

Ordinary Differential Equations : ODEs

Definition: A differential equation (D.E.) is an equation with a function and one or more of its derivatives.

Example: ① $y' + y = 2x$

② $(x+1) dy - 2 dx = 0$

① The ordinary differential equation is an equation where the function depends on one independent variable.

② Order: The highest derivative in the D.E. is called the order.

The degree of the D.E. is the exponent of the highest derivative

Example: ① $y'' + 2x y' = x^2$

order = 2

② $y''' + 2 (y'')^2 = e^x$

order = 3

③ $x dx - y dy = 0$

order = 1

③③ Linearity: The ODE is linear if it is linear in y and its derivatives

Ex: ① $y''' - 2x y' = \sin x$

3rd - order linear

② $y'' - (y')^2 = x$

Non linear second-order

③ $y^{(4)} - \sin y = x$

Non linear 4th-order

$$(4) \quad y'' + yy' = x$$

Nonlinear 2nd-order

Ex: show that $y(x) = e^{2x} - 1$ is a solution of

$$y'' - 4y = 4$$

The function y is a solution if it satisfies the D.E.

$$y = e^{2x} - 1$$

$$y' = 2e^{2x}$$

$$y'' = 4e^{2x}$$

By substitution in the D.E. we obtain:

$$4e^{2x} - 4(e^{2x} - 1) \stackrel{?}{=} 4$$

$$4e^{2x} - 4e^{2x} + 4 \stackrel{?}{=} 4$$

$$\Rightarrow 4 = 4$$

$\therefore y = e^{2x} - 1$ is a solution of $y'' - 4y = 4$

Separable Differential Equations

Def: A separable ODE is any first-order D.E.

which can be written as:

$$f(x) dx = g(y) dy$$

Example: Which of the following ODEs is sep.

$$\textcircled{1} \quad x \cos^2 y \, dx + e^x \, dy = 0$$

$$x \cos^2 y \, dx = -e^x \, dy$$

$$\frac{x}{e^x} \, dx = -\frac{1}{\cos^2 y} \, dy$$

$$\Rightarrow x e^{-x} \, dx = -\sec^2 y \, dy$$

$\therefore$ separable

$$\textcircled{2} \quad 2xy \, dx + (x^2 + 1) \, dy = 0$$

$$(3) (x+y^2) dx + x^2 e^y dy = 0$$

$$(4) (xy+x) dx = (x^2+1)y dy$$

$$x(y+1) dx = (x^2+1)y dy$$

$$\Rightarrow \frac{x}{x^2+1} dx = \frac{y}{y+1} dy$$

is separable

Example: solve the following ODEs

$$(1) y' = y^3 \cos^2 x$$

$$\frac{dy}{dx} = y^3 \cos^2 x$$

$$\Rightarrow \frac{dy}{y^3} = \cos^2 x dx$$

$$\Rightarrow \int y^{-3} dy = \frac{1}{2} \int (1 + \cos 2x) dx$$

$$= \frac{y^{-2}}{-2} + C_1 = \frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) + C_2$$

$$= -\frac{1}{2y^2} = \frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) + C$$

$$\textcircled{2} \quad e^{2x-y} dx = e^{x+y} dy$$

$$\frac{e^{2x}}{e^x} \frac{1}{e^y} dx = e^x \frac{1}{e^y} dy$$

$$\frac{e^{2x}}{e^x} dx = \frac{e^x}{e^y} dy$$

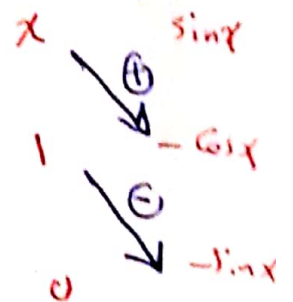
$$e^x dx = \frac{e^x}{e^y} dy$$

$$\Rightarrow e^x = \frac{1}{x} e^{2y} + C$$

$$\textcircled{3} \quad \frac{dy}{dx} = \frac{xy \sin x}{y+1}, \quad y(0) = 1$$

$$\frac{y+1}{y} dy = x \sin x dx$$

$$\int \left[1 + \frac{1}{y}\right] dy = \int x \sin x dx$$



$$\Rightarrow y + \ln|y| = -x \cos x + \sin x + C$$

$$y(0) = 1 \Rightarrow 1 + \ln 1 = C \Rightarrow \boxed{C=1}$$

$$\therefore y + \ln|y| = -x \cos x + \sin x + 1$$

$$(4) (x^2y + y) dy + (y+1) dx = 0$$

$$(5) \frac{dy}{dx} = x+1 + x^2y + y^2$$

$$dy = [x+1 + x^2y + y^2] dx$$

$$\Rightarrow dy = [x+1 + y^2(x+1)] dx$$

$$\Rightarrow dy = (x+1)(1+y^2) dx$$

$$\Rightarrow \frac{dy}{1+y^2} = (x+1) dx$$

$$\Rightarrow \tan^{-1} y = \frac{1}{2} x^2 + x + C$$

First-Order Linear D.E.

- ① If $p(x)$, $f(x)$ are continuous functions of x , then

$$y' + p(x)y = f(x)$$

is a first-order linear D.E.

- ② How can we solve such an equation

- ① The D.E. should be written as:

$$y' + p(x)y = f(x) \quad \text{--- (*)}$$

- ② Find the integrating factor $e^{\int p(x) dx}$

- ③ Multiply (*) by $e^{\int p(x) dx}$

- ④ simplify and integrate

Example: Solve the following ODEs

$$\textcircled{1} \quad x \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$$

$$y' + \frac{3}{x} y = \frac{\sin x}{x^3} \quad \dots\dots \textcircled{*}$$

$$\textcircled{2} \quad \int p(x) dx = \int \frac{3}{x} dx = 3 \int \frac{1}{x} dx = 3 \ln x \\ = \ln x^3 = \textcircled{x^3}$$

$\textcircled{3}$ Multiply $\textcircled{*}$ by x^3

$$x^3 y' + 3x^2 y = \sin x$$

$$(x^3 y)' = \sin x$$

$$x^3 y = -\cos x + C$$

$$(2) \quad y' + \frac{4x}{1+x^2} y = \frac{1}{(1+x^2)^3} \quad \text{--- (*)}$$

$$\begin{aligned} \textcircled{III} \quad \int p(x) dx &= \int \frac{4x}{1+x^2} dx = 2 \int \frac{2x}{1+x^2} dx \\ &= 2 \ln(1+x^2) = \ln(1+x^2)^2 \\ &= (1+x^2)^2 \end{aligned}$$

④ Multiply (*) by $(1+x^2)^2$

$$(1+x^2)^2 y' + 4x(1+x^2) y = \frac{1}{1+x^2}$$

$$\Rightarrow \left((1+x^2)^2 y \right)' = \frac{1}{1+x^2}$$

$$(1+x^2)^2 y = \tan^{-1} x + C$$

Remark: As a method of solution,
one can consider the following form

$$x' + p(y) x = f(y), \quad x' = \frac{dx}{dy}$$

$$\textcircled{3} \quad \frac{dy}{dx} = \frac{y}{4y^2 - 2x}$$

$$\frac{dx}{dy} = \frac{4y^2 - 2x}{y}$$

$$x' = 4y - \frac{2x}{y}$$

$$\Rightarrow x' + \frac{2}{y}x = 4y \quad \dots \textcircled{*}$$

$$\textcircled{ii} \quad \int p(y) dy = \int \frac{2}{y} dx = \frac{2 \ln y}{e} = y^2$$

$\textcircled{iii}$ Multiply $\textcircled{*}$ by y^2

$$\Rightarrow y^2 x' + 2yx = 4y^3$$

$$\Rightarrow (y^2 x)' = 4y^3$$

$$\Rightarrow y^2 x = y^4 + c$$

Bernoulli D.E.

- ① A Bernoulli D.E. is any D.E. of the form:

$$y' + p(x)y = f(x)y^n, \quad n \in \mathbb{R} - \{0, 1\}$$

where $p(x)$ and $f(x)$ are continuous

Remark: If $u(x) = (y(x))^n$, then

$$u'(x) = n (y(x))^{n-1} y'(x)$$

$$\text{i.e., } u' = n y^{n-1} y'$$

So, if $u = y^{1-n}$, then

$$u' = (1-n) y^{-n} y'$$

- ② Now, let us consider the D.E. *

$$y' + p(x)y = f(x)y^n \quad \text{---} * \quad n \neq 0, 1$$

This D.E. is nonlinear

To solve equation $(*)$, we assume

$$u = y^{1-n}$$

$$\Rightarrow u' = (1-n) y^{-n} y'$$

Multiply $(*)$ by $(1-n) y^{-n}$:

$$(1-n) y^{-n} y' + (1-n) y^{-n} p(x) y = (1-n) y^{-n} f(x) y^n$$

$$(1-n) y^{-n} y' + (1-n) p(x) y^{1-n} = (1-n) f(x)$$

$$\Rightarrow u' + (1-n) p(x) u = (1-n) f(x) \quad \dots (*)$$

which is linear !

Example : Solve the following ODEs :

$$\textcircled{1} \quad x^2 y' + 2xy = y^3$$

$$y' + \frac{2}{x} y = \frac{1}{x^2} y^3 \quad \text{---} \quad \textcircled{*}$$

$$\text{let } u = y^{1-3}, \text{ i.e., } u = y^{-2}$$

$\Rightarrow$

$$u' + (1-n) p(x) u = (1-n) f(x)$$

$$u' - 2 \cdot \frac{2}{x} u = -2 \cdot \frac{1}{x^2}$$

$$\therefore u' - \frac{4}{x} u = -\frac{2}{x^2} \quad \text{---} \quad \textcircled{**}$$

$$\textcircled{iii} \quad \frac{\int P(x) dx}{e} = \frac{\int \frac{-4}{x} dx}{e} = \frac{-4 \int \frac{1}{x} dx}{e} = \frac{-4 \ln x}{e} = x^{-4}$$

$\textcircled{iv}$ Multiply $\textcircled{**}$ by x^{-4}

$$\Rightarrow x^{-4} u' - 4 x^{-5} u = -2 x^{-6}$$

$$\Rightarrow (x^{-4} u)' = -2 x^{-6}$$

$$\therefore X^{-4} u = \frac{-2}{-5} X^{-5} + C$$

$$X^{-4} y^{-2} = \frac{2}{5} X^{-5} + C$$

$$\text{or } \frac{1}{x^4 y^2} = \frac{2}{5} \frac{1}{x^5} + C$$

$$\textcircled{2} \quad y' + x^2 y = \frac{e^{-x^3} \sinh x}{3 y^2}$$

$$y' + x^2 y = \frac{1}{3} e^{-x^3} \sinh x y^{-2} \quad \text{---} \textcircled{*}$$

$$\text{let } u = y^{1-(-2)} \Rightarrow u = y^3$$

$$\therefore u' + (1-n)P(x)u = (1-n)f(x)$$

$$u' + 3 \cdot x^2 u = 3 \cdot \frac{1}{3} e^{-x^3} \sinh x$$

$$\therefore u' + 3x^2 u = e^{-x^3} \sinh x \quad \text{---} \textcircled{**}$$

$$\int P(x) dx = \int 3x^2 dx = e^{x^3}$$

Multiply $\textcircled{**}$ by e^{x^3}

$$e^{x^3} u' + 3x^2 e^{x^3} u = e^{x^3} e^{-x^3} \sinh x$$

$$\therefore (e^{x^3} u)' = \sinh x$$

$$e^{x^3} u = \cosh x + c$$

$$\therefore e^{x^3} y^3 = \cosh x + c$$

Remark: Similarly, one can solve:

$$X' + p(x) X = f(x) X^n, \quad n \neq 0, 1$$

Let $u = X^{1-n}$. Then the D.E.

can be reduced to

$$u' + (1-n)p(x) u = (1-n)f(x)$$

$$(3) \quad \frac{dy}{dx} = \frac{y}{x^2 y^3 - x}$$

$$\frac{dx}{dy} = \frac{x^2 y^3 - x}{y}$$

$$\frac{dx}{dy} = x^2 y^2 - \frac{x}{y}$$

$$\therefore X' + \frac{1}{y} X = y^2 X^2 \quad \dots \textcircled{*}$$

$$\text{let } u = X^{1-2} \Rightarrow u = X^{-1}$$

Equation $\textcircled{*}$ is reduced to:

$$u' + (1-n) P(y) u = (1-n) f(y)$$

$$u' - \frac{1}{y} u = -y^2 \quad \dots \textcircled{**}$$

$$\int P(y) dy = \int \frac{1}{y} dy = \frac{-\ln y}{e} = \frac{\ln y^{-1}}{e} = \frac{1}{y}$$

Multiply $\textcircled{**}$ by $\frac{1}{y}$

$$\frac{1}{y} u' - \frac{1}{y^2} u = -y$$

$$\left(\frac{1}{y} u \right)' = -y$$

$$\frac{1}{y} u = -\frac{y^2}{2} + C$$

$$\frac{1}{y} \cdot \frac{1}{X} = -\frac{1}{2} y^2 + C$$

Remark: Consider the D.E.

$$\frac{dy}{dx} = \frac{x}{y - yx^2}$$

One can try $\frac{dx}{dy} = \frac{y - yx^2}{x}$

$$\Rightarrow \frac{dx}{dy} = \frac{y}{x} - yx$$

ie., $x' + yx = yx^{-1}$

or $\frac{dy}{dx} = \frac{x}{y - yx^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y(1-x^2)}$$

$$\Rightarrow y dy = \frac{x}{1-x^2} dx$$

$$\Rightarrow \frac{1}{2} y^2 = -\frac{1}{2} \ln|1-x^2| + C$$

Homogeneous D.E.

Def: The D.E. $y' = f(x, y)$ is called homogeneous if it can be written as

$$y' = g\left(\frac{y}{x}\right)$$

Ex: Which of the following CDEs is homogeneous

$$(1) \quad \frac{dy}{dx} = \frac{y-x}{x}$$

$$\frac{dy}{dx} = \left(\frac{y}{x}\right) - 1 \quad \therefore \text{homogeneous}$$

$$(2) \quad \frac{dy}{dx} = \frac{2x^2 - y^2}{xy + x^2}$$

$$\frac{dy}{dx} = \frac{\frac{2x^2}{x^2} - \frac{y^2}{x^2}}{\frac{xy}{x^2} + \frac{x^2}{x^2}}$$

$$\therefore \frac{dy}{dx} = \frac{2 - \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right) + 1}$$

$\therefore$ homogeneous

$$\textcircled{3} \quad \frac{dy}{dx} = \frac{x - 2y + 1}{y - x}$$

$$\frac{dy}{dx} = \frac{1 - 2\left(\frac{y}{x}\right) + \frac{1}{x}}{\left(\frac{y}{x}\right) - 1}$$

Not homogeneous

④ The homogeneous D.E. $y' = g\left(\frac{y}{x}\right)$ -----*

Can be solved by reducing it to a separable D.E. as follows:

$$\text{let } u = \frac{y}{x} \Rightarrow y = xu$$

$$\Rightarrow y' = xu' + u$$

The D.E. * becomes:

$$xu' + u = g(u)$$

$$x \frac{du}{dx} = g(u) - u$$

$$\Rightarrow \frac{du}{g(u) - u} = \frac{dx}{x} \quad \text{Separable}$$

Ex: Solve the following ODEs:

$$(1) \quad \frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2}$$

$$y' = \frac{xy}{x^2} + \frac{y^2}{x^2} + \frac{x^2}{x^2}$$

$$\therefore y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1 \quad \text{--- (*)}$$

$$\text{let } u = \frac{y}{x} \Rightarrow y = xu$$

$$\Rightarrow y' = xu' + u$$

Substitute in (*),

$$xu' + u = u + u^2 + 1$$

$$\Rightarrow xu' = u^2 + 1$$

$$\Rightarrow x \frac{du}{dx} = u^2 + 1$$

$$\Rightarrow \frac{du}{u^2 + 1} = \frac{dx}{x}$$

$$\Rightarrow \tan^{-1} u = \ln|x| + c$$

$$\Rightarrow u = \tan(\ln|x| + c) \Rightarrow \frac{y}{x} = \tan(\ln|x| + c)$$

$$(2) \quad (x+y) dy = (x-y) dx$$

$$\frac{dy}{dx} = \frac{x-y}{x+y}$$

$$\therefore \frac{dy}{dx} = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}} \quad \dots (*)$$

$$\text{let } u = \frac{y}{x} \Rightarrow y = xu$$

$$\Rightarrow y' = xu' + u$$

$$\Rightarrow xu' + u = \frac{1-u}{1+u}$$

$$\Rightarrow xu' = \frac{1-u}{1+u} - u$$

$$\therefore x \frac{du}{dx} = \frac{1-2u-u^2}{1+u}$$

$$\therefore \frac{1+u}{1-2u-u^2} du = \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2} \ln |1-2u-u^2| = \ln |x| + C$$

$$\Rightarrow \ln |1-2u-u^2| = -2 \ln |x| + C$$

$$\frac{1-u}{1+u} - u$$

$$\frac{1-u-u-u^2}{1+u}$$

$$\frac{1-2u-u^2}{1+u}$$

$$1 - 2u - u^2 = x^{-2} \cdot e^c$$

$$1 - 2 \frac{y}{x} - \frac{y^2}{x^2} = c x^{-2}$$

Multiply by x^2 :

$$x^2 - 2xy - y^2 = c$$

$$\textcircled{3} \quad \frac{dy}{dx} = \frac{y}{x} (\ln y - \ln x + 1)$$

$$y' = \frac{y}{x} \left(\ln \frac{y}{x} + 1 \right) \quad \text{---} \textcircled{*}$$

$$\text{let } u = \frac{y}{x} \Rightarrow y = xu$$

$$\Rightarrow y' = xu' + u$$

From $\textcircled{*}$ we have:

$$xu' + u = u (\ln u + 1)$$

$$xu' + u = u \ln u + u$$

$$xu' = u \ln u$$

$$x \frac{du}{dx} = u \ln u$$

$$\frac{du}{u \ln u} = \frac{dx}{x}$$

$$\frac{\frac{1}{u}}{\ln u} du = \frac{dx}{x}$$

$$\ln |\ln u| = \ln |x| + c$$

$$\ln u = e^{\ln |x| + c}$$

$$\ln u = e^c e^{\ln |x|}$$

$$\ln u = c x$$

$$\therefore u = \frac{e^{cx}}{e}$$

$$\frac{y}{x} = \frac{e^{cx}}{e}$$

$$\Rightarrow y = x \frac{e^{cx}}{e}$$

Remark: Note that $y = x$ is also a solution

H.W Solve the following ODEs:

$$(1) \quad (xy + y^2) dx - x^2 dy = 0$$

$$(2) \quad \frac{dy}{dx} = \frac{x \sec\left(\frac{y}{x}\right) + y}{x}$$

$$(3) \quad \frac{dy}{dx} = \frac{\sqrt{xy} + y}{x}, \quad x > 0$$

$$\frac{dy}{dx} = f(ax+by+c) \quad \dots \quad (*)$$

Q1) Examples of $f(ax+by+c)$

Q1) $f(x+y+z) = (x+y+z)^3 - 1$

Q2) $f(x+y) = \sin(x+y) - 1$

Q3) $f(x-y) = \frac{2x-2y+1}{x-y}$
 $= \frac{2(x-y)+1}{x-y}$

To solve (*) we set: $u = ax+by+c$

$$\Rightarrow u' = a + by'$$

$$\Rightarrow y' = \frac{u' - a}{b}$$

By this substitution, D.E. (*) is

reduced to separable.

