 1 - x/2$ provided that $x << 1$. The maximum displacement thus occurs at an angular frequency, which is slightly lower than the angular natural frequency of the undamped system. By substituting eq. (6.2.24a) in (6.2.23a) we get

$$\frac{|\xi_{max}|^2}{\xi_{stat}^2} = \frac{1}{4\zeta^2(1-\zeta^2)}. \qquad (6.2.25)$$

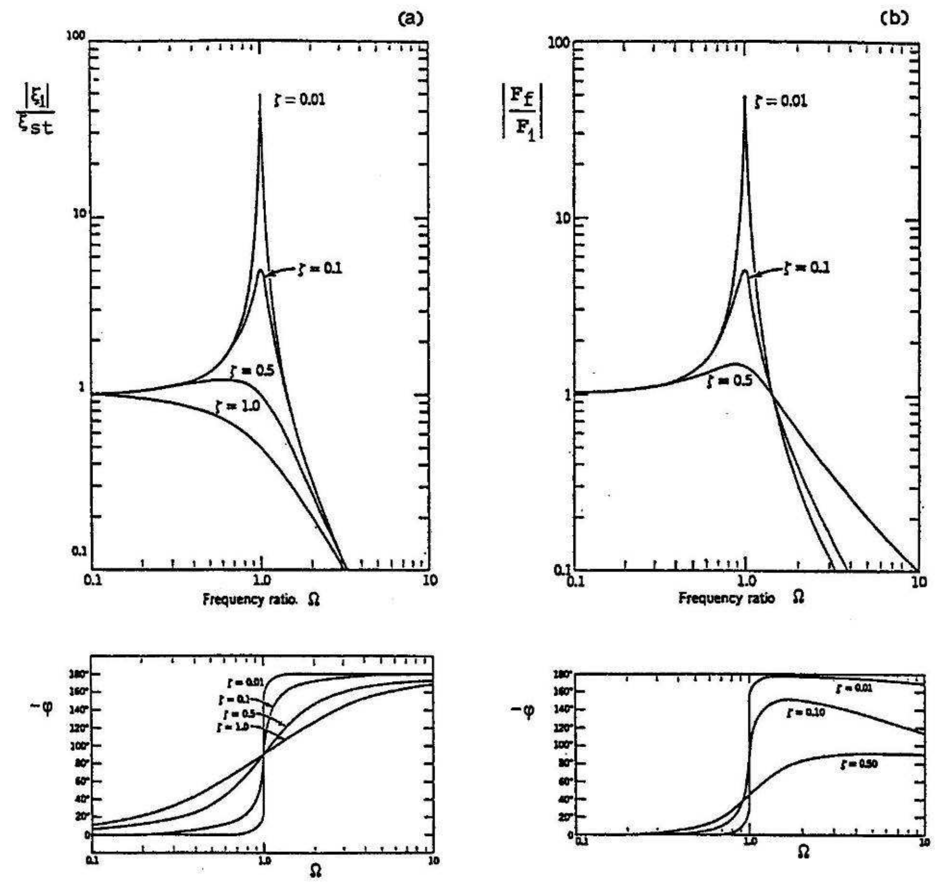


Figure 6.2.5 *Amplitude and phase characteristics for: (a) Displacement ratio* ξ_1/ξ_{stat} *, and (b) Force ratio* $F_f/|F_1|$ *. From ref. [2].*

However, when the damping is small (ζ << 0.05) the resonance frequency will nearly coincide with the natural frequency ω_0 of the undamped system, that is, $\omega_r \cong \omega_0$; the maximum displacement thus becomes

$$\left|\xi_{max}\right| \;\cong\; \xi_{stat}\,\frac{1}{2\zeta} \;=\; \frac{\left|F_1\right|}{\omega_0 r}\,. \tag{6.2.26}$$

The displacement at resonance is thus equal to ξ_{stat} divided by 2ζ.

Similarly, the vibration *velocity* of the system can be shown to take its maximum value $|v_{max}|$ at $\omega = \omega_0$, that is, at $\Omega = 1$. Since $|v| = \omega\,|\xi|$ this yields

$$\left|\xi_{max}\right| \cong \omega_0 \xi_{stat}\,\frac{1}{2\zeta} = \frac{\left|F_1\right|}{r}\,. \tag{6.2.27}$$

Relations for maximum acceleration can be derived in the same manner.

Finally, the modulus and phase of the frequency response functions for displacement and velocity, respectively, are sketched in log-log format in Figure 6.2.6.

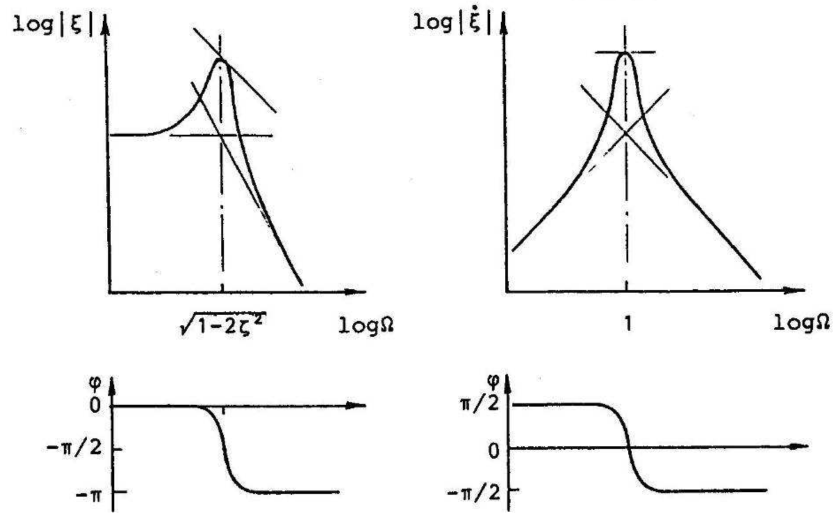


Figure 6.2.6 *Logarithmic plots of the frequency response functions of a simple resonator represented as displacement and velocity. A unit force excitation is assumed.*

<u>Characteristic properties.</u> As apparent from previous discussions the dynamic properties of the resonator are predominantly spring-like at low frequencies ($\Omega \ll 1$) and predominantly mass-like at high frequencies ($\Omega \gg 1$) ; the asymptotes shown in Figure 6.2.6 actually represent the dynamic properties of the individual elements s, m and r under the action of the force F_1. The dynamic properties of the resonator (ie, the combined system) are therefore characterised as being:

- Stiffness controlled for $\Omega \ll 1$, where $\left|\xi_1\right| \approx \dfrac{\left|F_1\right|}{s}$,
- Damping controlled at $\Omega \cong 1$, where $\left|\xi_1\right| \approx \dfrac{\left|F_1\right|}{\omega r}$, (6.2.28)
- Mass controlled for $\Omega \gg 1$, where $\left|\xi_1\right| \approx \dfrac{\left|F_1\right|}{m\omega^2}$.

These asymptotic values for the displacement response $|\xi_1|$ follow directly from eq. (6.2.16c). Similar relations can be determined for velocity and acceleration.

6.2.2.3 Structurally damped systems

So far we have only considered damping of the viscous type. A second type is *structural damping*, which is proportional to changes in elastic deformation, like the displacement of a spring. Such structural damping is therefore appropriately modelled by assigning the inherent losses to the spring element. For *harmonic* motion this can be represented by a complex stiffness $\underline{s} = s(1 + \mathrm{i}\eta)$ where η is the *damping loss factor* and s is the real part of the complex spring constant. The loss factor thus defines the phase lag (hysteresis) between harmonic driving force and spring displacement. By using the loss factor the equation of motion for a single mass-spring resonator becomes

$$m\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + s(1+\mathrm{i}\eta)\xi = F_1 e^{\mathrm{i}\omega t}, \qquad (6.2.29)$$

which, similar to eq. (6.2.15), has the solution $\xi(t) = \mathrm{Re}\{\xi_1 \mathrm{e}^{\mathrm{i}\omega t}\}$, where $\xi_1 = |\xi_1|\mathrm{e}^{\mathrm{i}\varphi}$ is the complex amplitude:

$$\xi_1 = \frac{F_1}{(s-\omega^2 m) + \mathrm{i}s\eta} = \frac{F_1}{m(\omega_0^2-\omega^2)+\mathrm{i}s\eta}. \qquad (6.2.30)$$

This 'complex stiffness' approach is very convenient, because the equation of motion can be formulated initially without regard to damping and finally the spring constant is replaced by its complex value $\underline{s} = s(1 + \mathrm{i}\eta)$.

