Bending Stresses in Curved Beams

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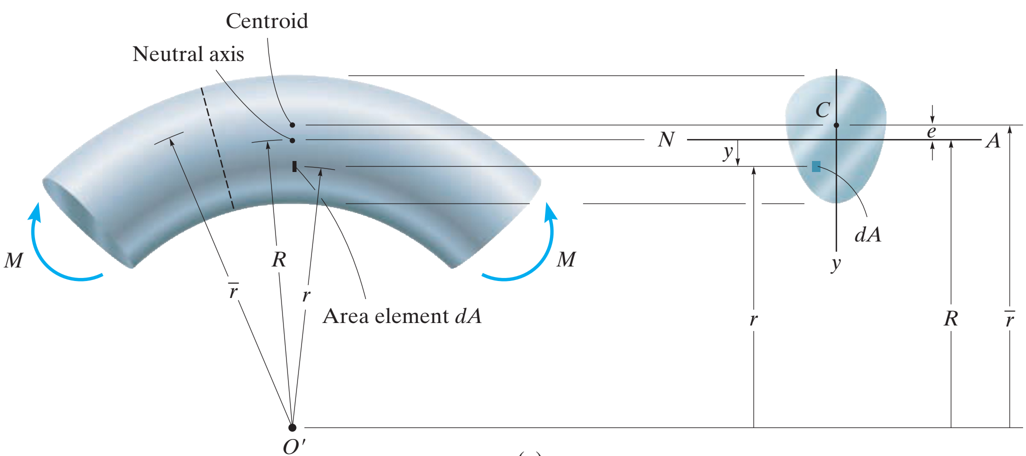
*6.8 Curved Beams

The flexure formula applies to a straight member, since it was shown that the normal strain within it varies linearly from the neutral axis. If the member is *curved*, however, this assumption becomes inaccurate, and so we must develop another method to describe the stress distribution. In this section we will consider the analysis of a *curved beam*, that is, a member that has a curved axis and is subjected to bending. Typical examples include hooks and chain links. In all cases, the members are not slender, but rather have a sharp curve, and their cross-sectional dimensions are large compared with their radius of curvature.

The following analysis assumes that the cross section is constant and has an axis of symmetry that is perpendicular to the direction of the applied moment **M**, Fig. 6–40*a*. Also, the material is homogeneous and isotropic, and it behaves in a linear-elastic manner when the load is applied. Like the case of a straight beam, we will also assume that the *cross sections* of the member *remain plane* after the moment is applied. Furthermore, any distortion of the cross section within its own plane will be neglected.

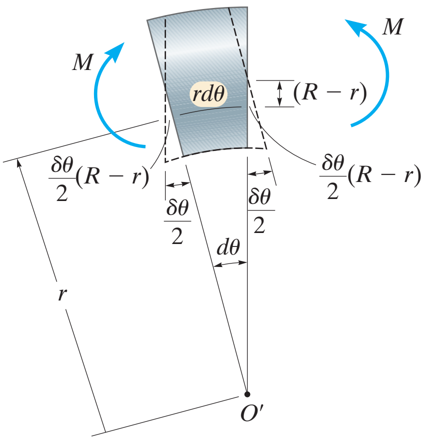
To perform the analysis, three radii, extending from the center of curvature O' of the member, are identified in Fig. 6–40*a*. Here $\bar{r}$ references the known location of the *centroid* for the cross-sectional area, R references the yet unspecified location of the *neutral axis*, and r locates the *arbitrary point* or area element dA on the cross section.

This crane hook represents a typical example of a curved beam.



(a)

Fig. 6–40



(b)

Fig. 6–40 (cont.)

If we isolate a differential segment of the beam, Fig. 6–40*b*, the stress tends to deform the material such that each cross section will rotate through an angle $\delta\theta/2$. The normal strain ϵ in the strip (or line) of material located at r will now be determined. This strip has an original length $r\,d\theta$, Fig. 6–40*b*. However, due to the rotations $\delta\theta/2$ the strip's total change in length is $\delta\theta(R - r)$. Consequently, $\epsilon = \delta\theta(R - r)/r\,d\theta$. If we let $k = \delta\theta/d\theta$, which is the same for any particular strip, we have $\epsilon = k(R - r)/r$. Unlike the case of straight beams, here it can be seen that the ***normal strain*** is a nonlinear function of r, in fact it varies in a ***hyperbolic fashion***. This occurs even though the cross section of the beam remains plane after deformation. If the material remains linearly elastic then $\sigma = E\epsilon$ and so

$$\sigma = Ek\left(\frac{R - r}{r}\right) \tag{6–22}$$

This variation is also hyperbolic, and since it has now been established, we can determine the location of the neutral axis and relate the stress distribution to the resultant internal moment M.

To obtain the location R of the neutral axis, we require the resultant internal force caused by the stress distribution acting over the cross section to be equal to zero; i.e.,

$$F_R = \Sigma F_x; \qquad \int_A \sigma\, dA = 0$$

$$\int_A Ek\left(\frac{R - r}{r}\right) dA = 0$$

Since Ek and R are constants, we have

$$R\int_A \frac{dA}{r} - \int_A dA = 0$$

Solving for R yields

$$R = \frac{A}{\displaystyle\int_A \frac{dA}{r}} \tag{6–23}$$

Here

R = the location of the neutral axis, specified from the center of curvature O' of the member

A = the cross-sectional area of the member

r = the arbitrary position of the area element dA on the cross section, specified from the center of curvature O' of the member

The integral in Eq. 6–23 has been evaluated for various cross-sectional geometries, and the results for some common cross sections are listed in Table 6–1.

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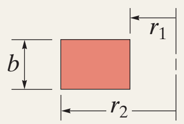
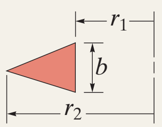
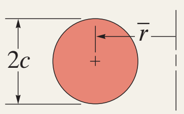
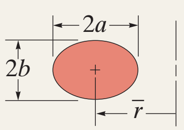
TABLE 6–1

Shape	$\int_A \frac{dA}{r}$
	$b \ln \frac{r_2}{r_1}$
	$\frac{b\,r_2}{(r_2 - r_1)}\left(\ln \frac{r_2}{r_1}\right) - b$
	$2\pi\left(\bar{r} - \sqrt{\bar{r}^2 - c^2}\right)$
	$\frac{2\pi b}{a}\left(\bar{r} - \sqrt{\bar{r}^2 - a^2}\right)$

In order to relate the stress distribution to the resultant bending moment, we require the resultant internal moment to be equal to the moment of the stress distribution calculated about the neutral axis. From Fig. 6–40*a*, the stress σ, acting on the area element dA and located a distance y from the neutral axis, creates a moment about the neutral axis of $dM = y(\sigma\, dA)$. For the entire cross section, we require $M = \int y\sigma\, dA$. Since $y = R - r$, and σ is defined by Eq. 6–22, we have

$$M = \int_A (R - r)Ek\left(\frac{R - r}{r}\right) dA$$

Expanding, realizing that Ek and R are constants, then

$$M = Ek\left(R^2 \int_A \frac{dA}{r} - 2R\int_A dA + \int_A r\, dA\right)$$

The first integral is equivalent to A/R as determined from Eq. 6–23, and the second integral is simply the cross-sectional area A. Realizing that the location of the centroid of the cross section is determined from $\bar{r} = \int r\, dA/A$, the third integral can be replaced by $\bar{r}A$. Thus,

$$M = EkA(\bar{r} - R)$$

Finally, solving for Ek in Eq. 6–22, substituting into the above equation, and solving for σ, we have

$$\sigma = \frac{M(R - r)}{Ar(\bar{r} - R)} \qquad (6\text{–}24)$$

Here

σ = the normal stress in the member

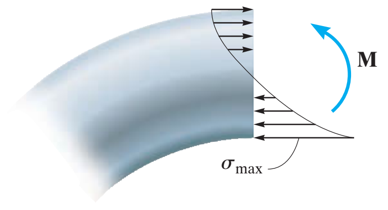
M = the internal moment, determined from the method of sections and the equations of equilibrium and calculated about the neutral axis for the cross section. This moment is *positive* if it tends to increase the member's radius of curvature, i.e., it tends to straighten out the member

A = the cross-sectional area of the member

R = the distance measured from the center of curvature to the neutral axis, determined from Eq. 6–23

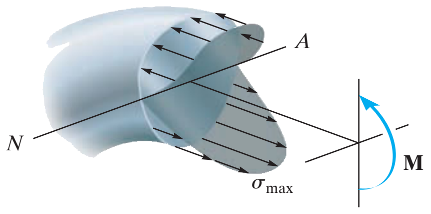
$\bar{r}$ = the distance measured from the center of curvature to the centroid of the cross section

r = the distance measured from the center of curvature to the point where the stress σ is to be determined

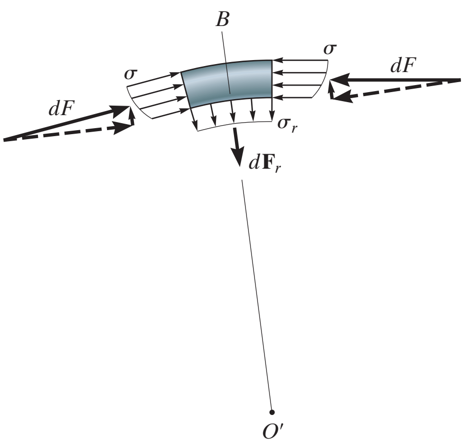


Bending stress variation
(profile view)

(c)



(d)



(e)

Fig. 6–40 (cont.)