Ex: Solve the following ODEs:

$$\textcircled{1} \quad \frac{dy}{dx} = (x+y+2)^2 \quad \dots \textcircled{*}$$

$$\text{let } u = x+y+2$$

$$\Rightarrow u' = 1+y' \Rightarrow y' = u' - 1$$

substitute in $\textcircled{*}$:

$$u' - 1 = u^2$$

$$\Rightarrow u' = u^2 + 1 \Rightarrow \frac{du}{dx} = u^2 + 1$$

$$\Rightarrow \frac{du}{u^2 + 1} = dx \Rightarrow \tan^{-1} u = x + c$$

$$\Rightarrow u = \tan(x+c)$$

$$\Rightarrow x+y+2 = \tan(x+c)$$

$$\textcircled{2} \quad \frac{dy}{dx} = \sqrt{x+y} - 1$$

$$\textcircled{3} \quad \frac{dy}{dx} = \sin(x-y)$$

$$(4) \quad y' = (2x+y)^2 - 2 \quad \dots \textcircled{*}$$

$$\text{let } u = 2x + y$$

$$\Rightarrow u' = 2 + y'$$

$$\Rightarrow y' = u' - 2$$

By substitution in $\textcircled{*}$ we obtain:

$$u' - 2 = u^2 - 2$$

$$\frac{du}{dx} = u^2 \Rightarrow \frac{du}{u^2} = dx$$

$$\Rightarrow -\frac{1}{u} = x + c$$

$$\Rightarrow \frac{1}{u} = -x + c$$

$$\Rightarrow u = \frac{1}{-x + c}$$

$$\therefore 2x + y = \frac{1}{-x + c}$$

$$⑤ \quad y' = \frac{x+y+2}{x+y}$$

$$y' = \frac{x+y}{x+y} + \frac{2}{x+y}$$

$$y' = 1 + \frac{2}{x+y} \quad \text{--- } \textcircled{*}$$

$$\text{Let } u = x+y \Rightarrow u' = 1+y' \\ \Rightarrow y' = u' - 1$$

From $\textcircled{*}$ we obtain

$$u' - 1 = 1 + \frac{2}{u}$$

$$u' = 2 + \frac{2}{u} \Rightarrow u' = \frac{2u+2}{u}$$

$$\frac{du}{dx} = \frac{2(u+1)}{u} \Rightarrow \frac{u}{u+1} du = 2 dx$$

$$\left(1 - \frac{1}{u+1}\right) du = 2 dx$$

$$u - \ln|u+1| = 2x + C$$

$$x+y - \ln|x+y+1| = 2x + C$$

Revision

Ex: If $f(x, y) = x^3 + 2y^2$, find

① $\frac{\partial f}{\partial y}$

② $\frac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial y} = 4y$$

$$\frac{\partial f}{\partial x} = 3x^2$$

Ex: If $f(x, y) = x^2 + xy + \sin y$,

find $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial y} = x + \cos y$$

$$\frac{\partial f}{\partial x} = 2x + y$$

Ex: If $f(x, y) = x \cos y - 2y$,

find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = \cos y$$

$$\frac{\partial f}{\partial y} = -x \sin y - 2$$

Ex: If $\frac{\partial f}{\partial y} = x \cos y - 2y$, find f .

Integrate with respect to y , gives

$$f(x, y) = x \sin y - y^2 + g(x)$$

Exact D.E.

We will be interested in D.E. of the form

$$M(x,y) dx + N(x,y) dy = 0$$

Ex: $(x-2y) dx + (y-2x) dy = 0$

Theorem: Let $M(x,y)$, $N(x,y)$, $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ be continuous. Then

$$M dx + N dy = 0 \quad \dots \textcircled{*}$$

is an exact D.E. iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

③ The solution of $\textcircled{*}$ is $u(x,y) = C$

where $\frac{\partial u}{\partial x} = M$, $\frac{\partial u}{\partial y} = N$

$$\Rightarrow u = \int M dx + g(y) , \quad u = \int N dy + h(x)$$

Ex: Solve:

$$(e^x + y) dx + (x - \sin y) dy = 0$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1 \Rightarrow \text{Exact}$$

$\therefore$ The solution is $u(x, y) = C$ where

$$\frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N$$

$$\frac{\partial u}{\partial x} = e^x + y \Rightarrow u(x, y) = \int (e^x + y) dx$$

$$\Rightarrow u(x, y) = e^x + yx + f(y)$$

Differentiate w.r. to y

$$\frac{\partial u}{\partial y} = x + f'(y)$$

$$\Rightarrow x - \sin y = x + f'(y) \Rightarrow f'(y) = -\sin y$$

$$\Rightarrow f(y) = \cos y + C_1$$

$$\therefore u(x, y) = e^x + yx + \cos y + C_1$$

The solution is $u(x, y) = C$

$$e^x + yx + \cos y + C_1 = C$$

$$\Rightarrow e^x + yx + \cos y = C$$

Ex: Solve :

$$(2x \cos y + 3x^2 y) dx + (x^3 - x^2 \sin y - y) dy = 0$$

$$\frac{\partial M}{\partial y} = -2x \sin y + 3x^2$$

$\therefore$ Exact

$$\frac{\partial N}{\partial x} = 3x^2 - 2x \sin y$$

The solution is $u(x, y) = c$ where

$$\frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x \cos y + 3x^2 y$$

$$\Rightarrow u(x, y) = x^2 \cos y + x^3 y + f(y)$$

$$\frac{\partial u}{\partial y} = -x^2 \sin y + x^3 + f'(y)$$

$$\Rightarrow x^3 - x^2 \sin y - y = -x^2 \sin y + x^3 + f'(y)$$

$$\Rightarrow -y = f'(y) \Rightarrow f(y) = -\frac{y^2}{2} + C$$

$$x^2 \cos y + x^3 y - \frac{y^2}{2} :$$

$\therefore$ The solution is: $u(x, y) = c$

$$\text{i.e., } x^2 \cos y + x^3 y - \frac{y^2}{2} = c$$

(H.w) Solve : $\frac{dy}{dx} = - \frac{\sin y + y \cos x}{\sin x + x \cos y}$

or $(\sin y + y \cos x) dx + (\sin x + x \cos y) dy = 0$

Ex: If $(2x + 6x^2y^2) dx + (4x^n y - 12y^3) dy = 0$

is exact, find n .

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$12x^2y = 4nx^{n-1}y$$

$$3x^2 = nx^{n-1}$$

$$\Rightarrow \boxed{n=3}$$

Integrating Factor

Suppose the D.E.

$$M(x,y) dx + N(x,y) dy = 0$$

is not exact, i.e., $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

③ If $\mu(x,y)$ is a function such that

$$\mu(x,y) M(x,y) dx + \mu(x,y) N(x,y) dy = 0$$

is exact, then $\mu(x,y)$ is called an integrating factor.

③ We will be interested in finding μ as a function of x alone or as a function of y alone

$$\textcircled{3} \quad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = R(x) \Rightarrow \mu(x) = e^{\int R(x) dx}$$

$$\textcircled{3} \quad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = R(y) \Rightarrow \mu(y) = e^{\int R(y) dy}$$

Ex solve the following ODEs :

$$(1) (y^2 - 3xy - 2x^2) dx + (xy - x^2) dy = 0$$

$$\frac{\partial M}{\partial y} = 2y - 3x$$

$\therefore$ not exact

$$\frac{\partial N}{\partial x} = y - 2x$$

$$\begin{aligned} \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} &= 2y - 3x - y + 2x \\ &= y - x \end{aligned}$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{y-x}{xy-x^2} = \frac{y-x}{x(y-x)} = \frac{1}{x} = R(x)$$

$$\mu(x) = e^{\int R(x) dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Multiply the D.E. by x :

$$(xy^2 - 3x^2y - 2x^3) dx + (x^2y - x^3) dy = 0$$

This D.E. is exact and its solution is given by

$$\frac{1}{2}x^2y^2 - x^3y - \frac{1}{2}x^4 = c$$

$$(2) (3x^2y + y^2)dx + (2x^3 + 3xy)dy = 0$$

$$\frac{\partial M}{\partial y} = 3x^2 + 2y$$

$\therefore$ Not exact

$$\frac{\partial N}{\partial x} = 6x^2 + 3y$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -3x^2 - y = -(3x^2 + y)$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{-(3x^2 + y)}{-(3x^2y + y^2)}$$

$$= \frac{(3x^2 + y)}{y(3x^2 + y)} = \frac{1}{y} = R(y)$$

$$\Rightarrow \mu(y) = e^{\int R(y) dy} = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$$

Multiply the D.E. by y :

$$(3x^2y^2 + y^3)dx + (2x^3y + 3xy^2)dy = 0$$

This D.E. is exact

The solution is:

$$x^3y^2 + xy^3 = C$$

$$x^3y^2 + y^3x$$

$$x^3y^2 + xy^3$$

H.W: Find the integrating factor:

$$(1) (3x^2y + 2xy + y^3) dx + (x^2 + y^2) dy = 0$$

$$\mu = e^{3x}$$

$$(2) y dx + (2xy - e^{-2y}) dy = 0$$

$$\mu = \frac{e^{2y}}{y}$$

$$(3) y dx + (2x - ye^x) dy = 0$$

$$\mu = y$$

$$(4) 2xy dx + (y^2 - x^2) dy = 0$$

$$\mu = \frac{1}{y^2}$$

H.W show that $\mu(x,y) = xy^2$ is an integrating factor of:

$$(2y - 6x) dx + (3x - 4x^2y^{-1}) dy = 0$$

Suppose $\mu(x)$ is an integrating factor of

$$M(x,y) dx + N(x,y) dy = 0$$

$$\Rightarrow \mu(x) M(x,y) dx + \mu(x) N(x,y) dy = 0 \text{ is } \underline{\underline{\text{exact}}}$$

$$\frac{\partial}{\partial y} [\mu(x) M(x,y)] = \frac{\partial}{\partial x} [\mu(x) N(x,y)]$$

$$\mu(x) \frac{\partial M}{\partial y} = \mu(x) \frac{\partial N}{\partial x} + N \mu'(x)$$

$$\mu(x) \frac{\partial M}{\partial y} - \mu(x) \frac{\partial N}{\partial x} = N \mu'(x)$$

$$\mu(x) \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = N \mu'(x)$$

$$\frac{\mu'(x)}{\mu(x)} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

$$\ln |\mu(x)| = \int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx$$

$$\Rightarrow \mu(x) = e^{\int R(x) dx} \quad \text{where } R(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

Second-order ODEs

Consider the D.E.:

$$y'' + p(x)y' + q(x)y = f(x) \text{ --- (1)}$$

where $p(x)$, $q(x)$ and $f(x)$ are continuous functions on some open interval I .

① If $f(x) = 0$, the D.E. (1) is called homogeneous, otherwise it is called nonhomogeneous

Theorem: Consider the homogeneous D.E.

$$y'' + p(x)y' + q(x)y = 0 \text{ --- (2)}$$

If y_1, y_2 are two solutions of (2) on I ,

then $y = c_1 y_1 + c_2 y_2$, c_1, c_2 constants,

is also a solution on I .

Def: Two functions $f_1(x)$, $f_2(x)$ are said to be linearly dependent on I if there exists constants c_1, c_2 , not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \quad \forall x \in I.$$

Otherwise, they are called independent, i.e.,

$$c_1 f_1(x) + c_2 f_2(x) = 0 \implies c_1 = c_2 = 0$$

① f_1, f_2 are linearly dependent $\implies f_1 = c f_2$
otherwise, they are linearly independent

Ex: $f(x) = \sin x$, $g(x) = \cos x$, $x \in (0, \frac{\pi}{2})$

$$\frac{f(x)}{g(x)} = \tan x \neq 0 \quad \forall x \in I \implies \text{independent}$$

Ex: $f(x) = x^2$, $g(x) = 3x^2$

$$\frac{f(x)}{g(x)} = \frac{x^2}{3x^2} = \frac{1}{3} \implies \text{Dependent}$$

Def: Suppose f, g are differentiable functions.

Then $W(f, g)$ is called the Wronskian of f and g and it is defined by

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'$$

③ If f, g and h are differentiable functions up to order 2, then

$$W(f, g, h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$$

Theorem: Let y_1, y_2 be two solutions of the homogeneous D.E.

$$y'' + p(x)y' + q(x)y = 0 \quad \dots (2)$$

on I . Then $\{y_1, y_2\}$ is linearly independent

iff $W(y_1, y_2) \neq 0 \quad \forall x \in I$, and

$\{y_1, y_2\}$ is called a fundamental set of solutions.

Def: let y_1, y_2 be two independent solutions of (2). Then $y = c_1 y_1 + c_2 y_2$, c_1, c_2 : constants is called the general solution of (2).

Theorem: (Abel's Theorem)

If y_1, y_2 are solutions of

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x), q(x)$ are continuous on some open interval I , then

$$W(y_1, y_2)(x) = C e^{-\int p(x) dx}$$

This means that:

$W(y_1, y_2)(x)$ either is zero for all $x \in I$ or else is never zero in I .

Ex: If y_1, y_2 are linearly independent

solutions of: $t y'' + 2 y' + t e^t y = 0$

and $W(y_1, y_2)(2) = 3$, find $W(y_1, y_2)(5)$.

