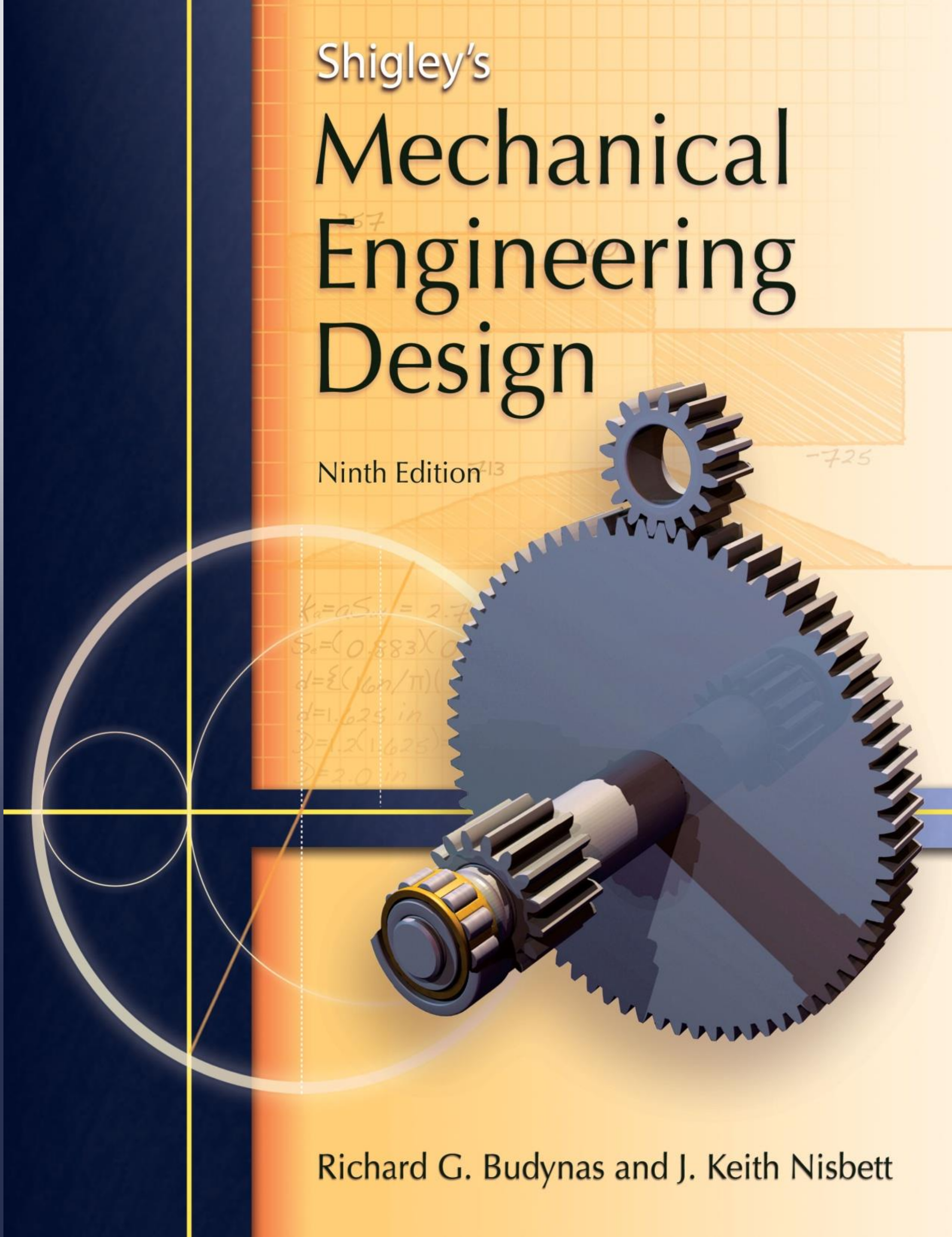
Lecture Slides

Chapter 6

Fatigue Failure Resulting from Variable Loading



Chapter Outline

Introduction to Fatigue in Metals

- Loading produces stresses that are variable, repeated, alternating, or fluctuating
- Maximum stresses well below yield strength
- Failure occurs after many stress cycles
- Failure is by sudden ultimate fracture
- No visible warning in advance of failure

Stages of Fatigue Failure

- *Stage I* – Initiation of micro-crack due to cyclic plastic deformation
- *Stage II* – Progresses to macro-crack that repeatedly opens and closes, creating bands called *beach marks*
- *Stage III* – Crack has propagated far enough that remaining material is insufficient to carry the load, and fails by simple ultimate failure

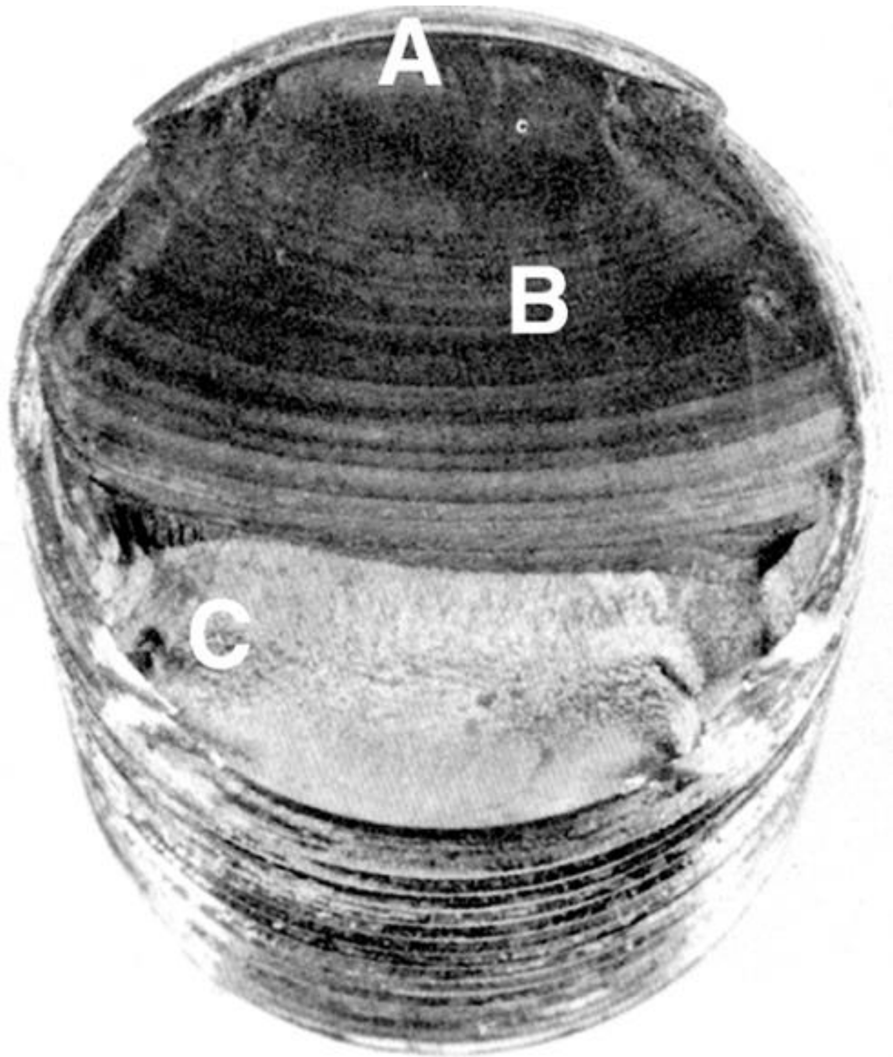


Fig. 6–1

Schematics of Fatigue Fracture Surfaces

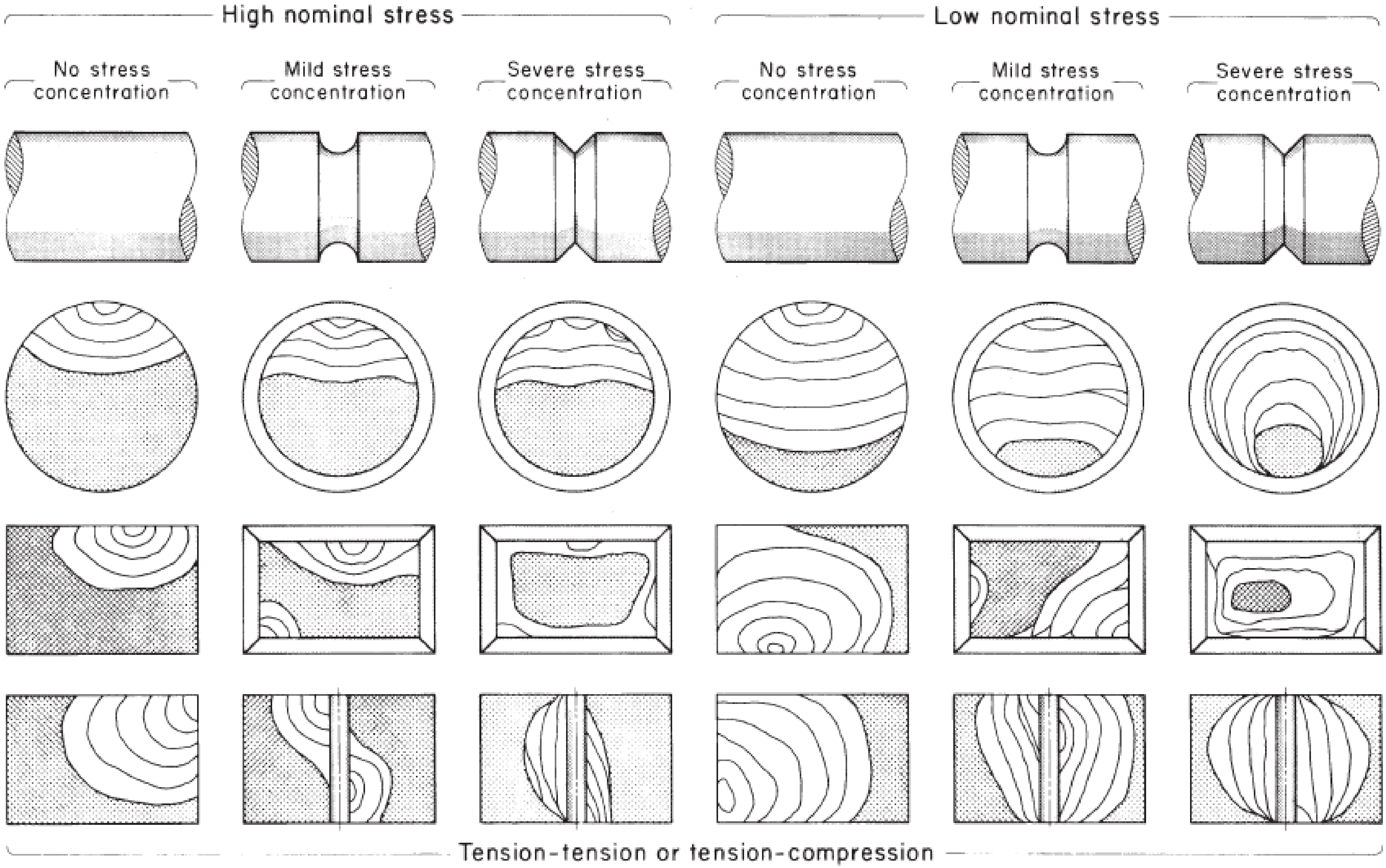


Fig. 6–2

Schematics of Fatigue Fracture Surfaces

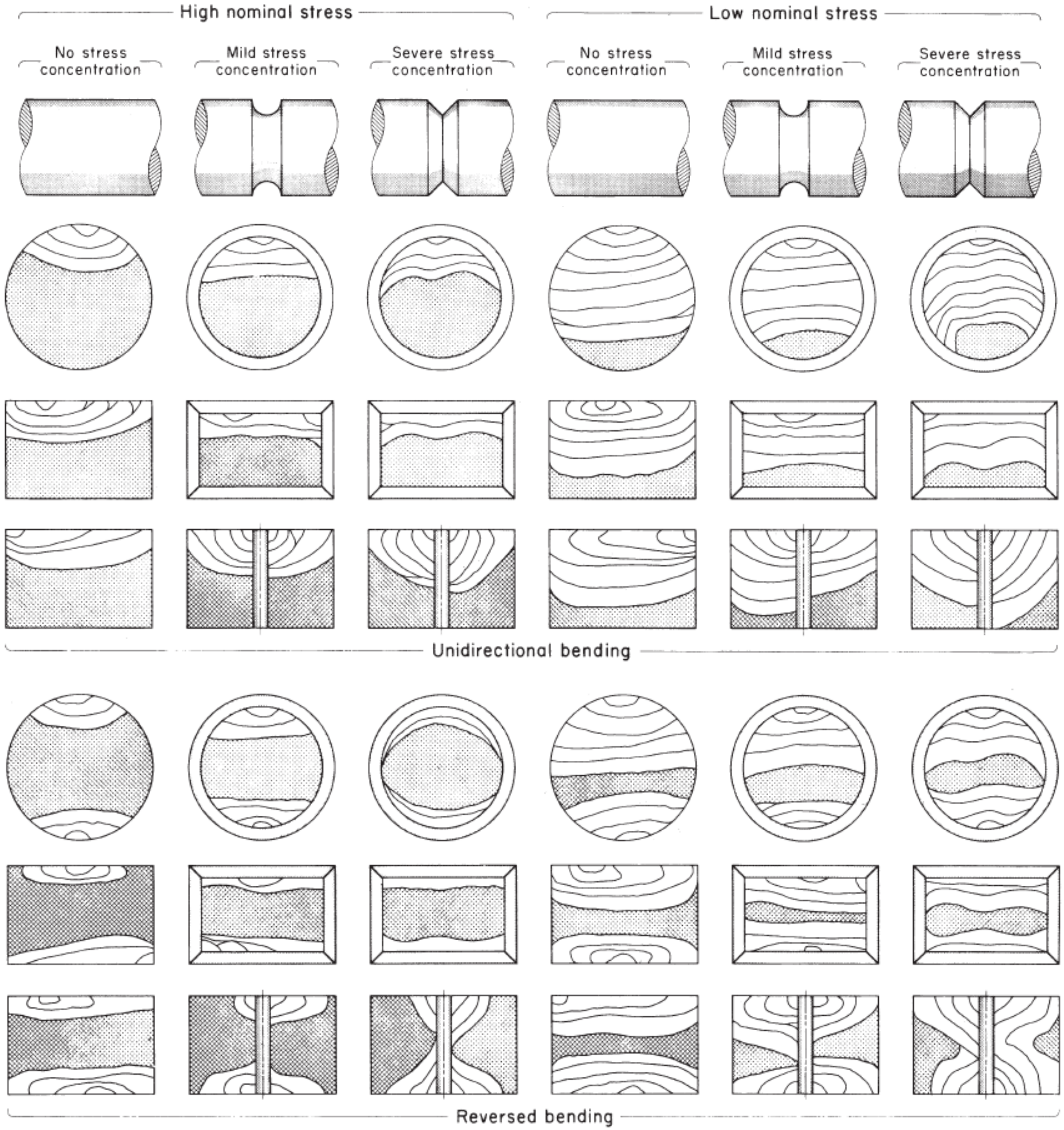


Fig. 6–2

Schematics of Fatigue Fracture Surfaces

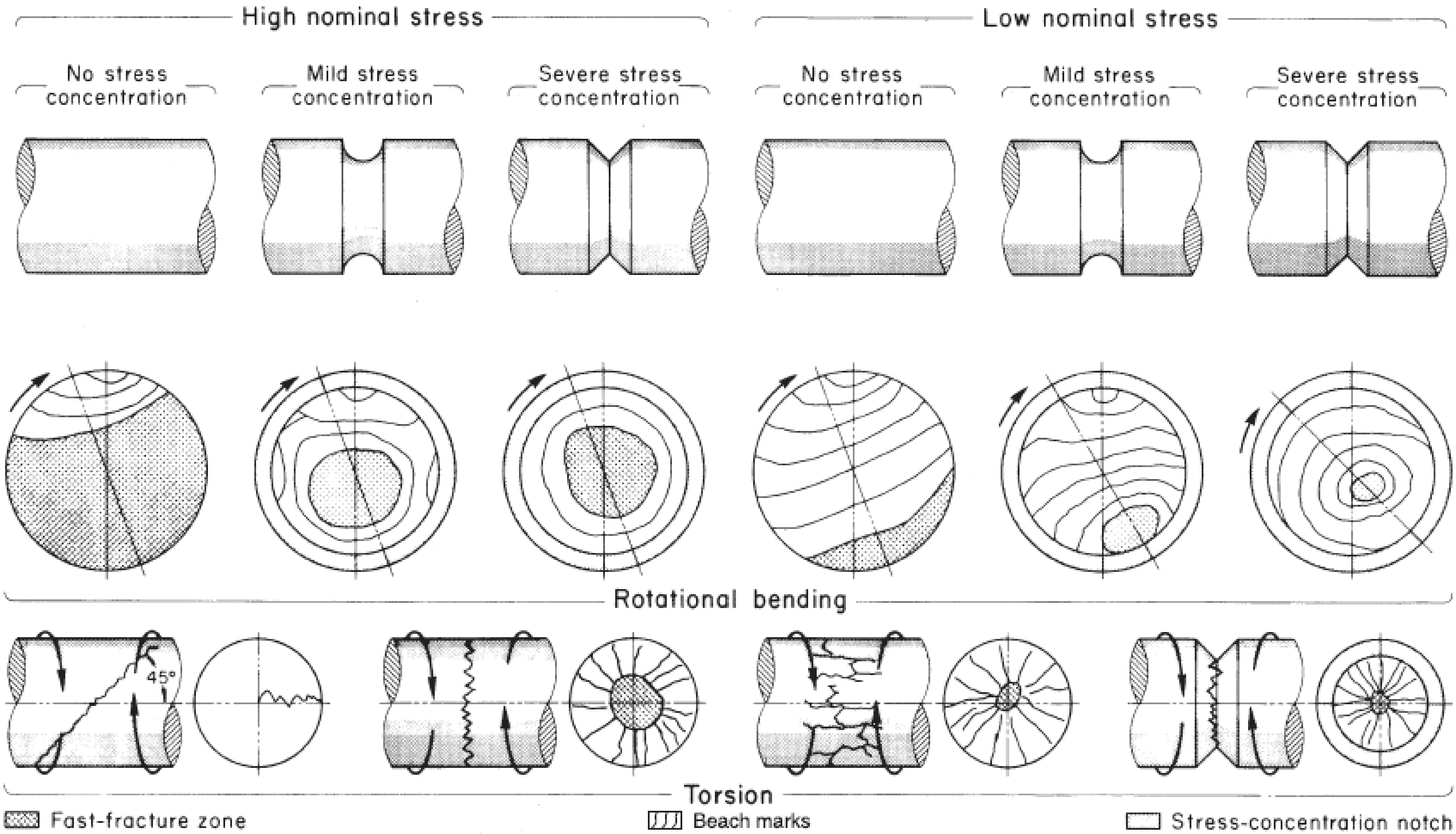


Fig. 6–2

Fatigue Fracture Examples

- AISI 4320 drive shaft
- B– crack initiation at stress concentration in keyway
- C– Final brittle failure

Fig. 6–3

Fatigue Fracture Examples

- Fatigue failure initiating at mismatched grease holes
- Sharp corners (at arrows) provided stress concentrations

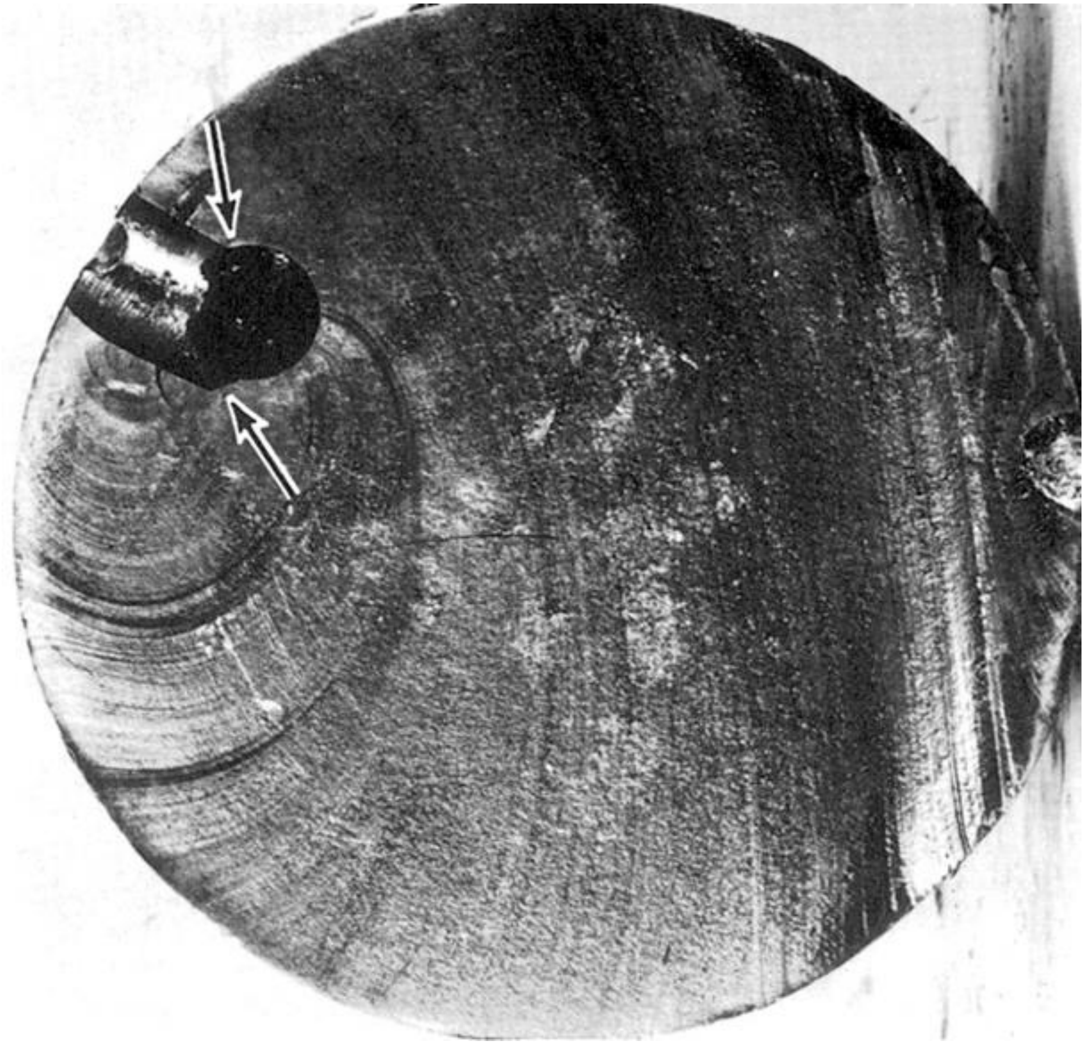


Fig. 6–4

Fatigue Fracture Examples

- Fatigue failure of forged connecting rod
- Crack initiated at flash line of the forging at the left edge of picture
- Beach marks show crack propagation halfway around the hole before ultimate fracture

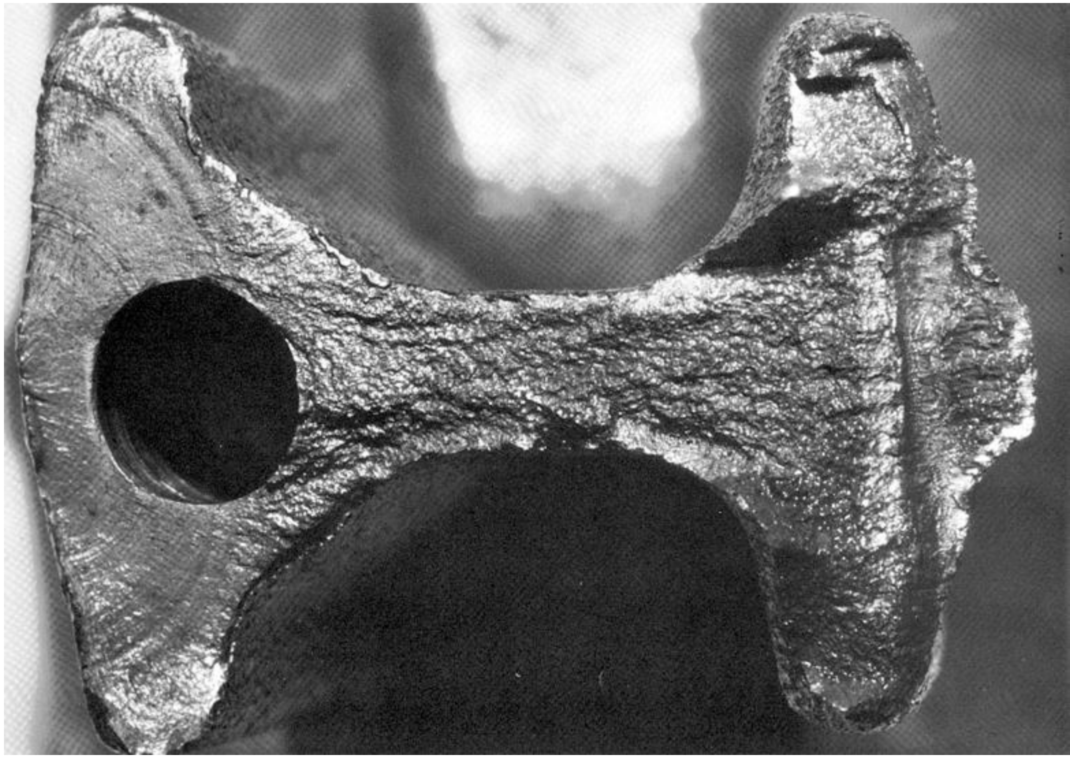


Fig. 6–5

Fatigue Fracture Examples

- Fatigue failure of a 200-mm diameter piston rod of an alloy steel steam hammer
- Loaded axially
- Crack initiated at a forging flake internal to the part
- Internal crack grew outward symmetrically

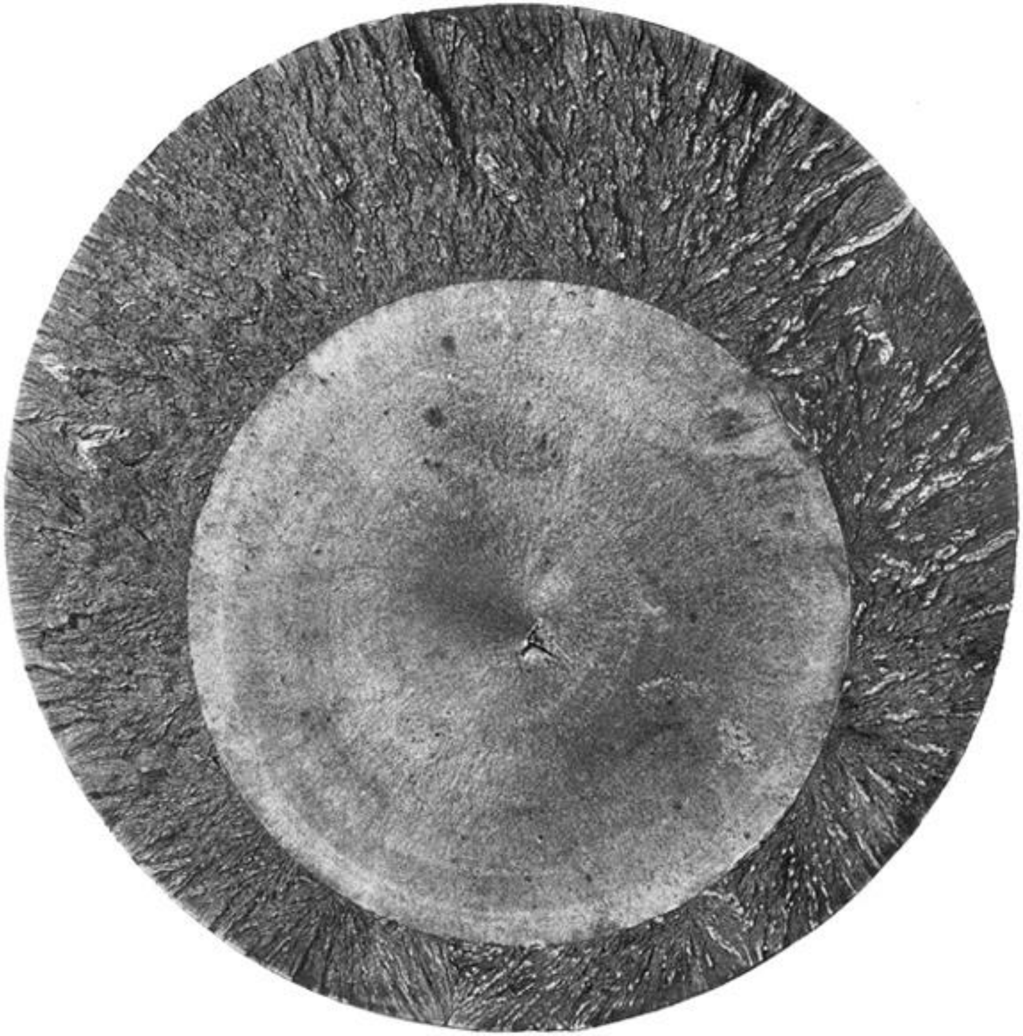
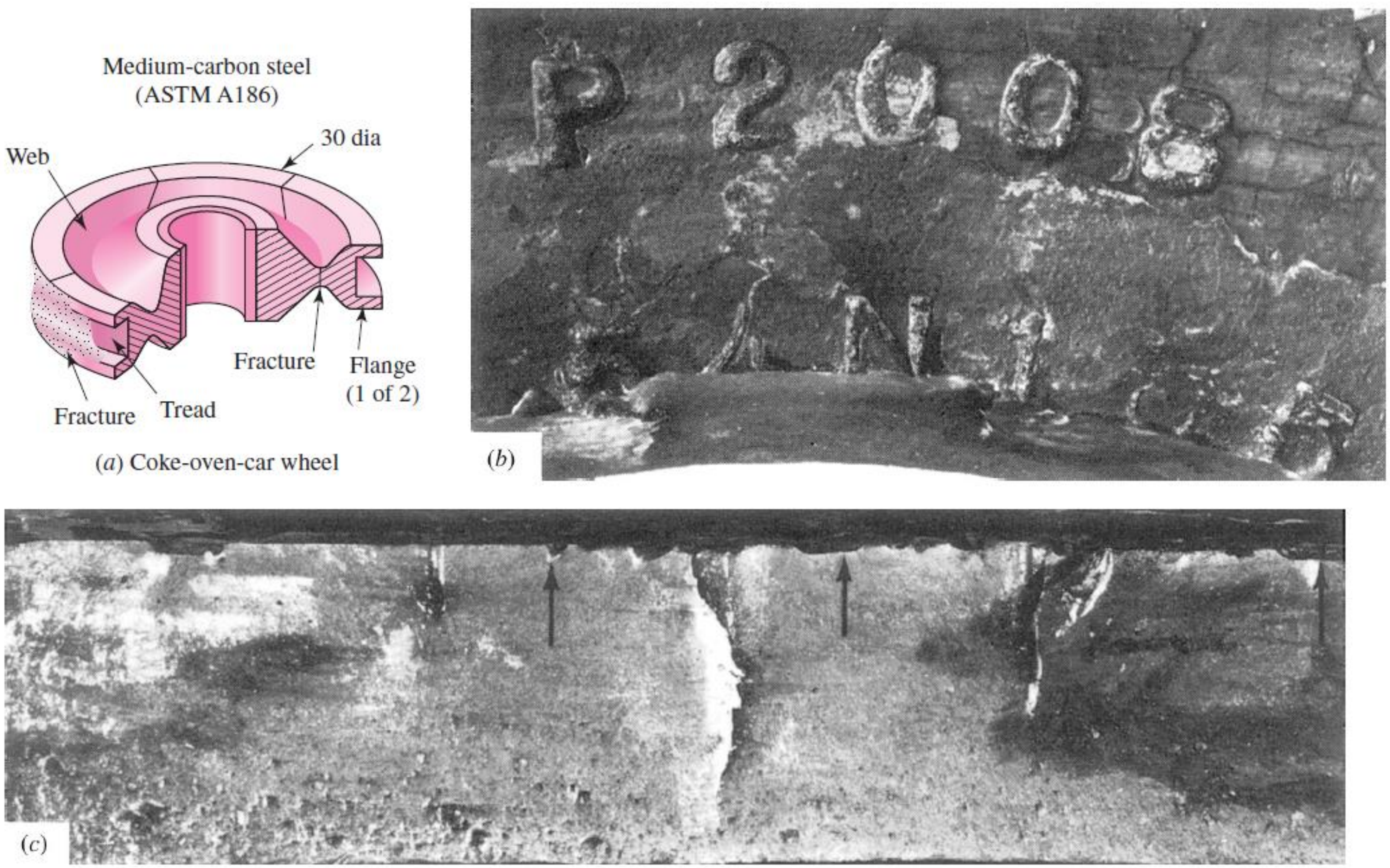


Fig. 6–6

Fatigue Fracture Examples

- Double-flange trailer wheel
- Cracks initiated at stamp marks



(*a*) Coke-oven-car wheel

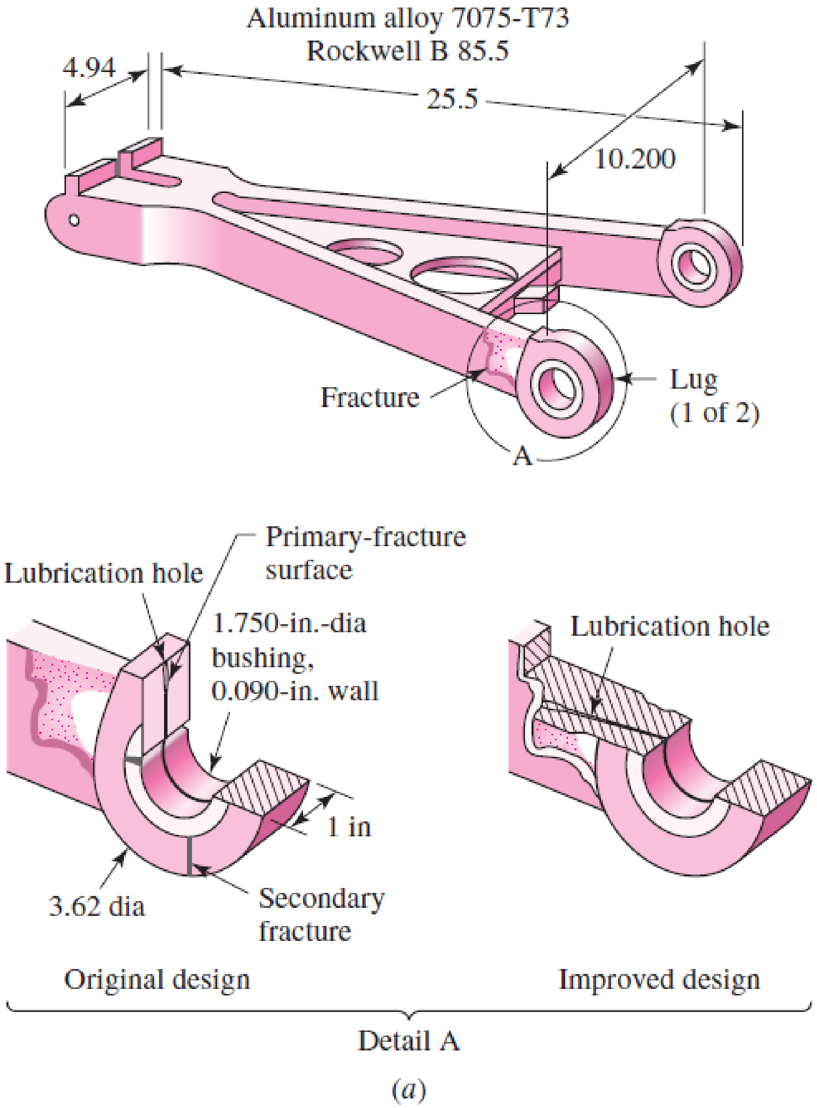
(*b*)

(*c*)

Fig. 6–7

Fatigue Fracture Examples

- Aluminum allow landing-gear torque-arm assembly redesign to eliminate fatigue fracture at lubrication hole



