Which of the following could be considered a model? Why? (select all that apply).
a. The ideal-gas law.
b. A set of instructions for using a Pitot-static tube to measure velocity.
c. An airplane built from a kit.
d. A computer program to predict the force on a pipe bend.

SOLUTION

A model is an idealization of something real (as a map is an idealization of a city)
a. The ideal-gas law: **Yes**, the equation is an idealization of the behavior of a gas.
b. A set of instructions for using a Pitot-static tube to measure velocity: **No**.
c. An airplane built from a kit: **No**.
d. A computer program to predict the force on a pipe bend: **Yes**, the equations in the computer program are an idealization of the behavior of the flowing fluid.

16.2: PROBLEM DEFINITION

Apply the modeling building process to the Balloon Payload problem.

The Balloon Payload Problem. Your team is designing a helium-filled balloon that will travel to at least 24,000 meters elevation in the atmosphere. The balloon will transport a payload comprising a camera and a data acquisition system. Right now, you choose to solve a simpler problem which is to develop a model that predicts the weight on the earth's surface (at your location) such that a helium-filled balloon is neutrally buoyant. This simpler problem can be easily tested with an experiment in your classroom.

- What are the relevant variables?

- How are the variables related? What are the relevant equations? How will you apply these equations to develop a single algebraic equation to solve for your goal?

- What is a simple and low cost way to test your math model using experimental data?

SOLUTION

- Relevant variables. Size of balloon (diameter), shape of balloon, volume of balloon, temperature, local atmospheric pressure, weight of balloon skin, weight of any structural materials.

- Equations. $\Sigma F = 0, W = mg, F_{\text{buoyant}} = \gamma_{\text{air}} \forall_D, m = \rho V$.

- To develop a single equation, do algebra. Straightforward.

- Testing the math model. Buy a container of helium and some balloons (e.g., at Wal-Mart). Then, inflate balloon with helium to a known volume, and measure how much weight can be lifted. Compare data (actual weight lifted) with the math model prediction.

16.3: PROBLEM DEFINITION

Apply the modeling building process to the Rocket Problem.

The Rocket Problem. Your team is designing a 2-stage solid-fuel rocket that intended to travel to 4500 meters and take photos. Right now, you choose to solve a simpler problem which is to develop a model that predicts the height that a small, low-cost rocket will fly because a small rocket can be purchased from Estes or Pitsco and it is relatively easy to measure elevation.

- What are the relevant variables?

- How are the variables related? What are the relevant equations?

- What is a simple and low cost way to test your math model using experimental data?

SOLUTION

- Relevant variables. Dimensions of rocket. Mass of rocket. Number of fins. Type of engine. Impulse of engine. How well the rocket is built. Cross wind. Local atmospheric pressure. Local temperature.

- Equations. Momentum equation applied to rocket. $W = mg$, Drag force equation.

- Testing the math model. Build a math model of a Estes/Pitsco rocket and predict the height it will fly. Then, build an actual rocket (or several) and fly them and measure the altitude reached. To measure altitude, apply ideas from trigonometry (you can Google this to find out the details).

16.4: PROBLEM DEFINITION

Why do you think that engineers make the effort to learn partial differential equations? What are the benefits to them?

SOLUTION

Some answers are listed below (others are possible).

- Gives engineers the ability to read the literature and improve one's ability to do fluid mechanics problems (life long skills).

- Gives engineers the ability to understand the foundation of computational fluid dynamics codes.

- Gives engineers the ability to understand the limitation of the PDEs.

- Many engineers enjoy learning new knowledge.

Consider the function.

$$f(x) = \frac{1}{1-x}$$

Find the Taylor series expansion for the function $f(x)$ about the point $x = 0$. Evaluate the numerical value of the Taylor series for $x = 0.1$ using 5 terms.

