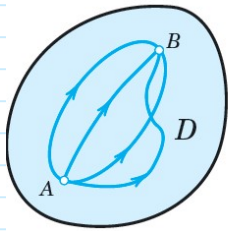


10.2 Path Independence of line Integrals



$$(1) \quad \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz) \quad (d\mathbf{r} = [dx, dy, dz])$$

The line integral (1) is said to be **path independent in a domain D in space** if for every pair of endpoints A, B in domain D , (1) has the same value for all paths in D that begin at A and end at B . This is illustrated in Fig. 224. (See Sec. 9.6 for "domain.")

Path independence is important. For instance, in mechanics it may mean that we have

Theorem 1.

Path Independence

A line integral (1) with continuous F_1, F_2, F_3 in a domain D in space is path independent in D if and only if $\mathbf{F} = [F_1, F_2, F_3]$ is the gradient of some function f in D ,

$$(2) \quad \mathbf{F} = \text{grad } f, \quad \text{thus,} \quad F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}.$$

In such a case f is called a potential of \vec{F} and we have

$$(3) \quad \int_A^B (F_1 dx + F_2 dy + F_3 dz) = f(B) - f(A) \text{ for } A \text{ and } B \text{ in } D.$$

Example. Evaluate the integral $\int_C 3x^2 dx + 2yz dy + y^2 dz$ from $A(0, 1, 2)$ to $B(1, -1, 7)$.

Solution. If there exists a function f such that $\vec{F} = \nabla f$ in some domain D containing A and B , then the line integral is path independent and f is a potential of \vec{F} and we can use (3).

Let us try to find f such that $\nabla f = \vec{F}$, that is

$$f_x = F_1 = 3x^2, \quad f_y = F_2 = 2yz, \quad f_z = F_3 = y^2.$$

$$f_x = 3x^2 \rightarrow f = x^3 + g(y, z) \quad (a)$$

$$f_y = g_y(y, z)$$

$$2yz = g_y(y, z) \rightarrow g = y^2 z + h(z) \quad (b)$$

$$\text{Putting (b) in (a)} \rightarrow f = x^3 + y^2 z + h(z) \quad \text{Constant} \quad (c)$$

$$f_z = y^2 + h'(z) \rightarrow y^2 = y^2 + h'(z) \rightarrow h'(z) = k$$

Putting (b) and (c) in (a) $\rightarrow f = x^3 + y^2z + k$
we can consider $k=0$ (why?) $\rightarrow f = x^3 + y^2z$

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It is clear that for $f = x^3 + y^2z$:

$$f_x = 3x^2 = F_1, \quad f_y = 2yz = F_2, \quad f_z = y^2 = F_3$$

in $D = \mathbb{R}^3$ (The whole space).

$$\begin{aligned} (3) \rightarrow \int_{(0,1,2)}^{(1,-1,7)} (3x^2 dx + 2yz dy + y^2 dz) &= f(1,-1,7) - f(0,1,2) \\ &= 1^3 + (-1)^2(7) - (0^3 + (1)^2(2)) \\ &= 6 \end{aligned}$$

Theorem 2.

Path Independence

The integral (1) is path independent in a domain D if and only if its value around every closed path in D is zero.

The differential form

$$(4) \quad \vec{F} \cdot d\vec{r} = F_1 dx + F_2 dy + F_3 dz$$

is exact in a domain D if and only if there exists a differentiable function f in D such that

$$(5) \quad \vec{F} = \nabla f \quad \text{in } D.$$

Hence Theorem 1. implies

Theorem 3*

Path Independence

The integral (1) is path independent in a domain D in space if and only if the differential form (4) has continuous coefficient functions F_1, F_2, F_3 and is exact in D .

This theorem is of practical importance because it leads to a useful exactness criterion. First we need the following concept, which is of general interest.

A domain D is called **simply connected** if every closed curve in D can be continuously shrunk to any point in D without leaving D .

For example, the interior of a sphere or a cube, the interior of a sphere with finitely many points removed, and the domain between two concentric spheres are simply connected. On the other hand, the interior of a torus, which is a doughnut as shown in Fig. 249 in Sec. 10.6 is not simply connected. Neither is the interior of a cube with one space diagonal removed.

The criterion for exactness (and path independence by Theorem 3*) is now as follows.

Theorem 3.