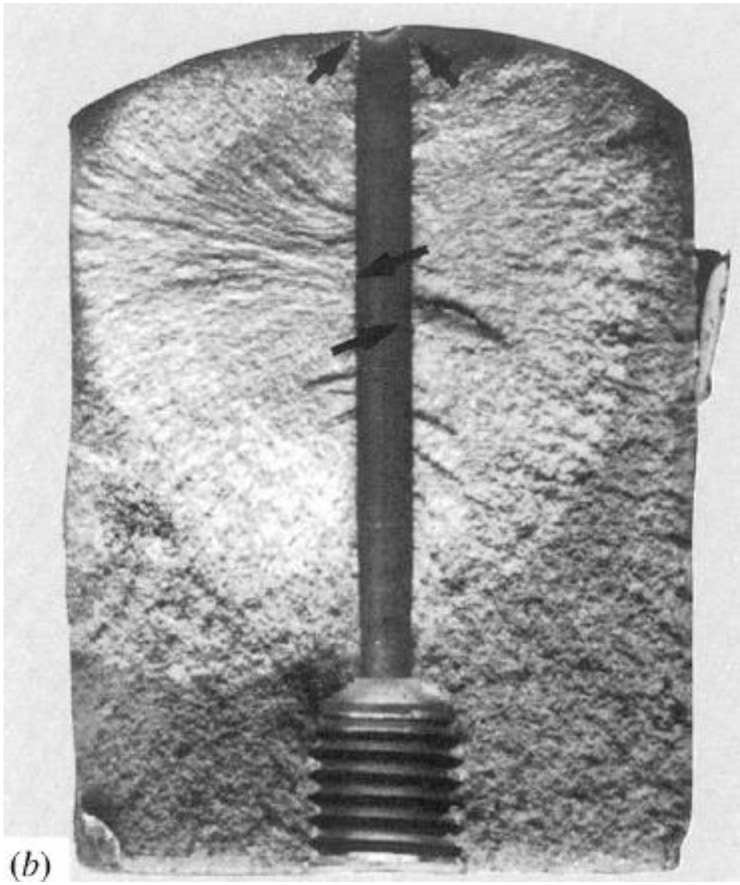


Fig. 6–8

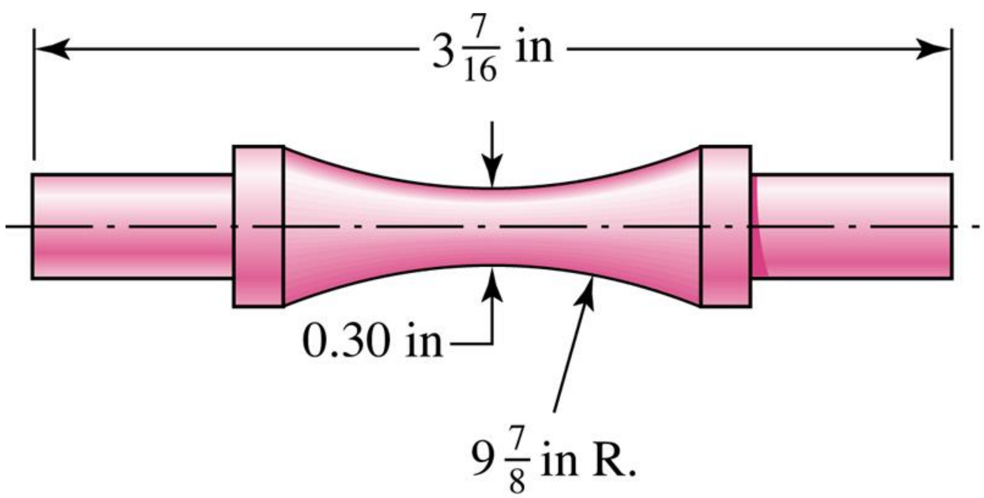
Fatigue-Life Methods

- Three major fatigue life models
- Methods predict life in number of cycles to failure, N, for a specific level of loading
- Stress-life method
  - Least accurate, particularly for low cycle applications
  - Most traditional, easiest to implement
- Strain-life method
  - Detailed analysis of plastic deformation at localized regions
  - Several idealizations are compounded, leading to uncertainties in results
- Linear-elastic fracture mechanics method
  - Assumes crack exists
  - Predicts crack growth with respect to stress intensity

Stress-Life Method

- Test specimens are subjected to repeated stress while counting cycles to failure
- Most common test machine is R. R. Moore high-speed rotating-beam machine
- Subjects specimen to pure bending with no transverse shear
- As specimen rotates, stress fluctuates between equal magnitudes of tension and compression, known as *completely reversed* stress cycling
- Specimen is carefully machined and polished

Fig. 6–9



S-N Diagram

- Number of cycles to failure at varying stress levels is plotted on log-log scale
- For steels, a knee occurs near 10^6 cycles
- Strength corresponding to the knee is called *endurance limit* S_e

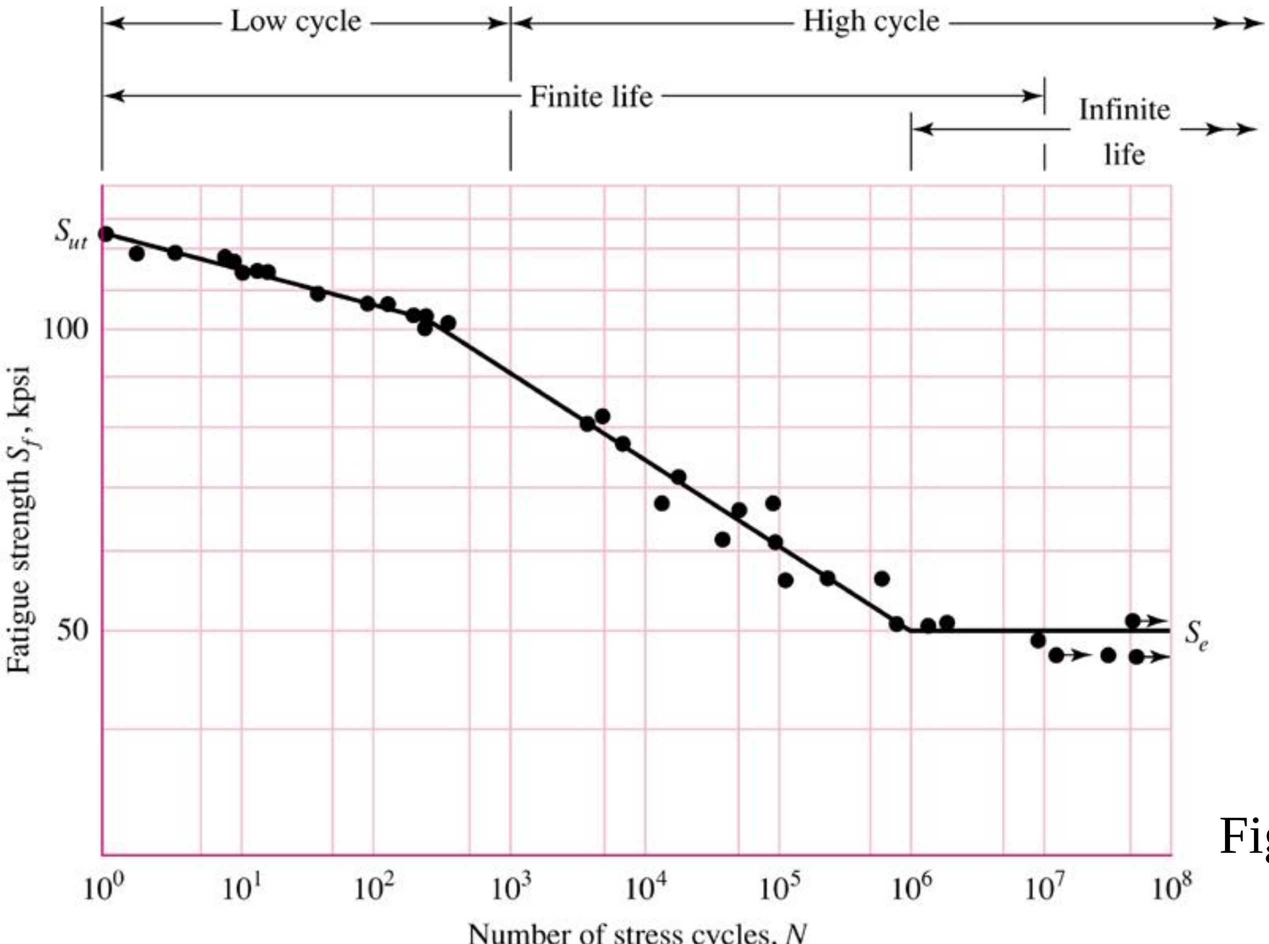


Fig. 6–10

S-N Diagram for Steel

- Stress levels below S_e predict infinite life
- Between 10^3 and 10^6 cycles, finite life is predicted
- Below 10^3 cycles is known as *low cycle,* and is often considered quasi-static. Yielding usually occurs before fatigue in this zone.

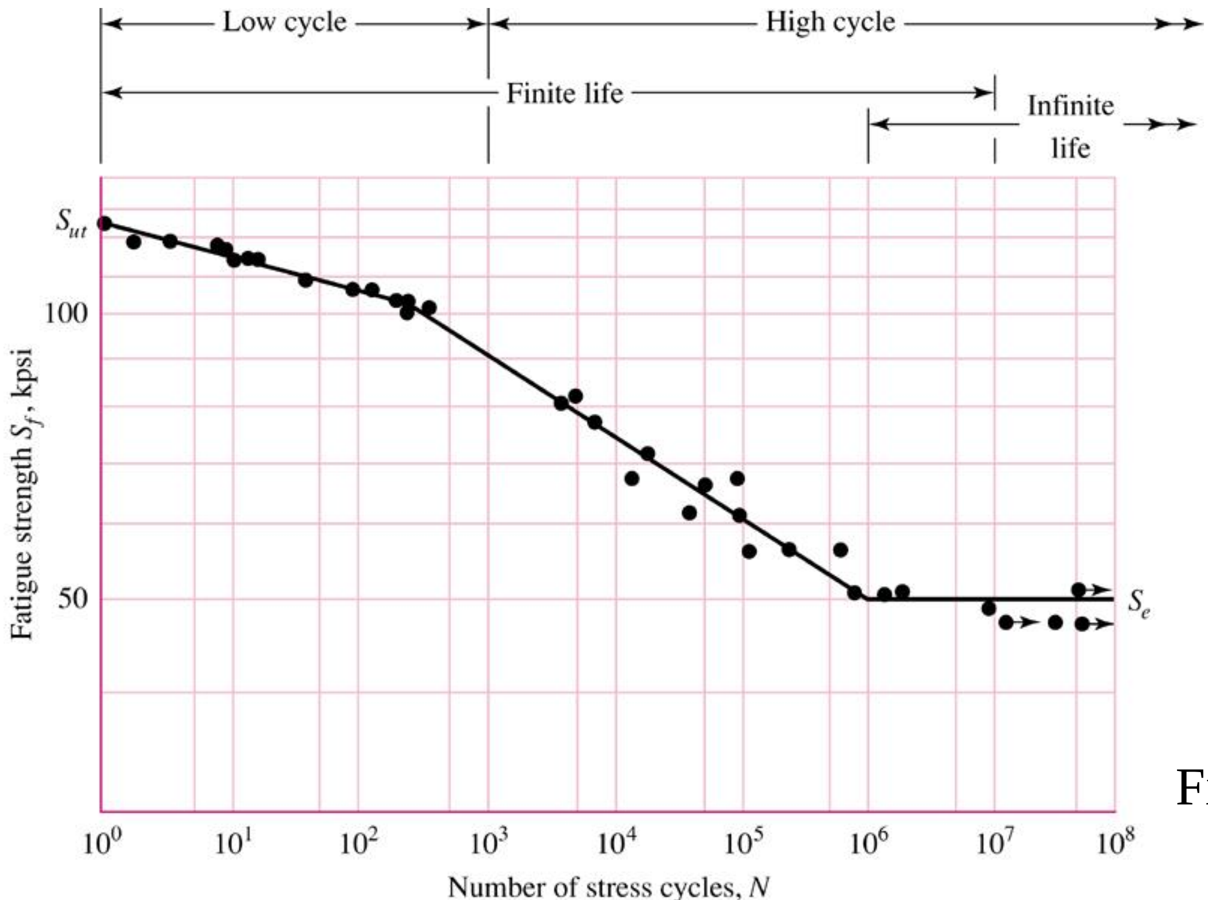


Fig. 6–10

S-N Diagram for Nonferrous Metals

- Nonferrous metals often do not have an endurance limit.
- Fatigue strength S_f is reported at a specific number of cycles
- Figure 6–11 shows typical *S-N* diagram for aluminums

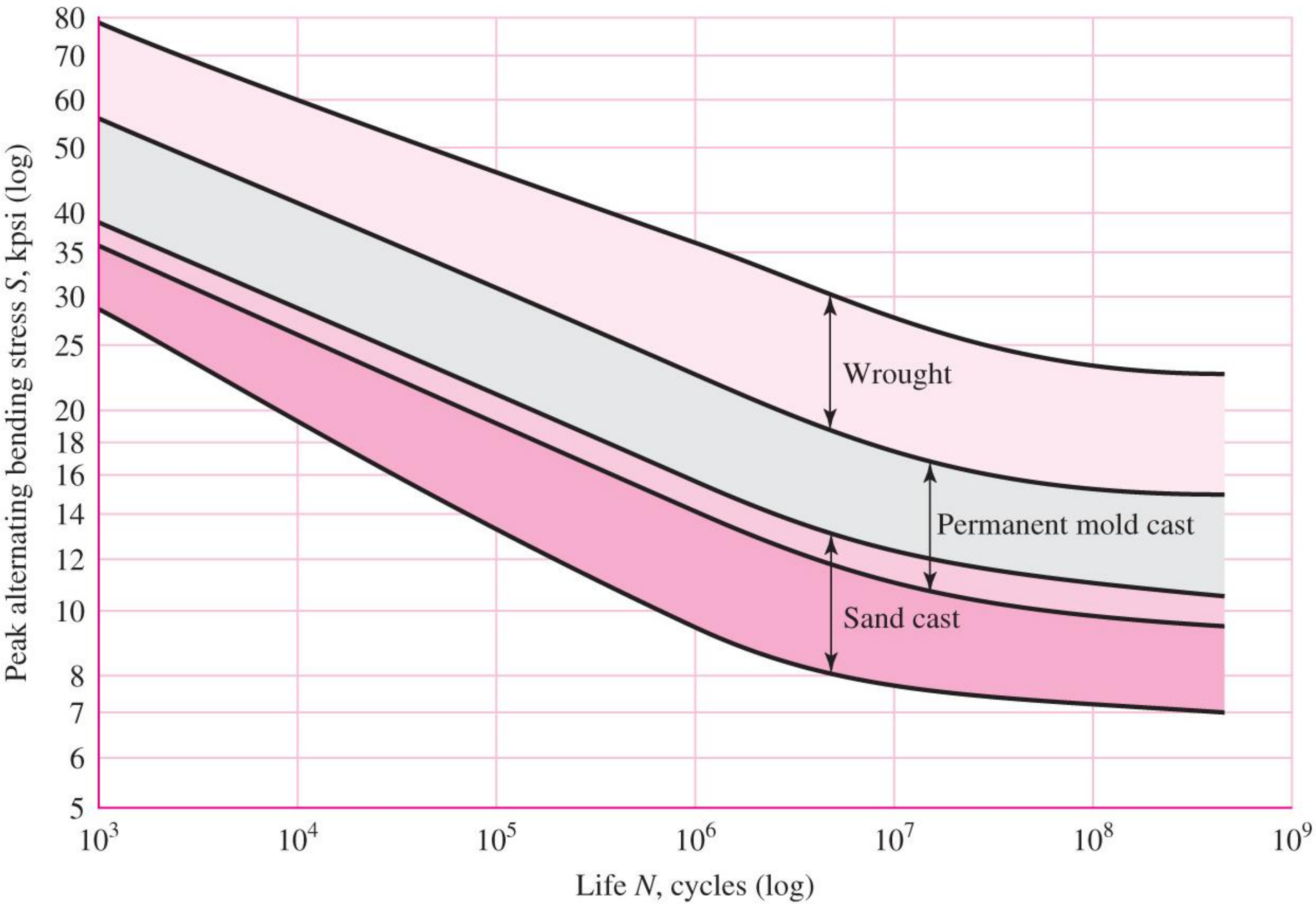


Fig. 6–11

Strain-Life Method

- Method uses detailed analysis of plastic deformation at localized regions
- Compounding of several idealizations leads to significant uncertainties in numerical results
- Useful for explaining nature of fatigue

Strain-Life Method

- Fatigue failure almost always begins at a local discontinuity
- When stress at discontinuity exceeds elastic limit, plastic strain occurs
- Cyclic plastic strain can change elastic limit, leading to fatigue
- Fig. 6–12 shows true stress-true strain hysteresis loops of the first five stress reversals

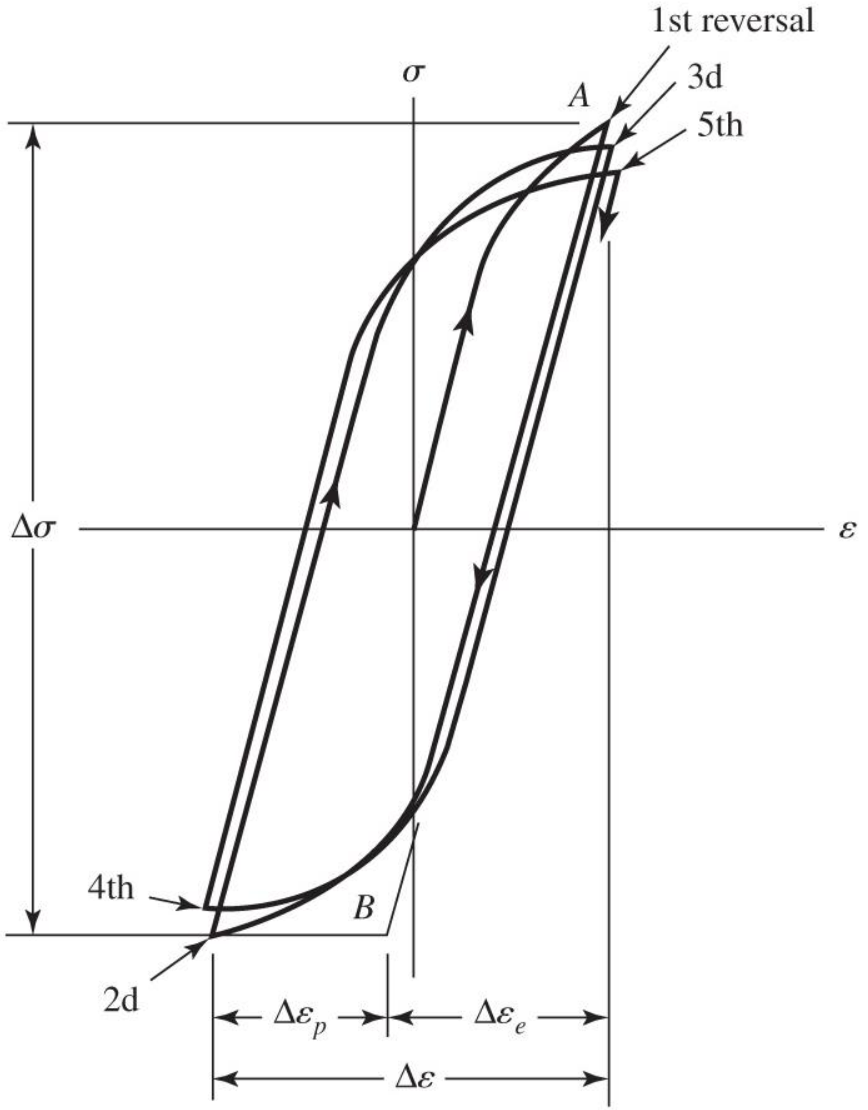


Fig. 6–12

Relation of Fatigue Life to Strain

- Figure 6–13 plots relationship of fatigue life to true-strain amplitude
- *Fatigue ductility coefficient* ε'_F is true strain corresponding to fracture in one reversal (point *A* in Fig. 6–12)
- *Fatigue strength coefficient* σ'_F is true stress corresponding to fracture in one reversal (point *A* in Fig. 6–12)

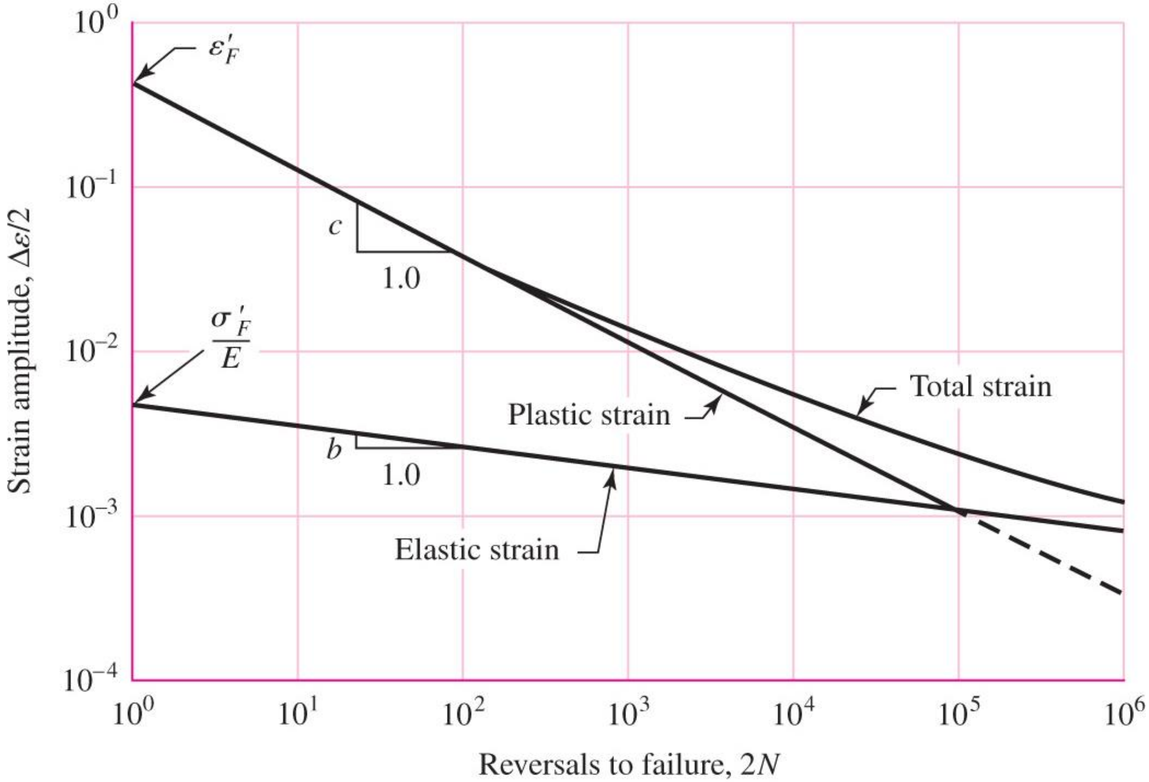


Fig. 6–13

Relation of Fatigue Life to Strain

- *Fatigue ductility exponent c* is the slope of plastic-strain line, and is the power to which the life $2N$ must be raised to be proportional to the true plastic-strain amplitude. Note that $2N$ stress reversals corresponds to N cycles.
- *Fatigue strength exponent b* is the slope of the elastic-strain line, and is the power to which the life $2N$ must be raised to be proportional to the true-stress amplitude.

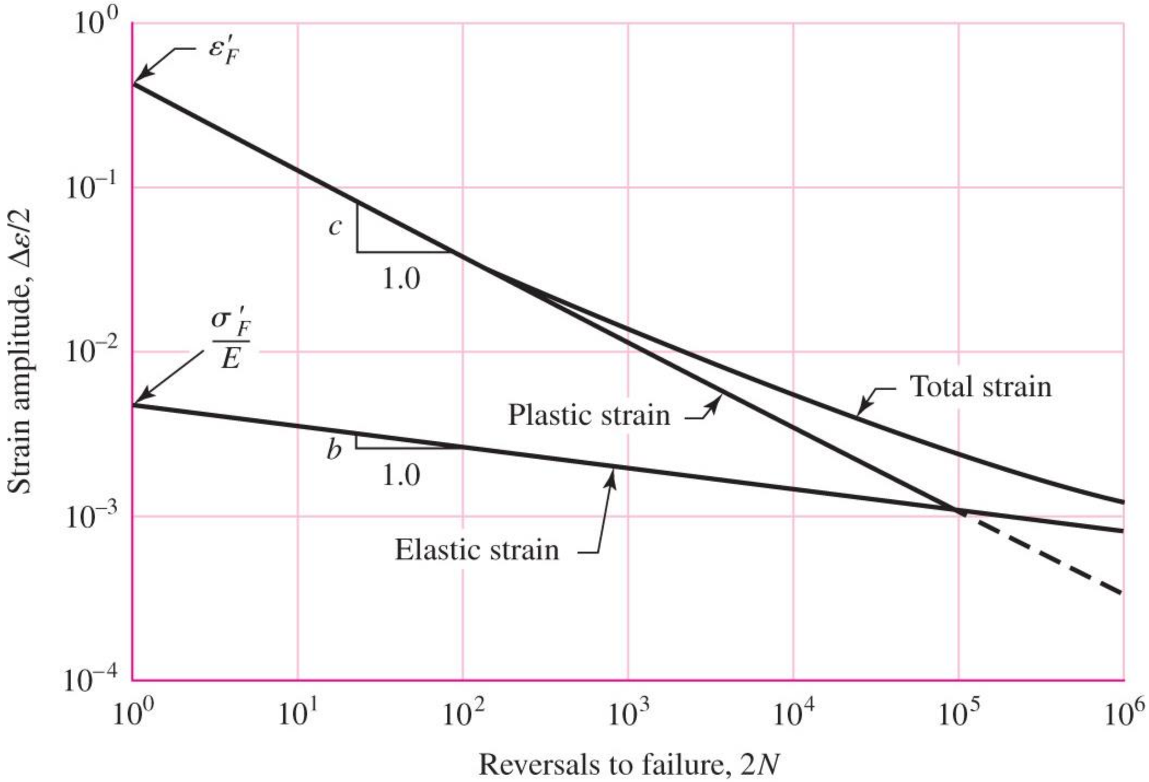


Fig. 6–13

Relation of Fatigue Life to Strain

- Total strain is sum of elastic and plastic strain
- Total strain amplitude is half the total strain range

$$\frac{\Delta\varepsilon}{2} = \frac{\Delta\varepsilon_e}{2} + \frac{\Delta\varepsilon_p}{2} \qquad (a)$$

- The equation of the plastic-strain line in Fig. 6–13

$$\frac{\Delta\varepsilon_p}{2} = \varepsilon'_F (2N)^c \qquad (6\text{–}1)$$

- The equation of the elastic strain line in Fig. 6–13

$$\frac{\Delta\varepsilon_e}{2} = \frac{\sigma'_F}{E}(2N)^b \qquad (6\text{–}2)$$

- Applying Eq. (*a*), the total-strain amplitude is

$$\frac{\Delta\varepsilon}{2} = \frac{\sigma'_F}{E}(2N)^b + \varepsilon'_F (2N)^c \qquad (6\text{–}3)$$

Relation of Fatigue Life to Strain

$$\frac{\Delta\varepsilon}{2} = \frac{\sigma'_F}{E}(2N)^b + \varepsilon'_F(2N)^c \qquad (6\text{–}3)$$

- Known as Manson-Coffin relationship between fatigue life and total strain
- Some values of coefficients and exponents given in Table A–23
- Equation has limited use for design since values for total strain at discontinuities are not readily available

Linear-Elastic Fracture Mechanics Method

- Assumes Stage I fatigue (crack initiation) has occurred
- Predicts crack growth in Stage II with respect to stress intensity
- Stage III ultimate fracture occurs when the stress intensity factor K_I reaches some critical level K_{Ic}

Crack Growth

- Stress intensity factor is given by

$$K_I = \beta\sigma\sqrt{\pi a} \qquad (5\text{–}37)$$

- For a stress range $\Delta\sigma$, the stress intensity range per cycle is

$$\Delta K_{\mathrm{I}} = \beta(\sigma_{\max} - \sigma_{\min})\sqrt{\pi a} = \beta\Delta\sigma\sqrt{\pi a} \qquad (6\text{–}4)$$

- Testing specimens at various levels of $\Delta\sigma$ provide plots of crack length vs. stress cycles

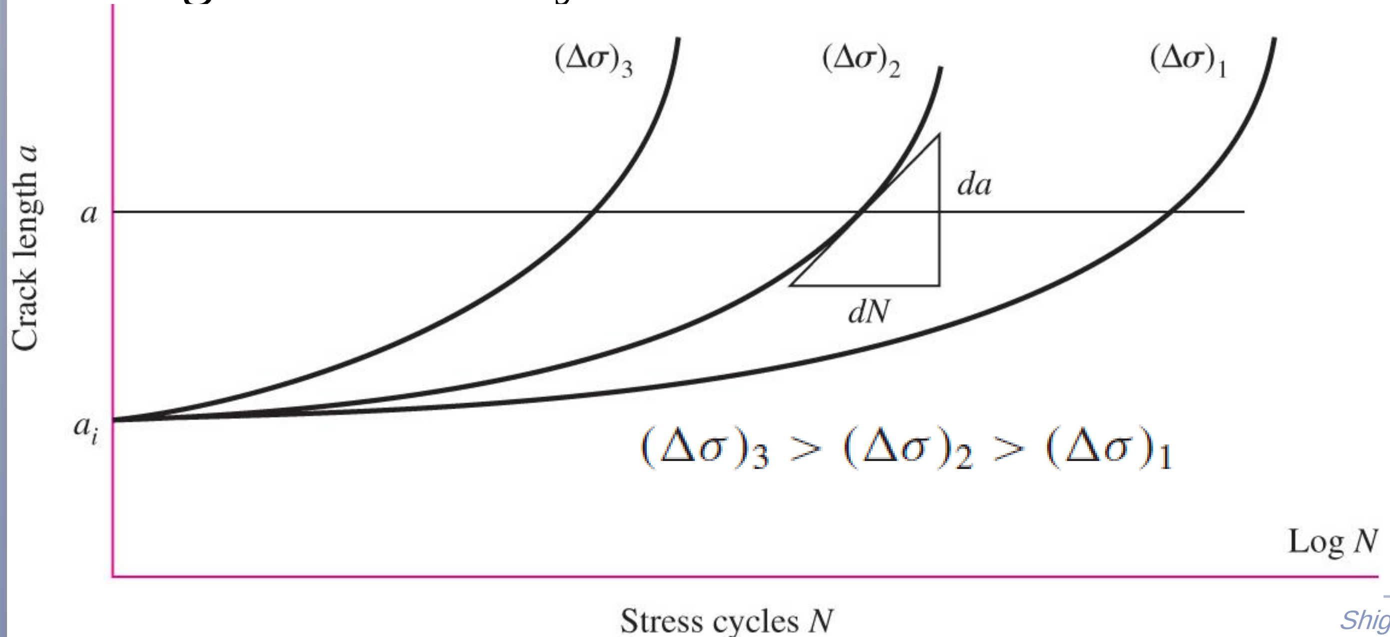


Fig. 6–14

Crack Growth

- Log-log plot of rate of crack growth, *da/dN*, shows all three stages of growth
- Stage II data are linear on log-log scale
- Similar curves can be generated by changing the stress ratio $R = \sigma_{min} / \sigma_{max}$

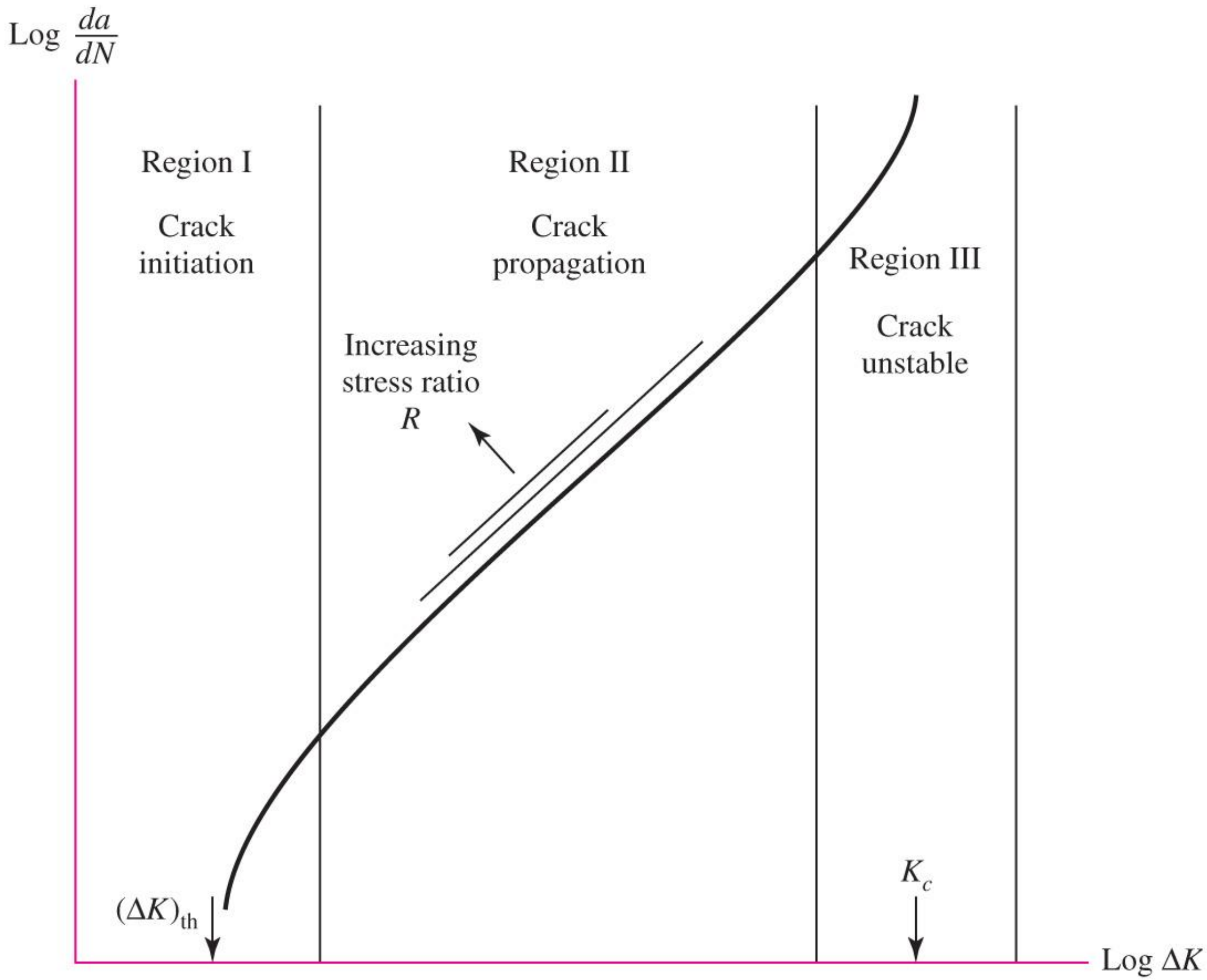


Fig. 6–15

Crack Growth

- Crack growth in Region II is approximated by the *Paris equation*

$$\frac{da}{dN} = C(\Delta K_{\mathrm{I}})^m \qquad (6\text{–}5)$$

- *C* and *m* are empirical material constants. Conservative representative values are shown in Table 6–1.

Table 6–1

Conservative Values of Factor C and Exponent m in Eq. (6–5) for Various Forms of Steel ($R = \sigma_{max}/\sigma_{min} \doteq 0$)

Material	C, $\frac{\text{m/cycle}}{(\text{MPa}\sqrt{\text{m}})^m}$	C, $\frac{\text{in/cycle}}{(\text{kpsi}\sqrt{\text{in}})^m}$	m
Ferritic-pearlitic steels	$6.89(10^{-12})$	$3.60(10^{-10})$	3.00
Martensitic steels	$1.36(10^{-10})$	$6.60(10^{-9})$	2.25
Austenitic stainless steels	$5.61(10^{-12})$	$3.00(10^{-10})$	3.25

From J. M. Barsom and S. T. Rolfe, *Fatigue and Fracture Control in Structures, 2nd ed.*, Prentice Hall, Upper Saddle River, NJ, 1987, pp. 288–291, Copyright ASTM International. Reprinted with permission.

Crack Growth

- Substituting Eq. (6–4) into Eq. (6–5) and integrating,

$$\int_0^{N_f} dN = N_f = \frac{1}{C}\int_{a_i}^{a_f} \frac{da}{(\beta\,\Delta\sigma\sqrt{\pi a})^m} \qquad (6\text{–}6)$$

- a_i is the initial crack length
- a_f is the final crack length corresponding to failure
- N_f is the estimated number of cycles to produce a failure after the initial crack is formed

Crack Growth

- If β is not constant, then the following numerical integration algorithm can be used.

$$
\begin{aligned}
\delta a_j &= C(\Delta K_I)_j^m (\delta N)_j \\
a_{j+1} &= a_j + \delta a_j \\
N_{j+1} &= N_j + \delta N_j \qquad (6\text{–}7) \\
N_f &= \sum \delta N_j
\end{aligned}
$$

Example 6-1

The bar shown in Fig. 6–16 is subjected to a repeated moment $0 \leq M \leq 1200$ lbf · in. The bar is AISI 4430 steel with $S_{ut} = 185$ kpsi, $S_y = 170$ kpsi, and $K_{Ic} = 73\ \text{kpsi}\sqrt{\text{in}}$. Material tests on various specimens of this material with identical heat treatment indicate worst-case constants of $C = 3.8(10^{-11})(\text{in/cycle})/(\text{kpsi}\sqrt{\text{in}})^m$ and $m = 3.0$. As shown, a nick of size 0.004 in has been discovered on the bottom of the bar. Estimate the number of cycles of life remaining.

