10.3 Calculus Review: Double Integrals.

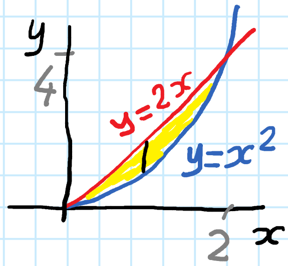
Example. Evaluate $\int_0^2 \int_{x^2}^{2x} (10-2xy)\,dy\,dx$

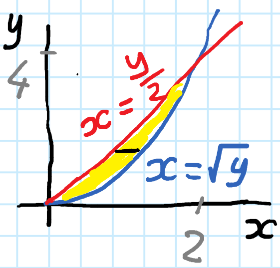
Solution.
$$= \int_0^2 (10y - xy^2)\Big|_{x^2}^{2x}\,dx$$
$$= \int_0^2 \left(10(2x) - x(2x)^2 - (10x^2 - x(x^2)^2\right)dx$$
$$= \int_0^2 (20x - 4x^3 - 10x^2 + x^5)\,dx$$
$$= 10x^2 - x^4 - \frac{10x^3}{3} + \frac{x^6}{6}\Big]_0^2 = 8.$$

$72 \quad -8+$

Example. Solve the above example with order reversed.

Solution $\int_0^2 \int_{x^2}^{2x} (10-2xy)\,dy\,dx$
$$= \int_0^4 \int_{y/2}^{\sqrt{y}} (10-2xy)\,dx\,dy$$

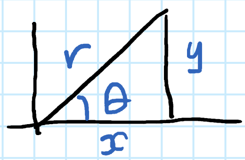




$$= \int_0^4 10x - x^2y\Big]_{y/2}^{\sqrt{y}}\,dy$$
$$= \int_0^4 \left(10\sqrt{y} - (\sqrt{y})^2 y - \left(10\left(\tfrac{y}{2}\right) - \left(\tfrac{y}{2}\right)^2 y\right)\right)dy$$
$$= \cdots = 8.$$

Polar Coordinates: For $x = r\cos\theta$, $y = r\sin\theta$

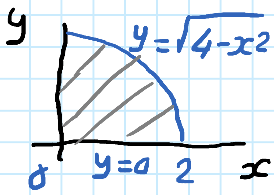
$r = \sqrt{x^2+y^2}$, $\tan\theta = \dfrac{y}{x}$.



$$\iint_R f(x,y)\,dy\,dx = \iint_{R^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where R^* is the region in the $r\theta$-plane corresponding to R in the xy-plane.

Example. Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx$



Solution. $= \int_0^{\pi/2} \int_0^2 e^{r^2}\,r\,dr\,d\theta$
$$= \int_0^{\pi/2} \int_0^4 e^u\, r\,\frac{du}{2r}\,d\theta = \cdots$$

$u = r^2 \to \dfrac{du}{dr} = 2r \to dr = \dfrac{du}{2r}$

10.4 Green's Theorem in the Plane.

Theorem 1.

Green's Theorem in the Plane[4]
(Transformation between Double Integrals and Line Integrals)

Let R be a closed bounded region (see Sec. 10.3) *in the xy-plane whose boundary C consists of finitely many smooth curves* (see Sec. 10.1). *Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives $\partial F_1/\partial y$ and $\partial F_2/\partial x$ everywhere in some domain containing R. Then*

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \oint_C (F_1\, dx + F_2\, dy). \tag{1}$$

Here we integrate along the entire boundary C of R in such a sense that R is on the left as we advance in the direction of integration (see Fig. 234).

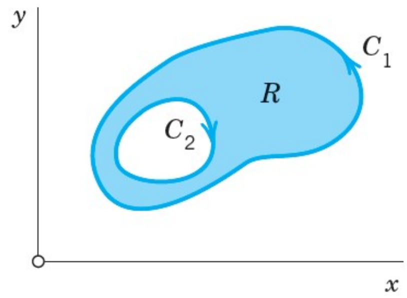


Fig. 234. Region R whose boundary C consists of two parts: C_1 is traversed counterclockwise, while C_2 is traversed clockwise in such a way that R is on the left for both curves

Verification of Green's Theorem in the Plane

Green's theorem in the plane will be quite important in our further work. Before proving it, let us get used to it by verifying it for $F_1 = y^2 - 7y$, $F_2 = 2xy + 2x$ and C the circle $x^2 + y^2 = 1$.