SOLUTION

Use Taylors series in this form.

$$f(x + \Delta x) = f(x) + \frac{\Delta x}{1!}\frac{df}{dx} + \frac{\Delta x^2}{2!}\frac{d^2 f}{dx^2} + \frac{\Delta x^3}{3!}\frac{d^3 f}{dx^3} + \frac{\Delta x^4}{4!}\frac{d^4 f}{dx^4} + \ldots$$

Let

$$\Delta x = 0.1$$
$$x = 0.0$$

The Taylor series becomes

$$f(0.1) = f(0) + \frac{0.1}{1!}\left(\frac{df}{dx}\right)_{x=0} + \frac{0.1^2}{2!}\left(\frac{d^2 f}{dx^2}\right)_{x=0} + \frac{0.1^3}{3!}\left(\frac{d^3 f}{dx^3}\right)_{x=0} + \frac{0.1^4}{4!}\left(\frac{d^4 f}{dx^4}\right)_{x=0} + \ldots$$

Evaluate $f(x)$ and its derivative terms

Function	Function in terms of x	Function evaluated at $x = 0$
$f(x)$	$\frac{1}{1-x}$	1
$\frac{df}{dx}$	$\frac{1}{(1-x)^2}$	1
$\frac{d^2 f}{dx^2}$	$\frac{2}{(1-x)^3}$	2
$\frac{d^3 f}{dx^3}$	$\frac{3\cdot 2}{(1-x)^4}$	$3 \cdot 2$
$\frac{d^4 f}{dx^4}$	$\frac{4\cdot 3\cdot 2}{(1-x)^5}$	$4 \cdot 3 \cdot 2$

Combine results

$$\begin{aligned}\left(\frac{1}{1-x}\right)_{x=0.1} &= 1 + \frac{0.1}{1!}1 + \frac{0.1^2}{2!}2 + \frac{0.1^3}{3!}3\cdot 2 + \frac{0.1^4}{4!}4\cdot 3\cdot 2 + \quad \ldots \\ &= 1 + 0.1 + 0.1^2 + 0.1^3 + 0.1^4 + \quad \ldots \\ &= 1.1111 \quad \ldots\end{aligned}$$

Consider the function f(x) = ln(x). Show how to find the Taylor series expansion for the function f(x) about the point x = a. Then, find the numerical value for x = 1.5 and six terms

SOLUTION

Taylor Series (general form)

$$f(x) = f(a) + \frac{df}{dx}_{(x=a)} \frac{(x-a)}{1!} + \frac{d^2 f}{dx^2}_{(x=a)} \frac{(x-a)^2}{2!} + \frac{d^3 f}{dx^3}_{(x=a)} \frac{(x-a)^3}{3!} + \ \ldots \quad (1)$$

Build a table to analyze the first six terms in the Taylor series

Function	Function in terms of x	Function evaluated at $x = a$
$f(x)$	$\ln(x)$	$\ln(a)$
$\frac{df}{dx}$	$\frac{1}{x}$	$\frac{1}{a}$
$\frac{d^2 f}{dx^2}$	$(-1)\frac{1}{x^2}$	$(-1)\frac{1}{a^2}$
$\frac{d^3 f}{dx^3}$	$(1 \cdot 2)\frac{1}{x^3}$	$(1 \cdot 2)\frac{1}{a^3}$
$\frac{d^4 f}{dx^4}$	$(-1 \cdot 2 \cdot 3)\frac{1}{x^4}$	$(-1 \cdot 2 \cdot 3)\frac{1}{a^4}$
$\frac{d^5 f}{dx^5}$	$(1 \cdot 2 \cdot 3 \cdot 4)\frac{1}{x^5}$	$(1 \cdot 2 \cdot 3 \cdot 4)\frac{1}{a^5}$

Substitute the table values into Eq. (1) and simplify

$$\ln(x) = \ln(a) + \frac{1}{a}\frac{(x-a)}{1} - \frac{1}{a^2}\frac{(x-a)^2}{2} + \frac{1}{a^3}\frac{(x-a)^3}{3} - \frac{1}{a^4}\frac{(x-a)^4}{4} + \frac{1}{a^5}\frac{(x-a)^5}{5} - \ \ldots \quad (2)$$

Let $a = 1$ in Eq. (2) because this simplifies the calculation. Then, Eq(2) becomes

$$\ln(x) = \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \ \ldots \quad (3)$$

Program Eq. (3) in a spreadsheet or similar program and let $x = 1.5$. The result is

$$\ln(1.5) \approx 0.4073 \text{ (using the first six terms of Taylor series example)}$$

REVIEW

The exact solution (calculator) is ln(1.5) = 0.4055, so the 6-term approximation overpredicts the true value by 0.45%. Pretty close indeed! In this case $x - a = 0.5$. If this difference were smaller, the result would be even more accurate.