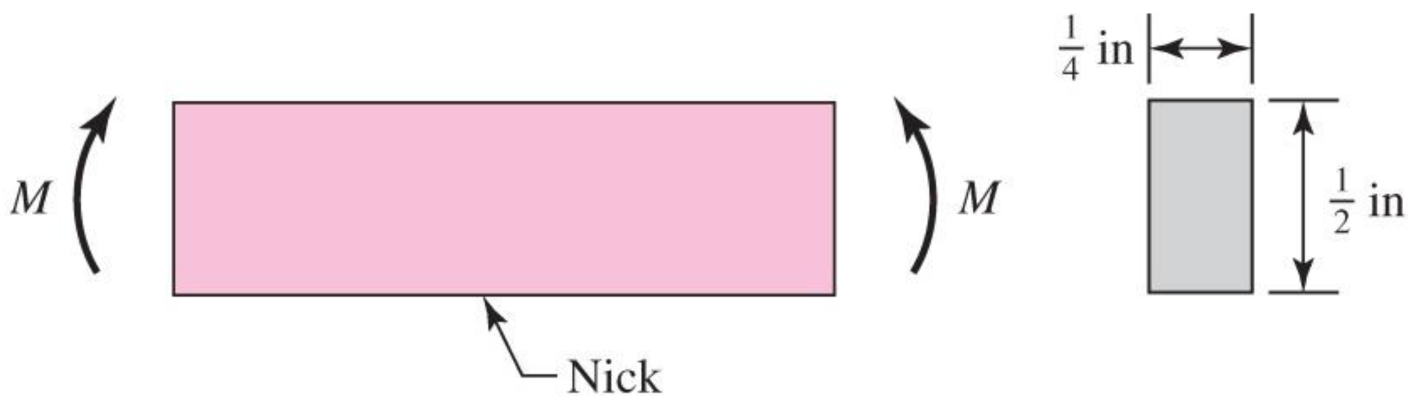


Fig. 6–16

Example 6-1

The stress range $\Delta\sigma$ is always computed by using the nominal (uncracked) area. Thus

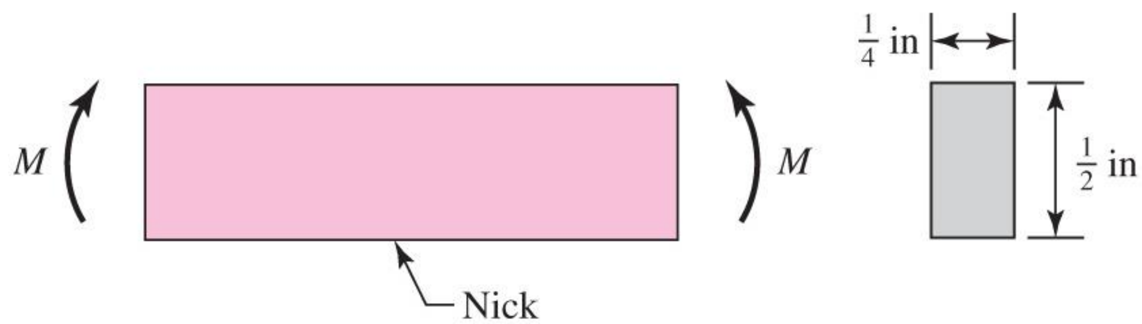
$$\frac{I}{c} = \frac{bh^2}{6} = \frac{0.25(0.5)^2}{6} = 0.010\,42\ \text{in}^3$$

Therefore, before the crack initiates, the stress range is

$$\Delta\sigma = \frac{\Delta M}{I/c} = \frac{1200}{0.010\,42} = 115.2(10^3)\ \text{psi} = 115.2\ \text{kpsi}$$

which is below the yield strength. As the crack grows, it will eventually become long enough such that the bar will completely yield or undergo a brittle fracture. For the ratio of S_y/S_{ut} it is highly unlikely that the bar will reach complete yield. For brittle fracture, designate the crack length as a_f. If $\beta = 1$, then from Eq. (5–37) with $K_{\text{I}} = K_{\text{I}c}$, we approximate a_f as

$$a_f = \frac{1}{\pi}\left(\frac{K_{\text{I}c}}{\beta\sigma_{\max}}\right)^2 \doteq \frac{1}{\pi}\left(\frac{73}{115.2}\right)^2 = 0.1278\ \text{in}$$



Example 6-1

From Fig. 5–27, we compute the ratio a_f/h as

$$\frac{a_f}{h} = \frac{0.1278}{0.5} = 0.256$$

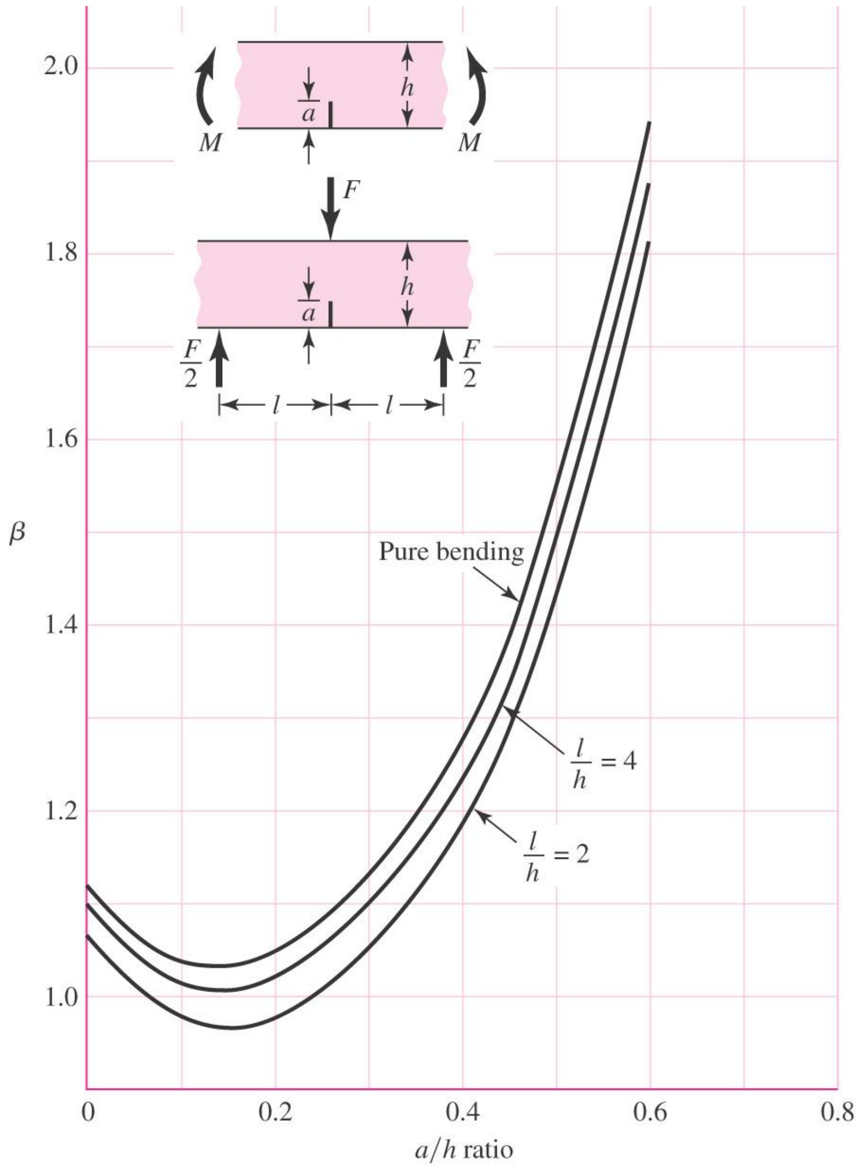


Fig. 5–27

Example 6-1

Thus a_f/h varies from near zero to approximately 0.256. From Fig. 5–27, for this range β is nearly constant at approximately 1.07. We will assume it to be so, and re-evaluate a_f as

$$a_f = \frac{1}{\pi}\left(\frac{73}{1.07(115.2)}\right)^2 = 0.112 \text{ in}$$

Thus, from Eq. (6–6), the estimated remaining life is

$$N_f = \frac{1}{C}\int_{a_i}^{a_f} \frac{da}{(\beta\Delta\sigma\sqrt{\pi a})^m} = \frac{1}{3.8(10^{-11})}\int_{0.004}^{0.112} \frac{da}{[1.07(115.2)\sqrt{\pi a}]^3}$$

$$= -\frac{5.047(10^3)}{\sqrt{a}}\bigg|_{0.004}^{0.112} = 64.7\ (10^3) \text{ cycles}$$

The Endurance Limit

- The endurance limit for steels has been experimentally found to be related to the ultimate strength

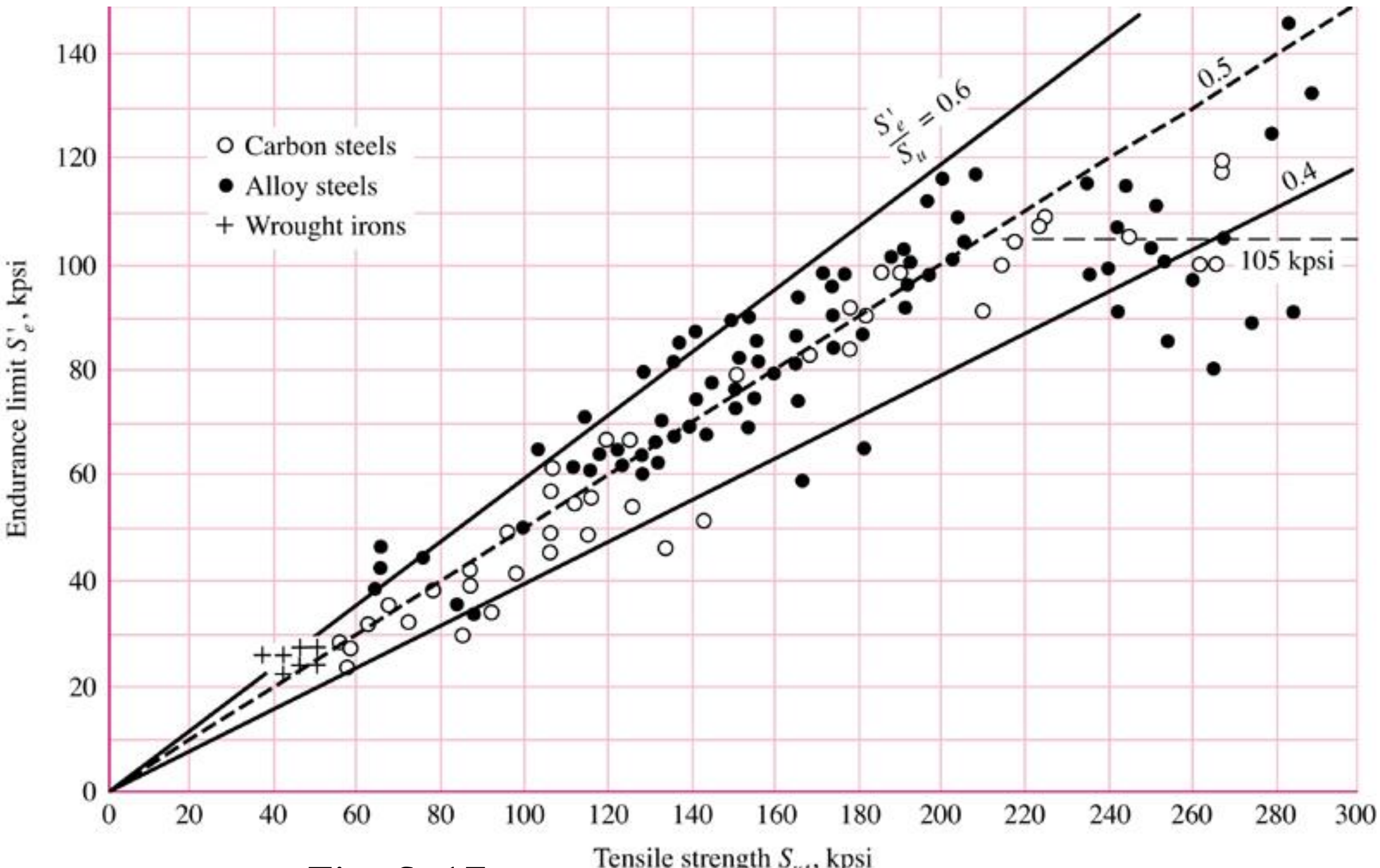


Fig. 6–17

The Endurance Limit

- Simplified estimate of endurance limit for steels for the rotating-beam specimen, S'_e

$$S'_e = \begin{cases} 0.5S_{ut} & S_{ut} \leq 200 \text{ kpsi } (1400 \text{ MPa}) \\ 100 \text{ kpsi} & S_{ut} > 200 \text{ kpsi} \\ 700 \text{ MPa} & S_{ut} > 1400 \text{ MPa} \end{cases} \qquad (6\text{–}8)$$

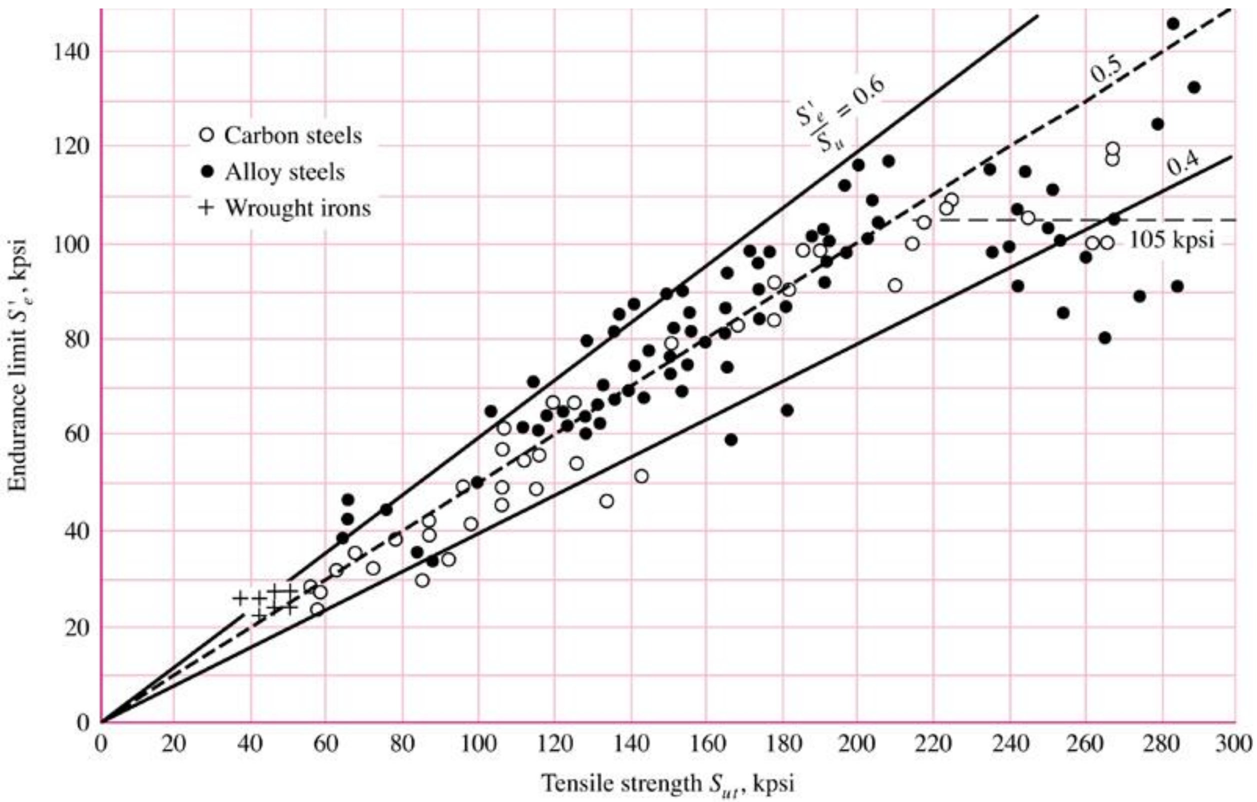


Fig. 6–17

Fatigue Strength

- For design, an approximation of the idealized *S-N* diagram is desirable.
- To estimate the fatigue strength at 10^3 cycles, start with Eq. (6-2)

$$\frac{\Delta \varepsilon_e}{2} = \frac{\sigma'_F}{E}(2N)^b \qquad (6\text{–}2)$$

- Define the specimen fatigue strength at a specific number of cycles as

$$(S'_f)_N = E\,\Delta\varepsilon_e/2$$

- Combine with Eq. (6–2),

$$(S'_f)_N = \sigma'_F(2N)^b \qquad (6\text{–}9)$$

Fatigue Strength

$$(S'_f)_N = \sigma'_F (2N)^b \qquad (6\text{–}9)$$

- At 10^3 cycles, $(S'_f)_{10^3} = \sigma'_F (2 \cdot 10^3)^b = f\, S_{ut}$
- f is the fraction of S_{ut} represented by $(S'_f)_{10^3}$
- Solving for f,

$$f = \frac{\sigma'_F}{S_{ut}} (2 \cdot 10^3)^b \qquad (6\text{–}10)$$

- The SAE approximation for steels with $H_B \leq 500$ may be used.

$$\sigma'_F = S_{ut} + 50 \text{ kpsi} \quad \text{or} \quad \sigma'_F = S_{ut} + 345 \text{ MPa} \qquad (6\text{–}11)$$

- To find b, substitute the endurance strength and corresponding cycles into Eq. (6–9) and solve for b

$$b = -\frac{\log\left(\sigma'_F / S'_e\right)}{\log\left(2N_e\right)} \qquad (6\text{–}12)$$

Fatigue Strength

$$(S'_f)_N = \sigma'_F (2N)^b \qquad (6\text{–}9)$$

$$f = \frac{\sigma'_F}{S_{ut}}(2 \cdot 10^3)^b \qquad (6\text{–}10)$$

$$\sigma'_F = S_{ut} + 50 \text{ kpsi} \quad \text{or} \quad \sigma'_F = S_{ut} + 345 \text{ MPa} \qquad (6\text{–}11)$$

$$b = -\frac{\log\left(\sigma'_F / S'_e\right)}{\log(2N_e)} \qquad (6\text{–}12)$$

- Eqs. (6–11) and (6–12) can be substituted into Eqs. (6–9) and (6–10) to obtain expressions for S'_f and f

Equations for *S*-*N* Diagram

- Write equation for *S*-*N* line from 10^3 to 10^6 cycles
- Two known points
- At $N = 10^3$ cycles, $S_f = f\,S_{ut}$
- At $N = 10^6$ cycles, $S_f = S_e$
- Equations for line:

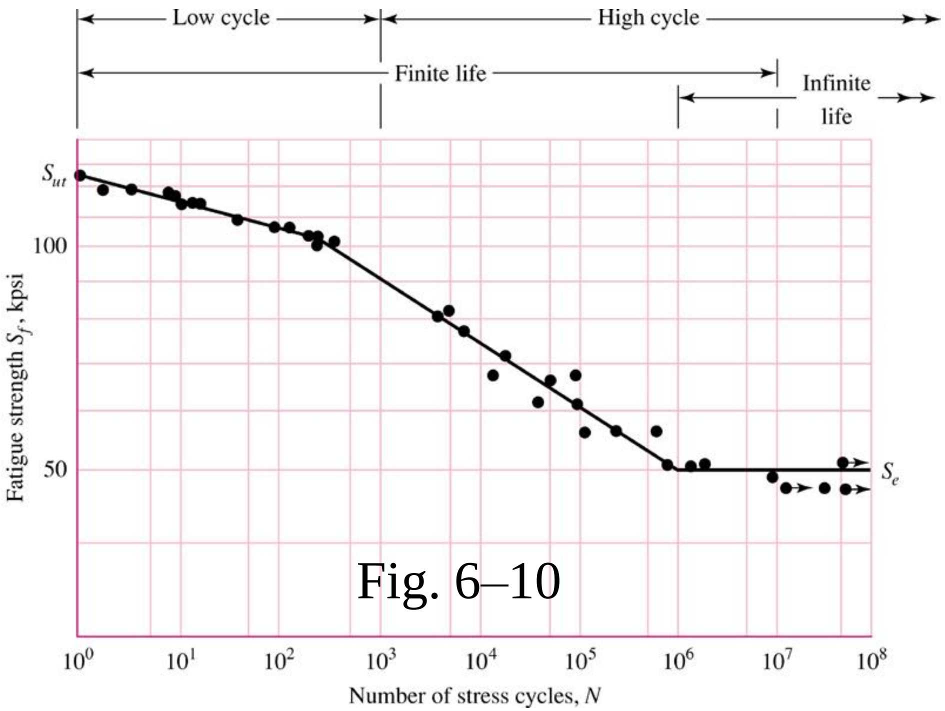


Fig. 6–10

$$S_f = a\,N^b \qquad (6\text{–}13)$$

$$a = \frac{(f\,S_{ut})^2}{S_e} \qquad (6\text{–}14)$$

$$b = -\frac{1}{3}\log\left(\frac{f\,S_{ut}}{S_e}\right) \qquad (6\text{–}15)$$

Equations for *S*-*N* Diagram

- If a completely reversed stress σ_{rev} is given, setting $S_f = \sigma_{\text{rev}}$ in Eq. (6–13) and solving for N gives,

$$N = \left(\frac{\sigma_{\text{rev}}}{a}\right)^{1/b} \qquad (6\text{–}16)$$

- Note that the typical *S*-*N* diagram is only applicable for completely reversed stresses
- For other stress situations, a completely reversed stress with the same life expectancy must be used on the *S*-*N* diagram

Fatigue Strength Fraction *f*

- Plot Eq. (6–10) for the fatigue strength fraction f of S_{ut} at 10^3 cycles
- Use f from plot for $S'_f = f\, S_{ut}$ at 10^3 cycles on S-N diagram
- Assumes $S_e = S'_e = 0.5 S_{ut}$ at 10^6 cycles

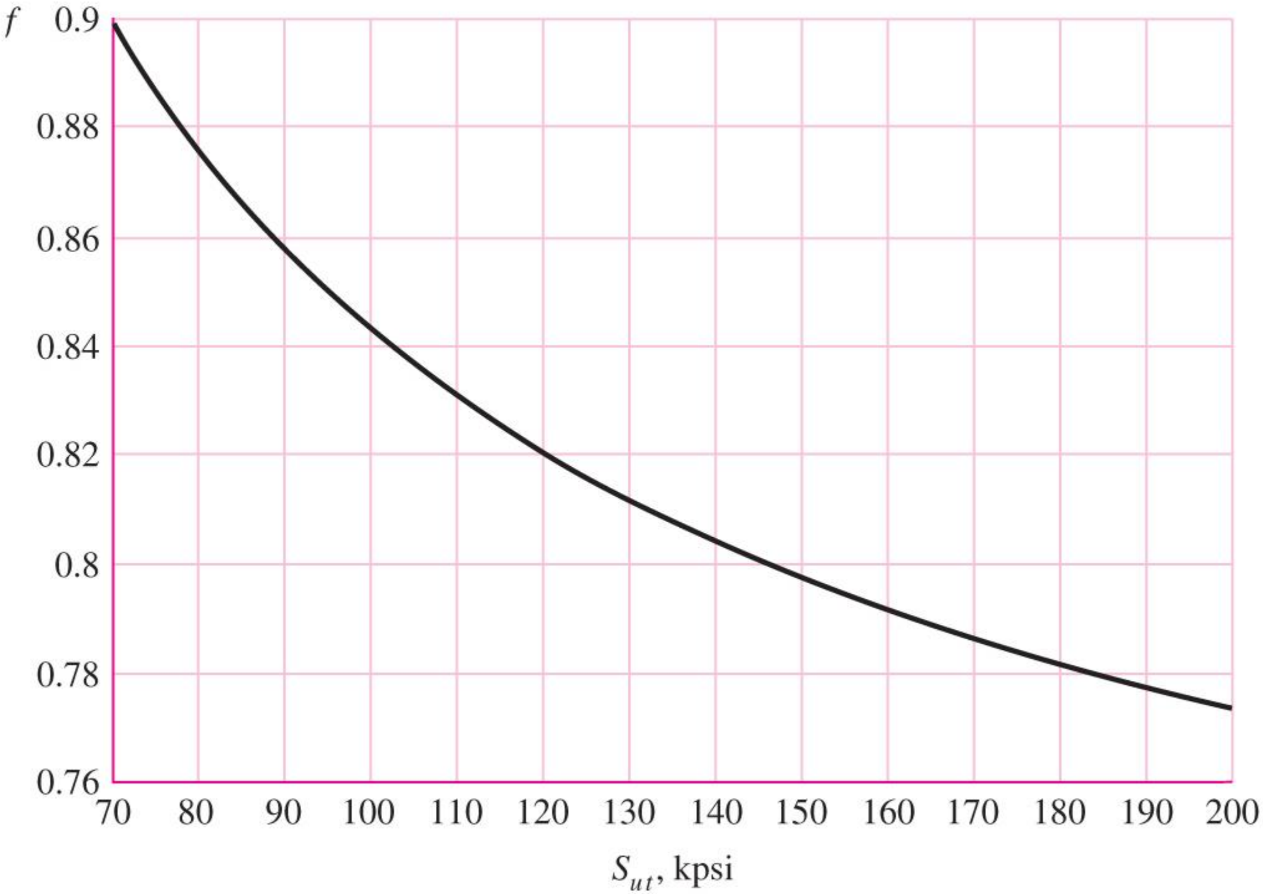


Fig. 6–18

Low-cycle Fatigue

- Low-cycle fatigue is defined for fatigue failures in the range $1 \leq N \leq 10^3$
- On the idealized *S-N* diagram on a log-log scale, failure is predicted by a straight line between two points $(10^3, f\,S_{ut})$ and $(1, S_{ut})$

$$S_f \geq S_{ut} N^{(\log f)/3} \qquad 1 \leq N \leq 10^3 \qquad (6\text{–}17)$$

Example 6-2

Given a 1050 HR steel, *estimate*

(a) the rotating-beam endurance limit at 10^6 cycles.

(b) the endurance strength of a polished rotating-beam specimen corresponding to 10^4 cycles to failure

(c) the expected life of a polished rotating-beam specimen under a completely reversed stress of 55 kpsi.

Solution

(a) From Table A–20, $S_{ut} = 90$ kpsi. From Eq. (6–8),

$$S'_e = 0.5(90) = 45 \text{ kpsi}$$

(b) From Fig. 6–18, for $S_{ut} = 90$ kpsi, $f \doteq 0.86$. From Eq. (6–14),

$$a = \frac{[0.86(90)]^2}{45} = 133.1 \text{ kpsi}$$

From Eq. (6–15),

$$b = -\frac{1}{3}\log\left[\frac{0.86(90)}{45}\right] = -0.0785$$

Example 6-2

Thus, Eq. (6–13) is

$$S'_f = 133.1\ N^{-0.0785}$$

For 10^4 cycles to failure, $S'_f = 133.1(10^4)^{-0.0785} = 64.6$ kpsi

(*c*) From Eq. (6–16), with $\sigma_{\text{rev}} = 55$ kpsi,

$$N = \left(\frac{55}{133.1}\right)^{1/-0.0785} = 77\,500 = 7.75(10^4) \text{ cycles}$$

Keep in mind that these are only *estimates*. So expressing the answers using three-place accuracy is a little misleading.

Endurance Limit Modifying Factors

- Endurance limit S'_e is for carefully prepared and tested specimen
- If warranted, S_e is obtained from testing of actual parts
- When testing of actual parts is not practical, a set of *Marin factors* are used to adjust the endurance limit

$$S_e = k_a k_b k_c k_d k_e k_f S'_e \qquad (6\text{–}18)$$

k_a = surface condition modification factor

k_b = size modification factor

k_c = load modification factor

k_d = temperature modification factor

k_e = reliability factor[13]

k_f = miscellaneous-effects modification factor

S'_e = rotary-beam test specimen endurance limit

S_e = endurance limit at the critical location of a machine part in the geometry and condition of use

Surface Factor k_a

- Stresses tend to be high at the surface
- Surface finish has an impact on initiation of cracks at localized stress concentrations
- Surface factor is a function of ultimate strength. Higher strengths are more sensitive to rough surfaces.

$$k_a = a S_{ut}^b \qquad (6\text{–}19)$$

Table 6–2

Parameters for Marin Surface Modification Factor, Eq. (6–19)

Surface Finish	Factor a S_{ut}, kpsi	Factor a S_{ut}, MPa	Exponent b
Ground	1.34	1.58	−0.085
Machined or cold-drawn	2.70	4.51	−0.265
Hot-rolled	14.4	57.7	−0.718
As-forged	39.9	272.	−0.995

From C.J. Noll and C. Lipson, "Allowable Working Stresses," *Society for Experimental Stress Analysis,* vol. 3, no. 2, 1946 p. 29. Reproduced by O.J. Horger (ed.) *Metals Engineering Design ASME Handbook,* McGraw-Hill, New York. Copyright © 1953 by The McGraw-Hill Companies, Inc. Reprinted by permission.

Example 6-4

A steel has a minimum ultimate strength of 520 MPa and a machined surface. Estimate k_a.

Solution From Table 6–2, $a = 4.51$ and $b = -0.265$. Then, from Eq. (6–19)

Answer

$$k_a = 4.51(520)^{-0.265} = 0.860$$

Size Factor k_b

- Larger parts have greater surface area at high stress levels
- Likelihood of crack initiation is higher
- Size factor is obtained from experimental data with wide scatter
- For bending and torsion loads, the trend of the size factor data is given by

$$k_b = \begin{cases} (d/0.3)^{-0.107} = 0.879d^{-0.107} & 0.11 \le d \le 2 \text{ in} \\ 0.91d^{-0.157} & 2 < d \le 10 \text{ in} \\ (d/7.62)^{-0.107} = 1.24d^{-0.107} & 2.79 \le d \le 51 \text{ mm} \\ 1.51d^{-0.157} & 51 < d \le 254 \text{ mm} \end{cases} \qquad (6\text{–}20)$$

- Applies only for round, rotating diameter
- For axial load, there is no size effect, so $k_b = 1$

Size Factor k_b

- For parts that are not round and rotating, an equivalent round rotating diameter is obtained.
- Equate the volume of material stressed at and above 95% of the maximum stress to the same volume in the rotating-beam specimen.
- Lengths cancel, so equate the areas.
- For a rotating round section, the 95% stress area is the area of a ring,

$$A_{0.95\sigma} = \frac{\pi}{4}[d^2 - (0.95d)^2] = 0.0766d^2 \qquad (6\text{–}22)$$

- Equate 95% stress area for other conditions to Eq. (6–22) and solve for d as the equivalent round rotating diameter

Size Factor k_b

- For non-rotating round,

$$A_{0.95\sigma} = 0.01046d^2 \qquad (6\text{–}23)$$

- Equating to Eq. (6-22) and solving for equivalent diameter,

$$d_e = 0.370d \qquad (6\text{–}24)$$

- Similarly, for rectangular section $h \times b$, $A_{95\sigma} = 0.05\, hb$. Equating to Eq. (6–22),

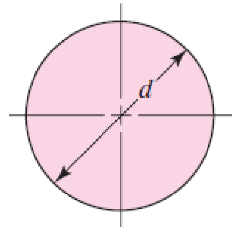
$$d_e = 0.808(hb)^{1/2} \qquad (6\text{–}25)$$

- Other common cross sections are given in Table 6–3

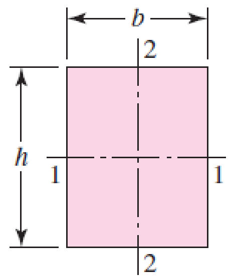
Size Factor k_b

Table 6–3

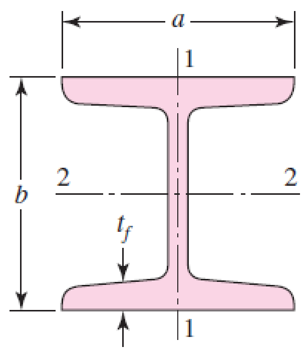
$A_{95\sigma}$ for common non-rotating structural shapes



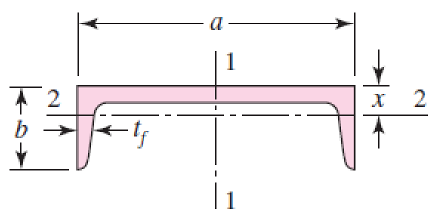
$$A_{0.95\sigma} = 0.01046d^2$$
$$d_e = 0.370d$$



$$A_{0.95\sigma} = 0.05hb$$
$$d_e = 0.808\sqrt{hb}$$



$$A_{0.95\sigma} = \begin{cases} 0.10at_f & \text{axis 1-1} \\ 0.05ba \quad t_f > 0.025a & \text{axis 2-2} \end{cases}$$



$$A_{0.95\sigma} = \begin{cases} 0.05ab & \text{axis 1-1} \\ 0.052xa + 0.1t_f(b - x) & \text{axis 2-2} \end{cases}$$

Example 6-4

A steel shaft loaded in bending is 32 mm in diameter, abutting a filleted shoulder 38 mm in diameter. The shaft material has a mean ultimate tensile strength of 690 MPa. Estimate the Marin size factor k_b if the shaft is used in

(*a*) A rotating mode.

(*b*) A nonrotating mode.

Solution (*a*) From Eq. (6–20)

Answer

$$k_b = \left(\frac{d}{7.62}\right)^{-0.107} = \left(\frac{32}{7.62}\right)^{-0.107} = 0.858$$

(*b*) From Table 6–3,

$$d_e = 0.37d = 0.37(32) = 11.84 \text{ mm}$$

From Eq. (6–20),

Answer

$$k_b = \left(\frac{11.84}{7.62}\right)^{-0.107} = 0.954$$

Loading Factor k_c

- Accounts for changes in endurance limit for different types of fatigue loading.
- Only to be used for single load types. Use Combination Loading method (Sec. 6–14) when more than one load type is present.

$$k_c = \begin{cases} 1 & \text{bending} \\ 0.85 & \text{axial} \\ 0.59 & \text{torsion}^{17} \end{cases} \tag{6–26}$$

Temperature Factor k_d

- Endurance limit appears to maintain same relation to ultimate strength for elevated temperatures as at room temperature
- This relation is summarized in Table 6–4

Table 6–4

Effect of Operating Temperature on the Tensile Strength of Steel.* (S_T = tensile strength at operating temperature; S_{RT} = tensile strength at room temperature; $0.099 \le \hat{\sigma} \le 0.110$)

Temperature, °C	S_T/S_{RT}	Temperature, °F	S_T/S_{RT}
20	1.000	70	1.000
50	1.010	100	1.008
100	1.020	200	1.020
150	1.025	300	1.024
200	1.020	400	1.018
250	1.000	500	0.995
300	0.975	600	0.963
350	0.943	700	0.927
400	0.900	800	0.872
450	0.843	900	0.797
500	0.768	1000	0.698
550	0.672	1100	0.567
600	0.549		

*Data source: Fig. 2–9.

Temperature Factor k_d

- If ultimate strength is known for operating temperature, then just use that strength. Let k_d= 1 and proceed as usual.
- If ultimate strength is known only at room temperature, then use Table 6–4 to estimate ultimate strength at operating temperature. With that strength, let k_d= 1 and proceed as usual.
- Alternatively, use ultimate strength at room temperature and apply temperature factor from Table 6–4 to the endurance limit.

$$k_d = \frac{S_T}{S_{RT}} \qquad (6\text{–}28)$$

- A fourth-order polynomial curve fit of the underlying data of Table 6–4 can be used in place of the table, if desired.

$$k_d = 0.975 + 0.432(10^{-3})T_F - 0.115(10^{-5})T_F^2 + 0.104(10^{-8})T_F^3 - 0.595(10^{-12})T_F^4 \qquad (6\text{–}27)$$

Example 6-5

A 1035 steel has a tensile strength of 70 kpsi and is to be used for a part that sees 450°F in service. Estimate the Marin temperature modification factor and $(S_e)_{450°}$ if

(*a*) The room-temperature endurance limit by test is $(S'_e)_{70°} = 39.0$ kpsi.

(*b*) Only the tensile strength at room temperature is known.

Solution (*a*) First, from Eq. (6–27),

$$k_d = 0.975 + 0.432(10^{-3})(450) - 0.115(10^{-5})(450^2) + 0.104(10^{-8})(450^3) - 0.595(10^{-12})(450^4) = 1.007$$

Thus,

Answer

$$(S_e)_{450°} = k_d(S'_e)_{70°} = 1.007(39.0) = 39.3 \text{ kpsi}$$

Example 6-5

(*b*) Interpolating from Table 6–4 gives

$$(S_T/S_{RT})_{450^\circ} = 1.018 + (0.995 - 1.018)\frac{450 - 400}{500 - 400} = 1.007$$

Thus, the tensile strength at 450°F is estimated as

$$(S_{ut})_{450^\circ} = (S_T/S_{RT})_{450^\circ}\,(S_{ut})_{70^\circ} = 1.007(70) = 70.5 \text{ kpsi}$$

From Eq. (6–8) then,

Answer $$(S_e)_{450^\circ} = 0.5\,(S_{ut})_{450^\circ} = 0.5(70.5) = 35.2 \text{ kpsi}$$

Part *a* gives the better estimate due to actual testing of the particular material.

Reliability Factor k_e

- From Fig. 6–17, $S'_e = 0.5\ S_{ut}$ is typical of the data and represents 50% reliability.
- Reliability factor adjusts to other reliabilities.
- *Only* adjusts Fig. 6–17 assumption. *Does not* imply overall reliability.

