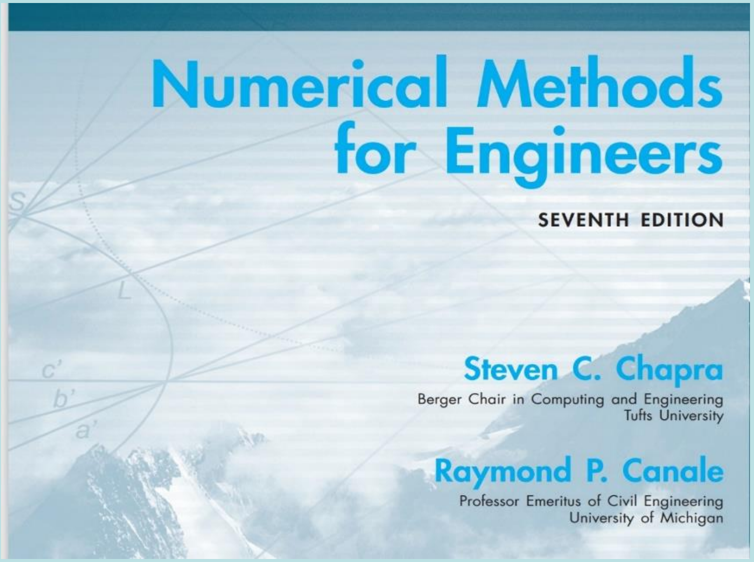

Lecture 8

Linear Algebraic Equations: Gauss Elimination

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Method of Gauss Elimination

- The basic strategy is to successively solve one of the equations of the set for one of the unknowns and to eliminate that variable from the remaining equations by substitution.

- The elimination of unknowns can be extended to systems with more than two or three equations; however, the method becomes extremely tedious to solve by hand.

Stage 1: (Forward Elimination Phase)

1. Search the first column of the augmented matrix [A | b] from the top to the bottom for the first non-zero entry, and then if necessary, the second column (the case where all the coefficients corresponding to the first variable are zero), and then the third column, and so on. The entry thus found is called the current pivot.

2. Interchange, if necessary, the row containing the current pivot with the first row.

3. Keeping the row containing the pivot (that is, the first row) untouched, subtract appropriate multiples of the first row from all the other rows to obtain all zeroes below the current pivot in its column.

4. Repeat the preceding steps on the sub-matrix consisting of all those elements which are below and to the right of the current pivot.

5. Stop when no further pivot can be found.

Stage 2:

(Backward substitution to find the solution)

Elimination process that does not **alter** the solution set:

1. Interchanging any two rows
2. Multiplying any row by a non-zero scalar
3. Adding or subtracting any scalar multiple of a row to another

This process is repeated until getting what is called "**echelon form**", that is until successively solve one of the equations of the set for one of the unknowns and to eliminate that variable from the remaining equations by substitution.

echelon form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & c_1 \\ a_{21} & a_{22} & a_{23} & \vdots & c_2 \\ a_{31} & a_{32} & a_{33} & \vdots & c_3 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \vdots & c_1 \\ & a'_{22} & a'_{23} & \vdots & c'_2 \\ & & a''_{33} & \vdots & c''_3 \end{bmatrix}$$

Forward elimination

$$\Downarrow$$

$$x_3 = c''_3 / a''_{33}$$

$$x_2 = (c'_2 - a'_{23} x_3) / a'_{22}$$

$$x_1 = (c_1 - a_{12} x_2 - a_{13} x_3) / a_{11}$$

Back substitution

Example: Solve By Gauss Elimination

$$-\frac{12}{6}\qquad \begin{pmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 12 \\ 34 \\ 27 \\ -38 \end{pmatrix}$$

$$\begin{array}{rrrrrr} & 12 & -8 & 6 & 10 & 34 \\ +)\ -\frac{12}{6}\times & 6 & -2 & 2 & 4 & 12 \\ \hline & 0 & -4 & 2 & 2 & 10 \end{array}$$

$$-\frac{3}{6}\qquad \begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ 27 \\ -38 \end{pmatrix}$$

$$\begin{array}{rrrrrr} & 3 & -13 & 9 & 3 & 27 \\ +)\ -\frac{3}{6}\times & 6 & -2 & 2 & 4 & 12 \\ \hline & 0 & -12 & 8 & 1 & 21 \end{array}$$

$$-\frac{-6}{6}\qquad \begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ -6 & 4 & 1 & -18 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ 21 \\ -38 \end{pmatrix}$$

$$
\begin{array}{rrrr|r}
-6 & 4 & 1 & -18 & -38 \\
+)\; -\dfrac{-6}{6}\times \quad 6 & -2 & 2 & 4 & 12 \\
\hline
0 & 2 & 3 & -14 & -26
\end{array}
$$

$$
-\frac{-12}{-4}\;(\text{row 2} \rightarrow \text{row 3})\qquad
\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{pmatrix}
\begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix}
=
\begin{pmatrix} 12 \\ 10 \\ 21 \\ -26 \end{pmatrix}
$$

$$
-\frac{-12}{-4}\;(\text{row 2} \rightarrow \text{row 3})\qquad
\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{pmatrix}
\begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix}
=
\begin{pmatrix} 12 \\ 10 \\ 21 \\ -26 \end{pmatrix}
\longleftarrow
$$