A flat horizontal plate is infinite in size in both dimensions.
The plate is at rest at $t = 0$, then sent into motion to the right.
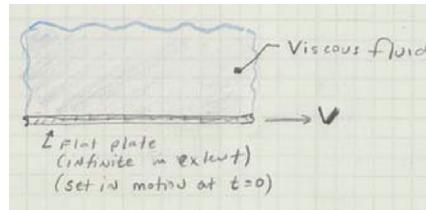
Answer the following questions.

- Which velocity components (u, v, w) are zero? Which are non-zero? Why?

- Which spatial variables (x, y, z) are parameters? Which can be ignored? Why?

- Is time a parameter? Or, can time be ignored? Why?

- What is the reduced equation that represents the velocity field?

SOLUTION

- Velocity components

  - u is nonzero because the moving plate drives flow in this direction
  - $v = w = 0$ because there nothing in the physical situation causes flow to occur in these component directions.

- Spatial variables

  - x is a parameter.
  - Both y and z can be ignored.
  - Rationale: Nothing in the physical situation causes velocity to vary spatially with y or z. However, viscous stresses cause velocity to vary in the x direction.

- Time

  - Time is a parameter because the velocity is initially zero at every point in the field. As time increases, then the velocity will change.

- Reduced equation

  - The velocity field is given by $\mathbf{V} = u(y)\mathbf{i}$

16.8: PROBLEM DEFINITION

Compare and contrast the integral-form of the continuity equation (Chapter 5 in EFM10e) with the PDE-form of the continuity equations (Chapter 16 in EFM10e). Address the following questions.

- Are the units and dimensions the same? Or different?

- How do the physics compare? What is the same? What is different?

- How do the derivations compare? What is the same? What is different?

- When would you want to apply the integral-form of the continuity equation (Chapter 5)? When would you want to apply the PDE-form of the continuity equation (Chapter 16)?

SOLUTION

- Units/Dimensions.

  - Integral form (Chapter 5, EFM10e). Units are kg/s. Dimensions are mass/time

  - Differential form (Chapter 16, EFM10e). Units are kg/s·m³. Dimensions are (mass/time)/volume

- Physics

  - Both express the same physics (accumulation of mass in a CV) = (inflow of mass into the CV) - (outflow of mass from the CV).

- Derivations

  - Similarity: Both start with the basic idea of conservation of mass (same physical law).

  - Difference: The equations start from different forms of the conservation law.

    * The integral equation starts with the mass conservation law for a closed system.

    * The partial differential equation starts with the mass conservation law for a control volume.

- Application (Purpose)

  - The integral form is selected for analyzing a control volume that is finite (i.e. not infintestimal)

  - The differential form is selected the goal is to find the velocity field.

16.9: PROBLEM DEFINITION

Start with the conservation form of the continuity equation in Cartesian coordinates and derive the non-conservation form.

SOLUTION

Conservation form

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

Apply the product rule (from calculus)

$$\frac{\partial \rho}{\partial t} + \left(\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} \right) + \left(\rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y} \right) + \left(\rho \frac{\partial w}{\partial z} + w \frac{\partial \rho}{\partial z} \right) = 0$$

Rearrange terms

$$\left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

Recognize the material derivative.

$$\frac{d\rho}{dt} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

Thus the proof is complete. The above equation is the continuity equation in the non-conservation form.

Start with the non-conservation form of the continuity equation in Cartesian coordinates and derive the conservation form.

SOLUTION

Non-conservation form

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Expand the material derivative

$$\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right) + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Rearrange terms

$$\frac{\partial \rho}{\partial t} + \left(\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x}\right) + \left(\rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y}\right) + \left(\rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z}\right) = 0$$

Apply the product rule (from calculus) to each bracketed term

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

Q.E.D. The above equation is the continuity equation in the conservation form.

Consider water draining out of round hole in the bottom of a tank. Assume constant density. Assume the water does not swirl. Then:

- Select the general form of the continuity equation that best applies to this problem.