From Fig. 6–40*a*, $r = R - y$. Also, the constant and usually very small distance between the neutral axis and the centroid is $e = \bar{r} - R$. When these results are substituted into Eq. 6–24, we can also write

$$\boxed{\sigma = \frac{My}{Ae(R - y)}} \qquad (6\text{–}25)$$

These two equations represent two forms of the so-called *curved-beam formula*, which like the flexure formula can be used to determine the normal-stress distribution in a curved member. This distribution is, as previously stated, hyperbolic; an example is shown in Fig. 6–40*c* and 6–40*d*. Since the stress acts along the circumference of the beam, it is sometimes called ***circumferential stress***. Note that due to the curvature of the beam, the circumferential stress will create a corresponding component of ***radial stress***, so called since this component acts in the radial direction. To show how it is developed, consider the free-body diagram of the segment shown in Fig. 6–40*e*. Here the radial stress σ_r is necessary since it creates the force dF_r that is required to balance the two components of circumferential forces dF which act along the line $O'B$.

Sometimes the radial stresses within curved members may become significant, especially if the member is constructed from thin plates and has, for example, the shape of an I-section. In this case the radial stress can become as large as the circumferential stress, and consequently the member must be designed to resist both stresses. For most cases, however, these stresses can be neglected, especially if the member has a *solid section.* Here the curved-beam formula gives results that are in very close agreement with those determined either by experiment or by a mathematical analysis based on the theory of elasticity.

The curved-beam formula is normally used when the curvature of the member is very pronounced, as in the case of hooks or rings. However, if the radius of curvature is greater than five times the depth of the member, the *flexure formula* can normally be used to determine the stress. For example, for rectangular sections for which this ratio equals 5, the maximum normal stress, when determined by the flexure formula, will be about 7% *less* than its value when determined by the curved-beam formula. This error is further reduced when the radius of curvature-to-depth ratio is more than 5.*

*See, for example, Boresi, A. P., et al., *Advanced Mechanics of Materials*, 3rd ed., p. 333, 1978, John Wiley & Sons, New York.

Important Points

- The *curved-beam formula* should be used to determine the circumferential stress in a beam when the radius of curvature is less than five times the depth of the beam.
- Due to the curvature of the beam, the normal strain in the beam *does not* vary linearly with depth as in the case of a straight beam. As a result, the neutral axis does not pass through the centroid of the cross section.
- The radial stress component caused by bending can generally be neglected, especially if the cross section is a solid section and not made from thin plates.

Procedure for Analysis

In order to apply the curved-beam formula the following procedure is suggested.

Section Properties.

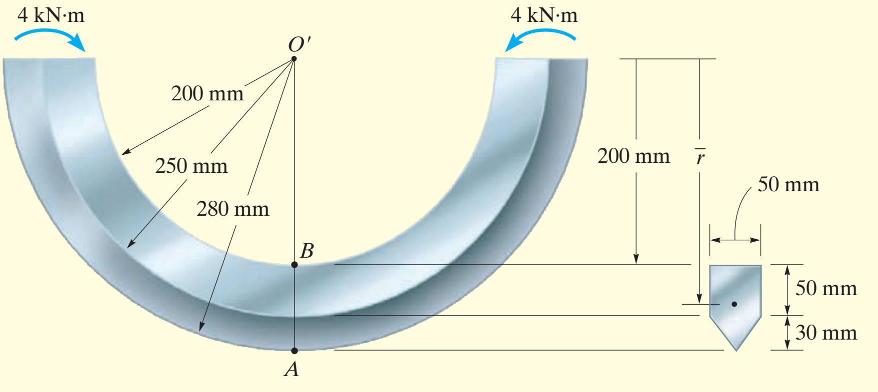
- Determine the cross-sectional area A and the location of the centroid, $\bar{r}$, measured from the center of curvature.
- Find the location of the neutral axis, R, using Eq. 6–23 or Table 6–1. If the cross-sectional area consists of n "composite" parts, determine $\int dA/r$ for *each part.* Then, from Eq. 6–23, for the entire section, $R = \Sigma A/\Sigma(\int dA/r)$. In all cases, $R < \bar{r}$.

Normal Stress.

- The normal stress located at a point r away from the center of curvature is determined from Eq. 6–24. If the distance y to the point is measured from the neutral axis, then find $e = \bar{r} - R$ and use Eq. 6–25.
- Since $\bar{r} - R$ generally produces a very *small number*, it is best to calculate $\bar{r}$ and R with sufficient accuracy so that the subtraction leads to a number e having at least four significant figures.
- If the stress is positive it will be tensile, whereas if it is negative it will be compressive.
- The stress distribution over the entire cross section can be graphed, or a volume element of the material can be isolated and used to represent the stress acting at the point on the cross section where it has been calculated.

EXAMPLE 6.19

The curved bar has a cross-sectional area shown in Fig. 6–41*a*. If it is subjected to bending moments of 4 kN · m, determine the maximum normal stress developed in the bar.



(a)

Fig. 6–41

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SOLUTION

Internal Moment. Each section of the bar is subjected to the same resultant internal moment of 4 kN · m. Since this moment tends to decrease the bar's radius of curvature, it is negative. Thus, $M = -4\ \text{kN}\cdot\text{m}$.

Section Properties. Here we will consider the cross section to be composed of a rectangle and triangle. The total cross-sectional area is

$$\Sigma A = (0.05\ \text{m})^2 + \frac{1}{2}(0.05\ \text{m})(0.03\ \text{m}) = 3.250(10^{-3})\ \text{m}^2$$

The location of the centroid is determined with reference to the center of curvature, point O', Fig. 6–41*a*.

$$\bar{r} = \frac{\Sigma \tilde{r} A}{\Sigma A}$$

$$= \frac{[0.225\ \text{m}](0.05\ \text{m})(0.05\ \text{m}) + [0.260\ \text{m}]\frac{1}{2}(0.050\ \text{m})(0.030\ \text{m})}{3.250(10^{-3})\ \text{m}^2}$$

$$= 0.23308\ \text{m}$$

We can find $\int_A dA/r$ for each part using Table 6–1. For the rectangle,

$$\int_A \frac{dA}{r} = 0.05\text{ m}\left(\ln\frac{0.250\text{ m}}{0.200\text{ m}}\right) = 0.011157\text{ m}$$

And for the triangle,

$$\int_A \frac{dA}{r} = \frac{(0.05\text{ m})(0.280\text{ m})}{(0.280\text{ m} - 0.250\text{ m})}\left(\ln\frac{0.280\text{ m}}{0.250\text{ m}}\right) - 0.05\text{ m} = 0.0028867\text{ m}$$

Thus the location of the neutral axis is determined from

$$R = \frac{\Sigma A}{\Sigma \int_A dA/r} = \frac{3.250(10^{-3})\text{ m}^2}{0.011157\text{ m} + 0.0028867\text{ m}} = 0.23142\text{ m}$$

Note that $R < \bar{r}$ as expected. Also, the calculations were performed with sufficient accuracy so that $(\bar{r} - R) = 0.23308\text{ m} - 0.23142\text{ m} = 0.00166\text{ m}$ is now accurate to three significant figures.

6

Normal Stress. The maximum normal stress occurs either at A or B. Applying the curved-beam formula to calculate the normal stress at B, $r_B = 0.200$ m, we have

$$\sigma_B = \frac{M(R - r_B)}{Ar_B(\bar{r} - R)} = \frac{(-4\text{ kN}\cdot\text{m})(0.23142\text{ m} - 0.200\text{ m})}{3.250(10^{-3})\text{ m}^2(0.200\text{ m})(0.00166\text{ m})}$$

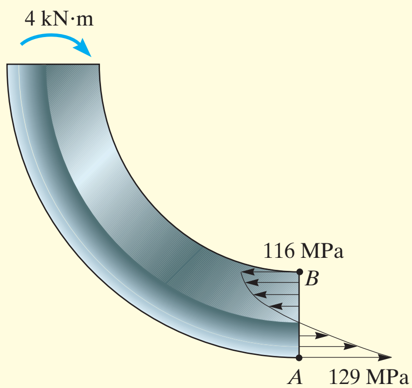
$$= -116\text{ MPa}$$

At point A, $r_A = 0.280$ m and the normal stress is

$$\sigma_A = \frac{M(R - r_A)}{Ar_A(\bar{r} - R)} = \frac{(-4\text{ kN}\cdot\text{m})(0.23142\text{ m} - 0.280\text{ m})}{3.250(10^{-3})\text{ m}^2(0.280\text{ m})(0.00166\text{ m})}$$

$$= 129\text{ MPa}$$ *Ans.*

By comparison, the maximum normal stress is at A. A two-dimensional representation of the stress distribution is shown in Fig. 6–41*b*.

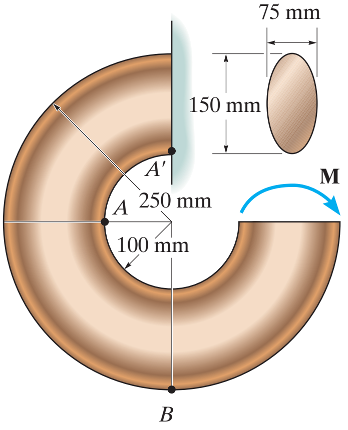


(b)

Fig. 6–41 (cont.)