Now, comparing eq. (6.2.30) with (6.2.15) shows that $s\eta$ corresponds to ωr. The *equivalent* damping 'constant' r_{eq} for a structurally damped spring thus becomes frequency dependent, and so does the *equivalent* damping ratio ζ_{eq}, ie

$$r_{eq} = r_{eq}(\omega) = s\eta/\omega \quad \text{and} \quad \zeta_{eq} = \zeta_{eq}(\omega) = \eta\omega_0/(2\omega). \qquad (6.2.31\text{a,b})$$

Alternatively, the loss factor of a parallel combination of an ideal spring and a viscous damper of constant r may be expressed as $\eta = r\omega/s$. Note also that the *equivalent* damping ratio eq. (6.2.31b) becomes $\zeta_{eq} = \eta/2$ *at resonance*. This relation may be used as an approximation for other frequencies that are close to resonance.

6.2.3 FREQUENCY RESPONSE FUNCTIONS

The *frequency response* of a system is defined as the ratio of complex amplitudes of two quantities representing the response to a certain excitation. This broad characterisation by the term 'frequency response' is often imprecise because the response quantity can be either displacement or one of its time derivatives: velocity and acceleration. It is therefore customary to assign specific names and symbols to the various types of frequency response functions.

6.2.3.1 Receptance

So far we have been dealing with ratios of response over force. When the system response is characterised by its displacement the complex frequency response is called the *receptance* $H(\omega)$. So, this is defined as

$$H(\omega) = |H(\omega)|e^{\mathrm{i}\varphi(\omega)} = \frac{\xi_1 e^{\mathrm{i}\omega t}}{F_1 e^{\mathrm{i}\omega t}}, \qquad (6.2.32)$$

where the notation with angular frequency dependence, $H(\omega)$, implies that the quantity is a continuous function of ω ; its amplitude spectrum $|H(\omega)|$ and phase spectrum $\varphi(\omega)$ can be determined from

$$|H(\omega)|^2 = H(\omega)H^*(\omega) \quad \text{and} \quad \tan\varphi(\omega) = \frac{\mathrm{Im}\{H(\omega)\}}{\mathrm{Re}\{H(\omega)\}}. \tag{6.2.33}$$

The definition eq. (6.2.32) states that $\xi_1 e^{i\omega t} = H(\omega)F_1 e^{i\omega t}$, which means that the time variation of the displacement for harmonic excitation is

$$\xi(t) = \mathrm{Re}\{H(\omega)F_1\, e^{i\omega t}\} = |H(\omega)|\,|F_1| \cos(\omega t + \varphi(\omega)), \tag{6.2.34}$$

where the force amplitude is assumed to be real.

Receptances of the discrete elements: spring s, damper r and mass m , follow respectively from the fundamental relations between harmonic force and the associated motion for such elements

$$H_s(\omega) = \frac{1}{s} \;,\; H_r(\omega) = \frac{1}{i\omega r} \quad \text{and} \quad H_m(\omega) = \frac{-1}{\omega^2 m}. \tag{6.2.35a,b,c}$$

Since the ideal spring and damper are massless it is assumed in the definition of their receptances that one of their terminals is blocked and that a harmonic force drives the other, free end.

It is sometimes useful to use the *reciprocal* of the receptance function; this is called *dynamic stiffness* [3].

<u>6.2.3.2 Mobility and Impedance</u>

The *velocity* response is often of interest in vibro-acoustics, for instance, because the radiated sound power from a vibrating structure is proportional to its surface velocity. The complex ratio between response velocity and driving force is called the *mobility* $Y(\omega)$ (or sometimes *admittance*) and is defined as

$$Y(\omega) = |Y(\omega)|\, e^{i\theta(\omega)} = \frac{v_1 e^{i\omega t}}{F_1 e^{i\omega t}}, \tag{6.2.36}$$

There is, of course, a very simple relation between mobility and receptance since the complex velocity amplitude is $v_1 = i\omega\xi_1$, ie

$$Y(\omega) = i\omega H(\omega). \tag{6.2.37}$$

The mobilities of the ideal components are therefore easily determined either from the fundamental relations or directly from eq. (6.2.35). Thus

$$Y_s(\omega) = \frac{i\omega}{s} \;,\; Y_r(\omega) = \frac{1}{r} \quad \text{and} \quad Y_m(\omega) = \frac{1}{i\omega m}. \tag{6.2.38}$$

The reciprocal of a mobility function is named the *impedance* $Z(\omega)$

$$Z(\omega) = \frac{1}{Y(\omega)}. \tag{6.2.39}$$

These different frequency response functions are summarized in Table 6.2.1 together with corresponding functions that involve acceleration response. The latter is called *accelerance* and its reciprocal, the *apparent mass.* The accelerance is sometimes used because acceleration is the response quantity that is usually measured directly.

Table 6.2.1 *Definition of frequency response functions R/F and F/R , where F is the force and R is the response that represents either displacement, velocity or acceleration.*

Response quantity R	Name of frequency response function R/F	F/R
Displacement ξ	Receptance $H(\omega)$	Dynamic stiffness $S(\omega)$
Velocity v	Mobility $Y(\omega)$	Impedance $Z(\omega)$
Acceleration a	Acceleration $A(\omega)$	Apparent mass $M(\omega)$

6.2.4 FORCED VIBRATION CAUSED BY MOTION EXCITATION

Vibratory disturbances like *motion excitation* is very common and occurs, for example, in transportation of any kind, in machinery and in certain cases also in buildings. In all these examples and in vibration isolation of delicate equipment from disturbing environments, the 'foundation' has a given motion $\xi_f = \xi_f(t)$ as shown in Figure 6.2.7. Thus we want to find the imposed/generated motion $\xi = \xi(t)$ of the mass.

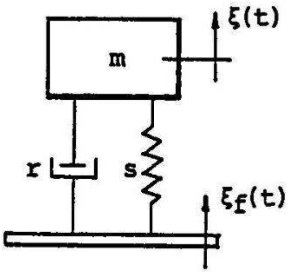


Figure 6.2.7 *Motion excitation of a damped simple resonator.*

There are two motion coordinates, but despite of this the system has only *one* degree of freedom, because the motion of the system is uniquely described by a so-called *generalized coordinate* $q = q(t)$; in this case by the motion differences

$$q = \xi_f - \xi \quad \text{and} \quad \dot{q} = \dot{\xi}_f - \dot{\xi} . \tag{6.2.40}$$

The quantities q and $\dot{q}$ describe, respectively, the compression (or elongation) of the spring and the velocity difference over the damper. Since the total force on the mass in Figure 6.2.7 readily can be written down, is it not necessary to use q explicitly. From eq. (6.2.3) follows directly

$$\sum F_i = s(\xi_f - \xi) + r(\dot{\xi}_f - \dot{\xi}) = m\,\ddot{\xi}\,. \qquad (6.2.41)$$

This gives the equation of motion

$$m\,\ddot{\xi} + r\,\dot{\xi} + s\xi = r\,\dot{\xi}_f + s\xi_f\,. \qquad (6.2.42)$$

It is seen that there is a clear analogy between this expression and eq. (6.2.4a), if the right-hand-side of eq. (6.2.42) simply is interpreted as a special 'forcing function'.