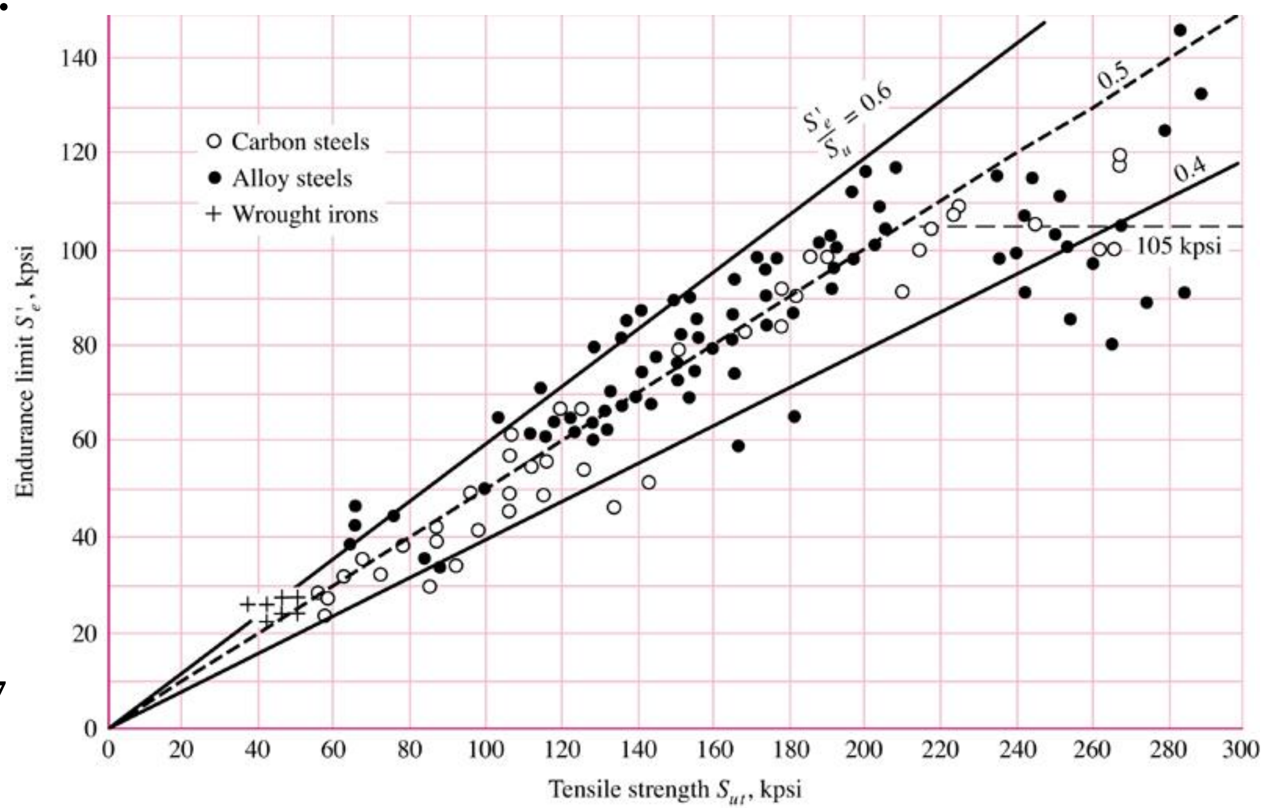


Fig. 6–17

Reliability

- *Reliability*, *R* – The statistical measure of the probability that a mechanical element will not fail in use
- *Probability of Failure*, p_f – the number of instances of failures per total number of possible instances

$$R = 1 - p_f \qquad (1\text{–}4)$$

- Example: If 1000 parts are manufactured, with 6 of the parts failing, the reliability is

$$R = 1 - \frac{6}{1000} = 0.994 \qquad \text{or } 99.4\ \%$$

Reliability Factor k_e

- Simply obtain k_e for desired reliability from Table 6–5.

Reliability, %	Transformation Variate z_a	Reliability Factor k_e
50	0	1.000
90	1.288	0.897
95	1.645	0.868
99	2.326	0.814
99.9	3.091	0.753
99.99	3.719	0.702
99.999	4.265	0.659
99.9999	4.753	0.620

Table 6–5

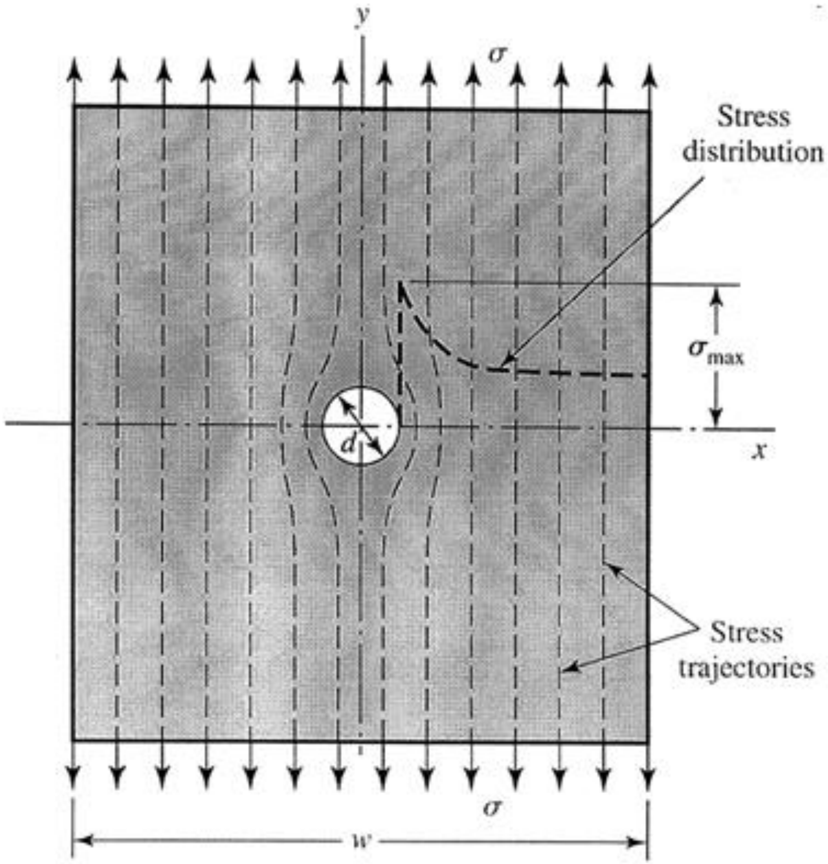
Miscellaneous-Effects Factor k_f

- Reminder to consider other possible factors.
  - Residual stresses
  - Directional characteristics from cold working
  - Case hardening
  - Corrosion
  - Surface conditioning, e.g. electrolytic plating and metal spraying
  - Cyclic Frequency
  - Frettage Corrosion
- Limited data is available.
- May require research or testing.

Stress Concentration

- Localized increase of stress near discontinuities
- K_t is Theoretical (Geometric) Stress Concentration Factor

$$K_t = \frac{\sigma_{max}}{\sigma_0} \qquad K_{ts} = \frac{\tau_{max}}{\tau_0} \qquad (3\text{–}48)$$



Theoretical Stress Concentration Factor

- Graphs available for standard configurations
- See Appendix A-15 and A-16 for common examples
- Many more in *Peterson's Stress-Concentration Factors*
- Note the trend for higher K_t at sharper discontinuity radius, and at greater disruption

Figure A–15–1

Bar in tension or simple compression with a transverse hole. $\sigma_0 = F/A$, where $A = (w - d)t$ and t is the thickness.

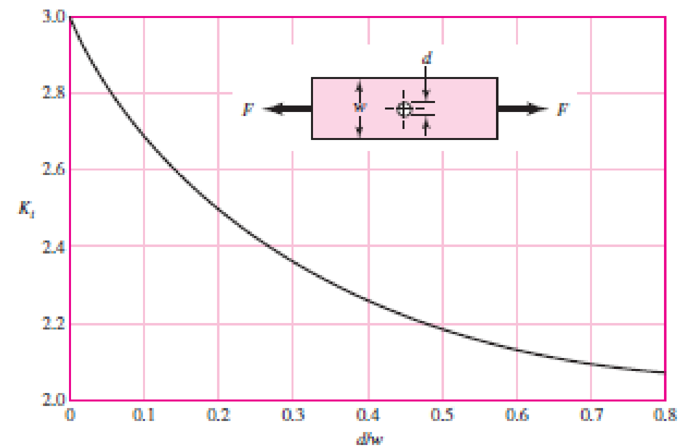
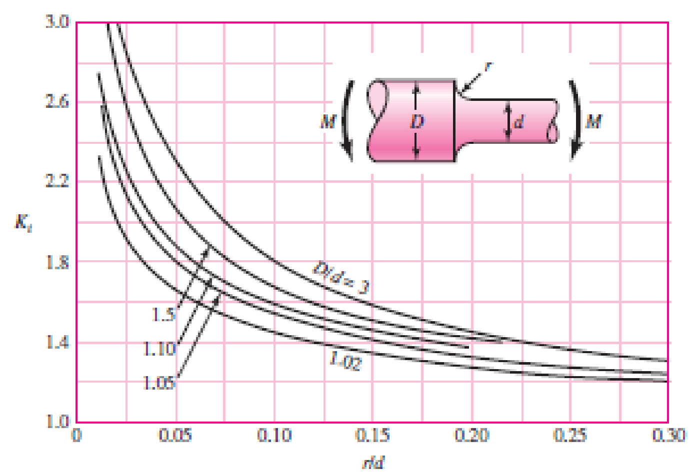


Figure A–15–9

Round shaft with shoulder fillet in bending. $\sigma_0 = Mc/I$, where $c = d/2$ and $I = \pi d^4/64$.



Stress Concentration and Notch Sensitivity

- For dynamic loading, stress concentration effects must be applied.
- Obtain K_t as usual (e.g. Appendix A–15)
- For fatigue, some materials are not fully sensitive to K_t so a reduced value can be used.
- Define K_f as the *fatigue stress-concentration factor.*
- Define q as *notch sensitivity,* ranging from 0 (not sensitive) to 1 (fully sensitive).

$$q = \frac{K_f - 1}{K_t - 1} \qquad \text{or} \qquad q_{\text{shear}} = \frac{K_{fs} - 1}{K_{ts} - 1} \qquad (6\text{–}31)$$

- For $q = 0$, $K_f = 1$
- For $q = 1$, $K_f = K_t$

Notch Sensitivity

- Obtain q for bending or axial loading from Fig. 6–20.
- Then get K_f from Eq. (6–32): $K_f = 1 + q(K_t - 1)$

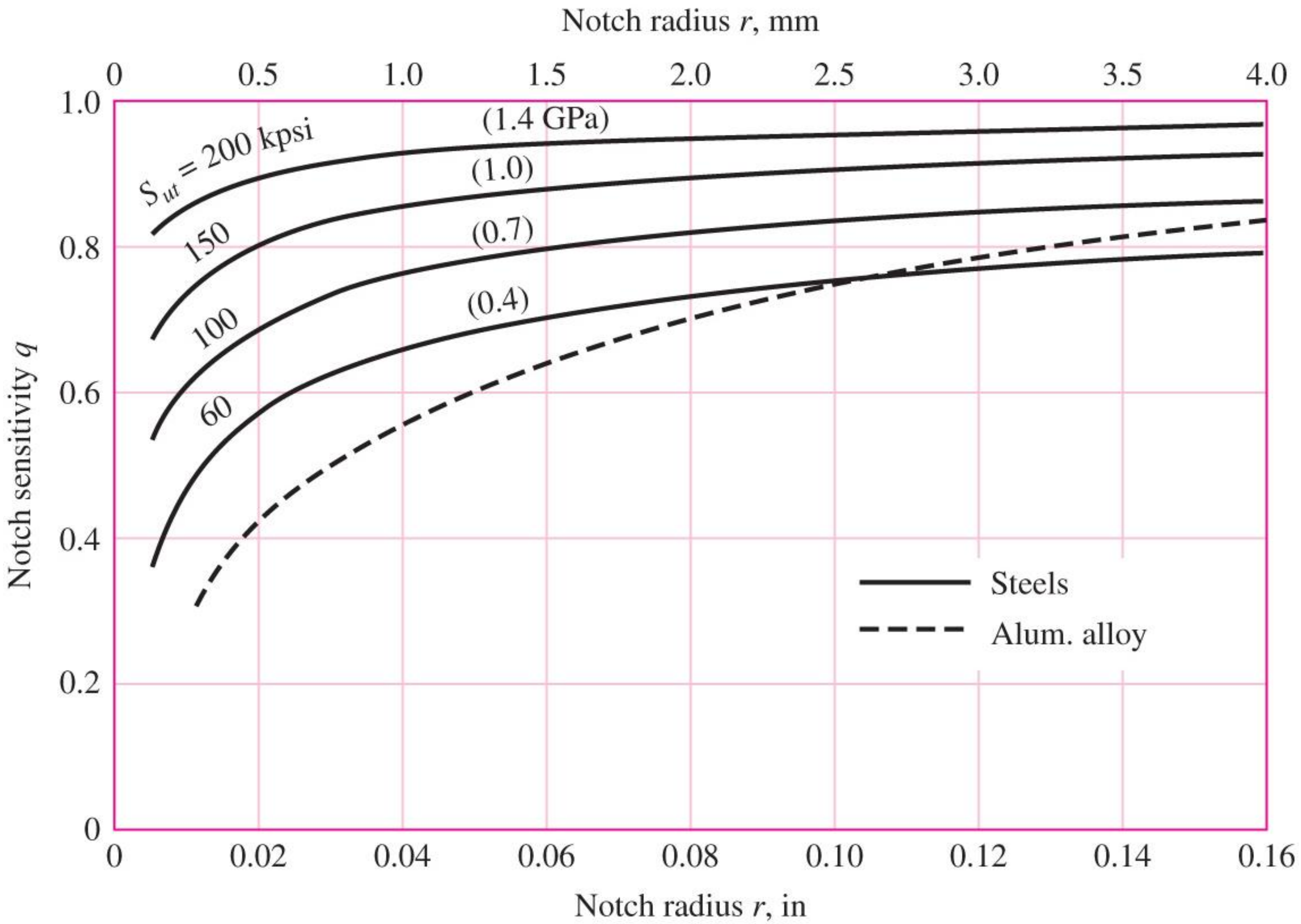


Fig. 6–20

Notch Sensitivity

- Obtain q_s for torsional loading from Fig. 6–21.
- Then get K_{fs} from Eq. (6–32): $K_{fs} = 1 + q_s(K_{ts} - 1)$
- Note that Fig. 6–21 is updated in 9th edition.

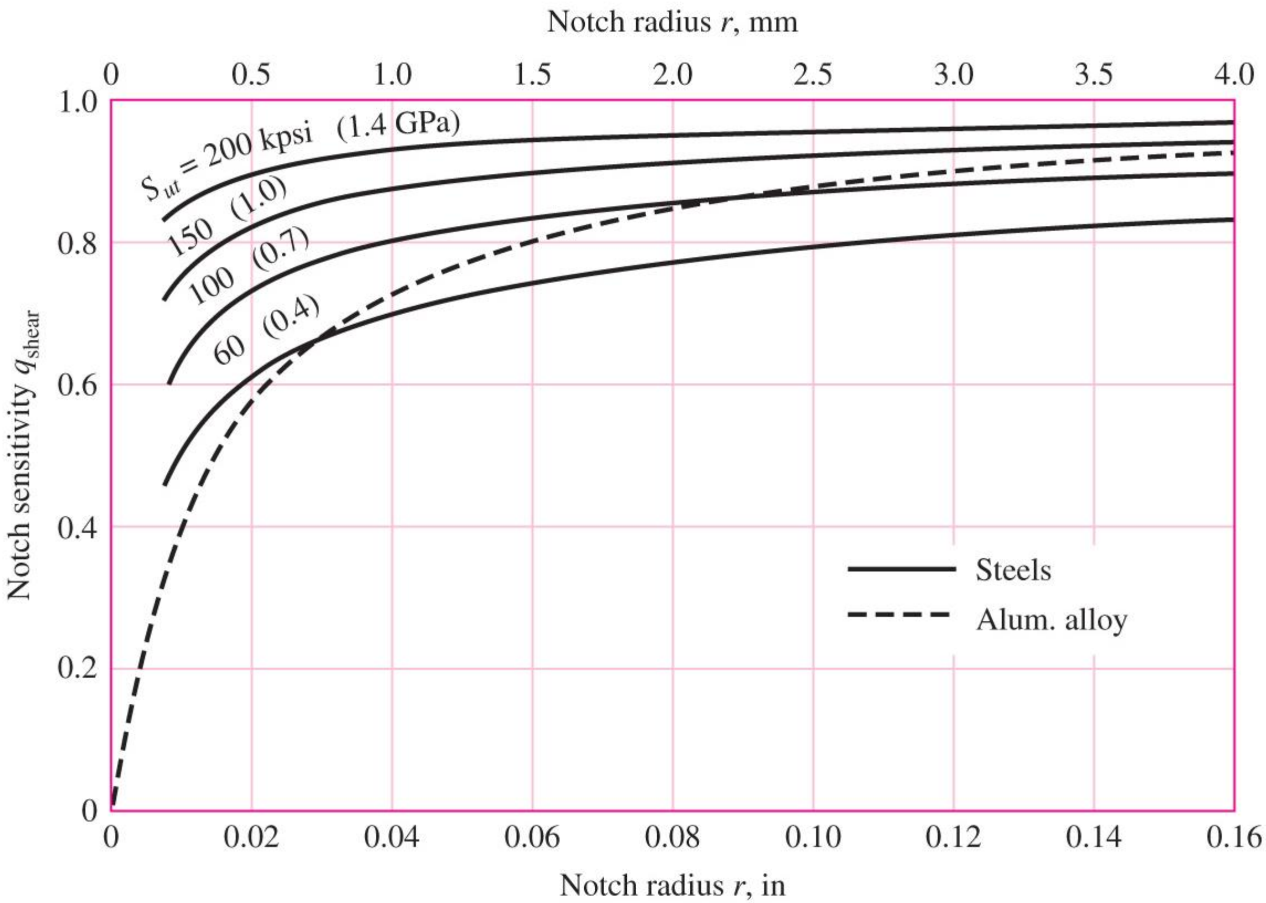


Fig. 6–21

Notch Sensitivity

- Alternatively, can use curve fit equations for Figs. 6–20 and 6–21 to get notch sensitivity, or go directly to K_f.

$$q = \frac{1}{1 + \frac{\sqrt{a}}{\sqrt{r}}} \qquad (6\text{–}34)$$

$$K_f = 1 + \frac{K_t - 1}{1 + \sqrt{a/r}} \qquad (6\text{–}33)$$

Bending or axial:

$$\sqrt{a} = 0.246 - 3.08(10^{-3})S_{ut} + 1.51(10^{-5})S_{ut}^2 - 2.67(10^{-8})S_{ut}^3 \qquad (6\text{–}35a)$$

Torsion:

$$\sqrt{a} = 0.190 - 2.51(10^{-3})S_{ut} + 1.35(10^{-5})S_{ut}^2 - 2.67(10^{-8})S_{ut}^3 \qquad (6\text{–}35b)$$

Notch Sensitivity for Cast Irons

- Cast irons are already full of discontinuities, which are included in the strengths.
- Additional notches do not add much additional harm.
- Recommended to use $q = 0.2$ for cast irons.

Example 6-6

A steel shaft in bending has an ultimate strength of 690 MPa and a shoulder with a fillet radius of 3 mm connecting a 32-mm diameter with a 38-mm diameter. Estimate K_f using:
(*a*) Figure 6–20.
(*b*) Equations (6–33) and (6–35).

Solution

From Fig. A–15–9, using $D/d = 38/32 = 1.1875$, $r/d = 3/32 = 0.093\,75$, we read the graph to find $K_t \doteq 1.65$.
(*a*) From Fig. 6–20, for $S_{ut} = 690$ MPa and $r = 3$ mm, $q \doteq 0.84$. Thus, from Eq. (6–32)

$$K_f = 1 + q(K_t - 1) \doteq 1 + 0.84(1.65 - 1) = 1.55$$

(*b*) From Eq. (6–35*a*) with $S_{ut} = 690 \text{ MPa} = 100 \text{ kpsi}$, $\sqrt{a} = 0.0622\sqrt{\text{in}} = 0.313\sqrt{\text{mm}}$. Substituting this into Eq. (6–33) with $r = 3$ mm gives

$$K_f = 1 + \frac{K_t - 1}{1 + \sqrt{a/r}} \doteq 1 + \frac{1.65 - 1}{1 + \dfrac{0.313}{\sqrt{3}}} = 1.55$$

Application of Fatigue Stress Concentration Factor

- Use K_f as a multiplier to increase the nominal stress.
- Some designers (and previous editions of textbook) sometimes applied $1/K_f$ as a Marin factor to reduce S_e.
- For infinite life, either method is equivalent, since

$$n_f = \frac{S_e}{K_f \sigma} = \frac{\left(1/K_f\right) S_e}{\sigma}$$

- For finite life, increasing stress is more conservative. Decreasing S_e applies more to high cycle than low cycle.

Example 6-7

For the step-shaft of Ex. 6–6, it is determined that the fully corrected endurance limit is $S_e = 280$ MPa. Consider the shaft undergoes a fully reversing nominal stress in the fillet of $(\sigma_{rev})_{nom} = 260$ MPa. Estimate the number of cycles to failure.

Solution

From Ex. 6–6, $K_f = 1.55$, and the ultimate strength is $S_{ut} = 690$ MPa $= 100$ kpsi. The maximum reversing stress is

$$(\sigma_{rev})_{max} = K_f(\sigma_{rev})_{nom} = 1.55(260) = 403 \text{ MPa}$$

From Fig. 6–18, $f = 0.845$. From Eqs. (6–14), (6–15), and (6–16)

$$a = \frac{(f S_{ut})^2}{S_e} = \frac{[0.845(690)]^2}{280} = 1214 \text{ MPa}$$

$$b = -\frac{1}{3}\log\frac{f S_{ut}}{S_e} = -\frac{1}{3}\log\left[\frac{0.845(690)}{280}\right] = -0.1062$$

$$N = \left(\frac{\sigma_{rev}}{a}\right)^{1/b} = \left(\frac{403}{1214}\right)^{1/-0.1062} = 32.3\left(10^3\right) \text{ cycles}$$

Example 6-8

A 1015 hot-rolled steel bar has been machined to a diameter of 1 in. It is to be placed in reversed axial loading for 70 000 cycles to failure in an operating environment of 550°F. Using ASTM minimum properties, and a reliability of 99 percent, estimate the endurance limit and fatigue strength at 70 000 cycles.

Solution

From Table A–20, $S_{ut} = 50$ kpsi at 70°F. Since the rotating-beam specimen endurance limit is not known at room temperature, we determine the ultimate strength at the elevated temperature first, using Table 6–4. From Table 6–4,

$$\left(\frac{S_T}{S_{RT}}\right)_{550^\circ} = \frac{0.995 + 0.963}{2} = 0.979$$

The ultimate strength at 550°F is then

$$(S_{ut})_{550^\circ} = (S_T/S_{RT})_{550^\circ}\,(S_{ut})_{70^\circ} = 0.979(50) = 49.0 \text{ kpsi}$$

The rotating-beam specimen endurance limit at 550°F is then estimated from Eq. (6–8) as

$$S_e' = 0.5(49) = 24.5 \text{ kpsi}$$

Example 6-8

Next, we determine the Marin factors. For the machined surface, Eq. (6–19) with Table 6–2 gives

$$k_a = aS_{ut}^b = 2.70(49^{-0.265}) = 0.963$$

For axial loading, from Eq. (6–21), the size factor $k_b = 1$, and from Eq. (6–26) the loading factor is $k_c = 0.85$. The temperature factor $k_d = 1$, since we accounted for the temperature in modifying the ultimate strength and consequently the endurance limit. For 99 percent reliability, from Table 6–5, $k_e = 0.814$. Finally, since no other conditions were given, the miscellaneous factor is $k_f = 1$. The endurance limit for the part is estimated by Eq. (6–18) as

$$\begin{aligned} S_e &= k_a k_b k_c k_d k_e k_f S_e' \\ &= 0.963(1)(0.85)(1)(0.814)(1)24.5 = 16.3 \text{ kpsi} \end{aligned}$$

Example 6-8

For the fatigue strength at 70 000 cycles we need to construct the S-N equation. From p. 285, since $S_{ut} = 49 < 70$ kpsi, then $f = 0.9$. From Eq. (6–14)

$$a = \frac{(f\,S_{ut})^2}{S_e} = \frac{[0.9(49)]^2}{16.3} = 119.3 \text{ kpsi}$$

and Eq. (6–15)

$$b = -\frac{1}{3}\log\left(\frac{f\,S_{ut}}{S_e}\right) = -\frac{1}{3}\log\left[\frac{0.9(49)}{16.3}\right] = -0.1441$$

Finally, for the fatigue strength at 70 000 cycles, Eq. (6–13) gives

$$S_f = a\,N^b = 119.3(70\,000)^{-0.1441} = 23.9 \text{ kpsi}$$

Example 6-9

Figure 6–22*a* shows a rotating shaft simply supported in ball bearings at *A* and *D* and loaded by a nonrotating force *F* of 6.8 kN. Using ASTM "minimum" strengths, estimate the life of the part.

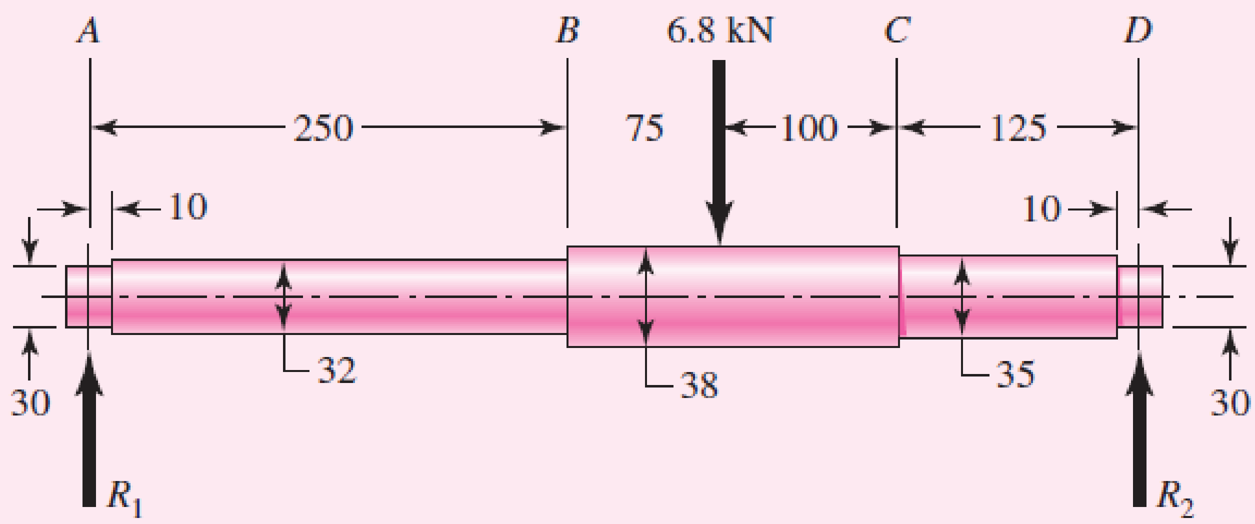


Fig. 6–22 (*a*)

Example 6-9

From Fig. 6–22*b* we learn that failure will probably occur at *B* rather than at *C* or at the point of maximum moment. Point *B* has a smaller cross section, a higher bending moment, and a higher stress-concentration factor than *C*, and the location of maximum moment has a larger size and no stress-concentration factor.

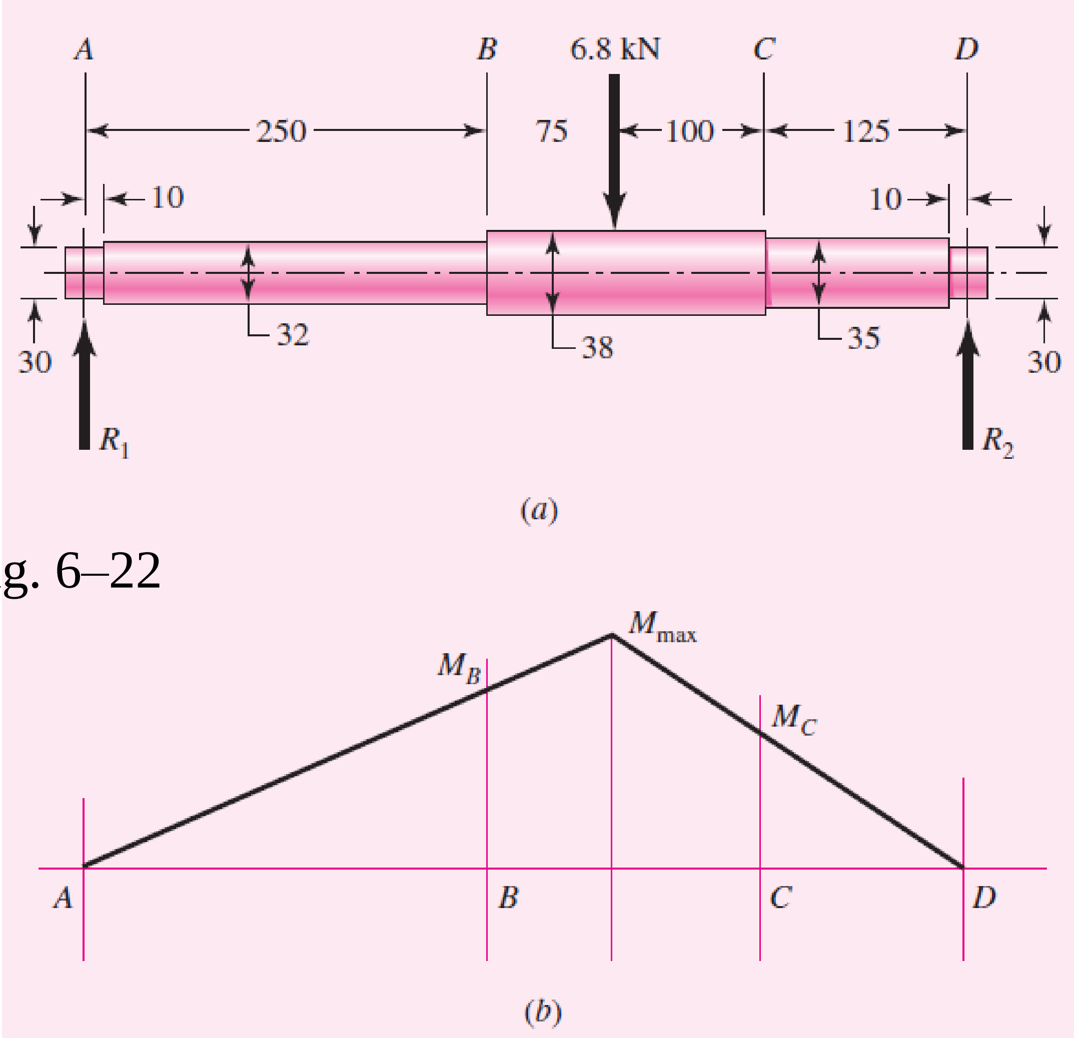


Fig. 6–22

Example 6-9

We shall solve the problem by first estimating the strength at point B, since the strength will be different elsewhere, and comparing this strength with the stress at the same point.

From Table A–20 we find $S_{ut} = 690$ MPa and $S_y = 580$ MPa. The endurance limit S'_e is estimated as

$$S'_e = 0.5(690) = 345 \text{ MPa}$$

From Eq. (6–19) and Table 6–2,

$$k_a = 4.51(690)^{-0.265} = 0.798$$

From Eq. (6–20),

$$k_b = (32/7.62)^{-0.107} = 0.858$$

Since $k_c = k_d = k_e = k_f = 1$,

$$S_e = 0.798(0.858)345 = 236 \text{ MPa}$$

Example 6-9

To find the geometric stress-concentration factor K_t we enter Fig. A–15–9 with $D/d = 38/32 = 1.1875$ and $r/d = 3/32 = 0.093\ 75$ and read $K_t \doteq 1.65$. Substituting $S_{ut} = 690/6.89 = 100$ kpsi into Eq. (6–35*a*) yields $\sqrt{a} = 0.0622\sqrt{\text{in}} = 0.313\sqrt{\text{mm}}$. Substituting this into Eq. (6–33) gives

$$K_f = 1 + \frac{K_t - 1}{1 + \sqrt{a/r}} = 1 + \frac{1.65 - 1}{1 + 0.313/\sqrt{3}} = 1.55$$

The next step is to estimate the bending stress at point B. The bending moment is

$$M_B = R_1 x = \frac{225F}{550}250 = \frac{225(6.8)}{550}250 = 695.5\ \text{N}\cdot\text{m}$$

Just to the left of B the section modulus is $I/c = \pi d^3/32 = \pi 32^3/32 = 3.217\ (10^3)\ \text{mm}^3$. The reversing bending stress is, assuming infinite life,

$$\sigma_{\text{rev}} = K_f \frac{M_B}{I/c} = 1.55\frac{695.5}{3.217}(10)^{-6} = 335.1(10^6)\ \text{Pa} = 335.1\ \text{MPa}$$

This stress is greater than S_e and less than S_y. This means we have both finite life and no yielding on the first cycle.