***6–144.** The member has an elliptical cross section. If it is subjected to a moment of $M = 50\ \text{N}\cdot\text{m}$, determine the stress at points A and B. Is the stress at point A', which is located on the member near the wall, the same as that at A? Explain.

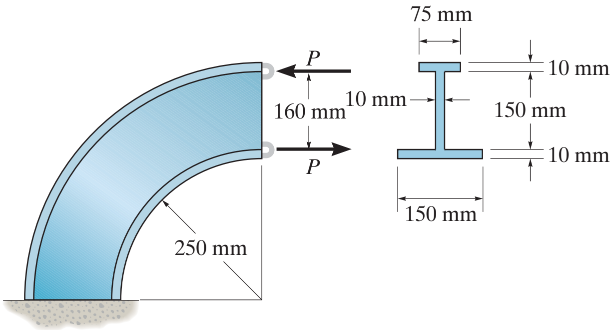
•6–145. The member has an elliptical cross section. If the allowable bending stress is $\sigma_{\text{allow}} = 125$ MPa, determine the maximum moment M that can be applied to the member.



Probs. 6–144/145

6–146. Determine the greatest magnitude of the applied forces P if the allowable bending stress is $(\sigma_{\text{allow}})_c = 50$ MPa in compression and $(\sigma_{\text{allow}})_t = 120$ MPa in tension.

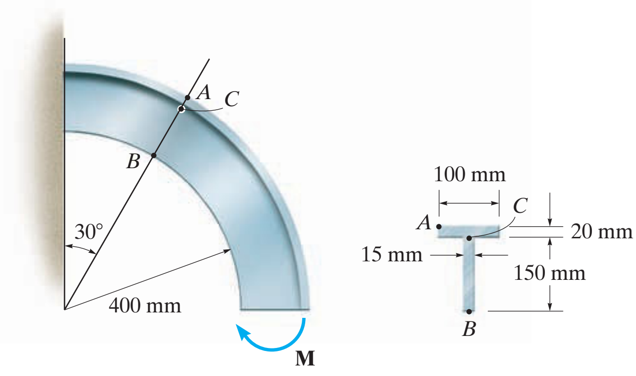
6–147. If $P = 6$ kN, determine the maximum tensile and compressive bending stresses in the beam.



Probs. 6–146/147

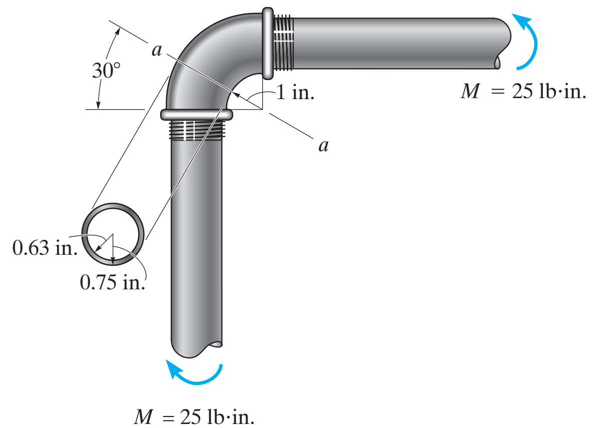
***6–148.** The curved beam is subjected to a bending moment of $M = 900\ \text{N}\cdot\text{m}$ as shown. Determine the stress at points A and B, and show the stress on a volume element located at each of these points.

•6–149. The curved beam is subjected to a bending moment of $M = 900\ \text{N}\cdot\text{m}$. Determine the stress at point C.



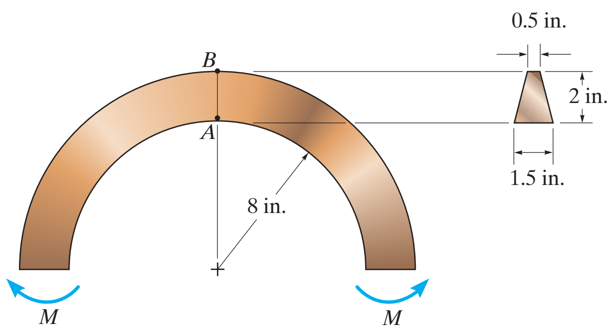
Probs. 6–148/149

6–150. The elbow of the pipe has an outer radius of 0.75 in. and an inner radius of 0.63 in. If the assembly is subjected to the moments of $M = 25\ \text{lb}\cdot\text{in.}$, determine the maximum stress developed at section $a-a$.



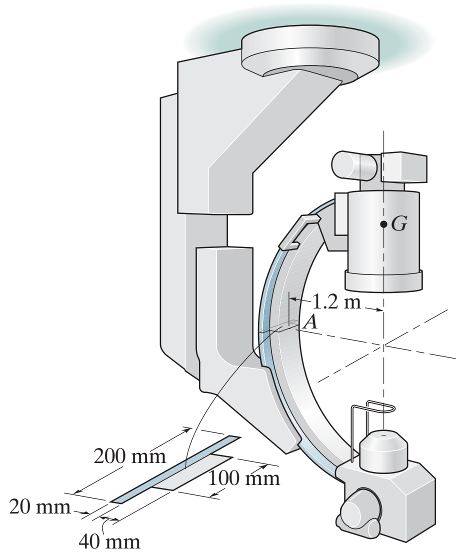
Prob. 6–150

6–151. The curved member is symmetric and is subjected to a moment of $M = 600\ \text{lb}\cdot\text{ft}$. Determine the bending stress in the member at points A and B. Show the stress acting on volume elements located at these points.



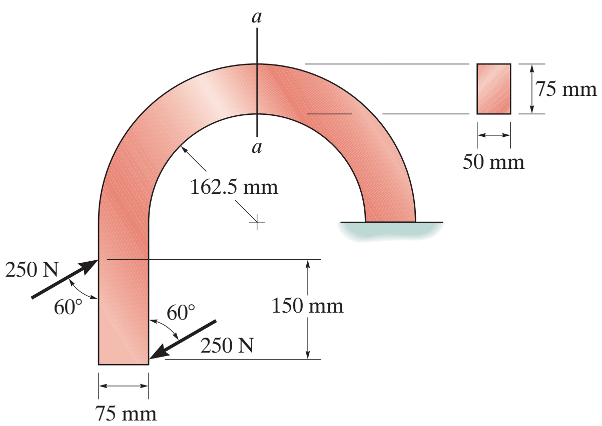
Prob. 6–151

•**6–153.** The ceiling-suspended C-arm is used to support the X-ray camera used in medical diagnoses. If the camera has a mass of 150 kg, with center of mass at G, determine the maximum bending stress at section A.



Prob. 6–153

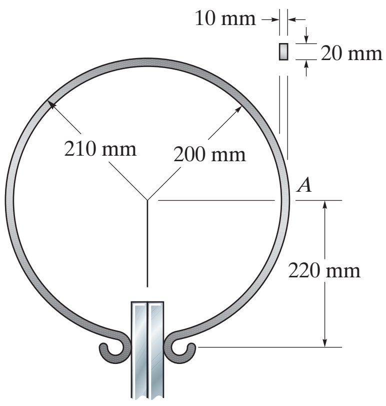
*__6–152.__ The curved bar used on a machine has a rectangular cross section. If the bar is subjected to a couple as shown, determine the maximum tensile and compressive stress acting at section a–a. Sketch the stress distribution on the section in three dimensions.



Prob. 6–152

6–154. The circular spring clamp produces a compressive force of 3 N on the plates. Determine the maximum bending stress produced in the spring at A. The spring has a rectangular cross section as shown.

6–155. Determine the maximum compressive force the spring clamp can exert on the plates if the allowable bending stress for the clamp is $\sigma_{\text{allow}} = 4$ MPa.



Probs. 6–154/155

6–143. Continued

$$= \frac{M\bar{r}}{AI}\left[\int_A\left(\frac{\frac{y}{\bar{r}}}{1+\frac{y}{\bar{r}}}\right)dA - \frac{y}{\bar{r}}\int_A\left(\frac{dA}{1+\frac{y}{\bar{r}}}\right)\right]$$

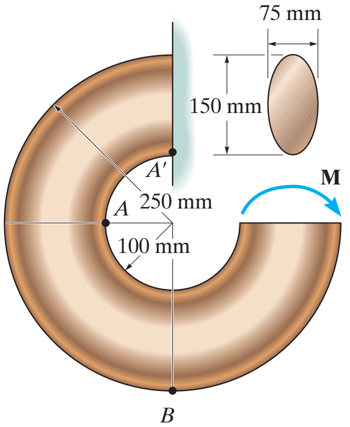
As $\frac{y}{\bar{r}} \to 0$

$$\int_A\left(\frac{\frac{y}{\bar{r}}}{1+\frac{y}{\bar{r}}}\right)dA = 0 \quad \text{and} \quad \frac{y}{\bar{r}}\int_A\left(\frac{dA}{1+\frac{y}{\bar{r}}}\right) = \frac{y}{\bar{r}}\int_A dA = \frac{yA}{\bar{r}}$$

Therefore, $\sigma = \frac{M\bar{r}}{AI}\left(-\frac{yA}{\bar{r}}\right) = -\frac{My}{I}$ **(Q.E.D.)**

$= \frac{M\bar{r}}{AI}$

***6–144.** The member has an elliptical cross section. If it is subjected to a moment of $M = 50\ \text{N}\cdot\text{m}$, determine the stress at points A and B. Is the stress at point A', which is located on the member near the wall, the same as that at A? Explain.