Sol:

$$y'' + \frac{2}{t} y' + e^t y = 0$$

$$\begin{aligned} e^{-\int p(x) dx} &= e^{-\int \frac{2}{t} dt} = e^{-2 \ln t} = e^{\ln t^{-2}} = t^{-2} \\ &= \frac{1}{t^2} \end{aligned}$$

$$\therefore W(y_1, y_2) = C e^{-\int p(x) dx}$$

$$\Rightarrow W(y_1, y_2)(x) = \frac{C}{t^2}$$

$$W(y_1, y_2)(2) = 3$$

$$\frac{C}{4} = 3 \Rightarrow C = 12$$

$$\therefore W(y_1, y_2)(x) = \frac{12}{t^2}$$

$$\therefore W(y_1, y_2)(5) = \frac{12}{25}$$

Ex: If $W(f, g) = 3e^{4t}$ and $f(t) = e^{2t}$
find $g(t)$

Sol: $W(f, g) = 3e^{4t}$

$$f g' - g f' = 3e^{4t}$$

$$e^{2t} g' - 2e^{2t} g = 3e^{4t}$$

$$g' - 2g = 3e^{2t} \quad \dots \textcircled{*}$$

$$\int p(t) dt = \int -2 dt = -2t$$

Multiply $\textcircled{*}$ by e^{-2t}

$$\therefore e^{-2t} g' - 2e^{-2t} g = 3$$

$$\therefore \left(e^{-2t} g \right)' = 3$$

$$\Rightarrow e^{-2t} g = 3t + c$$

$$\therefore g(t) = \frac{3t + c}{e^{-2t}}$$

Ex If $W(f, g) = 2$, find $W(f, f+2g)$

Sol

$$W(f, f+2g) = f(f+2g)' - (f+2g)f'$$

$$= f(f' + 2g') - ff' - 2gf'$$

$$= ff' + 2fg' - ff' - 2gf'$$

$$= 2(fg' - gf')$$

$$= 2W(f, g)$$

$$= 2(2)$$

$$= 4$$

$$F(x, y', y'') = 0 \quad : \quad y\text{-missing}$$

$$\text{let } u = y', \quad u' = y''$$

$$\Rightarrow F(x, u, u') = 0$$

$$\text{Ex; Solve: } y'' + \frac{2}{x} y' = \frac{1}{x^2}$$

$$\underline{\text{Sol:}} \quad y\text{-missed}$$

$$\text{let } u = y', \quad u' = y''$$

$$\Rightarrow u' + \frac{2}{x} u = \frac{1}{x^2} \quad \dots \textcircled{*}$$

$$\textcircled{1} \int p(x) dx = \int \frac{2}{x} dx = 2 \ln x = \ln x^2 = e^{\ln x^2} = x^2$$

Multiply $\textcircled{*}$ by x^2 :

$$\Rightarrow x^2 u' + 2x u = 1$$

$$\Rightarrow (x^2 u)' = 1$$

$$\Rightarrow x^2 u = x + C_1$$

$$\Rightarrow u = \frac{x + C_1}{x^2}$$

$$\therefore \frac{dy}{dx} = \frac{x+c_1}{x^2}$$

$$\Rightarrow dy = \frac{x+c_1}{x^2} dx$$

$$\Rightarrow dy = \left[\frac{1}{x} + \frac{c_1}{x^2} \right] dx$$

$$\Rightarrow y = \ln|x| - \frac{c_1}{x} + c_2$$

H.w: Solve the following ODEs:

$$(1) \quad x y'' + y' = 1$$

$$(2) \quad x^2 y'' + (y')^2 = 0$$

$$(3) \quad y'' + (y')^2 + 1 = 0$$

$$F(y, y', y'') = 0 \quad x\text{-missed}$$

$$u = y' \quad , \quad u' = y''$$

$$\Rightarrow F(y, u, u') = 0$$

$$u' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} \Rightarrow u' = u \frac{du}{dy}$$

$$\therefore F(y, u, u') = 0 \quad \text{where} \quad u' = u \frac{du}{dy}$$

$$\text{Ex: Solve: } y y'' + (y')^2 = 0$$

Q10 x -missed

$$\text{let } u = y' \quad , \quad u' = y''$$

$$\Rightarrow y u' + u^2 = 0$$

$$\Rightarrow y u \frac{du}{dy} + u^2 = 0$$

$$\Rightarrow y u \frac{du}{dy} = -u^2$$

$$\Rightarrow \frac{du}{u} = -\frac{dy}{y}$$

$$\Rightarrow \ln |u| = -\ln |y| + C_1$$

$$\therefore \ln|u| + \ln|y| = c_1$$

$$\Rightarrow \ln|uy| = c_1$$

$$\Rightarrow uy = e^{c_1} \quad \text{or} \quad uy = c_1$$

$$\therefore y \cdot \frac{dy}{dx} = c_1 \Rightarrow y dy = c_1 dx$$

$$\Rightarrow \frac{1}{2} y^2 = c_1 x + c_2$$

Ex: Solve: $(y+1)y'' = y'^2$

Let $u = y'$, $u' = y''$

$$\Rightarrow (y+1)u' = u^2 \quad \text{where } u' = u \frac{du}{dy}$$

$$\Rightarrow (y+1)u \frac{du}{dy} = u^2$$

$$\Rightarrow \frac{du}{u} = \frac{dy}{y+1}$$

$$\Rightarrow \ln|u| = \ln|y+1| + c_1$$

$$\Rightarrow u = (y+1) \cdot e^{c_1}$$

$$\text{or } u = c_1(y+1)$$

$$\therefore \frac{dy}{dx} = c_1 (y+1)$$

$$\Rightarrow \frac{dy}{y+1} = c_1 dx$$

$$\Rightarrow \ln |y+1| = c_1 x + c_2$$

$$\Rightarrow y+1 = \frac{c_1 x + c_2}{e}$$

$$\Rightarrow y+1 = c_2 e^{c_1 x}$$

$$\therefore y = c_2 e^{c_1 x} - 1$$

H.W: Solve:

$$(1) \quad y'' + 2y(y')^3 = 0$$

$$(2) \quad y'' = \frac{-1}{2y^2}, \quad y(0) = 1, \quad y'(0) = -1$$

Reduction of order

Consider

$$y'' + p(x)y' + q(x)y = 0 \quad \text{--- (2)}$$

Given y_1 a solution of (2), then we can find a 2nd - linearly independent solution y_2 using the formula:

$$y_2 = y_1 \int \frac{-\int p(x) dx}{y_1^2} dx$$

Ex: Given $y_1 = x$ a solution of

$$(x^2 - x)y'' - xy' + y = 0$$

find a 2nd - linearly independent solution y_2 .

sl. $y'' - \frac{x}{x^2 - x}y' + \frac{1}{x^2 - x}y = 0$

$$\therefore y'' - \frac{1}{x-1}y' + \frac{1}{x^2 - x}y = 0$$

$$\begin{aligned} \frac{-\int p(x) dx}{e^{\int p(x) dx}} &= e^{-\int \frac{1}{x-1} dx} = e^{\int \frac{1}{x-1} dx} \\ &= e^{\ln|x-1|} = |x-1| \end{aligned}$$

$$\begin{aligned} \therefore y_2 &= y_1 \int \frac{-\int p(x) dx}{y_1^2} dx \\ &= x \int \frac{x-1}{x^2} dx \\ &= x \int \left[\frac{1}{x} - \frac{1}{x^2} \right] dx \\ &= x \left(\ln|x| + \frac{1}{x} \right) \end{aligned}$$

$$\therefore y_2 = x \ln|x| + 1$$

Ex: Consider $x y'' + 2 y' + x y = 0$,

Given $y_1 = \frac{\cos x}{x}$ a solution, find

a 2nd - linearly independent solution.

Sol: $y'' + \frac{2}{x} y' + y = 0$

$$\frac{-\int p(x) dx}{e^{\int p(x) dx}} = \frac{-\int \frac{2}{x} dx}{e^{-2 \ln x}} = x^{-2}$$

$$y_2 = y_1 \int \frac{-\int p(x) dx}{y_1^2} dx$$

$$= \frac{\cos x}{x} \int \frac{x^{-2}}{\frac{\cos^2 x}{x^2}} dx$$

$$= \frac{\cos x}{x} \int \frac{x^{-2} \cdot x^2}{\cos^2 x} dx$$

$$= \frac{\cos x}{x} \int \sec^2 x dx$$

$$= \frac{\cos x}{x} \cdot \tan x$$

$$= \frac{\cos x}{x} \cdot \frac{\sin x}{\cos x} = \frac{\sin x}{x}$$

H.W: Consider:

$$(1-x^2)y'' - 2xy' + 2y = 0, \quad y_1 = x$$

Find a 2nd-independent solution y_2 .

$$\text{Hint: } \frac{1}{x^2(1-x)} = \frac{1}{x^2} + \frac{1/2}{x+1} - \frac{1/2}{x-1}$$

Homogeneous Linear ODEs with Constant Coefficients

$$y'' + ay' + by = 0 \quad \text{--- } (*)$$

Ex: Solve: $y' + by = 0$

$$y' = -by \Rightarrow \frac{y'}{y} = -b$$

$$\Rightarrow \ln|y| = -bx + c$$

$$\Rightarrow y = e^{-bx+c} \Rightarrow y = ce^{-bx}$$

Now, to solve $(*)$, we search for a solution of the form $y = e^{rx}$

$$\text{since } y' = re^{rx}, \quad y'' = r^2 e^{rx}$$

$$\Rightarrow r^2 e^{rx} + ar e^{rx} + be^{rx} = 0$$

$$\Rightarrow e^{rx} [r^2 + ar + b] = 0$$

$$\therefore r^2 + ar + b = 0$$

This equation is called characteristic equation or auxiliary equation.

② Distinct Real Roots

Ex: Solve $y'' + y' - 2y = 0$

$$r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0$$

$$\Rightarrow r = -2, 1$$

$$\therefore y_1 = e^{-2x}, \quad y_2 = e^x$$

$\therefore$ The general solution is:

$$y(x) = c_1 y_1 + c_2 y_2$$

$$y = c_1 e^{-2x} + c_2 e^x$$

② $\{e^{-2x}, e^x\}$ is called a fundamental set of solutions or a Basis of solutions

Ex: Solve: $2y'' + 3y' = 0$

$$2r^2 + 3r = 0 \Rightarrow r(2r+3) = 0$$

$$\Rightarrow r = 0, -\frac{3}{2}$$

$$\therefore y_1 = e^{0x} = 1, \quad y_2 = e^{-\frac{3}{2}x}$$

∴ The general solution is:

$$y = c_1 + c_2 e^{-\frac{3}{2}x}$$

Ex: Solve: $y'' + 4y' + 2y = 0$

$$r^2 + 4r + 2 = 0$$

$$b^2 - 4ac = 16 - 4(1)(2) = 8$$

$$\therefore r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{8}}{2} = \frac{-4 \pm 2\sqrt{2}}{2}$$

$$\therefore r = -2 \pm \sqrt{2}$$

$$\therefore y_1 = e^{(-2-\sqrt{2})x}, \quad y_2 = e^{(-2+\sqrt{2})x}$$

∴ The general solution is:

$$y = c_1 y_1 + c_2 y_2$$

$$\therefore y = c_1 e^{(-2-\sqrt{2})x} + c_2 e^{(-2+\sqrt{2})x}$$

Ex: Find a 2nd-order linear homogeneous
D.E. whose solution is: $y = C_1 e^{2x} + C_2 e^{3x}$

Sol: $r_1 = 2, \quad r_2 = 3$

$$\Rightarrow (r-2)(r-3) = 0$$

$$\therefore r^2 - 5r + 6 = 0$$

$$\therefore y'' - 5y' + 6y = 0$$

Real Double Roots

If the characteristic equation $r^2 + ar + b = 0$

has equal roots $r_1 = r_2 = r$, then

$$y_1 = e^{rx}, \quad y_2 = x e^{rx}$$

Ex: Solve: $y'' - 6y' + 9y = 0$

$$r^2 - 6r + 9 = 0$$

$$\Rightarrow (r-3)(r-3) = 0 \Rightarrow r = 3, 3$$

$$\therefore y_1 = e^{3x}, \quad y_2 = x e^{3x}$$

$\therefore$ The general solution is:

$$\begin{aligned} y &= C_1 e^{3x} + C_2 x e^{3x} \\ &= e^{3x} (C_1 + C_2 x) \end{aligned}$$

Ex: Solve $y'' + 4\pi y' + 4\pi^2 y = 0$

$$r^2 + 4\pi r + 4\pi^2 = 0$$

$$\Rightarrow (r + 2\pi)(r + 2\pi) = 0$$

$$\Rightarrow r = -2\pi, -2\pi$$

$$\Rightarrow y_1 = e^{-2\pi x}, \quad y_2 = x e^{-2\pi x}$$

$$\therefore y = C_1 e^{-2\pi x} + C_2 x e^{-2\pi x}$$

Ex: Find a 2nd-order linear homogeneous D.E.

whose general solution is: $y = (C_1 + C_2 x) e^{-2x}$

Sol: $y = C_1 e^{-2x} + C_2 x e^{-2x}$

$$\therefore r_1 = -2, \quad r_2 = -2$$

$$\Rightarrow (r+2)(r+2) = 0$$

$$\therefore r^2 + 4r + 4 = 0$$

$$\therefore y'' + 4y' + 4y = 0$$

Ex: Solve the IVP:

$$y'' - 9y = 0, \quad y(0) = -2, \quad y'(0) = -12$$

Sol: $r^2 - 9 = 0 \Rightarrow r^2 = 9 \Rightarrow r = 3, -3$

$$\therefore y_1 = e^{-3x}, \quad y_2 = e^{3x}$$

$$\therefore y = C_1 e^{-3x} + C_2 e^{3x}$$

$$y'(x) = -3C_1 e^{-3x} + 3C_2 e^{3x}$$

$$y(0) = -2 \Rightarrow C_1 + C_2 = -2 \text{ --- (1)}$$

$$y'(0) = -12 \Rightarrow -3C_1 + 3C_2 = -12$$

$$\therefore -C_1 + C_2 = -4 \text{ --- (2)}$$

$$\Rightarrow 2C_2 = -6 \Rightarrow C_2 = -3$$

$$C_1 = -2 - C_2 = -2 + 3 = 1$$

$$\therefore y(x) = e^{-3x} - 3e^{3x}$$

Revision : Complex Numbers

$$\textcircled{1} \sqrt{-1} = i, \quad i^2 = -1$$

$$\textcircled{2} \sqrt{4} = 2, \quad \sqrt{-4} = 2i$$

$$\textcircled{3} \sqrt{-7} = \sqrt{7}i$$

Complex Number:

The Complex number is any number of the form $a + bi$, $a, b \in \mathbb{R}$.

Ex: The numbers :

$$2 + 3i, \quad 2, \quad 5i$$

are all Complex numbers

Ex: Solve $x^2 + 2x + 2 = 0$

$$a=1 \quad b=2 \quad c=2$$

$$b^2 - 4ac = 4 - 4(1)(2) = -4$$

$$\therefore r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

Complex Roots:

$$r = \lambda \pm \mu i$$

$$y_1 = e^{(\lambda + \mu i)x}$$

$$, y_2 = e^{(\lambda - \mu i)x}$$

Remark: From the previous two complex solutions, we can obtain the following two real solutions.

$$y_1 = e^{\lambda x} \cos(\mu x)$$

$$y_2 = e^{\lambda x} \sin(\mu x)$$

Ex: Solve: $y'' + 2y' + 5y = 0$

Sol. $y^2 + 2y + 5 = 0$
 $a=1 \quad b=2 \quad c=5$

$$b^2 - 4ac = 4 - 4(1)(5) = -16$$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$\therefore y = \underbrace{-1}_{\lambda} \pm \underbrace{2i}_{\mu}$$

$$y_1 = e^{\lambda x} \cos \mu x$$

$$\therefore y_1 = e^{-x} \cos 2x$$

$$\therefore y_2 = e^{-x} \sin 2x$$

$\Rightarrow$ The general solution is given by:

$$\begin{aligned} y &= C_1 e^{-x} \cos 2x + C_2 e^{-x} \sin 2x \\ &= e^{-x} [C_1 \cos 2x + C_2 \sin 2x] \end{aligned}$$

Ex: Solve: $y'' + 16y = 0$

$$r^2 + 16 = 0 \implies r^2 = -16 \implies r = \pm \sqrt{-16}$$

$$\therefore r = \pm 4i$$

$$\therefore \lambda = 0, \mu = 4$$

$$y_1 = e^{\lambda x} \cos \mu x = \cos 4x$$

$$\therefore y_1 = \cos 4x$$

$$y_2 = \sin 4x$$

$\therefore$ The General Solution is given by:

$$y = C_1 \cos 4x + C_2 \sin 4x$$

Remark: let r_1, r_2 be two roots of

$$(r - r_1)(r - r_2) = 0.$$

Then

$$r^2 - (r_1 + r_2)r + r_1 r_2 = 0$$

Ex: Find a 2nd-order linear homogeneous

D.E. whose solution is :

$$y = e^{2x} [C_1 \cos 3x + C_2 \sin 3x]$$

Sol, $y = C_1 e^{2x} \cos 3x + C_2 e^{2x} \sin 3x$

$$\lambda = 2, \mu = 3$$

The roots of the char. eqn are:

$$\lambda \pm \mu i = 2 \pm 3i$$

$$\therefore r_1 \longrightarrow 2 + 3i$$

$$r_2 \longrightarrow 2 - 3i$$

$$r_1 + r_2 = \textcircled{4}$$

$$\begin{aligned} r_1 \cdot r_2 &= (2+3i)(2-3i) = 4 - (3i)^2 \\ &= 4 - (3)^2(i)^2 = 4 + 9 = \textcircled{13} \end{aligned}$$

$$r^2 - (r_1 + r_2)r + r_1 r_2 = 0$$

$$r^2 - 4r + 13 = 0$$

$$y'' - 4y' + 13y = 0$$

① Euler Formula: $e^{ib} = \cos b + i \sin b$
 $e^{-ib} = \cos b - i \sin b$

$$y_1 = e^{(\lambda + \mu i)x} = e^{\lambda x} e^{i\mu x}$$

$$= e^{\lambda x} [\cos \mu x + i \sin \mu x]$$

$$\therefore y_1 = e^{\lambda x} \cos \mu x + i e^{\lambda x} \sin \mu x \quad \text{--- (1)}$$

$$y_2 = e^{(\lambda - \mu i)x} = e^{\lambda x} e^{-i\mu x}$$

$$= e^{\lambda x} [\cos \mu x - i \sin \mu x]$$

$$\therefore y_2 = e^{\lambda x} \cos \mu x - i e^{\lambda x} \sin \mu x \quad \text{--- (2)}$$

$$\therefore y_1 + y_2 = 2 e^{\lambda x} \cos \mu x$$

$$\Rightarrow \frac{1}{2} (y_1 + y_2) = e^{\lambda x} \cos \mu x$$

Similarly, $\frac{1}{2i} (y_1 - y_2) = e^{\lambda x} \sin \mu x$

H.W

① If $\{e^x, e^{2x}\}$ is a basis of
 $y'' + ay' + by = 0$, find a, b .