Example 6-9

For finite life, we will need to use Eq. (6–16). The ultimate strength, $S_{ut} = 690$ MPa = 100 kpsi. From Fig. 6–18, $f = 0.844$. From Eq. (6–14)

$$a = \frac{(f\ S_{ut})^2}{S_e} = \frac{[0.844(690)]^2}{236} = 1437\ \text{MPa}$$

and from Eq. (6–15)

$$b = -\frac{1}{3}\log\left(\frac{f\ S_{ut}}{S_e}\right) = -\frac{1}{3}\log\left[\frac{0.844(690)}{236}\right] = -0.1308$$

From Eq. (6–16),

$$N = \left(\frac{\sigma_{\text{rev}}}{a}\right)^{1/b} = \left(\frac{335.1}{1437}\right)^{-1/0.1308} = 68(10^3)\ \text{cycles}$$

Characterizing Fluctuating Stresses

- The *S-N* diagram is applicable for *completely reversed* stresses
- Other fluctuating stresses exist
- Sinusoidal loading patterns are common, but not necessary

Fluctuating Stresses

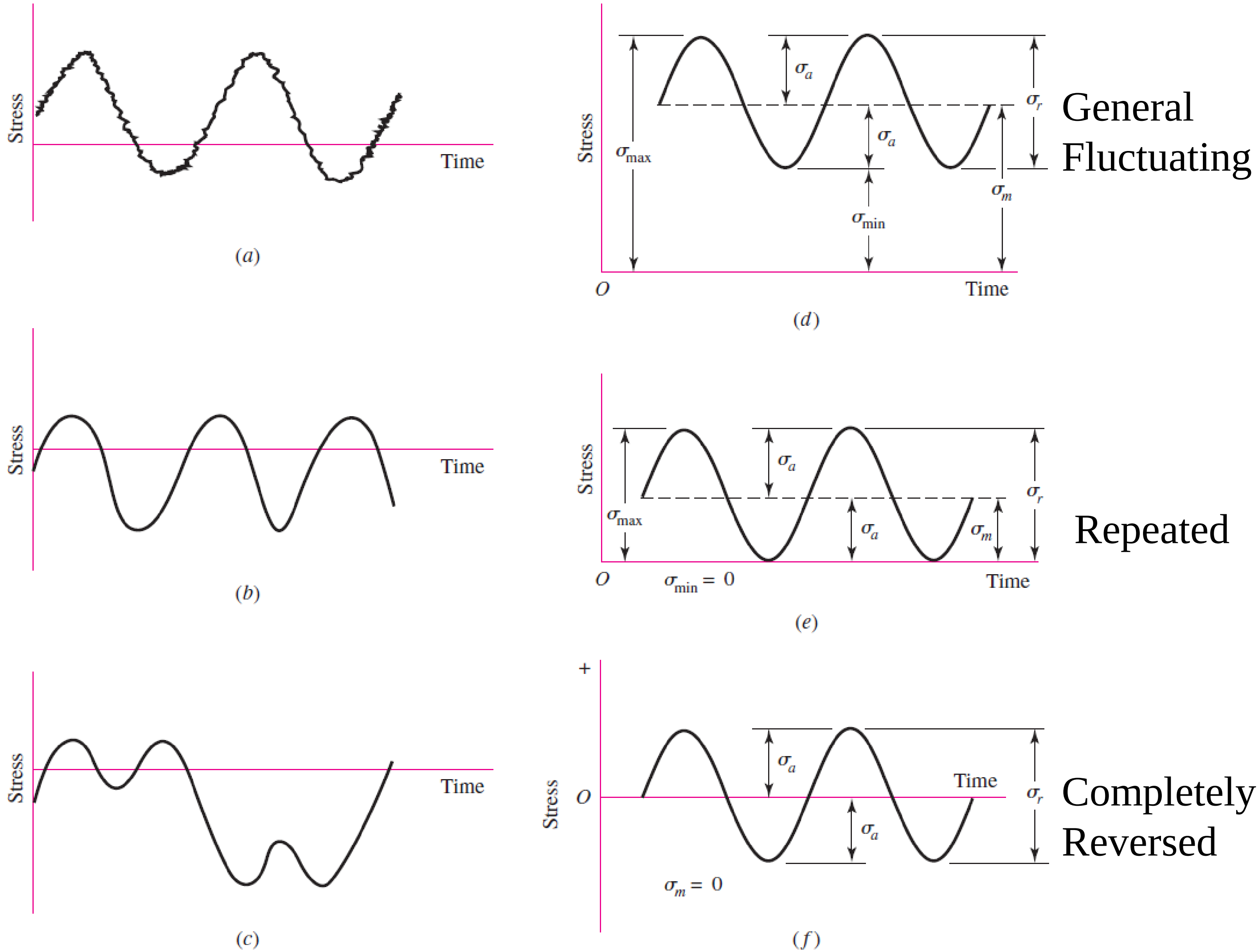
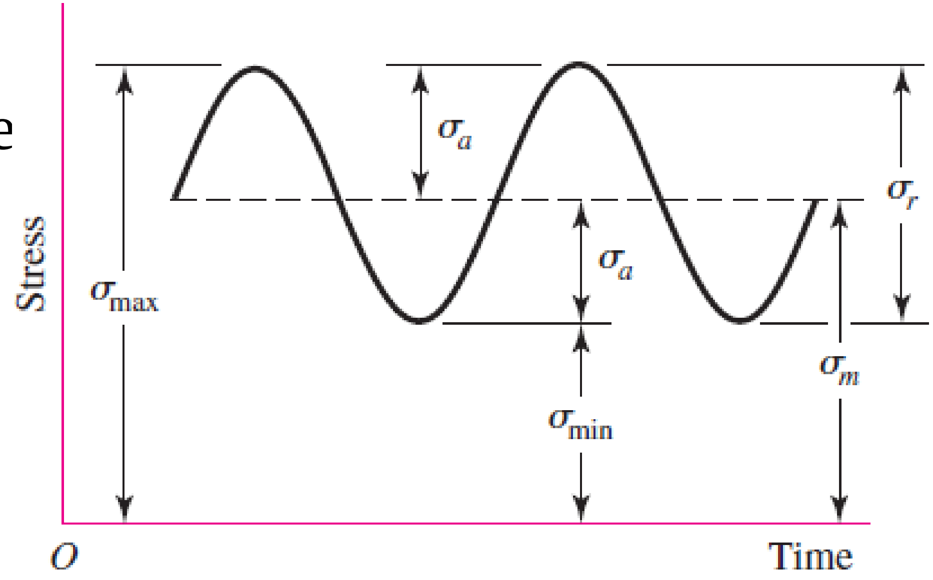


Fig. 6–23

Characterizing Fluctuating Stresses

- Fluctuating stresses can often be characterized simply by the minimum and maximum stresses, $\sigma_{\min}$ and $\sigma_{\max}$
- Define σ_m as *midrange* steady component of stress (sometimes called *mean* stress) and σ_a as amplitude of *alternating* component of stress



$$\sigma_m = \frac{\sigma_{\max} + \sigma_{\min}}{2}$$

$$\sigma_a = \left| \frac{\sigma_{\max} - \sigma_{\min}}{2} \right|$$

(6–36)

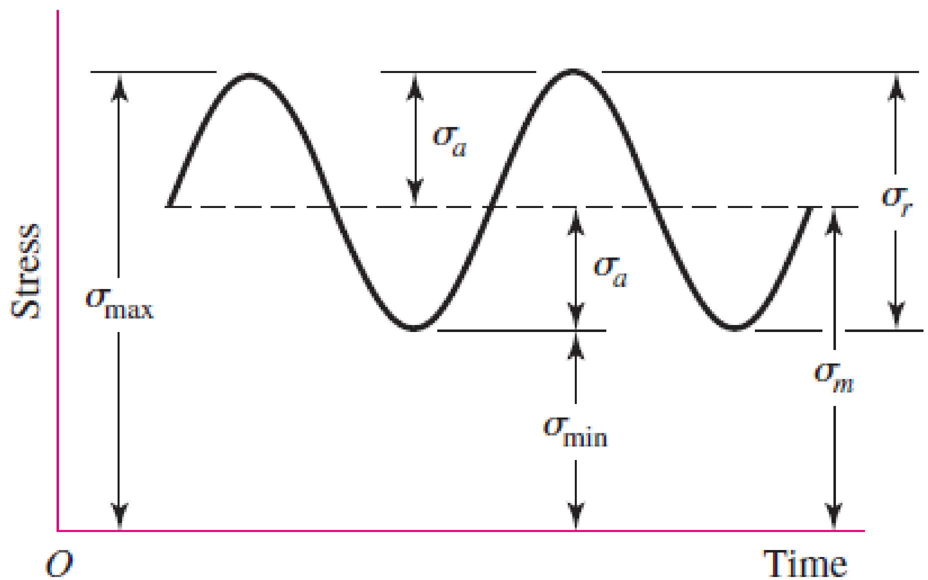
Characterizing Fluctuating Stresses

- Other useful definitions include *stress ratio*

$$R = \frac{\sigma_{\min}}{\sigma_{\max}} \qquad (6\text{–}37)$$

and *amplitude ratio*

$$A = \frac{\sigma_a}{\sigma_m} \qquad (6\text{–}38)$$



Application of K_f for Fluctuating Stresses

- For fluctuating loads at points with stress concentration, the best approach is to design to avoid all localized plastic strain.
- In this case, K_f should be applied to both alternating and midrange stress components.
- When localized strain does occur, some methods (e.g. *nominal mean stress* method and *residual stress* method) recommend only applying K_f to the alternating stress.
- The *Dowling method* recommends applying K_f to the alternating stress and K_{fm} to the mid-range stress, where K_{fm} is

$$
\begin{aligned}
K_{fm} &= K_f && K_f\,|\sigma_{\max,o}| < S_y \\
K_{fm} &= \frac{S_y - K_f\sigma_{ao}}{|\sigma_{mo}|} && K_f\,|\sigma_{\max,o}| > S_y \qquad (6\text{–}39) \\
K_{fm} &= 0 && K_f\,|\sigma_{\max,o} - \sigma_{\min,o}| > 2S_y
\end{aligned}
$$

Fatigue Failure for Fluctuating Stresses

- Vary the σ_m and σ_a to learn about the fatigue resistance under fluctuating loading
- Three common methods of plotting results follow.

Modified Goodman Diagram

- Midrange stress is plotted on abscissa
- All other components of stress are plotted on the ordinate

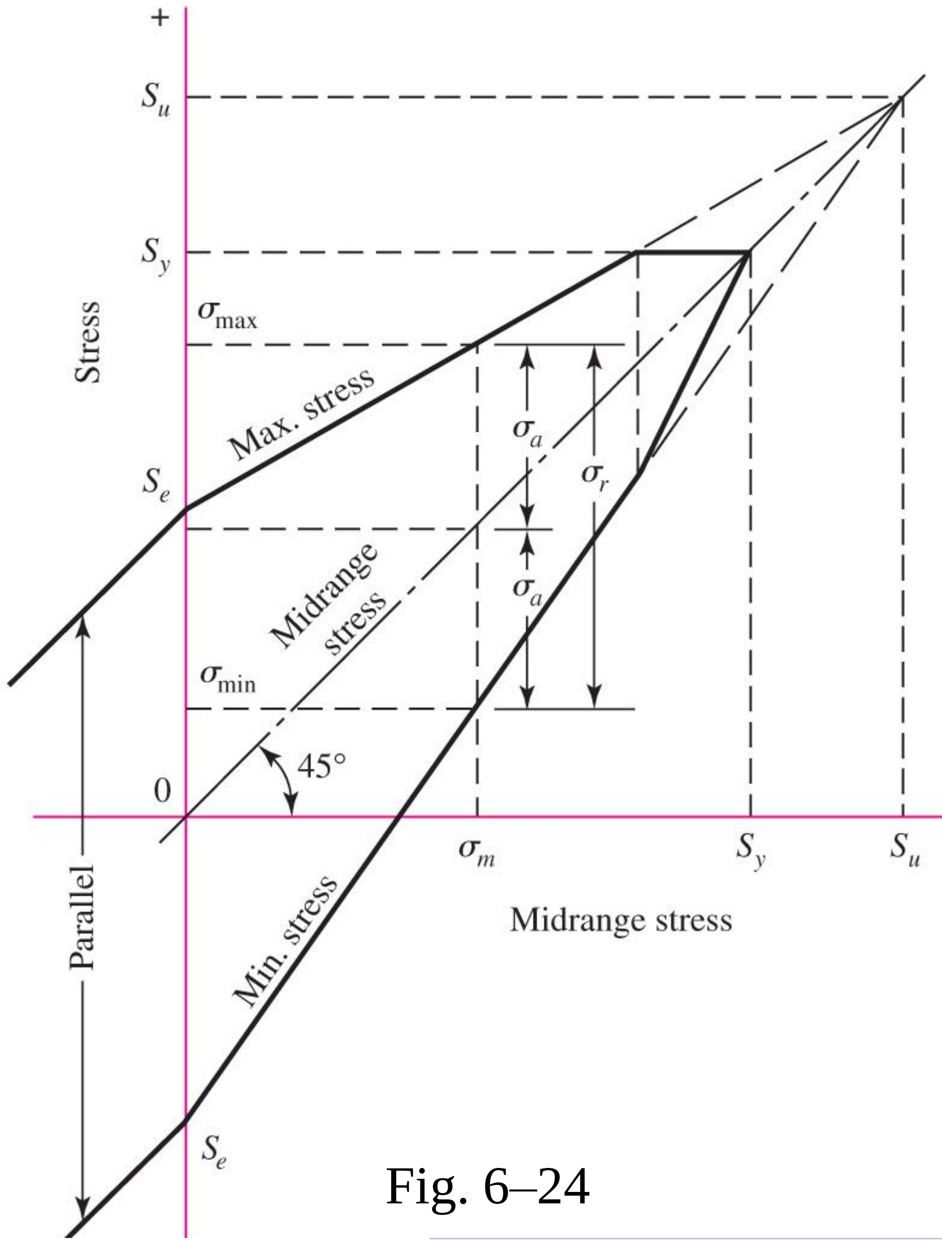


Fig. 6–24

Master Fatigue Diagram

- Displays four stress components as well as two stress ratios

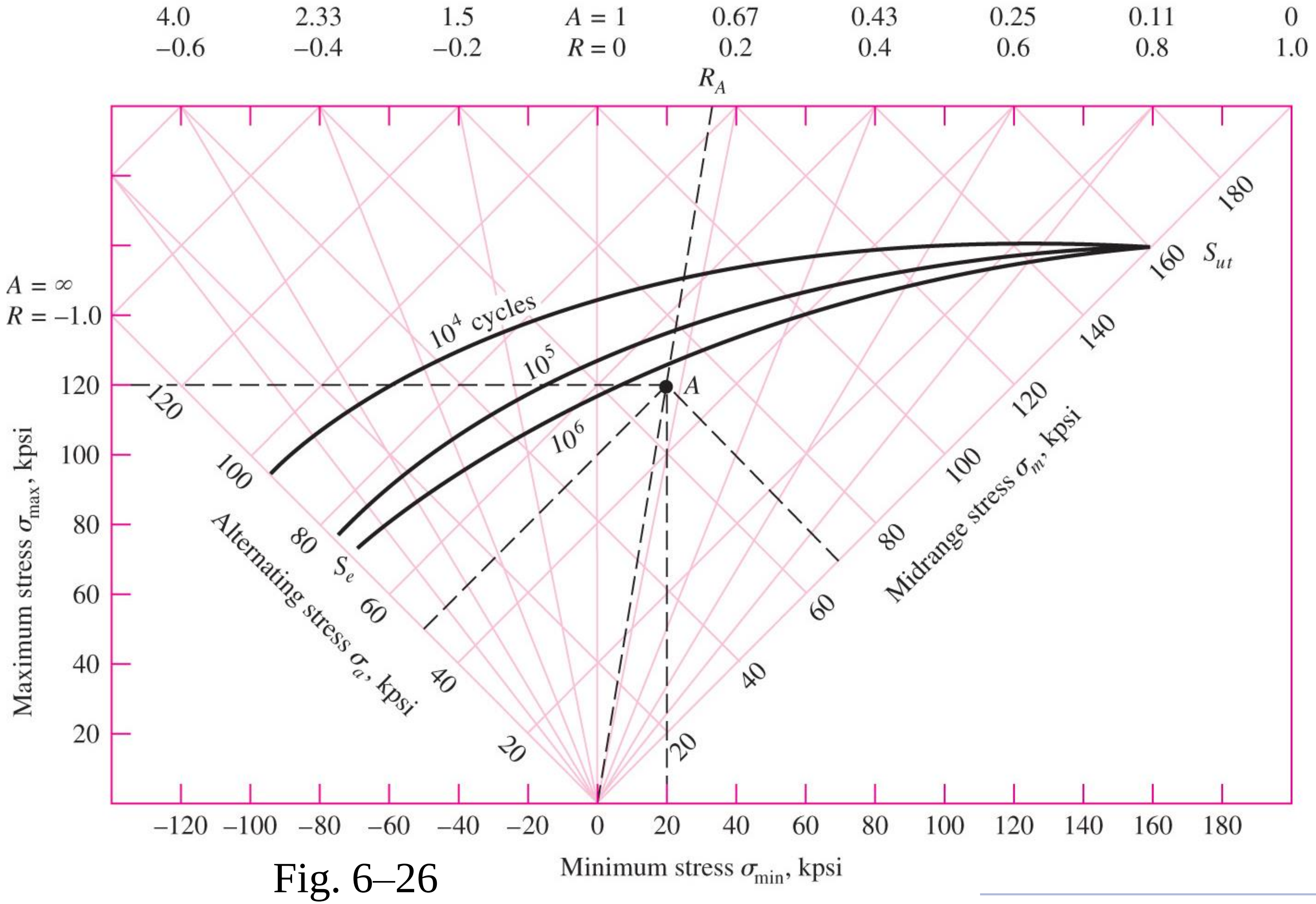
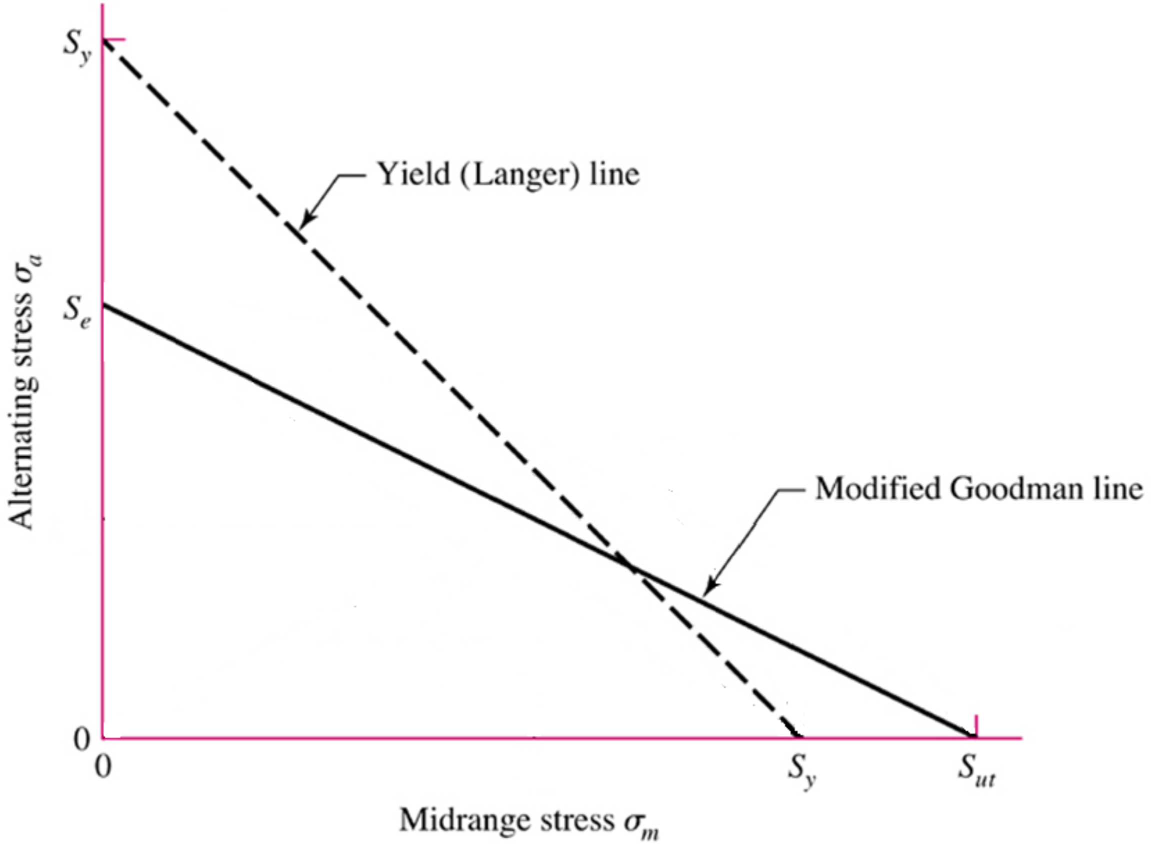


Fig. 6–26

Plot of Alternating vs Midrange Stress

- Probably most common and simple to use is the plot of σ_a vs σ_m
- Has gradually usurped the name of Goodman or Modified Goodman diagram
- Modified Goodman line from S_e to S_{ut} is one simple representation of the limiting boundary for infinite life



Plot of Alternating vs Midrange Stress

- Experimental data on normalized plot of σ_a vs σ_m
- Demonstrates little effect of negative midrange stress

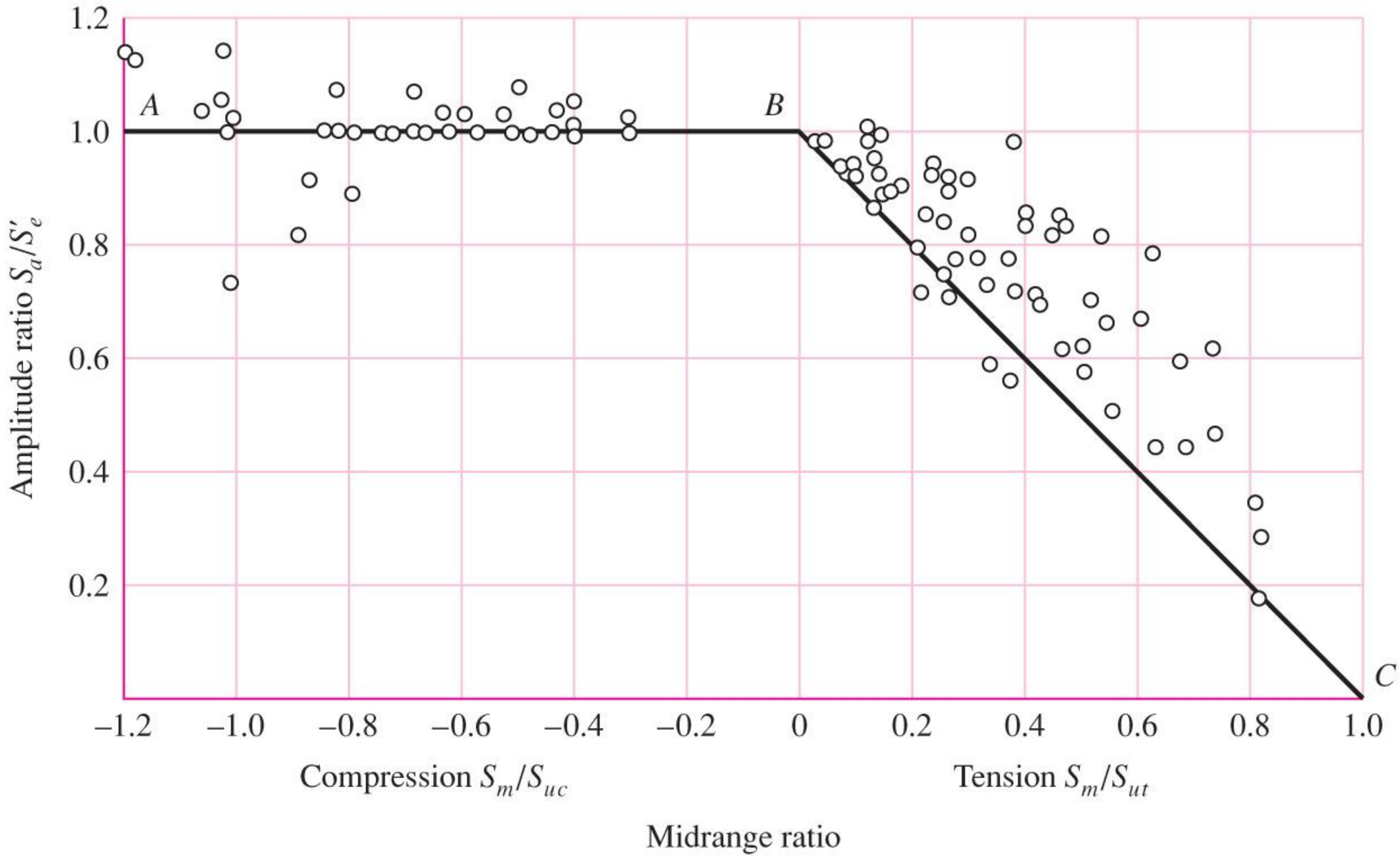


Fig. 6–25

Commonly Used Failure Criteria

- Five commonly used failure criteria are shown
- Gerber passes through the data
- ASME-elliptic passes through data and incorporates rough yielding check

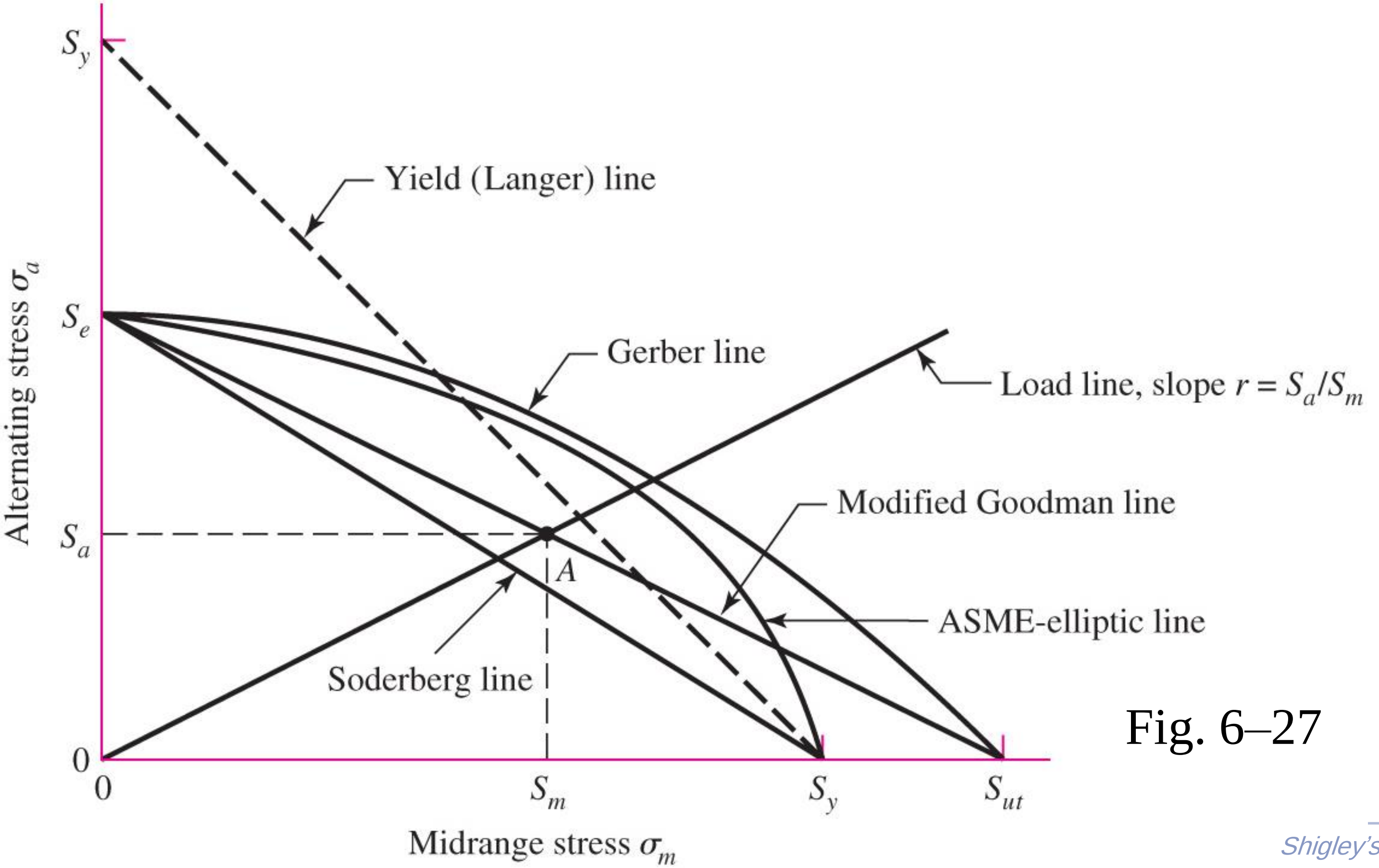


Fig. 6–27

Commonly Used Failure Criteria

- Modified Goodman is linear, so simple to use for design. It is more conservative than Gerber.
- Soderberg provides a very conservative single check of both fatigue and yielding.

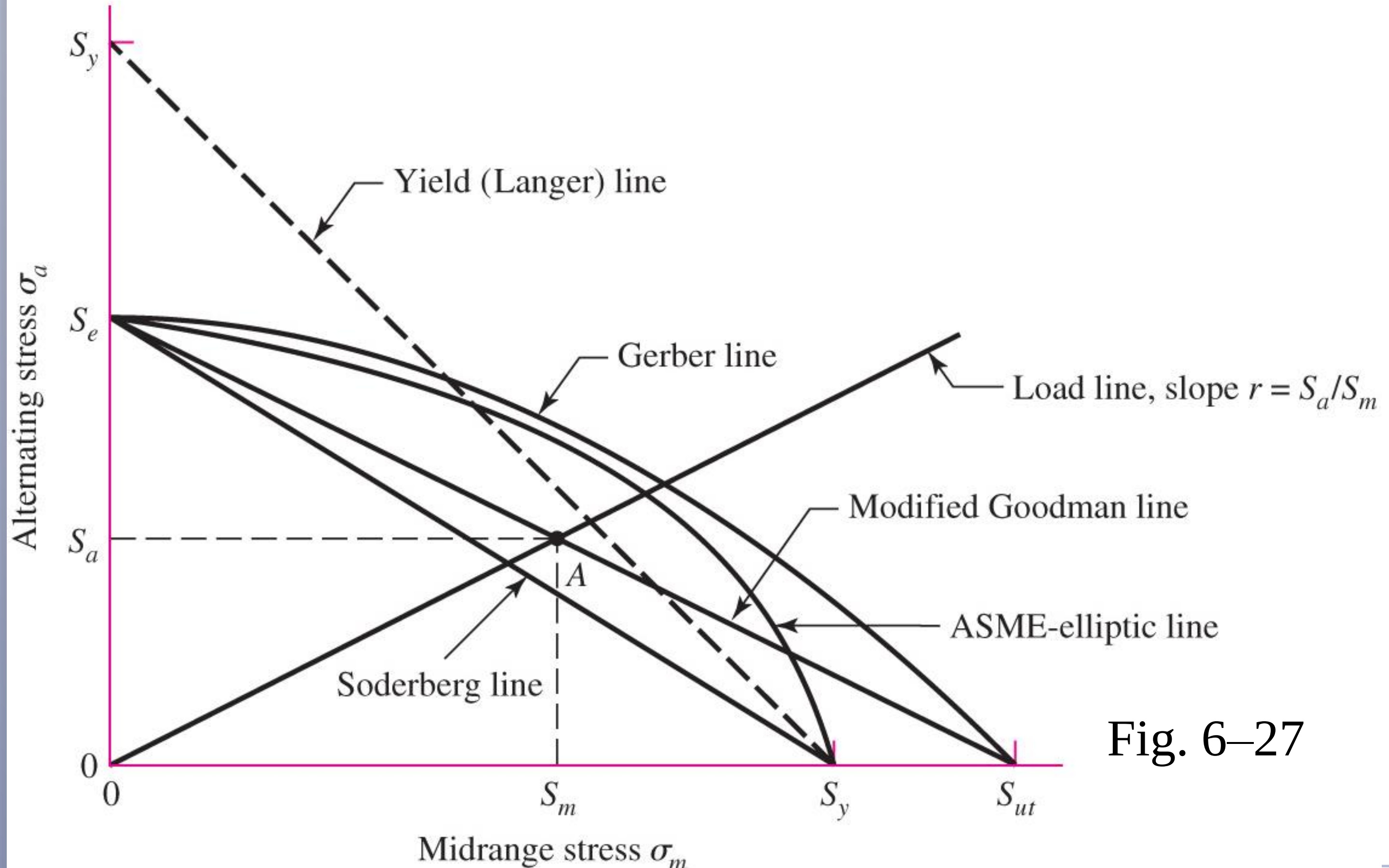


Fig. 6–27

Commonly Used Failure Criteria

- Langer line represents standard yield check.
- It is equivalent to comparing maximum stress to yield strength.

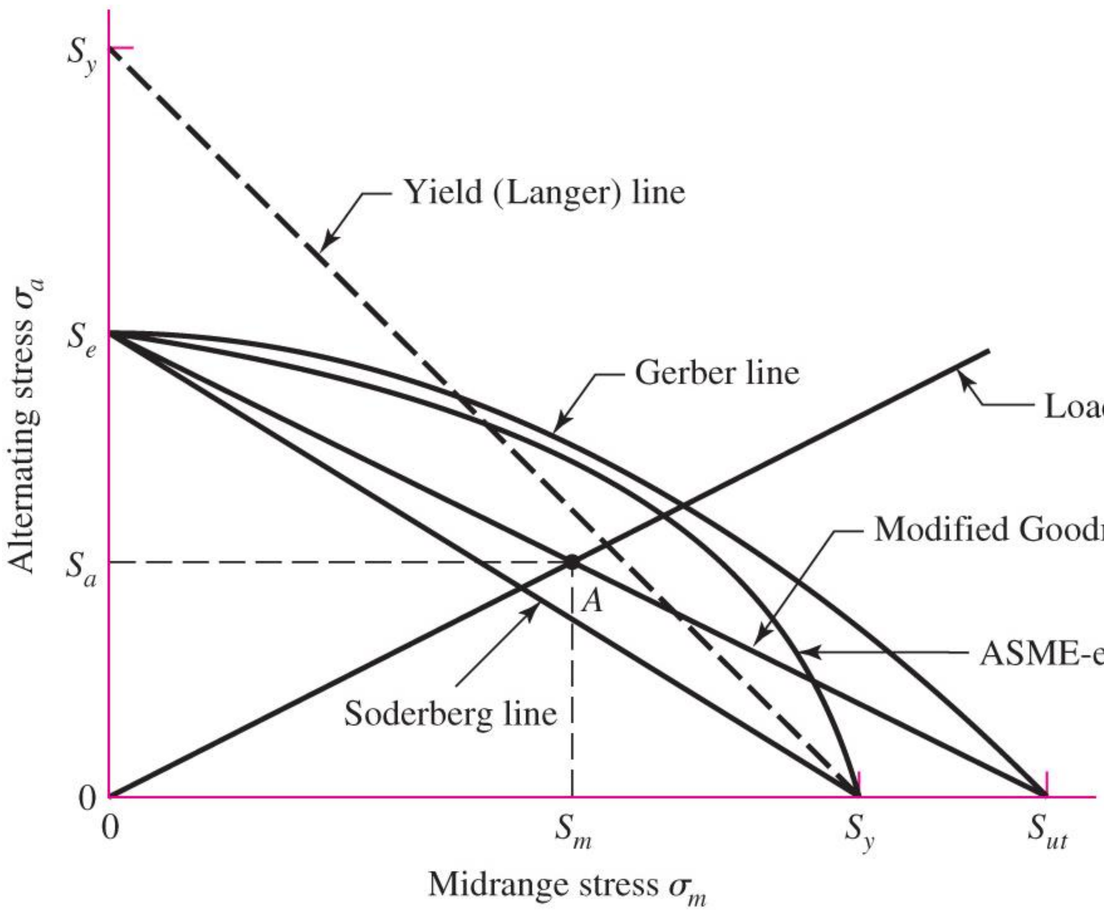


Fig. 6–27

Equations for Commonly Used Failure Criteria

- Intersecting a constant slope load line with each failure criteria produces design equations
- *n* is the design factor or factor of safety for infinite fatigue life

Soderberg $$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = \frac{1}{n} \qquad (6\text{–}45)$$

mod-Goodman $$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}} = \frac{1}{n} \qquad (6\text{–}46)$$

Gerber $$\frac{n\sigma_a}{S_e} + \left(\frac{n\sigma_m}{S_{ut}}\right)^2 = 1 \qquad (6\text{–}47)$$

ASME-elliptic $$\left(\frac{n\sigma_a}{S_e}\right)^2 + \left(\frac{n\sigma_m}{S_y}\right)^2 = 1 \qquad (6\text{–}48)$$

Summarizing Tables for Failure Criteria

- Tables 6–6 to 6–8 summarize the pertinent equations for Modified Goodman, Gerber, ASME-elliptic, and Langer failure criteria
- The first row gives fatigue criterion
- The second row gives yield criterion
- The third row gives the intersection of static and fatigue criteria
- The fourth row gives the equation for fatigue factor of safety
- The first column gives the intersecting equations
- The second column gives the coordinates of the intersection

Summarizing Table for Modified Goodman

Table 6–6

Amplitude and Steady Coordinates of Strength and Important Intersections in First Quadrant for Modified Goodman and Langer Failure Criteria

Intersecting Equations	Intersection Coordinates
$\frac{S_a}{S_e} + \frac{S_m}{S_{ut}} = 1$ Load line $r = \frac{S_a}{S_m}$	$S_a = \frac{r S_e S_{ut}}{r S_{ut} + S_e}$ $S_m = \frac{S_a}{r}$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$ Load line $r = \frac{S_a}{S_m}$	$S_a = \frac{r S_y}{1 + r}$ $S_m = \frac{S_y}{1 + r}$
$\frac{S_a}{S_e} + \frac{S_m}{S_{ut}} = 1$ $\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_m = \frac{(S_y - S_e) S_{ut}}{S_{ut} - S_e}$ $S_a = S_y - S_m,\ r_{\text{crit}} = S_a / S_m$

Fatigue factor of safety

$$n_f = \frac{1}{\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_{ut}}}$$

Summarizing Table for Gerber

Table 6–7

Amplitude and Steady Coordinates of Strength and Important Intersections in First Quadrant for Gerber and Langer Failure Criteria

Intersecting Equations	Intersection Coordinates
$\frac{S_a}{S_e} + \left(\frac{S_m}{S_{ut}}\right)^2 = 1$	$S_a = \frac{r^2 S_{ut}^2}{2S_e}\left[-1 + \sqrt{1 + \left(\frac{2S_e}{rS_{ut}}\right)^2}\right]$
Load line $r = \frac{S_a}{S_m}$	$S_m = \frac{S_a}{r}$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_a = \frac{rS_y}{1+r}$
Load line $r = \frac{S_a}{S_m}$	$S_m = \frac{S_y}{1+r}$
$\frac{S_a}{S_e} + \left(\frac{S_m}{S_{ut}}\right)^2 = 1$	$S_m = \frac{S_{ut}^2}{2S_e}\left[1 - \sqrt{1 + \left(\frac{2S_e}{S_{ut}}\right)^2 \left(1 - \frac{S_y}{S_e}\right)}\right]$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_a = S_y - S_m,\ r_{\text{crit}} = S_a/S_m$

Fatigue factor of safety

$$n_f = \frac{1}{2}\left(\frac{S_{ut}}{\sigma_m}\right)^2 \frac{\sigma_a}{S_e}\left[-1 + \sqrt{1 + \left(\frac{2\sigma_m S_e}{S_{ut}\sigma_a}\right)^2}\right] \qquad \sigma_m > 0$$

Summarizing Table for ASME-Elliptic

Table 6–8

Amplitude and Steady Coordinates of Strength and Important Intersections in First Quadrant for ASME-Elliptic and Langer Failure Criteria

Intersecting Equations	Intersection Coordinates
$\left(\frac{S_a}{S_e}\right)^2 + \left(\frac{S_m}{S_y}\right)^2 = 1$	$S_a = \sqrt{\frac{r^2 S_e^2 S_y^2}{S_e^2 + r^2 S_y^2}}$
Load line $r = S_a/S_m$	$S_m = \frac{S_a}{r}$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_a = \frac{r S_y}{1 + r}$
Load line $r = S_a/S_m$	$S_m = \frac{S_y}{1 + r}$
$\left(\frac{S_a}{S_e}\right)^2 + \left(\frac{S_m}{S_y}\right)^2 = 1$	$S_a = 0, \ \frac{2 S_y S_e^2}{S_e^2 + S_y^2}$
$\frac{S_a}{S_y} + \frac{S_m}{S_y} = 1$	$S_m = S_y - S_a, \ r_{\text{crit}} = S_a/S_m$

Fatigue factor of safety

$$n_f = \sqrt{\frac{1}{(\sigma_a/S_e)^2 + (\sigma_m/S_y)^2}}$$

Example 6-10

A 1.5-in-diameter bar has been machined from an AISI 1050 cold-drawn bar. This part is to withstand a fluctuating tensile load varying from 0 to 16 kip. Because of the ends, and the fillet radius, a fatigue stress-concentration factor K_f is 1.85 for 10^6 or larger life. Find S_a and S_m and the factor of safety guarding against fatigue and first-cycle yielding, using (*a*) the Gerber fatigue line and (*b*) the ASME-elliptic fatigue line.

