10.5 Surfaces for Surface Integrals

Representation of Surfaces

Representations of a surface S in xyz-space are

$$(1) \qquad z = f(x, y) \qquad \text{or} \qquad g(x, y, z) = 0.$$

For example, $z = +\sqrt{a^2 - x^2 - y^2}$ or $x^2 + y^2 + z^2 - a^2 = 0$ $(z \geqq 0)$ represents a hemisphere of radius a and center 0.

Now for *curves* C in line integrals, it was more practical and gave greater flexibility to use a *parametric* representation $\mathbf{r} = \mathbf{r}(t)$, where $a \leqq t \leqq b$. This is a mapping of the interval $a \leqq t \leqq b$, located on the t-axis, onto the curve C (actually a portion of it) in xyz-space. It maps every t in that interval onto the point of C with position vector $\mathbf{r}(t)$. See Fig. 241A.

Similarly, for surfaces S in surface integrals, it will often be more practical to use a *parametric* representation. Surfaces are *two*-dimensional. Hence we need *two* parameters,

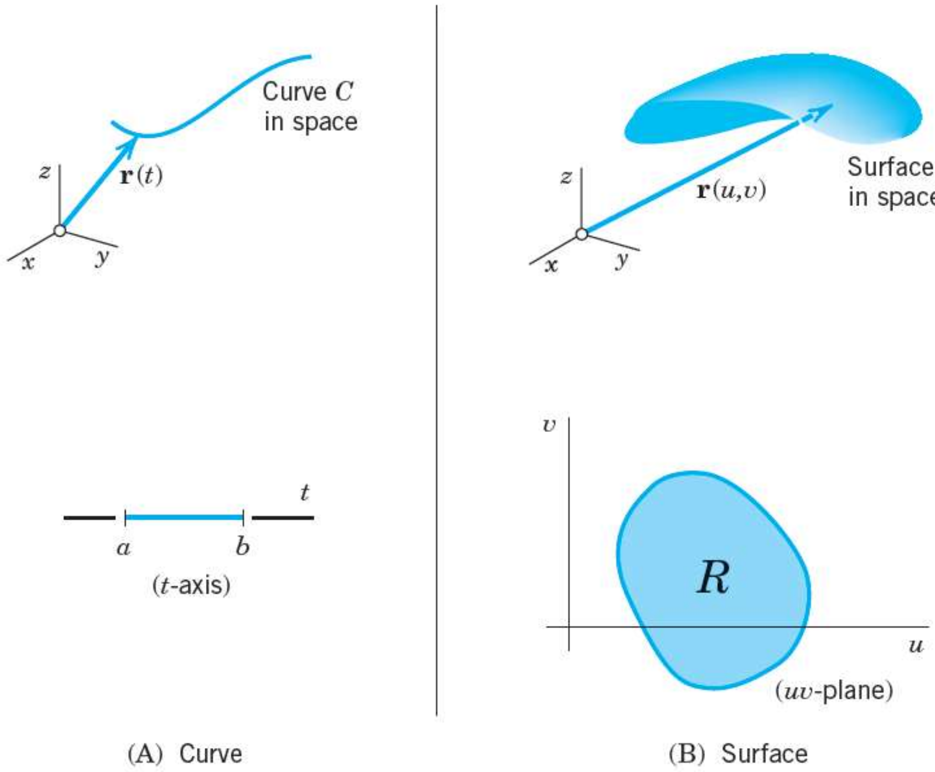


Fig. 241. Parametric representations of a curve and a surface

which we call u and v. Thus a **parametric representation** of a surface S in space is of the form

$$\textbf{(2)} \qquad \mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)] = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where (u, v) varies in some region R of the uv-plane. This mapping (2) maps every point (u, v) in R onto the point of S with position vector $\mathbf{r}(u, v)$. See Fig. 241B.

Example 1.

