9.5 Curves.

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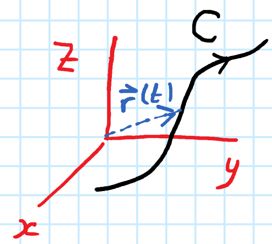
The parametric representation of a curve C in space with parameter t is given by

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$



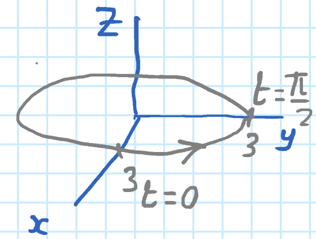
where x, y, and z are the Cartesian coordinates.
The sense of increasing t is called the positive sense on C, induces an orintation of C.

Example. Find the parametric representation of
The circle $x^2+y^2=9$, $z=0$.

Solution. $\vec{r}_1(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}_1(t) = \langle 3\cos t, 3\sin t, 0 \rangle$, $0 \le t \le 2\pi$.

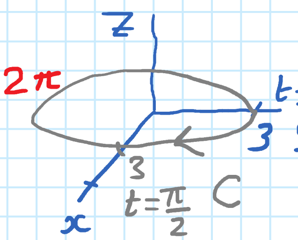
$\vec{r}_1(t) = 3\cos t\, i + 3\sin t\, j$



Note that $x^2+y^2 = (3\cos t)^2 + (3\sin t)^2 = 9$
Here the positive sense is counterclockwise.

Another Solution:

$\vec{r}_2(t) = \langle \overset{x}{3\sin t}, \overset{y}{3\cos t}, \overset{z}{0} \rangle$, $0 \le t \le 2\pi$

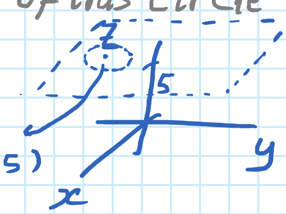


Note that $x^2+y^2 = (3\sin t)^2 + (3\cos t)^2 = 9$
Here the positive sense is clockwise

There are infinite number of representations of this circle

13/398 What curve is represented as

$\vec{r}(t) = \langle \underset{x}{2+\cos 3t}, \underset{y}{-2+\sin 3t}, \underset{z}{5} \rangle$



Solution. $(x-2)^2 + (y+2)^2 = (\cos 3t)^2 + (\sin 3t)^2 = 1$, $z=5$
It is a circle in the plane $z=5$ with center $(2,-2)$ and radius 1.

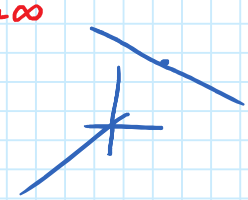
Example. The ellipse $\dfrac{(x+1)^2}{25} + \dfrac{y^2}{9} = 1$, $z=-2$
can be written in the parametric form:

$\vec{r}(t) = \langle -1+5\cos t, -3\sin t, -2 \rangle$ Why?

A straight line L through a point $A(a_1, a_2, a_3)$ in the direction of a constant vector $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is given by

$$\vec{r}(t) = \langle a_1, a_2, a_3 \rangle + t \langle b_1, b_2, b_3 \rangle \quad , \quad -\infty < t < +\infty$$

$$\vec{r}(t) = \langle a_1 + t b_1 , \ a_2 + t b_2 , \ a_3 + t b_3 \rangle .$$



$\frac{3}{398}$ Find a parametric representation of the line through $A(2, 0, 4)$ and $B(-3, 0, 9)$

$a_1 \ a_2 \ a_3$

Solution. $\vec{b} = \overrightarrow{AB} = \langle -3-2, 0-0, 9-4 \rangle = \langle \overset{b_1}{-5}, \overset{b_2}{0}, \overset{b_3}{5} \rangle$

$$\vec{r}(t) = \langle 2-5t, 0+0t, 4+5t \rangle$$

$$\vec{r}(t) = (2-5t)\,i + (4+5t)\,k. \qquad -\infty < t < \infty.$$

Circular Helix

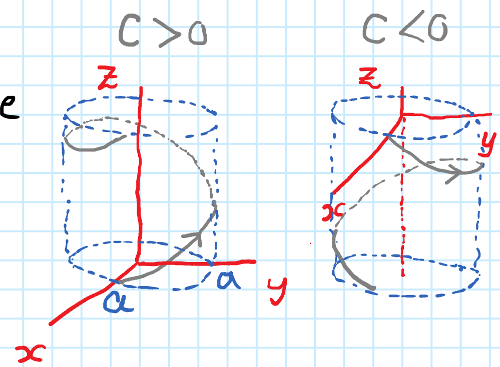
The curve C represented by the vector curve

$$\vec{r}(t) = \langle a\cos t, a\sin t, ct \rangle \quad , \quad c \neq 0$$

$$\vec{r}(t) = a\cos t\, i + a\sin t\, j + c\,t\, k$$

is called a circular helix.

