Chapter 10 Vector Integral Calculus. Integral Theorems.

10.1 Line Integrals.

We represent the curve C by the parametric representation

(2) $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)i + y(t)j + z(t)k, \quad a \le t \le b.$

The curve C is called the path of integration.
A: $\vec{r}(a)$ its initial point and B: $\vec{r}(b)$ its terminal point.

C is now oriented. The direction from A to B, in which t increases is called the positive direction on C and can be marked by an arrow (as in Fig 219a). The points A and B may coincide (as in Fig. 219b). Then C is called a closed path.

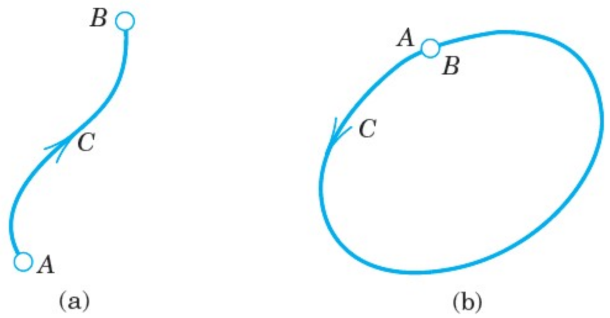


Fig. 219. Oriented curve

C is called a smooth curve if it has at each point a unique tangent whose direction varies continuously as we move along C. That is $\vec{r}(t)$ in (2) is differentiable and $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq \vec{0}$ at every point of C.

In this book, every path of integration of a line integral is assumed to be piecewise smooth; that is, it consists of finitely many smooth curves.

For example, the boundary of a square is piecewise smooth, consisting of four smooth curves (the four sides).

A line integral of a vector function $\vec{F}(\vec{r})$ over a curve $C: r(t)$, as in (2), is defined by

(3) $$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

In terms of components, with $d\vec{r} = \langle dx, dy, dz \rangle$, formula (3) becomes

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) = \int_a^b (F_1 x' + F_2 y' + F_3 z')\, dt$$

1-12 / 425 Calculate the line integral $\int_C \vec{F}(\vec{r})\cdot d\vec{r}$.
If $\vec{F}$ is a force, this gives the work done in the displacement along C.

$\frac{1}{425}$ $\vec{F} = \langle y^3, x^3 \rangle$, C the parabola $y = 5x^2$ from $A(0,0)$ to $B(2,20)$.

Solution. $C: \vec{r}(t) = \langle \overset{x}{t}, \overset{y}{5t^2} \rangle, \quad 0 \le t \le 2.$

$$\vec{r}\,'(t) = \langle 1, 10t \rangle$$

$$\vec{F} = \langle y^3, x^3 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle (5t^2)^3, t^3 \rangle = \langle 125t^3, t^3 \rangle.$$

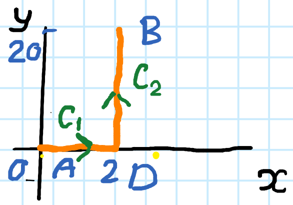
$$\int_C \vec{F}(\vec{r})\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^2 \langle 125t^3, t^3 \rangle \cdot \langle 1, 10t \rangle\,dt$$

$$= \int_0^2 (125t^3 + 10t^4)\,dt$$

$$= \frac{125t^4}{4} + \frac{10t^5}{5} \Big]_0^2 = 564.$$

$\frac{3}{425}$ $\vec{F}$ as in problem. C from $A(0,0)$ traight to $D(2,0)$, then vertically to $B(2,20)$



Solution. $C = C_1 \cup C_2$

where C_1: The straight line from $A(0,0)$ to $D(2,0)$

$C_1: \quad y = 0, \quad 0 \le x \le 2$

$C_1: \vec{r}(t) = \langle x(t), y(t) \rangle$

$C_1: \vec{r}(t) = \langle t, 0 \rangle, \quad 0 \le t \le 2$

$\vec{r}\,'(t) = \langle 1, 0 \rangle.$

$$\vec{F} = \langle y^3, x^3 \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle 0^3, t^3 \rangle$$

$$\int_{C_1} \vec{F}(\vec{r})\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^2 \langle 0, t^3 \rangle \cdot \langle 1, 0 \rangle\,dt$$

$$= \int_0^2 (0 + 0)\,dt = 0.$$

C_2: The straight line from $D(2,0)$ to $B(2,20)$

$C_2: x = 2, \quad 0 \le y \le 20$

$C_2: \vec{r}(t) = \langle x(t), y(t) \rangle$

$C_2: \vec{r}(t) = \langle 2, t \rangle \quad , 0 \le t \le 20.$

$\vec{r}'(t) = \langle 0, 1 \rangle$

$\vec{F} = \langle y^3, x^3 \rangle$

$\vec{F}(\vec{r}(t)) = \langle t^3, 2^3 \rangle$

$$\int_{C_2} \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$

$$= \int_0^{20} \langle t^3, 8 \rangle \cdot \langle 0, 1 \rangle\,dt$$

$$= \int_0^{20} 8\,dt = 8t\Big]_0^{20} = 160$$

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{C_1} \vec{F}(\vec{r}) \cdot d\vec{r} + \int_{C_2} \vec{F}(\vec{r}) \cdot dr = 160$$