In the case of steady-state harmonic motion excitation $\xi_f e^{i\omega t}$, the solution to eq. (6.2.42) can be assumed to be $\xi \equiv \xi_1 e^{i\omega t}$; by substituting these quantities we obtain the solution for the complex amplitude of the displacement $\xi = \mathrm{Re}\,\{\xi_1 e^{i\omega t}\}$

$$\xi_1 = \xi_f \frac{s + i\omega r}{m(\omega_0^2 - \omega^2) + i\omega r}\,. \qquad (6.2.43)$$

This expression has the same form as eq. (6.2.18a). In motion excitation the ratio between displacements is thus identical to the ratio between forces in the case of force excitation (Figure 6.2.1). The frequency variation of ξ_1/ξ_f is therefore exactly identical to that of F_f/F_1 shown in Figure 6.2.5b.

This finishes the analysis of simple sdof mechanical resonators. A treatment of free vibration of such systems and an analysis of more complicated multi-degree of freedom systems is outside the scope of this introductory note on discrete systems. We will therefore proceed with a brief introduction of continuous structures.

6.3 VIBRATION AND WAVES IN CONTINUOUS SYSTEMS

Distributed solid structures become 'dynamically elastic' and exhibit wave-type vibratory behaviour as the frequency is increased to an extent, where the wavelength become comparable to, or less than, the physical dimensions of the structure. Although discrete models can be used for analysing wave motion at the lower frequencies, it becomes expedient to use wave-type analysis in problems where the wavelength is short. Thus, a brief introduction will be given to vibration and wave motion in continuous systems. Only systems of one and two dimensions will be considered here, because most engineering structures have *at least one* dimension, which is small in comparison with the relevant structural wavelength of vibration. In the audible frequency range this is the case for basic engineering components, such as strings, rods, beams, membranes, plates, shells, pipes etc.

Equations of motion that describe different wave types and vibro-acoustic phenomena have been formulated for many types of continuous structures [4,5,6]. Usually each wave type is treated separately, although wave conversion between different types generally occurs at most structural discontinuities, such as edges, corners and cross sectional changes.

The most important wave types in structures are considered to be (a) longitudinal waves, (b) shear or torsional waves and (c) bending waves, which are also called flexural waves, see Figure 6.3.1. In the following an introduction of these waves in plane structures will be given.

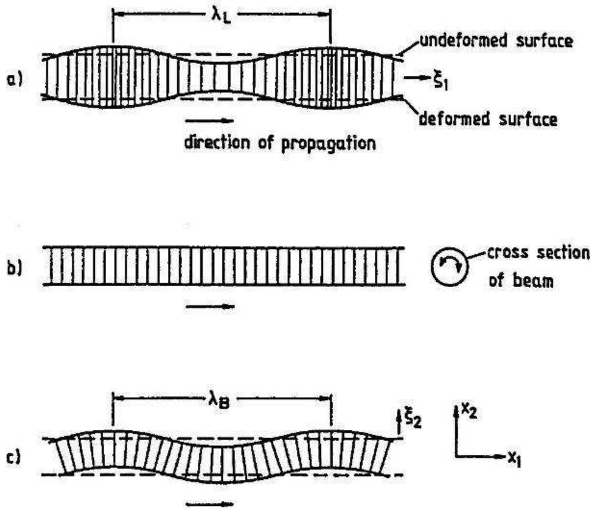


Figure 6.3.1 *Different wave types: (a) Longitudinal wave (the lateral deformations are exaggerated), (b) Torsional wave and (c) Bending wave. After ref. [7].*

6.3.1 LONGITUDINAL WAVES

Longitudinal waves in one-dimensional structures like rods and beams are compression-type waves that are similar to plane sound waves in a fluid. The local structural deformation in connection with longitudinal wave motion is primarily in the direction of wave propagation, although there is also a small lateral deformation normal to the structural surface. However, this deformation is generally so small that it can be neglected as a radiator of sound to the surrounding fluid. It should also be mentioned that the impedance of longitudinal waves in solids generally is very high.

The equation of motion for longitudinal waves in an undamped beam can be written in a compact form; the longitudinal displacement in the wave motion will be denoted by $u = u(x, t)$, where x represents its spatial dependence. If we assume purely harmonic excitation and harmonic wave motion $u = u(x)\mathrm{e}^{\mathrm{i}\omega t}$ this reads

$$L\{u(x)\} - \omega^2 m' u(x) = F'(x), \tag{6.3.1}$$

where $L\{\cdots\}$ is a differential operator that describes the force gradient in the beam, m' is its mass per unit length and $F'(x)$ is an external force excitation per unit length. For longitudinal waves the operator is given by $-ES\,\mathrm{d}^2/\mathrm{d}x^2$, where E in [N/m^2] is Young's modulus of elasticity of the beam material and S is the cross sectional area of the beam.

Two field variables are required for describing the longitudinal wave motion; these are the already mentioned displacement $u = u(x)\mathrm{e}^{\mathrm{i}\omega t}$ – or its time-derivative, the velocity $v = \mathrm{i}\omega u(x)\mathrm{e}^{\mathrm{i}\omega t} = v(x)\mathrm{e}^{\mathrm{i}\omega t}$ – and the internal force $F = F(x)\mathrm{e}^{\mathrm{i}\omega t}$ associated with the wave motion. This is given by

$$F = -ES\frac{\partial u}{\partial x}. \tag{6.3.2}$$

Moreover, the wave speed c_{l2} of a freely propagation longitudinal wave in the beam is

$$c_{l2} = \sqrt{\frac{E}{\rho}} \,, \qquad (6.3.3)$$

where ρ is the material mass density; index 2 on c_{l2} indicates that the structure has two surfaces that are small compared with the wavelength of the motion. The corresponding wave speed in a flat, homogenous *plate* is slightly higher (by about 5%):

$$c_{l1} = \sqrt{\frac{E}{\rho\,(1-\nu^2)}} \,, \qquad (6.3.4)$$

where ν is Poisson's ratio, which is a material constant that expresses the ratio between deformations in the lateral and length-wise directions of the structure. For common solid material $\nu \approx 0.3$, and for rubber-like materials $\nu \approx 0.5$.

A listing of material properties and wave speeds are given in Table 6.3.1. Note that the wave speed in metals is about 3000 to 5000 m/s, that is, a magnitude higher than for sound in air. Furthermore, the mass density for metals is seen to be up to 7000 times higher than for air. This means that the characteristic impedance (ρc_l) for compression waves in solid structures is much higher than for air; for example, the characteristic impedance for steel is 10^5 times higher than in air, but only 27 times higher than the impedance in water.

Table 6.3.1 *Material properties and wave speeds (phase speeds) for solid structures. After ref. [8].*

Material Properties[a] and Phase Speeds

Material	Young's modulus E (Nm^{-2})	Density ρ (kgm^{-3})	Poisson's ratio (ν)	c_{l1} (ms^{-1})	c_{l2} (ms^{-1})	c_S (ms^{-1})
Steel	2.0×10^{11}	7.8×10^3	0.28	5270	5060	3160
Aluminium	7.1×10^{10}	2.7×10^3	0.33	5434	5130	3145
Brass	10.0×10^{10}	8.5×10^3	0.36	3677	3430	2080
Copper	12.5×10^{10}	8.9×10^3	0.35	4000	3750	2280
Glass	6.0×10^{10}	2.4×10^3	0.24	5151	5000	3175
Concrete:						
light	3.8×10^9	1.3×10^3			1700	
dense	2.6×10^{10}	2.3×10^3			3360	
porous	2.0×10^9	6.0×10^2			1820	
Rubber:						
hard	2.3×10^9	1.1×10^3	0.4	1582	1450	867
soft	5.0×10^6	9.5×10^2	0.5		70	40
Brick	1.6×10^{10}	$1.9\text{–}2.2 \times 10^3$			2800	
Sand, dry	3.0×10^7	1.5×10^3			140	
Plaster	7.0×10^9	1.2×10^3			2420	
Chipboard[b]	4.6×10^9	6.5×10^2			2660	
Perspex[c]	5.6×10^9	1.2×10^3	0.4	2357	2160	1291
Plywood[b]	5.4×10^9	6.0×10^2			3000	
Cork	—	$1.2\text{–}2.4 \times 10^2$			430	
Asbestos cement	2.8×10^{10}	2.0×10^3			3700	

[a] Mean values from various sources.
[b] Temperature sensitive.
[c] Greatly variable from specimen to specimen.