Solution

We begin with some preliminaries. From Table A–20, $S_{ut} = 100$ kpsi and $S_y = 84$ kpsi. Note that $F_a = F_m = 8$ kip. The Marin factors are, deterministically,

$k_a = 2.70(100)^{-0.265} = 0.797$: Eq. (6–19), Table 6–2, p. 288

$k_b = 1$ (axial loading, see k_c)

$k_c = 0.85$: Eq. (6–26), p. 290

$k_d = k_e = k_f = 1$

$S_e = 0.797(1)0.850(1)(1)(1)0.5(100) = 33.9$ kpsi: Eqs. (6–8), (6–18), p. 282, p. 287

Example 6-10

The nominal axial stress components σ_{ao} and σ_{mo} are

$$\sigma_{ao} = \frac{4F_a}{\pi d^2} = \frac{4(8)}{\pi 1.5^2} = 4.53 \text{ kpsi} \qquad \sigma_{mo} = \frac{4F_m}{\pi d^2} = \frac{4(8)}{\pi 1.5^2} = 4.53 \text{ kpsi}$$

Applying K_f to both components σ_{ao} and σ_{mo} constitutes a prescription of no notch yielding:

$$\sigma_a = K_f \sigma_{ao} = 1.85(4.53) = 8.38 \text{ kpsi} = \sigma_m$$

(a) Let us calculate the factors of safety first. From the bottom panel from Table 6–7 the factor of safety for fatigue is

$$n_f = \frac{1}{2}\left(\frac{100}{8.38}\right)^2\left(\frac{8.38}{33.9}\right)\left\{-1 + \sqrt{1 + \left[\frac{2(8.38)33.9}{100(8.38)}\right]^2}\right\} = 3.66$$

From Eq. (6–49) the factor of safety guarding against first-cycle yield is

$$n_y = \frac{S_y}{\sigma_a + \sigma_m} = \frac{84}{8.38 + 8.38} = 5.01$$

Example 6-10

Thus, we see that fatigue will occur first and the factor of safety is 3.68. This can be seen in Fig. 6–28 where the load line intersects the Gerber fatigue curve first at point B. If the plots are created to true scale it would be seen that $n_f = OB/OA$.

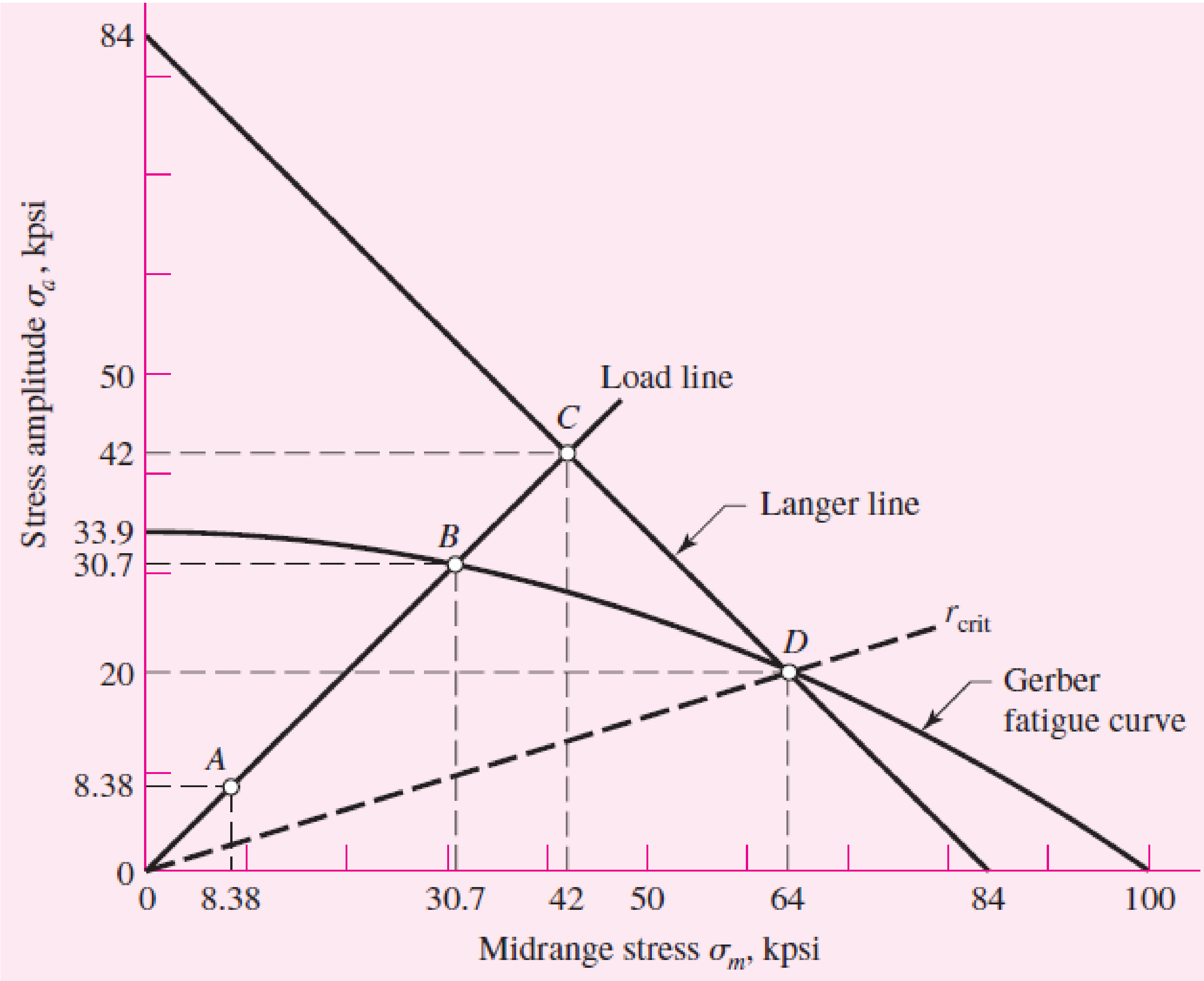


Fig. 6–28

Example 6-10

From the first panel of Table 6–7, $r = \sigma_a/\sigma_m = 1$,

$$S_a = \frac{(1)^2 100^2}{2(33.9)}\left\{-1+\sqrt{1+\left[\frac{2(33.9)}{(1)100}\right]^2}\right\} = 30.7 \text{ kpsi}$$

$$S_m = \frac{S_a}{r} = \frac{30.7}{1} = 30.7 \text{ kpsi}$$

As a check on the previous result, $n_f = OB/OA = S_a/\sigma_a = S_m/\sigma_m = 30.7/8.38 = 3.66$ and we see total agreement.

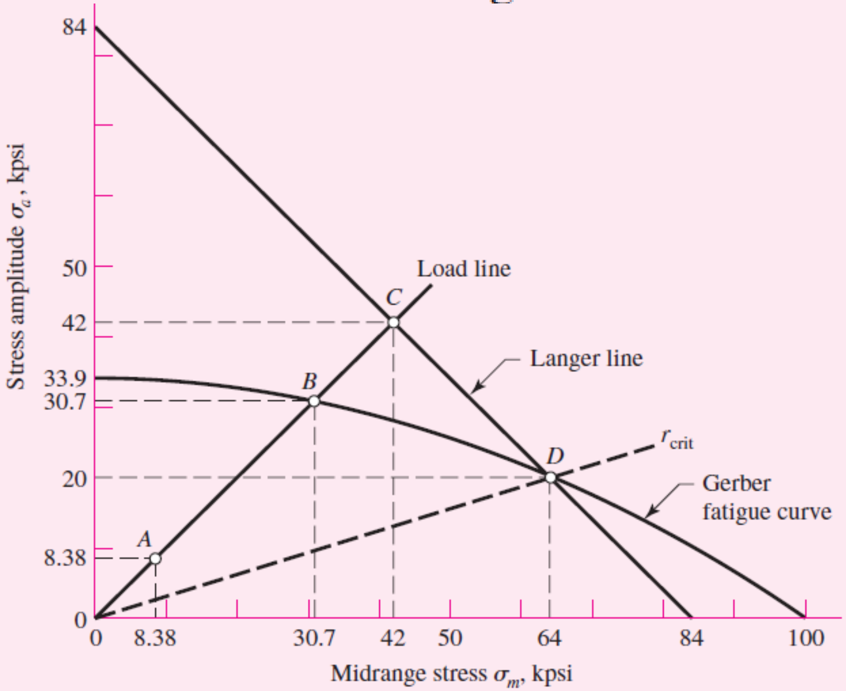


Fig. 6–28

Example 6-10

We could have detected that fatigue failure would occur first without drawing Fig. 6–28 by calculating r_{crit}. From the third row third column panel of Table 6–7, the intersection point between fatigue and first-cycle yield is

$$S_m = \frac{100^2}{2(33.9)}\left[1 - \sqrt{1 + \left(\frac{2(33.9)}{100}\right)^2 \left(1 - \frac{84}{33.9}\right)}\right] = 64.0 \text{ kpsi}$$

$$S_a = S_y - S_m = 84 - 64 = 20 \text{ kpsi}$$

The critical slope is thus

$$r_{\text{crit}} = \frac{S_a}{S_m} = \frac{20}{64} = 0.312$$

which is less than the actual load line of $r = 1$. This indicates that fatigue occurs before first-cycle-yield.

Example 6-10

(*b*) Repeating the same procedure for the ASME-elliptic line, for fatigue

$$n_f = \sqrt{\frac{1}{(8.38/33.9)^2 + (8.38/84)^2}} = 3.75$$

Again, this is less than $n_y = 5.01$ and fatigue is predicted to occur first. From the first row second column panel of Table 6–8, with $r = 1$, we obtain the coordinates S_a and S_m of point B in Fig. 6–29 as

$$S_a = \sqrt{\frac{(1)^2 33.9^2 (84)^2}{33.9^2 + (1)^2 84^2}} = 31.4 \text{ kpsi},$$

$$S_m = \frac{S_a}{r} = \frac{31.4}{1} = 31.4 \text{ kpsi}$$

To verify the fatigue factor of safety,

$$n_f = S_a/\sigma_a = 31.4/8.38 = 3.75.$$

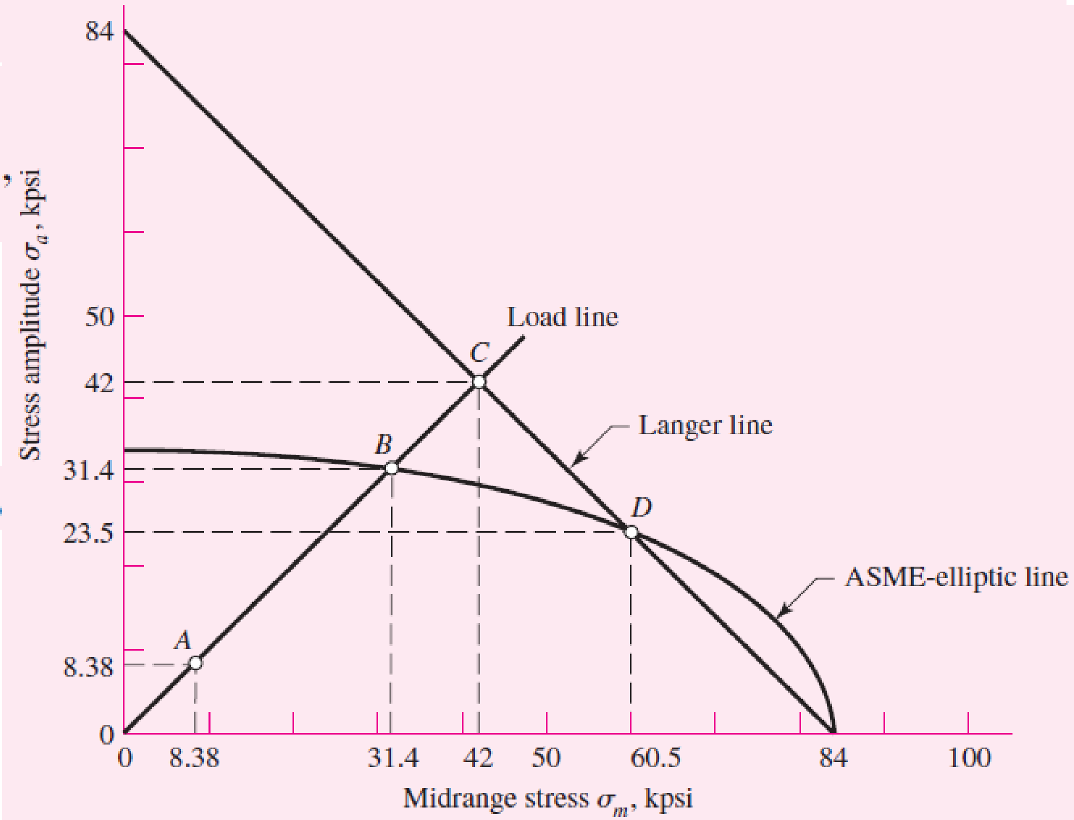


Fig. 6–29

Example 6-10

As before, let us calculate r_{crit}. From the third row second column panel of Table 6–8,

$$S_a = \frac{2(84)33.9^2}{33.9^2 + 84^2} = 23.5\ \text{kpsi}, \quad S_m = S_y - S_a = 84 - 23.5 = 60.5\ \text{kpsi}$$

$$r_{\text{crit}} = \frac{S_a}{S_m} = \frac{23.5}{60.5} = 0.388$$

which again is less than $r = 1$, verifying that fatigue occurs first with $n_f = 3.75$.

The Gerber and the ASME-elliptic fatigue failure criteria are very close to each other and are used interchangeably. The ANSI/ASME Standard B106.1M–1985 uses ASME-elliptic for shafting.

Example 6-11

A flat-leaf spring is used to retain an oscillating flat-faced follower in contact with a plate cam. The follower range of motion is 2 in and fixed, so the alternating component of force, bending moment, and stress is fixed, too. The spring is preloaded to adjust to various cam speeds. The preload must be increased to prevent follower float or jump. For lower speeds the preload should be decreased to obtain longer life of cam and follower surfaces. The spring is a steel cantilever 32 in long, 2 in wide, and $\frac{1}{4}$ in thick, as seen in Fig. 6–30*a*. The spring strengths are $S_{ut} = 150$ kpsi, $S_y = 127$ kpsi, and $S_e = 28$ kpsi fully corrected. The total cam motion is 2 in. The designer wishes to preload the spring by deflecting it 2 in for low speed and 5 in for high speed.

(*a*) Plot the Gerber-Langer failure lines with the load line.

(*b*) What are the strength factors of safety corresponding to 2 in and 5 in preload?

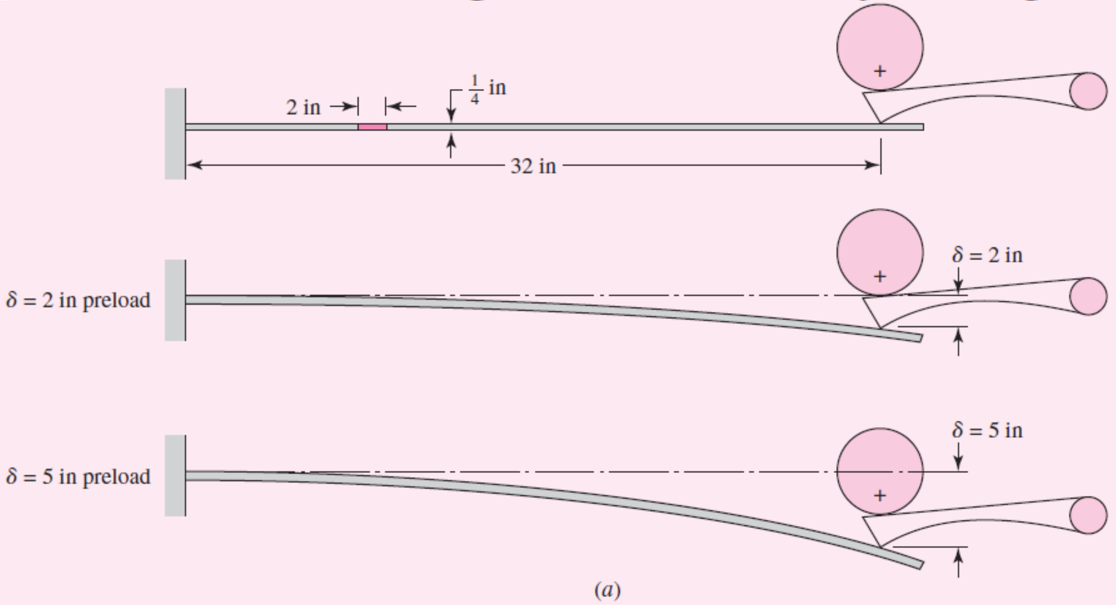


Fig. 6–30*a*

Example 6-11

We begin with preliminaries. The second area moment of the cantilever cross section is

$$I = \frac{bh^3}{12} = \frac{2(0.25)^3}{12} = 0.00260 \text{ in}^4$$

Since, from Table A–9, beam 1, force F and deflection y in a cantilever are related by $F = 3EIy/l^3$, then stress σ and deflection y are related by

$$\sigma = \frac{Mc}{I} = \frac{32Fc}{I} = \frac{32(3EIy)}{l^3}\frac{c}{I} = \frac{96Ecy}{l^3} = Ky$$

where $K = \dfrac{96Ec}{l^3} = \dfrac{96(30 \cdot 10^6)0.125}{32^3} = 10.99(10^3)$ psi/in $= 10.99$ kpsi/in

Now the minimums and maximums of y and σ can be defined by

$$y_{\text{min}} = \delta \qquad y_{\text{max}} = 2 + \delta$$

$$\sigma_{\text{min}} = K\delta \qquad \sigma_{\text{max}} = K(2 + \delta)$$

Example 6-11

The stress components are thus

$$\sigma_a = \frac{K(2+\delta) - K\delta}{2} = K = 10.99 \text{ kpsi}$$

$$\sigma_m = \frac{K(2+\delta) + K\delta}{2} = K(1+\delta) = 10.99(1+\delta)$$

For $\delta = 0$, $\quad \sigma_a = \sigma_m = 10.99 = 11$ kpsi

For $\delta = 2$ in, $\quad \sigma_a = 11$ kpsi, $\sigma_m = 10.99(1+2) = 33$ kpsi

For $\delta = 5$ in, $\quad \sigma_a = 11$ kpsi, $\sigma_m = 10.99(1+5) = 65.9$ kpsi

Example 6-11

(a) A plot of the Gerber and Langer criteria is shown in Fig. 6–30b. The three preload deflections of 0, 2, and 5 in are shown as points A, A', and A''. Note that since σ_a is constant at 11 kpsi, the load line is horizontal and does not contain the origin. The intersection between the Gerber line and the load line is found from solving Eq. (6–42) for S_m and substituting 11 kpsi for S_a:

$$S_m = S_{ut}\sqrt{1 - \frac{S_a}{S_e}} = 150\sqrt{1 - \frac{11}{28}} = 116.9 \text{ kpsi}$$

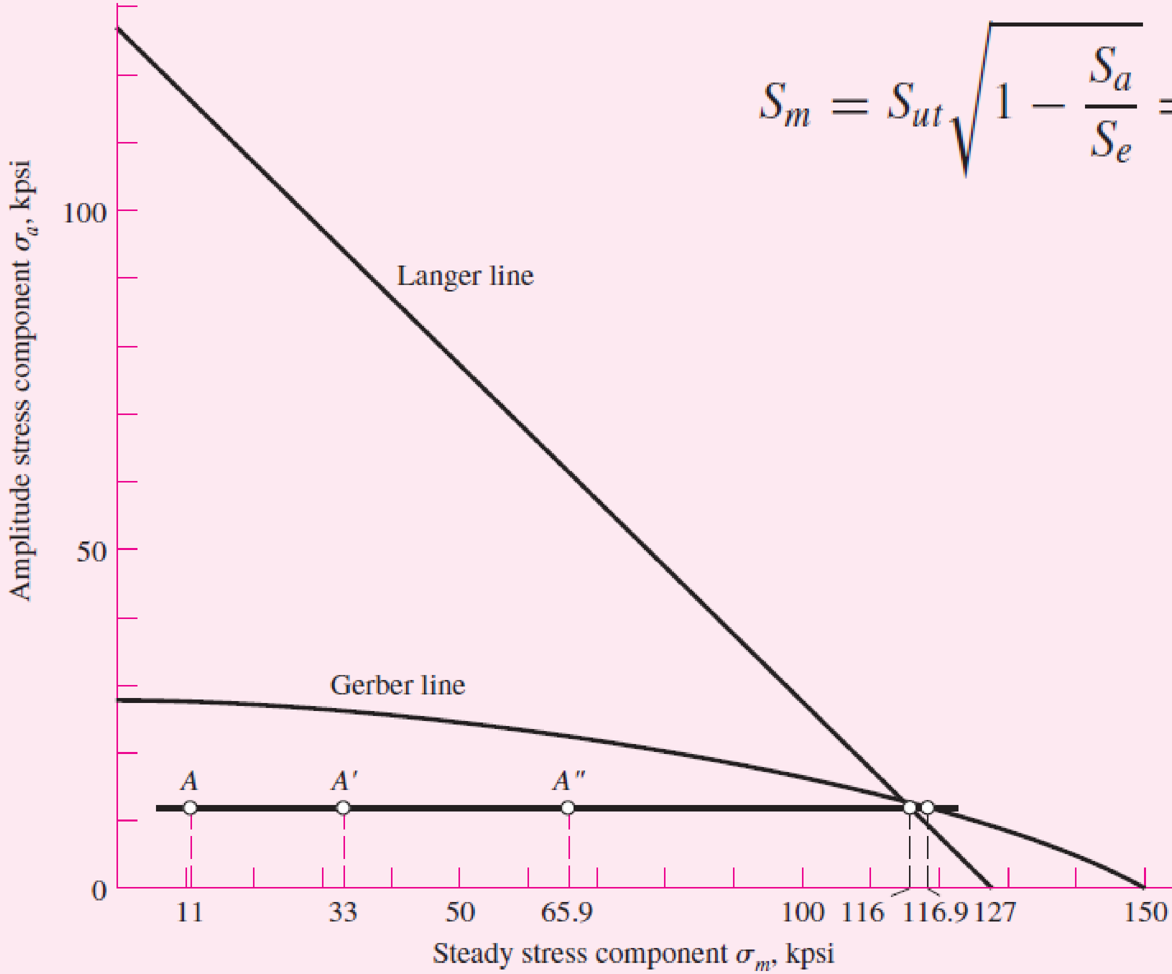


Fig. 6–30b

Example 6-11

The intersection of the Langer line and the load line is found from solving Eq. (6–44) for S_m and substituting 11 kpsi for S_a:

$$S_m = S_y - S_a = 127 - 11 = 116 \text{ kpsi}$$

The threats from fatigue and first-cycle yielding are approximately equal.

(*b*) For $\delta = 2$ in,

$$n_f = \frac{S_m}{\sigma_m} = \frac{116.9}{33} = 3.54 \qquad n_y = \frac{116}{33} = 3.52$$

and for $\delta = 5$ in,

$$n_f = \frac{116.9}{65.9} = 1.77 \qquad n_y = \frac{116}{65.9} = 1.76$$

Example 6-12

A steel bar undergoes cyclic loading such that $\sigma_{max} = 60$ kpsi and $\sigma_{min} = -20$ kpsi. For the material, $S_{ut} = 80$ kpsi, $S_y = 65$ kpsi, a fully corrected endurance limit of $S_e = 40$ kpsi, and $f = 0.9$. Estimate the number of cycles to a fatigue failure using:

(*a*) Modified Goodman criterion.

(*b*) Gerber criterion.

Solution

From the given stresses,

$$\sigma_a = \frac{60 - (-20)}{2} = 40 \text{ kpsi} \qquad \sigma_m = \frac{60 + (-20)}{2} = 20 \text{ kpsi}$$

(*a*) For the modified Goodman criterion, Eq. (6–46), the fatigue factor of safety based on infinite life is

$$n_f = \frac{1}{\dfrac{\sigma_a}{S_e} + \dfrac{\sigma_m}{S_{ut}}} = \frac{1}{\dfrac{40}{40} + \dfrac{20}{80}} = 0.8$$

Example 6-12

This indicates a finite life is predicted. The *S-N* diagram is only applicable for completely reversed stresses. To estimate the finite life for a fluctuating stress, we will obtain an equivalent completely reversed stress that is expected to be as damaging as the fluctuating stress. A commonly used approach is to assume that since the modified Goodman line represents all stress situations with a constant life of 10^6 cycles, other constant-life lines can be generated by passing a line through $(S_{ut}, 0)$ and a fluctuating stress point (σ_m, σ_a). The point where this line intersects the σ_a axis represents a completely reversed stress (since at this point $\sigma_m = 0$), which predicts the same life as the fluctuating stress.

This completely reversed stress can be obtained by replacing S_e with σ_{rev} in Eq. (6–46) for the modified Goodman line resulting in

$$\sigma_{\text{rev}} = \frac{\sigma_a}{1 - \dfrac{\sigma_m}{S_{ut}}} = \frac{40}{1 - \dfrac{20}{80}} = 53.3 \text{ kpsi}$$

Example 6-12

From the material properties, Eqs. (6–14) to (6–16), p. 285, give

$$a = \frac{(f\,S_{ut})^2}{S_e} = \frac{[0.9(80)]^2}{40} = 129.6 \text{ kpsi}$$

$$b = -\frac{1}{3}\log\left(\frac{f\,S_{ut}}{S_e}\right) = -\frac{1}{3}\log\left[\frac{0.9(80)}{40}\right] = -0.0851$$

$$N = \left(\frac{\sigma_{\text{rev}}}{a}\right)^{1/b} = \left(\frac{\sigma_{\text{rev}}}{129.6}\right)^{-1/0.0851} \tag{1}$$

Substituting σ_{rev} into Eq. (1) yields

$$N = \left(\frac{53.3}{129.6}\right)^{-1/0.0851} \doteq 3.4(10^4) \text{ cycles}$$

Example 6-12

(*b*) For Gerber, similar to part (*a*), from Eq. (6–47),

$$\sigma_{\text{rev}} = \frac{\sigma_a}{1 - \left(\dfrac{\sigma_m}{S_{ut}}\right)^2} = \frac{40}{1 - \left(\dfrac{20}{80}\right)^2} = 42.7 \text{ kpsi}$$

Again, from Eq. (1),

$$N = \left(\frac{42.7}{129.6}\right)^{-1/0.0851} \doteq 4.6(10^5) \text{ cycles}$$

Comparing the answers, we see a large difference in the results. Again, the modified Goodman criterion is conservative as compared to Gerber for which the moderate difference in S_f is then magnified by a logarithmic S, N relationship.

Fatigue Criteria for Brittle Materials

- For many brittle materials, the first quadrant fatigue failure criteria follows a concave upward Smith-Dolan locus,

$$\frac{S_a}{S_e} = \frac{1 - S_m/S_{ut}}{1 + S_m/S_{ut}} \qquad (6\text{–}50)$$

- Or as a design equation,

$$\frac{n\sigma_a}{S_e} = \frac{1 - n\sigma_m/S_{ut}}{1 + n\sigma_m/S_{ut}} \qquad (6\text{–}51)$$

- For a radial load line of slope *r*, the intersection point is

$$S_a = \frac{rS_{ut} + S_e}{2}\left[-1 + \sqrt{1 + \frac{4rS_{ut}S_e}{(rS_{ut} + S_e)^2}}\right] \qquad (6\text{–}52)$$

- In the second quadrant,

$$S_a = S_e + \left(\frac{S_e}{S_{ut}} - 1\right)S_m \qquad -S_{ut} \le S_m \le 0 \quad \text{(for cast iron)} \qquad (6\text{–}53)$$

Fatigue Criteria for Brittle Materials

- Table A–24 gives properties of gray cast iron, including endurance limit
- The endurance limit already includes k_a and k_b
- The average k_c for axial and torsional is 0.9

Example 6-13

A grade 30 gray cast iron is subjected to a load F applied to a 1 by $\frac{3}{8}$-in cross-section link with a $\frac{1}{4}$-in-diameter hole drilled in the center as depicted in Fig. 6–31*a*. The surfaces are machined. In the neighborhood of the hole, what is the factor of safety guarding against failure under the following conditions:

(*a*) The load $F = 1000$ lbf tensile, steady.

(*b*) The load is 1000 lbf repeatedly applied.

(*c*) The load fluctuates between -1000 lbf and 300 lbf without column action.

Use the Smith-Dolan fatigue locus.

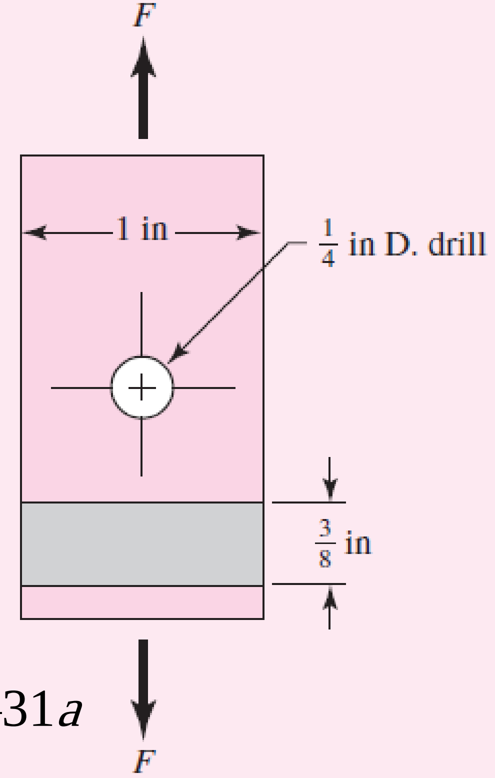


Fig. 6–31*a*

Example 6-13

Some preparatory work is needed. From Table A–24, $S_{ut} = 31$ kpsi, $S_{uc} = 109$ kpsi, $k_a k_b S'_e = 14$ kpsi. Since k_c for axial loading is 0.9, then $S_e = (k_a k_b S'_e)k_c = 14(0.9) = 12.6$ kpsi. From Table A–15–1, $A = t(w - d) = 0.375(1 - 0.25) = 0.281\ \text{in}^2$, $d/w = 0.25/1 = 0.25$, and $K_t = 2.45$. The notch sensitivity for cast iron is 0.20 (see p. 296), so

$$K_f = 1 + q(K_t - 1) = 1 + 0.20(2.45 - 1) = 1.29$$

$$(a)\quad \sigma_a = \frac{K_f F_a}{A} = \frac{1.29(0)}{0.281} = 0 \qquad \sigma_m = \frac{K_f F_m}{A} = \frac{1.29(1000)}{0.281}(10^{-3}) = 4.59\ \text{kpsi}$$

and

$$n = \frac{S_{ut}}{\sigma_m} = \frac{31.0}{4.59} = 6.75$$

Example 6-13

(b) $$F_a = F_m = \frac{F}{2} = \frac{1000}{2} = 500 \text{ lbf}$$

$$\sigma_a = \sigma_m = \frac{K_f F_a}{A} = \frac{1.29(500)}{0.281}(10^{-3}) = 2.30 \text{ kpsi}$$

$$r = \frac{\sigma_a}{\sigma_m} = 1$$

From Eq. (6–52),

$$S_a = \frac{(1)31 + 12.6}{2}\left[-1 + \sqrt{1 + \frac{4(1)31(12.6)}{[(1)31 + 12.6]^2}}\,\right] = 7.63 \text{ kpsi}$$

$$n = \frac{S_a}{\sigma_a} = \frac{7.63}{2.30} = 3.32$$

Example 6-13

(c) $F_a = \frac{1}{2}|300 - (-1000)| = 650 \text{ lbf}$ $\sigma_a = \frac{1.29(650)}{0.281}(10^{-3}) = 2.98 \text{ kpsi}$

$F_m = \frac{1}{2}[300 + (-1000)] = -350 \text{ lbf}$ $\sigma_m = \frac{1.29(-350)}{0.281}(10^{-3}) = -1.61 \text{ kpsi}$

$$r = \frac{\sigma_a}{\sigma_m} = \frac{3.0}{-1.61} = -1.86$$

From Eq. (6–53), $S_a = S_e + (S_e/S_{ut} - 1)S_m$ and $S_m = S_a/r$. It follows that

$$S_a = \frac{S_e}{1 - \frac{1}{r}\left(\frac{S_e}{S_{ut}} - 1\right)} = \frac{12.6}{1 - \frac{1}{-1.86}\left(\frac{12.6}{31} - 1\right)} = 18.5 \text{ kpsi}$$

$$n = \frac{S_a}{\sigma_a} = \frac{18.5}{2.98} = 6.20$$

Example 6-13

Figure 6–31*b* shows the portion of the designer's fatigue diagram that was constructed.

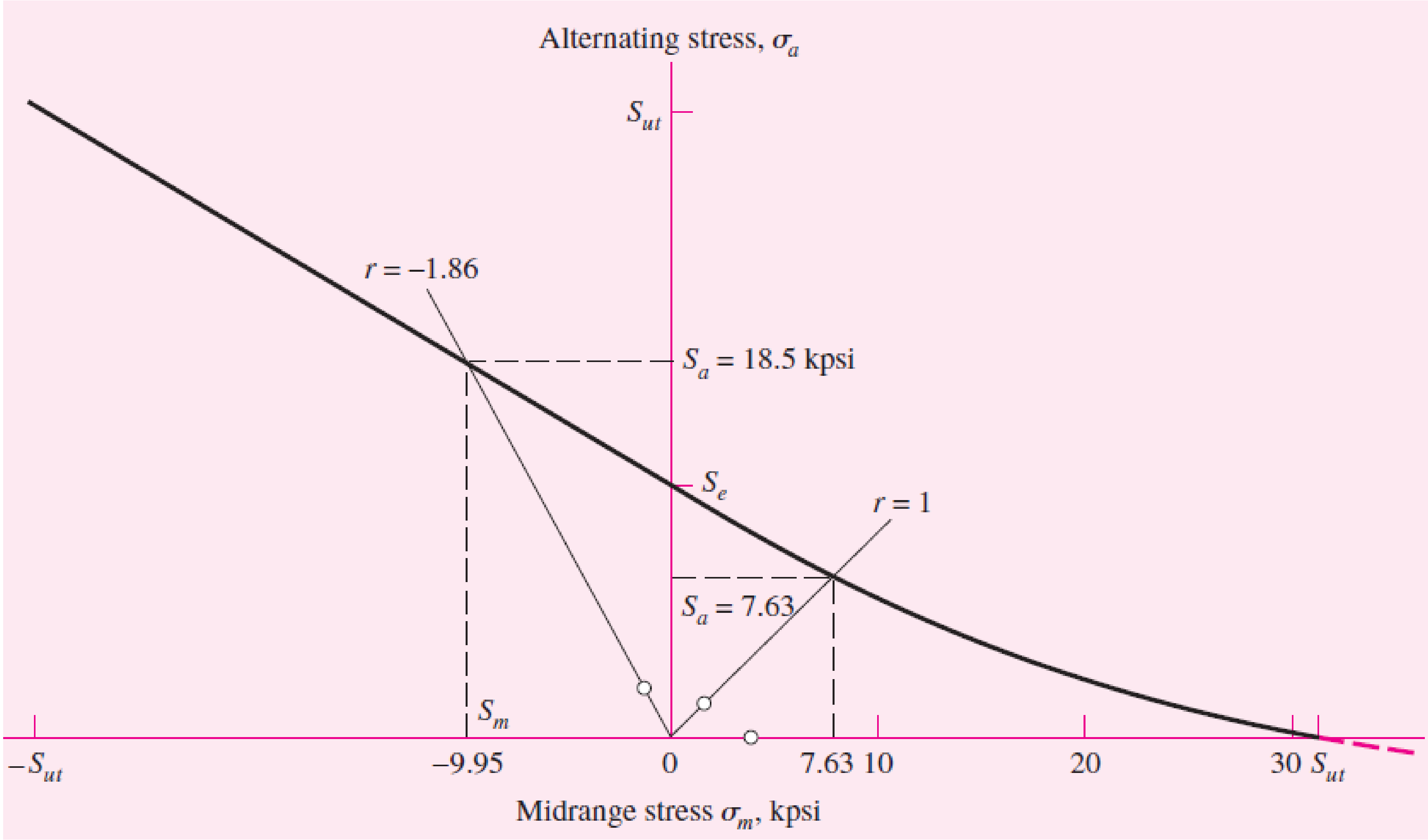


Fig. 6–31*b*

Torsional Fatigue Strength

- Testing has found that the steady-stress component has no effect on the endurance limit for torsional loading if the material is ductile, polished, notch-free, and cylindrical.
- However, for less than perfect surfaces, the modified Goodman line is more reasonable.
- For pure torsion cases, use $k_c = 0.59$ to convert normal endurance strength to shear endurance strength.
- For shear ultimate strength, recommended to use

$$S_{su} = 0.67 S_{ut} \qquad (6\text{–}54)$$

Combinations of Loading Modes

- When more than one type of loading (bending, axial, torsion) exists, use the Distortion Energy theory to combine them.
- Obtain von Mises stresses for both midrange and alternating components.
- Apply appropriate K_f to each type of stress.
- For load factor, use $k_c = 1$. The torsional load factor ($k_c = 0.59$) is inherently included in the von Mises equations.
- If needed, axial load factor can be divided into the axial stress.

$$\sigma'_a = \left\{ \left[(K_f)_{\text{bending}}(\sigma_a)_{\text{bending}} + (K_f)_{\text{axial}} \frac{(\sigma_a)_{\text{axial}}}{0.85} \right]^2 + 3\left[(K_{fs})_{\text{torsion}}(\tau_a)_{\text{torsion}}\right]^2 \right\}^{1/2}$$

(6–55)

$$\sigma'_m = \left\{ \left[(K_f)_{\text{bending}}(\sigma_m)_{\text{bending}} + (K_f)_{\text{axial}}(\sigma_m)_{\text{axial}} \right]^2 + 3\left[(K_{fs})_{\text{torsion}}(\tau_m)_{\text{torsion}}\right]^2 \right\}^{1/2}$$

(6–56)

Static Check for Combination Loading

- Distortion Energy theory still applies for check of static yielding
- Obtain von Mises stress for maximum stresses (sum of midrange and alternating)
- Stress concentration factors are not necessary to check for yielding at first cycle

$$\sigma'_{\max} = \left[\left(\sigma_a + \sigma_m \right)^2 + 3\left(\tau_a + \tau_m \right)^2 \right]^{1/2}$$

$$n_y = \frac{S_y}{\sigma'_{\max}}$$

- Alternate simple check is to obtain conservative estimate of $\sigma'_{\max}$ by summing σ'_a and σ'_m

$$\sigma'_{\max} \approx \sigma'_a + \sigma'_m$$

Example 6-14

A rotating shaft is made of 42- × 4-mm AISI 1018 cold-drawn steel tubing and has a 6-mm-diameter hole drilled transversely through it. Estimate the factor of safety guarding against fatigue and static failures using the Gerber and Langer failure criteria for the following loading conditions:

(*a*) The shaft is subjected to a completely reversed torque of 120 N · m in phase with a completely reversed bending moment of 150 N · m.