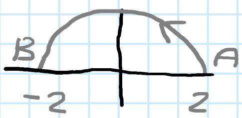
Comparing Problem 1 and 3, we have the same vector function $\vec{F} = \langle y^3, x^3 \rangle$ and the same initial and terminal points but we have different paths and different values for the line integrals.

In such a case we say that the line integral depends on path.

In the next section 10.2 we find that if some conditions are satisfied, the line integral will be independent of path.

$\frac{4}{425}$ $\vec{F} = \langle x^2, y^2 \rangle$ C the semicircle from $A(2,0)$ to $B(-2,0)$, $y \ge 0$.

Solution $C: x^2 + y^2 = 4, \quad -2 \le x \le 2, \ y \ge 0$



$C: \vec{r}(t) = \langle x(t), y(t) \rangle$

$C: \vec{r}(t) = \langle 2\cos t, 2\sin t \rangle, \quad 0 \le t \le \pi$ (Why?)

$\vec{r}'(t) = \langle -2\sin t, 2\cos t \rangle$

$\vec{F} = \langle x^2, y^2 \rangle \quad \to \vec{F}(\vec{r}(t)) = \langle (2\cos t)^2, (2\sin t)^2 \rangle$

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$

$$= \int_0^{\pi} \langle 4\cos^2 t, 4\sin^2 t \rangle \cdot \langle -2\sin t, 2\cos t \rangle\,dt = \cdots$$

$\frac{7}{425}$ $\vec{F} = \langle z, x, y \rangle$, $C: \vec{r} = \langle \overset{x}{\cos t}, \overset{y}{\sin t}, \overset{z}{t} \rangle$ from $(1,0,0)$ to $(1,0,4\pi)$

Solution. $\vec{r}(0) = \langle 1, 0, 0 \rangle$, $\vec{r}(4\pi) = \langle 1, 0, 4\pi \rangle$

$$\vec{F} = \langle z, x, y \rangle \rightarrow \vec{F}(\vec{r}(t)) = \langle t, \cos t, \sin t \rangle$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{4\pi} \langle t, \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t, 1 \rangle dt$$

$$= \int_0^{4\pi} (-t\sin t + \cos^2 t + \sin t)\, dt = \cdots$$

$\frac{9}{425}$ $\vec{F} = \langle \cosh x, \sinh y, e^z \rangle$, C the straight segment from $A\overset{a_1 a_2 a_3}{(0,0,0)}$ to $B\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$.

Solution: $\vec{AB} = \langle \overset{b_1}{\tfrac{1}{2}}, \overset{b_2}{\tfrac{1}{4}}, \overset{b_3}{\tfrac{1}{8}} \rangle$

$$C: \vec{r}(t) = \langle a_1 + tb_1, a_2 + tb_2, a_3 + tb_3 \rangle$$

$$\vec{r}(t) = \langle \underset{x}{0 + \tfrac{1}{2}t}, \underset{y}{0 + \tfrac{1}{4}t}, \underset{z}{0 + \tfrac{1}{8}t} \rangle, \quad 0 \le t \le 1.$$

$$\vec{r}\,'(t) = \langle \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8} \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle \cosh \tfrac{1}{2}t, \sinh \tfrac{1}{4}t, e^{\frac{1}{8}t} \rangle$$

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle \cosh \tfrac{1}{2}t, \sinh \tfrac{1}{4}t, e^{\frac{1}{8}t} \rangle \cdot \langle \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8} \rangle dt$$

$$= \int_0^1 \left(\tfrac{1}{2}\cosh \tfrac{1}{2}t + \tfrac{1}{4}\sinh \tfrac{1}{4}t + \tfrac{1}{8}e^{\frac{1}{8}t} \right) dt$$

$$= \sinh \tfrac{1}{2}t + \cosh \tfrac{1}{4}t + e^{\frac{1}{8}t} \Big]_0^1$$

$$= \sinh \tfrac{1}{2} + \cosh \tfrac{1}{4} + e^{\frac{1}{8}} - 2.$$