Parametric Representation of a Cylinder

The circular cylinder $x^2 + y^2 = a^2$, $-1 \leqq z \leqq 1$, has radius a, height 2, and the z-axis as axis. A parametric representation is

$$\mathbf{r}(u, v) = [a \cos u, a \sin u, v] = a \cos u\,\mathbf{i} + a \sin u\,\mathbf{j} + v\mathbf{k} \qquad \text{(Fig. 242).}$$

The components of $\mathbf{r}$ are $x = a \cos u$, $y = a \sin u$, $z = v$. The parameters u, v vary in the rectangle R: $0 \leqq u \leqq 2\pi$, $-1 \leqq v \leqq 1$in the uv-plane. The curves $u =$ const are vertical straight lines. The curves $v =$ const are parallel circles. The point P in Fig. 242 corresponds to $u = \pi/3 = 60°$, $v = 0.7$. ■

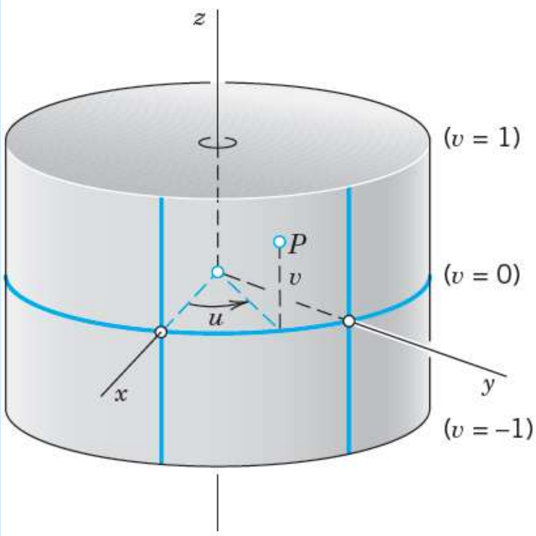


Fig. 242. Parametric representation of a cylinder

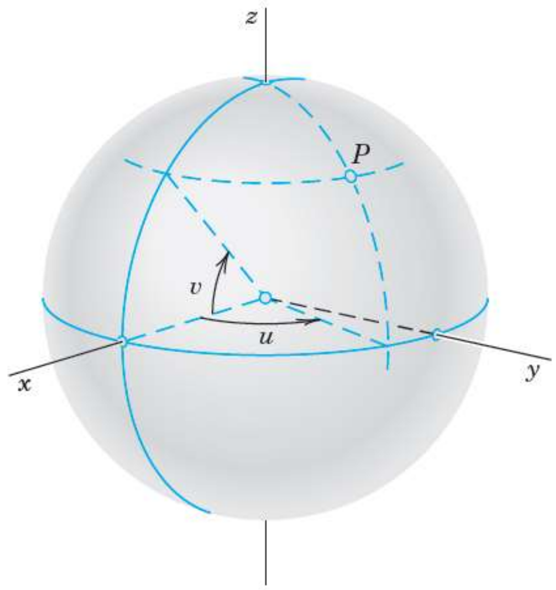


Fig. 243. Parametric representation of a sphere

Example 2

Parametric Representation of a Sphere

A sphere $x^2 + y^2 + z^2 = a^2$ can be represented in the form

(3) $$\mathbf{r}(u, v) = a \cos v \cos u\,\mathbf{i} + a \cos v \sin u\,\mathbf{j} + a \sin v\mathbf{k}$$

where the parameters u, v vary in the rectangle R in the uv-plane given by the inequalities $0 \leqq u \leqq 2\pi$, $-\pi/2 \leqq v \leqq \pi/2$. The components of $\mathbf{r}$ are

$$x = a \cos v \cos u, \qquad y = a \cos v \sin u, \qquad z = a \sin v.$$

The curves $u =$ const and $v =$ const are the "meridians" and "parallels" on S (see Fig. 243). *This representation is used in* ***geography*** *for measuring the latitude and longitude of points on the globe.*

Another parametric representation of the sphere also used in mathematics is

(3*) $$\mathbf{r}(u, v) = a \cos u \sin v\,\mathbf{i} + a \sin u \sin v\,\mathbf{j} + a \cos v\,\mathbf{k}$$

where $0 \leqq u \leqq 2\pi$, $0 \leqq v \leqq \pi$. ■

Example 3.