- Show how to simplify the general equation from part a to develop the reduced form.
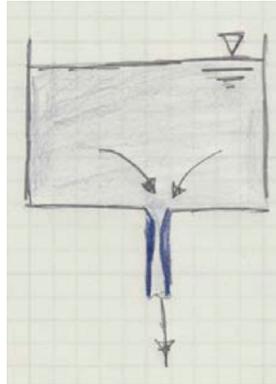
SOLUTION

Idealize water as incompressible.
Select cylindrical coordinates.
Write the general form of the continuity equation (cylindrical coordinates, incompressible flow)

$$\frac{1}{r}\frac{\partial\left(rv_r\right)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Because of the no-swirl assumption, $v_\theta = 0$. Thus, reduce the equation as follows.

$$\frac{1}{r}\frac{\partial\left(rv_r\right)}{\partial r} + \frac{\partial v_z}{\partial z} = 0$$

Answer each question below re: Eq (16.72) in EFM10e.

- Is the given equation in conservation form or non-conservation form? Why?

- Is the given equation in invariant form or coordinate specific form? Why?

SOLUTION

- Non-conservation form because the equation was derived by applying $\Sigma \mathbf{F} = m\mathbf{a}$ to a fluid particle.

- Invariant form because the terms in the equation are independent of any specific coordinate system.

What are the physics of the gradient of the pressure field? What are the units? What are the primary dimensions?

SOLUTION

- Physics (intepretation #1; developed from the textbook). The gradient of the pressure at a point gives the ratio of the (net pressure force acting a fluid particle situated at this point) to the (volume of the fluid particle).

- Physics (interpretation #2; developed from calculus)

  - Direction of grad(p): this vector points in the direction in which pressure increases the most rapidly.

  - Magnitude of grad(p): the magnitude of this vector describe the ratio of (change in pressure) to (incremental distance)

- Units. Newtons per cubic meter or Pascals per meter.

- Dimensions. (force/volume) or (pressure/distance).

What are the physics of the divergence of the shear stress tensor? What are the units? What are the dimensions.

SOLUTION

- Physics. The divergence of the shear stress tensor gives the ratio of (net shear force acting a fluid particle at location (x, y, z)) to the (volume of the fluid particle).

- Units. Newtons per cubic meter.

- Dimensions. (force/volume)

Compare the Navier-Stokes equation to Euler's equation (Chapter 4).

- What are the two important similarities?

- What are two important differences?

SOLUTION

- Two important similarities (there are other possibilities).

    - Both are derived by applying $\Sigma \mathbf{F} = m\mathbf{a}$ to a fluid particle.
    - Three of the terms are the same
        * (density)(acceleration of a fluid particle)
        * gradient of the pressure field
        * the weight term which is (gravity vector)(density)

- Two important differences (there are other possibilities).

    - The Navier-Stokes derivation includes the shear force (i.e. viscous force), but the Euler equation does not include the shear force.
    - The Navier-Stokes equation more general (i.e., applies to more flows) than the Euler equation.

16.16: PROBLEM DEFINITION

Stress, as introduced in the derivation of the Navier-Stokes equation, is a second-order tensor. Using the internet, finds some articles on tensors and answer the following questions:

- Why do people use tensors? What are the benefits?

- What does tensor mean? How is a tensor defined?

- What are five examples of tensors as they are applied in engineering and physics?

SOLUTION

- Benefit of using tensors (multiple possible answers; some given below)

  - Compact notation.
  - Invariant notation (i..e. independent of a specific coordinant system).
  - Useful for describing relationships between concepts.
    * Example. The stress tensor acting on the normal vector at a location on a surface gives the (surface force)/(area) at this point. That is, the stress tensor relates a normal vector and a surface force.

- Meaning of a tensor.

  - A mathematical entity analogous to but more general than a vector, represented by an array of components that are functions of the coordinates of a space.
    * A zeroth order tensor (scalar) has one component.
    * A first order tensor (vector) has three components.
    * A second order tensor (often just called a tensor) has nine components and is often expressed as a 3x3 rectangular matrix.
  - A 2nd order tensor is a mathematical entity that transform one vector into another vector using a linear transformation.