First row does not change thereafter

$$
\begin{array}{rrr|r}
-12 & 8 & 1 & 21 \\
+)\; -\dfrac{-12}{-4}\times \quad -4 & 2 & 2 & 10 \\
\hline
0 & 2 & -5 & -9
\end{array}
$$

$$
-\frac{2}{-4}\;(\text{row 2} \rightarrow \text{row 4})\qquad
\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 2 & 3 & -14 \end{pmatrix}
\begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix}
=
\begin{pmatrix} 12 \\ 10 \\ -9 \\ -26 \end{pmatrix}
$$

	2	3	-14	-26
$+)\ -\frac{2}{-4}\times$	-4	2	2	10
	0	4	-13	-21

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ -9 \\ -21 \end{pmatrix}$$

$-\frac{4}{2}$ (row 3 → row 4)

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ -9 \\ -21 \end{pmatrix}$$

	4	-13	-21
$+)\ -\frac{4}{2}\times$	2	-5	-9
	0	-3	-3

echelon form

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ -9 \\ -3 \end{pmatrix}$$

Backward substitution: inner-product-based

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ -9 \\ -3 \end{pmatrix}$$

$$-3x_4 = -3 \quad \Rightarrow \quad x_4 = 1$$

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ -9 \end{pmatrix}$$

$$2x_3 - 5x_4 = -9 \quad \Rightarrow \quad x_3 = -2$$

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \end{pmatrix}$$

$$-4x_2 + 2x_3 + 2x_4 = 10 \quad \Rightarrow \quad x_2 = -3$$

$$\begin{pmatrix} 6 & -2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 12 \end{pmatrix}$$

$$6x_1 - 2x_2 + 2x_3 + 4x_4 = 12 \quad \Rightarrow \quad x_1 = 1$$

Backward substitution: outer-product-based

$$\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \\ x4 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ -9 \\ -3 \end{pmatrix} \qquad -3x_4 = -3 \Rightarrow x_4 = 1$$

$$\begin{pmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix} = \begin{pmatrix} 12 \\ 10 \\ -9 \end{pmatrix} - x4 \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ -4 \end{pmatrix} \qquad 2x_3 = -4 \Rightarrow x_3 = -2$$

$$\begin{pmatrix} 6 & -2 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix} - x3 \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \end{pmatrix} \qquad -4x_2 = 12 \Rightarrow x_2 = -3$$

$$\begin{pmatrix} 6 \end{pmatrix} \begin{pmatrix} x1 \end{pmatrix} = \begin{pmatrix} 12 \end{pmatrix} - x2 \begin{pmatrix} -2 \end{pmatrix} = \begin{pmatrix} 6 \end{pmatrix} \qquad 6x_1 = 6 \Rightarrow x_1 = 1$$

Naive Gauss Elimination

- Extension of *method of elimination* to large sets of equations by developing a systematic scheme or algorithm to eliminate unknowns and to back substitute.

- As in the case of the solution of two equations, the technique for *n* equations consists of two phases:

  – Forward elimination of unknowns
  – Back substitution

Pseudo-code of Naive Gauss Elimination Method

(a) Forward Elimination

(b) Back substitution

(a)

```
DO k = 1, n − 1
  DO i = k + 1, n
    factor = a_{i,k} / a_{k,k}
    DO j = k + 1 to n
      a_{i,j} = a_{i,j} − factor · a_{k,j}
    END DO
    b_i = b_i − factor · b_k
  END DO
END DO
```

(b)

```
x_n = b_n / a_{n,n}
DO i = n − 1, 1, −1
  sum = 0
  DO j = i + 1, n
    sum = sum + a_{i,j} · x_j
  END DO
  x_i = (b_i − sum) / a_{i,i}
END DO
```

Pitfalls of Gauss Elimination Methods

1. Division by zero

$$
\begin{aligned}
2x_2 + 3x_3 &= 8 \\
4x_1 + 6x_2 + 7x_3 &= -3 \\
2x_1 + x_2 + 6x_3 &= 5
\end{aligned}
$$

$a_{11} = 0$
(the pivot element)

It is possible that during both elimination and back-substitution phases a division by zero can occur.