② If $\left\{ e^{(-1+i)x}, e^{(-1-i)x} \right\}$ is a basis
of $y'' + ay' + by = 0$, find a, b .

Euler - Cauchy Equations

$$x^2 y'' + a x y' + b y = 0 \quad x > 0 \quad \text{--- (*)}$$

③ Let us consider first $xy' + by = 0$

$$\Rightarrow xy' = -by \Rightarrow \frac{y'}{y} = \frac{-b}{x}$$

$$\ln|y| = -b \ln x + c$$

$$\Rightarrow \ln|y| = \ln x^{-b} + c$$

$$\Rightarrow |y| = e^{\ln x^{-b}} e^c$$

$$\Rightarrow y = e^c x^{-b}$$

So, we try a solution for (*) of the form

$$\boxed{y = x^r}$$

$$\therefore y' = r x^{r-1}, \quad y'' = r(r-1) x^{r-2}$$

Substitute in (*):

$$x^2 \cdot r(r-1) x^{r-2} + a x \cdot r x^{r-1} + b x^r = 0$$

$$\Rightarrow r(r-1)x^r + arx^r + bx^r = 0$$

$$\therefore r(r-1) + ar + b = 0$$

This 2nd-order algebraic equation has 3-cases:

① Different Real Roots $r_1 \neq r_2$

$$y_1 = x^{r_1}, \quad y_2 = x^{r_2}$$

② Equal Roots $r_1 = r_2 = r$

$$y_1 = x^r, \quad y_2 = x^r \ln x$$

③ Complex Roots $\lambda \pm \mu i$

$$y_1 = x^{\hat{\lambda}} \cos(\mu \ln x)$$

$$y_2 = x^{\hat{\lambda}} \sin(\mu \ln x)$$

Example: Solve the following ODEs:

$$\textcircled{1} \quad 2x^2 y'' + 3xy' - y = 0$$

Sol

$$2r(r-1) + 3r - 1 = 0$$

$$\Rightarrow 2r^2 - 2r + 3r - 1 = 0 \Rightarrow 2r^2 + r - 1 = 0$$

$$\Rightarrow (2r - 1)(r + 1) = 0$$

$$\Rightarrow 2r - 1 = 0, \quad r + 1 = 0$$

$$\Rightarrow r = \frac{1}{2}, \quad -1$$

$$\therefore y_1 = x^{1/2} = \sqrt{x}$$

$$y_2 = x^{-1} = \frac{1}{x}$$

$\therefore$ The General Solution is:

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 \sqrt{x} + \frac{c_2}{x}$$

$x > 0$

$$(2) \quad x^2 y'' - 5xy' + 9y = 0$$

Sol:

$$r(r-1) - 5r + 9 = 0$$

$$\Rightarrow r^2 - 6r + 9 = 0 \Rightarrow (r-3)(r-3) = 0$$

$$\therefore r = 3, 3$$

$$\therefore y_1 = x^3, \quad y_2 = x^3 \ln x$$

$\therefore$ The General Solution is:

$$\begin{aligned} y &= C_1 x^3 + C_2 x^3 \ln x \\ &= x^3 (C_1 + C_2 \ln x) \end{aligned}$$

$$(3) \quad x^2 y'' - 5xy' + 13y = 0$$

$$r(r-1) - 5r + 13 = 0$$

$$\Rightarrow r^2 - 6r + 13 = 0$$

$$a=1 \quad b=-6 \quad c=13$$

$$b^2 - 4ac = 36 - 4(1)(13)$$

$$= 36 - 52$$

$$= -16$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{-16}}{2}$$

$$\therefore r = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

$$\lambda = 3 \quad \mu = 2$$

$$\therefore y_1 = x^{\lambda} G_{\mu}(\mu \ln x)$$

$$\therefore y_1 = x^3 G_2(2 \ln x)$$

$$y_2 = x^3 \sin(2 \ln x)$$

$\therefore$ The General solution is:

$$\begin{aligned} y &= C_1 x^3 G_2(2 \ln x) + C_2 x^3 \sin(2 \ln x) \\ &= x^3 [C_1 G_2(2 \ln x) + C_2 \sin(2 \ln x)] \end{aligned}$$

$$\textcircled{4} \quad x y'' + 4 y' = 0, \quad x > 0$$

$$\underline{\text{Sol:}} \quad x^2 y'' + 4 x y' = 0$$

$$r(r-1) + 4r = 0$$

$$\Rightarrow r^2 + 3r = 0 \Rightarrow r(r+3) = 0$$

$$\therefore r = 0, -3$$

$$y_1 = x^0 = 1$$

$$y_2 = x^{-3} = \frac{1}{x^3}$$

$\therefore$ The General Solution is :

$$y = C_1 + \frac{C_2}{x^3}$$

Ex: If $\{x^2 \cos(\ln x), x^2 \sin(\ln x)\}$ is a fundamental set of solutions of

$$x^2 y'' + ax y' + by = 0, \text{ find } a, b.$$

Sol: $\lambda = 2, \mu = 1$

$$\therefore \lambda \pm \mu i = 2 \pm i$$

$$r_1 \longleftrightarrow 2+i$$

$$r_2 \longleftrightarrow 2-i$$

$$\therefore r_1 + r_2 = 4$$

$$r_1 r_2 = (2+i)(2-i)$$

$$= 4 - (i)^2 = 4 + 1 = \textcircled{5}$$

The Char. Eqn Can be written as:

$$r^2 - (r_1 + r_2)r + r_1 r_2 = 0$$

$$r^2 - 4r + 5 = 0$$

$$\therefore r(r-1) - 3r + 5 = 0$$

$$x^2 y'' - 3x y' + 5y = 0$$

$$\therefore a = -3, \quad b = 5$$

H.W: If $\{ \cos(2 \ln x), \sin(2 \ln x) \}$ is
a basis of solutions of:

$$a x^2 y'' + b x y' + c y = 0,$$

find a , b , and c .

H.W

① Solve the following ODEs:

$$① \quad 4x^2 y'' + y = 0$$

$$② \quad 4x^2 y'' + 4x y' - y = 0$$

$$③ \quad x y'' - 3x y' + 6y = 0$$

$$④ \quad y'' = \frac{y}{x^2}$$

Non homogeneous 2nd-order Linear ODEs.

Consider the 2nd-order linear ODE:

$$y'' + p(x)y' + q(x)y = r(x), \quad r(x) \neq 0 \quad \text{--- (1)}$$

② let $y_h(x) = c_1 y_1 + c_2 y_2$ be the general solution of $y'' + p(x)y' + q(x)y = 0$ --- (2)

③ let $y_p(x)$ be any solution of (1) containing no arbitrary constants

④ The general solution of (1) is given by

$$y = y_h + y_p$$

Remark: If z_1, z_2 are two solutions of (1), then $z_1 - z_2$ is a solution of (2)

Method of Undetermined Coefficients

The method of undetermined coefficients is used to find $y_p(x)$ for a linear D.E. with constant coefficients:

$$ay'' + by' + cy = r(x)$$

and $r(x)$ is one of the following

① polynomial

② e^{ax}

③ $\sin ax$ or $\cos ax$

④ product of 1, 2, 3

⑤ Linear Combination of 1, 2, 3, 4

Ex: Solve: $y'' + y = 2x^2 + 6$ $y(0) = 0$ $y'(0) = 1$ ----- (*)

① $y'' + y = 0$

$$r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i$$

$$\therefore y_1 = \cos x, \quad y_2 = \sin x$$

$$\therefore y_h(x) = C_1 \cos x + C_2 \sin x$$

Let $y_p(x) = Ax^2 + Bx + C$ be a solution of (*)

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

By substitution in (*) we obtain

$$2A + Ax^2 + Bx + C = 2x^2 + 6$$

$$\Rightarrow Ax^2 + Bx + 2A + C = 2x^2 + 6$$

$$\Rightarrow A = 2 \quad B = 0 \quad 2A + C = 6$$

$$\therefore C = 2 \quad \Rightarrow 4 + C = 6$$

$$\Rightarrow C = 2$$

$$\therefore y_p(x) = 2x^2 + 2$$

∴ The general solution of (a) is?

$$y(x) = C_1 \cos x + C_2 \sin x + 2x^2 + 2$$

$$y'(x) = -C_1 \sin x + C_2 \cos x + 4x$$

$$y(0) = 0 \Rightarrow C_1 + 2 = 0 \Rightarrow C_1 = -2$$

$$y'(0) = 1 \Rightarrow C_2 = 1$$

$$\therefore y(x) = -2 \cos x + \sin x + 2x^2 + 2$$

Ex: Solve: $y'' - 2y' = e^{3x}$ ----- (a)

$$\textcircled{ii} \quad y'' - 2y' = 0$$

$$r^2 - 2r = 0 \Rightarrow r(r-2) = 0 \Rightarrow r = 0, 2$$

$$y_1 = e^{0x} = 1 \quad y_2 = e^{2x}$$

$$\therefore y_h(x) = C_1 + C_2 e^{2x}$$

$\textcircled{iii}$ let $y_p(x) = A e^{3x}$ be a solution of (a)

$$y_p' = 3A e^{3x}$$

$$y_p'' = 9A e^{3x}$$

By substitution in * we obtain:

$$9Ae^{3x} - 2(3Ae^{3x}) = e^{3x}$$

$$3Ae^{3x} = e^{3x} \Rightarrow 3A = 1$$

$$\therefore A = \frac{1}{3}$$

$$\therefore y_p(x) = \frac{1}{3} e^{3x}$$

$\therefore$ The general solution of * is:

$$y = y_h + y_p$$

$$= c_1 + c_2 e^{2x} + \frac{1}{3} e^{3x}$$

Ex: Solve: $y'' + 2y' = 12 \sin x$

$$\textcircled{III} \quad y'' + 2y' = 0$$

$$r^2 + 2r = 0 \Rightarrow r(r+2) = 0 \Rightarrow r = 0, -2$$

$$y_1 = e^{0x} = 1, \quad y_2 = e^{-2x}$$

$$\therefore y_h(x) = c_1 + c_2 e^{-2x}$$

$$\textcircled{11} \text{ Let } y_p(x) = A \sin x + B \cos x$$

$$y'_p = A \cos x - B \sin x$$

$$y''_p = -A \sin x - B \cos x$$

By substitution in $\textcircled{8}$ we obtain:

$$-A \sin x - B \cos x + 2[A \cos x - B \sin x] = 12 \sin x$$

$$(-A - 2B) \sin x + (2A - B) \cos x = 12 \sin x$$

$$\Rightarrow -A - 2B = 12 \quad \text{---} \textcircled{1}$$

$$2A - B = 0 \quad \text{---} \textcircled{2}$$

$$2\textcircled{1} + \textcircled{2} \Rightarrow -5B = 24 \Rightarrow B = -\frac{24}{5}$$

$$\text{From } \textcircled{2} \quad A = \frac{B}{2} = -\frac{12}{5}$$

$$\therefore y_p(x) = -\frac{12}{5} \sin x - \frac{24}{5} \cos x$$

$\therefore$ The general solution of $\textcircled{8}$ is:

$$y(x) = C_1 + C_2 e^{-2x} - \frac{12}{5} \sin x - \frac{24}{5} \cos x$$

Ex: Solve: $y'' - 2y' = 2e^{2x}$ ---- (*)

② $y'' - 2y' = 0$

$$r^2 - 2r = 0 \Rightarrow r(r-2) = 0$$

$$\Rightarrow r = 0, 2$$

$$y_1 = e^{0x} = 1, \quad y_2 = e^{2x}$$

$$\therefore y_h(x) = C_1 + C_2 e^{2x}$$

③ Let $y_p(x) = A e^{2x}$

$$y_p' = 2A e^{2x}$$

$$y_p'' = 4A e^{2x}$$

Substitute in (*) we obtain:

$$4A e^{2x} - 2(2A e^{2x}) = 2e^{2x}$$

$$\therefore 0 = 2e^{2x} \quad \text{Contradiction!}$$

So, what is the problem!!

Try $y_p(x) = Ax e^{2x}$

$$y_p' = Ax (2e^{2x}) + A e^{2x}$$

$$\therefore y_p' = (2Ax + A) e^{2x}$$

$$\begin{aligned} y_p'' &= (2Ax + A) (2e^{2x}) + 2A e^{2x} \\ &= 4Ax e^{2x} + 4A e^{2x} \end{aligned}$$

Substitute in $\textcircled{*}$:

$$4Ax e^{2x} + 4A e^{2x} - 2(2Ax + A) e^{2x} = 2e^{2x}$$

$$4Ax e^{2x} + 4A e^{2x} - 4Ax e^{2x} - 2A e^{2x} = 2e^{2x}$$

$$\Rightarrow 2A e^{2x} = 2e^{2x}$$

$$\therefore 2A = 2 \Rightarrow A = 1$$

$$\therefore y_p(x) = x e^{2x}$$

$\therefore$ The General solution of $\textcircled{*}$ is:

$$y = c_1 + c_2 e^{2x} + x e^{2x}$$

Ex: Give the form of $y_p(x)$ if the method of undetermined coefficients is to be used:

$$\textcircled{1} \quad y'' - 5y' + 6y = x^2 + 2x + 4 + 2e^{-x}$$

$$\textcircled{2} \quad y'' - 5y' + 6y = 0$$

$$r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0$$

$$\Rightarrow r = 2, 3$$

$$\therefore y_1 = e^{2x}, \quad y_2 = e^{3x}$$

$$\text{Try } y_p(x) = Ax^2 + Bx + C + De^{-x}$$

$$\textcircled{2} \quad y'' + 4y = 2x \sin x + 4 \cos 2x$$

$$\textcircled{3} \quad y'' + 4y = 0$$

$$r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm 2i$$

$$y_1 = \cos 2x \quad y_2 = \sin 2x$$

$$y_p(x) = (Ax + B) \sin x + (Cx + D) \cos x + \overset{\downarrow}{x} E \cos 2x + \overset{\downarrow}{x} F \sin 2x$$

$$(3) y'' + 2y' + 2y = e^x \cos x + 2x e^{-x} \sin x$$

$$(11) y'' + 2y' + 2y = 0$$

$$y^2 + 2y + 2 = 0$$

$$a=1 \quad b=2 \quad c=2$$

$$b^2 - 4ac = 4 - 4(1)(2) = -4$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

$$y_1 = e^{-x} \cos x, \quad y_2 = e^{-x} \sin x$$

$$\therefore y_p(x) = A e^x \cos x + B e^x \sin x$$

$$+ \downarrow \times (A_2 x + A_1) e^{-x} \sin x + \downarrow \times (B_2 x + B_1) e^{-x} \cos x$$

$$(5) \quad y'' + 2y' = 3x e^{-2x} + x^2 + 4$$

$$\textcircled{11} \quad y'' + 2y' = 0$$

$$r^2 + 2r = 0 \Rightarrow r(r+2) = 0 \Rightarrow r = 0, -2$$

$$\therefore y_1 = 1, \quad y_2 = e^{-2x}$$

$$y_{p(x)} = \overset{\downarrow}{x} (Ax + B) e^{-2x} + \overset{\downarrow}{x} (Cx^2 + Dx + E)$$

$$(6) \quad y'' - 4y' + 4y = x e^{2x} + 3e^{-2x} + 5$$

$$\textcircled{11} \quad y'' - 4y' + 4y = 0$$

$$r^2 - 4r + 4 = 0 \Rightarrow (r-2)(r-2) = 0$$

$$\therefore r = 2, 2$$

$$\therefore y_1 = e^{2x}, \quad y_2 = x e^{2x}$$

$$y_{p(x)} = \overset{\downarrow}{x^2} (Ax + B) e^{2x} + C e^{-2x} + D$$

$$(7) \quad y'' + 4y = \cos^2 x$$

$$(8) \quad y'' + 4y = \sin x \cos x$$

$$(9) \quad y'' - 4y = e^{2(x+i)}$$

Variation of parameters

Consider :

$$y'' + p(x)y' + q(x)y = r(x), \quad r(x) \neq 0 \quad \text{--- (1)}$$

where $p(x)$, $q(x)$ and $r(x)$ are continuous functions on some open interval I .

⑦ let y_1, y_2 be independent solutions of

$$y'' + p(x)y' + q(x)y = 0 \quad \text{--- (2)}$$

i.e., $y_h(x) = c_1 y_1 + c_2 y_2$.