- Examples of tensors in engineering

  - Thermal conductivity (general form) is a 2nd order tensor.
  - Mass moment of intertial (general form) is a 2nd order tensor
  - Stress (general form) is a 2nd order tensor
  - The gradient of the velocity vector in fluid mechanics is a 2nd order tensor.
  - Strain (general form) is a 2nd order tensor.

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16.17: PROBLEM DEFINITION

Is someone asked you why are CFD codes useful for engineers how would you answer?
List your top three reasons in priority order.

SOLUTION

(Multiple possible answers exist; some are listed below)

- CFD provides a method when experimentation or analysis is not possible.

- CFD allows parameteric studies on design. For example, one can vary the design of the impellor of a centrifugal pump and see what happens.

- CFD provides a large amount of detail about a flow. In particular, CFD provides full details about the velocity field, the pressure field and other aspects of the flow.

- CFD provides plots and other visuals that reveal what is going on in the flow.

- CFD provides detailed understanding of product performance,

- CFD can provide a rapid evaluation of design alternatives.

- CFD provides a means for doing design optimization.

- CFD allows for easy what-if testing,

- CFD does not require any large capital facility,

16.18: PROBLEM DEFINITION

Would you prefer to write your own CFD programs or would you prefer to use codes that have been written by others? Discuss the advantages and disadvantages of each approach.

SOLUTION

- Advantages of writing your own code

    - Understand how the code works.
    - Learn how to do CFD by doing CFD.
    - Know exactly how the algorithms work.
    - Customize the code for specific applications.

- Disadvantages of writing your code

    - Resources required to develop a code
        * Requires a large amount of time (years).
        * Requires broad expertise; one has to understand multiple technical fields.
        * Validation and testing requires a large amount of time
    - Time is spent developing a code; not solving fluid mechanics problems.
    - Code may not be well accepted by others. They may not trust the results produced by the code.

16.19: PROBLEM DEFINITION

No solution provided.

16.20: PROBLEM DEFINITION

Briefly explain the meaning of each of the following ideas.

- Grid.

- Time Step.

- Solution time for a CFD program versus the accuracy.

- Boundary condition. A condition that is specified on the spatial boundaries of the CFD solution.

- Initial condition.

SOLUTION

- **Grid**. A grid which is a set of points in space at which a code solves for values of velocity and other variables of interest. The grid is often shown with an array of intersecting lines; hence the name.

- **Time Step**. The interval in unit of times between each solution time.

- Solution time for a CFD program versus the accuracy.

  – Solution time is the time interval that the computer needs in order to solve for all the variables.

  – Accuracy is how close the computer solution is to a true solution (what we would find in nature).

  – Generally, high levels of accuracy require long solution times.

- **Boundary condition**. Boundary conditions are specified values of the dependent variables (e.g. pressure, velocity) on the physical boundaries of the problem

- **Initial condition**. Specifying an initial condition involves giving numerical values for the dependent variables at all spatial points at the starting time of the solutions.

16.21: PROBLEM DEFINITION

Briefly explain the meaning of each of the following ideas.

- DNS.

- k-epsilon method.

- LES.

SOLUTION

- DNS. DNS is a way of modeling turbulent flows that involves setting the grid and time steps fine enough to resolve the features of the turbulent flow. The solution is very accurate but DNS is unrealistic for many flows because the required computational time is too large.

- k-epsilon method. The k-epsilon model (k-ϵ model) models turbulence by introducing two extra equations

- LES. Large Eddy Simulation (LES) involves direct simulation of the large-scale eddies in the turbulence and approximate solutions of the smaller eddies.

16.22: PROBLEM DEFINITION

Briefly explain the meaning of each of the following ideas.

- Post processor

- Verification

- Validation

SOLUTION

- **Post processor**. The post processor is a computer algorithm that uses the solution (from the solver) to generate plots and calculate other parameters of interest such as drag force and shear stress.

- **Verification**. Verification examines the degree to which the numerical methods used by the code result in accurate answers.Verification involves testing of the numerical methods of the code.

- **Validation**. Validation assesses the degree to which CFD predictions agree with real world observations.