(*b*) The shaft is subjected to a pulsating torque fluctuating from 20 to 160 N · m and a steady bending moment of 150 N · m.

Example 6-14

Here we follow the procedure of estimating the strengths and then the stresses, followed by relating the two.

From Table A–20 we find the minimum strengths to be $S_{ut} = 440$ MPa and $S_y = 370$ MPa. The endurance limit of the rotating-beam specimen is $0.5(440) = 220$ MPa. The surface factor, obtained from Eq. (6–19) and Table 6–2, p. 287, is

$$k_a = 4.51 S_{ut}^{-0.265} = 4.51(440)^{-0.265} = 0.899$$

From Eq. (6–20) the size factor is

$$k_b = \left(\frac{d}{7.62}\right)^{-0.107} = \left(\frac{42}{7.62}\right)^{-0.107} = 0.833$$

The remaining Marin factors are all unity, so the modified endurance strength S_e is

$$S_e = 0.899(0.833)220 = 165 \text{ MPa}$$

Example 6-14

(a) Theoretical stress-concentration factors are found from Table A–16. Using $a/D = 6/42 = 0.143$ and $d/D = 34/42 = 0.810$, and using linear interpolation, we obtain $A = 0.798$ and $K_t = 2.366$ for bending; and $A = 0.89$ and $K_{ts} = 1.75$ for torsion. Thus, for bending,

$$Z_{\text{net}} = \frac{\pi A}{32D}(D^4 - d^4) = \frac{\pi(0.798)}{32(42)}[(42)^4 - (34)^4] = 3.31\,(10^3)\text{mm}^3$$

and for torsion

$$J_{\text{net}} = \frac{\pi A}{32}(D^4 - d^4) = \frac{\pi(0.89)}{32}[(42)^4 - (34)^4] = 155\,(10^3)\text{mm}^4$$

Next, using Figs. 6–20 and 6–21, pp. 295–296, with a notch radius of 3 mm we find the notch sensitivities to be 0.78 for bending and 0.81 for torsion. The two corresponding fatigue stress-concentration factors are obtained from Eq. (6–32) as

$$K_f = 1 + q(K_t - 1) = 1 + 0.78(2.366 - 1) = 2.07$$

$$K_{fs} = 1 + 0.81(1.75 - 1) = 1.61$$

Example 6-14

The alternating bending stress is now found to be

$$\sigma_{xa} = K_f \frac{M}{Z_{\text{net}}} = 2.07 \frac{150}{3.31(10^{-6})} = 93.8(10^6)\text{Pa} = 93.8 \text{ MPa}$$

and the alternating torsional stress is

$$\tau_{xya} = K_{fs} \frac{TD}{2J_{\text{net}}} = 1.61 \frac{120(42)(10^{-3})}{2(155)(10^{-9})} = 26.2(10^6)\text{Pa} = 26.2 \text{ MPa}$$

The midrange von Mises component σ'_m is zero. The alternating component σ'_a is given by

$$\sigma'_a = \left(\sigma_{xa}^2 + 3\tau_{xya}^2\right)^{1/2} = [93.8^2 + 3(26.2^2)]^{1/2} = 104.2 \text{ MPa}$$

Since $S_e = S_a$, the fatigue factor of safety n_f is

$$n_f = \frac{S_a}{\sigma'_a} = \frac{165}{104.2} = 1.58$$

Example 6-14

The first-cycle yield factor of safety is

$$n_y = \frac{S_y}{\sigma'_a} = \frac{370}{105.6} = 3.50$$

There is no localized yielding; the threat is from fatigue. See Fig. 6–32.

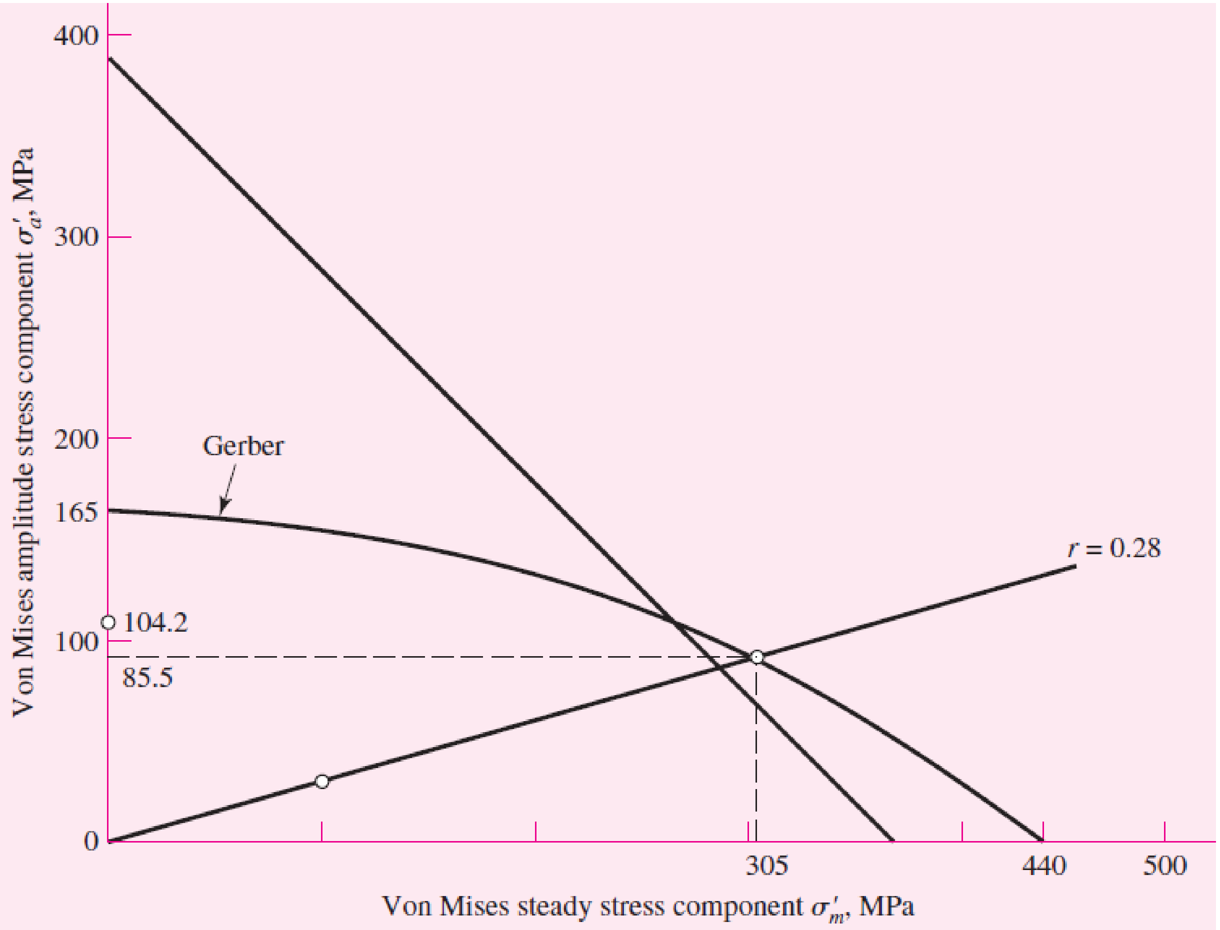


Fig. 6–32

Example 6-14

(*b*) This part asks us to find the factors of safety when the alternating component is due to pulsating torsion, and a steady component is due to both torsion and bending. We have $T_a = (160 - 20)/2 = 70$ N · m and $T_m = (160 + 20)/2 = 90$ N · m. The corresponding amplitude and steady-stress components are

$$\tau_{xya} = K_{fs}\frac{T_a D}{2J_{\text{net}}} = 1.61\frac{70(42)(10^{-3})}{2(155)(10^{-9})} = 15.3(10^6)\text{Pa} = 15.3 \text{ MPa}$$

$$\tau_{xym} = K_{fs}\frac{T_m D}{2J_{\text{net}}} = 1.61\frac{90(42)(10^{-3})}{2(155)(10^{-9})} = 19.7(10^6)\text{Pa} = 19.7 \text{ MPa}$$

The steady bending stress component σ_{xm} is

$$\sigma_{xm} = K_f\frac{M_m}{Z_{\text{net}}} = 2.07\frac{150}{3.31(10^{-6})} = 93.8(10^6)\text{Pa} = 93.8 \text{ MPa}$$

The von Mises components σ'_a and σ'_m are

$$\sigma'_a = [3(15.3)^2]^{1/2} = 26.5 \text{ MPa}$$

$$\sigma'_m = [93.8^2 + 3(19.7)^2]^{1/2} = 99.8 \text{ MPa}$$

Example 6-14

From Table 6–7, p. 307, the fatigue factor of safety is

$$n_f = \frac{1}{2}\left(\frac{440}{99.8}\right)^2 \frac{26.5}{165}\left\{-1 + \sqrt{1 + \left[\frac{2(99.8)165}{440(26.5)}\right]^2}\right\} = 3.12$$

From the same table, with $r = \sigma'_a/\sigma'_m = 26.5/99.8 = 0.28$, the strengths can be shown to be $S_a = 85.5$ MPa and $S_m = 305$ MPa. See the plot in Fig. 6–32.

The first-cycle yield factor of safety n_y is

$$n_y = \frac{S_y}{\sigma'_a + \sigma'_m} = \frac{370}{26.5 + 99.8} = 2.93$$

There is no notch yielding. The likelihood of failure may first come from first-cycle yielding at the notch. See the plot in Fig. 6–32.

Varying Fluctuating Stresses

- Loading patterns may be complex
- Simplifications may be necessary
- Small fluctuations may be negligible compared to large cycles

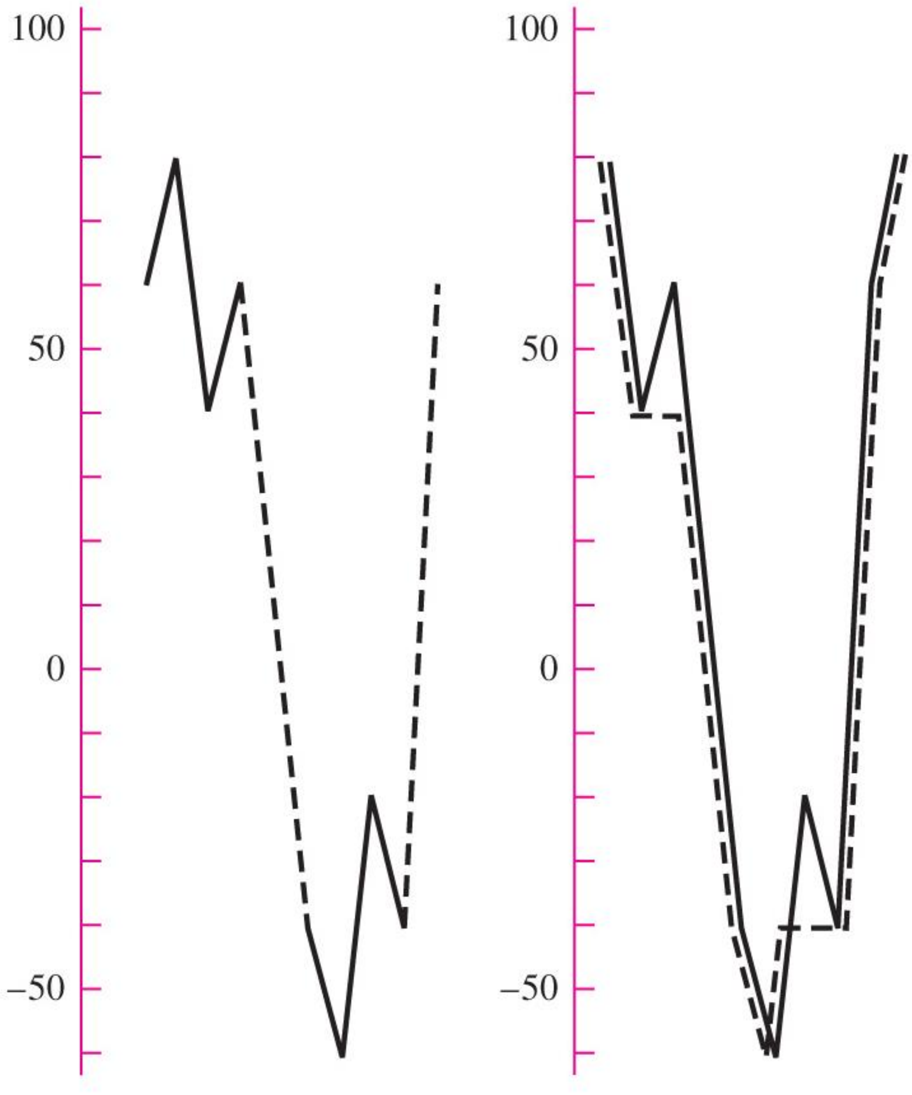


Fig. 6–33 (*a*) (*b*)

Cumulative Fatigue Damage

- A common situation is to load at σ_1 for n_1 cycles, then at σ_2 for n_2 cycles, etc.
- The cycles at each stress level contributes to the fatigue damage
- Accumulation of damage is represented by the *Palmgren-Miner cycle-ratio summation rule,* also known as *Miner's rule*

$$\sum \frac{n_i}{N_i} = c \qquad (6\text{–}57)$$

where n_i is the number of cycles at stress level σ_i and N_i is the number of cycles to failure at stress level σ_i

- c is experimentally found to be in the range $0.7 < c < 2.2$, with an average value near unity
- Defining D as the accumulated damage,

$$D = \sum \frac{n_i}{N_i} \qquad (6\text{–}58)$$

Example 6-15

Given a part with $S_{ut} = 151$ kpsi and at the critical location of the part, $S_e = 67.5$ kpsi. For the loading of Fig. 6–33, estimate the number of repetitions of the stress-time block in Fig. 6–33 that can be made before failure.

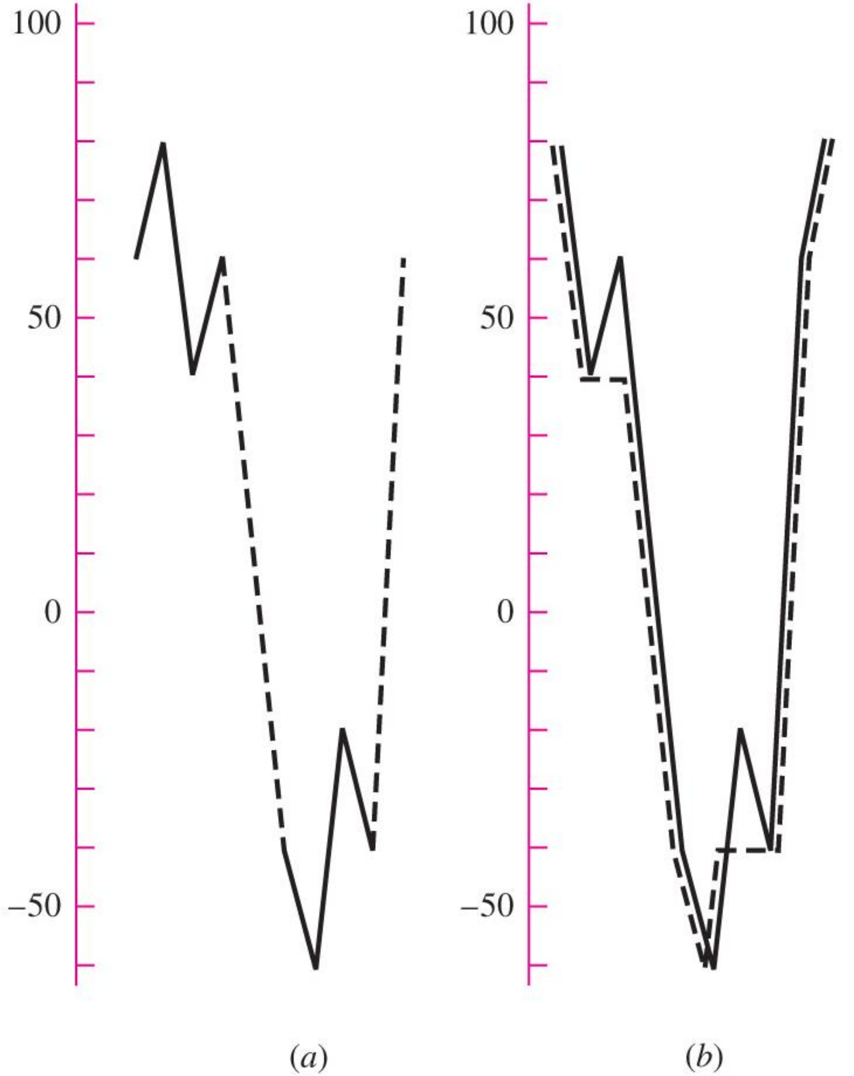


Fig. 6–33

Example 6-15

From Fig. 6–18, p. 285, for $S_{ut} = 151$ kpsi, $f = 0.795$. From Eq. (6–14),

$$a = \frac{(f S_{ut})^2}{S_e} = \frac{[0.795(151)]^2}{67.5} = 213.5 \text{ kpsi}$$

From Eq. (6–15),

$$b = -\frac{1}{3}\log\left(\frac{f S_{ut}}{S_e}\right) = -\frac{1}{3}\log\left[\frac{0.795(151)}{67.5}\right] = -0.0833$$

So,

$$S_f = 213.5 N^{-0.0833} \qquad N = \left(\frac{S_f}{213.5}\right)^{-1/0.0833} \qquad (1), (2)$$

We prepare to add two columns to the previous table. Using the Gerber fatigue criterion, Eq. (6–47), p. 306, with $S_e = S_f$, and $n = 1$, we can write

$$S_f = \begin{cases} \dfrac{\sigma_a}{1 - (\sigma_m / S_{ut})^2} & \sigma_m > 0 \\ S_e & \sigma_m \le 0 \end{cases} \qquad (3)$$

where S_f is the fatigue strength associated with a completely reversed stress, σ_{rev}, equivalent to the fluctuating stresses [see Ex. 6–12, part (*b*)].

Example 6-15

Cycle 1: $r = \sigma_a/\sigma_m = 70/10 = 7$, and the strength amplitude from Table 6–7, p. 307, is

$$S_a = \frac{7^2 151^2}{2(67.5)}\left\{-1 + \sqrt{1 + \left[\frac{2(67.5)}{7(151)}\right]^2}\right\} = 67.2 \text{ kpsi}$$

Since $\sigma_a > S_a$, that is, $70 > 67.2$, life is reduced. From Eq. (3),

$$S_f = \frac{70}{1 - (10/151)^2} = 70.3 \text{ kpsi}$$

and from Eq. (2)

$$N = \left(\frac{70.3}{213.5}\right)^{-1/0.0833} = 619(10^3) \text{ cycles}$$

Example 6-15

Cycle 2: $r = 10/50 = 0.2$, and the strength amplitude is

$$S_a = \frac{0.2^2 151^2}{2(67.5)}\left\{-1 + \sqrt{1 + \left[\frac{2(67.5)}{0.2(151)}\right]^2}\right\} = 24.2 \text{ kpsi}$$

Since $\sigma_a < S_a$, that is $10 < 24.2$, then $S_f = S_e$ and indefinite life follows. Thus, $N \rightarrow \infty$.

Cycle 3: $r = 10/-30 = -0.333$, and since $\sigma_m < 0$, $S_f = S_e$, indefinite life follows and $N \rightarrow \infty$

Cycle Number	S_f, kpsi	N, cycles
1	70.3	$619(10^3)$
2	67.5	∞
3	67.5	∞

Example 6-15

From Eq. (6–58) the damage per block is

$$D = \sum \frac{n_i}{N_i} = N\left[\frac{1}{619(10^3)} + \frac{1}{\infty} + \frac{1}{\infty}\right] = \frac{N}{619(10^3)}$$

Setting $D = 1$ yields $N = 619(10^3)$ cycles.

Illustration of Miner's Rule

- Figure 6–34 illustrates effect of Miner's rule on endurance limit and fatigue failure line.
- Note that the damaged material line is predicted to be parallel to original material line.

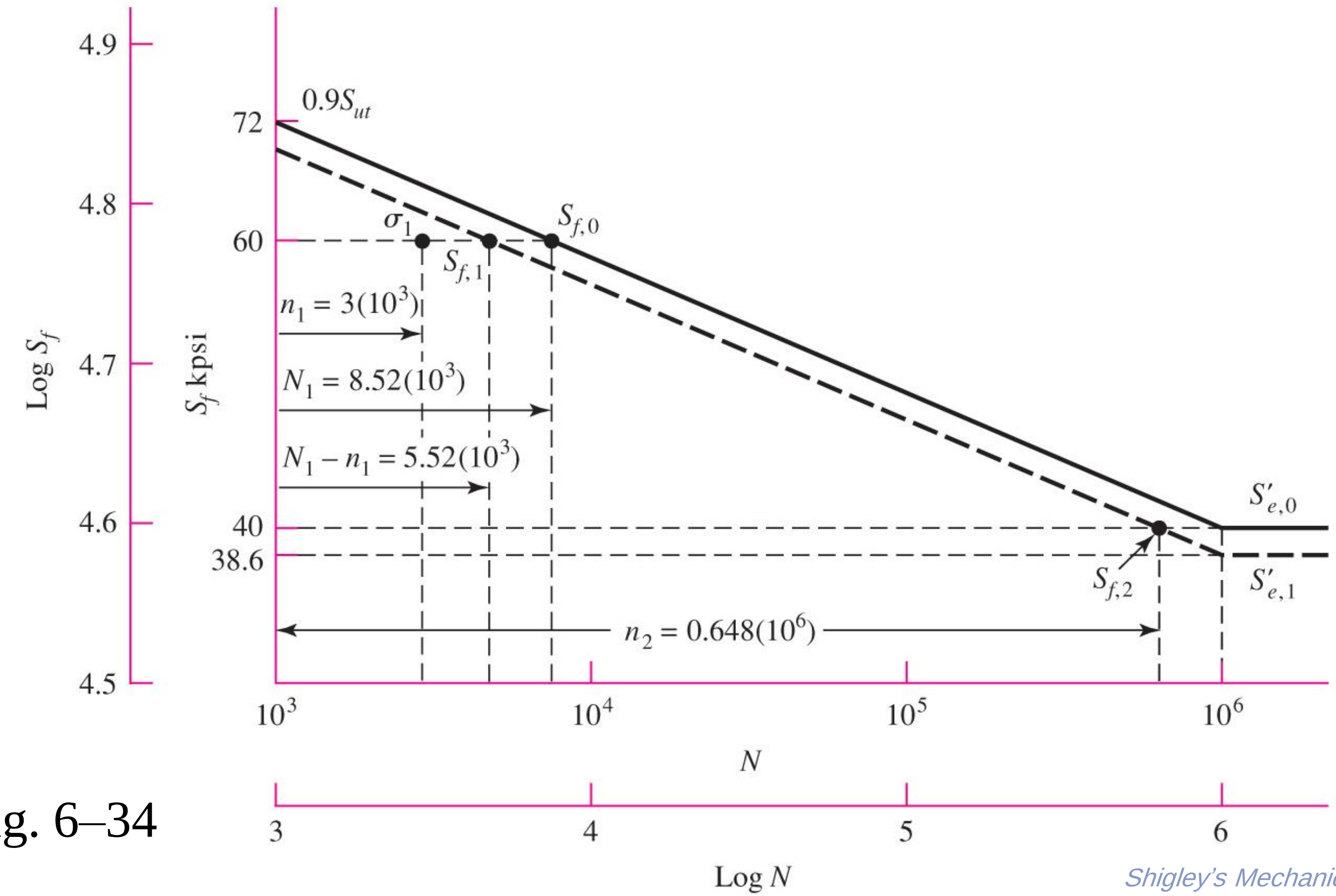


Fig. 6–34

Weaknesses of Miner's Rule

- Miner's rule fails to agree with experimental results in two ways
  - It predicts the static strength S_{ut} is damaged.
  - It does not account for the order in which the stresses are applied

Manson's Method

- Manson's method overcomes deficiencies of Miner's rule.
- It assumes all fatigue lines on the *S*-*N* diagram converge to a common point at $0.9S_{ut}$ at 10^3 cycles.
- It requires each line to be constructed in the same historical order in which the stresses occur.

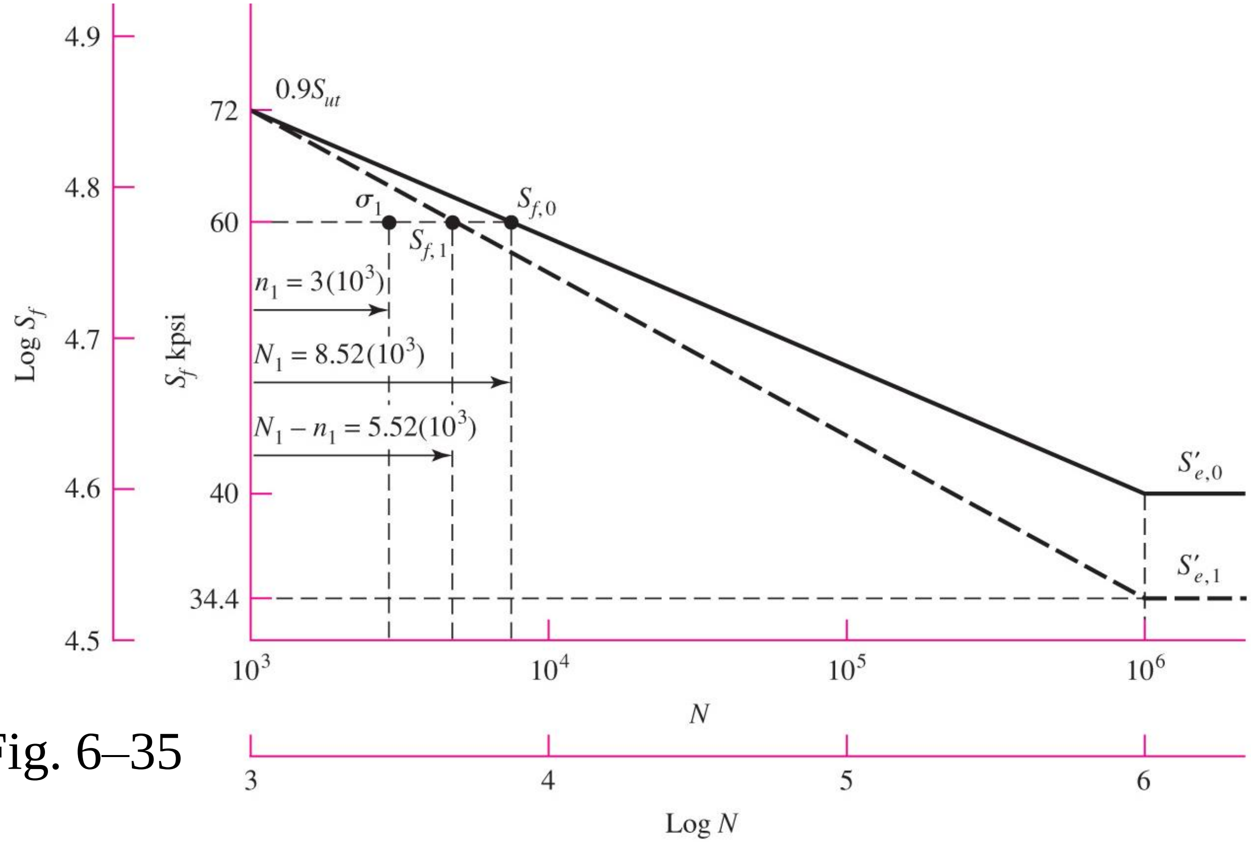


Fig. 6–35

Surface Fatigue Strength

- When two surfaces roll or roll and slide against one another, a pitting failure may occur after a certain number of cycles.
- The surface fatigue mechanism is complex and not definitively understood.
- Factors include Hertz stresses, number of cycles, surface finish, hardness, lubrication, and temperature

Surface Fatigue Strength

- From Eqs. (3–73) and (3–74), the pressure in contacting cylinders,

$$b = \sqrt{\frac{2F}{\pi l} \frac{(1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2}{(1/d_1) + (1/d_2)}} \quad (6\text{–}59)$$

$$p_{\max} = \frac{2F}{\pi b l} \quad (6\text{–}60)$$

- Converting to radius *r* and width *w* instead of length *l*,

$$b^2 = \frac{4F}{\pi w} \frac{\left(1-\nu_1^2\right)/E_1 + \left(1-\nu_2^2\right)/E_2}{1/r_1 + 1/r_2} \quad (6\text{–}61)$$

$$p_{\max} = \frac{2F}{\pi b w} \quad (6\text{–}62)$$

- Define p_{max} as *surface endurance strength* (also called contact strength, contact fatigue strength, or Hertzian endurance strength)

$$S_C = \frac{2F}{\pi b w} \quad (6\text{–}63)$$

Surface Fatigue Strength

- Combining Eqs. (6–61) and (6–63),

$$\frac{F}{w}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)=\pi S_C^2\left[\frac{1-\nu_1^2}{E_1}+\frac{1-\nu_2^2}{E_2}\right]=K_1 \qquad (6\text{–}64)$$

- K_1 is known as *Buckingham's load-stress factor*, or *wear factor*
- In gear studies, a similar factor is used,

$$K_g=\frac{K_1}{4}\sin\phi \qquad (6\text{–}65)$$

- From Eq. (6–64), with material property terms incorporated into an elastic coefficient C_P

$$S_C=C_P\sqrt{\frac{F}{w}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)} \qquad (6\text{–}66)$$

Surface Fatigue Strength

- Experiments show the following relationships

$$K_1 = \alpha_1 N^{\beta_1} \qquad K_g = a N^b \qquad S_C = \alpha N^\beta$$

$$\beta_1 = \frac{\log(K_1/K_2)}{\log(N_1/N_2)} \qquad b = \frac{\log(K_{g1}/K_{g2})}{\log(N_1/N_2)} \qquad \beta = \frac{\log(S_{C1}/S_{C2})}{\log(N_1/N_2)} \qquad (6\text{–}67)$$

Data on induction-hardened steel on steel give $(S_C)_{10^7} = 271$ kpsi and $(S_C)_{10^8} =$ 239 kpsi, so β, from Eq. (6–67), is

$$\beta = \frac{\log(271/239)}{\log(10^7/10^8)} = -0.055$$

Surface Fatigue Strength

- A longstanding correlation in steels between S_C and H_B at 10^8 cycles is

$$(S_C)_{10^8} = \begin{cases} 0.4H_B - 10 \text{ kpsi} \\ 2.76H_B - 70 \text{ MPa} \end{cases} \tag{6–68}$$

- AGMA uses

$$_{0.99}(S_C)_{10^7} = 0.327H_B + 26 \text{ kpsi} \tag{6–69}$$

Surface Fatigue Strength

- Incorporating design factor into Eq. (6–66),

$$\sigma_C = C_P \sqrt{\frac{F}{w n_d}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)} = \frac{C_P}{\sqrt{n_d}} \sqrt{\frac{F}{w}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)} = \frac{S_C}{\sqrt{n_d}}$$

- Since this is nonlinear in its stress-load transformation, the definition of n_d depends on whether load or stress is the primary consideration for failure.
- If the loss of function is focused on the load,

$$n_d = (S_C/\sigma_C)^2$$

- If the loss of function is focused on the stress,

$$n_d = S_C/\sigma_C$$

Stochastic Analysis

- *Fatigue ratio*

$$\boldsymbol{\phi} = \mathbf{S}'_e / \bar{S}_{ut}$$

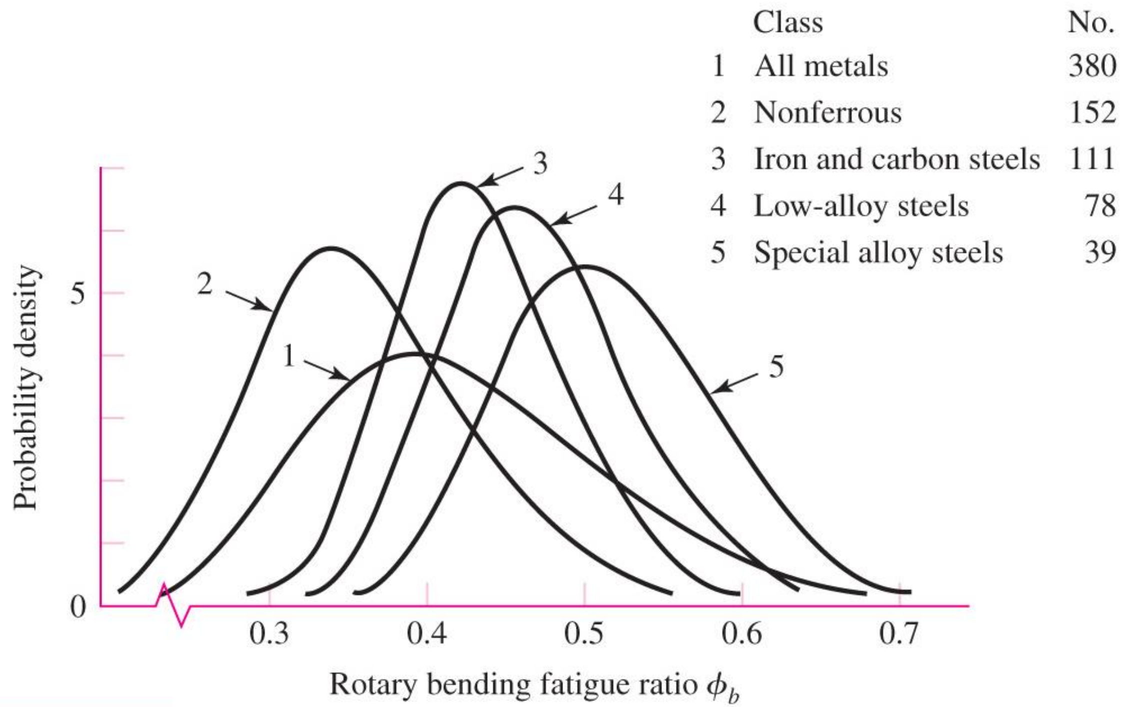


Fig. 6–36

Table 6–9

Material Class	$\bar{\phi}_{0.30}$
Wrought steels	0.50
Cast steels	0.40
Powdered steels	0.38
Gray cast iron	0.35
Malleable cast iron	0.40
Normalized nodular cast iron	0.33

Stochastic Analysis

- Endurance Limit

$$\mathbf{S}'_e = \begin{cases} 0.506\bar{S}_{ut}\mathbf{LN}(1, 0.138) \text{ kpsi or MPa} & \bar{S}_{ut} \le 212 \text{ kpsi (1460 MPa)} \\ 107\mathbf{LN}(1, 0.139) \text{ kpsi} & \bar{S}_{ut} > 212 \text{ kpsi} \\ 740\mathbf{LN}(1, 0.139) \text{ MPa} & \bar{S}_{ut} > 1460 \text{ MPa} \end{cases} \quad (6\text{–}70)$$

$$\mathbf{S}_e = \mathbf{k}_a k_b \mathbf{k}_c \mathbf{k}_d \mathbf{k}_f \mathbf{S}'_e \quad (6\text{–}71)$$

Stochastic Analysis

- Surface Factor

$$\mathbf{k}_a = a\bar{S}_{ut}^{b}\mathbf{LN}(1, C) \qquad (\bar{S}_{ut} \text{ in kpsi or MPa}) \qquad (6\text{–}72)$$

Table 6–10

Parameters in Marin Surface Condition Factor

$\mathbf{k}_a = aS_{ut}^b\,\mathbf{LN}(1, C)$				
	a			
Surface Finish	**kpsi**	**MPa**	***b***	**Coefficient of Variation, *C***
Ground*	1.34	1.58	−0.086	0.120
Machined or Cold-rolled	2.67	4.45	−0.265	0.058
Hot-rolled	14.5	58.1	−0.719	0.110
As-forged	39.8	271	−0.995	0.145

*Due to the wide scatter in ground surface data, an alternate function is $\mathbf{k_a} = 0.878\mathbf{LN}(1, 0.120)$.
Note: S_{ut} in kpsi or MPa.

