9.7 Gradient of a Scalar Function

Def. The gradient of a Scalar function $f(x,y,z)$ is given by

$$\operatorname{grad} f = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Here x, y, z are Cartesian coordinates in a domain in 3-space in which f is defined and differentiable.

Example. Let $f = x^2 + 4y^3$. Then

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x, 12y^2 \rangle.$$

For the applications of the gradient see the textbook.

Directional Derivative:

The directional derivative of the scalar function $f(x,y,z)$ in the direction of $\vec{a}$ $(\vec{a} \neq \vec{0})$ is given by

$$D_{\vec{a}} f = \frac{1}{|\vec{a}|} \vec{a} \cdot \nabla f.$$

$\frac{35}{409}$ Find the directional derivative of $f = xyz$ at $P(-1,1,3)$ in the direction of $\vec{a} = \langle 1, -2, 2 \rangle$.

Solution. $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle yz, xz, xy \rangle =$

$\nabla f(P) = \langle 3, -3, -1 \rangle$

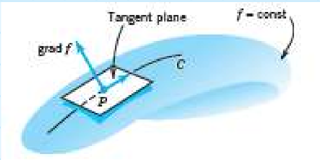
$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3.$

$$D_{\vec{a}} f(P) = \frac{1}{|\vec{a}|} \vec{a} \cdot \nabla f(P) = \frac{1}{3} (1 \times 3 + -2 \times -3 + 2 \times -1) = \frac{7}{3}.$$

Theorem. Let f be differentiable scalar function. Let

$$f(x,y,z) = c \quad (c \text{ is a constant})$$

represent a surface S. Then if $\nabla f \neq \vec{0}$ at a point P of S, $\nabla f(P)$ is normal vector of S at P.



$\frac{32}{409}$ Find a normal vector of the surface $z = x^2 + y^2$ at $P(3,4,25)$.

Solution. $x^2 + y^2 - z = 0 \rightarrow f(x,y,z) = x^2 + y^2 - z$

$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 2x, 2y, -1 \rangle$

Normal vector $\nabla f(3,4,25) = \langle 6, 8, -1 \rangle$

9.8 Divergence of a Vector Field.

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Let $\vec{v}(x,y,z) = \langle v_1(x,y,z),\ v_2(x,y,z),\ v_3(x,y,z) \rangle$ be a differentiable vector function. Then

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is the divergence of $\vec{v}$.

For the applications of the divergence see the textbook.

$\underline{1-7}$ 413 Find the divergence of the following vector functions.

$\underline{2}$ 413 $\vec{v} = \langle \underbrace{e^{2x}\cos 2y}_{v_1},\ \underbrace{e^{2x}\sin 2y}_{v_2},\ \underbrace{5e^{2x}}_{v_3} \rangle$

Solution: $\operatorname{div} \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 2e^{2x}\cos 2y + 2e^{2x}\cos 2y + 0$
$= 4e^{2x}\cos 2y.$

$\underline{7}$ 413 $\vec{v} = x^2y^2z^2\langle x, y, z\rangle$

Solution. $\vec{v} = \langle \underbrace{x^3y^2z^2}_{v_1},\ \underbrace{x^2y^3z^2}_{v_2},\ \underbrace{x^2y^2z^3}_{v_3} \rangle$

$$\operatorname{div} \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 3x^2y^2z^2 + 3x^2y^2z^2 + 3x^2y^2z^2 = 9x^2y^2z^2.$$

Given a scalar function f, the Laplacian of f is given by

$$\nabla^2 f = \operatorname{div}(\nabla f).$$

$\underline{14}$ 414 $f = \frac{xy}{z^2}$. Find $\nabla^2 f$.

Solution. $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle \underbrace{\frac{y}{z^2}}_{v_1}, \underbrace{\frac{x}{z^2}}_{v_2}, \underbrace{-\frac{2xy}{z^3}}_{v_3} \rangle$

$$\nabla^2 f = \operatorname{div}(\nabla f) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0 + 0 + \frac{6xy}{z^4}.$$

$\frac{13}{414}$ Useful formulas for the divergence.

Prove:

(a) $\operatorname{div}(k\vec{v}) = k \operatorname{div}\vec{v}$ (k constant)

(b) $\operatorname{div}(f\vec{v}) = f \operatorname{div}\vec{v} + \vec{v}\cdot\nabla f$

(c) $\operatorname{div}(f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$

Proof of (b). Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\operatorname{div}(f\vec{v}) = \operatorname{div}\langle f v_1, f v_2, f v_3 \rangle$$

$$= \frac{\partial}{\partial x}(f v_1) + \frac{\partial}{\partial y}(f v_2) + \frac{\partial}{\partial z}(f v_3)$$

$f\,\partial v$ $\quad v_1 \frac{\partial f}{} = + f \frac{\partial v_2}{}, v_2 \partial f \quad \partial y - \quad \frac{}{\partial y} + f\frac{\partial v_3}{\partial z} + v_3 \frac{\partial f}{\partial}$

$$= f\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) + v_1\frac{\partial f}{\partial x} + v_2\frac{\partial f}{\partial y} + v_3\frac{\partial f}{\partial z}$$

$$= f \operatorname{div}\vec{v} + \vec{v}\cdot\nabla f$$

9.9 Curl of a Vector Field.

Let $\vec{v}(x,y,z) = \langle v_1, v_2, v_3 \rangle$ be a differentiable vector function. Then the curl of $\vec{v}$ is defined by

$$\operatorname{curl} \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)i - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right)j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)k$$

This is the formula when x, y, z are right handed. If they are left handed, the deteminent has a minus sign in front.

For the applications of the curl see the text book.

6/416 Find curl $\vec{v}$ for $\vec{v} = \langle \sin y, \cos z, -\tan x \rangle$ with respect to right handed Cartesian coordinates.

Solution.

$$\operatorname{curl} \vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & \cos z & -\tan x \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-\tan x) - \frac{\partial}{\partial z}\cos z\right)i - \left(\frac{\partial}{\partial x}(-\tan x) - \frac{\partial}{\partial z}\sin y\right)j + \left(\frac{\partial}{\partial x}\cos z - \frac{\partial}{\partial y}\sin y\right)k$$

$$= (0 + \sin z)\,i + \sec^2 x\, j - \cos y\, k.$$

16/416 Assuming sufficient differentiability, show that

(a) $\operatorname{curl}(\vec{u}+\vec{v}) = \operatorname{curl}\vec{u} + \operatorname{curl}\vec{v}$

(b) $\operatorname{div}(\operatorname{curl}\vec{v}) = 0$

(c) $\operatorname{curl}(f\vec{v}) = (\operatorname{grad} f) \times \vec{v} + f \operatorname{curl}\vec{v}$

(d) $\operatorname{curl}(\operatorname{grad} f) = \vec{0}$

(e) $\operatorname{div}(\vec{u} \times \vec{v}) = \vec{v}\cdot\operatorname{curl}\vec{u} - \vec{u}\cdot\operatorname{curl}\vec{v}$.

Proof of (b). Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\operatorname{curl}\vec{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)i - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right)j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)k$$

$$\operatorname{div}(\operatorname{curl}\vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$= \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_1}{\partial y \partial z} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}$$

$\partial x \partial z \quad \partial y \partial x \quad \partial y \partial z \quad \partial z \partial x \quad \partial z \partial y$

$= 0.$