$$\int_A \frac{dA}{r} = \frac{2\pi b}{a}\left(\bar{r} - \sqrt{\bar{r}^2 - a^2}\right)$$

$$= \frac{2\pi(0.0375)}{0.075}\left(0.175 - \sqrt{0.175^2 - 0.075^2}\right) = 0.053049301 \text{ m}$$

$$A = \pi ab = \pi(0.075)(0.0375) = 2.8125(10^{-3})\pi$$

$$R = \frac{A}{\int_A \frac{dA}{r}} = \frac{2.8125(10^{-3})\pi}{0.053049301} = 0.166556941$$

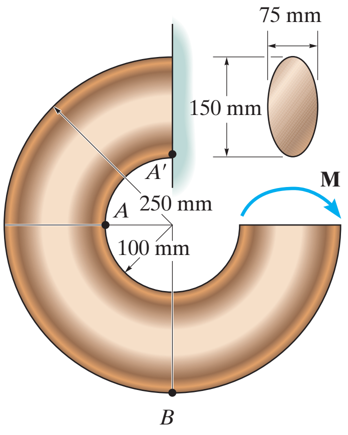
$$\bar{r} - R = 0.175 - 0.166556941 = 0.0084430586$$

$$\sigma_A = \frac{M(R - r_A)}{Ar_A(\bar{r} - R)} = \frac{50(0.166556941 - 0.1)}{2.8125(10^{-3})\pi\,(0.1)(0.0084430586)} = 446\text{k Pa (T)}$$ **Ans.**

$$\sigma_B = \frac{M(R - r_B)}{Ar_B(\bar{r} - R)} = \frac{50(0.166556941 - 0.25)}{2.8125(10^{-3})\pi\,(0.25)(0.0084430586)} = 224 \text{ kPa (C)}$$ **Ans.**

No, because of localized stress concentration at the wall. **Ans.**

•6–145. The member has an elliptical cross section. If the allowable bending stress is $\sigma_{allow} = 125$ MPa determine the maximum moment M that can be applied to the member.



$a = 0.075$ m; $\quad b = 0.0375$ m

$A = \pi(0.075)(0.0375) = 0.0028125\,\pi$

$$\int_A \frac{dA}{r} = \frac{2\pi b}{a}\left(\bar{r} - \sqrt{\bar{r}^2 - a^2}\right) = \frac{2\pi(0.0375)}{0.075}\left(0.175 - \sqrt{0.175^2 - 0.075^2}\right)$$

$$= 0.053049301 \text{ m}$$

$$R = \frac{A}{\int_A \frac{dA}{r}} = \frac{0.0028125\pi}{0.053049301} = 0.166556941 \text{ m}$$

$$\bar{r} - R = 0.175 - 0.166556941 = 8.4430586(10^{-3}) \text{ m}$$

$$\sigma = \frac{M(R - r)}{Ar(\bar{r} - R)}$$

Assume tension failure.

$$125(10^6) = \frac{M(0.166556941 - 0.1)}{0.0028125\pi(0.1)(8.4430586)(10^{-3})}$$

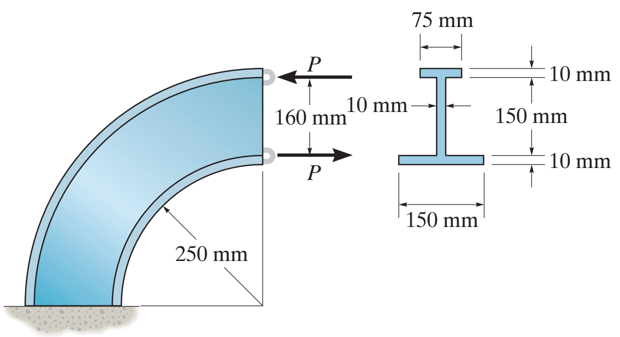
$M = 14.0 \text{ kN} \cdot \text{m}$ (controls) **Ans.**

Assume compression failure:

$$-125(10^6) = \frac{M(0.166556941 - 0.25)}{0.0028125\pi(0.25)(8.4430586)(10^{-3})}$$

$M = 27.9 \text{ kN} \cdot \text{m}$

6–146. Determine the greatest magnitude of the applied forces P if the allowable bending stress is $(\sigma_{\text{allow}})_c = 50$ MPa in compression and $(\sigma_{\text{allow}})_t = 120$ MPa in tension.



Internal Moment: $M = 0.160P$ is positive since it tends to increase the beam's radius of curvature.

Section Properties:

$$\bar{r} = \frac{\Sigma \bar{y}A}{\Sigma A}$$

$$= \frac{0.255(0.15)(0.01) + 0.335(0.15)(0.01) + 0.415(0.075)(0.01)}{0.15(0.01) + 0.15(0.01) + 0.075(0.01)}$$

$$= 0.3190 \text{ m}$$

$$A = 0.15(0.01) + 0.15(0.01) + 0.075(0.01) = 0.00375 \text{ m}^2$$

$$\Sigma \int_A \frac{dA}{r} = 0.15 \ln \frac{0.26}{0.25} + 0.01 \ln \frac{0.41}{0.26} + 0.075 \ln \frac{0.42}{0.41}$$

$$= 0.012245 \text{ m}$$

$$R = \frac{A}{\Sigma \int_A \frac{dA}{r}} = \frac{0.00375}{0.012245} = 0.306243 \text{ m}$$

$$\bar{r} - R = 0.319 - 0.306243 = 0.012757 \text{ m}$$

Allowable Normal Stress: Applying the curved-beam formula

Assume tension failure

$$(\sigma_{\text{allow}})_t = \frac{M(R - r)}{Ar(\bar{r} - R)}$$

$$120\left(10^6\right) = \frac{0.16P(0.306243 - 0.25)}{0.00375(0.25)(0.012757)}$$

$$P = 159482 \text{ N} = 159.5 \text{ kN}$$

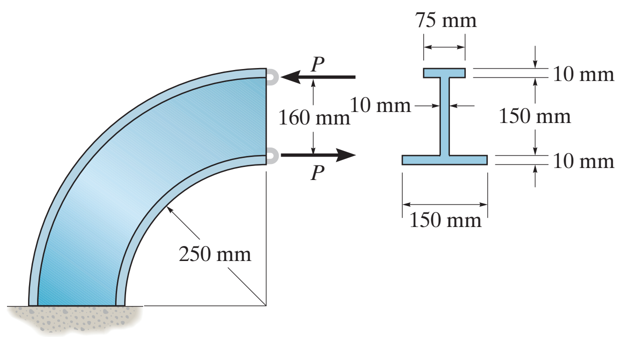
Assume compression failure

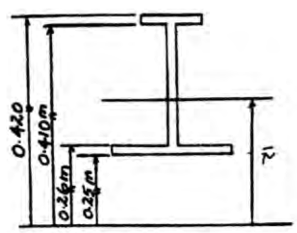
$$(\sigma_{\text{allow}})_t = \frac{M(R - r)}{Ar(\bar{r} - R)}$$

$$-50\left(10^6\right) = \frac{0.16P(0.306243 - 0.42)}{0.00375(0.42)(0.012757)}$$

$$P = 55195 \text{ N} = 55.2 \text{ kN} \ (\textit{Controls !}) \qquad \textbf{Ans.}$$

6–147. If $P = 6$ kN, determine the maximum tensile and compressive bending stresses in the beam.





Internal Moment: $M = 0.160(6) = 0.960\ \text{kN}\cdot\text{m}$ is positive since it tends to increase the beam's radius of curvature.

Section Properties:

$$\bar{r} = \frac{\Sigma \bar{y} A}{\Sigma A}$$

$$= \frac{0.255(0.15)(0.01) + 0.335(0.15)(0.01) + 0.415(0.075)(0.01)}{0.15(0.01) + 0.15(0.01) + 0.075(0.01)}$$

$$= 0.3190\ \text{m}$$

$$A = 0.15(0.01) + 0.15(0.01) + 0.075(0.01) = 0.00375\ \text{m}^2$$

$$\Sigma \int_A \frac{dA}{r} = 0.15 \ln \frac{0.26}{0.25} + 0.01 \ln \frac{0.41}{0.26} + 0.075 \ln \frac{0.42}{0.41}$$

$$= 0.012245\ \text{m}$$

$$R = \frac{A}{\Sigma \int_A \frac{dA}{r}} = \frac{0.00375}{0.012245} = 0.306243\ \text{m}$$

$$\bar{r} - R = 0.319 - 0.306243 = 0.012757\ \text{m}$$

Normal Stress: Applying the curved-beam formula

$$(\sigma_{\max})_t = \frac{M(R - r)}{Ar(\bar{r} - R)}$$

$$= \frac{0.960(10^3)(0.306243 - 0.25)}{0.00375(0.25)(0.012757)}$$

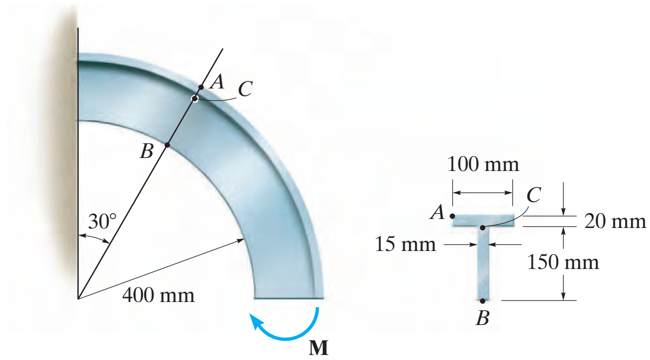
$$= 4.51\ \text{MPa} \qquad \textbf{Ans.}$$

$$(\sigma_{\max})_c = \frac{M(R - r)}{Ar(\bar{r} - R)}$$

$$= \frac{0.960(10^3)(0.306243 - 0.42)}{0.00375(0.42)(0.012757)}$$

$$= -5.44\ \text{MPa} \qquad \textbf{Ans.}$$

***6–148.** The curved beam is subjected to a bending moment of $M = 900\ \text{N} \cdot \text{m}$ as shown. Determine the stress at points A and B, and show the stress on a volume element located at each of these points.



