10.6 Surface Integrals.

To define a surface integral, we take a surface S, given by a parametric representation as just discussed,

$$\text{(1)} \qquad \mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)] = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where (u, v) varies over a region R in the uv-plane. We assume S to be piecewise smooth (Sec. 10.5), so that S has a normal vector

$$\text{(2)} \qquad \mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \qquad \text{and unit normal vector} \qquad \mathbf{n} = \frac{1}{|\mathbf{N}|}\mathbf{N}$$

at every point (except perhaps for some edges or cusps, as for a cube or cone). For a given vector function $\mathbf{F}$ we can now define the **surface integral** over S by

$$\textbf{(3)} \qquad \iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iint_R \mathbf{F}(\mathbf{r}(u, v)) \cdot \mathbf{N}(u, v)\, du\, dv.$$

Here $\mathbf{N} = |\mathbf{N}|\mathbf{n}$ by (2), and $|\mathbf{N}| = |\mathbf{r}_u \times \mathbf{r}_v|$ is the area of the parallelogram with sides $\mathbf{r}_u$ and $\mathbf{r}_v$, by the definition of cross product. Hence

$$\textbf{(3*)} \qquad \mathbf{n}\, dA = \mathbf{n}|\mathbf{N}|\, du\, dv = \mathbf{N}\, du\, dv.$$

And we see that $dA = |\mathbf{N}|du\, dv$ is the element of area of S.

We can write (3) in components, using $\mathbf{F} = [F_1, \ F_2, \ F_3]$, $\mathbf{N} = [N_1, \ N_2, \ N_3]$, and $\mathbf{n} = [\cos\alpha, \cos\beta, \cos\gamma]$. Here, α, β, γ are the angles between $\mathbf{n}$ and the coordinate axes; indeed, for the angle between $\mathbf{n}$ and $\mathbf{i}$, formula (4) in Sec. 9.2 gives $\cos\alpha = \mathbf{n} \cdot \mathbf{i}/|\mathbf{n}||\mathbf{i}| = \mathbf{n} \cdot \mathbf{i}$, and so on. We thus obtain from (3)

$$\textbf{(4)} \qquad \begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n}\, dA &= \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA \\ &= \iint_R (F_1 N_1 + F_2 N_2 + F_3 N_3)\, du\, dv. \end{aligned}$$

In (4) we can write $\cos\alpha\, dA = dy\, dz$, $\cos\beta\, dA = dz\, dx$, $\cos\gamma\, dA = dx\, dy$. Then (4) becomes the following integral for the flux:

$$\textbf{(5)} \qquad \iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy).$$

Example 1

Flux Through a Surface

Compute the flux of water through the parabolic cylinder S: $y = x^2$, $0 \leqq x \leqq 2$, $0 \leqq z \leqq 3$ (Fig. 245) if the velocity vector is $\mathbf{v} = \mathbf{F} = [3z^2, \quad 6, \quad 6xz]$, speed being measured in meters/sec. (Generally, $\mathbf{F} = \rho\mathbf{v}$, but water has the density $\rho = 1 \text{ g/cm}^3 = 1 \text{ ton/m}^3$.)

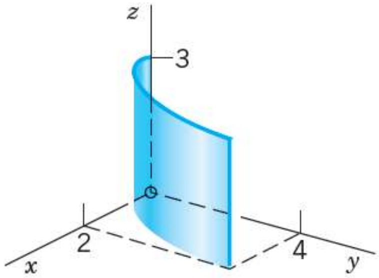

Fig. 245. Surface S in Example 1

Solution. Writing $x = u$ and $z = v$, we have $y = x^2 = u^2$. Hence a representation of S is

$$S: \qquad \mathbf{r} = [u, u^2, v] \qquad\qquad (0 \leqq u \leqq 2, 0 \leqq v \leqq 3).$$

By differentiation and by the definition of the cross product,

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [1, \quad 2u, \quad 0] \times [0, \quad 0, \quad 1] = [2u, \quad -1, \quad 0].$$

On S, writing simply $\mathbf{F}(S)$ for $\mathbf{F}[\mathbf{r}(u, v)]$, we have $\mathbf{F}(S) = [3v^2, 6, 6uv]$. Hence $\mathbf{F}(S) \bullet \mathbf{N} = 6uv^2 - 6$. By integration we thus get from (3) the flux

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \int_0^3 \int_0^2 (6uv^2 - 6)\, du\, dv = \int_0^3 (3u^2v^2 - 6u)\Big|_{u=0}^{2} dv$$

$$= \int_0^3 (12v^2 - 12)\, dv = (4v^3 - 12v)\Big|_{v=0}^{3} = 108 - 36 = 72 \; [\text{m}^3/\text{sec}]$$

or 72,000 liters/sec. Note that the y-component of $\mathbf{F}$ is positive (equal to 6), so that in Fig. 245 the flow goes from left to right.

Example 2.

Evaluate (3) when $\mathbf{F} = [x^2, 0, 3y^2]$ and S is the portion of the plane $x + y + z = 1$ in the first octant (Fig. 246).

Solution. Writing $x = u$ and $y = v$, we have $z = 1 - x - y = 1 - u - v$. Hence we can represent the plane $x + y + z = 1$ in the form $\mathbf{r}(u, v) = [u, v, 1 - u - v]$. We obtain the first-octant portion S of this plane by restricting $x = u$ and $y = v$ to the projection R of S in the xy-plane. R is the triangle bounded by the two coordinate axes and the straight line $x + y = 1$, obtained from $x + y + z = 1$ by setting $z = 0$. Thus $0 \leqq x \leqq 1 - y$, $0 \leqq y \leqq 1$.