2. Round-off errors

In the some examples you may kept 6 digits during the calculations. However we may end up with closer results to the real solution. For example

$x_3 = 7.00003$, instead of $x_3 = 7.0$

3. Ill-conditioned systems

$$x_1 + 2x_2 = 10$$
$$1.1x_1 + 2x_2 = 10.4$$

→ $x_1 = 4.0$ & $x_2 = 3.0$

$$x_1 + 2x_2 = 10$$
$$1.05x_1 + 2x_2 = 10.4$$

→ $x_1 = 8.0$ & $x_2 = 1.0$

Ill conditioned systems are those where small changes in coefficients result in large change in solution. Alternatively, it happens when two or more equations are nearly identical, resulting a wide ranges of answers to approximately satisfy the equations. Since round off errors can induce small changes in the coefficients, these changes can lead to large solution errors.

4. Singular systems.

- When two equations are identical, we would loose one degree of freedom and be dealing with case of *n-1* equations for *n* unknowns.

To check for singularity:

- After getting the forward elimination process and getting the triangle system, then the determinant for such a system is the product of all the diagonal elements. If a zero diagonal element is created, the **determinant is Zero** then we have a singular system.
- The determinant of a singular system is zero.

Techniques for Improving Solutions

1. **Use of more significant figures** to solve for the round-off error.
2. **Pivoting**. If a pivot element is zero, elimination step leads to division by zero. The same problem may arise, when the pivot element is close to zero. This Problem can be avoided by:
   - Partial pivoting. Switching the rows so that the largest element is the pivot element.
   - Complete pivoting. Searching for the largest element in all rows and columns then switching.
3. **Scaling**
   - Solve problem of ill-conditioned system.
   - Minimize round-off error

Use of more significant figures to solve for the round-off error :Example.

Use Gauss Elimination to solve these 2 equations: (keeping only 4 sig. figures)

$$0.0003\, x_1 + 3.0000\, x_2 = 2.0001$$
$$1.0000\, x_1 + 1.0000\, x_2 = 1.000$$

$$0.0003\, x_1 + 3.0000\, x_2 = 2.0001$$
$$-\ 9999.0\, x_2 = -6666.0$$

Solve: $x_2 = 0.6667$ & $x_1 = 0.0$

The exact solution is

$x_2 = 2/3$ & $x_1 = 1/3$ ⇒ Large round-off error

Use of more significant figures to solve for the round-off error :Example (cont'd).

$$x_2 = \frac{2}{3}$$

$$x_1 = \frac{2.0001 - 3(2/3)}{0.0003}$$

Checking the effect of the # of significant digits

Significant Figures	x_2	x_1
3	0.667	-3.33
4	0.6667	0.000
5	0.66667	0.3000
6	0.666667	0.33000
7	0.6666667	0.333000

Pivoting: Example

Now, solving the pervious example using the partial pivoting technique:

$$1.0000\, x_1 + 1.0000\, x_2 = 1.000$$
$$0.0003\, x_1 + 3.0000\, x_2 = 2.0001$$

The pivot is 1.0

$$1.0000\, x_1 + 1.0000\, x_2 = 1.000$$
$$2.9997\, x_2 = 1.9998$$

$x_2 = 0.6667$ & $x_1 = 0.3333$

Checking the effect of the # of significant digits:

# of dig	x_2	x_1	Ea% in x1
4	0.6667	0.3333	0.01
5	0.66667	0.33333	0.001

Scaling: Example

- Solve the following equations using naïve gauss elimination:
  (keeping only 3 sig. figures)

$$2\,x_1 + 100{,}000\,x_2 = 100{,}000$$

$$x_1 + x_2 = 2.0$$

- Forward elimination:

$$2\,x_1 + 100.000\,x_2 = 100.000$$

$$-\,50{,}0000\,x_2 = -50{,}0000$$

Solve $x_2 = 1.00$ & $x_1 = 0.00$

- **The exact solution is $x_1 = 1.00002$ & $x_2 = 0.99998$**

Scaling: Example (cont'd)

B) Using the scaling algorithm to solve:

$$2x_1 + 100{,}000\,x_2 = 100{,}000$$
$$x_1 + x_2 = 2.0$$

Scaling the first equation by dividing by 100,000:

$$0.00002\,x_1 + x_2 = 1.0$$
$$x_1 + x_2 = 2.0$$

Rows are pivoted:

$$x_1 + x_2 = 2.0$$
$$0.00002\,x_1 + x_2 = 1.0$$

Forward elimination yield:

$$x_1 + x_2 = 2.0$$
$$x_2 = 1.00$$

Solve: $x_2 = 1.00$ & $x_1 = 1.00$

The exact solution is $x_1 = 1.00002$ & $x_2 = 0.99998$

Scaling: Example (cont'd)

C) The scaled coefficient indicate that pivoting is necessary.