⑧ We can find $y_p(x)$ for (1) as follows:

$$y_p(x) = -y_1 \int \frac{y_2}{W} r + y_2 \int \frac{y_1}{W} r$$

Ex: Solve: $y'' + y = \sec x$

$$\textcircled{1} y'' + y = 0$$

$$r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i$$

$$y_1 = \cos x, \quad y_2 = \sin x$$

$$\therefore y_h(x) = C_1 \cos x + C_2 \sin x$$

$$\begin{aligned} \textcircled{2} W(\cos x, \sin x) &= \cos x \cdot \cos x - \sin x (-\sin x) \\ &= \cos^2 x + \sin^2 x \\ &= 1 \end{aligned}$$

$$\textcircled{3} r(x) = \sec x$$

$$\begin{aligned} \therefore y_p(x) &= -y_1 \int \frac{y_2}{W} r + y_2 \int \frac{y_1}{W} r \\ &= -\cos x \int \sin x \sec x dx + \sin x \int \cos x \sec x dx \\ &= -\cos x \int \frac{\sin x}{\cos x} dx + \sin x \int \cos x \cdot \frac{1}{\cos x} dx \end{aligned}$$

$$\therefore y_p(x) = \cos x \ln |\cos x| + x \sin x$$

∴ The General solution is :

$$y = y_h + y_p$$

$$= C_1 \cos x + C_2 \sin x + C_3 x \ln |C_1 x| + x \sin x$$

Ex: Solve: $x^2 y'' - 2x y' - 4y = 12x^{-3}$ ---- ①

② $x^2 y'' - 2x y' - 4y = 0$

$$r(r-1) - 2r - 4 = 0$$

$$r^2 - 3r - 4 = 0 \implies (r+1)(r-4) = 0$$

$$\implies r = -1, 4$$

$$\therefore y_1 = x^{-1}, \quad y_2 = x^4$$

$$\therefore y_h(x) = \frac{C_1}{x} + C_2 x^4$$

③ $W\left(\frac{1}{x}, x^4\right) = \frac{1}{x} (4x^3) - x^4 \cdot \left(-\frac{1}{x^2}\right)$

$$= 4x^2 + x^2 = 5x^2$$

④ $y(x) = \frac{12x^{-3}}{x^2} = 12x^{-5}$

$$\textcircled{11} y_p(x) = -y_1 \left(\frac{y_2}{W} x + y_2 \right) \frac{y_1}{W} x$$

$$= -\frac{1}{x} \left(\frac{x^4}{5x^2} \cdot 12x^{-5} dx + x^4 \left(\frac{1}{x} \cdot 12x^{-5} dx \right) \right)$$

$$= -\frac{12}{5} \cdot \frac{1}{x} \left(x^{-3} dx + \frac{12}{5} x^4 \right) x^{-8} dx$$

$$= \frac{-12}{5x} \cdot \frac{x^{-2}}{-2} + \frac{12}{5} x^4 \cdot \frac{x^{-7}}{-7}$$

$$= \frac{12}{10} x^{-3} - \frac{12}{35} x^{-3}$$

$$= \frac{6}{7} x^{-3}$$

$\textcircled{11}$ The general solution is:

$$y = y_h + y_p$$

$$= \frac{C_1}{x} + C_2 x^4 + \frac{6}{7} x^{-3}$$

H.w Solve: $y'' + 2y' + y = 4e^{-x} \ln x$

Revision

Consider: $x^3 - 2x^2 - 5x + 6$

How to factor such expression!

Try $x=1$. Then

$$x^3 - 2x^2 - 5x + 6 = (x-1) (\text{-----})$$

1	1	-2	-5	6
		1	-1	-6
	1	-1	-6	0

$$\therefore x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 - x - 6)$$

$$= (x-1)(x-3)(x+2)$$

Ex: solve the following ODEs:

$$\textcircled{1} \quad y''' - 2y'' - 5y' + 6y = 0$$

$$r^3 - 2r^2 - 5r + 6 = 0$$

$$\Rightarrow (r-1)(r-3)(r+2) = 0$$

$$\Rightarrow r = 1, 3, -2$$

$$\therefore y_1 = e^x, \quad y_2 = e^{3x}, \quad y_3 = e^{-2x}$$

$$\therefore y(x) = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x}$$

$$\textcircled{2} \quad y^{(5)} - 3y^{(4)} + 3y''' - y'' = 0$$

$$r^5 - 3r^4 + 3r^3 - r^2 = 0$$

$$\Rightarrow r^2 (r^3 - 3r^2 + 3r - 1) = 0$$

$$\Rightarrow r^2 (r-1)(r-1)(r-1) = 0$$

$$\Rightarrow r = 0, 0, 1, 1, 1$$

$$\therefore y_1 = 1, y_2 = x, y_3 = e^x, y_4 = xe^x$$

$$y_5 = x^2 e^x$$

$$\therefore y(x) = C_1 + C_2 x + C_3 e^x + C_4 x e^x + C_5 x^2 e^x$$

Remark:

$$y^3 - 3y^2 + 3y - 1 = y^3 - 1 - 3y^2 + 3y$$

$$= (y-1)(y^2 + y + 1) - 3y(y-1)$$

$$= (y-1)(y^2 + y + 1 - 3y)$$

$$= (y-1)(y^2 - 2y + 1)$$

$$= (y-1)(y-1)(y-1)$$

$$\textcircled{3} \quad y^{(4)} + 2y'' + y = 0$$

$$y^4 + 2y^2 + 1 = 0$$

$$\therefore (y^2)^2 + 2(y^2) + 1 = 0$$

$$\Rightarrow (y^2 + 1)(y^2 + 1) = 0$$

$$\Rightarrow y^2 + 1 = 0 \quad y^2 + 1 = 0$$

$$\therefore y = \pm i$$

$$y = \pm i$$

$$y_1 = \cos x$$

$$y_3 = x \cos x$$

$$y_2 = \sin x$$

$$y_4 = x \sin x$$

$$\therefore y(x) = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$$

$$\textcircled{4} \quad x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$$

$$r(r-1)(r-2) - 3r(r-1) + 6r - 6 = 0$$

$$r(r-1)(r-2) - 3r(r-1) + 6(r-1) = 0$$

$$(r-1)(r(r-2) - 3r + 6) = 0$$

$$\therefore (r-1)(r^2 - 5r + 6) = 0$$

$$\therefore (r-1)(r-2)(r-3) = 0$$

$$\Rightarrow r = 1, 2, 3$$

$$\therefore y_1 = x, \quad y_2 = x^2, \quad y_3 = x^3$$

$$\therefore y(x) = C_1 x + C_2 x^2 + C_3 x^3$$

Ex: Find a 3rd-order linear homogeneous

D.E. whose solution is:

$$y(x) = c_1 + c_2 x + c_3 e^{2x}$$

Sol:

$$r = 0, 0, 2$$

$$\therefore r^2(r-2) = 0$$

$$r^3 - 2r^2 = 0$$

$$\therefore y''' - 2y'' = 0$$

Ex: Solve: $y''' + y'' = 2x + 1$ --- (*)

② $y''' + y'' = 0$

$$r^3 + r^2 = 0 \implies r^2(r+1) = 0$$

$$\implies r = 0, 0, -1$$

$$\therefore y_1 = 1, \quad y_2 = x, \quad y_3 = e^{-x}$$

$$\therefore y_h(x) = C_1 + C_2 x + C_3 e^{-x}$$

③ let $y_p(x) = (Ax + B) \cdot x^2$

$$\therefore y_p(x) = Ax^3 + Bx^2$$

$$y_p' = 3Ax^2 + 2Bx$$

$$y_p'' = 6Ax + 2B$$

$$y_p''' = 6A$$

By substitution in (*) we get:

$$6A + 6Ax + 2B = 2x + 1$$

$$\Rightarrow 6Ax + 6A + 2B = 2x + 1$$

$$\Rightarrow 6A = 2 \quad \text{and} \quad 6A + 2B = 1$$

$$\Rightarrow A = \frac{1}{3}, \quad B = -\frac{1}{2}$$

$$\therefore y_p(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2$$

$\therefore$ The general solution of $\textcircled{*}$ is given by:

$$y = y_h + y_p$$

$$= C_1 + C_2x + C_3e^{-x} + \frac{1}{3}x^3 - \frac{1}{2}x^2$$

Ex: Consider the CDE:

$$y^{(4)} + 2y''' + 2y'' = x^2 + 3 + 2e^{-x} + xe^x \sin x$$

Write a suitable form for $y_p(x)$ if the undetermined coefficient, is to be used.

$$\textcircled{111} \quad y^{(4)} + 2y''' + 2y'' = 0$$

$$r^4 + 2r^3 + 2r^2 = 0$$

$$\Rightarrow r^2(r^2 + 2r + 2) = 0$$

$$\therefore r = 0, 0, -1 \pm i$$

$$y_1 = 1, \quad y_2 = x, \quad y_3 = e^{-x} \cos x$$

$$y_4 = e^{-x} \sin x$$

$$\textcircled{112} \quad y_p(x) = (A_2 x^2 + A_1 x + A_0) \cdot \downarrow x^2$$

$$+ B e^{-x}$$

$$+ (Cx + D) e^x \sin x + (Ex + F) e^x \cos x$$

Higher-order Non homogeneous ODEs

"Variation of Parameters"

Consider the 3rd-order linear ODE:

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = r(x) \quad \text{--- } (*)$$

where $p_2(x)$, $p_1(x)$, $p_0(x)$ and $r(x)$ are continuous on some open interval I .

Ⓐ The general solution of $*$ can be written as:

$$y = y_h + y_p$$

where $y_p(x) = y_1 \int \frac{W_1}{W} r + y_2 \int \frac{W_2}{W} r + y_3 \int \frac{W_3}{W} r$

Ⓑ y_1, y_2, y_3 are independent solutions of the homogeneous part.

$$\textcircled{1} \quad W = W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$\textcircled{1}$ W_J is obtained from W by replacing the J^{th} -column of W by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Ex: Find the following?

① $W(x, x^2, x^3)$

② $W_1(x, x^2, x^3)$

③ $W_2(x, x^2, x^3)$

④ $W_3(x, x^2, x^3)$

$$\textcircled{1} \quad W(x, x^2, x^3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x \begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} - (1) \begin{vmatrix} x^2 & x^3 \\ 2 & 6x \end{vmatrix} + 0 \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= x (12x^2 - 6x^2) - (6x^3 - 2x^3)$$

$$= 6x^3 - 4x^3 = 2x^3$$

$$\textcircled{2} \quad W_1(x, x^2, x^3) = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix}$$

$$= (1) \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= 3x^4 - 2x^4 = x^4$$

$$\textcircled{3} \quad W_2(x, x^2, x^3) = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix}$$

$$= (-1) \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = -(3x^3 - x^3)$$

$$= -2x^3$$

$$(4) \quad W_3(x, x^2, x^3) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= (1) \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

Ex: Solve the ODE:

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 24x^4, \quad x > 0$$

$$\textcircled{1} \quad x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$$

$$\Rightarrow y_1 = x, \quad y_2 = x^2, \quad y_3 = x^3$$

$$\therefore y_h(x) = C_1 x + C_2 x^2 + C_3 x^3$$

$$\textcircled{2} \quad y_p(x) = y_1 \int \frac{W_1}{W} r + y_2 \int \frac{W_2}{W} r + y_3 \int \frac{W_3}{W} r$$

$$\textcircled{3} \quad W(x, x^2, x^3) = 2x^3$$

$$\textcircled{4} \quad W_1 = x^4 \quad W_2 = -2x^3 \quad W_3 = x^2$$

$$\textcircled{5} \quad y(x) = \frac{24x^4}{x^3} = 24x$$

$$\therefore y_p(x) = x \int \frac{x^4}{2x^3} 24x \, dx + x^2 \int \frac{-2x^3}{2x^3} 24x \, dx \\ + x^3 \int \frac{x^2}{2x^3} 24x \, dx$$

$$= 12x \int x^2 \, dx - 24x^2 \int x \, dx + 12x^3 \int dx$$

$$= 12x \cdot \frac{x^3}{3} - 24x^2 \cdot \frac{x^2}{2} + 12x^3 \cdot x$$

$$= 4x^4 - 12x^4 + 12x^4$$

$$= 4x^4$$

$\therefore$ The General solution of (2) is :

$$y = y_h + y_p$$

$$= C_1 x + C_2 x^2 + C_3 x^3 + 4x^4$$

H.W: Solve: $y''' + y' = \sec x$.

Homogeneous Linear System with Constant Coefficients

Q1) Consider the linear system,

$$a y_1 + b y_2 = f_1$$

$$c y_1 + d y_2 = f_2$$

Then this system can be written as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

or $A \underline{y} = \underline{f}$ where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \underline{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

Q2) If $\underline{f} = \underline{0}$, then the system is called homogeneous, otherwise it is called nonhomogeneous.

① A^{-1} exists $\Leftrightarrow |A| \neq 0$

② If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| \neq 0$

then $A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Ex: If $A = \begin{bmatrix} 4 & 2 \\ 3 & 4 \end{bmatrix}$, then

$$|A| = -2 \quad \text{and} \quad A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \\ = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

③ $AA^{-1} = A^{-1}A = I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

④ If $|A| \neq 0$, then $A\underline{y} = \underline{0}$ has only the trivial solution $\underline{y} = \underline{0}$

① If $|A| = 0$, then $A\underline{v} = \underline{0}$ has infinitely many solutions.

② If $\underline{\xi}$ is a non zero vector such that

$$A\underline{\xi} = \lambda \underline{\xi}$$

then $\underline{\xi}$ is called an eigenvector of A corresponding to the eigenvalue λ

How to find eigenvectors and eigenvalues!!

$$A\underline{\xi} = \lambda \underline{\xi} \Leftrightarrow A\underline{\xi} - \lambda \underline{\xi} = \underline{0}$$

$$\Leftrightarrow [A - \lambda I] \underline{\xi} = \underline{0}$$

$$\text{Since } \underline{\xi} \neq \underline{0} \Rightarrow |A - \lambda I| = 0$$

Ex: Find the eigenvalues and the corresponding eigenvectors to $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$$\begin{aligned} [A - \lambda I] &= \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} \end{aligned}$$

$$\textcircled{1} |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda) - 4 = 0$$

$$\Rightarrow (\lambda-1)^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$\Rightarrow (\lambda+1)(\lambda-3) = 0$$

$$\Rightarrow \lambda = -1, 3$$

$$\textcircled{1} [A - \lambda I] \underline{\xi} = \underline{0}$$

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda = -1 \Rightarrow \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2y_1 + y_2 = 0 \Rightarrow y_2 = -2y_1$$

$$\therefore \underline{\xi}^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\textcircled{2} \lambda = 3 \Rightarrow \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2y_1 + y_2 = 0 \Rightarrow y_2 = 2y_1$$

$$\therefore \underline{\xi}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Now, consider the system:

$$y_1' = ay_1 + by_2$$

$$y_2' = cy_1 + dy_2$$

$$\text{or } \underline{y}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\therefore \underline{y}' = A \underline{y} \quad \dots \textcircled{*}$$

To solve $\textcircled{*}$, we try a solution of the form

$$\underline{y} = \underline{\xi} e^{\lambda t}$$

$$\Rightarrow \underline{y}' = \lambda \underline{\xi} e^{\lambda t}$$

substitute in $\textcircled{*}$:

$$\lambda \underline{\xi} e^{\lambda t} = A \underline{\xi} e^{\lambda t}$$

$$\Rightarrow \lambda \underline{\xi} = A \underline{\xi}$$

Hence, $\underline{y} = \underline{\xi} e^{\lambda t}$ is a solution of $\textcircled{*}$

if $\underline{\xi}$ is an eigenvector corresponding to the eigenvalue λ .