Solution. In (1) on the left we get

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \iint_R [(2y + 2) - (2y - 7)]\, dx\, dy = 9 \iint_R dx\, dy = 9\pi$$

since the circular disk R has area π.

We now show that the line integral in (1) on the right gives the same value, 9π. We must orient C counterclockwise, say, $\mathbf{r}(t) = [\cos t, \sin t]$. Then $\mathbf{r}'(t) = [-\sin t, \cos t]$, and on C,

$$F_1 = y^2 - 7y = \sin^2 t - 7 \sin t, \qquad F_2 = 2xy + 2x = 2 \cos t \sin t + 2 \cos t.$$

Hence the line integral in (1) becomes, verifying Green's theorem,

$$\begin{aligned}\oint_C (F_1 x' + F_2 y')\, dt &= \int_0^{2\pi} [(\sin^2 t - 7 \sin t)(-\sin t) + 2(\cos t \sin t + \cos t)(\cos t)]\, dt \\ &= \int_0^{2\pi} (-\sin^3 t + 7 \sin^2 t + 2 \cos^2 t \sin t + 2 \cos^2 t)\, dt \\ &= 0 + 7\pi - 0 + 2\pi = 9\pi.\end{aligned}$$

Area of a Plane Region as a Line Integral Over the Boundary

In (1) we first choose $F_1 = 0$, $F_2 = x$ and then $F_1 = -y$, $F_2 = 0$. This gives

$$\iint_R dx\, dy = \oint_C x\, dy \qquad \text{and} \qquad \iint_R dx\, dy = -\oint_C y\, dx$$

respectively. The double integral is the area A of R. By addition we have

$$A = \frac{1}{2} \oint_C (x\, dy - y\, dx) \tag{4}$$

where we integrate as indicated in Green's theorem. This interesting formula expresses the area of R in terms of a line integral over the boundary. It is used, for instance, in the theory of certain **planimeters** (mechanical instruments for measuring area). See also Prob. 11.

For an ***ellipse*** $x^2/a^2 + y^2/b^2 = 1$ or $x = a \cos t$, $y = b \sin t$ we get $x' = -a \sin t$, $y' = b \cos t$; thus from (4) we obtain the familiar formula for the area of the region bounded by an ellipse,

$$A = \frac{1}{2} \int_0^{2\pi} (xy' - yx')\, dt = \frac{1}{2} \int_0^{2\pi} [ab \cos^2 t - (-ab \sin^2 t)]\, dt = \pi ab.$$

1–12/444 Using Green's Theorem evaluate $\int_C \vec{F}(\vec{r})\cdot d\vec{r}$ counterclockwise around the boundary curve C of the region R, where

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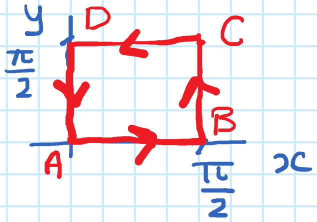
2/444 $\vec{F} = \langle \underbrace{y\sin x}_{F_1}, \underbrace{2x\cos y}_{F_2} \rangle$, R the square with vertices $A(0,0)$, $B(\frac{\pi}{2},0)$, $C(\frac{\pi}{2},\frac{\pi}{2})$, $D(0,\frac{\pi}{2})$

Solution. From (1):



$$\oint_C \vec{F}(\vec{r})\cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

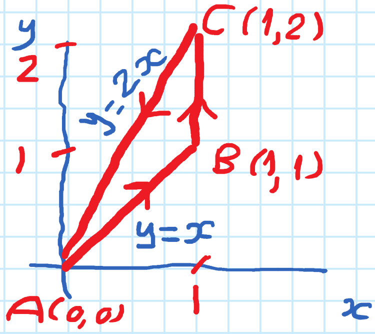
$$= \int_0^{\pi/2}\int_0^{\pi/2} (2\cos y - \sin x)\,dx\,dy$$

$$= \int_0^{\pi/2} 2x\cos y + \cos x \Big]_0^{\pi/2} dy$$

$$= \int_0^{\pi/2} (\pi\cos y - 1)\,dy = \pi\sin y - y\Big]_0^{\pi/2} = \pi/2.$$

5/444 $\vec{F} = \langle \underbrace{e^{x+y}}_{F_1}, \underbrace{e^{x-y}}_{F_2} \rangle$, R the triangle with vertices $A(0,0)$, $B(1,1)$, $C(1,2)$