We therefore pivot but retain the original coefficient to give:

$$x_1 + x_2 = 2.0$$

$$2x_1 + 100{,}000\, x_2 = 100{,}000$$

Forward elimination yields:

$$x_1 + x_2 = 2.0$$

$$100{,}000\, x_2 = 100{,}000$$

Solve: $x_2 = 1.00$ & $x_1 = 1.00$

Thus, scaling was useful in determining whether pivoting was necessary, but the equation themselves did not require scaling to arrive at a correct result.

Example: Gauss Elimination

$$\begin{aligned}
&& +2x_2 && && +x_4 &= 0 \\
2x_1 && +2x_2 && +3x_3 && +2x_4 &= -2 \\
4x_1 && -3x_2 && && +x_4 &= -7 \\
6x_1 && +x_2 && -6x_3 && -5x_4 &= 6
\end{aligned}$$

a) Forward Elimination

$$\left[\begin{array}{cccc|c} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{array}\right] \quad R1 \rightleftarrows R4 \quad \left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

Example: Gauss Elimination (cont'd)

$$\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R2 - 0.33333 \cdot R1 \\ R3 - 0.66667 \cdot R1 \end{array}$$

$$\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right] \quad R2 \rightleftarrows R3$$

$$\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

Example: Gauss Elimination (cont'd)

$$\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ R3 + 0.45455 \cdot R2 \\ R4 + 0.54545 \cdot R2 \end{array}$$

$$\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8182 & 5.6364 & -9.0001 \\ 0 & 0 & 2.1818 & 3.3636 & -5.9999 \end{array}\right] \begin{array}{l} \\ \\ \\ R4 - 0.32000 \cdot R3 \end{array}$$

$$\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8182 & 5.6364 & -9.0001 \\ 0 & 0 & 0 & 1.5600 & -3.1199 \end{array}\right]$$

Example: Gauss Elimination (cont'd)

$$\left[\begin{array}{cccc|c} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8182 & 5.6364 & -9.0001 \\ 0 & 0 & 0 & 1.5600 & -3.1199 \end{array}\right]$$

→ **echelon form**

b) Back Substitution

$$x_4 = \frac{-3.1199}{1.5600} = -1.9999$$

$$x_3 = \frac{-9.0001 - 5.6364(-1.9999)}{6.8182} = 0.33325$$

$$x_2 = \frac{-11 - 4.3333(-1.9999) - 4(0.33325)}{-3.6667} = 1.0000$$

$$x_1 = \frac{6 + 5(-1.9999) + 6(0.33325) - 1(1.0000)}{6} = -0.50000$$

Gauss-Jordan Elimination

- It is a variation of Gauss elimination. The major differences are:
  - When an unknown is eliminated, it is eliminated from all other equations rather than just the subsequent ones.
  - All rows are normalized by dividing them by their pivot elements.
  - Elimination step results in an identity matrix.
  - It is not necessary to employ back substitution to obtain solution.

Gauss-Jordan Elimination- Example

$$\left[\begin{array}{cccc|c} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{array}\right] \begin{array}{c} R1 \rightleftarrows R4 \\ R4/6.0 \end{array} \left[\begin{array}{cccc|c} 1 & 0.16667 & -1 & -0.83335 & 1 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0.16667 & -1 & -0.83335 & 1 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ R2 - 2 \cdot R1 \\ R3 - 4 \cdot R1 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0.16667 & -1 & -0.83335 & 1 \\ 0 & 1.6667 & 5 & 3.6667 & -2 \\ 0 & -3.6667 & 4 & 4.3334 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right]$$

Dividing the 2nd row by 1.6667 and reducing the second column. (operating above the diagonal as well as below) gives:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1.5 & -1.2000 & 1.4000 \\ 0 & 1 & 2.9999 & 2.2000 & -2.4000 \\ 0 & 0 & 15.000 & 12.400 & -19.800 \\ 0 & 0 & -5.9998 & -3.4000 & 4.8000 \end{array}\right]$$

Divide the 3rd row by 15.000 and make the elements in the 3rd Column zero.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0.04000 & -0.58000 \\ 0 & 1 & 0 & -0.27993 & 1.5599 \\ 0 & 0 & 1 & 0.82667 & -1.3200 \\ 0 & 0 & 0 & 1.5599 & -3.1197 \end{array}\right]$$

Divide the 4th row by 1.5599 and create zero above the diagonal in the fourth column.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -0.49999 \\ 0 & 1 & 0 & 0 & 1.0001 \\ 0 & 0 & 1 & 0 & -0.33326 \\ 0 & 0 & 0 & 1 & -1.9999 \end{array}\right]$$

Note: Gauss-Jordan method requires almost 50 % more operations than Gauss elimination; therefore it is not recommended to use it.