VIBRATIONS

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Mech Family
Notebooks

CHAPTER (1) - Introduction To Mechanical Vibrations

→ Vibration (oscillations) :- Any motion that repeats it's self after an interval of time

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* Classifications of Vibrations :-

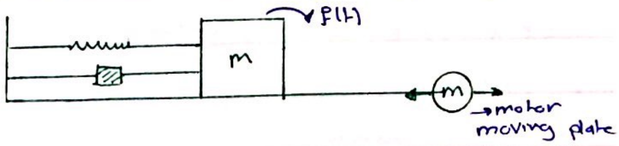
[1] Free Vibration

No external forces acting on your system

example → Pendulum

[2] Forced Vibration

An external force acting on your system

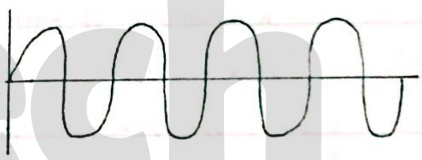


[3] Damped Vibration

If there is energy lost during vibration

[4] Undamped Vibration

No energy lost during vibration

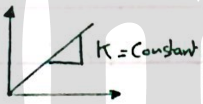


→ it doesn't exist in real life

[5] Linear Vibration

If all the elements are linear

for example (stiffeness)



[6] Non - linear Vibration

If any of the elements is non - linear

example → If the stiffeness changes with time

[7] Deterministic Vibration

If the magnitude of the external force is known (can be constant or represented as a function of time)

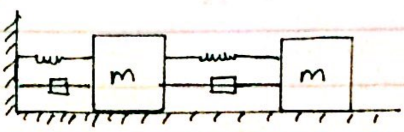
[8] Undeterministic Vibration

If the magnitude of the external force is unknown (Random vibration)

→ can't be represented as a function of (t)

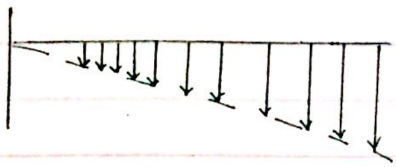
[9] Descrete Vibration

Has a finite number of degrees of freedom.



[10] Continuous Vibration

Has an infinite number of degrees of freedom.



→ each point has a different deflection.

* Modeling in vibrations :-

1. understand the problem (system)
2. write the mathematical model.
3. Solve the mathematical model
4. Interpret results

→ mathematical model is finding the equation of motion $x(t)$.

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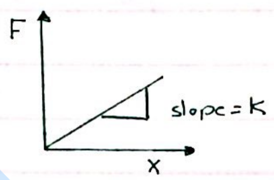
* Elements of vibrations (spring - mass - damper) :-

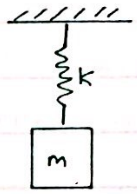
[1] Spring

$F = kx$; k (stiffeness) → N/m

$U = \frac{1}{2}kx^2$

↳ potential energy





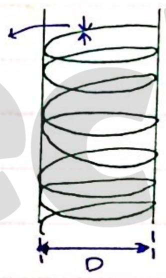
→ To Find (K) for a coil spring :-

$$\boxed{K = \frac{Gd^4}{8nD^3}}$$

n → number of turns

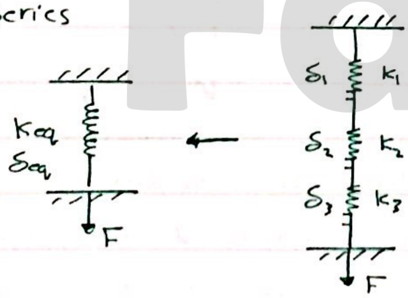
d → diameter of the rod that the spring is made of.

G → modulus of elasticity



→ Spring Combinations :-

[1] In Series



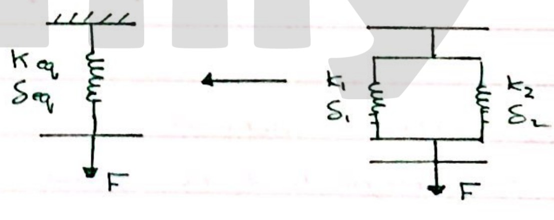
$F_1 = F_2 = F_3$

$\delta_{eq} = \delta_1 + \delta_2 + \delta_3$

So:

$$* \ \frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

[2] In parallel :-



$F = F_1 + F_2$

$k_1\delta_1 + k_2\delta_2 = k_{eq}\,\delta_{eq}$

But $\delta_1 = \delta_2 = \delta_{eq}$

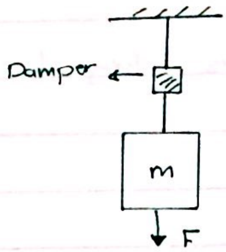
So:

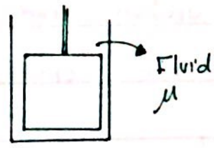
$$* \ k_1 + k_2 = k_{eq}$$

[2] Dampers :-

$F = c\dot{x}$; $\dot{x}$ (velocity)

c (

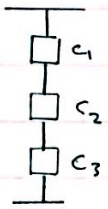




→ Dampers combinations:

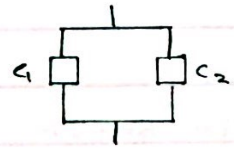
[1] In series

$$\frac{1}{C_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}$$



[2] In Parallel

$$C_{eq} = C_1 + C_2 + C_3 + \dots$$



Example :-

Find K_{rod} :-

uniform rod ; same cross section along it's length.

solution :

$$\delta = \frac{PL}{EA} \rightarrow P = \frac{\delta EA}{L}$$

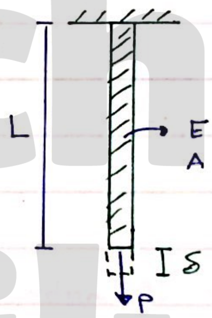
$F = kx$

$P = F$

$\delta \equiv x$

$$k\underline{\underline{x}} = \frac{EA}{L}\underline{\underline{\delta}}$$

$$\therefore K_{rod} = \frac{EA}{L}$$



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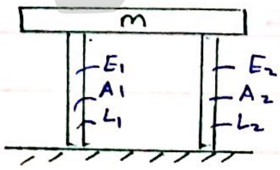
* Find k_{eq} :

[1] Two uniform rods (1, 2) :

(Parallel)

$$K_{eq} = k_1 + k_2 \quad ; \quad K_{rod} = \frac{EA}{L}$$

$$= \frac{E_1 A_1}{L_1} + \frac{E_2 A_2}{L_2}$$

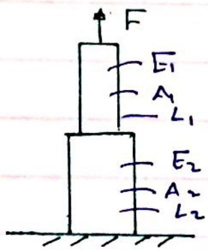


[1]

[2] (series) :-

$$\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2} \qquad K_{rod} = \frac{EA}{L}$$

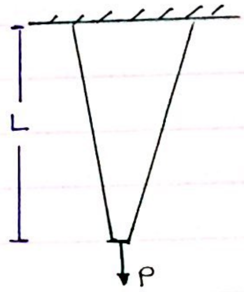
$$\frac{1}{K_{eq}} = \frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2}$$



[2]

* Non - uniform rod:

→ Different cross sections along it's length



$$\delta = \int_0^L \frac{P}{EA}\, dx$$

$$\delta = ?\ldots?$$

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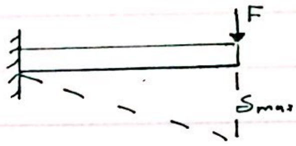
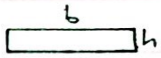
* For Beams :-

$\delta_{max} = \dfrac{F L^3}{3 E I}$; I : moment of Inertia



$K_{Beam} = \dfrac{3EI}{L^3}$



$I = \frac{1}{12} b h^3$

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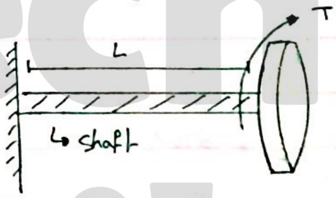
* Torsional Spring :-



$$\phi = \frac{TL}{GJ}$$

where :

J → Polar moment of Inertia

G → Shear modulus of elasticity

T as a function of ϕ :

$$T = \underbrace{\frac{JG}{L}}_{\hookrightarrow k} \phi$$

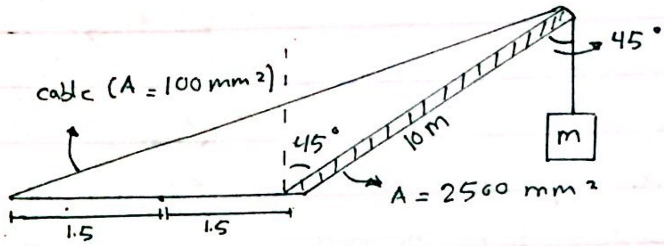
note → The term (spring) goes for anything that has stiffeness not just coil springs ()

* Example (1.8) :-

$E_{steel} = 200$ GPa

Find k_{eq}

Solution :-



→ First you have to find θ and L_{cable}

$L_{cable}^2 = 3^2 + 10^2 - 2(3)(10)\cos(135°)$

$L_{cable} = 12.3055$ m

→ $10^2 = 3^2 + L_{cable}^2 - 2(3) L_{cable} \cos\theta$

$\theta = 35.075°$

$\frac{1}{2} k_{eq} x^2 = \frac{1}{2} k_{rod} (x \cos 45°) + \frac{1}{2} k_{cable} (x \cos(90 - 35.075))^2$

$k_{rod} = \frac{EA_1}{L_1} = 51.75 * 10^3$ N/m

$k_{cable} = \frac{EA_2}{L_2} = 1.6822 * 10^6$ N/m

↳ $\frac{1}{2} k_{eq} \cancel{x^2} = (\quad) \cancel{x^2}$

$k_{eq} = 26.43 * 10^6$ N/m

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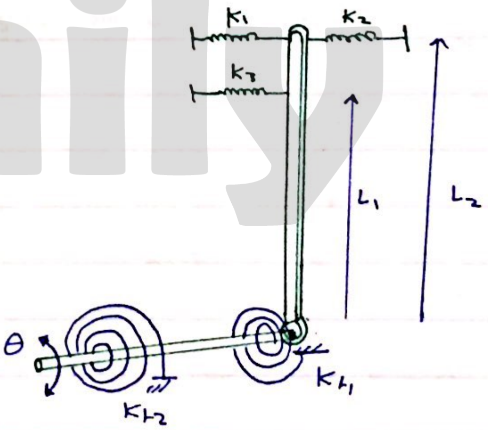
* Problem (1.9) :-

Find the equivilant spring constant of the system in the direction of θ.

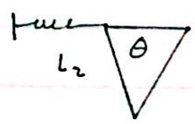
Solution :



$U_{eq} = U_{k1} + U_{k2} + U_{kt1} + U_{kt2} + U_{k3}$

↳ $\frac{1}{2} k_{eq} \theta^2 = \frac{1}{2} k_{t1} \theta^2 + \frac{1}{2} k_{t2} \theta^2 + \frac{1}{2} k_3 (L_2\theta)^2 + \frac{1}{2} k_1 (L_1\theta)^2 + \frac{1}{2} k_2 (L_1\theta)^2$

$k_{eq} = k_{t1} + k_{t2} + k_3 L_2^2 + (k_1 + k_2) L_1^2$

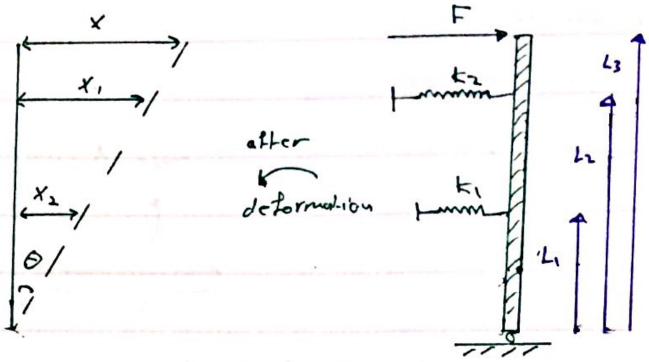


note → Always use potential energy when the assembly is complicated and you don't know whether the springs are parallel or series

Example (1.9)

Find k_{eq}

Solution:



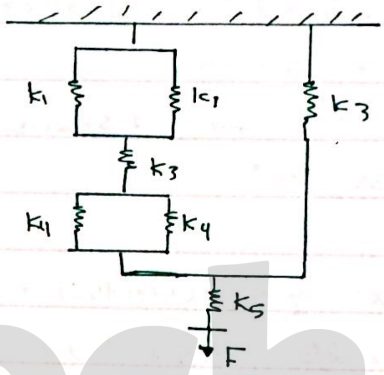
$U = U_{k1} + U_{k2}$

$\frac{1}{2} k_{eq} x^2 = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2$

→ we don't consider k for the rod because it is free to move.

Example (for you):

Find k_{eq}



* Mass or Inertia Elements :-

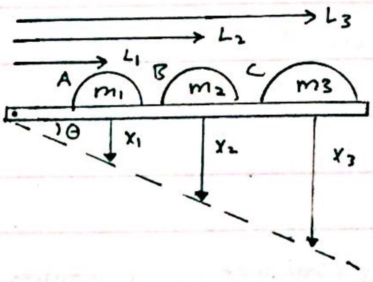
→ First Case : Translating masses connected by a rigid bar.

$$\left.\begin{aligned} x_1 &= L_1 \theta \\ x_2 &= L_2 \theta \\ x_3 &= L_3 \theta \end{aligned}\right] \xrightarrow{\text{Differentiate}} \begin{aligned} \dot{x}_1 &= L_1 \dot{\theta} \\ \dot{x}_2 &= L_2 \dot{\theta} \\ \dot{x}_3 &= L_3 \dot{\theta} \end{aligned} \xrightarrow{\text{By substitution}} \begin{aligned} \dot{x}_2 &= \frac{L_2}{L_1}(\dot{x}_1) \\ \dot{x}_3 &= \frac{L_3}{L_1}(\dot{x}_1) \end{aligned}$$

$\tan\theta = \frac{x_1}{L_1}$

$\tan\theta \simeq \theta$ (small θ)

$x_1 = L_1 \theta$



→ The goal is to find the equivalent mass and locate it somewhere.

in this case we will locate in the position of (m_1)

using kinetic energy:

$$T_{eq} = T_1 + T_2 + T_3$$

$$\frac{1}{2} m_{eq} \dot{x}_1^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \left(\frac{L_2}{L_1}\right)^2 \dot{x}_1^2 + \frac{1}{2} m_3 \left(\frac{L_3}{L_1}\right)^2 \dot{x}_1^2$$

$$m_{eq} = m_1 + m_2 \left(\frac{L_2}{L_1}\right)^2 + m_3 \left(\frac{L_3}{L_1}\right)^2$$

↳ # This is the transational equivalent mass at (A) - it would have a different value if it was at (B) or (C)

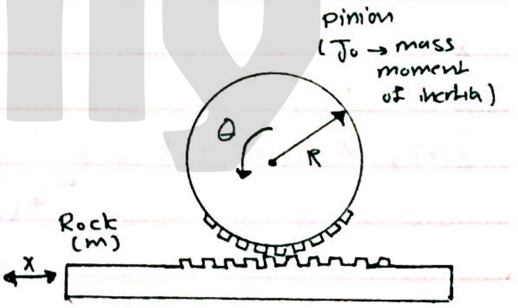
note → $\dot{x}_1 = \dot{x}$

same L, same θ

→ Second Case : Transational & Rotational masses coupled

$$x = R\theta \longrightarrow \dot{x} = R\dot{\theta}$$

$$T_{eq} = T_{pinion} + T_{Rack}$$



(Rack-Pinion system)

[1] Equivalant Transational mass:

$$\frac{1}{2} m_{eq} \dot{x}_{eq}^2 = \frac{1}{2} J_o \dot{\theta}^2 + \frac{1}{2} m \dot{x}^2$$

$$\frac{1}{2} m_{eq} \dot{x}_{eq}^2 = \frac{1}{2} J_o \left(\frac{1}{R}\right)^2 \dot{x}^2 + \frac{1}{2} m \dot{x}^2$$

$$m_{eq} = J_o / R^2 + m$$

[2] Equivalant Rotational mass:

$$\frac{1}{2} J_{eq} \dot{\theta}^2 = \frac{1}{2} J_o \dot{\theta}^2 + \frac{1}{2} m R^2 \dot{\theta}^2$$

$$J_{eq} = J_o + m R^2$$

note → The pinion is not translating, only Rotating (not G.P.M)

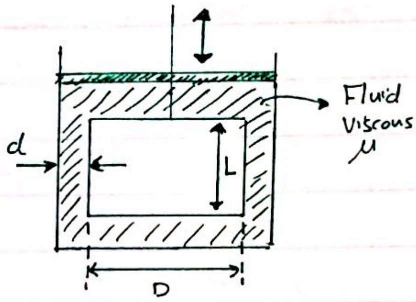
* Damping Element :-

[1] c → Damping coefficient

$$c = \frac{\mu 3\pi D^3 L}{4d^3}\left(1 + \frac{2d}{D}\right)$$

$F = c\dot{x}$

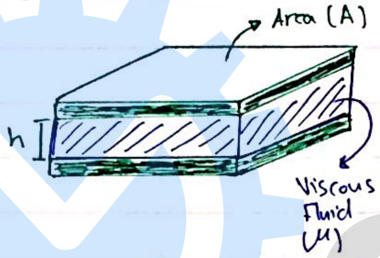
unit of c → $\left(\frac{N \cdot s}{m}\right)$

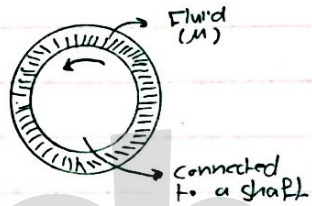


(Piston cylinder dash pot)

[2] Clearance in a Bearing.

$$c = \frac{\mu A}{h}$$





* Any oscillatory motion that repeats it's self after equal intervals of time is called a Periodic Motion.