6.3.2 SHEAR WAVES

In this wave type only shear deformations occur, but no volume changes. Moreover, the direction of the 'particle' motion is perpendicular to the direction of propagation. Shear waves are of importance in plates that are built-up of several layers of material with different properties, eg sandwich honeycomb panels.

The equation of motion for shear waves is governed by a second order partial differential equation [5] of a general form similar to that of longitudinal waves; the details shall not be given here, though. The wave speed c_s for shear waves in a plate is given by

$$c_s = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{E}{\rho\, 2(1+\nu)}}\,, \tag{6.3.5}$$

where G is the shear modulus of the material. From the right-hand-side of this equation it is clear that there is a unique relation between Young's modulus E and the shear modulus G, ie

$$G = \frac{E}{2(1+\nu)}\,. \tag{6.3.6}$$

Shear waves in rods are called torsional waves. This type of wave motion that involves twisting of the cross section of the rod was shown in Figure 6.3.1b. If the rod has a circular cross section then the wave speed is as given by eq. (6.3.5); otherwise the wave speed will be lower.

The two wave types discussed so far have high characteristic impedances. These waves may therefore be important for the wave transmission over large distances (eg in buildings and ships) and in wave conversion to bending waves, which is the dominant wave type when it comes to sound radiation to the surrounding fluid media, being air or water.

6.3.3 BENDING WAVES

Bending waves in beams and plates are characterised by the motion being perpendicular to both the direction of propagation, and the surface of the structure, see Figure 6.3.1c. Bending waves do therefore play a dominant role in sound radiation from structures. The reasons for this are that the wave motion has a good 'match' to the adjacent air, and that bending waves are easily generated, because of their low characteristic impedance.

The equation of motion for bending waves in an undamped beam can be written in the previous compact form, but with the transverse displacement of the bending wave motion being denoted by w=$w(x, t)$. If we again assume purely harmonic excitation and harmonic wave motion w=$w(x)e^{i\omega t}$, we get

$$L\{w(x)\} - \omega^2 m'\, w(x) = F'(x)\,, \tag{6.3.7}$$

where the differential operator $L\{\cdots\}$ that describes the shear force gradient in the beam now takes the form $B\, d^4/dx^4$. Here, B is the bending stiffness of the beam, m' is its mass per unit length and $F'(x)$ is an external force excitation per unit length. The operator is of fourth order, and *four* field variables are thus required for describing the bending wave motion. There are two motion variables, the transverse displacement $w = w(x)e^{i\omega t}$ and the angular displacement $\beta = \beta(x)e^{i\omega t}$, which is the first spatial derivative of w, ie dw/dx. Two force variables are associated with the wave motion, the internal shear force $F_y = F_y(x)e^{i\omega t}$ and the internal bending moment $M_z = M_z(x)e^{i\omega t}$; these are given by

$$F_y = B\frac{\partial^3 w}{\partial x^3} \quad \text{and} \quad M_z = -B\frac{\partial^2 w}{\partial x^2}\,. \tag{6.3.8}$$

Moreover, the wave speed c_b of a freely propagation bending wave in the beam is

$$c_b = \omega^{\frac{1}{2}} \left(\frac{B}{m'} \right)^{\frac{1}{4}} , \tag{6.3.9}$$

which is seen to depend upon *frequency*; this special phenomenon is called *dispersion*. Such dependence results in complicated sound radiation properties for plates and built-up structures. The wave speed or phase speed is furthermore noticed to depend upon the bending stiffness and the mass per unit length.

The phase speed of bending waves in a thin homogeneous *beam* with a rectangular cross-section and of thickness h in the direction of the motion, is given by

$$c_b \cong \sqrt{1.8 \, c_{l2} \, h \, f} , \tag{6.3.10}$$

where f is the frequency (in Hz) and c_{l2} is given by eq. (6.3.3).

Moreover, the phase speed in a thin homogeneous *plate* of thickness h is given by

$$c_b \cong \sqrt{1.8 \, c_{l1} \, h \, f} , \tag{6.3.11}$$

where c_{l1} is given by eq. (6.3.4).

6.3.4 INPUT MOBILITY OF INFINITE SYSTEMS

Finally, in this brief introduction it is appropriate to list some input mobilities for *point force* excitation. Or more specifically, input mobilities relating translational velocity $v \, e^{i\omega t}$ to translational force $F \, e^{i\omega t}$, both at the same point and in the same coordinate (direction). The corresponding point impedances are the reciprocal of the given point mobilities.

6.3.4.1 Beam or rod

Longitudinal vibration. In the case of a semi-infinite ($s\infty$) beam driven axially at the end, the input mobility is

$$Y_{s\infty} = \frac{1}{m' \, c_{l2}} . \tag{6.3.12}$$

where m' is mass per unit length and c_{l2} is given by eq. (6.3.3).

Bending vibration. The input mobility of a semi-infinite beam driven at the end is

$$Y_{s\infty} = \frac{1 - i}{m' \, c_b} . \tag{6.3.13}$$

where m' is mass per unit length and c_b is given by eq. (6.3.9), or by eq. (6.3.10), provided that the beam is of rectangular cross-section and is vibrating in the direction in which the beam thickness h is measured.

The input mobility of an infinite beam driven in the 'middle' is given by

$$Y_{\infty} = \frac{1 - i}{4 m' \, c_b} . \tag{6.3.14}$$

Note that this is four times lower than the input mobility of the semi-infinite beam, eq. (6.3.13).

6.3.4.2 Plate

Bending vibration. The input mobility of a semi-infinite plate driven normal to its surface and at the end (edge) is

$$Y_{s\infty} = \frac{1}{3.5\sqrt{B' m''}}, \tag{6.3.15}$$

where m'' is the mass per unit area, and for a homogeneous plate of thickness h the bending stiffness B' is

$$B' = \frac{E h^3}{12(1-\nu^2)}. \tag{6.3.16}$$

It is noted that this input mobility, eq. (6.3.15), is purely real, provided that the plate is undamped as is assumed here.

The input mobility of an infinite plate driven in the 'middle' is also real and is given by:

$$Y_{\infty} = \frac{1}{8\sqrt{B' m''}}. \tag{6.3.17}$$

Other point mobilities relating angular velocity to moment excitation, as well as cross mobilities, are given in ref. [5].

6.4 VIBRATION ISOLATION AND POWER TRANSMISSION

Vibration isolation is one of the most effective ways of reducing the transmission of audio frequency vibration from a disturbing *source* (machine, apparatus, etc) to a connected *receiver* structure. This is generally accomplished by 'disconnecting' the transmission paths between the two systems. In practice vibration isolation is done by inserting resilient mechanical connections or rubber elements that are much more compliant (ie, dynamically soft), than both the source structure and the receiving structure. Such vibration isolators have spring-like properties and are often made of vulcanised rubber elements, metal springs or combinations thereof. The isolation principle is depicted in Figure 4.1a, and Figure 4.1b shows an example of measured mobilities of a rubber isolator, engine source and elastic receiver.

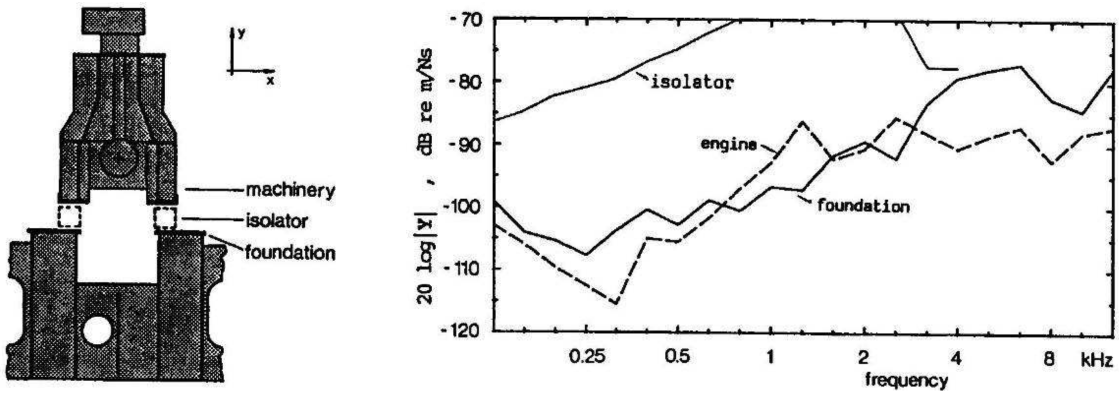


Figure 6.4.1 *(a) Vibration isolated diesel engine on elastic ship foundation; (b) Mobilities of isolator, engine and ship foundation. From ref. [9].*

The principle of vibration isolation has already been described in Chapter 6.2. Thus, in the case of a harmonically driven simple source of mass m resting on a spring s attached to an idealised rigid foundation, it was found that vibration isolation is achieved when the angular natural frequency ω_0 of the system is somewhat lower than the frequency component ω of the excitation force.