Internal Moment: $M = -900\ \text{N} \cdot \text{m}$ is negative since it tends to decrease the beam's radius curvature.

Section Properties:

$$\Sigma A = 0.15(0.015) + 0.1(0.02) = 0.00425\ \text{m}^2$$

$$\Sigma \bar{r} A = 0.475(0.15)(0.015) + 0.56(0.1)(0.02) = 2.18875(10^{-3})\ \text{m}^3$$

$$\bar{r} = \frac{\Sigma \bar{r} A}{\Sigma A} = \frac{2.18875\ (10^{-3})}{0.00425} = 0.5150\ \text{m}$$

$$\Sigma \int_A \frac{dA}{r} = 0.015 \ln \frac{0.55}{0.4} + 0.1 \ln \frac{0.57}{0.55} = 8.348614(10^{-3})\ \text{m}$$

$$R = \frac{A}{\Sigma \int_A \frac{dA}{r}} = \frac{0.00425}{8.348614(10^{-3})} = 0.509067\ \text{m}$$

$$\bar{r} - R = 0.515 - 0.509067 = 5.933479(10^{-3})\ \text{m}$$

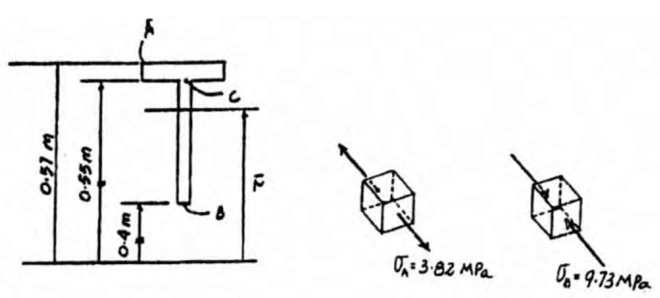
Normal Stress: Applying the curved-beam formula

$$\sigma_A = \frac{M(R - r_A)}{A r_A (\bar{r} - R)} = \frac{-900(0.509067 - 0.57)}{0.00425(0.57)(5.933479)(10^{-3})}$$

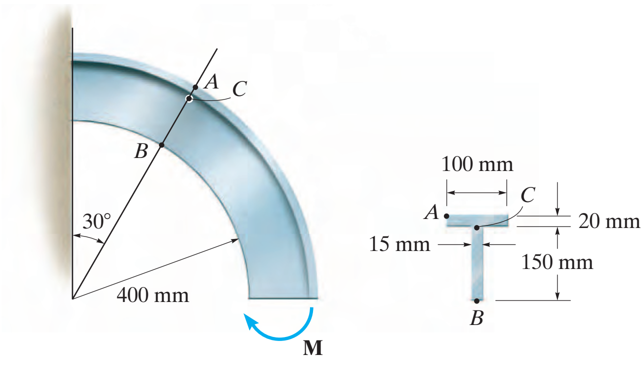
$$= 3.82\ \text{MPa (T)} \qquad \textbf{Ans.}$$

$$\sigma_B = \frac{M(R - r_B)}{A r_B (\bar{r} - R)} = \frac{-900(0.509067 - 0.4)}{0.00425(0.4)(5.933479)(10^{-3})}$$

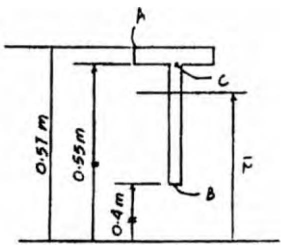
$$= -9.73\ \text{MPa} = 9.73\ \text{MPa (C)} \qquad \textbf{Ans.}$$



•6–149. The curved beam is subjected to a bending moment of $M = 900\ \text{N}\cdot\text{m}$. Determine the stress at point C.



Internal Moment: $M = -900\ \text{N}\cdot\text{m}$ is negative since it tends to decrease the beam's radius of curvature.



Section Properties:

$$\Sigma A = 0.15(0.015) + 0.1(0.02) = 0.00425\ \text{m}^2$$

$$\Sigma \bar{r} A = 0.475(0.15)(0.015) + 0.56(0.1)(0.02) = 2.18875(10^{-3})\ \text{m}$$

$$\bar{r} = \frac{\Sigma \bar{r} A}{\Sigma A} = \frac{2.18875\ (10^{-3})}{0.00425} = 0.5150\ \text{m}$$

$$\Sigma \int_A \frac{dA}{r} = 0.015 \ln \frac{0.55}{0.4} + 0.1 \ln \frac{0.57}{0.55} = 8.348614(10^{-3})\ \text{m}$$

$$R = \frac{A}{\Sigma \int_A \frac{dA}{r}} = \frac{0.00425}{8.348614(10^{-3})} = 0.509067\ \text{m}$$

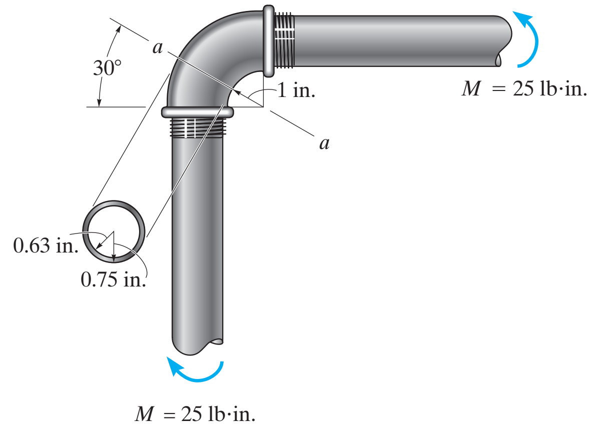
$$\bar{r} - R = 0.515 - 0.509067 = 5.933479(10^{-3})\ \text{m}$$

Normal Stress: Applying the curved-beam formula

$$\sigma_C = \frac{M(R - r_C)}{A r_C(\bar{r} - R)} = \frac{-900(0.509067 - 0.55)}{0.00425(0.55)(5.933479)(10^{-3})}$$

$$= 2.66\ \text{MPa (T)}$$ **Ans.**

6–150. The elbow of the pipe has an outer radius of 0.75 in. and an inner radius of 0.63 in. If the assembly is subjected to the moments of $M = 25$ lb · in., determine the maximum stress developed at section $a-a$.



$$\int_A \frac{dA}{r} = \Sigma 2\pi\left(\bar{r} - \sqrt{\bar{r}^2 - c^2}\right)$$

$$= 2\pi\left(1.75 - \sqrt{1.75^2 - 0.75^2}\right) - 2\pi\left(1.75 - \sqrt{1.75^2 - 0.63^2}\right)$$

$$= 0.32375809 \text{ in.}$$

$$A = \pi(0.75^2) - \pi(0.63^2) = 0.1656\,\pi$$

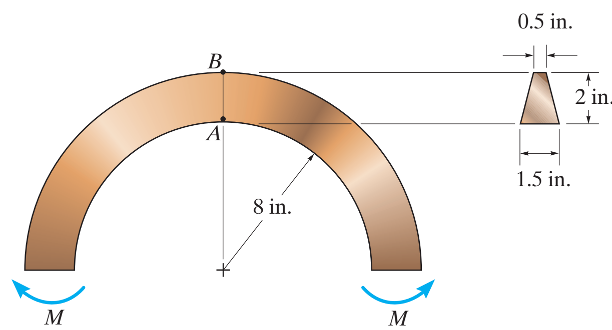
$$R = \frac{A}{\int_A \frac{dA}{r}} = \frac{0.1656\,\pi}{0.32375809} = 1.606902679 \text{ in.}$$

$$\bar{r} - R = 1.75 - 1.606902679 = 0.14309732 \text{ in.}$$

$$(\sigma_{max})_t = \frac{M(R - r_A)}{Ar_A(\bar{r} - R)} = \frac{25(1.606902679 - 1)}{0.1656\,\pi(1)(0.14309732)} = 204 \text{ psi (T)}$$ **Ans.**

$$(\sigma_{max})_c = = \frac{M(R - r_B)}{Ar_B(\bar{r} - R)} = \frac{25(1.606902679 - 2.5)}{0.1656\pi(2.5)(0.14309732)} = 120 \text{ psi (C)}$$ **Ans.**

6–151. The curved member is symmetric and is subjected to a moment of $M = 600$ lb · ft. Determine the bending stress in the member at points A and B. Show the stress acting on volume elements located at these points.



$$A = 0.5(2) + \frac{1}{2}(1)(2) = 2 \text{ in}^2$$

$$\bar{r} = \frac{\Sigma\bar{r}A}{\Sigma A} = \frac{9(0.5)(2) + 8.6667\left(\frac{1}{2}\right)(1)(2)}{2} = 8.83333 \text{ in.}$$

$$\int_A \frac{dA}{r} = 0.5 \ln\frac{10}{8} + \left[\frac{1(10)}{(10 - 8)}\left[\ln\frac{10}{8}\right] - 1\right] = 0.22729 \text{ in.}$$