Ex: Solve: $y'_1 = y_1 + y_2$
 $y'_2 = 4y_1 + y_2$

Sol: $\underline{y}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \underline{y}$

The solution is given by $\underline{y} = e^{\lambda t} \underline{\xi}$

③ $[A - \lambda I] = \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2 - 4 = 0 \Rightarrow \lambda = -1, 3$

③ $\lambda = -1 \iff \underline{\xi}^{(1)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

③ $\lambda = 3 \iff \underline{\xi}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore \underline{y}^{(1)} = \underline{\xi}^{(1)} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$

$\underline{y}^{(2)} = \underline{\xi}^{(2)} e^{\lambda_2 t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$

The general solution is $\underline{y} = c_1 \underline{y}^{(1)} + c_2 \underline{y}^{(2)}$

Ex: Solve: $\underline{y}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \underline{y}$

① $[A - \lambda I] = \begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix}$

② $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)(-4-\lambda) + 6 = 0$$

$$(\lambda-1)(\lambda+4) + 6 = 0$$

$$\lambda^2 + 3\lambda - 4 + 6 = 0$$

$$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda+1)(\lambda+2) = 0$$

$$\Rightarrow \lambda = -1, -2$$

③ $\lambda = -1 \Rightarrow \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow 2y_1 - 2y_2 = 0 \Rightarrow y_2 = y_1$$

$$\therefore \underline{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \underline{y^{(1)}} = \sum_{\underline{i}} \xi_{i1} e^{\lambda_i t}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

$$\textcircled{1} \quad \lambda = -2 \Rightarrow \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3y_1 - 2y_2 = 0$$

$$\Rightarrow y_2 = \frac{3}{2} y_1$$

$$\therefore \underline{\xi^{(2)}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\therefore \underline{y^{(2)}} = \sum_{\underline{i}} \xi_{i2} e^{\lambda_i t} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}$$

$\therefore$ The general solution is:

$$\underline{y} = c_1 \underline{y^{(1)}} + c_2 \underline{y^{(2)}}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}$$

Ex: Solve: $\underline{y}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \underline{y}$

① $[A - \lambda I] = \begin{bmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{bmatrix}$

② $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)(-4-\lambda) + 6 = 0$$

$$(\lambda-1)(\lambda+4) + 6 = 0$$

$$\lambda^2 + 3\lambda - 4 + 6 = 0$$

$$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda+1)(\lambda+2) = 0$$

$$\Rightarrow \lambda = -1, -2$$

③ $\lambda = -1 \Rightarrow \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow 2y_1 - 2y_2 = 0 \Rightarrow y_2 = y_1$$

$$\therefore \underline{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore y^{(1)} = \sum e^{\lambda_1 t}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

① $\lambda = -2 \Rightarrow \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow 3y_1 - 2y_2 = 0$$

$$\Rightarrow y_2 = \frac{3}{2}y_1$$

$$\therefore \xi^{(2)} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\therefore y^{(2)} = \sum e^{\lambda_2 t} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}$$

$\therefore$ The General solution is:

$$\underline{y} = c_1 \underline{y}^{(1)} + c_2 \underline{y}^{(2)}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t}$$

③ Complex eigenvalues

Consider

$$\underline{y}' = A \underline{y} \quad \text{---} (*)$$

If $\underline{y}(t) = \underline{u}(t) + i \underline{v}(t)$ is a complex solution of $(*)$, then both $\underline{u}(t)$ and $\underline{v}(t)$ are real solutions.

Ex: solve:

$$\begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned}$$

sol:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\therefore \underline{y}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \underline{y}$$

$$\therefore \underline{y}' = A \underline{y}$$

③ To find the eigenvalues we set

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)(-1-\lambda) + 1 = 0$$

$$\Rightarrow (\lambda+1)(\lambda+1) + 1 = 0$$

$$(\lambda+1)^2 = -1$$

$$\Rightarrow \lambda+1 = \pm i$$

$$\therefore \lambda = -1 \pm i$$

③ To find the eigenvectors we set

$$[A - \lambda I] \underline{\xi} = \underline{0}$$

$$\begin{bmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = -1 + i$:

$$\begin{bmatrix} -1+1-i & 1 \\ -1 & -1+1-i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -i y_1 + y_2 = 0$$

$$\Rightarrow y_2 = i y_1$$

$$\therefore \vec{y}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\therefore y^{(1)} = e^{(-1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{-t} e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{-t} (\cos t + i \sin t) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{-t} \begin{bmatrix} \cos t + i \sin t \\ -\sin t + i \cos t \end{bmatrix}$$

$$= e^{-t} \left(\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \right)$$

$$\therefore \underline{y}^{(1)}(t) = e^{-t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + i e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$$\therefore \underline{y}^{(1)}(t) = e^{-t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}, \quad \underline{y}^{(2)}(t) = e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

$\therefore$ The General solution is :

$$\underline{y}(t) = c_1 e^{-t} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

H.W: Solve: $\underline{y}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \underline{y}$

③ Equal Eigen values

Ex: Solve: $\underline{y}' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \underline{y}$

Sol: To find eigen values we set:

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(2-\lambda) + 1 = 0$$

$$\Rightarrow (\lambda-4)(\lambda-2) + 1 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 9 = 0$$

$$\Rightarrow (\lambda-3)(\lambda-3) = 0$$

$$\therefore \lambda = 3, 3$$

⑩ To find eigenvectors we set:

$$[A - \lambda I] \underline{\xi} = \underline{0}$$

$$\Rightarrow \begin{bmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\lambda = 3$:

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1 + y_2 = 0 \Rightarrow y_2 = -y_1$$

$$\therefore \underline{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore y^{(1)}(t) = e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

⑪ To find a 2nd-independent solution,

we try a solution of the form

$$\underline{y}^{(2)}(t) = \underline{\xi} t e^{3t} + \underline{\eta} e^{3t}$$

② $\underline{\eta}$ is called a generalized eigenvector,
and it can be obtained as follows:

$$[A - \lambda I] \underline{\eta} = \underline{\xi}$$

$$\text{Set } \lambda = 3$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \eta_1 + \eta_2 = 1$$

$$\Rightarrow \eta_2 = 1 - \eta_1$$

$$\therefore \underline{\eta} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \underline{y}^{(2)}(t) = \underline{\xi} t e^{3t} + \underline{\eta} e^{3t}$$

$$= t e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \left(t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t}$$

$\therefore$ The generalized solution is :

$$\underline{y} = c_1 \underline{y}^{(1)} + c_2 \underline{y}^{(2)}$$

$$= c_1 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t}$$

H.w: Solve: $\underline{y}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \underline{y}$

Non homogeneous system

$$\underline{y}' = A\underline{y} + \underline{F}(t)$$

③ Undetermined Coefficients

Ex: Solve : $\underline{y}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \underline{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$

③ First we solve $\underline{y}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \underline{y}$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-3-\lambda)(-3-\lambda) - 1 = 0$$

$$\Rightarrow (\lambda+3)(\lambda+3) - 1 = 0$$

$$\therefore (\lambda+3)^2 = 1$$

$$\therefore \lambda+3=1, \quad \lambda+3=-1$$

$$\lambda = -2, -4$$

③ To find the eigenvectors, we solve

$$[A - \lambda I] \underline{\xi} = \underline{0}$$

$$\begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

⑦ For $\lambda = -2$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-y_1 + y_2 = 0 \Rightarrow y_2 = y_1 \quad \therefore \underline{\xi^{(1)}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \underline{y^{(1)}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

⑧ For $\lambda = -4$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y_1 + y_2 = 0 \Rightarrow y_2 = -y_1 \quad \therefore \underline{\xi^{(2)}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore \underline{y^{(2)}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$\therefore \underline{y^{(h)}} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

② To find $\underline{y}^{(p)}$ we assume:

$$\underline{y}^{(p)} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

substitute in the nonhomogeneous system:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$0 = -3a_1 + a_2 - 6 \quad \dots (1)$$

$$0 = a_1 - 3a_2 + 2 \quad \dots (2)$$

$$3(1) + (2) \Rightarrow 0 = -8a_1 - 16$$

$$\Rightarrow \boxed{a_1 = -2}$$

$$\therefore \boxed{a_2 = 0}$$

$$\therefore \underline{y}^{(p)} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$\therefore$ The General solution is:

$$\underline{y} = \underline{y}^{(h)} + \underline{y}^{(p)}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Ex1 Consider $\underline{y}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \underline{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$

Determine a suitable form for the particular solution $\underline{y}^{(p)}$

Sol: Since $\underline{y}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$, $\underline{y}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$

$$\begin{aligned} \therefore \underline{y}^{(p)}(t) &= (\underline{a}t + \underline{b}) e^{-2t} \\ &= \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) e^{-2t} \end{aligned}$$

Ex 1 Determine a suitable form for the particular solution $\underline{y}^{(p)}$ of the system:

$$y_1' = 5y_1 + 3y_2 - 2e^{-t} + 1$$

$$y_2' = -y_1 + y_2 + e^{-t} - 5t + 7$$

Sol

$$\underline{y}' = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix} \underline{y} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 0 \\ -5 \end{bmatrix} t + \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Since the eigenvalues : $\lambda = 2, \lambda = 4$

We can write $\underline{y}^{(p)}$ in the form:

$$\underline{y}^{(p)} = \underline{a} e^{-t} + \underline{b} t + \underline{c}$$

$$= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{-t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Variation of parameter

③ Let $\underline{y}^{(1)}, \underline{y}^{(2)}$ be two independent solutions of $\underline{y}' = A \underline{y}$

$$\text{Then } \underline{\Phi}(t) = \begin{bmatrix} \vdots & \vdots \\ \underline{y}^{(1)} & \underline{y}^{(2)} \\ \vdots & \vdots \end{bmatrix}$$

is called a fundamental matrix of the system

④ Consider $\underline{y}' = A \underline{y} + \underline{F}(t) \dots (*)$

The particular solution of $(*)$ is given by

$$\underline{y}^{(p)} = \underline{\Phi}(t) \int \underline{\Phi}^{-1}(t) \underline{F}(t) dt$$

Ex: Solve the system:

$$\underline{y}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \underline{y} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

⑤ First we solve: $\underline{y}' = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} \underline{y}$

One can find easily the homogeneous solutions :

$$\underline{y}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t \quad \underline{y}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

$$\therefore \underline{y}^{(h)} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

$$= \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\textcircled{III} \quad \underline{\Phi}(t) = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}$$

$$\underline{\Phi}^{-1} = \frac{1}{e^{3t}} \begin{bmatrix} 2e^{2t} & -e^{2t} \\ -e^t & 2e^t \end{bmatrix} = \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix}$$

$$\textcircled{IV} \quad \underline{\Phi}^{-1} \underline{f} = \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{bmatrix} \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ -3e^t \end{bmatrix}$$

$$\int \underline{\Phi}^{-1} \underline{F} = \int \begin{bmatrix} 2 \\ -3e^{-t} \end{bmatrix} dt$$

$$= \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix}$$

$$\underline{\Phi} \int \underline{\Phi}^{-1} \underline{F} = \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^t \end{bmatrix} \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} 4te^t + 3e^t \\ 2te^t + 3e^t \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix} t e^t + \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^t$$

$$\therefore \underline{y}^{(p)} = \left(\begin{bmatrix} 4 \\ 2 \end{bmatrix} t + \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right) e^t$$

$$\therefore \underline{y} = \underline{y}^{(h)} + \underline{y}^{(p)}$$

Power Series Method

Def: A function f is said to be analytic at x_0 if its Taylor Series about x_0

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$
$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots$$

exists for all x in some open interval I including x_0 .

Ex: ① All polynomials are analytic everywhere.

② e^x , $\sin x$, $\cos x$ are analytic everywhere

③ Rational Functions are analytic except where the denominator is zero

$$f(x) = \frac{1}{x^2 - 3x + 2} \longrightarrow f \text{ is analytic except at } x=1, x=2$$

Def: The point x_0 is called an ordinary point of $A(x)y'' + B(x)y' + C(x)y = 0$ --- (1)

if $\frac{B(x)}{A(x)}$, $\frac{C(x)}{A(x)}$ are analytic at x_0 .

(iii) If at least one of $\frac{B(x)}{A(x)}$, $\frac{C(x)}{A(x)}$ is not analytic at x_0 , then x_0 is called a singular point of (1)

Ex: $y'' + xy' + (x^2+1)y = 0$

The ordinary points are all $x \in \mathbb{R}$.

Ex: $(x-1)y'' + xy' + \frac{1}{x}y = 0$

$\Rightarrow y'' + \frac{x}{x-1}y' + \frac{1}{x(x-1)}y = 0$

singular points : $x=0$ and $x=1$

ordinary points : $\mathbb{R} - \{0, 1\}$

Theorem: let x_0 be an ordinary point of

$$A(x)y'' + B(x)y' + C(x)y = 0$$

Then the above D.E. has two nontrivial linearly independent power series solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

which converges in some open interval I including x_0 .

Remark: $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

$$\Rightarrow y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

$$\sum_{n=2} n(n-1) (x-x_0)^{n-2}$$



$$\sum_{n+2=2} (n+2)[n+2-1] (x-x_0)^{n+2-2}$$

$$\sum_{n=0} (n+2)(n+1) (x-x_0)^n$$

Theorem (Identity principle)

If $\sum_{n=0} a_n x^n = \sum_{n=0} b_n x^n$ for all x in some open interval I

then $a_n = b_n \quad \forall n \geq 0$

In particular, if

$\sum_{n=0} a_n x^n = 0$, then $a_n = 0 \quad \forall n \geq 0$

Ex1 Find a power series solution of

$$y'' + x^2 y = 0 \quad \text{about } x_0 = 0$$

Sol. $x_0 = 0$ is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be a solution.

$$\Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute y, y'' in the D.E.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow 2a_2 + 6a_3x + \sum_{n=2} \left[(n+2)(n+1)a_{n+2} + a_{n-2} \right] x^n = 0$$

$$\Rightarrow 2a_2 = 0 \Rightarrow a_2 = 0$$

$$6a_3 = 0 \Rightarrow a_3 = 0$$

$$(n+2)(n+1)a_{n+2} + a_{n-2} = 0 \quad n \geq 2$$

Recurrence Relation

$$a_{n+2} = \frac{-a_{n-2}}{(n+2)(n+1)} \quad n \geq 2$$

$$n=2 \quad a_4 = \frac{-a_0}{(4)(3)} = \boxed{\frac{-a_0}{12}}$$

$$n=3 \quad a_5 = \frac{-a_1}{(5)(4)} = \boxed{\frac{-a_1}{20}}$$

$$n=4 \quad a_6 = \frac{-a_2}{(6)(5)} = \boxed{0}$$

$$n=5 \quad a_7 = \frac{-a_3}{(7)(6)} = \boxed{0}$$

$$n=6 \quad a_8 = \frac{-a_4}{(8)(7)} = \frac{a_0}{(8)(7)(12)} = \boxed{\frac{a_0}{672}}$$

$$n=7 \quad a_9 = \frac{-a_5}{(9)(8)} = \frac{a_1}{(9)(8)(20)} = \boxed{\frac{a_1}{1440}}$$

The solution of the D.E. is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

$$= a_0 + a_1 x - \frac{a_0}{12} x^4 - \frac{a_1}{20} x^5 - \dots$$

$$= a_0 \left[1 - \frac{x^4}{12} - \dots \right] + a_1 \left[x - \frac{x^5}{20} - \dots \right]$$

Ex: Find a power series solution of

$$y'' - xy = 0 \text{ about } x_0 = 2$$

Sol: $x_0 = 2$ is an ordinary point.

Let $y = \sum_{n=0}^{\infty} a_n (x-2)^n$ be a solution

$$y' = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2}$$

Substitute in $y'' - xy = 0$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} - x \sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$- [(x-2)+2]$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} - \sum_{n=0}^{\infty} a_n (x-2)^{n+1} - \sum_{n=0}^{\infty} 2 a_n (x-2)^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-2)^n - \sum_{n=1}^{\infty} a_{n-1} (x-2)^n - \sum_{n=0}^{\infty} 2 a_n (x-2)^n = 0$$

$$2a_2 - 2a_0 + \sum_{n=1} \left[(n+2)(n+1)a_{n+2} - a_{n-1} - 2a_n \right] (x-2)^n = 0$$

$$\Rightarrow 2a_2 - 2a_0 = 0 \Rightarrow a_2 = a_0$$

$$(n+2)(n+1)a_{n+2} - a_{n-1} - 2a_n = 0 \quad n \geq 1$$

$$\therefore a_{n+2} = \frac{a_{n-1} + 2a_n}{(n+2)(n+1)} \quad n \geq 1$$

$$\boxed{n=1} \quad a_3 = \frac{a_0 + 2a_1}{6} = \frac{a_0}{6} + \frac{a_1}{3}$$

$$\boxed{n=2} \quad a_4 = \frac{a_1 + 2a_2}{(4)(3)} = \frac{a_1}{12} + \frac{a_0}{6}$$

The solution is:

$$y = \sum_{n=0} a_n (x-2)^n$$

$$= a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + \dots$$

$$= a_0 + a_1(x-2) + a_0(x-2)^2 + \left(\frac{a_0}{6} + \frac{a_1}{3}\right)(x-2)^3 + \dots$$

$$= a_0 \left[1 + (x-2)^2 + \frac{(x-2)^3}{6} + \dots \right] + a_1 \left[(x-2) + \frac{(x-2)^3}{3} + \dots \right]$$

Regular singular points

Consider $A(x)y'' + B(x)y' + C(x)y = 0$

where A , B , and C are analytic functions.

Let $x = x_0$ be a singular point, i.e., $A(x_0) = 0$

then x_0 is said to be regular singular if

$$(1) \quad \lim_{x \rightarrow x_0} \frac{B(x)}{A(x)} (x - x_0) = p_0 < \infty$$

$$(2) \quad \lim_{x \rightarrow x_0} \frac{C(x)}{A(x)} (x - x_0)^2 = q_0 < \infty$$

otherwise, x_0 is called an irregular singular point

Def: If x_0 is regular singular, then

we define the indicial equation as:

$$V(r-1) + p_0 r + q_0 = 0$$

Ex: Find all regular singular points.

$$(1) (x^2 - 4x + 3)y'' + 4xy' + 2xy = 0$$

$$x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \\ \Rightarrow x = 1, 3 \text{ (Singular points)}$$

At $\boxed{x_0 = 1}$

$$(1) \lim_{x \rightarrow 1} \frac{4x}{x^2 - 4x + 3} (x-1)$$

$$= \lim_{x \rightarrow 1} \frac{4x}{\cancel{(x-1)}(x-3)} (\cancel{x-1}) = \lim_{x \rightarrow 1} \frac{4x}{x-3} = \frac{4}{-2} = \boxed{-2} \underline{\underline{p_0}}$$

$$(2) \lim_{x \rightarrow 1} \frac{2x}{x^2 - 4x + 3} (x-1)^2$$

$$= \lim_{x \rightarrow 1} \frac{2x}{\cancel{(x-1)}(x-3)} (\cancel{x-1})^2 = \lim_{x \rightarrow 1} \frac{2x(x-1)}{x-3} = \frac{0}{-2} = \boxed{0} \underline{\underline{q_0}}$$

$x_0 = 1$ is a regular singular point.