→ one of the most simple forces is called (Harmonic Motion force)

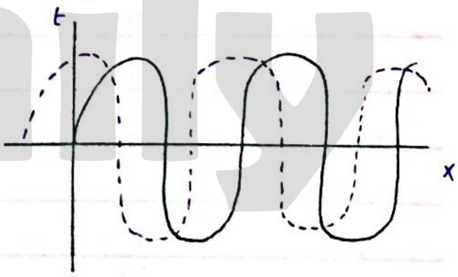
To Represent That motion:

$x(t) = A\sin(\omega t)$

$\dot{x}(t) = A\omega$

$\ddot{x}(t) = -A\omega^2 \sin(\omega t) = \underline{-\omega^2 x(t)}$

↳ (we can represent the acceleration directly with displacement).



+ → shift to left

− → shift to Right

shifting :

$x(t) = A\sin(\omega t \mp \phi)$

A → Amplitude

ω → Frequency

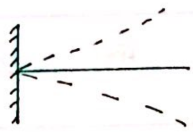
ϕ → shift angle

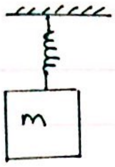
$x(t) = A(\cos\omega t + i\sin\omega t) = Ae^{i\omega t}$

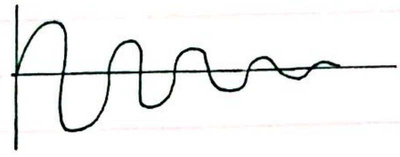
* Important terms we will be using :-

→ Natural Frequency (ω_n) :-

Frequency of oscillation with no external force acting on your system.







→ Frequency (Hz) :-

Number of cycles per unit time (Hertz)

f (Hz)

ω (rad/s)

$f = \frac{\omega}{2\pi}$ (unit → $\frac{1}{s}$)

→ Amplitude (A) :-

Maximum Displacement of the body.

→ Periodic Time (τ) :-

The required time to complete one cycle

$\tau = \frac{2\pi}{\omega}$ unit → (s)

If you have $A\cos(\omega t + \phi)$:-

$$A\cos(\omega t + \phi) = A(\cos(\omega t)\cos\phi - \sin(\omega t)\sin\theta)$$

$$= B_1 (\cos(\omega t)) - B_2 \sin(\omega t)$$

where → $B_1 = A\cos\phi$, $B_2 = A\sin\phi$

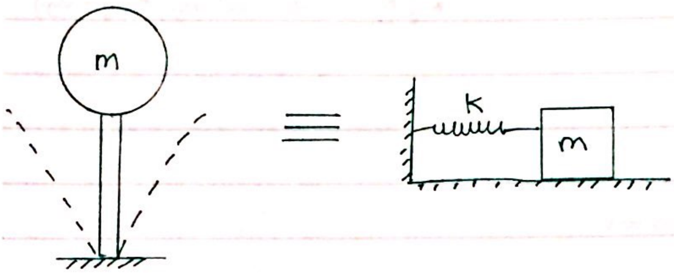
$$A^2 = B_1^2 + B_2^2$$

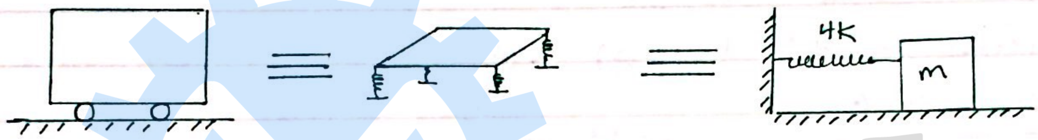
$$A(\sin(\omega t + \phi)) = A\cos(\omega t + (\phi - 90))$$

CHAPTER (2)

Free vibration of Single Degree of Freedom System.

→ Several mechanical and structural systems can be idealized as a single degree of freedom system :- - examples -

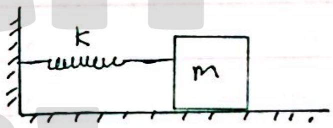




2.2 → Free vibration of an undamped translational system.

_ We can use two methods to derive the equation of motion:

[1] Newton's second law:



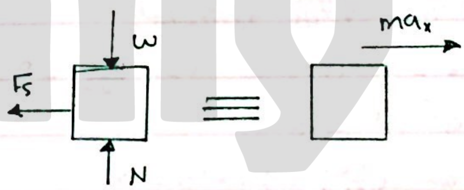
$\sum F_x = m a_x$

$\sum F_x = m\ddot{x}$

$F_s = m\ddot{x} \qquad (F_s = -kx)$

$-kx = m\ddot{x}$

$m\ddot{x} + kx = 0$ → Equation of motion.



F.B.D K.D

[2] Conservation of Energy:

$T + U = 0$

$\frac{d}{dt}(T+U) = 0$

$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$

$\frac{1}{2}m\dot{x}\ddot{x}(2) + \frac{1}{2}k(2x\dot{x}) = 0$

$m\ddot{x} + kx = 0$ → (same equation, different approach)

Finding the Solution :-

$m\ddot{x} + kx = 0 \rightarrow$ 2nd - order linear homo. diff. Equation.

solution:

$x(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

$\quad = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$

$\quad = A \cos(\omega_n t - \phi)$

where $\rightarrow A = \sqrt{C_1^2 + C_2^2}$

$\phi = \tan^{-1}\left(\frac{C_2}{C_1}\right)$

* $r^2 + \frac{k}{m} = 0 \rightarrow r^2 = \frac{-k}{m}$
* $\sqrt{\frac{k}{m}} = \omega_n$ (natural frequency)

To find C_1 & C_2 use initial conditions:

$x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$ $\rightarrow$ $C_1 = x_0$, $\omega_n C_2 = \dot{x}_0 \rightarrow C_2 = \frac{\dot{x}_0}{\omega_n}$

$\rightarrow x(t) = \sqrt{C_1^2 + C_2^2}\, \cos\left(\omega_n t - \tan^{-1}\left(\frac{C_2}{C_1}\right)\right) \quad - (*)$

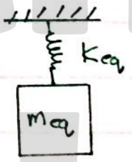
substitute C_1, C_2 to $(*)$:-

$x(t) = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega_n^2}}\, \cos\left(\omega_n t - \tan^{-1}\left(\frac{\dot{x}_0}{x_0 \omega_n}\right)\right)$

Example :-

$K_{eq} = 4000$ N/m $\qquad x_0 = (1)$cm

$M_{eq} = 40$ Kg $\qquad \dot{x}_0 = 0$



Find the equation of motion :-

$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{4000}{40}} = 10$ rad/s

$C_1 = x_0 = 0.01$m $\qquad C_2 = \frac{\dot{x}_0}{\omega_n} = 0$

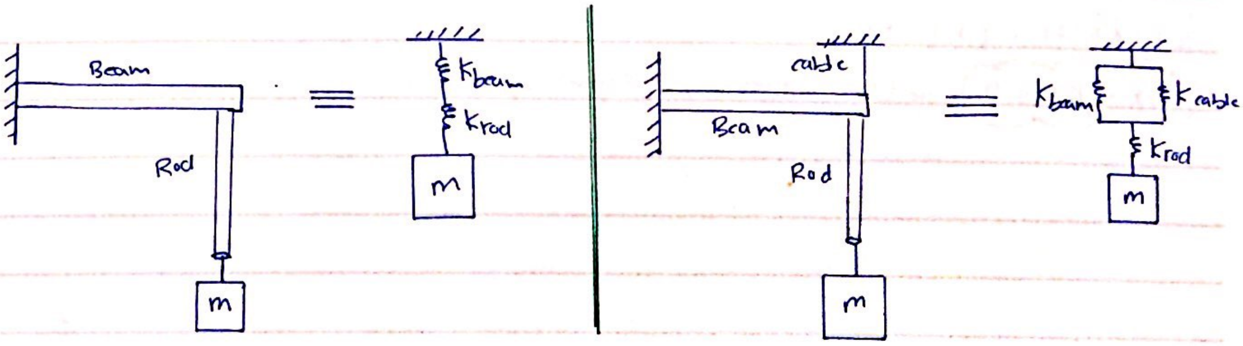
$A = \sqrt{C_1^2 + C_2^2} = 0.01$

$\phi = \tan^{-1}\left(\frac{C_2}{C_1}\right) = \tan^{-1} 0 = 0$

$x(t) = \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega_n^2}}\, \cos\left(\omega_n t - \tan^{-1}\left(\frac{\dot{x}_0}{x_0 \omega_n}\right)\right)$

$\quad = 0.01 \cos(10t - 0) = 0.01 \cos(10t)$

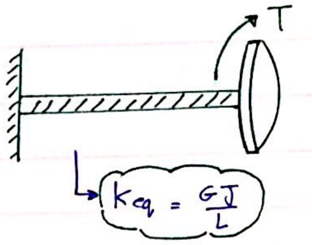
Examples of systems represented as a mass-spring system:-



2.3 → Free vibration of an undamped torsional system:

To find the equation of motion:

→ Conservation of energy:



$K.E + P.E = C$

$\frac{1}{2} J_o \dot{\phi}^2 + \frac{1}{2} k_t \phi^2 = C$

$\frac{d}{dt} \left(\frac{1}{2} J_o \dot{\phi}^2 + \frac{1}{2} k_t \phi^2 \right) = 0$

$\frac{1}{2} J_o (2 \dot{\phi} \ddot{\phi}) + \frac{1}{2} k_t (2 \phi \dot{\phi}) = 0$

$\boxed{J_o \ddot{\phi} + k_t \phi = 0}$

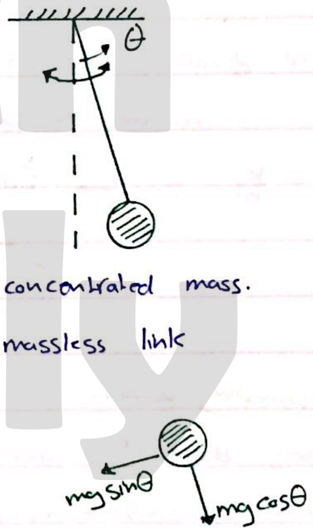
---

* Pendulums:-

→ we have two types of Pendulums
- → simple pendulum.
- → Compound pendulum.

[1] Simple pendulum:-

To find the equation of motion:-



$\Sigma M_o = J_o \ddot{\theta}$

$-(mg \sin\theta) L = J_o \ddot{\theta}$

$J_o \ddot{\theta} + mg \sin\theta \, L = 0$

→ $J_o = mL^2$

$mL^2 \ddot{\theta} + mgL \sin\theta = 0$

divide by (mL^2) :-

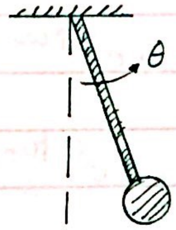
$\ddot{\theta} + \frac{g}{L} \sin\theta = 0$

→ For small angles $\sin\theta \simeq \theta$:

$\boxed{\ddot{\theta} + \frac{g}{L} \theta = 0}$ (notice that it doesn't depend on the mass).

[2] Compound Pendulum:-

we use the same procedure as the simple pendulum with one difference:



$J_o = mL^2 + mk_{CG}^2$; k → Radius of gyration

mass of the link is considered

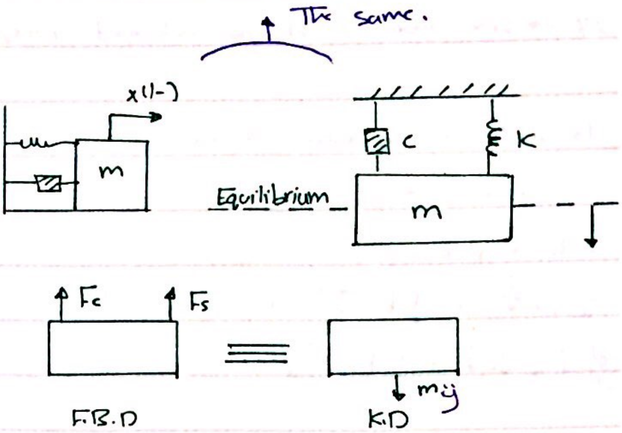
2.6 → Free vibration with viscous damping

Equation of motion (newton's 2nd law) :-

$+\downarrow \sum F_y = m\ddot{y}$

$-ky - c\dot{y} = m\ddot{y}$

$m\ddot{y} + c\dot{y} + ky = 0$



Finding the solution :-

$m\ddot{x} + c\dot{x} + kx = 0$

$ms^2 + cs + k = 0$

$$S_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$= \frac{-c}{2m} \mp \sqrt{\frac{c^2}{4m^2} - \frac{4mk}{4m^2}} = \frac{-c}{2m} \mp \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}$$

$$x(t) = C_1 e^{\left(\frac{-c}{2m} + \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}\right)t} + C_2 e^{\left(\frac{-c}{2m} - \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}\right)t}$$

* critical damping constant (C_{cr})

→ C_{cr} is the value of the damping constant for which $\left(\sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}\right)$ becomes zero.

$$\frac{C^2}{4m^2} - \frac{k}{m} = 0 \longrightarrow C_{cr} = 2m\sqrt{\frac{k}{m}}$$

$$C_{cr} = 2m\,\omega_n$$

* The Damping Ratio :-

For any damped system, the damping ratio (ξ) is defined as the ratio of the damping constant to the critical damping constant :

$$\xi = \frac{c}{C_{cr}}$$

$$\rightarrow \frac{c}{2m} = \frac{c}{C_{cr}} \cdot \frac{C_{cr}}{2m} = \frac{c}{C_{cr}} \cdot \frac{2m\omega_n}{2m} = \frac{c}{C_{cr}} \cdot \omega_n = \xi\omega_n \rightarrow \frac{c}{2m} = \xi\omega_n$$

→ rewrite the solution by substituting ($\frac{c}{2m} = \xi\omega_n$) :-

$$x(t) = C_1 e^{\left(-\xi\omega_n + \sqrt{\xi^2\omega_n^2 - \omega_n^2}\right)t} + C_2 e^{\left(-\xi\omega_n - \sqrt{\xi^2\omega_n^2 - \omega_n^2}\right)t}$$

$$x(t) = C_1 e^{\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n t}$$ → * General form of the solution.

There are four cases depending on the value of ζ:

$\zeta < 1 \rightarrow$ underdamped vibration

$\zeta = 1 \rightarrow$ Critically Damped Vibration

$\zeta > 1 \rightarrow$ Over damped vibration

$\zeta = 0 \rightarrow$ Undamped vibration
↳ already discussed.

Case (1) $\rightarrow \zeta < 1$, $c < c_{cr}$

* Underdamped vibration

$\sqrt{\zeta^2 - 1}$ is negative $\xrightarrow{i = \sqrt{-1}}$ $\sqrt{\zeta^2 - 1} = i\sqrt{1 - \zeta^2}$ (to make it positive)

$S_{1,2}$:- (solutions)

$$S_{1,2} = \left(-\zeta \pm i\sqrt{1-\zeta^2}\right)\omega_n$$

$$\begin{aligned}
x(t) &= C_1 e^{\left(-\zeta + i\sqrt{1-\zeta^2}\right)\omega_n t} + C_2 e^{\left(-\zeta - i\sqrt{1-\zeta^2}\right)\omega_n t} \\
&= e^{-\zeta\omega_n t}\left(C_1 e^{\left(i\sqrt{1-\zeta^2}\right)\omega_n t} + C_2 e^{\left(-i\sqrt{1-\zeta^2}\right)\omega_n t}\right) \\
&= e^{-\zeta\omega_n t}\left(C_1 \cos\sqrt{1-\zeta^2}\,\omega_n t + C_2 \sin\sqrt{1-\zeta^2}\,\omega_n t\right) \\
&= A e^{-\zeta\omega_n t} \cos\left(\underbrace{\sqrt{1-\zeta^2}\,\omega_n t}_{\omega_d} - \phi\right)
\end{aligned}$$

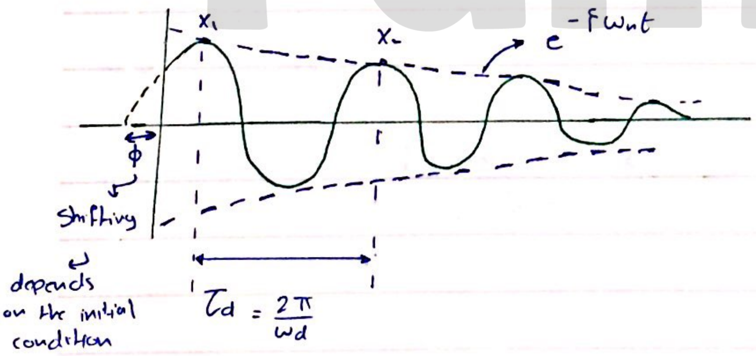
$\rightarrow \left[\begin{array}{l} A = \sqrt{C_1^2 + C_2^2} \\ \phi = \tan^{-1}\left(\frac{C_2}{C_1}\right) \end{array}\right]$

$\rightarrow \omega_d = \sqrt{1-\zeta^2}\,\omega_n$ (Frequency of a damped vibration) $\rightarrow \omega_d < \omega_n$ (always)

use initial conditions to find C_1, C_2

$x(0) = x_0 \rightarrow C_1 = x_0$ $\rightarrow$ plug them into A & ϕ

$\dot{x}(0) = \dot{x}_0 \rightarrow C_2 = \dfrac{\dot{x}_0 + \zeta\omega_n x_0}{\left(\sqrt{1-\zeta^2}\right)\omega_n}$



* notice that because of the factor $\left(e^{-\zeta\omega_n t}\right)$ the amplitude decreases exponentially with time.

Case (2) $\rightarrow \zeta = 1$, $c = c_{cr}$

* Critically damped vibration.