6.4.1 ESTIMATION OF SPRING STIFFNESS AND NATURAL FREQUENCY

It is often easy[2] to determine the important quantities (m, s and $\omega_0 = \sqrt{s/m}$) for uncritical arrangements of simple machinery sources that are mounted on vibration isolators (springs). Usually the mass m of the machine is known. For a vertically loaded spring the static force $F_0 = mg$ from the mass results in a static deflection (compression) of the spring of magnitude $\xi_0 = F_0/s$. These two relations enable the determination of the *static* stiffness of the spring, ie

$$s = \frac{m\,g}{\xi_0}, \qquad (6.4.1)$$

where g (= 9.81m/s^2) is the gravitational acceleration. The designed natural frequency of the system can therefore be determined by a very simple formula:

$$\omega_0 = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\xi_0}} \quad \text{[rad/s]}, \qquad \Leftrightarrow \qquad f_0 = \frac{\omega_0}{2\pi} \cong \frac{0.5}{\sqrt{\xi_0}} \quad \text{[Hz]}. \qquad (6.4.2)$$

If the spring element is slender and rod-like with a cross sectional area S , length (height) d and made from a material of Young's modulus E , then the *static* spring constant s can be calculated from

$$s = ES/d\,. \qquad (6.4.3)$$

Note that the *dynamic* stiffness of rubber-like material generally differs from this value of static spring constant or stiffness s. This will be treated in more details in Section 6.4.4.

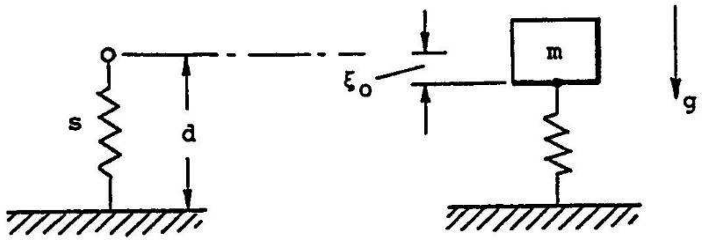


Figure 6.4.2 *Static deflection of spring, which in the unloaded condition has the length d .*

It was mentioned previously that the vibration isolation can be improved by reducing ω_0, that is to say, by increasing the static deflection ξ_0 . This can be accomplished by reducing s, but this results in a more laterally unstable arrangement. As a compromise for a number of practical source cases it is therefore often 'common' to choose values in the approximate range of 0.004 m < ξ_0 < 0.01 m, which corresponds to $8 > f_0 > 5$ Hz.

---

[2]) It should be recalled, however, that the simple oscillator model is a coarse simplification of the reality, where an extended rigid body on springs will have six degrees of freedom and thus six natural frequencies, eg see ref. [2].

6.4.2 TRANSMISSION OF POWER IN RIGIDLY COUPLED SYSTEMS

In contrast to the idealised model of a simple source on a rigid foundation we shall now examine the more realistic case of source and foundation or receiving structure of finite mobilities or impedances. It is reasonable to expect that the dynamic properties of the source and receiver will effect the vibration isolation that is achievable in practice.

For reference we shall initially address the situation where the vibration source is *rigidly* connected to the receiving structure, and it is assumed that source and receiver are connected via a *single* motion coordinate (or terminal). First consider the source in a *free uncoupled* state in which the vibration activity of the source can be characterised by its *free terminal velocity* v_{free} and its ability to transmit power by its terminal mobility Y_S, see Figure 6.4.3a . These source quantities are suitably combined into a single descriptor [10] called the terminal *source strength* $|J_{term}|$:

$$|J_{term}| = \frac{\overline{v_{free}^2}}{|Y_S|}, \tag{6.4.4}$$

where $\overline{v_{free}^2}$ is the time-average mean-square value of the free velocity $v_{free} = v_{free}(t)$. This source strength $|J_{term}|$, with units of power [W], is useful when comparing different vibratory sources.

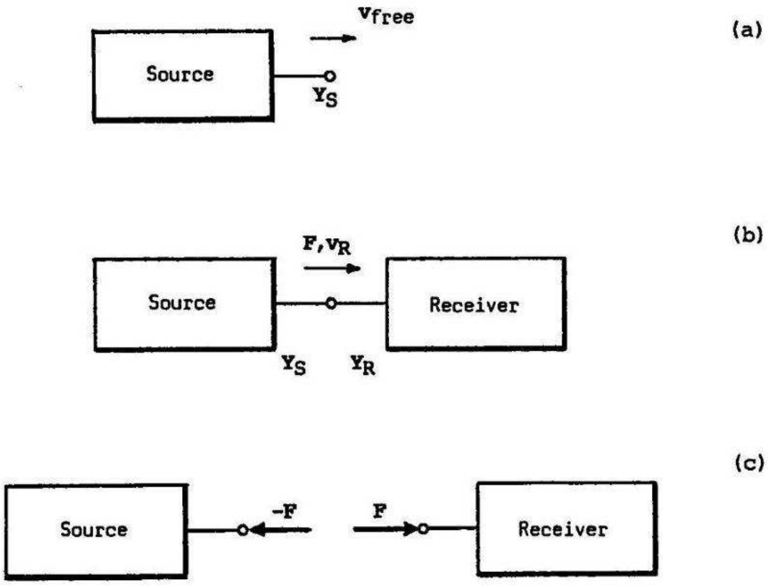


Figure 6.4.3 *Systems with a single coupling coordinate: (a) Free vibration source, (b) Source coupled rigidly to receiving structure, (c) Reaction forces on systems.*

In the analysis that follows we assume harmonic vibration $v_{free} \equiv v_{free}\, e^{i\omega t}$. The source is now being connected to a receiving structure, which is characterised by the input mobility Y_R . This loading of the source causes the free velocity to change to v_R , because of the force reaction $(-F)$ on the source, ie

$$v_R = v_{free} - Y_S F\,. \tag{6.4.5}$$

Since per definition $v_R = Y_R F$, we find directly for the rigid coupled system:

$$F = (Y_S + Y_R)^{-1} v_{free} \qquad \text{and} \qquad v_R = Y_R (Y_S + Y_R)^{-1} v_{free}\,. \tag{6.4.6a,b}$$

The force and velocity at the coupling point have hereby been determined for this case of rigid coupling.

The power that is transmitted to the receiving structure is given by the well-known relations:

$$\overline{P} = \tfrac{1}{2}\mathrm{Re}\{Fv_R^*\} = \tfrac{1}{2}|F|^2\,\mathrm{Re}\{Y_R\} = \tfrac{1}{2}|v_R|^2\,\mathrm{Re}\{Z_R\}. \tag{6.4.7}$$

By substituting the expressions from eq. (6.4.6) herein yields

$$\overline{P} = \tfrac{1}{2}|v_{free}|^2 \frac{\mathrm{Re}\{Y_R\}}{|Y_S+Y_R|^2} = \tfrac{1}{2}|v_R|^2 \frac{\mathrm{Re}\{Y_R\}}{|Y_R|^2}. \tag{6.4.8}$$

For further evaluation of the transmitted power this can be written in a convenient alternative form. Introducing the terminal source strength $|J_{term}|$, eq. (6.4.4), and a power coupling factor C_P yields

$$\overline{P} = |J_{term}|C_P, \tag{6.4.9}$$

where

$$C_P = \frac{|Y_S|\,|Y_R|}{|Y_S+Y_R|^2}\cos\varphi_R = \frac{\cos\varphi_R}{\left|\,|Y_S|/|Y_R| + |Y_R|/|Y_S| + 2\cos\theta\right|}, \tag{6.4.10}$$

and φ_R is the phase angle of the receiver mobility and $\theta = \varphi_R - \varphi_S$ is the phase difference between receiver and source mobilities. This takes values in the interval: $0 \le |\theta| \le \pi$. The power coupling factor is noted to be symmetric with respect to the logarithm of the mobility ratio $|Y_R|/|Y_S|$. For further details see ref. [10, 11].