Criterion for Exactness and Path Independence

Let F_1, F_2, F_3 in the line integral (1),

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz),$$

be continuous and have continuous first partial derivatives in a domain D in space. Then:

(a) If the differential form (4) is exact in D —and thus (1) is path independent by Theorem 3*—, then in D ,

$$(6) \quad \text{curl } \mathbf{F} = \mathbf{0};$$

in components (see Sec. 9.9)

$$(6') \quad \frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}.$$

(b) If (6) holds in D and D is simply connected, then (4) is exact in D —and thus (1) is path independent by Theorem 3*.

1-8
432 Show that the form under the integral sign is exact in the plane (Probs. 1-4) or in space (Probs. 5-8) and evaluate the integral.

$$\frac{3}{432} \int_{(-1,-1)}^{(1,1)} e^{-x^2-y^2} (x dx + y dy)$$

Solution. $\vec{F} = \langle \underset{F_1}{x e^{-x^2-y^2}}, \underset{F_2}{y e^{-x^2-y^2}} \rangle$

We want to find f such that $\vec{F} = \nabla f$ in the plane, that is
 $f_x = x e^{-x^2-y^2}$, $f_y = y e^{-x^2-y^2}$

By inspection, $f(x, y) = -\frac{1}{2} e^{-x^2-y^2}$ (How?)

$$\text{By (3)} \rightarrow \int_{(-1,-1)}^{(1,1)} e^{-x^2-y^2} (x dx + y dy) = f(1,1) - f(-1,-1) \\ = \frac{1}{2} e^{-2} - \frac{1}{2} e^{-2} = 0$$

$$\frac{7}{432} \int_{(1,0,0)}^{(7,8,0)} (2xy dx + x^2 dy + \sinh z dz).$$

Solution. We want to find f such that $\vec{F} = \nabla f$ in space, that is

$$f_x = F_1 = 2xy, \quad f_y = F_2 = x^2, \quad f_z = F_3 = \sinh z$$

$$\downarrow$$

$$f = x^2y + g(y, z) \quad (a)$$

$$\downarrow f_y = x^2 + g_y = x^2 \rightarrow g_y(y, z) = 0 \rightarrow g(y, z) = h(z) \quad (b)$$

$$\text{Putting (b) in (a)} \rightarrow f = x^2y + h(z) \quad (c)$$

$$f_z = h'(z) = \sinh z$$

$$h(z) = \cosh z + k$$

$$\text{Putting (c) in (b)} \rightarrow f(x, y, z) = x^2y + \cosh z + k$$

We can take $k=0$ (Why?)

$f(x, y, z) = x^2y + \cosh z$ (In fact it is easy to find it by inspection).

$$(3) \rightarrow \int_{(1,0,0)}^{(7,8,0)} (2xy dx + x^2 dy + \sinh z dz) = f(7,8,0) - f(1,0,0)$$

$$= (7^2)(8) \cosh 0 - (1^2)(0) \cosh 0$$

$$= 392.$$

11-19 Check for path independence, and if independent, integrate
432 from $(0,0,0)$ to (a,b,c) .

13 $3x^2y dx + x^3 dy + y dz$

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Solution. Here $\vec{F} = \langle 3x^2y, x^3, y \rangle$ and

$$\text{curl } \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & x^3 & y \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y} y - \frac{\partial}{\partial z} x^3 \right) \mathbf{i} - \left(\frac{\partial}{\partial x} y - \frac{\partial}{\partial z} (3x^2) \right) \mathbf{j} + \left(\frac{\partial}{\partial x} x^3 - \frac{\partial}{\partial y} (3x^2) \right) \mathbf{k}$$

$$= \mathbf{i} + 3x^2 \mathbf{j}$$

$\text{curl } \vec{F} \neq 0 \rightarrow$ The line integral is not path independent.

18 $yz \cosh x dx + z \sinh x dy + y \sinh x dz$.

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Solution. $\vec{F} = \langle yz \cosh x, z \sinh x, y \sinh x \rangle$

By inspection for $f(x, y, z) = yz \sinh x$, we have

$$\vec{F} = \nabla f \text{ in space,}$$

that is $F_1 = f_x = yz \cosh x, F_2 = f_y = z \sinh x, F_3 = f_z = y \sinh x$

The line integral is path independent

$$(3) \rightarrow \int_{(0,0,0)}^{(a,b,c)} (yz \cosh x dx + z \sinh x dy + y \sinh x dz) = f(a,b,c) - f(0,0,0) = bc \sinh a$$