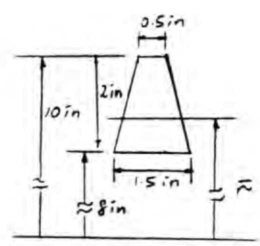
$$R = \frac{A}{\int_A \frac{dA}{r}} = \frac{2}{0.22729} = 8.7993 \text{ in.}$$

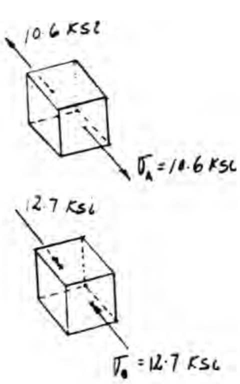
$$\bar{r} - R = 8.83333 - 8.7993 = 0.03398 \text{ in.}$$

$$\sigma = \frac{M(R - r)}{Ar(\bar{r} - R)}$$

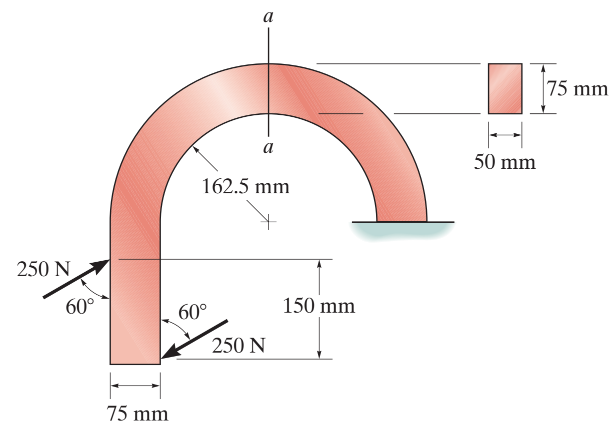
$$\sigma_A = \frac{600(12)(8.7993 - 8)}{2(8)(0.03398)} = 10.6 \text{ ksi (T)}$$ **Ans.**

$$\sigma_B = \frac{600(12)(8.7993 - 10)}{2(10)(0.03398)} = -12.7 \text{ ksi} = 12.7 \text{ ksi (C)}$$ **Ans.**





***6–152.** The curved bar used on a machine has a rectangular cross section. If the bar is subjected to a couple as shown, determine the maximum tensile and compressive stress acting at section a–a. Sketch the stress distribution on the section in three dimensions.



$$\circlearrowleft + \Sigma M_O = 0; \qquad M - 250 \cos 60° (0.075) - 250 \sin 60° (0.15) = 0$$

$$M = 41.851 \text{ N} \cdot \text{m}$$

$$\int_A \frac{dA}{r} = b \ln \frac{r_2}{r_1} = 0.05 \ln \frac{0.2375}{0.1625} = 0.018974481 \text{ m}$$

$$A = (0.075)(0.05) = 3.75(10^{-3}) \text{ m}^2$$

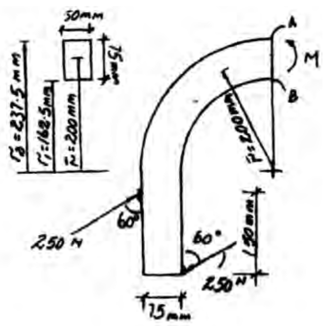
$$R = \frac{A}{\int_A \frac{dA}{r}} = \frac{3.75(10^{-3})}{0.018974481} = 0.197633863 \text{ m}$$

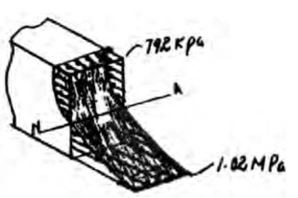
$$\bar{r} - R = 0.2 - 0.197633863 = 0.002366137$$

$$\sigma_A = \frac{M(R - r_A)}{Ar_A(\bar{r} - R)} = \frac{41.851(0.197633863 - 0.2375)}{3.75(10^{-3})(0.2375)(0.002366137)} = -791.72 \text{ kPa}$$

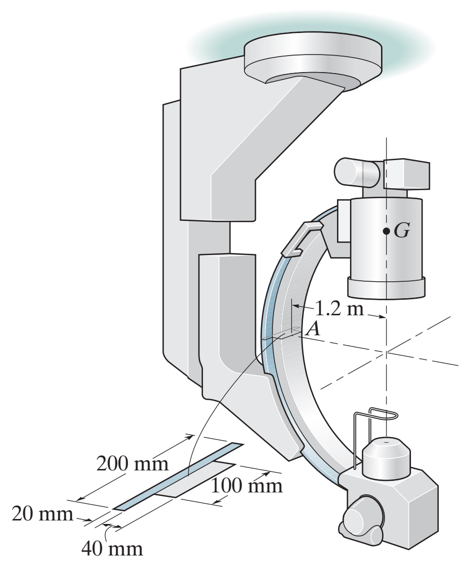
$$= 792 \text{ kPa (C)}$$ **Ans.**

$$\sigma_B = \frac{M(R - r_B)}{Ar_B(\bar{r} - R)} = \frac{41.851\,(0.197633863 - 0.1625)}{3.75(10^{-3})(0.1625)(0.002366137)} = 1.02 \text{ MPa (T)}$$ **Ans.**

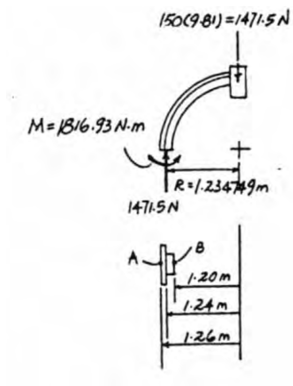




•6–153. The ceiling-suspended C-arm is used to support the X-ray camera used in medical diagnoses. If the camera has a mass of 150 kg, with center of mass at G, determine the maximum bending stress at section A.



Section Properties:



$$\bar{r} = \frac{\Sigma \bar{r} A}{\Sigma A} = \frac{1.22(0.1)(0.04) + 1.25(0.2)(0.02)}{0.1(0.04) + 0.2(0.02)} = 1.235 \text{ m}$$

$$\Sigma \int_A \frac{dA}{r} = 0.1 \ln \frac{1.24}{1.20} + 0.2 \ln \frac{1.26}{1.24} = 6.479051\left(10^{-3}\right)\text{m}$$

$$A = 0.1(0.04) + 0.2(0.02) = 0.008 \text{ m}^2$$

$$R = \frac{A}{\int_A \frac{dA}{r}} = \frac{0.008}{6.479051\,(10^{-3})} = 1.234749 \text{ m}$$

$$\bar{r} - R = 1.235 - 1.234749 = 0.251183\left(10^{-3}\right) \text{ m}$$

Internal Moment: The internal moment must be computed about the neutral axis as shown on FBD. $M = -1816.93 \text{ N}\cdot\text{m}$ is negative since it tends to decrease the beam's radius of curvature.

Maximum Normal Stress: Applying the curved-beam formula

$$\sigma_A = \frac{M(R - r_A)}{A r_A (\bar{r} - R)}$$

$$= \frac{-1816.93(1.234749 - 1.26)}{0.008(1.26)(0.251183)(10^{-3})}$$

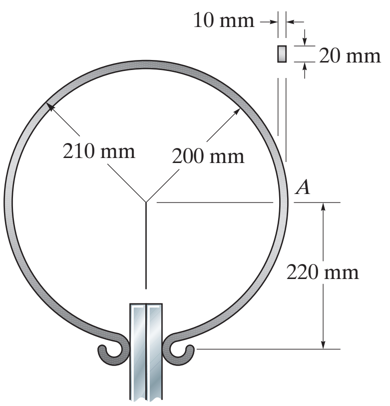
$$= 18.1 \text{ MPa (T)}$$

$$\sigma_B = \frac{M(R - r_B)}{A r_B (\bar{r} - R)}$$

$$= \frac{-1816.93(1.234749 - 1.20)}{0.008(1.20)(0.251183)(10^{-3})}$$

$$= -26.2 \text{ MPa} = 26.2 \text{ MPa (C)} \qquad \textbf{(Max)} \qquad \textbf{Ans.}$$

6–154. The circular spring clamp produces a compressive force of 3 N on the plates. Determine the maximum bending stress produced in the spring at A. The spring has a rectangular cross section as shown.



Internal Moment: As shown on FBD, $M = 0.660\ \text{N}\cdot\text{m}$ is positive since it tends to increase the beam's radius of curvature.