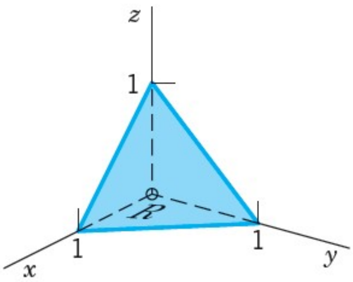

Fig. 246. Portion of a plane in Example 2

By inspection or by differentiation,

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = [1, 0, -1] \times [0, 1, -1] = [1, 1, 1].$$

Hence $\mathbf{F}(S) \bullet \mathbf{N} = [u^2, 0, 3v^2] \bullet [1, 1, 1] = u^2 + 3v^2$. By (3),

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dA = \iint_R (u^2 + 3v^2)\, du\, dv = \int_0^1 \int_0^{1-v} (u^2 + 3v^2)\, du\, dv$$

$$= \int_0^1 \left[\frac{1}{3}(1 - v)^3 + 3v^2(1 - v)\right] dv = \frac{1}{3}.$$

A smooth surface S (see Sec. 10.5) is called **orientable** if the positive normal direction, when given at an arbitrary point P_0 of S, can be continued in a unique and continuous way to the entire surface. In many practical applications, the surfaces are smooth and thus orientable.

See the Textbook for piecewise smooth orientable surfaces. and for an example of a nonorietable surface (Möbius Strip).

For problems 1-3, evaluate the flux integrals (3) $\int_S \vec{F} \cdot \vec{n}\, dA$ for the given data

Problem 1. $\vec{F} = \langle \cosh(yz), 0, y^4 \rangle$

S: $y^2 + z^2 = 1, \quad 0 \le x \le 20, \quad z \ge 0.$

Solution. $S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$= \langle u, \cos v, \sin v \rangle, \quad 0 \le u \le 20, \quad 0 \le v \le \pi.$

$\vec{r}_u = \langle 1, 0, 0 \rangle, \qquad \vec{r}_v = \langle 0, -\sin v, \cos v \rangle.$

$\vec{N}(u,v) = \vec{r}_u \times \vec{r}_v = \langle 0, -\cos v, -\sin v \rangle.$

$\vec{F} = \langle \cosh yz, 0, y^4 \rangle$

$\vec{F}(\vec{r}(u,v)) = \langle \cosh(\cos v \sin v), 0, \cos^4 v \rangle.$

$(3) \to \iint_S \vec{F} \cdot \vec{n}\, dA = \iint_R \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(u,v)\, du\, dv.$

$= \int_0^{\pi} \int_0^{20} \langle \cosh(\tfrac{1}{2}\sin 2v), 0, \cos^4 v \rangle \cdot \langle 0, -\cos v, -\sin v \rangle\, du\, dv$

Let $w = \cos v$.

$= \int_0^{\pi} \int_0^{20} -\sin v \cos^4 v\, du\, dv = -20 \int_0^{\pi} \sin v \cos^4 v\, dv \overset{\uparrow}{=} -8.$

Problem 2. $\vec{F} = \langle z^2, 0, 8\rangle$, $S: z = 3\sqrt{x^2+4y^2}$, $0 \le z \le 15$ $y \ge 0$.

Solution. $S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

$= \langle u\cos v, \frac{1}{2}u\sin v, 3u\rangle$, $0 \le u \le 5$, $0 \le v \le \pi$.

$\vec{r}_u = \langle \cos v, \frac{1}{2}\sin v, 3\rangle$, $\vec{r}_v = \langle -u\sin v, \frac{1}{2}u\cos v, 0\rangle$

$\vec{N}(u,v) = \vec{r}_u \times \vec{r}_v = \langle -\frac{3}{2}u\cos v, 3u\sin v, \frac{1}{2}u\rangle$

$\vec{F} = \langle z^2, 0, 8\rangle$

$\vec{F}(\vec{r}(u,v)) = \langle (3u)^2, 0, 8\rangle$

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}(\vec{r}(u,v)).\vec{N}(u,v)\,du\,dv$$

$$= \int_0^\pi\int_0^5 \langle 9u^2, 0, 8\rangle\cdot\langle -\frac{3}{2}u\cos v, 3u\sin v, \frac{1}{2}u\rangle\,du\,dv$$

$$= \int_0^\pi\int_0^5 \left(-\frac{27}{2}u^3\cos v + 4u\right)du\,dv$$

$$= \int_0^\pi -\frac{27u^4}{8}\cos v + 2u^2\Big]_0^5\,dv$$

$$= \int_0^\pi \left(-\frac{16875}{8}\cos v + 50\right)dv = 50\pi.$$

Problem 3. $\vec{F} = \langle \cosh y, 0, \sinh x\rangle$.

$S: z = x + y^2$, $0 \le y \le x$, $0 \le x \le 1$.

Solution. $S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

$= \langle u, v, u+v^2\rangle$, $0 \le v \le u$, $0 \le u \le 1$.

$\vec{r}_u = \langle 1, 0, 1\rangle$, $\vec{r}_v = \langle 0, 1, 2v\rangle$.

$\vec{N}(u,v) = \vec{r}_u \times \vec{r}_v = \langle -1, -2v, 1\rangle$

$\vec{F} = \langle \cosh y, 0, \sinh x\rangle$

$\vec{F}(\vec{r}(u,v)) = \langle \cosh v, 0, \sinh u\rangle$

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}(\vec{r}(u,v)).\vec{N}(u,v)\,dv\,du$$

$$= \int_0^1\int_0^u \langle \cosh v, 0, \sinh u\rangle\cdot\langle -1, -2v, 1\rangle\,dv\,du$$

$$= \int_0^1\int_0^u (-\cosh v + \sinh u)\,dv\,du = 1 - \sinh 1.$$