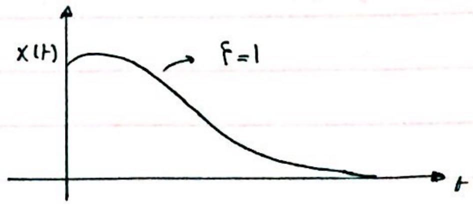
$$S_{1,2} = (-\zeta \pm \sqrt{\zeta^2 - 1}\,)\,\omega_n$$
$$= (-1 \pm 0)\,\omega_n$$
$$S_{1,2} = S = -\omega_n$$
$$x(t) = c_1 e^{-\omega_n t} + c_2 t e^{-\omega_n t}$$
$$= e^{-\omega_n t}(c_1 + c_2 t)$$

I.C's (to find c_1 & c_2)

$x(0) = x_0$ $\qquad c_1 = x_0$

$\dot{x}(0) = \dot{x}_0$ $\qquad c_2 = \dot{x}_0 + x_0\omega_n$

$$x(t) = (x_0 + (\dot{x}_0 + x_0\omega_n)t)\,e^{-\omega_n t}$$



$\rightarrow$ This motion is called Aperiodic $\rightarrow$ not periodic

---

Case (3) $\rightarrow \zeta > 1$, $c > c_{cr}$

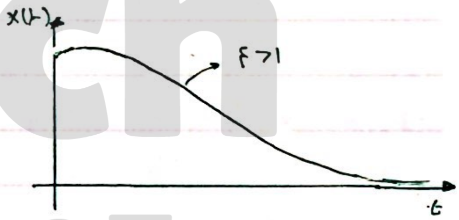
* Overdamped vibration.

$\sqrt{\zeta^2 - 1} \rightarrow$ positive

$S_1 = (-\zeta + \sqrt{\zeta^2 - 1}\,)\,\omega_n$

$S_2 = (-\zeta - \sqrt{\zeta^2 - 1}\,)\,\omega_n$

$\rightarrow$ Both will always be negative.



$$x(t) = c_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + c_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

IC'S :-

$x(0) = x_0$

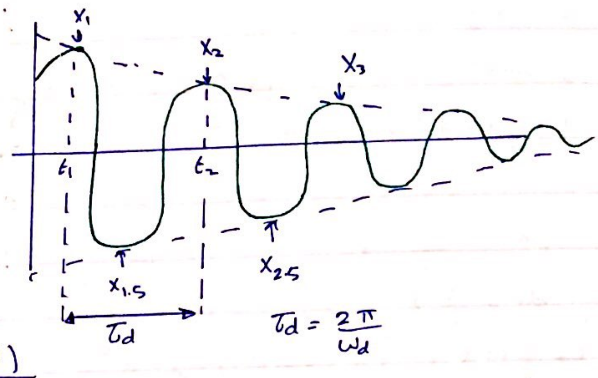
$\dot{x}(0) = \dot{x}_0$

From the IC'S :

$$c_1 = \frac{x_0\omega_n(\zeta + \sqrt{\zeta^2 - 1}) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}} \qquad c_2 = \frac{-x_0\omega_n(\zeta - \sqrt{\zeta^2 - 1}) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

* Logarithmic Decrement :- (Under damped)

$A = \sqrt{c_1^2 + c_2^2}$ (constant)



$$\frac{x_1}{x_2} = \frac{A e^{-\zeta \omega_n t_1} \cos(\omega_d t_1 - \phi)}{A e^{-\zeta \omega_n t_2} \cos(\omega_d t_2 - \phi)}$$

$$t_2 = t_1 + \tau_d = t_1 + \frac{2\pi}{\omega_d}$$

$$\frac{x_1}{x_2} = \frac{\cancel{A} e^{-\zeta \omega_n t_1} \cos(\omega_d t_1 - \phi)}{\cancel{A} e^{-\zeta \omega_n (t_1 + \frac{2\pi}{\omega_d})} \cos\left(\omega_d (t_1 + \frac{2\pi}{\omega_d}) - \phi\right)}$$

$$= e^{-\zeta \omega_n t_1 + \zeta \omega_n (\tau_d + t_1)} \frac{\cos(\omega_d (t_1 - \phi))}{\cos(\omega_d t_1 + 2\pi - \phi)}$$

$\rightarrow$ This is equal to $\cos(\omega_d t_1 - \phi)$

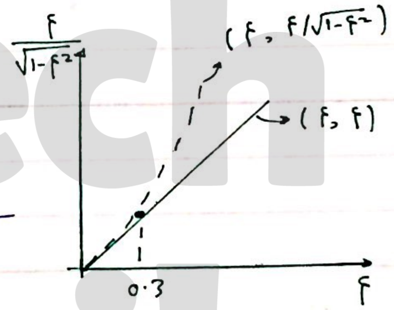
$$\frac{x_1}{x_2} = e^{\zeta \omega_n \tau_d}$$

$$\ln \frac{x_1}{x_2} = \zeta \omega_n \tau_d$$

$\rightarrow \delta$

$$\delta = \zeta \omega_n \tau_d = \zeta \omega_n \frac{2\pi}{\omega_d} = \zeta \frac{\omega_n 2\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$\delta = \frac{2\pi \zeta}{\sqrt{1-\zeta^2}}$$



For :

$0 < \zeta < 0.3$

$$\frac{\zeta}{\sqrt{1-\zeta^2}} \simeq \zeta$$

$$\delta = 2\pi \zeta$$

$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdots \frac{x_m}{x_{m+1}}$$

$$\frac{x_1}{x_{m+1}} = e^{m \zeta \omega_n \tau_d}$$

$$\ln\left(\frac{x_1}{x_{m+1}}\right) = m \zeta \omega_n \tau_d$$

$$= m (2\pi \zeta) \rightarrow \delta$$

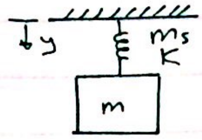
$$\ln\left(\frac{x_1}{x_{m+1}}\right) = m \delta$$

$$\delta = \frac{1}{m} \ln\left(\frac{x_1}{x_{m+1}}\right)$$

Example :-

Find the natural frequency of the system shown

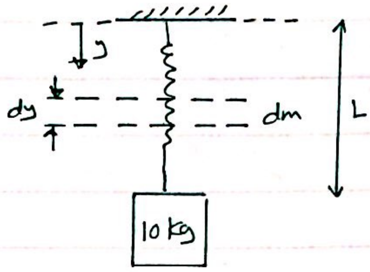
$m = 10\ kg$



Solution :

each point has a different velocity.

$$T = \frac{1}{2} m \dot{x}^2 + \int \frac{1}{2} dm\, dv^2$$



$$dm = \frac{dy}{L} m_s$$

$$dv = \frac{y}{L} \dot{x}$$

$$\frac{1}{2} m \dot{x}^2 + \int_0^L \frac{1}{2} \left(\frac{m_s}{L} dy\right)\left(\frac{y^2 \dot{x}^2}{L^2}\right)$$

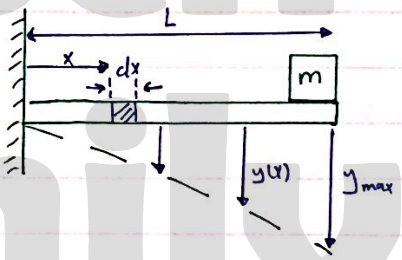
$$\frac{1}{2} m_{eq} \dot{x}^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2}\left(\frac{m_s}{3}\right)\dot{x}^2$$

$$m_{eq} = m + \frac{m_s}{3}$$

$$\text{now} \longrightarrow \omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{k}{m + \frac{m_s}{3}}}$$

Example:-

Find

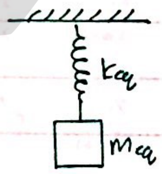


Solution:-

$$y(x) = \frac{Px^2}{6EI}(3L - x)$$

$$y_{max} = \frac{PL^3}{3EI}$$

from strength of materials



we take it at ymax

$$y(x) = y_{max}\left(\frac{3Lx^2 - x^3}{2L^3}\right)$$

$$\boxed{dm = \frac{dx}{L} m_B}$$

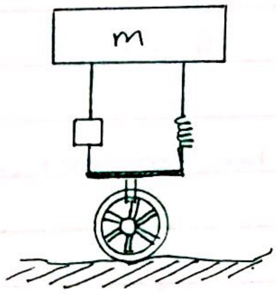
$$\dot{y}(x) = \dot{y}_{max}\left(\frac{3Lx^2 - x^3}{2L^3}\right)$$

$$T = \frac{1}{2} M \dot{y}_{max}^2 + \int_0^L \frac{1}{2} \frac{dx}{L} m_B \left(\frac{3Lx^2 - x^3}{2L^3}\right) \dot{y}_{max}^2$$

$$\frac{1}{2} m_{eq} \dot{y}_{max}^2 = \frac{1}{2} m \dot{y}_{max}^2 + \frac{1}{2}\left(\frac{33}{140}\right) m_B \dot{y}_{max}^2$$

$$m_{eq} = m + \left(\frac{33}{140}\right) m_B$$

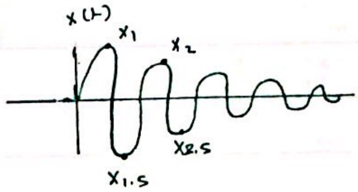
Example (2.11) :-



$m = 200\ kg$

1) Find (c) & (k) if $\tau_d = 2\ sec$ and x_1 is to be reduced to one-fourth in one half cycle ($x_{1.5} = \frac{x_1}{4}$)

2) Find $V_{initial}$ that leads to a maximum displacement of 250 mm ($x_{max} = 250mm$)



solution :-

[1] $\frac{x_1}{x_{1.5}}$ (I can't use this one directly)

$\frac{x_1}{x_2} = \frac{x_1}{x_{1.5}} * \frac{x_{1.5}}{x_2} = 4 * 4 = 16$

→ we can find (c) using $f = \frac{c}{C_{cr}}$ → we need C_{cr}

$\ln \frac{x_1}{x_2} = \frac{2\pi f}{\sqrt{1-f^2}} \longrightarrow \ln(16) = \frac{2\pi f}{\sqrt{1-f^2}} \longrightarrow f = 0.4037$

$C_{cr} = 2m\,\omega_n$

$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-f^2}}$

$\leftrightarrow \omega_n = \frac{2\pi}{2\sqrt{1-f^2}} = \frac{\pi}{\sqrt{1-f^2}}$

$\omega_n = \frac{\pi}{\sqrt{1-f^2}} = 3.4338\ rad/s$

↳ $C_{cr} = 2(200)(3.4338) = 1373.54\ N.S/m$

now → $c = (f)(C_{cr}) = (0.4037)(1373.54) = 554.4981$ $\boxed{c = 554.4981}$ ans.

* To find k:

$\omega_n = \sqrt{\frac{k}{m}} \rightarrow k = m\,\omega_n^2 = (200)(3.4338)^2 = 2358.2652\ N/m$ → $\boxed{k = 2358.2652}$ ans.

[2] x_{max} will be at $t = t_1$

$\sin \omega_d t_1 = \sqrt{1-f^2}$; $\omega_d = \pi$ / $f = 0.4037$

$t_1 = \sin^{-1}(0.9144)/\pi = 0.3678\ sec$

* note → we can't use $(2\pi f)$ instead of $\frac{2\pi f}{\sqrt{1-f^2}}$ here because we don't know if $0 \leq f \leq 0.3$

$x(t) = A e^{-f\omega_n t} \sin(\omega_d t)$

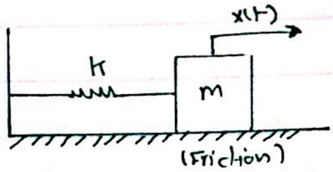
$\dot{x}(t) = A(-f\omega_n e^{-f\omega_n t} \sin(\omega_d t) + e^{-f\omega_n t}\omega_d \cos(\omega_d t))$

to find A: $x(t_1) = 0.25$ t_1 ✓ ω_n ✓ f ✓ ω_d ✓

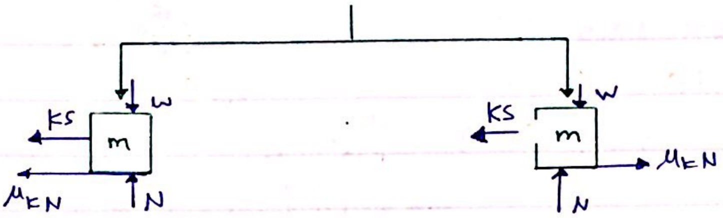
$0.25 = A e^{-f\omega_n t_1} \sin \omega_d t$ → A = ✓

29 → Free vibration with Colomb Damping.

(Dry Friction)



→ We'll have two cases for the motion of the mass :



(To Right) (To left)

Equation of motion; (general)

$m\ddot{x} + Kx = \pm \mu_k N$

solution :-

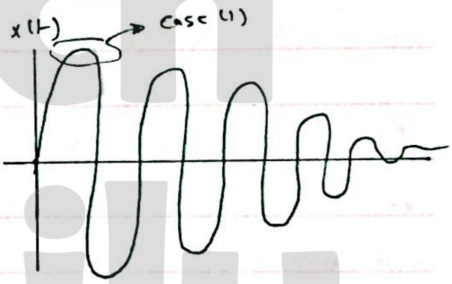
$x(t) = X_h + X_p$

* Case (1) - To left :-

$X_h = C_1 \cos\omega_n t + C_2 \sin\omega_n t$

$X_p = Y$ ↳ constant $\rightarrow \dot{X}_p = 0$, $\ddot{X}_p = 0$ → substitute into : $m\ddot{x} + Kx = \mu_k N$

$0 + KY = \mu_k N \rightarrow X_p = Y = \frac{\mu_k N}{K}$



$0 < t < \frac{\tau}{2}$

$\rightarrow x(t) = X_h + X_p$

$x(t) = C_1 \cos\omega_n t + C_2 \sin\omega_n t + \frac{\mu_k N}{K}$

I.C's :-

$x(0) = X_0 \rightarrow C_1 = X_0 - \frac{\mu_k N}{K}$

$\dot{x}(0) = 0 \rightarrow C_2 = 0$

$\rightarrow x(t) = \left(X_0 - \frac{\mu_k N}{K}\right)\cos\omega_n t + \frac{\mu_k N}{K}$

case (2) - to Right :-

$m\ddot{x} + kx = -\mu_k N$

$x(t) = x_h + x_p$

$x_p = Y$

$kY = -\mu_k N \rightarrow Y = -\frac{\mu_k N}{k}$

↳ $x(t) = C_3 \cos(\omega_n t) + C_4(\omega_n t) - \frac{\mu_k N}{k}$

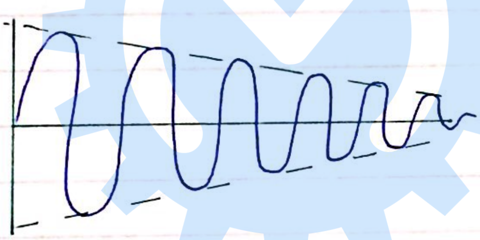
$x(\frac{\tau}{2}) = x_0 + \frac{2\mu_k}{k} \rightarrow C_3 = x_0 - \frac{3\mu_k N}{k}$

$\dot{x}(\frac{\tau}{2}) = 0 \rightarrow C_4 =$

↳ $x(t) = (x_0 - \frac{3\mu_k N}{k}) \cos \omega_n t - \frac{\mu_k N}{k}$ (This is valid for $\tau/2 < t < \tau$)

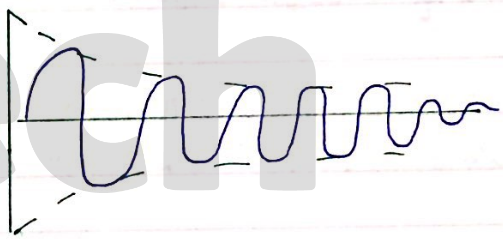
---

* In case of friction



→ Linearly Reducing Amplitude

* In case of viscous damping.

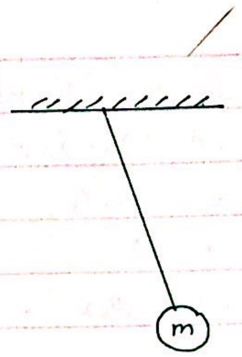


→ Exponentially Reducing Amplitude.

Example:-

Simple pendulum is found to vibrate at a frequency of (5 Hz) in vaccum ($f_{vacum} = 0.5$) and it vibrates at ($f_{fluid} = 0.45$ Hz), assume ($m = 1$ kg)

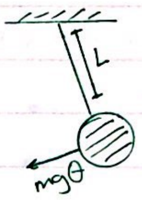
Find → Damping constant (c).



Solution :-

$\omega_n = 2\pi f_n = 2\pi(0.5) = \pi$ rad/s

$\tau = 2$ sec



$\theta \simeq \sin\theta$

$\Sigma M_o = J\ddot{\theta}$

$-mg\theta L = J\ddot{\theta}$

$J\ddot{\theta} + mg\theta L = 0$ #

Equation of motion (derived previously) :-

$J\ddot{\theta} + mgL\theta = 0$

$mL^2\ddot{\theta} + mgL\theta = 0$

$\ddot{\theta} + \frac{g}{L}\theta = 0$

$\omega_n = \sqrt{g/L}$

$\omega_n^2 = \frac{g}{L} \rightarrow L = g f_n^2 = 9.81(0.5)^2 = \_\_$ m

→ $\omega_d = (0.45)(2\pi) = 0.9\pi$ rad/s

$\omega_d = \omega_n\sqrt{1 - \xi^2} \longrightarrow \xi = 0.4$

$\xi = \frac{c}{C_{cr}}$

To find C_{cr} :-

$J\ddot{\theta} + c\dot{\theta} + mgL\theta = 0$

$S_{1,2} = \frac{-c \pm \sqrt{c^2 - 4JmgL}}{2J}$

$c^2 - 4JmgL = 0$

$C_{cr} = \sqrt{4JmgL} = \_\_$ N.S/m

$C = \xi\, C_{cr}$ (Damping coefficient of the fluid).