6.4.3 VIBRATION ISOLATED SOURCE

The effect of a vibration isolator is now considered. The source is connected to the receiver via a vibration isolator as schematically shown in Figure 6.4.4a. For simplicity it is assumed that the isolator can be modelled as an ideal spring with a spring constant s . Thus, because the spring is assumed massless, this implies that the force on the left-hand-side of the spring

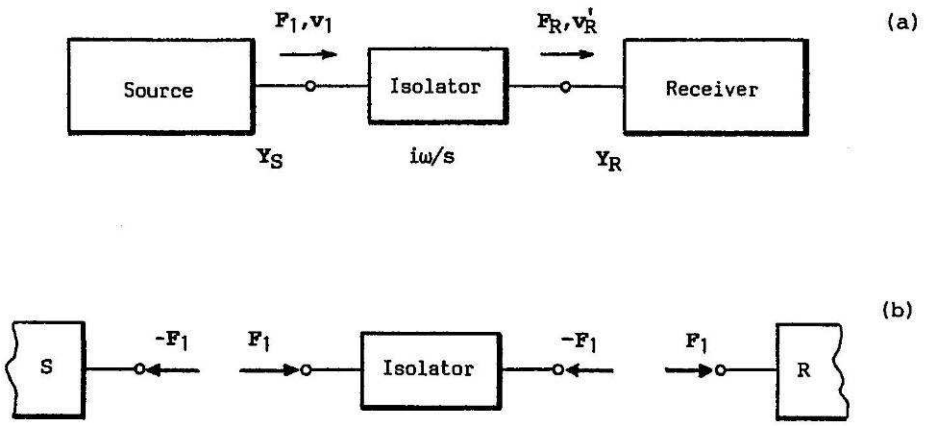


***Figure** 6.4.4 (a) Block diagram of vibration isolation of a source with a single coupling coordinate. (b) Diagram that shows the forces on the system elements.*

is identical to the force on the receiver, $F_1 = F_R$. The velocities are different, of course, and similar to before given by

$$v_1 = v_{free} - Y_S F_1 \qquad \text{and} \qquad v'_R = Y_R F_1 . \qquad (6.4.11a,b)$$

The force and the velocities are related according to Hooke's law as

$$F_1 = s\left(\frac{v_1}{\mathrm{i}\omega} - \frac{v'_R}{\mathrm{i}\omega}\right), \qquad (6.4.12)$$

which, together with eq. (6.4.11) give

$$F_1 = \left(\mathrm{i}\omega/s + Y_S + Y_R\right)^{-1} v_{free} \qquad \text{and} \qquad v'_R = Y_R\left(\mathrm{i}\omega/s + Y_S + Y_R\right)^{-1} v_{free} . \quad (6.4.13a,b)$$

By substituting F_1 into the general relation, eq. (6.4.7b), gives the transmitted power to the receiver

$$\overline{P'} = \tfrac{1}{2}\left|v_{free}\right|^2 \frac{\mathrm{Re}\{Y_R\}}{\left|\mathrm{i}\omega/s + Y_S + Y_R\right|^2} . \qquad (6.4.14)$$

So, this gives

$$\overline{P'} = \left|J_{term}\right| C'_P , \qquad (6.4.15)$$

where

$$C'_P = \frac{\left|Y_S\right|\left|Y_R\right|}{\left|\mathrm{i}\omega/s + Y_S + Y_R\right|^2} \cos\varphi_R . \qquad (6.4.16)$$

These results for the vibration-isolated source have to be compared with those for the rigid coupled case in order to realistically evaluate the influence of the vibration isolator. This influence is most suitably described by the *effectiveness* $E_{iso} = E_{iso}(\omega)$ of the vibration isolator, also called its *insertion loss*. This is defined as the ratio between the squared magnitudes of the receiver velocities *before* and *after* the installation of the vibration isolator - or for that matter - as the ratio of the corresponding injected powers. Eqs. (6.4.6b) and (6.4.13b) thus give

$$E_{iso} = \frac{\left|v_R\right|^2}{\left|v'_R\right|^2} = \left|\, 1 + \frac{\mathrm{i}\omega/s}{Y_S + Y_R}\right|^2 . \qquad (6.4.17)$$

From this equation it is evident that a *high* effectiveness (ie, large number) requires that the isolator mobility $\mathrm{i}\omega/s \equiv Y_I$ is much higher (ie, much more mobile or compliant) than the *sum* of the source and receiver mobilities, that is,

$$Y_I = \mathrm{i}\omega/s >> Y_S + Y_R . \qquad (6.4.18)$$

Such a large value of inequality is not easily accomplished over the broad audible frequency range, because lightly damped resonance in elastic source and receiving structures will occur and limit the effectiveness of the isolator. Furthermore, at high frequencies the mass of the isolator can no longer be ignored and resonance occur in the isolator itself, which also limit the effectiveness. In the case of a symmetric vibration isolator, such modal behaviour can be accounted for in a prediction by replacing $\mathrm{i}\omega/s$ in eq. (6.4.17) with the actual mobility of the isolator Y_I , see also ref. [12].

At first, the definition of the isolator effectiveness in eq. (6.4.17) does not seem to apply to the ideal case of a rigid (immoveable) foundation that was assumed in Chapter 6.2. However, this is not so, because E_{iso} might as well be defined as the ratios of forces acting on the receiver, whether this is moving or not. This follows from the fact that velocities and forces are related via the receiver mobility. So, for the general elastic receiver the effectiveness also reads $E_{iso} = |F_R|^2 / |F_R'|^2$, where the dash refers to the case with the source resiliently connected to the receiver. Hence, by substituting the derived expressions for the corresponding forces, eq. (6.4.6a) and (6.4.13a), respectively, we obtain exactly eq. (6.4.17).

<u>*Example 6.4.1*</u> The isolation effectiveness E_{iso} is to be determined for a harmonically driven mass-spring resonator, which is connected to a *rigid* foundation, similar to the systems in Figure 6.2.1 or 6.4.2. The undamped natural frequency of the resonator is $\omega_0 = \sqrt{s/m}$, where m is its mass and s is the spring stiffness. It is here assumed that the system is structurally damped and that this is accounted for by taken the spring stiffness to be complex $\underline{s} = s(1+i\eta)$.

The source, being the mass m , has the mobility $Y_S = (i\omega m)^{-1}$ and the mobility of the receiver in the form of a rigid foundation is $Y_R = 0$. Substituting these into eq. (6.4.15) gives

$$E_{iso} = \left| 1 - \omega^2 \frac{m}{s(1+i\eta)} \right|^2 \cong \left| 1 - \left(\frac{\omega}{\omega_0}\right)^2 + i\eta\left(\frac{\omega}{\omega_0}\right)^2 \right|^2 ; \qquad (6.4.19)$$

in the last approximation it is assumed that $\eta << 1$, so that $1 + \eta^2 \approx 1$. By comparison it is seen that eq.(6.4.19) is equal to the *reciprocal* of the results for $|F_f|^2 / |F_1|^2$ in Figure 6.2.5b. This can also be deduced from eq. (6.2.18), if the damping constant r is replaced by the equivalent constant r_{eq} for a structurally damped spring $r_{eq} = s\eta/\omega$.