Section Properties:

$$\bar{r} = \frac{0.200 + 0.210}{2} = 0.205\ \text{m}$$

$$\int_A \frac{dA}{r} = b \ln \frac{r_2}{r_1} = 0.02 \ln \frac{0.21}{0.20} = 0.97580328\left(10^{-3}\right)\ \text{m}$$

$$A = (0.01)(0.02) = 0.200\left(10^{-3}\right)\ \text{m}^2$$

$$R = \frac{A}{\int_A \frac{dA}{r}} = \frac{0.200(10^{-3})}{0.97580328(10^{-3})} = 0.204959343\ \text{m}$$

$$\bar{r} - R = 0.205 - 0.204959343 = 0.040657\left(10^{-3}\right)\ \text{m}$$

Maximum Normal Stress: Applying the curved-beam formula

$$\sigma_C = \frac{M(R - r_2)}{Ar_2(\bar{r} - R)}$$

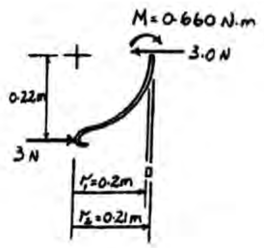
$$= \frac{0.660(0.204959343 - 0.21)}{0.200(10^{-3})(0.21)(0.040657)(10^{-3})}$$

$$= -1.95\text{MPa} = 1.95\ \text{MPa (C)}$$

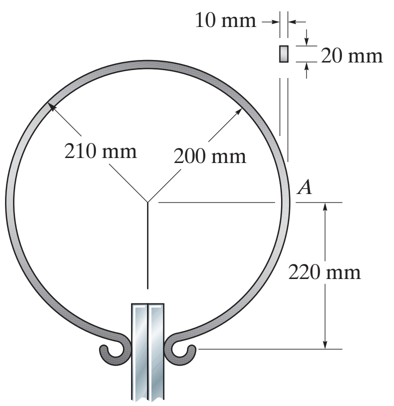
$$\sigma_t = \frac{M(R - r_1)}{Ar_1(\bar{r} - R)}$$

$$= \frac{0.660(0.204959343 - 0.2)}{0.200(10^{-3})(0.2)(0.040657)(10^{-3})}$$

$$= 2.01\ \text{MPa}\ (\textbf{T}) \qquad \text{(Max)}$$ **Ans.**



6–155. Determine the maximum compressive force the spring clamp can exert on the plates if the allowable bending stress for the clamp is $\sigma_{allow} = 4$ MPa.



Section Properties:

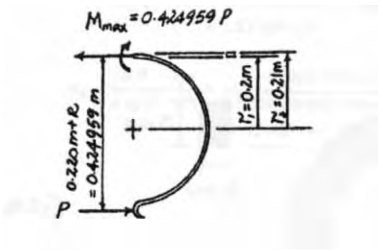
$$\bar{r} = \frac{0.200 + 0.210}{2} = 0.205 \text{ m}$$

$$\int_A \frac{dA}{r} = b \ln \frac{r_2}{r_1} = 0.02 \ln \frac{0.21}{0.20} = 0.97580328\left(10^{-3}\right) \text{ m}$$

$$A = (0.01)(0.02) = 0.200\left(10^{-3}\right) \text{ m}^2$$

$$R = \frac{A}{\int_A \frac{dA}{r}} = \frac{0.200(10^{-3})}{0.97580328(10^{-3})} = 0.204959 \text{ m}$$

$$\bar{r} - R = 0.205 - 0.204959343 = 0.040657\left(10^{-3}\right) \text{ m}$$



Internal Moment: The internal moment must be computed about the neutral axis as shown on FBD. $M_{max} = 0.424959P$ is positive since it tends to increase the beam's radius of curvature.

Allowable Normal Stress: Applying the curved-beam formula

Assume compression failure

$$\sigma_c = \sigma_{allow} = \frac{M(R - r_2)}{Ar_2(\bar{r} - R)}$$

$$-4\left(10^6\right) = \frac{0.424959P(0.204959 - 0.21)}{0.200(10^{-3})(0.21)(0.040657)(10^{-3})}$$

$$P = 3.189 \text{ N}$$

Assume tension failure

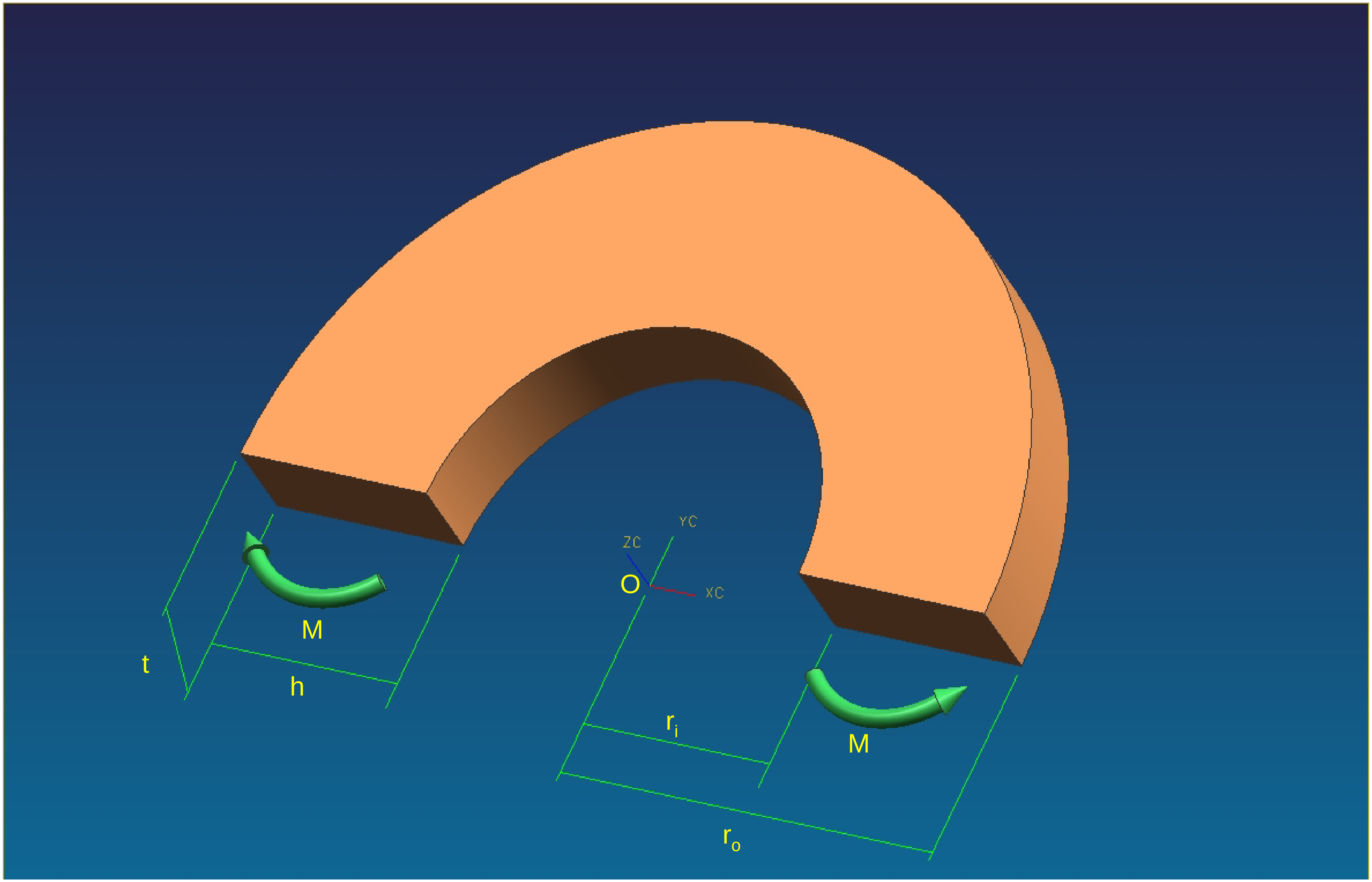
$$\sigma_t = \sigma_{allow} = \frac{M(R - r_1)}{Ar_1(\bar{r} - R)}$$

$$4\left(10^6\right) = \frac{0.424959P(0.204959 - 0.2)}{0.200(10^{-3})(0.2)(0.040657)(10^{-3})}$$