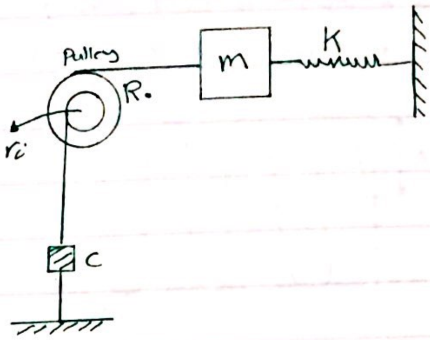
Example :-

If m, J_o, r, R are given

and $(f_n = 5\,Hz)$, in 10 cycles

the displacement is reduced by 80%.

Find $\rightarrow$ k, c

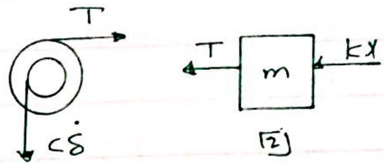


Solution :-

Equations of motion :-

[1] $J\ddot{\theta} = -c\dot{\delta}r + TR$

$J\ddot{\theta} + \dot{\delta}cr - TR = 0 \quad -[1]$



[2] $\Sigma F = m\ddot{x}$

$-kx - T = m\ddot{x}$

$-T = m\ddot{x} + kx$ (plug (-T) into eqn [1])

$\hookrightarrow J\ddot{\theta} + \dot{\delta}cr + R(m\ddot{x} + kx) = 0$

$J\left(\frac{\ddot{x}}{R}\right) + c\left(\frac{r\dot{x}}{R}\right) + Rm\ddot{x} + RKX = 0$

$\left(\frac{J}{R} + Rm\right)\ddot{x} + \frac{Cr^2}{R}\dot{x} + RKX = 0 \quad - (*)$

$x = R\theta$

$\theta = \frac{x}{R}$

$\delta = r\theta$

$\theta = \frac{\delta}{r}$

Plug-into [1]

$\frac{x}{R} = \frac{\delta}{r}$

$\delta = \frac{r}{R}x$

$\dot{\delta} = \frac{r}{R}\dot{x}$

$\hookrightarrow \omega_n = \sqrt{\frac{RK}{J/R + Rm}}$ $\rightarrow$ Solve for k, $k = \checkmark$

To find c:-

$\delta = \frac{1}{N}\ln\left(\frac{X_1}{X_{n+1}}\right) = \frac{1}{10}\ln\left(\frac{1}{0.2}\right) = \frac{2\pi f}{\sqrt{1-f^2}} \rightarrow f = \checkmark$

$C_{cr} \rightarrow$ you can find C_{cr} from $(*)$; $A = \frac{J}{R} + Rm$ $\quad B = \frac{Cr^2}{R}$ $\rightarrow \sqrt{B^2 - 4AD} = 0$

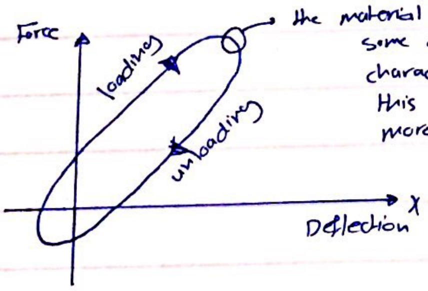
$D = RK$

$C_{cr} = \checkmark$

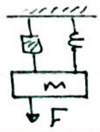
Then $\rightarrow$ $c = f\,C_{cr} = \checkmark$

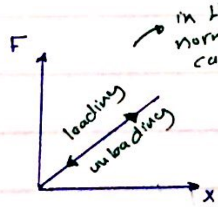
---

* Hysteretic Damping :-



the material loses some of it's characteristics at this point which causes more deflection at less values of (F).

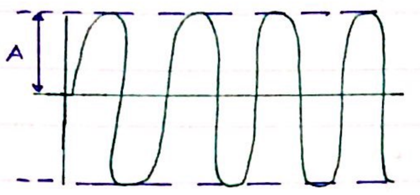




in the normal case

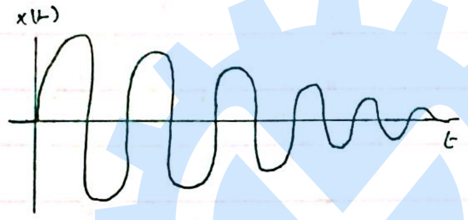
* Stability :-

→ A system is called (stable) if it's free vibration response neither decays nor grows time (A = Constant)



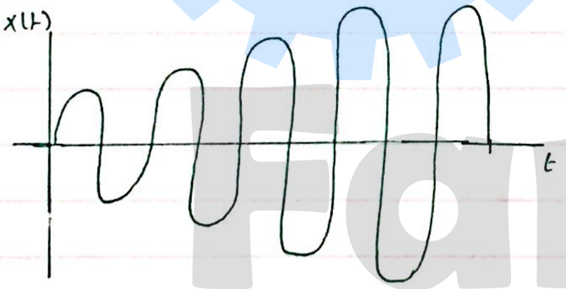
(stable)

→ A system is called (Asymptotically Stable) if it's free vibration response decreases with time (A not a constant) DECREASE



(Asymptotically Stable)

→ A system is called (Unstable) if it's free vibration response grows with time (A increases with time)



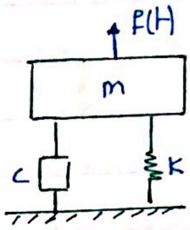
CHAPTER (3) - Harmonically Excited System.

Equation of motion :-

$m\ddot{x} + c\dot{x} + kx = 0$ → Free Vibration

$m\ddot{x} + c\dot{x} + kx = F(t)$ → Harmonically excited vibration

↳ where $F(t)$ is a harmonic force ($A\cos\omega t$)

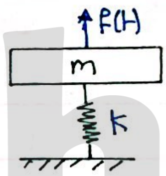


note → the force could be of any kind but in this chapter we only care about harmonic forces.

→ Response of undamped vibration under harmonic force (excitation) :-

$m\ddot{x} + kx = F(t)$

$m\ddot{x} + kx = F_0\cos\omega t$; ω → excitation frequency
F_0 → Force amplitude.



↓

Solution :

$x = x_h + x_p$

$x_h = C_1\cos\omega_n t + C_2\sin\omega_n t$

$x_p = X\cos\omega t$

$\dot{x}_p = -\omega X\sin\omega t$

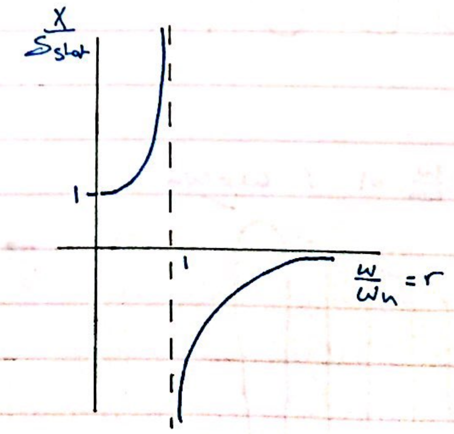
$\ddot{x}_p = -\omega^2 X\cos\omega t = -\omega^2 x_p$

$m(-\omega^2 X\cancel{\cos\omega t}) + k(X\cancel{\cos\omega t}) = F_0\cancel{\cos\omega t}$

$X(k - m\omega^2) = F_0$

$X = \dfrac{F_0}{k - m\omega^2} \xrightarrow[\text{by } k]{\text{Devide}} X = \dfrac{F_0/k}{1 - \frac{m}{k}\omega^2} = \dfrac{F_0/k}{1 - \frac{\omega^2}{\omega_n^2}} = \dfrac{\delta_{stat}}{1 - \frac{\omega^2}{\omega_n^2}}$ → (static deflection - constant)

$$\frac{X}{\delta_{stat}} = \frac{1}{1 - \frac{\omega^2}{\omega_n^2}}$$



$\frac{X}{\delta_{stat}}$ → This Ratio is called the magnification factor.
(the ratio of the dynamic to the static amplitude of motion)

note → If you choose (ω) to be very big the ratio ($\frac{\omega}{\omega_n}$) will be very big as well and the response of your system will be close to zero (no vibration)

* using initial conditions to find C_1 & C_2 :-

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{\delta_{st}}{1-\frac{\omega^2}{\omega_n^2}} \cos \omega t$$

→ I'C's → $x(0) = x_0$ → $C_1 = x_0 - \frac{\delta_{st}}{1-\omega^2/\omega_n^2}$

$\dot{x}(0) = \dot{x}_0$ → $C_2 = \frac{\dot{x}_0}{\omega_n}$

→ Total response of an undamped harmonically excited system:

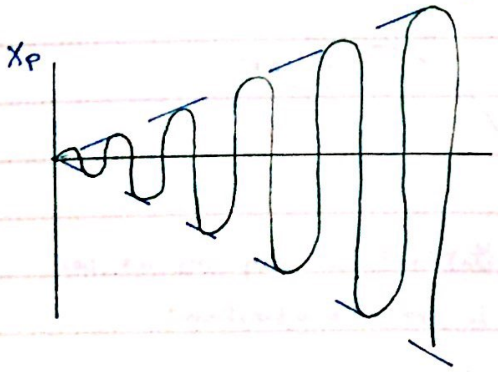
$$\boxed{x(t) = \left(x_0 - \frac{\delta_{st}}{1-\frac{\omega^2}{\omega_n^2}}\right) \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right) \sin \omega_n t + \left(\frac{\delta_{st}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right) \cos \omega t}$$

According to $\left(\frac{X}{\delta_{st}} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$, the response of a system can be identified to be one of 3 types:-

* Case (1) $\frac{\omega}{\omega_n} < 1$ or $\omega < \omega_n$ "In phase"

* Case (2) $\frac{\omega}{\omega_n} > 1$ or $\omega > \omega_n$ "Out of phase"

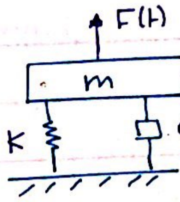
* Case (3) $\frac{\omega}{\omega_n} = 1$ / $\omega = \omega_n$



→ I this case "Resonance" occurs and you'll have an (unstable) response where the displacement increases with time.

→ $X = \frac{\delta_{st}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \xrightarrow{\omega = \omega_n} X = \frac{\delta_{st}}{1-1} = \frac{\delta}{0} = \infty$

→ Responce of a Damped System Under Harmonic Excitation:-



Equation of motion:-

$m\ddot{x} + c\dot{x} + kx = F_0 \cos\omega t$

$x(t) = x_h + x_p$

x_h → Already covered in ch2 (overdamped, under ~, critically ..)

x_p:

$x_p = X\cos(\omega t - \phi)$ → same as $(c_1\cos\omega_n t + c_2\sin\omega_n t)$

↳ we have to include $(\sin\omega_n t)$ here because we have (sin) in the equation of motion. (not like undamped)

$\dot{x}_p = -X\omega\sin(\omega t - \phi)$

$\ddot{x}_p = -X\omega^2\cos(\omega t - \phi)$

↳ plug into the EOM:

$m(-X\omega^2\cos(\omega t-\phi)) + c(-X\omega\sin(\omega t-\phi)) + k(X\cos(\omega t-\phi)) = F_0\cos\omega t$

$X\left((k-m\omega^2)\cos(\omega t-\phi) - (c\omega)\sin(\omega t-\phi)\right) = F_0\cos\omega t$

similar to the right side

But:-

$\cos(\omega t-\phi) = \cos\omega t\cos\phi + \sin\omega t\sin\phi$

$\sin(\omega t-\phi) = \sin\omega t\cos\phi - \cos\omega t\sin\phi$

→ we did this because we need $\cos\omega t$, $\sin\omega t$ not $\cos(\omega t-\phi)$

↓

$X\left((k-m\omega^2)(\cos\omega t\cos\phi + \sin\omega t\sin\phi) - c\omega(\sin\omega t\cos\phi - \cos\omega t\sin\phi)\right) = F_0\cos\omega t$

$X\left((k-m\omega^2)\cos\phi + c\omega\sin\phi\right)\cos\omega t + \left((k-m\omega^2)\sin\phi - c\omega\cos\phi\right)\sin\omega t = F_0\cos\omega t$

$X\left((k-m\omega^2)\cos\phi + c\omega\sin\phi\right) = F_0$

$X\left((k-m\omega^2)\sin\phi - c\omega\cos\phi\right) = 0$

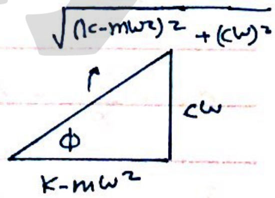
solving for X:

→ $X = \dfrac{F_0}{(k-m\omega^2)\cos\phi + c\omega\sin\phi}$

→ $(k-m\omega^2)\sin\phi = c\omega\cos\phi$

↳ $\tan\phi = \dfrac{c\omega}{k-m\omega^2}$

you can find $\sin\phi$, $\cos\phi$



$$\boxed{X = \frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}}$$

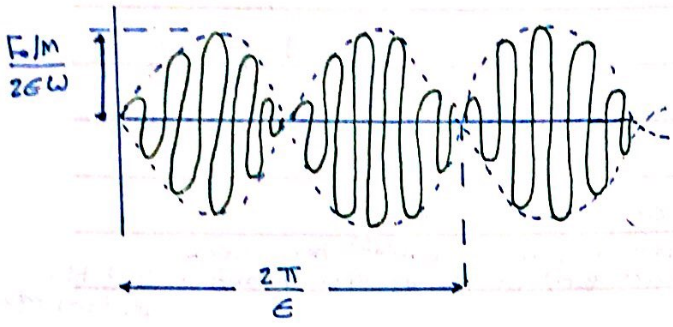
$$\boxed{\phi = \tan^{-1}\frac{c\omega}{k-m\omega^2}}$$

If you Devide X by k :- $\boxed{\dfrac{X}{\delta_{st}} = \dfrac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}}$; $\delta_{st} = \dfrac{F_0}{k}$

$\mu = \dfrac{X}{\delta_{st}}$ (magnification factor)

* Beating Phenomenon :-

→ This phenomenon occurs when (ω) is close (not equal) to ω_n



$\epsilon = \frac{\omega - \omega_n}{2}$ (it tells you how close is the forcing frequency to the natural frequency)

note → The closer ω to ω_n the wider is your cycle.

Example (3.1)

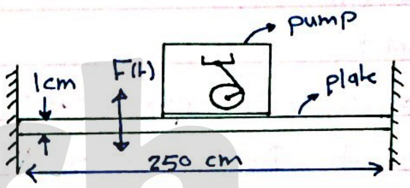
The plate is subjected to harmonic force due to the operation of the pump

$F(t) = 220 \cos 62.832t$

$m_{pump} = 68$ Kg , plate is massless

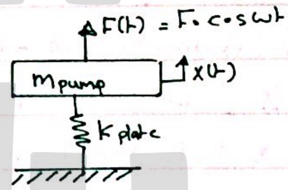
$E = 200$ GPa

Find → the amplitude of vibration (X).



width = 50 cm

Solution :-



. Undamped, harmonically excited system.