Example 6-16

A steel has a mean ultimate strength of 520 MPa and a machined surface. Estimate $\mathbf{k}_a$.

Solution

From Table 6–10,

$$\mathbf{k}_a = 4.45(520)^{-0.265}\mathbf{LN}(1, 0.058)$$

$$\bar{k}_a = 4.45(520)^{-0.265}(1) = 0.848$$

$$\hat{\sigma}_{ka} = C\bar{k}_a = (0.058)4.45(520)^{-0.265} = 0.049$$

so $\mathbf{k}_a = \mathbf{LN}(0.848, 0.049)$.

Stochastic Analysis

- Size factor, k_b
- Use same deterministic approach as before

Stochastic Analysis

- Load factor

$$(\mathbf{k}_c)_{\text{axial}} = 1.23\bar{S}_{ut}^{-0.0778}\mathbf{LN}(1, 0.125) \quad (6\text{–}73)$$

$$(\mathbf{k}_c)_{\text{torsion}} = 0.328\bar{S}_{ut}^{0.125}\mathbf{LN}(1, 0.125) \quad (6\text{–}74)$$

Table 6–11

Parameters in Marin Loading Factor

Mode of Loading	α kpsi	α MPa	β	C	Average k_c
	$\mathbf{k}_c = \alpha\bar{S}_{ut}^{-\beta}\,\mathbf{LN}(1, C)$				
Bending	1	1	0	0	1
Axial	1.23	1.43	−0.0778	0.125	0.85
Torsion	0.328	0.258	0.125	0.125	0.59

Stochastic Analysis

Table 6–12

Average Marin Loading Factor for Axial Load

$\bar{S}_{ut}$, kpsi	k_c^*
50	0.907
100	0.860
150	0.832
200	0.814

*Average entry 0.85.

Table 6–13

Average Marin Loading Factor for Torsional Load

$\bar{S}_{ut}$, kpsi	k_c^*
50	0.535
100	0.583
150	0.614
200	0.636

*Average entry 0.59.

Table 6–14

Average Marin Torsional Loading Factor k_c for Several Materials

Material	Range	n	$\bar{k}_c$	$\hat{\sigma}_{kc}$
Wrought steels	0.52–0.69	31	0.60	0.03
Wrought Al	0.43–0.74	13	0.55	0.09
Wrought Cu and alloy	0.41–0.67	7	0.56	0.10
Wrought Mg and alloy	0.49–0.60	2	0.54	0.08
Titanium	0.37–0.57	3	0.48	0.12
Cast iron	0.79–1.01	9	0.90	0.07
Cast Al, Mg, and alloy	0.71–0.91	5	0.85	0.09

Example 6-17

Estimate the Marin loading factor $\mathbf{k}_c$ for a 1–in-diameter bar that is used as follows.

(a) In bending. It is made of steel with $\mathbf{S}_{ut} = 100\mathbf{LN}(1, 0.035)$ kpsi, and the designer intends to use the correlation $\mathbf{S}'_e = \boldsymbol{\phi}_{0.30}\bar{S}_{ut}$ to predict $\mathbf{S}'_e$.

(b) In bending, but endurance testing gave $\mathbf{S}'_e = 55\mathbf{LN}(1, 0.081)$ kpsi.

(c) In push-pull (axial) fatigue, $\mathbf{S}_{ut} = \mathbf{LN}(86.2, 3.92)$ kpsi, and the designer intended to use the correlation $\mathbf{S}'_e = \boldsymbol{\phi}_{0.30}\bar{S}_{ut}$.

(d) In torsional fatigue. The material is cast iron, and $\mathbf{S}'_e$ is known by test.

Example 6-17

Solution (*a*) Since the bar is in bending,

Answer
$$\mathbf{k}_c = (1, 0)$$

(*b*) Since the test is in bending and use is in bending,

Answer
$$\mathbf{k}_c = (1, 0)$$

(*c*) From Eq. (6–73),

Answer
$$(\mathbf{k}_c)_{ax} = 1.23(86.2)^{-0.0778}\mathbf{LN}(1, 0.125)$$

$$\bar{k}_c = 1.23(86.2)^{-0.0778}(1) = 0.870$$

$$\hat{\sigma}_{kc} = C\bar{k}_c = 0.125(0.870) = 0.109$$

(*d*) From Table 6–15, $\bar{k}_c = 0.90$, $\hat{\sigma}_{kc} = 0.07$, and

Answer
$$C_{kc} = \frac{0.07}{0.90} = 0.08$$

Stochastic Analysis

- Temperature factor

$$\mathbf{k}_d = \bar{k}_d \mathbf{LN}(1, 0.11) \qquad (6\text{–}75)$$

Stochastic Analysis

- Stress concentration and Notch Sensitivity

$$\mathbf{q} = \frac{\mathbf{K}_f - 1}{K_t - 1} \qquad (6\text{–}76)$$

$$\mathbf{q} = \mathbf{LN}\left(\frac{\bar{K}_f - 1}{K_t - 1}, \frac{C\bar{K}_f}{K_t - 1}\right)$$

$$\bar{q} = \frac{\bar{K}_f - 1}{K_t - 1}$$

$$\hat{\sigma}_q = \frac{C\bar{K}_f}{K_t - 1} \qquad (6\text{–}77)$$

$$C_q = \frac{C\bar{K}_f}{\bar{K}_f - 1}$$

Stochastic Analysis

- Stress concentration and Notch Sensitivity

$$\bar{K}_f = \frac{K_t}{1 + \dfrac{2(K_t - 1)}{K_t} \dfrac{\sqrt{a}}{\sqrt{r}}} \tag{6–78}$$

$$\mathbf{K}_f = \bar{K}_f \mathbf{LN}\left(1, C_{K_f}\right) \tag{6–79}$$

Example 6-18

Estimate $\mathbf{K}_f$ and $\mathbf{q}$ for the steel shaft given in Ex. 6–6, p. 296.

Solution

From Ex. 6–6, a steel shaft with $S_{ut} = 690$ MPa and a shoulder with a fillet of 3 mm was found to have a theoretical stress-concentration-factor of $K_t \doteq 1.65$. From Table 6–15,

$$\sqrt{a} = \frac{139}{S_{ut}} = \frac{139}{690} = 0.2014\sqrt{\text{mm}}$$

From Eq. (6–78),

$$K_f = \frac{K_t}{1 + \dfrac{2(K_t - 1)}{K_t}\dfrac{\sqrt{a}}{\sqrt{r}}} = \frac{1.65}{1 + \dfrac{2(1.65 - 1)}{1.65}\dfrac{0.2014}{\sqrt{3}}} = 1.51$$

which is 2.5 percent lower than what was found in Ex. 6–6.

Example 6-18

From Table 6–15, $C_{Kf} = 0.11$. Thus from Eq. (6–79),

Answer
$$\mathbf{K}_f = 1.51\ \mathbf{LN}(1, 0.11)$$

From Eq. (6–77), with $K_t = 1.65$

$$\bar{q} = \frac{1.51 - 1}{1.65 - 1} = 0.785$$

$$C_q = \frac{C_{K_f}\bar{K}_f}{\bar{K}_f - 1} = \frac{0.11(1.51)}{1.51 - 1} = 0.326$$

$$\hat{\sigma}_q = C_q\bar{q} = 0.326(0.785) = 0.256$$

So,

Answer
$$\mathbf{q} = \mathbf{LN}(0.785, 0.256)$$

Example 6-19

The bar shown in Fig. 6–37 is machined from a cold-rolled flat having an ultimate strength of $\mathbf{S}_{ut} = \mathbf{LN}(87.6, 5.74)$ kpsi. The axial load shown is completely reversed. The load amplitude is $\mathbf{F}_a = \mathbf{LN}(1000, 120)$ lbf.

(*a*) Estimate the reliability.

(*b*) Reestimate the reliability when a rotating bending endurance test shows that $\mathbf{S}'_e = \mathbf{LN}(40, 2)$ kpsi.

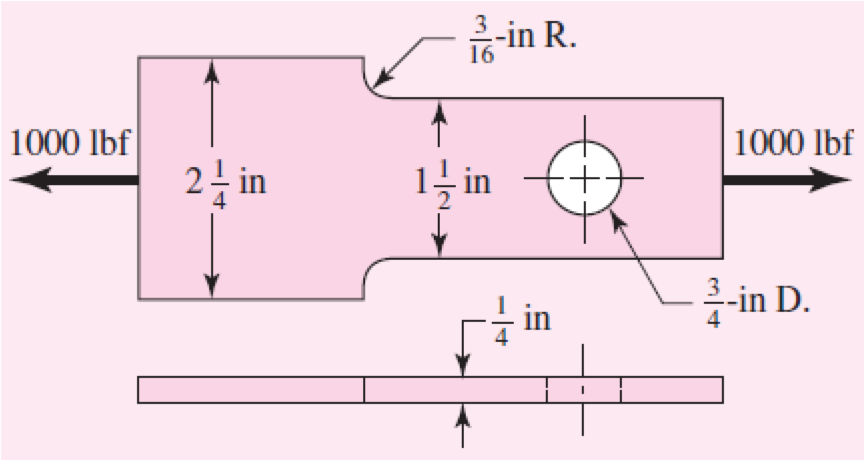


Fig. 6–37

Example 6-19

(a) From Eq. (6–70), $\mathbf{S}'_e = 0.506\bar{S}_{ut}\mathbf{LN}(1, 0.138) = 0.506(87.6)\mathbf{LN}(1, 0.138)$

$$= 44.3\mathbf{LN}(1, 0.138) \text{ kpsi}$$

From Eq. (6–72) and Table 6–10,

$$\mathbf{k}_a = 2.67\bar{S}_{ut}^{-0.265}\mathbf{LN}(1, 0.058) = 2.67(87.6)^{-0.265}\mathbf{LN}(1, 0.058)$$

$$= 0.816\mathbf{LN}(1, 0.058)$$

$$k_b = 1 \qquad \text{(axial loading)}$$

From Eq. (6–73),

$$\mathbf{k}_c = 1.23\bar{S}_{ut}^{-0.0778}\mathbf{LN}(1, 0.125) = 1.23(87.6)^{-0.0778}\mathbf{LN}(1, 0.125)$$

$$= 0.869\mathbf{LN}(1, 0.125)$$

$$\mathbf{k}_d = \mathbf{k}_f = (1, 0)$$

Example 6-19

The endurance strength, from Eq. (6–71), is

$$\mathbf{S}_e = \mathbf{k}_a k_b \mathbf{k}_c \mathbf{k}_d \mathbf{k}_f \mathbf{S}'_e$$

$$\mathbf{S}_e = 0.816\mathbf{LN}(1, 0.058)(1)0.869\mathbf{LN}(1, 0.125)(1)(1)44.3\mathbf{LN}(1, 0.138)$$

The parameters of $\mathbf{S}_e$ are

$$\bar{S}_e = 0.816(0.869)44.3 = 31.4 \text{ kpsi}$$

$$C_{Se} = (0.058^2 + 0.125^2 + 0.138^2)^{1/2} = 0.195$$

so $\mathbf{S}_e = 31.4\mathbf{LN}(1, 0.195)$ kpsi.

In computing the stress, the section at the hole governs. Using the terminology of Table A–15–1 we find $d/w = 0.50$, therefore $K_t \doteq 2.18$. From Table 6–15, $\sqrt{a} = 5/S_{ut} = 5/87.6 = 0.0571$ and $C_{kf} = 0.10$. From Eqs. (6–78) and (6–79) with $r = 0.375$ in,

$$\mathbf{K}_f = \frac{K_t}{1 + \dfrac{2(K_t - 1)}{K_t}\dfrac{\sqrt{a}}{\sqrt{r}}}\mathbf{LN}\left(1, C_{K_f}\right) = \frac{2.18}{1 + \dfrac{2(2.18 - 1)}{2.18}\dfrac{0.0571}{\sqrt{0.375}}}\mathbf{LN}(1, 0.10)$$

$$= 1.98\mathbf{LN}(1, 0.10)$$

Example 6-19

The stress at the hole is

$$\boldsymbol{\sigma} = \mathbf{K}_f \frac{\mathbf{F}}{A} = 1.98\mathbf{LN}(1, 0.10)\frac{1000\mathbf{LN}(1, 0.12)}{0.25(0.75)}$$

$$\bar{\sigma} = 1.98\frac{1000}{0.25(0.75)}10^{-3} = 10.56 \text{ kpsi}$$

$$C_\sigma = (0.10^2 + 0.12^2)^{1/2} = 0.156$$

so stress can be expressed as $\boldsymbol{\sigma} = 10.56\mathbf{LN}(1, 0.156)$ kpsi.[34]

Example 6-19

The endurance limit is considerably greater than the load-induced stress, indicating that finite life is not a problem. For interfering lognormal-lognormal distributions, Eq. (5–43), p. 250, gives

$$z = -\frac{\ln\left(\dfrac{\bar{S}_e}{\bar{\sigma}}\sqrt{\dfrac{1+C_\sigma^2}{1+C_{S_e}^2}}\right)}{\sqrt{\ln\left[\left(1+C_{S_e}^2\right)\left(1+C_\sigma^2\right)\right]}} = -\frac{\ln\left(\dfrac{31.4}{10.56}\sqrt{\dfrac{1+0.156^2}{1+0.195^2}}\right)}{\sqrt{\ln[(1+0.195^2)(1+0.156^2)]}} = -4.37$$

From Table A–10 the probability of failure $p_f = \Phi(-4.37) = .000\,006\,35$, and the reliability is

$$R = 1 - 0.000\,006\,35 = 0.999\,993\,65$$

Example 6-19

(*b*) The rotary endurance tests are described by $\mathbf{S}'_e = 40\mathbf{LN}(1, 0.05)$ kpsi whose mean is *less* than the predicted mean in part *a*. The mean endurance strength $\bar{S}_e$ is

$$\bar{S}_e = 0.816(0.869)40 = 28.4 \text{ kpsi}$$

$$C_{Se} = (0.058^2 + 0.125^2 + 0.05^2)^{1/2} = 0.147$$

so the endurance strength can be expressed as $\mathbf{S}_e = 28.3\mathbf{LN}(1, 0.147)$ kpsi. From Eq. (5–43),

$$z = -\frac{\ln\left(\dfrac{28.4}{10.56}\sqrt{\dfrac{1+0.156^2}{1+0.147^2}}\right)}{\sqrt{\ln[(1+0.147^2)(1+0.156^2)]}} = -4.65$$

Example 6-19

Using Table A–10, we see the probability of failure $p_f = \Phi(-4.65) = 0.000\,001\,71$, and

$$R = 1 - 0.000\,001\,71 = 0.999\,998\,29$$

an increase! The reduction in the probability of failure is $(0.000\,001\,71 - 0.000\,006\,35)/0.000\,006\,35 = -0.73$, a reduction of 73 percent. We are analyzing an existing design, so in part (*a*) the factor of safety was $\bar{n} = \bar{S}/\bar{\sigma} = 31.4/10.56 = 2.97$. In part (*b*) $\bar{n} = 28.4/10.56 = 2.69$, a *decrease*. This example gives you the opportunity to see the role of the design factor. Given knowledge of $\bar{S}$, C_S, $\bar{\sigma}$, C_σ, and reliability (through z), the mean factor of safety (as a design factor) separates $\bar{S}$ and $\bar{\sigma}$ so that the reliability goal is achieved. Knowing $\bar{n}$ alone *says nothing about the probability of failure*. Looking at $\bar{n} = 2.97$ and $\bar{n} = 2.69$ says nothing about the respective probabilities of failure. The tests did not reduce $\bar{S}_e$ significantly, but reduced the variation C_S such that the reliability was *increased*.

When a mean design factor (or mean factor of safety) defined as $\bar{S}_e/\bar{\sigma}$ is said to be *silent* on matters of frequency of failures, it means that a scalar factor of safety by itself does not offer any information about probability of failure. Nevertheless, some engineers let the factor of safety speak up, and they can be wrong in their conclusions.

Stochastic Analysis

- Gerber equations

$$\bar{S}_a = \frac{r^2 \bar{S}_{ut}^2}{2\bar{S}_e}\left[-1 + \sqrt{1 + \left(\frac{2\bar{S}_e}{r\bar{S}_{ut}}\right)^2}\,\right] \qquad (6\text{–}80)$$

$$C_{Sa} = \frac{(1 + C_{Sut})^2}{1 + C_{Se}} \frac{\left\{-1 + \sqrt{1 + \left[\frac{2\bar{S}_e(1 + C_{Se})}{r\bar{S}_{ut}(1 + C_{Sut})}\right]^2}\,\right\}}{\left[-1 + \sqrt{1 + \left(\frac{2\bar{S}_e}{r\bar{S}_{ut}}\right)^2}\,\right]} - 1 \qquad (6\text{–}81)$$

Stochastic Analysis

- ASME-elliptic equations

$$\bar{S}_a = \frac{r\bar{S}_y\bar{S}_e}{\sqrt{r^2\bar{S}_y^2 + \bar{S}_e^2}} \qquad (6\text{–}82)$$

$$C_{Sa} = (1 + C_{Sy})(1 + C_{Se})\sqrt{\frac{r^2\bar{S}_y^2 + \bar{S}_e^2}{r^2\bar{S}_y^2(1 + C_{Sy})^2 + \bar{S}_e^2(1 + C_{Se})^2}} - 1 \qquad (6\text{–}83)$$

Stochastic Analysis

- Smith-Dolan equations for brittle materials

$$\frac{n\sigma_a}{S_e} = \frac{1 - n\sigma_m/S_{ut}}{1 + n\sigma_m/S_{ut}} \tag{6–84}$$

$$\frac{\bar{S}_a}{\bar{S}_e} = \frac{1 - \bar{S}_m/\bar{S}_{ut}}{1 + \bar{S}_m/\bar{S}_{ut}} \tag{6–85}$$

$$\bar{S}_a = \frac{r\bar{S}_{ut} + \bar{S}_e}{2}\left[-1 + \sqrt{1 + \frac{4r\bar{S}_{ut}\bar{S}_e}{(r\bar{S}_{ut} + \bar{S}_e)^2}}\,\right] \tag{6–86}$$

$$C_{Sa} = \frac{r\bar{S}_{ut}(1 + C_{Sut}) + \bar{S}_e(1 + C_{Se})}{2\bar{S}_a} \cdot \left\{-1 + \sqrt{1 + \frac{4r\bar{S}_{ut}\bar{S}_e(1 + C_{Se})(1 + C_{Sut})}{[r\bar{S}_{ut}(1 + C_{Sut}) + \bar{S}_e(1 + C_{Se})]^2}}\right\} - 1 \tag{6–87}$$

Example 6-20

A rotating shaft experiences a steady torque $\mathbf{T} = 1360\mathbf{LN}(1, 0.05)$ lbf · in, and at a shoulder with a 1.1-in small diameter, a fatigue stress-concentration factor $\mathbf{K}_f = 1.50\mathbf{LN}(1, 0.11)$, $\mathbf{K}_{fs} = 1.28\mathbf{LN}(1, 0.11)$, and at that location a bending moment of $\mathbf{M} = 1260\mathbf{LN}(1, 0.05)$ lbf · in. The material of which the shaft is machined is hot-rolled 1035 with $\mathbf{S}_{ut} = 86.2\mathbf{LN}(1, 0.045)$ kpsi and $\mathbf{S}_y = 56.0\mathbf{LN}(1, 0.077)$ kpsi. Estimate the reliability using a stochastic Gerber failure zone.

Example 6-20

Establish the endurance strength. From Eqs. (6–70) to (6–72) and Eq. (6–20), p. 288,

$$\mathbf{S}'_e = 0.506(86.2)\mathbf{LN}(1, 0.138) = 43.6\mathbf{LN}(1, 0.138) \text{ kpsi}$$

$$\mathbf{k}_a = 2.67(86.2)^{-0.265}\mathbf{LN}(1, 0.058) = 0.820\mathbf{LN}(1, 0.058)$$

$$k_b = (1.1/0.30)^{-0.107} = 0.870$$

$$\mathbf{k}_c = \mathbf{k}_d = \mathbf{k}_f = \mathbf{LN}(1, 0)$$

$$\mathbf{S}_e = 0.820\mathbf{LN}(1, 0.058)0.870(43.6)\mathbf{LN}(1, 0.138)$$

$$\bar{S}_e = 0.820(0.870)43.6 = 31.1 \text{ kpsi}$$

$$C_{Se} = (0.058^2 + 0.138^2)^{1/2} = 0.150$$

and so $\mathbf{S}_e = 31.1\mathbf{LN}(1, 0.150)$ kpsi.

Example 6-20

Stress (in kpsi):

$$\sigma_a = \frac{32\mathbf{K}_f \mathbf{M}_a}{\pi d^3} = \frac{32(1.50)\mathbf{LN}(1, 0.11)1.26\mathbf{LN}(1, 0.05)}{\pi(1.1)^3}$$

$$\bar{\sigma}_a = \frac{32(1.50)1.26}{\pi(1.1)^3} = 14.5 \text{ kpsi}$$

$$C_{\sigma a} = (0.11^2 + 0.05^2)^{1/2} = 0.121$$

$$\boldsymbol{\tau}_m = \frac{16\mathbf{K}_{fs}\mathbf{T}_m}{\pi d^3} = \frac{16(1.28)\mathbf{LN}(1, 0.11)1.36\mathbf{LN}(1, 0.05)}{\pi(1.1)^3}$$

$$\bar{\tau}_m = \frac{16(1.28)1.36}{\pi(1.1)^3} = 6.66 \text{ kpsi}$$

$$C_{\tau m} = (0.11^2 + 0.05^2)^{1/2} = 0.121$$

$$\bar{\sigma}'_a = \left(\bar{\sigma}_a^2 + 3\bar{\tau}_a^2\right)^{1/2} = [14.5^2 + 3(0)^2]^{1/2} = 14.5 \text{ kpsi}$$

$$\bar{\sigma}'_m = \left(\bar{\sigma}_m^2 + 3\bar{\tau}_m^2\right)^{1/2} = [0 + 3(6.66)^2]^{1/2} = 11.54 \text{ kpsi}$$

$$r = \frac{\bar{\sigma}'_a}{\bar{\sigma}'_m} = \frac{14.5}{11.54} = 1.26$$

Example 6-20

Strength: From Eqs. (6–80) and (6–81),

$$\bar{S}_a = \frac{1.26^2 86.2^2}{2(31.1)}\left\{-1+\sqrt{1+\left[\frac{2(31.1)}{1.26(86.2)}\right]^2}\right\} = 28.9 \text{ kpsi}$$

$$C_{Sa} = \frac{(1+0.045)^2}{1+0.150}\,\frac{-1+\sqrt{1+\left[\dfrac{2(31.1)(1+0.15)}{1.26(86.2)(1+0.045)}\right]^2}}{-1+\sqrt{1+\left[\dfrac{2(31.1)}{1.26(86.2)}\right]^2}} - 1 = 0.134$$

Example 6-20

Reliability: Since $\mathbf{S}_a = 28.9\mathbf{LN}(1, 0.134)$ kpsi and $\boldsymbol{\sigma}'_a = 14.5\mathbf{LN}(1, 0.121)$ kpsi, Eq. (5–43), p. 250, gives

$$z = -\frac{\ln\left(\dfrac{\bar{S}_a}{\bar{\sigma}_a}\sqrt{\dfrac{1+C_{\sigma_a}^2}{1+C_{S_a}^2}}\right)}{\sqrt{\ln\left[\left(1+C_{S_a}^2\right)\left(1+C_{\sigma_a}^2\right)\right]}} = -\frac{\ln\left(\dfrac{28.9}{14.5}\sqrt{\dfrac{1+0.121^2}{1+0.134^2}}\right)}{\sqrt{\ln[(1+0.134^2)(1+0.121^2)]}} = -3.83$$

From Table A–10 the probability of failure is $p_f = 0.000\,065$, and the reliability is, against fatigue,

$$R = 1 - p_f = 1 - 0.000\,065 = 0.999\,935$$

Example 6-20

The chance of first-cycle yielding is estimated by interfering $\mathbf{S}_y$ with $\boldsymbol{\sigma}'_{\max}$. The quantity $\boldsymbol{\sigma}'_{\max}$ is formed from $\boldsymbol{\sigma}'_a + \boldsymbol{\sigma}'_m$. The mean of $\boldsymbol{\sigma}'_{\max}$ is $\bar{\sigma}'_a + \bar{\sigma}'_m = 14.5 + 11.54 = 26.04$ kpsi. The coefficient of variation of the sum is 0.121, since both COVs are 0.121, thus $C_{\sigma\,\max} = 0.121$. We interfere $\mathbf{S}_y = 56\mathbf{LN}(1, 0.077)$ kpsi with $\boldsymbol{\sigma}'_{\max} = 26.04\mathbf{LN}\,(1, 0.121)$ kpsi. The corresponding z variable is

$$z = -\frac{\ln\left(\dfrac{56}{26.04}\sqrt{\dfrac{1+0.121^2}{1+0.077^2}}\right)}{\sqrt{\ln[(1+0.077^2)(1+0.121^2)]}} = -5.39$$

which represents, from Table A–10, a probability of failure of approximately 0.0^7358 [which represents $3.58(10^{-8})$] of first-cycle yield in the fillet.

Example 6-20

The probability of observing a fatigue failure exceeds the probability of a yield failure, something a deterministic analysis does not foresee and in fact could lead one to expect a yield failure should a failure occur. Look at the $\boldsymbol{\sigma}'_a\mathbf{S}_a$ interference and the $\boldsymbol{\sigma}'_{\max}\mathbf{S}_y$ interference and examine the z expressions. These control the relative probabilities. A deterministic analysis is oblivious to this and can mislead. Check your statistics text for events that are not mutually exclusive, but are independent, to quantify the probability of failure:

$$p_f = p(\text{yield}) + p(\text{fatigue}) - p(\text{yield and fatigue})$$

$$= p(\text{yield}) + p(\text{fatigue}) - p(\text{yield})\,p(\text{fatigue})$$

$$= 0.358(10^{-7}) + 0.65(10^{-4}) - 0.358(10^{-7})0.65(10^{-4}) = 0.650(10^{-4})$$

$$R = 1 - 0.650(10^{-4}) = 0.999\,935$$

against either or both modes of failure.

Example 6-20

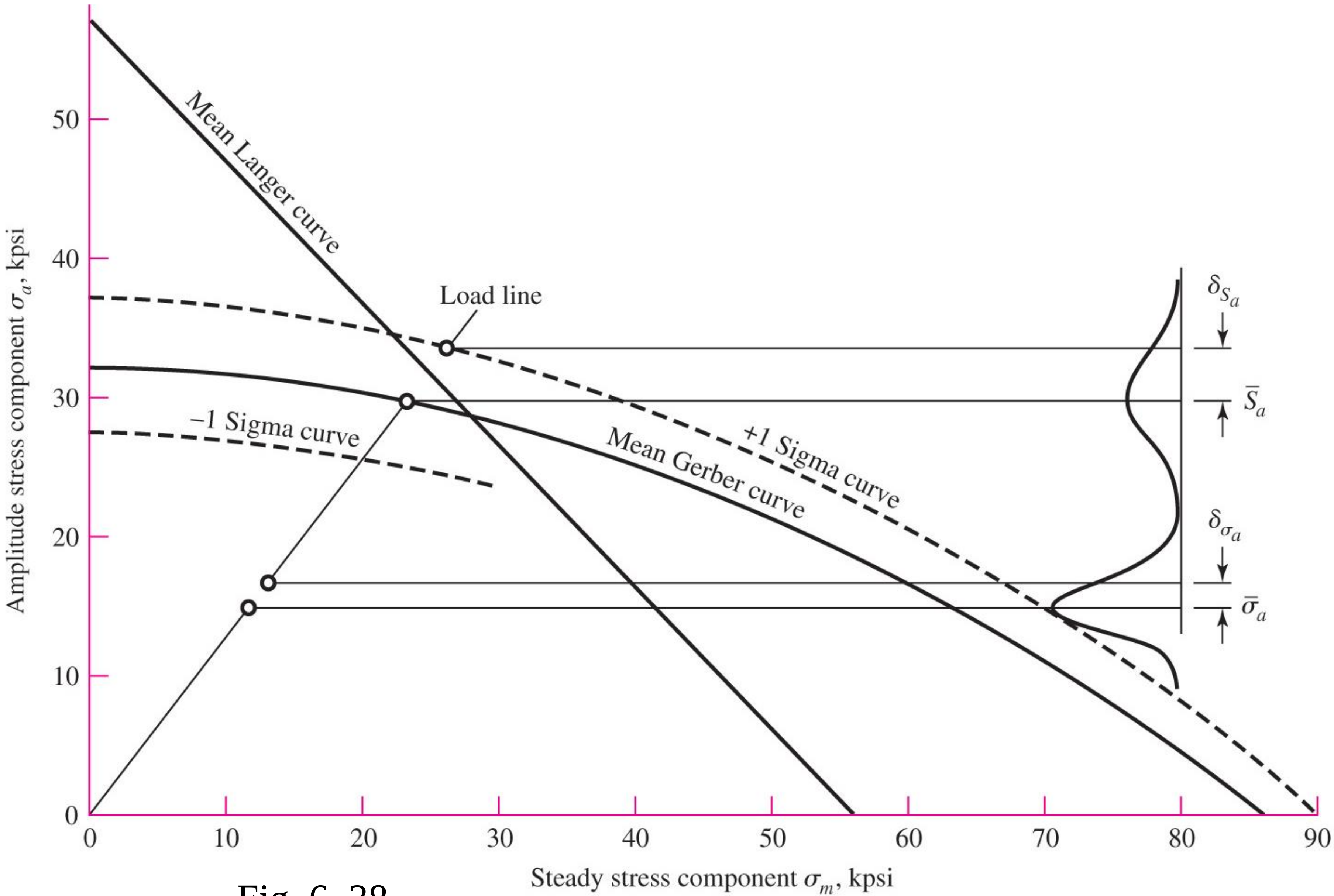


Fig. 6–38

Stochastic Analysis

- Design Factor in Fatigue

$$\bar{n} = \exp\left[-z\sqrt{\ln\left(1 + C_n^2\right)} + \ln\sqrt{1 + C_n^2}\right] \doteq \exp[C_n(-z + C_n/2)] \qquad (6\text{–}88)$$

$$C_n = \sqrt{\frac{C_S^2 + C_\sigma^2}{1 + C_\sigma^2}}$$

$$C_{\sigma'_a} = \left(C_{Kf}^2 + C_F^2\right)^{1/2}$$

$$C_{Se} = \left(C_{ka}^2 + C_{kc}^2 + C_{kd}^2 + C_{kf}^2 + C_{Se'}^2\right)^{1/2}$$

Example 6-21

A strap to be made from a cold-drawn steel strip workpiece is to carry a fully reversed axial load $\mathbf{F} = \mathbf{LN}(1000, 120)$ lbf as shown in Fig. 6–39. Consideration of adjacent parts established the geometry as shown in the figure, except for the thickness t. Make a decision as to the magnitude of the design factor if the reliability goal is to be 0.999 95, then make a decision as to the workpiece thickness t.

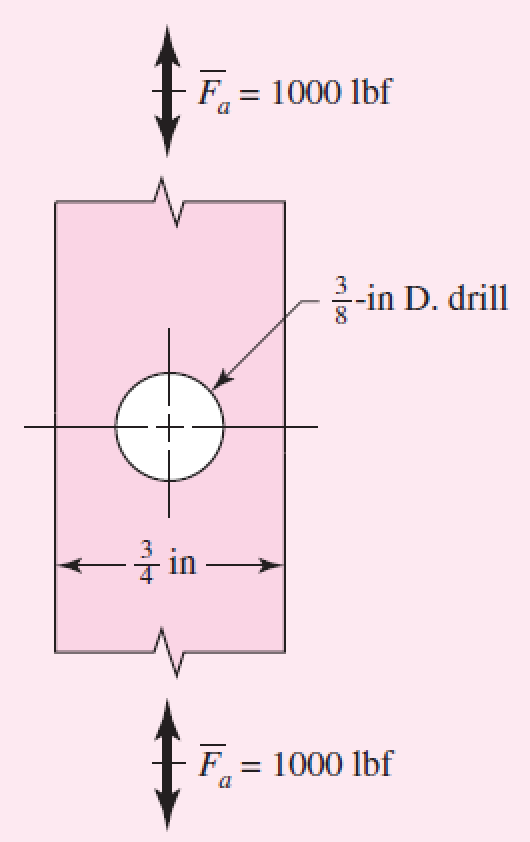


Fig. 6–39

Example 6-21

Let us take each a priori decision and note the consequence:

A Priori Decision	Consequence
Use 1018 CD steel	$\bar{S}_{ut} = 87.6$ kpsi, $C_{Sut} = 0.0655$
Function:	
Carry axial load	$C_F = 0.12$, $C_{kc} = 0.125$
$R \geq 0.999\ 95$	$z = -3.891$
Machined surfaces	$C_{ka} = 0.058$
Hole critical	$C_{Kf} = 0.10$, $C_{\sigma'_a} = (0.10^2 + 0.12^2)^{1/2} = 0.156$
Ambient temperature	$C_{kd} = 0$
Correlation method	$C_{S'_e} = 0.138$
Hole drilled	$C_{Se} = (0.058^2 + 0.125^2 + 0.138^2)^{1/2} = 0.195$

$$C_n = \sqrt{\frac{C_{Se}^2 + C_{\sigma'_a}^2}{1 + C_{\sigma'_a}^2}} = \sqrt{\frac{0.195^2 + 0.156^2}{1 + 0.156^2}} = 0.2467$$

$$\bar{n} = \exp\left[-(-3.891)\sqrt{\ln(1 + 0.2467^2)} + \ln\sqrt{1 + 0.2467^2}\right]$$

$$= 2.65$$

Example 6-21

These eight a priori decisions have quantified the mean design factor as $\bar{n} = 2.65$. Proceeding deterministically hereafter we write

$$\sigma'_a = \frac{\bar{S}_e}{\bar{n}} = \bar{K}_f \frac{\bar{F}}{(w-d)t}$$

from which

$$t = \frac{\bar{K}_f \bar{n} \bar{F}}{(w-d)\bar{S}_e} \qquad (1)$$

To evaluate the preceding equation we need $\bar{S}_e$ and $\bar{K}_f$. The Marin factors are

$$\mathbf{k}_a = 2.67\bar{S}_{ut}^{-0.265}\mathbf{LN}(1, 0.058) = 2.67(87.6)^{-0.265}\mathbf{LN}(1, 0.058)$$

$$\bar{k}_a = 0.816$$

$$k_b = 1$$

$$\mathbf{k}_c = 1.23\bar{S}_{ut}^{-0.078}\mathbf{LN}(1, 0.125) = 0.868\mathbf{LN}(1, 0.125)$$

$$\bar{k}_c = 0.868$$

$$\bar{k}_d = \bar{k}_f = 1$$

and the endurance strength is

$$\bar{S}_e = 0.816(1)(0.868)(1)(1)0.506(87.6) = 31.4 \text{ kpsi}$$

Example 6-21

The hole governs. From Table A–15–1 we find $d/w = 0.50$, therefore $K_t = 2.18$. From Table 6–15 $\sqrt{a} = 5/\bar{S}_{ut} = 5/87.6 = 0.0571$, $r = 0.1875$ in. From Eq. (6–78) the fatigue stress-concentration factor is

$$\bar{K}_f = \frac{2.18}{1 + \dfrac{2(2.18-1)}{2.18}\dfrac{0.0571}{\sqrt{0.1875}}} = 1.91$$

The thickness t can now be determined from Eq. (1)

$$t \geq \frac{\bar{K}_f \bar{n} \bar{F}}{(w-d)S_e} = \frac{1.91(2.65)1000}{(0.75-0.375)31\,400} = 0.430 \text{ in}$$

Use $\frac{1}{2}$-in-thick strap for the workpiece. The $\frac{1}{2}$-in thickness attains and, in the rounding to available nominal size, exceeds the reliability goal.