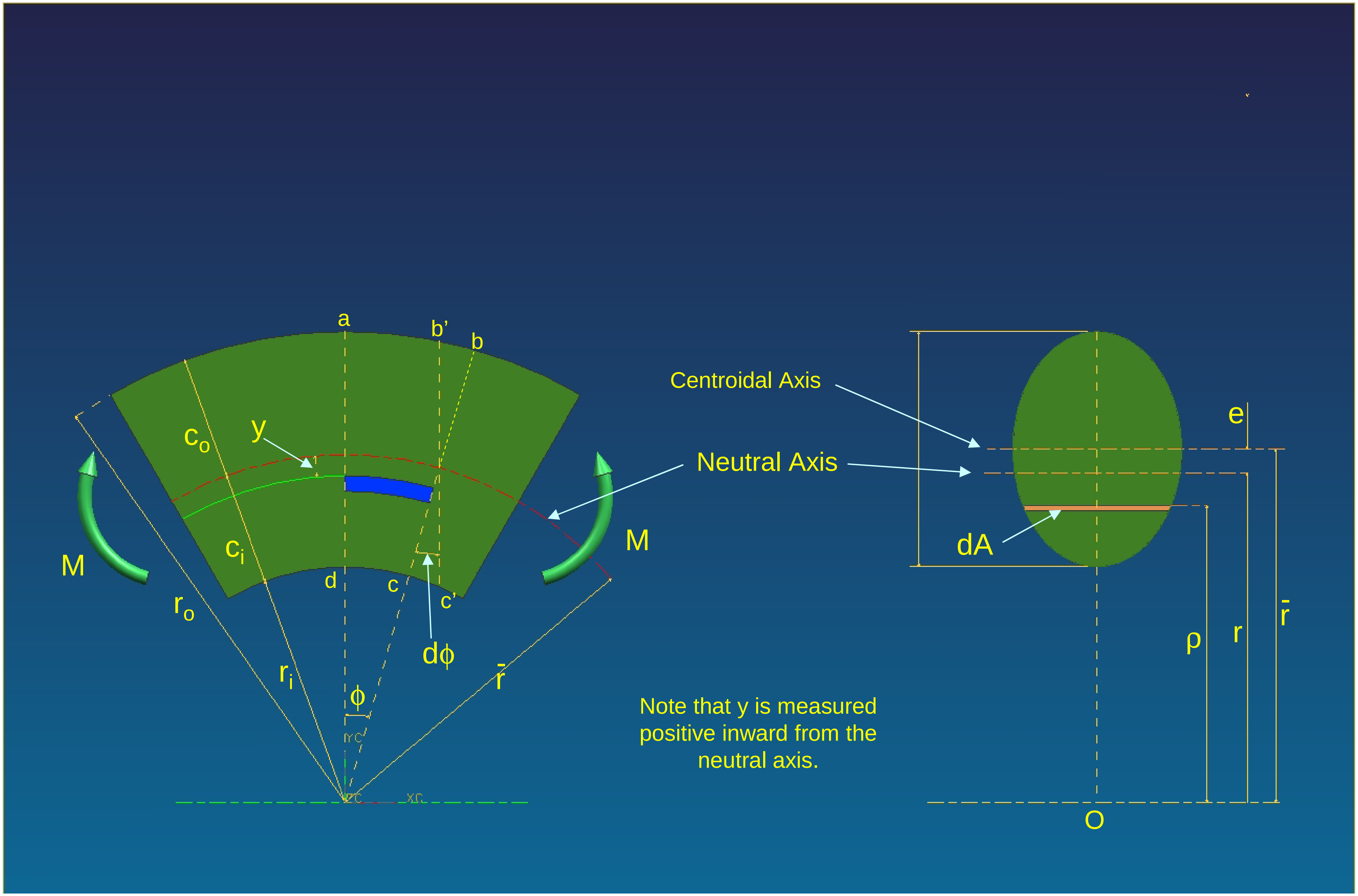
$$P = 3.09 \text{ N } (\textit{Controls !})$$ **Ans.**

Curved Beams

Derivation of stress equations



YC
ZC
O
XC
M
t
h
r_i
M
r_o



a
b'
b
c_o
y
c_i
M
d
c
c'
r_o
r_i
$d\phi$
ϕ
$\bar{r}$
M
Centroidal Axis
Neutral Axis
e
dA
ρ
r
$\bar{r}$
O
Note that y is measured positive inward from the neutral axis.

CURVED MEMBERS IN FLEXURE

The distribution of stress in a curved flexural member is determined by using the following assumptions.

1 The cross section has an axis of symmetry in a plane along the length of the beam.
2 Plane cross sections remain plane after bending.
3 The modulus of elasticity is the same in tension as in compression.

It will be found that the neutral axis and the centroidal axis of a curved beam, unlike a straight beam, are not coincident and also that the stress does not vary linearly from the neutral axis. The notation shown in the above figures is defined as follows:

r_o =	radius of outer fiber
r_i =	radius of inner fiber
h =	depth of section
c_o =	distance from neutral axis to outer fiber
c_i =	distance from neutral axis to inner fiber
r =	radius of neutral axis
$\bar{r}$ =	radius of centroidal axis
e =	distance from centroidal axis to neutral axis

To begin, we define the element *abcd* by the angle ϕ. A bending moment *M* causes section *bc* to rotate through $d\phi$ to *b'c'.* The strain on any fiber at distance ρ from the center *0* is

$$\varepsilon = \frac{\delta l}{l} = \frac{(r-\rho)\,d\phi}{\rho\phi}$$

The normal stress corresponding to this strain is

$$\sigma = \varepsilon E = \frac{E\left(r-\rho\right)d\phi}{\rho\phi} \tag{1}$$

Since there are no axial external forces acting on the beam, the sum of the normal forces acting on the section must be zero. Therefore

$$\int \sigma dA = E\frac{d\phi}{\phi}\int \frac{\left(r-\rho\right)dA}{\rho} = 0 \tag{2}$$

Now arrange Eq. (2) in the form

$$E\frac{d\phi}{\phi}\left(r\int \frac{dA}{\rho} - \int dA\right) = 0 \tag{3}$$

and solve the expression in parentheses. This gives

$$r\int \frac{dA}{\rho} - A = 0 \qquad \text{or} \qquad r = \frac{A}{\int \frac{dA}{\rho}} \tag{4}$$

This important equation is used to find the location of the neutral axis with respect to the center of curvature *0* of the cross section. The equation indicates that the neutral and the centroidal axes are not coincident.

Our next problem is to determine the stress distribution. We do this by balancing the external applied moment against the internal resisting moment. Thus, from Eq. (2),

$$\int (r-\rho)(\sigma dA) = E\frac{d\phi}{\phi}\int \frac{(r-\rho)^2 dA}{\rho} = M \qquad (5)$$

Since $(r-\rho)^2 = r^2 - 2\rho r + \rho^2$, Eq. (5) can be written in the form

$$M = E\frac{d\phi}{\phi}\left(r^2 \int \frac{dA}{\rho} - r\int dA - r\int dA + \int \rho dA \right) \qquad (6)$$

Note that *r* is a constant; then compare the first two terms in parentheses with Eq. (4). These terms vanish, and we have left

$$M = E\frac{d\phi}{\phi}\left(-r\int dA + \int \rho dA\right)$$

The first integral in this expression is the area *A,* and the second is the product *rA.* Therefore

$$M = E\frac{d\phi}{\phi}(\bar{r} - r)A = E\frac{d\phi}{\phi}eA$$

Now, using Eq. (1) once more, and rearranging, we finally obtain $\sigma = \dfrac{My}{Ae(r-y)}$

This equation shows that the stress distribution is hyperbolic. The algebraic *maximum* stresses occur at the inner and outer fibers and are

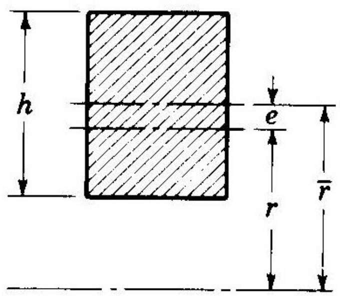
$$\sigma_i = \frac{Mc_i}{Aer_i} \qquad \sigma_o = \frac{Mc_o}{Aer_o} \qquad (7)$$

The sign convention used is that M is positive if it acts to straighten on the beam. The distance y is positive inwards to the center of curvature and is measured from the neutral axis. It follows that c_i is positive and c_o is negative.

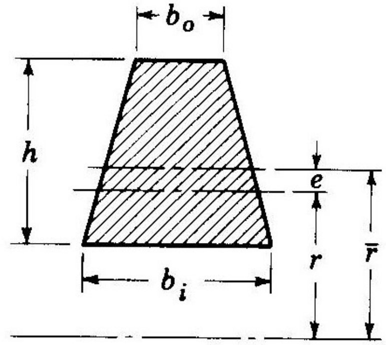
These equations are valid for pure bending. In the usual and more general case such as a crane hook, the U frame of a press, or the frame of a clamp, the bending moment is due to forces acting to one side of the cross section under consideration. In this case the bending moment is computed about the *centroidal axis, not* the neutral axis. Also, an additional axial tensile (*P/A*) or compressive (-*P/A*) stress must be added to the bending stress given by Eq. (7) to obtain the resultant stress acting on the section.

Formulas for Some Common Sections

Sections most frequently encountered in the stress analysis of curved beams are shown below.



(a)



(b)

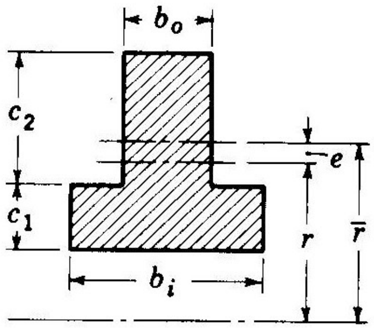
For the rectangular section shown in (a), the formulae are

$$\bar{r} = r_i + \frac{h}{2} \qquad and \qquad r = \frac{h}{\ln(r_o / r_i)}$$

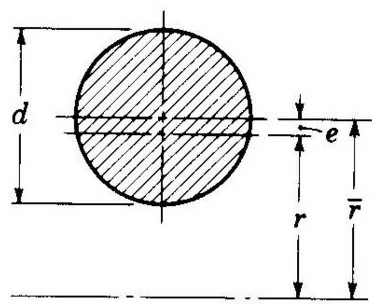
For the trapezoidal section in (b), the formulae are

$$\bar{r} = r_i + \frac{h}{3}\frac{b_i + 2b_o}{b_i + b_o}$$

$$r = \frac{A}{b_o - b_i + [(b_i r_o - b_o r_i)/h]\ln(r_o / r_i)}$$



(c)



(d)

For the T section in we have

$$\bar{r} = r_i + \frac{b_i c_1^2 + 2b_o c_1 c_2 + b_o c_2^2}{2(b_o c_2 + b_i c_1)}$$

$$r = \frac{b_i c_1 + b_o c_2}{b_i \ln[(r_i + c_1)/r_i] + b_o \ln[r_o/(r_i + c_1)]}$$

The equations for the solid round section of Fig. *(d)* are

$$\bar{r} = r_i + \frac{d}{2}$$

$$r = \frac{d^2}{4\left(2\bar{r} - \sqrt{4\bar{r}^2 - d^2}\right)}$$