$K = \frac{192 EI}{L^3} = \checkmark$

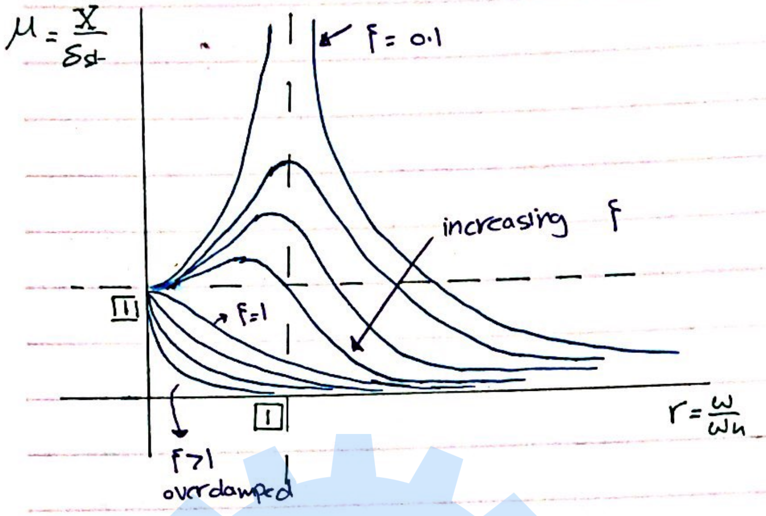
$\omega_n = \sqrt{\frac{K}{m}} = \checkmark$

$r = \frac{\omega}{\omega_n} = \checkmark$

$\delta_{st} = \frac{F_o}{K} = \frac{220}{K} = \checkmark$

$X = \frac{\delta_{st}}{1 - r^2} = -1.32$ → the negative sign indicates that the system is out of phase and ($\omega > \omega_n$) - case (2)

Plotting $\left(\frac{X}{\delta_{st}}, \frac{\omega}{\omega_n}\right)$:-



* Total Response :-

$x(t) = x_h + x_p$

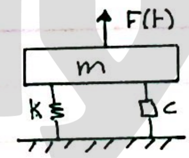
$$\left(x(t) = A e^{-\xi \omega_n t} \cos(\omega_d t - \phi_o) + X \cos(\omega t - \phi) \right) \qquad \phi \neq \phi_o$$

$X = \delta_{st} / \sqrt{(1-r^2)^2 + (2\xi r)^2}$

$\phi = \tan^{-1}\left(c\omega / (k - m\omega^2) \right)$

$\left.\begin{matrix} A \\ \phi_o \end{matrix}\right] \rightarrow$ can be found using initial conditions (DON'T use the relations we used in (CH2) it's different)

Example (3.3) :-



$m = 10\,kg$, $c = 20\ N.s/m$, $k = 4000\ N/m$

$x_o = 0.01$, $\dot{x}_o = 0$

If $F(t) = 200 \cos 10t$, Find total Response.

solution:-

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20\ rad/s$

$c_{cr} = 2\sqrt{mk} = 2\sqrt{(10)(4000)} = 400 \longrightarrow \xi = \frac{c}{c_{cr}} = \frac{20}{400} = 0.05 < 1$ (underdamped)

$\omega_d = \omega_n \sqrt{1 - \xi^2} = \sqrt{1 - (0.05)^2}\,(20) = 19.9749\ rad/s$, $r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$

$X = \delta_{st} / \left(\sqrt{(1-r^2)^2 + (2\xi r)^2} \right) = 0.03326\ m$, $\phi = \tan^{-1}(c\omega / k - m\omega^2) = 3.814°$

$x(0) = A \cos(\phi_o) + X \cos(\phi) = 0.01$ $\qquad (\cos - \phi = \cos \phi)$

$A \cos(\phi_o) = 0.01 - X \cos(\phi)$ ... $\boxed{1}$ $\qquad (\sin - \phi = -\sin \phi)$

$\dot{x}(t) = -\xi \omega_n A e^{-\xi \omega_n t} (\cos(\omega_d t - \phi_o)) - A e^{-\xi \omega_n t} \sin(\omega_d t - \phi_o)(\omega_d) - X\omega \sin(\omega t - \phi)$

$0 = -\xi \omega_n A \cos \phi_o + A \omega_d \sin(\phi_o) + X\omega$ --- $\boxed{2}$

solve for A & ϕ_o

* Response of a Damped System Under $F(t) = F_0 e^{i\omega t}$

Equation of Motion :-

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t}$$

$$x(t) = x_h + x_p$$

x_p:

$$x_p = X e^{i\omega t}$$

$$\dot{x}_p = X(i\omega) e^{i\omega t}$$

$$\ddot{x}_p = -X(\omega)^2 e^{i\omega t}$$

$$m(-X(\omega)^2 e^{i\omega t}) + c(X(i\omega) e^{i\omega t}) + k(X e^{i\omega t}) = F_0 e^{i\omega t}$$

$$X(k - m\omega^2 + i\omega c) = F_0$$

$$X = \frac{F_0}{(k-m\omega^2) + i\omega c} * \frac{(k-m\omega^2) - i\omega c}{(k-m\omega^2) - i\omega c}$$

$$X = \frac{F_0((k-m\omega^2) - i\omega c)}{(k-m\omega^2)^2 + (\omega c)^2}$$

using that ←

We know that:

$$x + iy = A e^{i\phi}$$

$$A = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

$$X = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} e^{i\phi}$$

$$\phi = \tan^{-1}\frac{c\omega}{k - m\omega^2}$$

* Response of a Damped System under the Harmonic motion of the Base :-

Equation of Motion :-

$$m\ddot{x} + c(\dot{x}-\dot{y}) + k(x-y) = 0$$

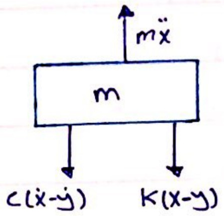
$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

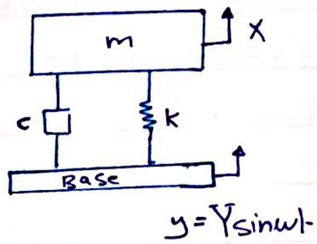
$$m\ddot{x} + c\dot{x} + kx = c(Y\omega\cos\omega t) + k(Y\sin\omega t)$$

$$m\ddot{x} + c\dot{x} + kx = A\cos(\omega t - \alpha)$$

$\hookrightarrow$ $A = \sqrt{(cY\omega)^2 + (kY)^2}$

$\alpha = \tan^{-1}\dfrac{k}{c\omega}$





$$x(t) = x_h + x_p$$

$$x_p = X\cos((\omega t - \alpha) - \phi)$$

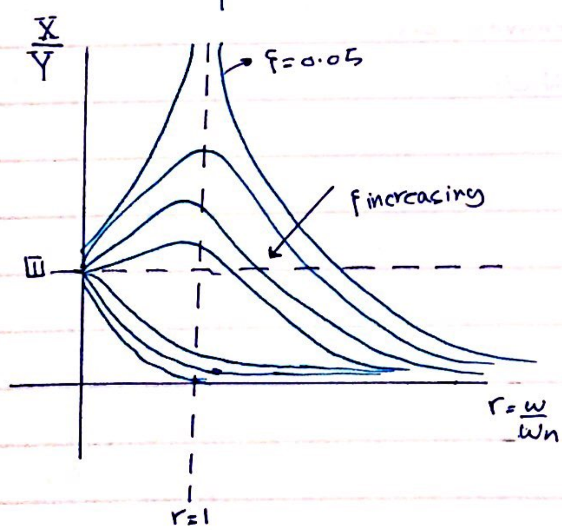
$$X = Y\sqrt{\frac{(c\omega)^2 + k^2}{(k-m\omega^2)^2 + (c\omega)^2}}$$

$$\frac{X}{Y} = \sqrt{\frac{(c\omega)^2 + k^2}{(k-m\omega^2)^2 + (c\omega)^2}}$$

$$\frac{X}{Y} = \sqrt{\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}}$$

$\dfrac{X}{Y}$ → This Ratio is called "Displacement Transimissibility" it indicates how much of the base displacement has transmitted to the motion of the mass.
(if it was (1) it means that we'll have a strong vibration)

* Plotting $(\frac{X}{Y}, r)$:-



* To get the force transmitted to the mass from the base :-

$$\frac{F_T}{kY} = r^2\left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

note that → if $r = 0 \rightarrow \omega = 0 \rightarrow$ no vibration

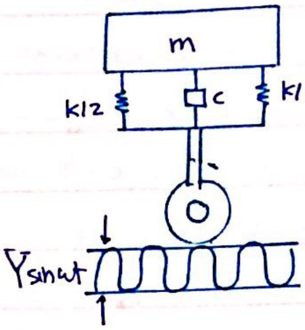
* Example :-

$m = 1200\ kg$, $K = 400\ kN/m$

$\zeta = 0.5$, $V = 20 km/h$

$Y = 0.05$, wave length $= 6m$

Find $\rightarrow X$



Solution :-

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{400(10)^3}{1200}} = 18.25\ rad/s$$

$$f = \frac{V\ (m/s)}{\text{wave length } (m)} = \frac{1}{s}\ (Hz)$$

$$= \left(\frac{20(10)^3}{3600}\right) m/s \Big/ 6\ m = \_\_ \ \frac{1}{s}$$

$$\omega = 2\pi f\ rad/s$$

$$= 5.817\ rad/s$$

$$r = \frac{\omega}{\omega_n} = \frac{5.817}{18.25}$$

▲ For you → Try different speeds and see how it affects the answer.

$$\frac{X}{Y} = \sqrt{\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}} \rightarrow \text{Solve for } X = \_\_ \ m$$

note :-

$x(t) = x_h + x_p$

↳ For Damped vibration the amplitude of (X_h) will keep on decreasing then end to be zero so, the amplitude of (X_p) is what remains and it is called "The steady state amplitude"

* Response of a Damped System Under Rotating unbalance :-

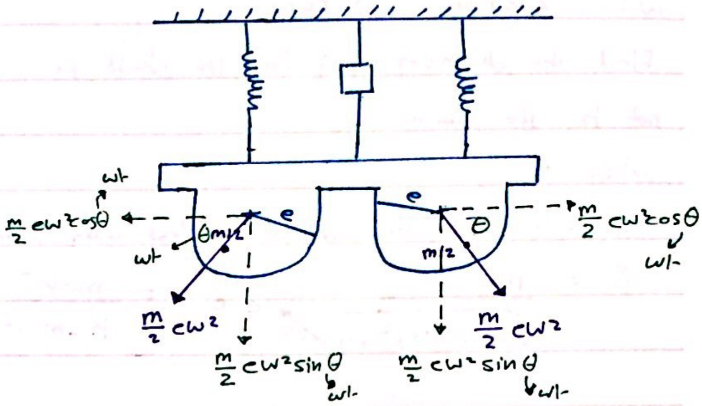
Equation of motion:

$\mu \ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t$

↳ This is forced, damped, Harmonically excited vibration.

$x = x_h + x_p$

$x_p = X \sin(\omega t - \phi)$



$$X = \frac{me\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

or

$$X = \frac{mer^2}{\mu\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\frac{\mu X}{me} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\phi = \tan^{-1} \frac{2\zeta r}{1-r^2}$$

→ since θ is a variable at each position of $(\frac{m}{2})$ during the circular motion, you can replace it with (ωt) - a function of time -

→ From Dynamics :-

$F = ma$

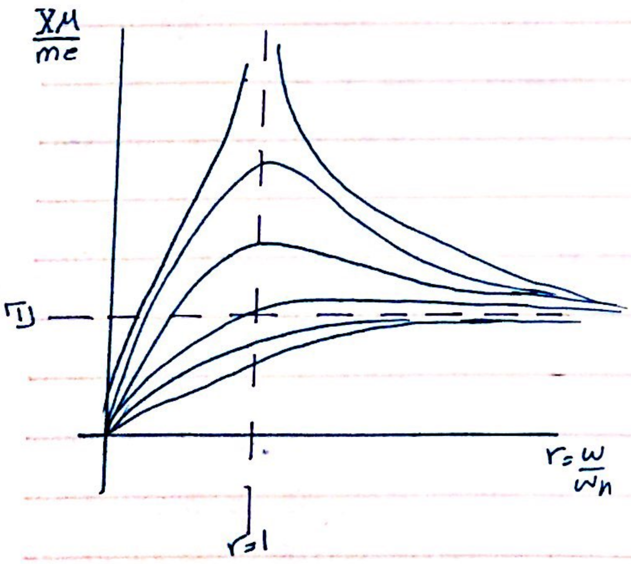
$= mr\omega^2$

→ We assume that both $(m/2, m/2)$ have the same starting point and the same rotating angle (θ)

* Plotting $(\frac{X\mu}{me}, r)$:-

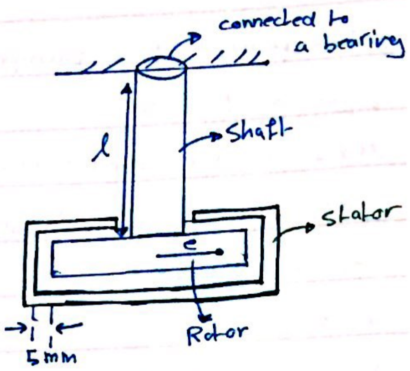
For $0 < \zeta < \frac{1}{\sqrt{2}}$ → we have a max. value

For $\zeta > \frac{1}{\sqrt{2}}$ → Doesn't attain a max. value.



$$\boxed{\left(\frac{X\mu}{me}\right)_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}} \rightarrow 0 < \zeta < \frac{1}{\sqrt{2}}$$

Example :-



$M = ?$

$me = 5$ kg.mm

ω : 600 rpm - 6000 rpm

Find the diameter (d) for the shaft to not hit the stator.

Solution:-

The max. Deflection should be less than 5 mm

$$X = \frac{me\omega^2}{\sqrt{(K-m\omega^2)^2+\cancel{(c\omega)^2}}} \quad c=0 \quad = \frac{me\omega^2}{K-m\omega^2}$$

$$0.05 = \frac{(5(10)^3\ kg.m)\left(\frac{600(2\pi)}{60}\right)^2}{K_1 - 250\left(\frac{600 * 2\pi}{60}\right)^2}$$

For $\omega = 600$ rpm $\rightarrow K_1 = ?$

$\omega = 6000$ rpm $\rightarrow K_2 = ?$

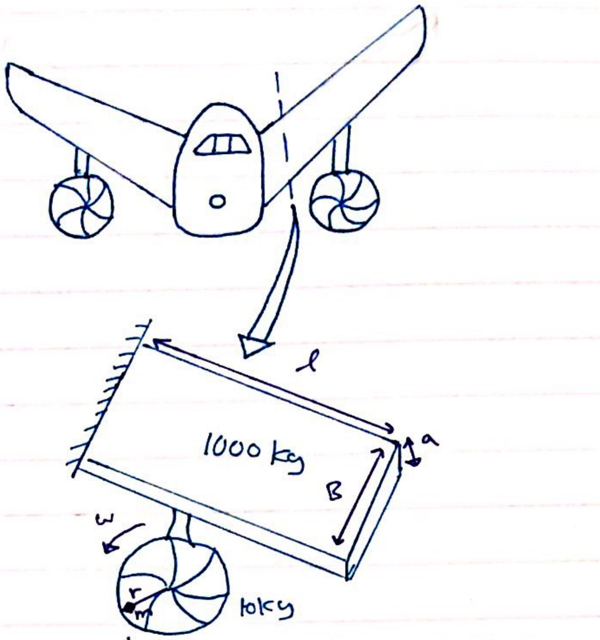
$K = \frac{3EI}{l^3} \longrightarrow$ Solve for $I \longrightarrow$ Find (d) for K_1 & K_2

↳ $K = \frac{3E}{l^3}\left(\frac{\pi d^4}{64}\right)$

Example :-

$\omega = ✓$ rpm

Find the max. deflection of the engine.



Solution :-

EOM $\rightarrow m\ddot{y} + kX = mr\omega^2 \sin\omega t$

$$X = \frac{mr\omega^2}{K - \mu\omega^2}$$

$$K = \frac{3EI}{l^3} = ✓$$

$$\mu = \rho(a)(b)(L)$$

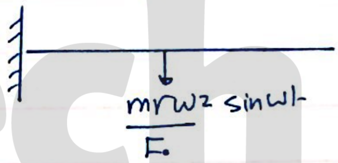
↳ I can't use this directly in (X)

I have to find (μ_{eq}), because it is a distributed mass, From (CH2) :-

$$\mu_{eq} = \frac{33}{140}\mu$$

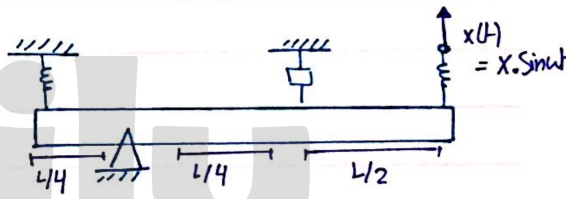
$$\therefore X = \frac{F_0}{K - \mu_{eq}\omega^2}$$

↳ we neglect the mass of the motor because it is too small compared to the wing.



Example :-

Find amplitude of x_p ($\theta_p(t)$)

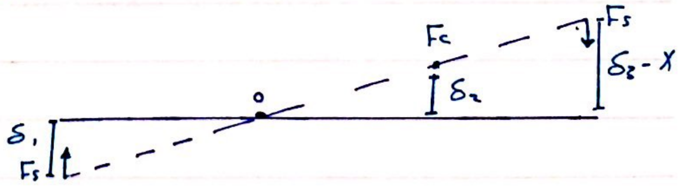


Solution :-

$$\Sigma M_o = J_o \ddot{\theta}$$

$$J_o\ddot{\theta} = -K\left(\tfrac{L}{4}\theta\right)\left(\tfrac{L}{4}\right) - c\left(\tfrac{L}{4}\right)\dot{\theta}\left(\tfrac{L}{4}\right) - K\left(\tfrac{3L\theta}{4} - x\right)\tfrac{3L}{4}$$

$$\underbrace{J_o}_{\mu}\ddot{\theta} + \underbrace{\frac{CL^2}{16}}_{C}\dot{\theta} + \underbrace{\frac{10KL^2}{16}}_{K}\theta = \underbrace{K\left(\frac{3L}{4}\right)X_0}_{F_0} \sin\omega t$$



like X but in Rotational

$$\theta_p(t) = H\sin(\omega t - \phi)$$

$\delta_1 = \frac{L}{4}\theta$, $\delta_2 = \frac{L}{4}\theta$

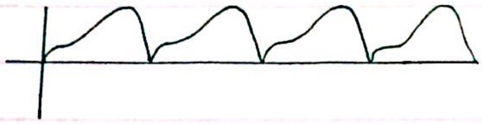
$\delta_3 = \frac{3L}{4}\theta$

$$H = \frac{F_0}{\sqrt{(k - \mu\omega^2)^2 + (c\omega)^2}}, \quad \phi = \tan^{-1}\left(\frac{c\omega}{K - \mu\omega^2}\right)$$

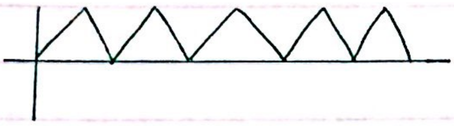
CHAPTER (4) :-

Vibration under general Forcing conditions

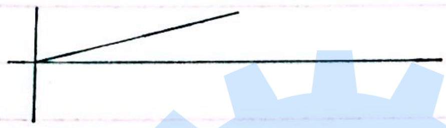
→ Examples of general forces:



→ Irregular periodic



→ Regular periodic

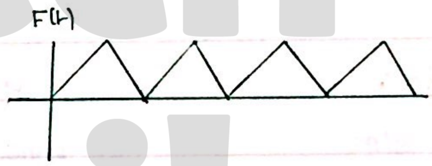


→ Aperiodic

* For regular periodic Force:

We will be using Fourier series to approximate it to a harmonic function:

$$F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t$$



$$a_j = \frac{2}{\tau} \int_0^{\tau} F(t) \cos j\omega t \, dt$$

$$b_j = \frac{2}{\tau} \int_0^{\tau} F(t) \sin j\omega t \, dt$$

* Response of second order system:

$$m\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} a_j \sin j\omega t$$

The Solution:

$$X = X_h + X_p \rightarrow X_p = X_{p1} + X_{p2} + X_{p3}$$

1) $m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2}$

2) $m\ddot{x} + c\dot{x} + kx = \sum_{j=1}^{\infty} a_j \cos j\omega t$

3) $m\ddot{x} + c\dot{x} + kx = \sum_{j=1}^{\infty} a_j \cos j\omega t$

$X_p = X_{p_1} + X_{p_2} + X_{p_3}$

$X_{p_1} = \frac{a_0}{2k}$

$$X_{p_2} = \sum_{j=1}^{\infty} \frac{a_j k}{\sqrt{(1-j^2r^2)^2+(2j\xi r)^2}} \cos(j\omega t - \phi_j)$$

$$X_{p_3} = \sum_{j=1}^{\infty} \frac{b_j k}{\sqrt{(1-(jr)^2)^2+(2\xi r)^2}} \sin(j\omega t - \phi_j)$$

$$\phi_j = \tan^{-1}\left(\frac{2jr\xi}{1-j^2r^2}\right) \qquad r = \frac{\omega}{\omega_n}$$

* Even Functions:

$b_j = 0$

$b_0 = b_1 = b_2 = \dots b_j = 0$

* odd Functions:

$a_j = 0$

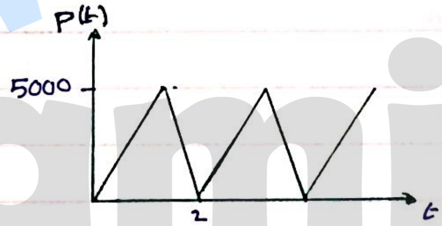
$a_0 = a_1 = a_2 \dots a_j = 0$

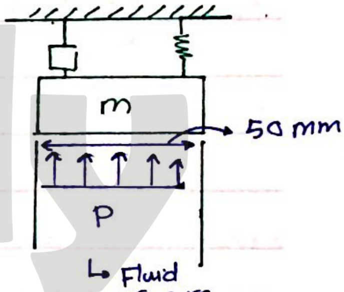
* Example:-

$k = 2500$ N/m

$c = 10$ N.s/m

$m = 0.25$





Solution:-

$F(t) = A\,P(t)$

$$P(t) = \begin{cases} 5000\,A(t), & 0 \le t \le \tau/2 \\ 5000\,A(2-t), & \tau/2 < t < \tau \end{cases}$$

$\Rightarrow$ This is an even function.