(iii) Similarly $x_0 = 3$ is a regular singular point.

$\therefore$ The set of all regular singular points
is $\{1, 3\}$

Remark - in the previous example:
we can determine the indicial equation at

$$x_0 = 1$$

$$r(r-1) + p_0 r + q_0 = 0$$

$$r(r-1) - 2r + 0 = 0$$

$$\boxed{r^2 - 3r = 0}$$

The roots of the indicial equation (exponents)

are $r = 0, 3$

$$(2) \quad x(x+2)^2 y'' + (x+1)y' + 2xy = 0$$

$$(1) \quad x(x+2)^2 = 0 \Rightarrow x = 0, -2$$

$$\text{At } \boxed{x_0 = 0}$$

$$(1) \quad \lim_{x \rightarrow 0} \frac{x+1}{x(x+2)^2} \cdot x = \lim_{x \rightarrow 0} \frac{x+1}{(x+2)^2} = \frac{1}{4} = \underline{\underline{p_0}}$$

$$(2) \quad \lim_{x \rightarrow 0} \frac{2x}{x(x+2)^2} \cdot x^2 = \lim_{x \rightarrow 0} \frac{2x^2}{(x+2)^2} = \frac{0}{4} = 0 = q_0$$

$\therefore x_0 = 0$ is a regular singular point.

$$\text{At } \boxed{x_0 = -2}$$

$$(1) \quad \lim_{x \rightarrow -2} \frac{x+1}{x(x+2)^2} (x+2) = \lim_{x \rightarrow -2} \frac{x+1}{x(x+2)} \quad \text{D.N.E}$$

$\therefore x_0 = -2$ is irregular singular.

$\therefore$ The set of all regular singular points is

$$\{0\}$$

Frobenius' Theorem

If $x = x_0$ is a regular singular point of

$$A(x)y'' + B(x)y' + C(x)y = 0$$

then there exists at least one solution of

the form: $y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+r}$, $a_0 \neq 0$

where the number r is a constant to be determined

Remark: $y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+r}$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n (x-x_0)^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n (x-x_0)^{n+r-2}$$

Ex: Solve:

$$2x^2 y'' + (x^2 - x) y' + y = 0 \quad \text{near } x_0 = 0$$

③ x_0 is regular singular.

Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_n \neq 0$ be a solution.

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

After substitution in the D.E. we obtain

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

$$+ (x^2 - x) \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r+1}$$

$$- \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0} 2(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=1} (n+r-1)a_{n-1} x^{n+r}$$

$$- \sum_{n=0} (n+r)a_n x^{n+r} + \sum_{n=0} a_n x^{n+r} = 0$$

$$\Rightarrow 2r(r-1)a_0 x^r - r a_0 x^r + a_0 x^r$$

$$+ \sum_{n=1} \left[2(n+r)(n+r-1)a_n + (n+r-1)a_{n-1} - (n+r)a_n + a_n \right] x^{n+r} = 0$$

$$\Rightarrow (2r(r-1) - r + 1)a_0 x^r$$

$$+ \sum_{n=1} \left\{ \left[2(n+r)(n+r-1) - (n+r) + 1 \right] a_n + (n+r-1)a_{n-1} \right\} x^{n+r} = 0$$

$$\Rightarrow (2r(r-1) - r + 1)a_0 = 0 \Rightarrow 2r^2 - 3r + 1 = 0$$

$$\left[2(n+r)(n+r-1) - (n+r) + 1 \right] a_n + (n+r-1)a_{n-1} = 0 \quad n \geq 1$$

$$\Downarrow$$

$$\left[2(n+r)(n+r-1) - (n+r) + 1 \right] a_n = -(n+r-1)a_{n-1}$$

$$(n+r-1)(2(n+r)-1)a_n = -(n+r-1)a_{n-1}$$

$$(n+r-1)(2(n+r)-1)a_n = -(n+r-1)a_{n-1} \quad n \geq 1$$

$$\Rightarrow \left(a_n = \frac{-a_{n-1}}{2(n+r)-1} \right) \quad n \geq 1$$

$$2r^2 - 3r + 1 = 0 \Rightarrow (2r-1)(r-1) = 0$$

$$\Rightarrow r = 1, \frac{1}{2}$$

$$\boxed{r=1} \Rightarrow \left(a_n = \frac{-a_{n-1}}{2n+1} \right) \quad n \geq 1$$

$$a_1 = -\frac{a_0}{3}$$

$$a_2 = -\frac{a_1}{5} = \frac{a_0}{15}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$= a_0 x + a_1 x^2 + a_2 x^3 + \dots$$

$$= a_0 x - \frac{a_0}{3} x^2 + \frac{a_0}{15} x^3 + \dots$$

$$\therefore y_1 = a_0 \left[x - \frac{x^2}{3} + \frac{x^3}{15} - \dots \right]$$

$$\boxed{r = \frac{1}{2}} \Rightarrow \left(a_n = \frac{-a_{n-1}}{2n} \right) \quad n \geq 1$$

$$a_1 = -\frac{a_0}{2}$$

$$a_2 = \frac{-a_1}{4} = \frac{a_0}{8}$$

$$\begin{aligned} \therefore y_2 &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\ &= a_0 x^{\frac{1}{2}} + a_1 x^{\frac{3}{2}} + a_2 x^{\frac{5}{2}} + \dots \\ &= a_0 x^{\frac{1}{2}} - \frac{a_0}{2} x^{\frac{3}{2}} + \frac{a_0}{8} x^{\frac{5}{2}} + \dots \\ &= a_0 \left[x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2} + \frac{x^{\frac{5}{2}}}{8} - \dots \right] \end{aligned}$$

Remark: $a_1 = -\frac{a_0}{2 \cdot 1}$

$$a_2 = \frac{-a_1}{2 \cdot 2} = \frac{a_0}{2^2 \cdot (2)(1)}$$

$$a_3 = \frac{-a_2}{2 \cdot 3} = \frac{-a_0}{2^3 \cdot (3)(2)(1)}$$

$$\Rightarrow a_n = \frac{(-1)^n a_0}{2^n n!}$$

Consider $A(x)y'' + B(x)y' + C(x)y = 0$ --- (1)

Let $x_0 = 0$ be a regular singular point and that the indicial roots r_1 and r_2 of the singularity are real.

We distinguish three cases:

1 $r_1 > r_2$ and $r_1 - r_2 \notin \mathbb{Z}$

There exist two linearly independent solutions:

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}, \quad a_0 \neq 0$$

$$y_2(x) = \sum_{n=0}^{\infty} a_n x^{n+r_2}, \quad a_0 \neq 0$$

2 $r_1 > r_2$ and $r_1 - r_2 \in \mathbb{Z}$

There exist two linearly independent solutions

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}, \quad a_0 \neq 0$$

$$y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} a_n x^{n+r_2} \quad a_0 \neq 0$$

C : a constant which may be zero.

$\therefore$ The second solution may contain a logarithm

$$\boxed{3} \quad r_1 = r_2 = r$$

There exist two linearly independent solutions

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \quad a_0 \neq 0$$

$$y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} a_n x^{n+r}$$

$\therefore$ The second solution will always contain a solution.

Ex: Consider $xy'' - xy' + y = 0$

① Show that $x_0 = 0$ is regular singular

② Determine the roots of the indicial equation at $x_0 = 0$

③ Find the Frobenius series solution corresponding to the larger root.

④ What would be the form of the second linearly independent solution.

$$A(x) = x \Rightarrow A(0) = 0$$

$\therefore x_0 = 0$ is singular

$$\lim_{x \rightarrow 0} \frac{-x}{x} \cdot x = \boxed{0}$$

$$\lim_{x \rightarrow 0} \frac{1}{x} \cdot x^2 = \boxed{0}$$

$\therefore x_0 = 0$

is regular singular

Q10 The indicial equation is: $r(r-1) + p_0r + q_0 = 0$

$$\Rightarrow \boxed{r(r-1) = 0}$$

$\therefore r = 1, 0$

$$\text{let } y = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$y' = \sum_{n=0}^{\infty} (n+1) a_n x^n$$

$$y'' = \sum_{n=1}^{\infty} n(n+1) a_n x^{n-1}$$

Substitute in : $xy'' - xy' + y = 0$

$$x \sum_{n=1}^{\infty} n(n+1) a_n x^{n-1} - x \sum_{n=0}^{\infty} (n+1) a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=1}^{\infty} n(n+1) a_n x^n - \sum_{n=0}^{\infty} (n+1) a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+1} x^{n+1} - \sum_{n=0}^{\infty} (n+1) a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+1} - (n+1) a_n + a_n \right] x^{n+1} = 0$$

$$\Rightarrow (n+2)(n+1) a_{n+1} - \underbrace{(n+1) a_n + a_n}_{-n a_n - a_n + a_n} = 0 \quad n \geq 0$$

$$\Rightarrow (n+2)(n+1) a_{n+1} - n a_n = 0$$

$$\therefore a_{n+1} = \frac{n a_n}{(n+2)(n+1)} \quad n \geq 0$$

$$\boxed{n=0} \quad a_1 = 0$$

$$\boxed{n=1} \quad a_2 = \frac{a_1}{(3)(2)} = 0$$

⋮

$$a_n = 0, \quad n \geq 1$$

$$\therefore y_1 = \sum_{n=0} a_n x^{n+1}$$

$$= a_0 x + a_1 x^2 + a_2 x^3 + \dots$$

$$\therefore y_1 = a_0 x$$

$$\therefore y_2 = C x \ln x + \sum_{n=0} a_n x^n$$

Laplace Transform

Def: let f be a function of t , $0 \leq t < \infty$.

The function (if it exists)

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt$$

is called the Laplace transform of f .

Ex: Find $\mathcal{L}\{1\}$.

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st} dt = \lim_{p \rightarrow \infty} \int_0^p e^{-st} dt$$

$$\int_0^p e^{-st} dt = \left. \frac{e^{-st}}{-s} \right|_0^p = -\frac{1}{s} [e^{-ps} - 1]$$

$$\begin{aligned} \lim_{p \rightarrow \infty} -\frac{1}{s} [e^{-ps} - 1] &= -\frac{1}{s} [0 - 1] \\ &= \frac{1}{s}, \quad s > 0 \end{aligned}$$

$$\textcircled{1} \mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$$

$$\textcircled{2} \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

$$\textcircled{3} \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$$

$$\textcircled{4} \mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}, \quad s > 0$$

$$\textcircled{5} \mathcal{L}\{\cos(at)\} = \frac{s}{s^2 + a^2}, \quad s > 0$$

$$\textcircled{6} \mathcal{L}\{\sinh(at)\} = \frac{a}{s^2 - a^2}, \quad s > |a|$$

$$\textcircled{7} \mathcal{L}\{\cosh(at)\} = \frac{s}{s^2 - a^2}, \quad s > |a|$$

EX: Find

$$\textcircled{1} \mathcal{L}\{5\} = \frac{5}{s}$$

$$\textcircled{2} \mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$$

$$\textcircled{3} \mathcal{L}\{\sin 2t\} = \frac{2}{s^2 + 4}$$

Remark: $\mathcal{L}$ is linear

$$\mathcal{L}\{a f(t) + b g(t)\} = a \mathcal{L}\{f(t)\} + b \mathcal{L}\{g(t)\}$$

Ex: Find

$$\textcircled{1} \mathcal{L}\{2t + e^{3t} + \sin t\}$$

$$= \mathcal{L}\{2t\} + \mathcal{L}\{e^{3t}\} + \mathcal{L}\{\sin t\}$$

$$= 2 \cdot \frac{1}{s^2} + \frac{1}{s-3} + \frac{1}{s^2+1}$$

$$\textcircled{2} \mathcal{L}\{G_s^2 t\}$$

$$= \mathcal{L}\left\{\frac{1}{2}(1 + G_s 2t)\right\}$$

$$= \frac{1}{2} \left[\mathcal{L}\{1\} + \mathcal{L}\{G_s 2t\} \right]$$

$$= \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2+4} \right]$$

$$\text{Theorem: } \mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Ex: Find

$$\textcircled{1} \mathcal{L}\{t^3 e^{2t}\}$$

$$= \mathcal{L}\{t^3\} \Big|_{s-2} = \frac{3!}{s^4} \Big|_{s-2} = \frac{6}{(s-2)^4}$$

$$\textcircled{2} \mathcal{L}\{e^{-3t} \cos 2t\}$$

$$= \mathcal{L}\{\cos 2t\} \Big|_{s+3} = \frac{s}{s^2+4} \Big|_{s+3}$$

$$= \frac{s+3}{(s+3)^2 + 4}$$

The Inverse Laplace Transform

If $\mathcal{L}\{f(t)\} = F(s)$, then the inverse Laplace transform of $F(s)$ is:

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

③ The inverse Laplace $\mathcal{L}^{-1}$ is a linear operator

$$\begin{aligned}\mathcal{L}^{-1}\{a F(s) + b G(s)\} &= a \mathcal{L}^{-1}\{F(s)\} + b \mathcal{L}^{-1}\{G(s)\} \\ &= a f(t) + b g(t)\end{aligned}$$

$f(t)$

$F(s)$

1

$\frac{1}{s}$

e^{at}

$\frac{1}{s-a}$

t^n

$\frac{n!}{s^{n+1}}$

$\sin at$

$\frac{a}{s^2+a^2}$

$\cos at$

$\frac{s}{s^2+a^2}$

$e^{at} f(t)$

$F(s-a)$

Ex: Find

$$\textcircled{1} \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} = t$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\textcircled{2} \mathcal{L}^{-1} \left\{ \frac{1}{s^3} \right\}$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}$$

$$= \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^3} \right\} = \frac{1}{2} t^2$$

$$\textcircled{3} \mathcal{L}^{-1} \left\{ \frac{2}{s^2+4} \right\} = \sin 2t$$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\textcircled{4} \mathcal{L}^{-1} \left\{ \frac{3}{s^3} + \frac{s}{s^2+9} \right\}$$

$$\mathcal{L}\{\cos 3t\} = \frac{s}{s^2+9}$$

$$= 3 \mathcal{L}^{-1} \left\{ \frac{1}{s^3} \right\} + \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\}$$

$$= \frac{3}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^3} \right\} + \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\}$$

$$= \frac{3}{2} t^2 + \cos 3t$$

Remark: $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$$\Rightarrow \mathcal{L}^{-1}\{F(s-a)\} = e^{at} \mathcal{L}^{-1}\{F(s)\} \\ = e^{at} f(t)$$

Ex: Find

$$\textcircled{1} \mathcal{L}^{-1} \left\{ \frac{1}{(s-2)^4} \right\}$$

$$\mathcal{L}\{t^3\} = \frac{3!}{s^4}$$

$$= e^{2t} \mathcal{L}^{-1} \left\{ \frac{1}{s^4} \right\}$$

$$= \frac{e^{2t}}{6} \mathcal{L}^{-1} \left\{ \frac{6}{s^4} \right\} = \frac{e^{2t}}{6} \cdot t^3 = \frac{t^3}{6} e^{2t}$$

$$\textcircled{2} \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^2 + 4} \right\}$$

$$= e^t \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4} \right\}$$

$$= \frac{e^t}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 4} \right\} = \frac{1}{2} e^t \sin 2t$$

$$\textcircled{3} \mathcal{L}^{-1} \left\{ \frac{s-2}{(s-2)^2 + 1} \right\}$$

$$= e^{2t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1} \right\}$$

$$= e^{2t} \cos t.$$

Ex: Find

$$\textcircled{1} \mathcal{L}^{-1} \left\{ \frac{s}{(s-2)^2+9} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{(s-2)+2}{(s-2)^2+9} \right\} = \mathcal{L}^{-1} \left\{ \frac{s-2}{(s-2)^2+9} \right\} + \mathcal{L}^{-1} \left\{ \frac{2}{(s-2)^2+9} \right\}$$

$$= e^{2t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\} + e^{2t} \mathcal{L}^{-1} \left\{ \frac{2}{s^2+9} \right\}$$

$$= e^{2t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+9} \right\} + \frac{2}{3} e^{2t} \mathcal{L}^{-1} \left\{ \frac{3}{s^2+9} \right\}$$

$$= e^{2t} \cos 3t + \frac{2}{3} e^{2t} \sin 3t$$

$$\textcircled{2} \mathcal{L}^{-1} \left\{ \frac{s+1}{s^2+4s+5} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{s+1}{(s+2)^2+1} \right\} = \mathcal{L}^{-1} \left\{ \frac{s+2-1}{(s+2)^2+1} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{s+2}{(s+2)^2+1} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{(s+2)^2+1} \right\}$$