Parametric Representation of a Cone

A circular cone $z = \sqrt{x^2 + y^2}, 0 \leqq \cancel{x}\, z \leqq H$ can be represented by

$$\mathbf{r}(u, v) = [u \cos v, u \sin v, u] = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + u\mathbf{k},$$

in components $x = u \cos v$, $y = u \sin v$, $z = u$. The parameters vary in the rectangle R: $0 \leqq u \leqq H$, $0 \leqq v \leqq 2\pi$. Check that $x^2 + y^2 = z^2$, as it should be. What are the curves $u = \text{const}$ and $v = \text{const}$?

Given a surface S in the parametric representation $\vec{r} = \vec{r}(u,v)$ in (2),

(4) $$\vec{N} = \vec{r}_u \times \vec{r}_v \neq \vec{0},$$

and

(5) $$\vec{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}.$$

We have the following Theorem.

Theorem 1.

Tangent Plane and Surface Normal

If a surface S is given by (2) *with continuous* $\mathbf{r}_u = \partial\mathbf{r}/\partial u$ *and* $\mathbf{r}_v = \partial\mathbf{r}/\partial v$ *satisfying* (4) *at every point of S, then S has, at every point P, a unique tangent plane passing through P and spanned by* $\mathbf{r}_u$ *and* $\mathbf{r}_v$, *and a unique normal whose direction depends continuously on the points of S. A normal vector is given by* (4) *and the corresponding unit normal vector by* (5). (See Fig. 244.)

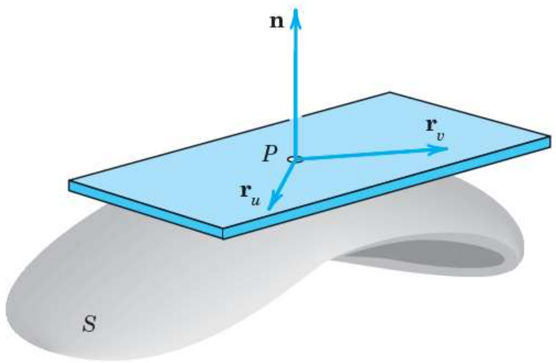


Fig. 244. Tangent plane and normal vector

A surface S is called a **smooth surface** if its surface normal depends continuously on the points of S.

S is called **piecewise smooth** if it consists of finitely many smooth portions.

For instance, a sphere is smooth, and the surface of a cube is piecewise smooth

12–19 449 Find a parametric representation and a normal vector. (The answer gives one of them, there are many.)

12/449 Plane $5x+y-3z=30$

Solution. $(2)\to \vec{r}(u,v)=\langle x(u,v),\, y(u,v),\, z(u,v)\rangle$

$$\vec{r}(u,v)=\left\langle u,\ v,\ \frac{5u+v-30}{3}\right\rangle,\quad -\infty<u<+\infty,\ -\infty<v<+\infty.$$

$$\vec{r}_u=\left\langle 1,0,\frac{5}{3}\right\rangle,\quad \vec{r}_v=\left\langle 0,1,\frac{1}{3}\right\rangle.$$

$$(4)\to \vec{N}=\vec{r}_u\times\vec{r}_v=\begin{vmatrix} i & j & k\\ 1 & 0 & 5/3\\ 0 & 1 & 1/3\end{vmatrix}$$

$$=\begin{vmatrix}0 & 5/3\\ 1 & 1/3\end{vmatrix} i-\begin{vmatrix}1 & 5/3\\ 0 & 1/3\end{vmatrix} j+\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} k$$