To find ω:-

$\tau = 2$ sec

$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{2} = \pi$ rad/s

continued →

$a_0 = \frac{2}{\tau}\int_0^{\tau} F(t)\,dt = \frac{2}{2}\left[\int_0^1 5000At\,dt + \int_1^2 5000A(2-t)\,dt\right] = 5000A$

$a_1 = \frac{2}{\tau}\int_0^{\tau} F(t)\cos\omega t = \frac{2}{2}\left[\int_0^1 5000A\cos\pi t\,dt + \int_1^2 5000A(2-t)\cos\pi t\,dt\right] = -\frac{2(10)^5}{\pi^2}A$

$a_2 = \frac{2}{\tau}\int_0^{\tau} F(t)\cos 2\pi t\,dt = 0$

$a_3 = \frac{2}{\tau}\int_0^{\tau} F(t)\cos 3\pi t\,dt = \frac{2}{2}\left[\int_0^1 5000At\cos\pi t\,dt + \int_1^2 5000A(2-t)\cos\pi t\,dt\right] = \frac{-2(10)^5}{9\pi^2}A$

$F(t) \approx 2500A - \frac{2(10)^5}{\pi^2}A\cos\overset{\pi}{\omega}t - \frac{2(10)^5}{9\pi^2}A\cos(3\overset{\pi}{\omega}t)$

$\frac{a_0}{2k} = \frac{5000A}{2k}$

$$x_p(t) = \frac{2500A}{k} - \frac{\frac{2(10)^5}{k\pi^2}\cos(\pi t - \phi_1)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} - \frac{\frac{2(10)^5}{9k\pi^2}\cos(\pi t - \phi_2)}{\sqrt{(1-9r^2)^2 + (6\xi r)^2}}$$

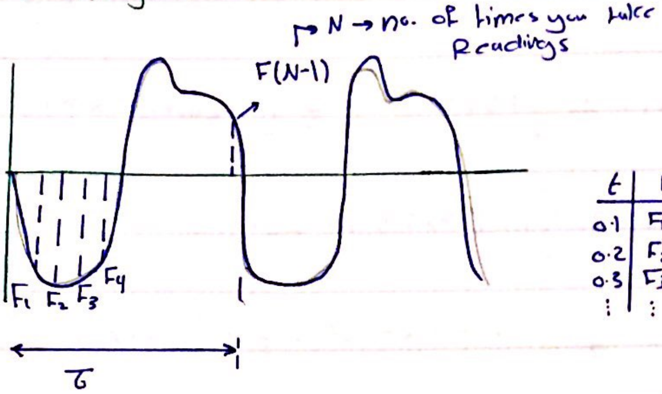
$r = \frac{\omega}{\omega_n} = \_\_ \qquad \omega_n = \sqrt{\frac{k}{m}} = \_\_ \qquad \xi = \frac{c}{c_{cr}} = \frac{c}{2m\omega_n} = \_\_$

* Response under a periodic force of Irregular form:

$$a_0 = \frac{2}{N} \sum_{i=1}^{N} F_i$$

$$a_j = \frac{2}{N} \sum_{i=1}^{N} F_i \cos \frac{j 2\pi t_i}{\tau}$$

$$b_j = \frac{2}{N} \sum_{i=1}^{N} F_i \sin \frac{j 2\pi t_i}{\tau}$$



t	F
0.1	F_1
0.2	F_2
0.3	F_3
⋮	⋮

→ we have a measuring Device that plots the magnitude of the force on the system every small interval of time like (0.1).

Example (4.6) :- (Page 400 – Book)

Time	0	0.01	0.02	0.03	...	0.12
P	0	20	34	42	...	0

Solution:

$\tau = 0.12 \qquad N = 12$

$$a_0 = \frac{2}{N} \sum_{i=1}^{N} F_i = \frac{2}{12} \sum_{i=1}^{12} P_i = \frac{1}{6}(0+20+34+42+\dots+0) = 68.166 \text{ kN/m}^2$$

$$a_1 = \frac{2}{N} \sum_{i=1}^{N} P_i \cos \frac{2\pi t_i}{\tau} = \frac{1}{6}\left(0 + 20 \cos \frac{2\pi(0.01)}{0.12} + 34 \cos \frac{2\pi(0.02)}{0.12} + \dots\right) = -26.9960 \text{ kN/m}^2$$

$$a_2 = \frac{2}{N} \sum_{i=1}^{N} P_i \cos\left(\frac{4\pi t_i}{\tau}\right) = \frac{1}{6}\left(0 + 20 \cos \frac{4\pi(0.01)}{0.12}\right.$$

$$a_3 = \frac{2}{N} \sum_{i=1}^{N} P_i \cos\left(\frac{6\pi t_i}{\tau}\right) = 5.833 \text{ kN/m}^2$$

$$b_1 = \frac{2}{N} \sum_{i=1}^{N} P_i \sin\left(\frac{2\pi t_i}{\tau}\right) = \frac{1}{6}\left(0 + 20 \sin\left(\frac{2\pi(0.01)}{0.12}\right) + 34 \sin\left(\frac{2\pi(0.02)}{0.12}\right)\right)$$

$b_2 = 3.608 \text{ kN/m}^2 \qquad b_3 = -2.333 \text{ kN/m}^2 \qquad \omega = \frac{2\pi}{0.12} = 52.36 \text{ rad/s}$

$$P(t) = \frac{68166}{2} - 269960 \cos(\overset{\omega}{52.36t}) + 8307.7 \sin(\overset{\omega}{52.36t}) + 14167 \cos(\overset{2\omega}{104.72t}) + 3608 \sin(104.72t) - 5833 \cos(157.08t) - 2333 \sin(157.08t) + \dots$$

$F(t) = P(t) \cdot A$

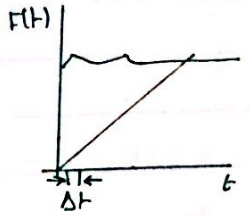
$x_p = x_{p1} + x_{p3} + \dots + x_{p7}$

→ you still have to find x_p using the formulas

* Response Under a non-periodic force:

→ Convulation Integral

→ Laplace Transform

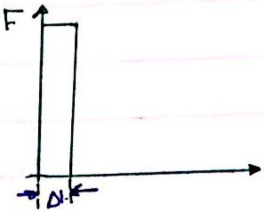


→ non-periodic forces.

* Impulse:

$I = mV_2 - mV_1$

$F\Delta t = m\dot{x}_2 - m\dot{x}_1$



→ we assume (F) is constant during (Δt) because Δt is very small.

→ Unit Impulse

$I = 1$

$I = I\,\delta(t)$

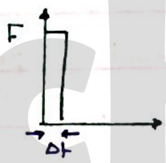
* Response To Impulse:

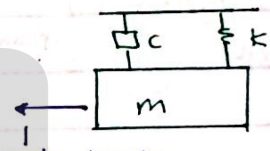
$m\ddot{x} + c\dot{x} + kx = 0$

$x(t) = e^{-\zeta w_n t}\left(X_0 \cos w_d t + \left(\frac{\dot{X}_0 + \zeta w_n X_0}{w_d}\right)\sin w_d t\right)$

$w_d = w_n\sqrt{1-\zeta^2}$

$\zeta = \frac{c}{c_{cr}} \qquad w_n = \sqrt{\frac{k}{m}}$





apply impulse to this system.

↓

For $X_0 = 0 \quad \dot{X}_0 = \frac{1}{m}$:

$x(t) = e^{-\zeta w_n t}\left(\frac{1/m}{w_d}\sin w_d t\right)$

$$\boxed{x(t) = \frac{e^{-\zeta w_n t}}{m w_d}\sin w_d t} \rightarrow g(t)$$

$* \; I = 1 = m\dot{x}_2 - m\dot{x}_1$

$= m\dot{x}_{(t=0^+)} - \cancel{m\dot{x}_{(t=0^-)}}^{\,0}$

$1 = m\dot{x}_0$

$\dot{X}_0 = \frac{1}{m}$ → This is The Initial Condition.

↳ − → just before we apply the Impulse.

+ → just after we apply the impulse.

→ For the magnitude of Impulse $= I$ (not unit impulse):

$x(t) = I\,g(t)$

→ If the Impulse was applied at $t = T_0$:

$x(t) = I\,g(t - T_0)$

$= \frac{I}{m w_d} e^{-\zeta w_n (t - T_0)} \sin(w_d(t - T_0))$

Example :

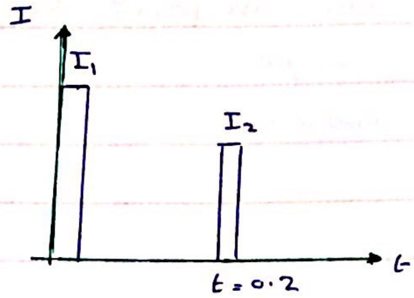
$I_1 = 20$

$I_2 = 10$

$I = 20\delta(t) + 10\delta(t-0.2)$

$m = 5\,kg$, $k = 2000\ N/m$

$c = 10\ N.s/m$



Solution :-

$$\omega_n = \sqrt{\frac{2000}{5}} = 20\ rad/s$$

$$\zeta = \frac{c}{c_{cr}} = \frac{10}{\underbrace{2(5)(20)}_{2m\omega_n}} = 0.05$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 19.975\ rad/s$$

→ $0<t<0.2$

$$x_1(t) = \frac{I_1}{m\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t = \frac{20}{5(19.975)} e^{-0.05\times20\times t}\sin(19.975t) = 0.20025\,e^{-t}\sin(19.975t)$$

→ $t>0.2$

$$x_2(t) = \frac{I_2}{m\omega_d} e^{-\zeta\omega_n t\,(t-0.2)}\sin(\omega_d(t-0.2))$$

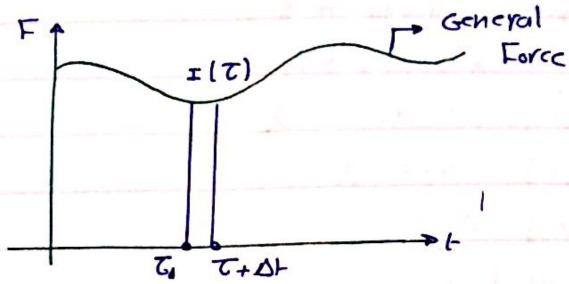
$$= \frac{10}{19.975} e^{-0.05(20)(t-0.2)}\sin(19.975(t-0.2)) = 0.100125\,e^{-(t-0.2)}$$

$$x(t) = \begin{cases} 0.20025\,e^{-t}\sin(19.975)(t-0.2) & 0<t<0.2 \\ 0.20025\,e^{-t}\sin(19.975)(t-0.2) + 0.100125\,e^{-(t-0.2)}\sin(19.975(t-0.2)) & t>0.2 \end{cases}$$

* General Forcing Conditions:

$$x(t) = \int_0^t I(\tau)\, g(t-\tau)\, d\tau$$

$$x(t) = \int_0^t \frac{I(\tau)}{m\omega_d} e^{-\xi\omega_n(t-\tau)} \sin(\omega_d(t-\tau))\, d\tau$$

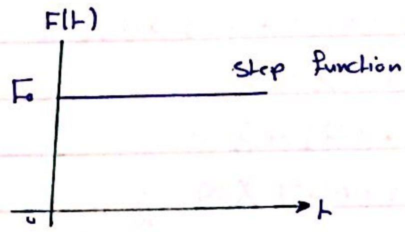


Example:

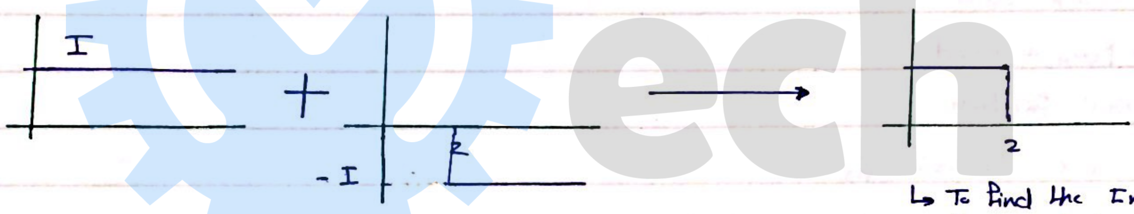
Determine the Response of the system

Solution:

$$x(t) = \frac{I}{m\omega_d} \int_0^t e^{-\xi\omega_n(t-\tau)} \sin(\omega_d(t-\tau))\, d\tau$$



* For a step function, If you want to find I for a certain period of time:



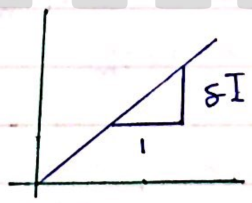
↳ To find the Impulse during $(0 \le t \le 2)$ only. For a constant Force.

$$x(t) = \frac{I}{m\omega_d} \int_0^t e^{-\xi\omega_n(t-\tau-2)} \sin(\omega_d(t-\tau-2))\, d\tau$$

* In case of a linear Force:

$I(\tau) = \delta I\, \tau$ → slope

$$x(t) = \frac{\delta I}{m\omega_d} \int_0^t \tau e^{-\xi\omega_n(t-\tau)} \sin(\omega_d(t-\tau))\, d\tau$$



$\text{Slope} = \frac{\delta I}{1} = \delta I$

note:

→ to find the response we devide the force function into equal intervals of time and find the Impulse for each one then integrate, we integrate from $0 \to t$ to have a function of time to be capable of finding the response at any time we want.