Figure 6.4.5 shows an example of measured and predicted values of the isolation effectiveness for a complicated vibration source (the diesel engine in Figure 6.4.1a), which is resiliently mounted on an elastic foundation. The source is mounted on ten multi-directional isolators; note that these isolators have a much higher mobility than the isolator example shown in Figure 6.4.1b. The effectiveness is seen to be rather good, about 25 dB on average. Also shown are two course estimations based upon, respectively, a simple mass-spring-mass model (LF-prediction of resemblance to eq. (6.4.19)), and a simple mono-coupled model, where measured isolator mobility and average point mobilities of source and receiver have

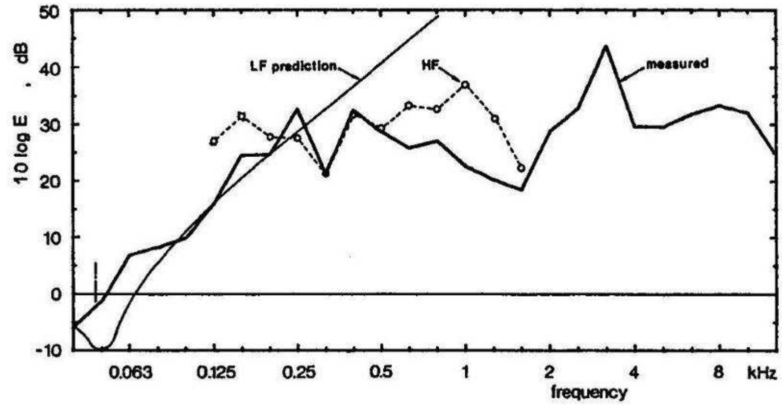


Figure 6.4.5 *Effectiveness of vibration isolation* $10 \log E_{iso}$ *of a multi-coupled machinery source on an elastic receiving structure.*

been used in eq. (6.4.17). Despite of the coarse simplifications in these models, a reasonable agreement with measurement is found in the frequency range up to 800 Hz.

Another example of predicted isolation effectiveness is shown in Figure 6.4.6. Here, a 105 m tall building structure is mounted on large, flat rubber pads that allow thermal expansion or contraction of the huge building. Calculations were carried out in order to estimate their isolation effectiveness against structureborne sound transmission from disturbing underground rail traffic. It is apparent from Figure 6.4.6 that these thermal expansion devises are not very useful as vibration isolators; their static deformation is simply too small – in other words – the stiffness of the isolators is too high. At the fundamental natural frequency of the system vibration amplification is observed and in the frequency range above 90 Hz the effectiveness is seen to become very small at certain frequencies. These correspond to the natural frequencies of the foundation columns (≈ 'source'), on which the rubber pads and building structure rest.

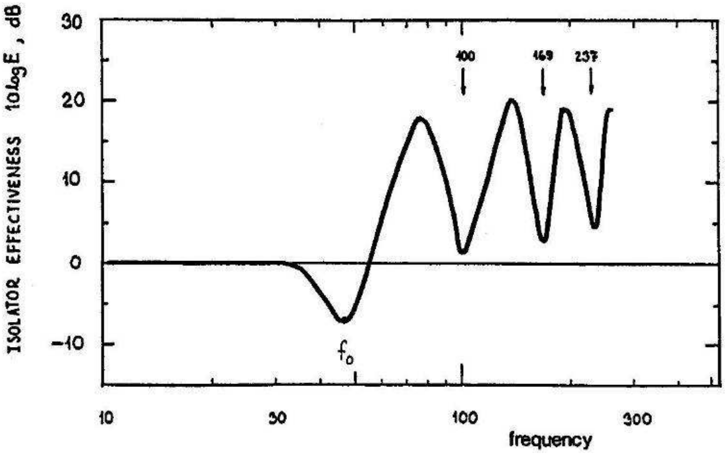


Figure 6.4.6 *Isolation effectiveness of rubber expansion devises that support a tall building.*

6.4.4 DESIGN CONSIDERATIONS FOR RESILIENT ELEMENTS

It was mentioned in Section 6.4.1 that the dynamic stiffness of rubber-like material generally differs from the static spring stiffness s determined by static measurement. When such isolators are used it is therefore necessary to insert the *dynamic* stiffness value s_{dyn} instead of s in the equation for the natural frequency, eg eq. (6.4.2a).

6.4.4.1 Rubber-like materials

The dynamic stiffness of rubber isolators depends upon a number parameters. An important parameter is the rubber hardness, which is usually characterised in °Shore A of hardness. The typical hardness-range of commercial rubber isolators is from about 40°Shore A (for soft isolators) to 80°Shore A , which is rather hard. Table 6.4.1 presents a coarse guide that shows approximate, empirical values for the relation between rubber hardness, static Young's modulus E and dynamic Young' modulus E_{dyn} , or more specifically their ratio E_{dyn}/E .

Thus, for a slender rubber isolator (of static stiffness given by eq. (6.4.3)), the appropriate *dynamic* stiffness s_{dyn} becomes

$$s_{dyn} = E_{dyn} S/d . \qquad (6.4.20)$$

However, this is generally not the final estimate, because the stiffness of rubber isolators also depends upon another important parameter, which is basically the compactness of the isolator. Generally, the stiffness of a *short* rubber block is found to be much higher than the stiffness of a *long* slender sample. (Note, that this effect of course is accounted for when estimations are based on a static load-deflection test, ie on eq. (6.4.1).)

Table 6.4.1 *Approximate values for the relation between rubber hardness, static Young's modulus E and dynamic Young's modulus E_{dyn}. The results apply to natural rubber.*

Rubber hardness °Shore A	Static Young's modulus E 10^6 N/m^2	Ratio: E_{dyn}/E --
40	1.5	1.2
50	2.5	1.4
60	4.0	1.8
70	6.0	2.2

The stiffness expressed by eq. (6.4.18) therefore has to be corrected for the 'bulkiness' of the rubber isolator. This can be characterised by an area ratio (or shape factor) $R_S = S_{const}/S_{free}$, in which the area S_{constr} represents the total *constrained* or loaded area of the isolator, and S_{free} is the total *free* surface area of the isolator. Figure 6.4.7 shows the stiffness correction factor C_s to be used for a given area ratio R_S . Thus, s_{dyn} is to be multiplied with C_s to give the actual, corrected dynamic stiffness.

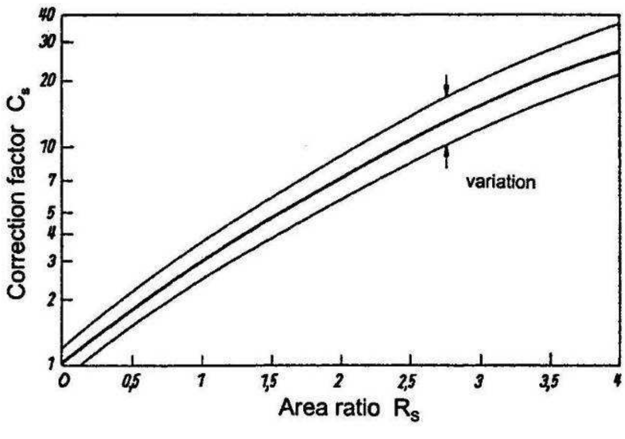


Figure 6.4.7 *Stiffness correction C_s to be used as a function of the area ratio R_S of the vibration isolator. After ref. [13].*

6.4.4.2 Metal and other elastic solids

As oppose to the rubber-like materials, the static and dynamic elastic properties for most engineering materials are found to be practical identical. For a given elastic material this means that its Young's modulus $E \cong E_{dyn}$ and its shear modulus $G \cong G_{dyn}$. Furthermore, since $\nu \approx 0.3$ for most solid materials, we have $E \approx 3G$.

Resilient elements of metal may take many different forms. Usually they are extended, continuous components with distributed mass and stiffness, and basically they are designed to achieve a specified small stiffness at low frequencies. However, at mid and high frequencies such a resilient element can support different wave types, and resonances will occur in the resilient element because it is of finite size. This will diminish the isolator effectiveness, unless damping and/or rubber elements are incorporated into the final design of the resilient element.