Solution.



$$\oint_C \vec{F}(\vec{r})\cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dy\,dx$$

$$= \int_0^1\int_x^{2x} \left(e^{x-y} - e^{x+y}\right) dy\,dx$$

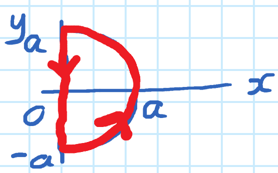
$$= \int_0^1 -e^{x-y} - e^{x+y}\Big]_x^{2x} dx = \int_0^1 -e^{x-2x} - e^{x+2x} - (-e^{x-x} - e^{x+x})\,dx$$

$$= \int_0^1 (1 + e^{2x} - e^{-x} - e^{3x})\,dx = x + \frac{e^{2x}}{2} + e^{-x} - \frac{e^{3x}}{3}\Big]_0^1$$

$$= 1 + \frac{e^2}{2} + e^{-1} - \frac{e^3}{3} - \frac{7}{6}.$$

8/444 $\vec{F} = \langle \underbrace{e^x\cos y}_{F_1}, \underbrace{-e^x\sin y}_{F_2} \rangle$, R the semidesk $x^2+y^2 \le a^2$, $x \ge 0$

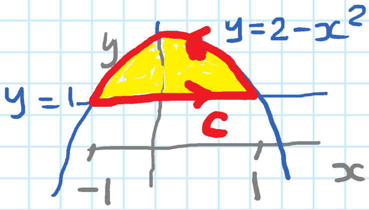
Solution.



$$\oint_C \vec{F}(\vec{r})\cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \iint_R (-e^x\sin y + e^x\sin y)\,dx\,dy = 0.$$

$\frac{7}{444}$ $\vec{F} = \langle \underbrace{x^2+y^2}_{F_1}, \underbrace{x^2-y^2}_{F_2} \rangle$, $R: 1 \le y \le 2-x^2$, sketch R.



Solution. $\oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$

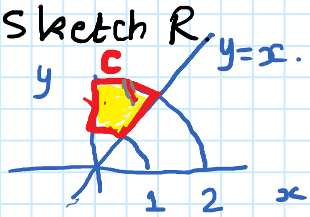
$$= \int_{-1}^{1}\int_{1}^{2-x^2} (2x - 2y)\,dy\,dx = \int_{-1}^{1} \left[2xy - y^2\right]_1^{2-x^2} dx$$

$2-x^2 = 1 \rightarrow x = \pm 1$

$$= \int_{-1}^{1} \left(2x(2-x^2) - (2-x^2)^2\right) dx$$

$$= \int_{-1}^{1} \left(4x - 2x^3 - 4 + 4x^2 - x^4\right) dx = \left[2x^2 - \frac{x^4}{2} - 4x + \frac{4x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = -\frac{86}{15}$$

$\frac{12}{444}$ $\vec{F} = \langle \underbrace{x^2y^2}_{F_1}, \underbrace{-\frac{x}{y^2}}_{F_2} \rangle$, $R: 1 \le x^2+y^2 \le 4$, $x \ge 0$, $y \ge x$. Sketch R.



Solution $\oint_C \vec{F}(\vec{r}) \cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$

$$= \iint_R \left(-\frac{1}{y^2} - 2x^2y\right) dx\,dy$$

In Polar

$$= \int_{\pi/4}^{\pi/2}\int_1^2 \left(-\frac{1}{r^2\sin^2\theta} - 2r^2\cos^2\theta(r\sin\theta)\right) r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\int_1^2 \left(-\frac{1}{r\sin^2\theta} - 2r^4\cos^2\theta\sin\theta\right) dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[-\frac{1}{\sin^2\theta}\ln r - \frac{2r^5}{5}\cos^2\theta\sin\theta\right]_1^2 d\theta$$

$$= -\ln 2 - \frac{31}{15\sqrt{2}}$$