* Laplace Transform :-

$m\ddot{x} + c\dot{x} + kx = \delta(t)$

$L[\ddot{x}] = (s^2 - sx_0 - \dot{x}_0)X(s)$

$L[\dot{x}] = (s - x_0)X(s)$

$L[x] = X(s)$

$L[\delta(t)] = 1$

↓ plug-in the equation of motion

$m(s^2 - sx_0 - \dot{x}_0)X(s) + c(s - x_0)X(s) + kX(s) = 1$ (assume x_0 & $\dot{x}_0 = 0$)

$(ms^2 + cs + k)X(s) = 1$

$(s^2 + 2\zeta\omega_n s + \omega_n^2)mX(s) = 1$

$(s^2 + 2\zeta\omega_n s + \omega_n^2)X(s) = \frac{1}{m} \rightarrow X(s) = \frac{\frac{1}{m}}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$X(s) = \frac{c_1}{s - s_1} + \frac{c_2}{s - s_2}$; s_1 & s_2 are solutions for $(s^2 + 2\zeta\omega_n s + \omega_n^2)$

$s_1 = -\zeta\omega_n - i\omega_d$

$s_2 = -\zeta\omega_n + i\omega_d$

↓ By partial fraction:

$c_1(s - s_1) + c_2(s - s_2) = \frac{1}{m}$

$(c_1 + c_2)s - (c_1 s_1 + c_2 s_2) = \frac{1}{m} + (0)s$

$c_1 + c_2 = 0 \rightarrow (c_1 = -c_2)$

$c_1 s_1 + c_2 s_2 = -\frac{1}{m}$

↓ plug-in s_1 & s_2:-

$c_1(-\zeta\omega_n - i\omega_d) + c_2(-\zeta\omega_n + i\omega_d) = -\frac{1}{m}$ (But $c_1 = c_2$)

$c_2(\zeta\omega_n + i\omega_d - \zeta\omega_n + i\omega_d) = -\frac{1}{m}$

$\boxed{c_2 = \frac{-1}{2im\omega_d}} \rightarrow c_1 = \frac{1}{2im\omega_d}$

$X(s) = \frac{1}{2im\omega_d}\left[\frac{1}{s - s_1} - \frac{1}{s - s_2}\right]$

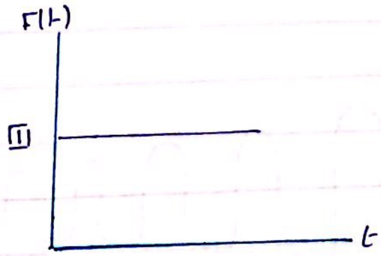
↓ take laplace inverse:

$x(t) = \frac{1}{2im\omega_d}\left(e^{(-\zeta\omega_n - i\omega_d)t} - e^{(-\zeta\omega_n + i\omega_d)t}\right) = \frac{1}{2im\omega_d}\left(2i\sin\omega_d t * e^{-\zeta\omega_n t}\right)$

$x(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d} * \sin\omega_d t$ where → $I = 1$ Recall $g(t)$

* Step Response of Underdamped System :-

$m\ddot{x} + c\dot{x} + kx = 1$



Assume x_0 and $\dot{x}_0 = 0$

$L[\ddot{x}] = [s^2 - s x_0 - \dot{x}_0]\,\bar{X}(s)$

$L[\dot{x}] = [s - x_0]\,\bar{X}(s)$

$L[x] = \bar{X}(s)$

$L[1] = \frac{1}{s}$

$(ms^2 + cs + k)\,\bar{X}(s) = \frac{1}{s}$

$$\bar{X}(s) = \frac{\frac{1}{m}}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{\frac{1}{m}}{s-(-\zeta\omega_n - i\omega_d))(s-(-\zeta\omega_n + i\omega_d))}$$

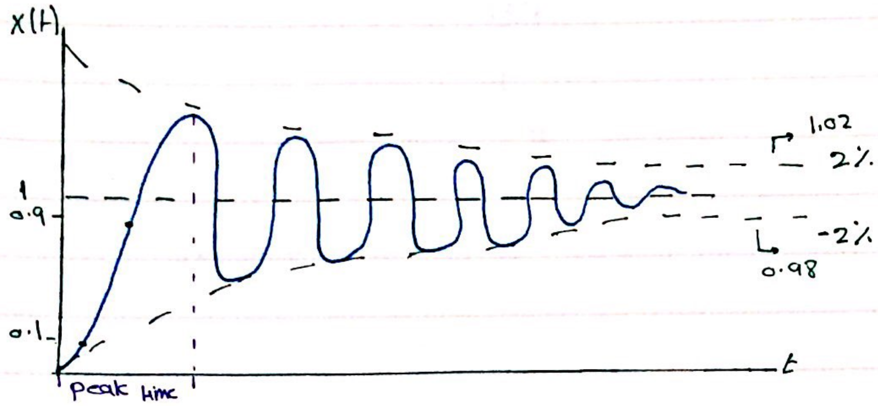
$$\bar{X}(s) = \frac{c_1}{s - s_1} + \frac{c_2}{s - s_2} + \frac{c_3}{s} \quad — \circledast$$

$$c_1 = \frac{\frac{1}{m}}{2i\omega_d(\zeta\omega_n + i\omega_d)}, \quad c_2 = \frac{\frac{1}{m}}{2i\omega_d(-\zeta\omega_n + i\omega_d)}$$

$$c_3 = \frac{\frac{1}{m}}{\zeta^2\omega_n + \underbrace{\omega_d^2}_{\omega_n^2(1-\zeta^2)}}$$

↓ substitute in $\circledast$ then take laplace Inverse :-

$$x(t) = \frac{\frac{1}{m}}{2i\omega_n\zeta\omega_d - 2\omega_d^2} * e^{(-\zeta\omega_n - i\omega_d)t} + \frac{\frac{1}{m}}{-2i\omega_n\zeta\omega_d - 2\omega_d^2} * e^{(-\zeta\omega_n + i\omega_d)t} + \frac{\frac{1}{m}}{\omega_n^2}$$



* Peak Time → The time required for the response to attain the first peak

* Rise Time → Time required for the response to rise from (10%) to (90%) of the final or steady-state value
↳ (some say 0 → 100)

* Maximum overshoot → The max. peak value of the response compared to the final or steady-state value.

* Settling Time → The time required for the response to reach and stay within ± 2% of the steady-state value.

* Delay Time → The time required to reach 50% of the final or steady-state value.

To understand → If the overshoot is (1.1) then compared to the steady-state (1) the max. overshoot would be 10% → (0.1)

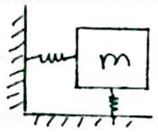
* In this case ↑(the figure), the delay time would be the time needed to reach (0.5)

* Rise time would be the time needed from (0.1) to (0.9)

CHAPTER (5) - Two Degrees of Freedom System

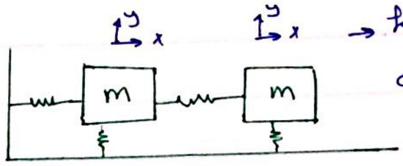
number of degrees of freedom of the system = number of masses in the system $\times$ number of possible types of motion of each mass.

* Examples :-



(two degrees of freedom)

(1×2)

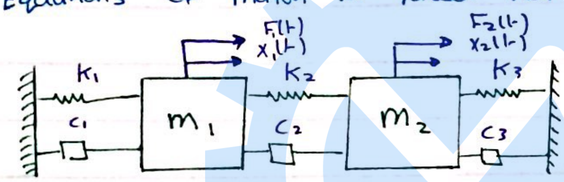


$\rightarrow$ four degrees of freedom

(2×2)

* Equations of motion for forced vibration:

note $\rightarrow$ Here it is assumed that $X_2 > X_1$, you can assume the opposite.



For (m_1) :-

$$\sum F = m_1\ddot{X}_1 = K_2(X_2-X_1) + C_2(\dot{X}_2-\dot{X}_1) - K_1X_1 - C_1\dot{X}_1 + F_1(t)$$

$$m_1\ddot{X}_1 + (C_1+C_2)\dot{X}_1 - C_2\dot{X}_2 + (K_1+K_2)X_1 - K_2X_2 = F_1(t)$$

For (m_2) :-

$$\sum F = m_2\ddot{X}_2 = -K_3X_2 - C_3\dot{X}_2 - K_2(X_2-X_1) - C_2(\dot{X}_2-\dot{X}_1) + F_2(t)$$

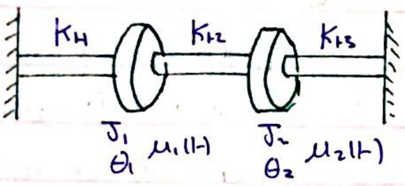
$$m_2\ddot{X}_2 + (C_2+C_3)\dot{X}_2 - C_2\dot{X}_1 + (K_2+K_3)X_2 - K_2X_1 = F_2(t)$$

$$M[\ddot{X}(t)] + C[\dot{X}(t)] + K[X(t)] = [F(t)]$$

where $\rightarrow$ $\ddot{X}(t) = \begin{bmatrix}\ddot{X}_1\\ \ddot{X}_2\end{bmatrix}$ $\quad \dot{X}(t) = \begin{bmatrix}\dot{X}_1\\ \dot{X}_2\end{bmatrix}$ $\quad X(t) = \begin{bmatrix}X_1\\ X_2\end{bmatrix}$

$$\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}\begin{bmatrix}\ddot{X}_1\\ \ddot{X}_2\end{bmatrix} + \begin{bmatrix}C_1+C_2 & -C_2\\ -C_2 & C_2+C_3\end{bmatrix}\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\end{bmatrix} + \begin{bmatrix}K_1+K_2 & -K_2\\ -K_2 & K_2+K_3\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix} = \begin{bmatrix}F_1(t)\\ F_2(t)\end{bmatrix}$$

* Tortional System :-



Equation of motion:

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} K_{t1}+K_{t2} & -K_{t2} \\ -K_{t2} & K_{t2}+K_{t3} \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \mu_1(t) \\ \mu_2(t) \end{bmatrix}$$

---

* Free Vibration Analysis of Undamped System :-

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2+K_3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$m_1\ddot{x}_1 + (K_1+K_2)X_1 - K_2X_2 = 0$$
$$m_2\ddot{x}_2 + (K_2+K_3)X_2 - K_2X_1 = 0$$

note → we assume that both masses have the same (ω, ϕ) - First mode of vibration.

$x_1(t) = \bar{X}_1\cos(\omega t + \phi) \rightarrow \dot{x}_1(t) = -\omega\bar{X}_1\sin(\omega t + \phi) \rightarrow \ddot{X}_1(t) = -\omega^2\bar{X}_1\cos(\omega t + \phi)$

$X_2(t) = \bar{X}_2\cos(\omega t + \phi) \rightarrow \dot{x}_2(t) = -\omega\bar{X}_2\sin(\omega t + \phi) \rightarrow \ddot{X}_2(t) = -\omega^2\bar{X}_2\cos(\omega t + \phi)$

↳ plug into the matrix, you'll get :

$$\begin{bmatrix} (-m_1\omega^2 + (K_1+K_2)) & (-K_2) \\ (-K_2) & (-m_2\omega^2 + (K_2+K_3)) \end{bmatrix}\begin{bmatrix} \bar{X}_1\cos(\omega t + \phi) \\ \bar{X}_2\cos(\omega t + \phi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We'll have:
- → Trivial solution $(\bar{X}_1 = \bar{X}_2 = 0)$ - But that means we'll have no vibration.
- → non trivial solution (we find it by (det = 0))

For non-trivial solution :

$$\begin{bmatrix} -m_1\omega^2 + (K_1+K_2) & -K_2 \\ -K_2 & -m_2\omega^2 + (K_2+K_3) \end{bmatrix} \rightarrow$$ (find the det and equal it to zero)

continued →

$$\underbrace{[m_1 m_2]}_{a}\omega^4 - \underbrace{[m_1(k_2+k_3)+m_2(k_1+k_2)]}_{b}\omega^2 + \underbrace{[(k_1+k_2)(k_2+k_3)-k_2^2]}_{c} = 0$$

↳ solve for ω^2 :-

↳ This equation is called the (frequency characteristic eqn)

→ natural frequencies of the system.

$$\omega_1^2, \omega_2^2 = \frac{m_1(k_2+k_3)+m_2(k_1+k_2)}{2m_1m_2} \mp \frac{\sqrt{\left(m_1(k_2+k_3)+m_2(k_1+k_2)\right)^2 - 4m_1m_2\left((k_1+k_2)(k_2+k_3)-k_2^2\right)}}{2m_1m_2}$$

* For each ω (ω_1, ω_2) we'll have $(\bar{X}_1, \bar{X}_2)$ - depends on the initial conditions.
* We'll consider both modes of vibration :- $\omega_1 \rightarrow \bar{X}_1^{(1)}, \bar{X}_2^{(1)}$

  $\omega_2 \rightarrow \bar{X}_1^{(2)}, \bar{X}_2^{(2)}$
* For each mode we can find (r: the amplitude ratio between $\bar{X}_1$ & $\bar{X}_2$) :-

The same

$$r_1 = \frac{\bar{X}_2^{(1)}}{\bar{X}_1^{(1)}} = \frac{k_2}{-m_2\omega_1^2+(k_2+k_3)} = \frac{-m_1\omega_1^2+(k_1+k_2)}{k_2}$$

$$r_2 = \frac{\bar{X}_2^{(2)}}{\bar{X}_1^{(2)}} = \frac{-m_1\omega_2^2+(k_1+k_2)}{k_2} = \frac{k_2}{-m_2\omega_2^2+(k_2+k_3)}$$

For (ω_1) :-

$$X^{(1)} = \begin{bmatrix} x_1^{(1)} \\ r_1 x_1^{(1)} \end{bmatrix} = \begin{bmatrix} \bar{X}_1^{(1)} \cos(\omega_1 t + \phi) \\ r_1 \bar{X}_1^{(1)} \cos(\omega_1 t + \phi) \end{bmatrix}$$

For (ω_2) :-

$$X^{(2)} = \begin{bmatrix} x_1^{(2)} \\ r_2 x_1^{(2)} \end{bmatrix} = \begin{bmatrix} \bar{X}_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_2 \bar{X}_1^{(2)} \cos(\omega_2 t + \phi_2) \end{bmatrix}$$

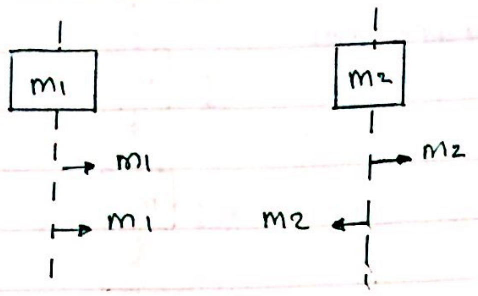
$$x_1(t) = \bar{X}_1^{(1)}\cos(\omega_1 t + \phi_1) + \bar{X}_1^{(2)}\cos(\omega_2 t + \phi_2)$$
$$x_2(t) = \bar{X}_2^{(1)}\cos(\omega_1 t + \phi_1) + \bar{X}_2^{(2)}\cos(\omega_2 t + \phi_2)$$

→ If you have 1st mode these terms would be zero, you control this by initial conditions.

note → now we have four unknowns so we need four initial conditions, two for each mass x_1 x_2

$\dot{x}_1$ $\dot{x}_2$

→ To understand modes of vibration:



(First mode)

(second mode)

→ We'll see how to excite the system for 1st mode only or 2nd mode only.

→ note: modes of vibration depends on the number of degrees of freedom.

* Initial conditions:

$$x_1(t=0) = x_1(0)$$
$$\dot{x}_1(t=0) = \dot{x}_1(0)$$
$$x_2(t=0) = x_2(0)$$
$$\dot{x}_2(t=0) = \dot{x}_2(0)$$

$$x_1(0) = X_1^{(1)} \cos\phi_1 + X^{(2)} \cos\phi_2$$
$$\dot{x}_1(0) = -\omega_1 X_1^{(1)} \sin\phi_1 - \omega_2 X_1^{(2)} \sin\phi_2$$
$$x_2(0) = r_1 X_1^{(1)} \cos\phi_1 + r_2 \left(- X_1^{(2)} \cos\phi_2\right)$$
$$\dot{x}_2(0) = -r_1\omega_1 X_1^{(1)} \sin\phi_1 - r_2\omega_2 X_1^{(2)} \sin\phi_2)$$

$$X_{(1)}^{(1)} = \frac{1}{r_2-r_1}\left(\left(r_2 x_1(0) - x_2(0)\right)^2 + \frac{\left(-r_2\dot{x}_1(0) + \dot{x}_2(0)\right)^2}{\omega_1^2}\right)^{\frac{1}{2}}$$

$$X_{(1)}^{(2)} = \frac{1}{r_2-r_1}\left(\left(-r_1 x_1(0) + x_2(0)\right)^2 + \frac{\left(r_1\dot{x}_1(0) - \dot{x}_2(0)\right)^2}{\omega_2^2}\right)^{\frac{1}{2}}$$

$$\phi_1 = \tan^{-1}\left(\frac{-r_2\dot{x}_1(0) + \dot{x}_2(0)}{\omega_1\left(r_2 x_1(0) - x_2(0)\right)}\right)$$

$$\phi_2 = \tan^{-1}\left(\frac{r_1\dot{x}_1(0) - \dot{x}_2(0)}{\omega_2\left(-r_1 x_1(0) + x_2(0)\right)}\right)$$

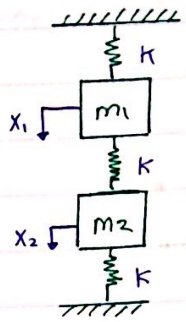
* To excite only 1st mode I have to set $X_1^{(2)} = 0$ $(\omega_2 = 0)$ $\xrightarrow{so}$ $r_1 x_1(0) = x_2(0)$

$\dot{x}_2(0) = r_1 \dot{x}_1(0)$

If I want 2nd mode only $(X_1^{(1)} = 0)$

Example :-

Find the natural frequencies and mode shapes of a spring-mass system



Solution:

$m_1 \ddot{x}_1 + 2k(x_1) - kx_2 = 0$

$m_2 \ddot{x}_2 + 2kx_2 - kx_1 = 0$

$x_1(t) = X_1 \cos(\omega t + \phi)$

$x_2(t) = X_2 \cos(\omega t + \phi)$

$$\begin{bmatrix} -m\omega^2 + 2k & -k \\ -k & -m\omega^2 + 2k \end{bmatrix} = 0$$

$$m^2\omega^4 - 4mk\omega^2 + 3k^2 = 0$$

$$\omega_1^2 = \frac{4km - \sqrt{16k^2m^2 - 12k^2m^2}}{2m^2} = \frac{4km - 2km}{2m^2} = \frac{2km}{2m^2} = \frac{k}{m} \rightarrow \omega_1 = \sqrt{\frac{k}{m}}$$

$$\omega_2^2 = \frac{4km + \sqrt{16k^2m^2 - 12k^2m^2}}{2m^2} = \frac{4km + 2km}{2m^2} = \frac{6km}{2m^2} = \frac{3k}{m} \rightarrow \omega_2 = \sqrt{\frac{3k}{m}}$$

$$r_1 = \frac{k}{-m\omega_1^2 + 2k} = \frac{-m\omega_1^2 + 2k}{k} = \frac{k}{-k + 2k} = \frac{k}{k} = 1$$

$$r_2 = \frac{k}{-m\omega_2^2 + 2k} = \frac{-m\omega_2^2 + 2k}{k} = \frac{k}{-3k + 2k} = \frac{k}{-k} = -1$$

$$x_1(t) = X_1^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) + X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right)$$

$$x_2(t) = X_2^{(1)} \cos\left(\sqrt{\frac{k}{m}}\, t + \phi_1\right) - X_1^{(2)} \cos\left(\sqrt{\frac{3k}{m}}\, t + \phi_2\right)$$