It lies on the cylinder $x^2 + y^2 = a^2$.



A simple curve is a curve without multiple points, that is without points at which the curve touches or intersects itself.

An arc of a curve is the portion between any two points of the curve. For simplicity we say a curve for both curves and arcs.

Note that the circle and helix are simple.

Tangent to a curve:

If C is given by $\vec{r}(t)$, then the following limits becomes the derivative

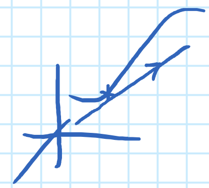
$$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

provided that this limit exists.

If $\vec{r}'(t) \neq \vec{0}$, we call $\vec{r}'(t)$ a tangent vector of C at P where P correspond to t. In such a case

If $\vec{r}'(t) \neq \vec{0}$, we call $\vec{r}'(t)$ a tangent vector of C at P where P correspond to t. In such a case

$\vec{u} = \dfrac{\vec{r}'}{|\vec{r}'|}$ is the unit tangent vector.



and the tangent to C at P is given by

$$\vec{q}(w) = \vec{r} + w\vec{r}'$$

Example. Find the tangent to the circular helix

$$\vec{r}(t) = \langle 2\sin t, -2\cos t, -3t \rangle \quad \text{at } P(1, -\sqrt{3}, -\tfrac{\pi}{2})$$

Solution. $P(1, -\sqrt{3}, -\frac{\pi}{2})$ corresponds to $t = \frac{\pi}{6}$.

$$\vec{r}'(t) = \langle 2\cos t, 2\sin t, -3 \rangle$$

$$\vec{r}'(\tfrac{\pi}{6}) = \langle \sqrt{3}, 1, -3 \rangle$$

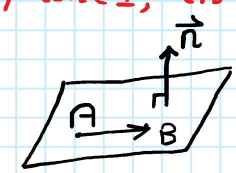
The tangent to C at P is given by

$$\vec{q}(w) = \vec{r}(\tfrac{\pi}{6}) + w\vec{r}'(\tfrac{\pi}{6})$$

$$\vec{q}(w) = \langle 1, -\sqrt{3}, -\tfrac{\pi}{2} \rangle + w\langle \sqrt{3}, 1, -3 \rangle$$

The plane.

Let $A:(a_1, a_2, a_3)$ be a point on the plane P,
and $\vec{n} = \langle n_1, n_2, n_3 \rangle$ be a normal vector to the plane P, $(\vec{n} \neq \vec{0})$
then the equation of the plane P is given by



$$\vec{n} \cdot \vec{AB} = 0 \quad \text{where } B:(x, y, z).$$

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

which can be written as

$$ax + by + cz = d,$$ (Here a, b, and c, are not all zeros).

The following are planes

$$-3x + y - 5z = 4$$

$$2y - z = 0$$

$$z = -2.$$

$\underline{1\text{–}10}$ Find a parametric representation of the following curves
398

$\underline{6}$ Ellipse $x^2 + y^2 = 1$, $z = y$

6/398 Ellipse $x^2+y^2=1$, $z=y$

Solution: $\vec{r}(t) = \langle \underset{x}{\cos t}, \underset{y}{\sin t}, \underset{z}{\sin t} \rangle$. $0 \le t \le 2\pi$

8/398 Intersection of $x+y-z=2$, $2x-5y+z=3$.

Solution. Let $x=t$ (1)

$x+y-z=2 \to y-z=2-t$ (2)

$2x-5y+z=3 \to -5y+z=3-2t$ (3)

$(2)+(3) \to -4y = 5-3t \to y = \frac{3t-5}{4}$ (4)

Putting (4) in (2) $\to \frac{3t-5}{4} - z = 2-t \to z = \frac{7t-13}{4}$

$\to \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}(t) = \langle t, \frac{3t-5}{4}, \frac{7t-13}{4} \rangle$, $-\infty < t < \infty$

11–18/398 What curves are represented as follows?

11/398 $\vec{r}(t) = \langle \underset{x}{2+r\cos 4t}, \underset{y}{6+r\sin 4t}, \underset{z}{2t} \rangle$

Solution. It is a circular helix, it lies on the cylinder

$(x-2)^2+(y-6)^2=r^2$

Since $z=2t$, $2>0$, it is shaped right handed screw.

12/398 $\vec{r}(t) = \langle 4-2t, 8t, -3+5t \rangle$

Solution: It is a straight line through $(4,0,-3)$

and in the direction of $\vec{b} = \langle -2, 8, 5 \rangle$

14/398 $\vec{r}(t) = \langle \underset{x}{t}, \underset{y}{t^2}, \underset{z}{t^3} \rangle$

Solution. It is the curve of intersection of the two surfaces $y=x^2$ and $z=x^3$ in space.