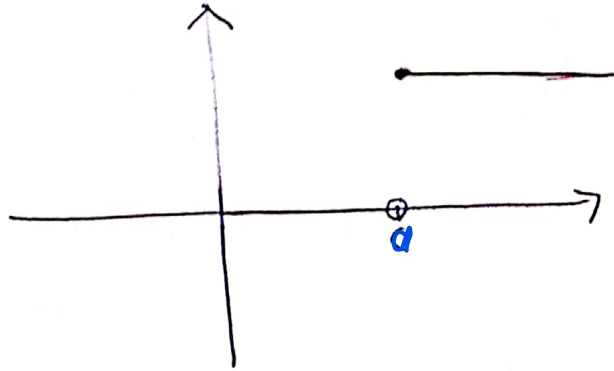
$$= e^{-2t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+1} \right\} - e^{-2t} \mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\}$$

$$= e^{-2t} \cos t - e^{-2t} \sin t = e^{-2t} [\cos t - \sin t]$$

unit step function

"Heaviside function"

$$u(t-a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq a \end{cases} \quad a \geq 0$$



⑦⑦ $u(t) = 1$

Ex: Let $f(t) = t^2 + u(t-1)$, Find $f(2)$

$$f(2) = (2)^2 + 1 = 5$$

⑦⑦ $L\{u(t-a)\} = \frac{e^{-as}}{s}$

⑦⑦ $L\{u(t-a)f(t)\} = e^{-as}L\{f(t+a)\}$

Remark:

$$\sin(\pi + t) = -\sin t$$

$$\sin(\pi - t) = \sin t$$

$$\cos(\pi + t) = -\cos t$$

$$\cos(\pi - t) = -\cos t$$

$$\sin\left(\frac{\pi}{2} + t\right) = \cos t$$

$$\sin\left(\frac{\pi}{2} - t\right) = \cos t$$

$$\cos\left(\frac{\pi}{2} + t\right) = -\sin t$$

$$\cos\left(\frac{\pi}{2} - t\right) = \sin t$$

Ex: Find

$$\textcircled{1} \mathcal{L}\{u(t-2)\} = \frac{e^{-2s}}{s}$$

$$\begin{aligned} \textcircled{2} \mathcal{L}\{t u(t-2)\} &= \parallel \\ &= e^{-2s} \mathcal{L}\{t+2\} = e^{-2s} \left[\frac{1}{s^2} + \frac{2}{s} \right] \end{aligned}$$

$$\begin{aligned} \textcircled{3} \mathcal{L}\{t^2 u(t-1)\} &= \\ &= e^{-s} \mathcal{L}\{(t+1)^2\} = e^{-s} \mathcal{L}\{t^2 + 2t + 1\} \\ &= e^{-s} \left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right] \end{aligned}$$

$$(4) \mathcal{L} \left\{ \sin t \, u\left(t - \frac{\pi}{2}\right) \right\}$$

$$= e^{-\frac{\pi}{2}s} \mathcal{L} \left\{ \sin\left(t + \frac{\pi}{2}\right) \right\}$$

$$= e^{-\frac{\pi}{2}s} \mathcal{L} \{ \cos t \} = e^{-\frac{\pi}{2}s} \cdot \frac{s}{s^2 + 1}$$

$$(5) \mathcal{L} \left\{ e^{2t} u(t-3) \right\}$$

$$= \frac{e^{-3s}}{s} \bigg|_{s-2} = \frac{e^{-3(s-2)}}{s-2} = \frac{e^{6-3s}}{s-2}$$

$$(6) \mathcal{L} \left\{ t e^{5t} u(t-2) \right\}$$

$$\begin{aligned} \mathcal{L} \{ t u(t-2) \} &= e^{-2s} \mathcal{L} \{ t+2 \} \\ &= e^{-2s} \left[\frac{1}{s^2} + \frac{2}{s} \right] \end{aligned}$$

$$\therefore \mathcal{L} \left\{ e^{5t} t u(t-2) \right\} = e^{-2(s-5)} \left[\frac{1}{(s-5)^2} + \frac{2}{s-5} \right]$$

$$= e^{10-2s} \left[\frac{1}{(s-5)^2} + \frac{2}{s-5} \right]$$

Remark: let $f(t) = \begin{cases} f_1(t) & 0 \leq t < a \\ f_2(t) & a \leq t < b \\ f_3(t) & t \geq b \end{cases}$

Then we can rewrite $f(t)$ as follows:

$$f(t) = f_1(t) [u(t) - u(t-a)] + f_2(t) [u(t-a) - u(t-b)] + f_3(t) u(t-b)$$

Ex: let $f(t) = \begin{cases} t & 0 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$

Find $F(s)$

$$\begin{aligned} f(t) &= t [u(t) - u(t-2)] + 0 u(t-2) \\ &= t - t u(t-2) \end{aligned}$$

$$\begin{aligned} \therefore \mathcal{L}\{f(t)\} &= \mathcal{L}\{t\} - \mathcal{L}\{t u(t-2)\} \\ &= \frac{1}{s^2} - e^{-2s} \mathcal{L}\{t+2\} \\ &= \frac{1}{s^2} - e^{-2s} \left[\frac{1}{s^2} + \frac{2}{s} \right] \end{aligned}$$

$$\textcircled{1} \mathcal{L}\{u(t-a)f(t)\} = e^{-as} \mathcal{L}\{f(t+a)\}$$

$$\begin{aligned} \Rightarrow \mathcal{L}^{-1}\{e^{-as} \overline{f(s)}\} &= u(t-a) \mathcal{L}^{-1}\{\overline{f(s)}\}_{t-a} \\ &= u(t-a) f(t-a) \end{aligned}$$

Ex: Find

$$\textcircled{1} \mathcal{L}^{-1}\left\{\frac{s e^{-s}}{s^2-1}\right\}$$

$$= u(t-1) \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\}_{t-1}$$

$$= u(t-1) \cosh(t-1)$$

$$\textcircled{2} \mathcal{L}^{-1}\left\{\frac{4 e^{-2s}}{s-7}\right\}$$

$$= 4 u(t-2) \mathcal{L}^{-1}\left\{\frac{1}{s-7}\right\}_{t-2}$$

$$= 4 u(t-2) \boxed{e^{7t}}_{t-2}$$

$$= 4 u(t-2) e^{7(t-2)}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Remark: $\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1) + Bs}{s(s+1)}$

$$\Rightarrow 1 = A(s+1) + Bs$$

$$\boxed{s=0} \quad A=1$$

$$\boxed{s=-1} \quad B=-1$$

$$\therefore \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

Ex: Find

$$\textcircled{1} \mathcal{L}^{-1} \left\{ \frac{1}{s(s+1)} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{s+1} \right\}$$

$$= 1 - e^{-t}$$

$$\textcircled{2} \mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s(s+1)} \right\}$$

$$= u(t-1) \mathcal{L}^{-1} \left\{ \frac{1}{s(s+1)} \right\}_{t-1}$$

$$= u(t-1) [1 - e^{-t}]_{t-1}$$

$$= u(t-1) [1 - e^{-(t-1)}] = u(t-1) [1 - e^{1-t}]$$

$$\textcircled{1} \mathcal{L}\{y'(t)\} = s \underline{Y}(s) - y(0)$$

$$\textcircled{2} \mathcal{L}\{y''(t)\} = s^2 \underline{Y}(s) - s y(0) - y'(0)$$

Ex: solve: $y' + y = u(t-1)$, $y(0) = 0$

$$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{u(t-1)\}$$

$$= s \underline{Y}(s) - y(0) + \underline{Y}(s) = \frac{e^{-s}}{s}$$

$$\Rightarrow (s+1) \underline{Y}(s) = \frac{e^{-s}}{s}$$

$$\Rightarrow \underline{Y}(s) = \frac{e^{-s}}{s(s+1)}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left\{ \frac{e^{-s}}{s(s+1)} \right\}$$

$$\therefore y(t) = u(t-1) [1 - e^{1-t}]$$

Dirac Delta Function

The Dirac Delta $\delta(t-a)$ is a generalized function (distribution) which can be characterized by the following properties:

$$\textcircled{1} \quad \delta(t-a) = \begin{cases} \infty & t=a \\ 0 & t \neq a \end{cases}$$

$$\textcircled{2} \quad \int_{-\infty}^{\infty} \delta(t-a) dt = 1$$

$$\therefore \delta(t) = \begin{cases} \infty & t=0 \\ 0 & t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

Theorem: If $g(t)$ is a cts function, then

$$\int_{-\infty}^{\infty} \delta(t-a) g(t) dt = g(a)$$

Ex: Find

$$\textcircled{1} \int_{-\infty}^{\infty} \delta(t) \cos t \, dt = \cos 0 = 1$$

$$\textcircled{2} \int_0^{\infty} (1 + e^{-t}) \delta(t-2) \, dt = 1 + e^{-2}$$

$$\textcircled{3} \int_0^{\infty} \delta(t - \frac{\pi}{2}) \sin t \, dt = \sin \frac{\pi}{2} = 1$$

Theorem: For $a \geq 0$ and $g(t)$ cts:

$$\textcircled{1} \mathcal{L}\{\delta(t-a)\} = e^{-as}$$

$$\textcircled{2} \mathcal{L}\{g(t) \delta(t-a)\} = e^{-as} g(a)$$

$$\textcircled{3} \mathcal{L}\{\delta(t-a)\} = \int_0^{\infty} \delta(t-a) e^{-st} \, dt \\ = e^{-as}$$

$$\textcircled{1} \mathcal{L}\{g(t) \delta(t-a)\} = \int_0^{\infty} \delta(t-a) g(t) e^{-st} dt$$

$$= g(a) e^{-as}$$

Ex: Find

$$\textcircled{1} \mathcal{L}\{\delta(t-3)\} = e^{-3s}$$

$$\textcircled{2} \mathcal{L}\{(2t+1) \delta(t-1)\}$$

$$= (2(1)+1) e^{-s} = 3 e^{-s}$$

Ex: Solve: $y'' + \pi^2 y = \delta(t-1)$

$y(0)=1, y'(0)=0$

$$\mathcal{L}\{y''\} + \pi^2 \mathcal{L}\{y\} = \mathcal{L}\{\delta(t-1)\}$$

$$s^2 Y(s) - s y(0) - y'(0) + \pi^2 Y(s) = e^{-s}$$

$$s^2 Y(s) - s + \pi^2 Y(s) = e^{-s}$$

$$(s^2 + \pi^2) Y(s) = s + e^{-s}$$

$$\therefore Y(s) = \frac{s}{s^2 + \pi^2} + \frac{e^{-s}}{s^2 + \pi^2}$$

$$\begin{aligned} \therefore y(t) &= \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + \pi^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s^2 + \pi^2} \right\} \\ &= \cos(\pi t) + u(t-1) \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + \pi^2} \right\}_{t-1} \\ &= \cos(\pi t) + u(t-1) \cdot \boxed{\frac{1}{\pi} \sin \pi t}_{t-1} \\ &= \cos(\pi t) + \frac{1}{\pi} u(t-1) \sin \pi(t-1) \end{aligned}$$

H.W. Solve:

$$\textcircled{1} \quad y' - 3y = \delta(t-2), \quad y(0) = 0$$

$$\textcircled{2} \quad y'' + 16y = \delta(t-2\pi) \quad \begin{matrix} y(0) = 0, \\ y'(0) = 1 \end{matrix}$$

$$\begin{aligned} \textcircled{3} \quad y'' + y' &= \delta(t-1) - \delta(t-2) \\ y(0) &= 0, \quad y'(0) = 0 \end{aligned}$$

$$\textcircled{///} \mathcal{L}\{t f(t)\} = -F'(s) = -\frac{d}{ds}(F(s))$$

$$\mathcal{L}^{-1}\{F'(s)\} = -t f(t)$$

Ex: Find

$$\textcircled{1} \mathcal{L}\{t \sin 2t\}$$

$$= -\frac{d}{ds} \left[\frac{2}{s^2+4} \right] = -\frac{-2(2s)}{(s^2+4)^2}$$

$$= \frac{4s}{(s^2+4)^2}$$

$$\textcircled{2} \text{ H.W } \mathcal{L}\{t e^{2t} \cos 2t\}$$

$$\textcircled{3} \mathcal{L}\{t \sin^2 t\}$$

$$= \mathcal{L}\left\{ \frac{1}{2} t (1 - \cos 2t) \right\} = \frac{1}{2} [\mathcal{L}\{t\} - \mathcal{L}\{t \cos 2t\}]$$

$$= \frac{1}{2} \left[\frac{1}{s^2} - \frac{d}{ds} \left(\frac{s}{s^2+4} \right) \right]$$

$$= \frac{1}{2} \left[\frac{1}{s^2} + \frac{s^2+4 - s(2s)}{(s^2+4)^2} \right] = \frac{1}{2} \left[\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2} \right]$$

Ex: Find

$$\textcircled{1} \mathcal{L}^{-1} \left\{ \ln \left(\frac{s^2+4}{s^2} \right) \right\}$$

$$\text{let } F(s) = \ln \left(\frac{s^2+4}{s^2} \right)$$

$$F(s) = \ln(s^2+4) - \ln s^2$$

$$F(s) = \ln(s^2+4) - 2 \ln s$$

$$F'(s) = \frac{2s}{s^2+4} - \frac{2}{s}$$

$$\mathcal{L}^{-1} \{ F'(s) \} = 2 \mathcal{L}^{-1} \left\{ \frac{s}{s^2+4} \right\} - 2 \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\}$$

$$\text{--- } t f(t) = 2 \cos 2t - 2$$

$$\Rightarrow f(t) = \frac{2 - 2 \cos 2t}{t}$$

Q2 H.W $\textcircled{a} \mathcal{L}^{-1} \left\{ \ln \left(\frac{s}{s-1} \right) \right\}$

$$\textcircled{b} \mathcal{L}^{-1} \left\{ \ln \left(\frac{s^2+1}{(s-1)^2} \right) \right\}$$

$$(3) \mathcal{L}^{-1} \left\{ \cot^{-1} \frac{s}{\pi} \right\}$$

$$\text{let } F(s) = \cot^{-1} \frac{s}{\pi}$$

$$\Rightarrow F'(s) = - \frac{\frac{1}{\pi}}{1 + \frac{s^2}{\pi^2}}$$

$$\Rightarrow F'(s) = - \frac{\pi}{\pi^2 + s^2}$$

$$\Rightarrow \mathcal{L}^{-1} \{ F'(s) \} = - \mathcal{L}^{-1} \left\{ \frac{\pi}{s^2 + \pi^2} \right\}$$

$$\Rightarrow -t f(t) = - \sin \pi t$$

$$\Rightarrow f(t) = \frac{\sin \pi t}{t}$$

$$(4) \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + 16)^2} \right\}$$

$$\text{let } F(s) = \frac{1}{s^2 + 16} \Rightarrow f(t) = \frac{1}{4} \sin 4t$$

$$\therefore F'(s) = \frac{-2s}{(s^2 + 16)^2} \Rightarrow \mathcal{L}^{-1} \{ F'(s) \} = \mathcal{L}^{-1} \left\{ \frac{-2s}{(s^2 + 16)^2} \right\}$$

$$\Rightarrow -t f(t) = -2 \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + 16)^2} \right\}$$

$$\Rightarrow \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + 16)^2} \right\} = \frac{t}{2} f(t) = \left(\frac{t}{8} \sin 4t \right)$$

$$\textcircled{///} \mathcal{L} \left\{ \frac{f(u)}{t} \right\} = \int_s^\infty \bar{f}(u) du$$

Ex: Find $\mathcal{L} \left\{ \frac{\sin t}{t} \right\}$

$$\begin{aligned}
 &= \int_s^\infty \frac{1}{u^2+1} du = \tan^{-1} u \Big|_s^\infty \\
 &= \tan^{-1} \infty - \tan^{-1} s \\
 &= \frac{\pi}{2} - \tan^{-1} s
 \end{aligned}$$

$$\textcircled{///} \mathcal{L} \left\{ \int_0^t f(u) du \right\} = \frac{\bar{f}(s)}{s}$$

$$\mathcal{L}^{-1} \left\{ \frac{\bar{f}(s)}{s} \right\} = \int_0^t f(u) du$$

Ex: $\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2+1)} \right\}$

$$= \mathcal{L}^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{s^2+1} \right\} = \int_0^t \sin u du$$

$$= -\cos u \Big|_0^t = -[\cos t - 1] = 1 - \cos t$$

$$\underline{\underline{\text{Ex: } \mathcal{L}^{-1} \left\{ \frac{1}{s^2(s^2+1)} \right\}}}$$

$$= \mathcal{L}^{-1} \left\{ \frac{\frac{1}{s(s^2+1)}}{s} \right\}$$

$$= \int_0^t [1 - \cos u] du = u - \sin u \Big|_0^t$$

$$= t - \sin t$$

$$\underline{\underline{\text{Ex: } \mathcal{L}^{-1} \left\{ \frac{1}{s(s+1)} \right\}}}$$

$$= \mathcal{L}^{-1} \left\{ \frac{\frac{1}{s+1}}{s} \right\}$$

$$= \int_0^t e^{-u} du = -e^{-u} \Big|_0^t$$

$$= -[e^{-t} - 1]$$

$$= 1 - e^{-t}$$