$r_2 = -1$

From the previous example:

$r_1 = 1 \qquad r_2 = -1$

$$x_1(t) = X_1^{(1)} \cos\left(\sqrt{\tfrac{k}{m}}\, t + \phi_1\right) + X_1^{(2)} \cos\left(\sqrt{\tfrac{3k}{m}} + \phi_2\right)$$

$$x_2(t) = X_2^{(1)} \cos\left(\sqrt{\tfrac{k}{m}}\, t + \phi_1\right) + \bar{X} \cos\left(\sqrt{\tfrac{3k}{m}} + \phi_2\right)$$

For first mode of vibration ONLY :-

$$X_1^{(2)} = 0$$

$$X_1^{(2)} = \frac{-1}{2}\left[\big(\underbrace{-x_1(0) + x_2(0)}_{0}\big)^2 + \frac{m}{3k}\big((\underbrace{\dot{x}_1(0) - \dot{x}_2(0)}_{0}\big)^2\right]^{\frac{1}{2}}$$

$x_2(0) = x_1(0)$

$\dot{x}_2(0) = \dot{x}_1(0)$

To Excite second mode of vibration ONLY :-

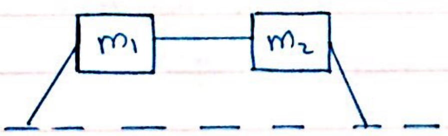
$$X_1^{(1)} = 0$$

$$X_1^{(1)} = \frac{-1}{2}\left[\big(\underbrace{-X_1(0) - X_2(0)}_{0}\big)^2 + \frac{m}{k}\big((\underbrace{\dot{x}_1(0) + \dot{x}_2(0)}_{0}\big)^2\right]^{\frac{1}{2}}$$

$x_2(0) = -x_1(0)$

$\dot{x}_2(0) = -\dot{x}_1(0)$

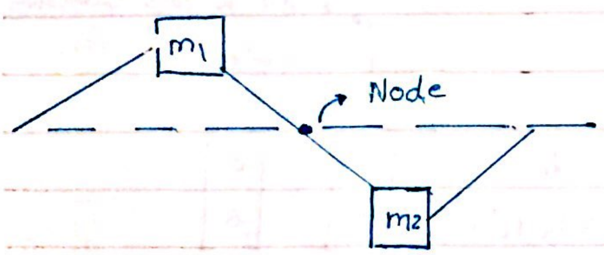
For the 1st mode → Amplitudes of the two masses remains the same so the length of the middle spring remains constant



Equilibrium.

For the 2nd mode → displacements of the two masses have the same mag. with opposite directions / the middle point of the middle spring remains stationary for all time (t) & it's called a (node)

↳ " center of oscillation"



DYNAMIC & STATIC COUPLING

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

This matrix is not coupled

This matrix is coupled by (k_2) and this coupling is called "Static or Elastic coupling"

→ If $(k_2 = 0)$ it's not coupled.

note → If the mass or inertia matrix is coupled it's called "Dynamic coupling"

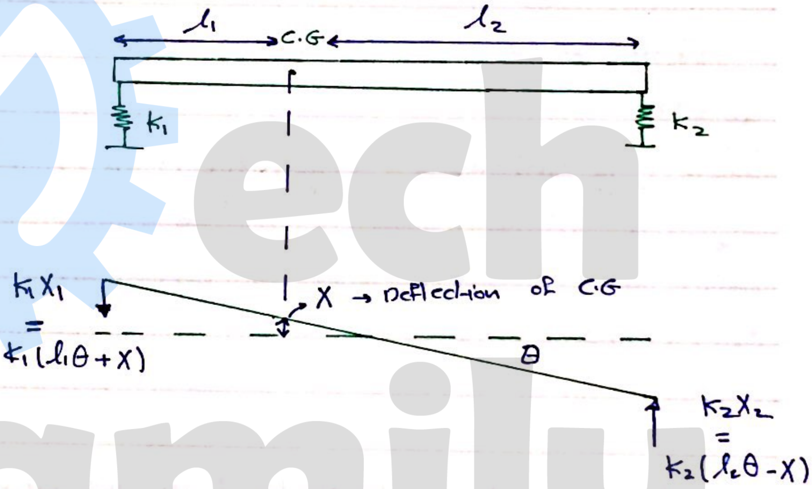
So this system is only Statically coupled.

---

* There are many sets of coordinates that can be used to describe the motion of the two-degree of freedom system that is shown.
* we'll be describing it's motion using:

$x(t)$ → Deflection of C.G

$\theta(t)$ → Rotation



Equations of motion:

$\Sigma F_y = m\ddot{x}$

→ We assumed the rotation to be about C.G.

$m\ddot{x} = -k_1(l_1\theta + X) + k_2(l_2\theta - X)$

$m\ddot{x} + (k_1 + k_2)X + (k_1 l_1 - k_2 l_2)\theta = 0 \quad \ldots \boxed{1}$

$\Sigma M = J\ddot{\theta}$

$-k_1 l_1(l_1\theta + X) + k_2 l_2(l_2\theta - X) = J\ddot{\theta}$

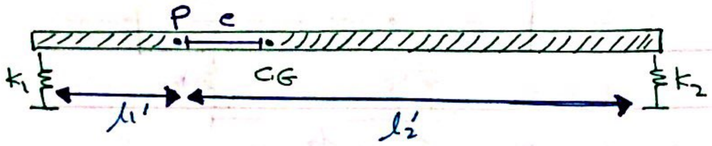
$J\ddot{\theta} + (k_1 l_1^2 + k_2 l_2^2)\theta - (k_1 l_1 - k_2 l_2)X = 0 \quad \ldots \boxed{2}$

$\boxed{1}$ & $\boxed{2}$ in matrix form:

* This is also statically coupled only.

$$\begin{bmatrix} m & 0 \\ 0 & J_o \end{bmatrix}\begin{bmatrix} \ddot{X} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & (k_1 l_1 - k_2 l_1) \\ (k_1 l_1 - k_2 l_2) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix}\begin{bmatrix} X \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now, assume the rotation to be about an arbitrary point (P) :-



$$\sum F = m\ddot{y}$$

$$m\ddot{y} = -k(y + l_1'\theta) + k_2(l_2'\theta - y) - me\ddot{\theta}$$

$$m\ddot{y} + me\ddot{\theta} + (k_1 + k_2)y + (k_1 l_1' - k_2 l_2')\theta = 0 \quad --- \boxed{1}$$

$$\sum M = J_P\ddot{\theta}$$

$$-k_1 l_1'(l_1'\theta + y) - k_2 l_2'(l_2'\theta - y) - m\ddot{y}e = J_P\ddot{\theta}$$

$$J_P\ddot{\theta} + me\ddot{y} + (k_1 l_1'^2 + k_2 l_2'^2)\theta + (k_1 l_1' - k_2 l_2')y = 0 \quad --- \boxed{2}$$

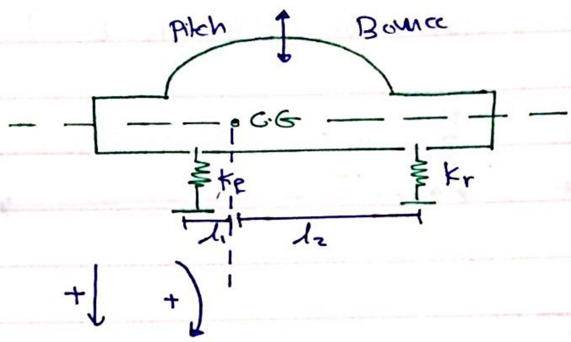
$\boxed{1}$ & $\boxed{2}$ in matrix form:

$$\underbrace{\begin{bmatrix} m & me \\ me & J_P \end{bmatrix}}_{\substack{\text{Dynamic} \\ \text{(mass, inertia)} \\ \text{coupling.}}} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} (k_1 + k_2) & (k_1 l_1' - k_2 l_2') \\ (k_1 l_1' - k_2 l_2') & (k_1 l_1'^2 + k_2 l_2'^2) \end{bmatrix} \begin{bmatrix} y \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Static (Elastic) coupling (the off-diagonal terms $(k_1 l_1' - k_2 l_2')$)

So → The System is Dynamically and Statically coupled.

EXAMPLE :-



$m = 1000$ kg

$k_G = 0.9$ m (radius of gyration)

$L_1 = 1$ m

$L_2 = 1.5$ m

$k_f = 18$ kN/m $\qquad k_r = 22$ kN/m

Solution :-

$x(t) = X \cos\omega t$

$\theta(t) = \Theta \cos\omega t$

$$\begin{vmatrix} m & 0 \\ 0 & J \end{vmatrix} \begin{vmatrix} \ddot{X} \\ \ddot{\theta} \end{vmatrix} + \begin{vmatrix} (k_1 + k_2) & (k_2 l_2 - k_1 l_1) \\ (k_2 l_2 - k_1 l_1) & (k_1 l_1^2 + k_2 l_2^2) \end{vmatrix} \begin{vmatrix} X \\ \theta \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} (-m\omega^2) & (k_2 l_2 - k_1 l_1) \\ (k_2 l_2 - k_1 l_1) & (-J\omega^2 + (k_1 l_1^2 + k_2 l_2^2)) \end{vmatrix} \begin{vmatrix} X \\ \Theta \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} -1000\omega^2 + 40000 & 15000 \\ 15000 & -810\omega^2 + 67500 \end{vmatrix} \begin{vmatrix} X \\ \Theta \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

↳ Find the (det) to get the characteristic equation :

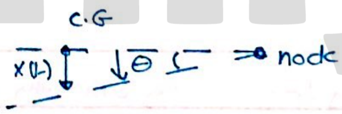
$-810\omega^4 - 999\omega^2 + 2475^0 = 0$

$\omega_1 = 5.8593$ rad/s

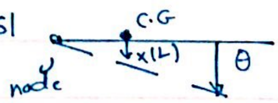
$\omega_2 = 9.4341$ rad/s

negative sign means they move opposite to each other)

$r_1 = \dfrac{X_1^{(1)}}{\Theta_1^{(1)}} = \dfrac{15000}{-1000\omega_1^2 + 40000} = -2.6461$



$r_2 = \dfrac{X_1^{(2)}}{\Theta_1^{(2)}} = \dfrac{-810\omega^2 + 67500}{15000} = 0.3061$



FORCED VIBRATION

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{vmatrix} \begin{vmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix} \begin{vmatrix} \dot{X}_1 \\ \dot{X}_2 \end{vmatrix} + \begin{vmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} F_1(t) \\ F_2(t) \end{vmatrix}$$

↳ This is the general way to write the equations of motion of a two degree of freedom system under external forces.

* We will assume the external forces $(F_1(t), F_2(t))$ to be harmonic

$$\left.\begin{aligned} F_1(t) &= F_{10}\, e^{i\omega t} \\ F_2(t) &= F_{20}\, e^{i\omega t} \end{aligned}\right] \rightarrow \omega: \text{excitation frequency.}$$

* To find the solution:

$x_1(t) = X_1 e^{i\omega t} \longrightarrow \dot{x}_1(t) = i\omega X_1 e^{i\omega t} \longrightarrow \ddot{x}_1(t) = -\omega^2 X_1 e^{i\omega t}$

$x_2(t) = X_2 e^{i\omega t} \longrightarrow \dot{x}_2(t) = i\omega X_2 e^{i\omega t} \longrightarrow \ddot{x}_2(t) = -\omega^2 X_2 e^{i\omega t}$

substitute into the matrix, you'll get:

$$\begin{vmatrix} \overbrace{-m_{11}\omega^2 + i\omega c_{11} + k_{11}}^{Z_{11}} & \overbrace{-m_{12}\omega^2 + i\omega c_{12} + k_{12}}^{Z_{12}} \\ \underbrace{-m_{12}\omega^2 + i\omega c_{12} + k_{12}}_{Z_{12}} & \underbrace{-m_{22}\omega^2 + i\omega c_{22} + k_{22}}_{Z_{22}} \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} F_{10} \\ F_{20} \end{vmatrix}$$

we can write the equation above using the (mechanical impedance Z_{ij})
where $(Z_{ij} = -\omega^2 m_{ij} + i\omega c_{ij} + k_{ij})$, the equation will become:

$$\begin{vmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} F_{10} \\ F_{20} \end{vmatrix}$$

$$\begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \frac{1}{Z_{11}Z_{22} - (Z_{12})^2} \begin{vmatrix} Z_{22} & -Z_{12} \\ -Z_{12} & Z_{11} \end{vmatrix} \begin{vmatrix} F_{10} \\ F_{20} \end{vmatrix}$$

$$\boxed{X_1 = \frac{1}{Z_{11}Z_{22} - (Z_{12})^2}\left(Z_{22}F_{10} - Z_{12}F_{20}\right)}$$

$$\boxed{X_2 = \frac{1}{Z_{11}Z_{22} - (Z_{12})^2}\left(Z_{11}F_{20} - Z_{12}F_{10}\right)}$$

note → From Diff. we know that the method to find X_1, X_2, should be:

$$[X] = [Z]^{-1}[F]$$

where $[Z]^{-1}$:

$$\frac{1}{\det[Z]}\begin{vmatrix} Z_{22} & -Z_{12} \\ -Z_{12} & Z_{11} \end{vmatrix}$$

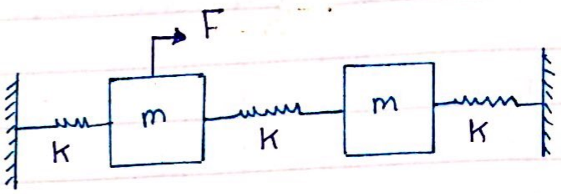
and

$$[X] = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$[F] = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix}$$

EXAMPLE

Find the steady-state response of the system shown and plot it's frequency-response curve.



Solution :-

$$\begin{vmatrix} m & 0 \\ 0 & m \end{vmatrix} \begin{vmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{vmatrix} + \begin{vmatrix} 2k & -k \\ -k & 2k \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} F_{10}\, e^{i\omega t} \\ 0 \end{vmatrix}$$

$$x_1(t) = X_1 e^{i\omega t}$$
$$x_2(t) = X_2 e^{i\omega t}$$

$$\begin{vmatrix} -m\omega^2 + 2k & -k \\ -k & -m\omega^2 + 2k \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} F_{10} \\ 0 \end{vmatrix}$$

we know that:

$\omega_1^2 = \frac{k}{m}$
$\omega_2^2 = \frac{3k}{m}$] natural frequencies of the system

$$X_1 = \frac{(-m\omega^2 + 2k)F_{10}}{(-m\omega^2 + 2k)^2 - k^2} = \frac{(-m\omega^2 + 2k)\,F_{10}}{(-m\omega^2 + 3k)(-m\omega^2 + k)}$$

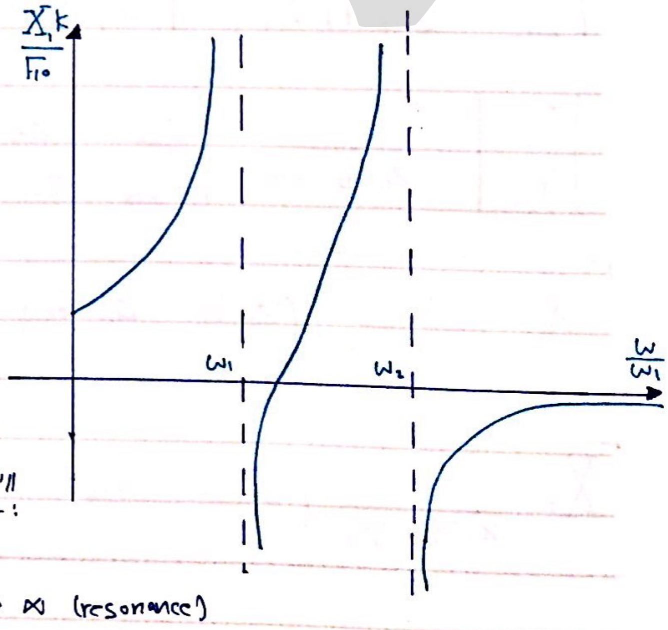
$$X_2 = \frac{kF_0}{(-m\omega^2 + 2k)^2 - k^2} = \frac{kF_{10}}{(-m\omega^2 + 3k)(-m\omega^2 + k)}$$

now, to plot the frequency responce (To clarify Resonance conditions):

By Deviding X_1 by (m) and using ($\omega_1^2 = k/m$ & $\omega_2^2 = \frac{3k}{m}$) you can get:

$$X_1 = \frac{\left(2 - \left(\frac{\omega}{\omega_1}\right)^2\right) F_{10}}{k\left(\left(\frac{\omega_2}{\omega_1}\right)^2 - \left(\frac{\omega}{\omega_1}\right)^2\right)\left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)}$$

$$\frac{kX_1}{F_{10}} = \frac{2 - \left(\frac{\omega}{\omega_1}\right)^2}{\left(\left(\frac{\omega_2}{\omega_1}\right)^2 - \left(\frac{\omega}{\omega_1}\right)^2\right)\left(1 - \left(\frac{\omega}{\omega_1}\right)^2\right)}$$



* In this case we'll have two resonance conditions when ($\omega_1 \to \omega$) & ($\omega_2 \to \omega$)
* to understand set the Denominator to = 0, you'll get:

$\frac{\omega_2}{\omega_1} = \frac{\omega}{\omega_1} \to \omega_2 = \omega$
$1 = \frac{\omega}{\omega_1} \to \omega_1 = \omega$] in both cases the mag. of $X \to \infty$ (resonance)