$$=-\frac{5}{3}i-\frac{1}{3}j+k$$

14/449 Sphere $(x-1)^2+(y+2)^2+z^2=25$

Solution. $(2)\to \vec{r}(u,v)=\langle x(u,v),\, y(u,v),\, z(u,v)\rangle$

(See Example 2) $=\langle 1+5\cos v\cos u,\ -2+5\cos v\sin u,\ 5\sin v\rangle$

$$0\le u\le 2\pi,\quad -\frac{\pi}{2}\le v\le\frac{\pi}{2}.$$

$$\vec{r}_u=\langle -5\cos v\sin u,\ 5\cos v\cos u,\ 0\rangle$$

$$\vec{r}_v=\langle -5\sin v\cos u,\ -5\sin v\sin u,\ 5\cos v\rangle$$

$$(4)\to \vec{N}_1=\begin{vmatrix} i & j & k\\ -5\cos v\sin u & 5\cos v\cos u & 0\\ -5\sin v\cos u & -5\sin v\sin u & 5\cos v\end{vmatrix}$$

$$\vec{N}_1=25\cos^2 v\cos u\, i+25\cos^2 v\sin u\, j+25(\cos v\sin v\sin^2 u+\cos v\sin v\cos^2 u)k$$

$$\vec{N}_1=25\cos^2 v\cos u\, i+25\cos^2 v\sin u\, j+25\cos v\sin v\, k.$$

Another method to find $\vec{N}$ (By Theorem 2 in section 9.7).

Here $f(x,y,z)=C \leftrightarrow (x-1)^2+(y+2)^2+z^2=25$

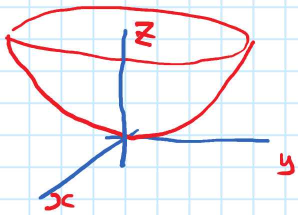
$$\vec{N}_2=\nabla f=\langle f_x, f_y, f_z\rangle=\langle 2(x-1),\ 2(y+2),\ 2z\rangle$$

$\vec{N}_2(u,v) = \langle 10\cos v \cos u,\ 10\cos v \sin u,\ 10 \sin v \rangle$

Note that $\vec{N}_1$ and $\vec{N}_2$ are normal vectors of the sphere.

In fact $\vec{N}_1(u,v) = \frac{25}{10} \cos v\ \vec{N}_2(u,v)$

$\frac{16}{449}$ Elliptic paraboloid $z = 4x^2 + y^2$



Solution. (2) $\rightarrow r(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$.

$r(u,v) = \langle u\cos v,\ 2u\sin v,\ 4u^2 \rangle$, $-\infty < u < +\infty, 0 \le v \le 2\pi$.

$\vec{r}_u = \langle \cos v,\ 2\sin v,\ 8u \rangle$, $\vec{r}_v = \langle -u\sin v,\ 2u\cos v,\ 0 \rangle$

(4) $\rightarrow$ $\vec{N}_1 = \vec{r}_u \times \vec{r}_v = \cdots = \langle -16u^2\cos v,\ -8u^2 \sin v,\ 2u \rangle$

Another Method to find a normal vector (Using Theorem 2 of section 9.7)

Here $f(x,y,z) = C \longleftrightarrow 4x^2 + y^2 - z = 0$

$\vec{N}_2 = \nabla f = \langle f_x, f_y, f_z \rangle = \langle 8x, 2y, -1 \rangle$

$\vec{N}_2(u,v) = \langle 8u\cos v,\ 4u\sin v,\ -1 \rangle$

Note that $\vec{N}_1(u,v) = -2u\vec{N}_2(u,v)$.

$\frac{18}{449}$ Hyperbolic Cylinder $9x^2 - 4y^2 = 36$.

Solution. (2) $\rightarrow \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$\vec{r}(u,v) = \langle 2\cosh u,\ 3\sinh u,\ v \rangle$, $-\infty < u < +\infty$, $-\infty < v < +\infty$.

$\vec{r}_u = \langle 2\sinh u,\ 3\cosh u,\ 0 \rangle$, $\vec{r}_v = \langle 0, 0, 1 \rangle$

(4) $\rightarrow$ $\vec{N} = \vec{r}_u \times \vec{r}_v = \langle 3\cosh u, -2\sinh u,\ 0 \rangle$.