The most common resilient element of metal is probably the helical spring, which is often made of harden steel. The static and low frequency stiffness in the axial direction of the spring is

$$s = \frac{G\,d^4}{8\,n\,D^3}, \qquad (6.4.21)$$

where G is the shear modulus of the material, D is the average diameter of the spring, d is the diameter of the coil and n is the number of coils or windings.

Other types of resilient elements are leaf springs, which may be thin metal beams or plates. One example is a so-called cantilever beam, which is rigidly built-in at the receiver-end and is completely free at the other end, where it supports the source to be isolated. For a beam with constant thickness h and constant rectangular cross-section S the spring stiffness is

$$s = \frac{E\,S\,h^2}{4L^3}, \qquad (6.4.22)$$

in which E is Young's modulus and L is the length of the beam. However, usually the source will be bolted to the beam and this will hinder angular motion at its 'free' end. Thereby the spring stiffness of the resilient element will increase by a factor of *four*, to become $s = E\,S\,h^2/L^3$. This clearly illustrates the importance of the boundary conditions at mounting positions.

6. 5 REFERENCES

1. Resonators by M. Heckl, Chapter 20 in ‘Modern methods in analytical acoustics: Lecture Notes’ (ed. D. G. Crighton, A. P. Dowling, J. E. Ffowcs Williams, M. Heckl and F. G. Leppington), Springer-Verlag, London Ltd. 1994.

2. C. M. Harris and C. E. Crede: Shock and vibration handbook, 2nd ed. McGraw-Hill, New York 1976.

3. D. J. Mead: Passive vibration control. John Wiley & Sons, Chichester 1999.

4. E. Skudrzyk: Simple and complex vibratory systems. Penn State University Press 1968.

5. L. Cremer , M. Heckl and E. E. Ungar: Structure-borne sound, 2nd ed., Springer Verlag, Berlin 1988.

6. L. Cremer und M. Heckl: Körperschall, 2nd ed. Springer Verlag, Berlin 1996.

7. Vibration of one- and two-dimensional continuous systems by M. Heckl, Chapter 64 in ‘Encyclopedia of acoustics’ (ed. M.J. Crocker), John Wiley & Sons, Ney York 1997.

8. F. J. Fahy: Sound and structural vibration. Academic Press, London 1985.

9. M. Ohlrich and F. Jacobsen: Isolation of structural vibration from machinery. Proceedings of Nordic Acoustical Meeting, NAM-82, Stockholm, 1982, pp. 309-312.

10. M. Ohlrich: Vibrational source strength as a prerequisite for response prediction by SEA. NOVEM 2000, Proceedings of Intern. Conf. on Noise & Vibration Pre-design and Characterisation using Energy Methods, Lyon, 2000, on CD-ROM, pp.12.

11. J. M. Mondot and B. Petersson: Characterisation of structure-borne sound sources: The source descriptor and the coupling function. Journal of Sound and Vibration 114, 1987, pp. 507-518.

12. E. E. Ungar and C. W. Dietrich: High-frequency vibration isolation. Journal of Sound and Vibration 4(2), 1966, pp. 224-241.

13. VDI 2062 Blatt 2 Vibration isolation: Resilient elements (In German), 1976.

LIST OF SYMBOLS

a	radius of sphere [m]; acceleration [m/s^2]
A	equivalent absorption area [m^2]; accelerance [m/Ns^2]
A_0	reference area [m^2]
B	bending stiffness per unit length [Nm]; bending stiffness [Nm^2]
B'	bending stiffness per unit width [Nm]
c	speed of sound [m/s]
c_b	speed of bending waves [m/s]
c_L	speed of longitudinal waves [m/s]
C_P	power coupling factor [dimensionless]
d	length [m]
D	directivity [dimensionless]
DI	directivity index [dB]
E	total acoustic energy [J]; Young's modulus of elasticity [N/ m^2]
E_{iso}	vibration isolation effectiveness; insertion loss [dimensionless]
f	frequency [Hz]
f_0	resonance frequency [Hz]
f_c	critical frequency [Hz]
F	force [N]
G	shear modulus [N/m^2]
h	distance [m]; plate thickness [m]
H	receptance [m/N]
H_1	Struve function
$\mathbf{I}$	sound intensity [W/m^2]
I_{ref}	reference sound intensity [W/m^2]
I_x	component of sound intensity [W/m^2]
J_m	Bessel fuction
$\lvert J_{term}\rvert$	terminal source strength [W]
k	wavenumber [m^{-1}]
K	stiffness constant [N/m]
K_s	adiabatic bulk modulus [N/m^2]
l	length [m]
l_m	mean free path [m]
L	loudness level [phone]; total length of edges [m]; length [m]
L_A	A-weighted sound pressure level [dB re p_{ref}]
L_{Aeq}	equivalent A-weighted sound pressure level [dB re p_{ref}]
L_{AE}	sound exposure level [dB re p_{ref}]
L_C	C-weighted sound pressure level [dB re p_{ref}]
L_{eq}	equivalent sound pressure level [dB re p_{ref}]
L_I	sound intensity level [dB re I_{ref}]
L_n	impact sound pressure level [dB re p_{ref}]
L_p	sound pressure level [dB re p_{ref}]
L_W	sound power level [dB re P_{ref}]
L_Z	sound pressure level measured without frequency weighting [dB re p_{ref}]
m	air attenuation factor [m^{-1}]; mass [kg]; mass per unit area [kg/m^2]
m'	mass per unit length [kg/m]

m'' mass per unit area [kg/m^2]
M mass [kg]
n natural number [dimensionless]
N loudness [sone]; number of modes [dimensionless]
p sound pressure [Pa]
$p_A(t)$ instantaneous A-weighted sound pressure [Pa]
p_{ref} reference sound pressure [Pa]
p_{rms} rms value of sound pressure [Pa]
p_0 static pressure [Pa]
P power [W]
P_a sound power [W]
P_{ref} reference sound power [W]
q volume velocity associated with a fictive surface [m^3/s]; generalised coordinate [m]
Q volume velocity of source [m^3/s]; directivity factor [dimensionless]
r radial distance in spherical coordinate system [m]; damping constant of viscous damper [kg/s]
r_{rev} reverberation distance in a room [m]
R gas constant [m^{2}s^{-2}K^{-1}]; reflection factor [dimensionless]; transmission loss [dB]
R_0 transmission loss at normal incidence [dB]
s standing wave ratio [dimensionless]; spring constant [N/m]
S surface area [m^2]; cross sectional area [m^2]
t time [s]
T absolute temperature [K]; averaging time [s]
T_{60} reverberation time [s]
u longitudinal displacement [m]
$\mathbf{u}$ particle velocity [m/s]
u_x component of the particle velocity [m/s]
U velocity [m/s]
v velocity [m/s]
V volume [m^3]
w transverse displacement [m]
w_{kin} kinetic energy density [J/m^3]
w_{pot} potential energy density [J/m^3]
x, y, z Cartesian coordinates [m]
Z_a acoustic impedance [kg m^{-4}s^{-1}]
$Z_{a,r}$ acoustic radiation impedance [kg m^{-4}s^{-1}]
Z_m mechanical impedance [kg/s]
$Z_{m,r}$ mechanical radiation impedance [kg/s]
Z_w separation impedance [kg m^{-2}s^{-1}]
Y mobility (mechanical admittance) [s/kg]

α absorption coefficient [dimensionless]
α_m mean absorption coefficient [dimensionless]
β angular displacement [radian]
γ ratio of specific heats [dimensionless]
δ damping coefficient [s^{-1}]; end correction [m]
ΔL insertion loss [dB]

ΔV volume displacement [m^3]
ζ viscous damping ratio [dimensionless]
η loss factor [dimensionless]
θ polar angle in spherical coordinate system [dimensionless]
λ wavelength [m]
ν Poisson's ratio [dimensionless]
ξ displacement [m]
ρ density [kgm^{-3}]
τ time constant [s]; transmission coefficient [dimensionless]
φ phase angle [radian]; azimuth angle in spherical coordinate system [radian]
ω angular frequency [radian/s]
Ω frequency ratio [dimensionless]

^ indicates complex representation of a harmonic variable

INDEX