Last Lecture of the Course
We are still in section

12.7 Heat Equation;
Modeling Very Long Bar

Example4 (Page 573 in the Book)
Solve the Heat Equation

$$u_t = c^2 u_{xx}, \qquad 0 < x < +\infty, \ t > 0, \qquad (1)$$

with boundary condition

$$u(0,t) = 0, \qquad t \geq 0, \qquad (2)$$

and initial Condition

$$u(x,0) = f(x), \qquad 0 \leq x < +\infty \qquad (3)$$

Solution. Here we use the Fourier sine transform.
with respect to x.

Let $\hat{u}_s(w,t) = F_s(u(x,t))$

See (1a) Page 518 $\quad = \sqrt{\frac{2}{\pi}} \int_0^\infty u(p,t) \sin wp \, dp. \qquad (4)$

Then

$$F_s(u_t) = \frac{\partial}{\partial t} \hat{u}_s \quad \text{(Why?)} \qquad (5)$$

$$F_s(u_{xx}) = -w^2 F_s(u) + \sqrt{\frac{2}{\pi}}\, w \underbrace{u(0,t)}_{\substack{\parallel \\ 0 \text{ from } (2)}} = -w^2 \hat{u}_s \qquad (6)$$

where we used (5b) page 521 from the book

$$F_s(g''(x)) = -w^2 F_s(g(x)) + \sqrt{\frac{2}{\pi}}\, w\, g(0)$$

$(1) \longrightarrow F_s(u_t) = c^2 F_s(u_{xx})$

$(5) \& (6) \longrightarrow \frac{\partial}{\partial t} \hat{u}_s = -c^2 w^2 \hat{u}_s \qquad (7)$

(7) is similar to the ODE

$$\frac{dy}{dt} = -c^2\omega^2 y$$

whose solution is

$$y(t) = K e^{-c^2\omega^2 t}, \quad K \in \mathbb{R}, \quad (\text{Why?})$$

Therefore the solution of (7) is

$$\hat{u}_s(\omega, t) = K(\omega) e^{-c^2\omega^2 t} \qquad (8)$$

$$(3) \rightarrow F_s(u(x,0)) = F_s(f(x))$$

$$\hat{u}_s(\omega, 0) = \hat{f}_s(\omega) \qquad (9)$$

$$(8) \rightarrow \hat{u}_s(\omega, 0) = K(\omega) \qquad (10)$$

$$(9) \,\&\, (10) \rightarrow K(\omega) = \hat{f}_s(\omega)$$

and (8) becomes

$$\hat{u}_s(\omega, t) = \hat{f}(\omega) e^{-c^2\omega^2 t}$$

Taking the inverse Fourier sine transform:

$$\xrightarrow{(2b)} u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(\omega) e^{-c^2\omega^2 t} \sin \omega x \, d\omega \quad (11)$$

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But $\hat{f}(\omega) = F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(p) \sin \omega p \, dp$ (12)

Putting (12) into (11), we get

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(p) \sin \omega p \, e^{-c^2\omega^2 t} \sin \omega x \, dp \, d\omega$$

Example 5. Solve the Heat equation

$$u_t = 4\, u_{xx}\,, \quad 0 < x < +\infty, \quad t > 0 \qquad (1)$$

$$u_x(0,t) = 0, \qquad t \geq 0 \qquad (2)$$

$$u(x,0) = e^{-3x} \qquad 0 < x < +\infty \qquad (3)$$

Solution. Here we use the Fourier cosine transform with respect to x

Let $\hat{u}_c(w,t) = F_c(u(x,t))$

See (2a) Pge 518 $= \sqrt{\frac{2}{\pi}} \int_0^\infty u(p,t)\cos wp\,dp \quad (4)$

Then

$$F_c(u_t) = \frac{\partial}{\partial t}\hat{u}_c \qquad (5)$$

and $F_c(u_{xx}) = -w^2 F_c(u) - \sqrt{\frac{2}{\pi}}\, \underbrace{u_x(0,t)}_{=0 \text{ from } (2)} = -w^2 \hat{u}_c \quad (6)$

where we used (5a) page 521:

$$F_c(g''(x)) = -w^2 F_c(g(x)) - \sqrt{\frac{2}{\pi}}\, g'(0)$$

$$(1) \rightarrow F_c(u_t) = 4\, F_c(u_{xx}) \qquad (7)$$

Putting (5) and (6) into (7), we get

$$\frac{\partial}{\partial t}\hat{u}_c = -4\, w^2 \hat{u}_c \qquad (8)$$

whose solution is

$$\hat{u}_c(w,t) = K(w)\, e^{-4w^2 t}, \qquad (9)$$

$$(3) \rightarrow F_c(u(x,0)) = F_c(e^{-3x})$$

$$\hat{u}_c(w,0) = \sqrt{\frac{2}{\pi}}\, \frac{3}{9+w^2} \qquad (10)$$

See Table 1 of Sec. 11.10

$$(9) \rightarrow \hat{u}_c(w,0) = K(w) \qquad (11)$$

From (1

From (10) and (11) $\longrightarrow$ $K(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{3}{9+\omega^2}$

and (9) becomes

$$\hat{u}_c(\omega,t) = \sqrt{\frac{2}{\pi}}\,\frac{3}{9+\omega^2}\,e^{-4\omega^2 t}$$

Taking the inverse Fourier cosine transform

$$\xrightarrow[\text{Page 518}]{(1b)} \quad u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{u}_c(\omega,t)\cos\omega x\,d\omega$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{3}{9+\omega^2}\,e^{-4\omega^2 t}\cos\omega x\,d\omega$$

$$= \frac{6}{\pi}\int_0^\infty \frac{1}{9+\omega^2}\,e^{-4\omega^2 t}\cos\omega x\,d\omega.$$